Thematisches Seminar – FS 2020

Topics in Fractal Geometry

Topic	Description	Date	Bibliography	Names
1	Space filling curves Basic Measure theory	21.02.	[4, chp. 2.1, 2.2] [1, chp. 1.1]	T. Soultanis S. Wenger
2	Hausdorff measure Hausdorff dimension	28.02.	[1, chp. 1.2, 1.3, 1.5] [3, chp. 2, 4], [4, chp. 2.3, 2.4]	T. Soultanis S. Wenger
3	Box counting dimension, packing measures/dimension	06.03.	[2, chp. 2.1, 2.2] [3, chp. 5]	?? ??
4	Mass distribution principle Basic examples of fractals	13.03	[2, chp. 4.1]	J. Hilken ??
5	Densities of sets and measures	20.03	[3, chp. 6], [4, chp. 2.5]	M. Blaise D. Clerc
6	Self-similar sets and iterated function systems	27.03	[2, chp. 9.1, 9.2]	M. Cotting A. Corrent
7	Capacity, Frostman's lemma product formulae	03.04.	[3, chp. 8] [2, chp. 4.3, 7], [1, chp. 5]	J. Münch D. Marti
8	Structure of sets of non-integral dimension	24.04.	[1, chp. 4]	M. Blaise D. Clerc
9	Structure of sets of integral dimension	01.05.	[1, chp. 3]	M. Cotting A. Corrent
10	Projections of fractals	08.05.	[3, chp. 3, 9], [1, chp. 6.4]	J. Hilken ??
11	Julia and Mandelbrot sets	15.05.	[2, chp. 14]	J. Münch D. Marti

References

[1] K. Falconer: *The geometry of fractal sets.* Cambridge Tracts in Mathematics, 85. Cambridge University Press, Cambridge, 1986.

[2] K. Falconer: *Fractal geometry. Mathematical foundations and applications.* Third edition. John Wiley & Sons, 2014.

[3] P. Mattila: *Geometry of sets and measures in Euclidean spaces.* Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

[4] S. Wenger: *Introduction to Geometric Measure Theory.* Lecture notes for course at the University of Fribourg.

Outlines

Topic 1:

- Hölder maps and a Hölder continuous space filling curve
- Outer measures and metric outer measures
- Measurable sets
- Basic properties of outer measures
- Borel sets

Topic 2:

- Definition of Hausdorff measure
- Basic properties such as metric outer measure and Borel regularity
- Behavior under Hölder mappings
- Hausdorff measure of a rectifiable Jordan curve
- Definition of Hausdorff dimension
- Standard middle third Cantor set
- $5r$–covering theorem
- Vitali covering theorem
- Lebesgue differentiation theorem

Topic 3:

- Box dimension
  - Definition using balls and equivalent definition using boxes
  - Examples: Box dimension of Cantor set, Sierpinski triangle, $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$
  - Basic properties of box dimension: monotonicity, finite stability, open sets, behavior under Lipschitz maps, $\dim_B(\overline{F}) = \dim_B(F)$
  - Comparison with Hausdorff dimension
- Packing measure and dimension

Topic 4:

- Mass distribution principle
- Middle third Cantor set revisited
- Product of middle third Cantor set with line segment.
- Variants of Cantor set
- Sierpinski triangle and porosity
- (Von Koch curve is biLipschitz equivalent to snowflaked interval)

Topic 5:

- Definition of upper and lower density of a set
- Inequalities for upper density of a set
- Maybe example of a very porous set
- Densities of measures and comparison with Hausdorff measure
- Density theorems for packing measures

Topic 6:

- Iterated function systems IFS
- Existence and uniqueness of invariant sets (attractors) of IFS
- Examples: von Koch curve, Sierpinski triangle, etc
- Hausdorff dimension of invariant sets of IFS

Topic 7:

- Riesz capacity and capacitary dimension
- Comparison of Hausdorff and capacitary dimension
- Frostman's lemma in $\mathbb{R}^n$
- Dimension of products
- (Mention) Existence of subsets of finite Hausdorff measure
- If time permits, a few words on Frostman's lemma in metric spaces

Topic 8:

- Non-existence of densities
- Zero lower angular density
- Non-existence of weak tangents
- A few words on sets in higher dimensions

Topic 9:

- Rectifiable curves and countably 1-rectifiable sets (Y-sets)
- Existence of densities and weak tangents for rectifiable curves and countably 1-rectifiable sets
- Structure of continua with finite Hausdorff 1-measure
- Regular 1-sets are countably 1-rectifiable

Topic 10:

- The Grassmannian of m-planes and the natural measure on it
- Projections of sets of Hausdorff dimension at most m onto m-planes
- Projections of sets of Hausdorff dimension larger than m onto m-planes
- Projections of planar regular 1-sets onto lines

Topic 11:

- Normal families and Montel's theorem
- Basic complex dynamics and the Julia set of a polynomial
- Basic properties of Julia sets
- Dimension estimates of the Julia set of a quadratic function
- The Mandelbrot set and some basic properties