Ideally, all infinite type surfaces can be triangulated.

Alan McLeay and Hugo Parlier[*]

Abstract. We show that any surface of infinite type admits an ideal triangulation. Furthermore, we show that a set of disjoint arcs can be completed into a triangulation if and only if, as a set, they intersect every simple closed curve a finite number of times.

1. Introduction

Topological surfaces are often thought of as the result of pasting together polygons. Provided you have enough topology, pants decompositions are a natural way of decomposing (orientable) surfaces, or conversely one can build a surface by pasting together 3 holed spheres (pants) along their cuffs. This latter point of view has the advantage of being naturally related to underlying moduli spaces via the so-called length and twist Fenchel-Nielsen coordinates which parametrize the space of (marked) hyperbolic metrics of the topological surface (Teichmüller space). Alternatively, provided an underlying finite type surface has punctures, ideal triangulations are also used to parametrize the same space of metrics using shear coordinates. The corresponding topological construction is the process of pasting triangles along their sides and then forgetting the vertices which become ideal points (which geometrically correspond to cusps).

When passing to (orientable) surfaces of infinite type, the pants decomposition description is often used for describing how to construct these surfaces: one simply pastes together infinitely many pairs of pants to obtain a connected surface. This natural point of view has been very useful in the Teichmüller theory of these surfaces (see for instance [1, 3, 11]). When wanting to do the same thing for ideal triangulations, an immediate difficulty appears. In the finite type, an ideal triangulation is a maximal collection (with respect to inclusion) of disjoint simple arcs with end points in the ends [8, 9]. In particular any collection of disjoint arcs can be completed into a triangulation. As we shall see, this no longer works in the infinite type case.

In this short note, we observe that one can overcome this apparent difficulty.

Theorem 1.1. *Any orientable surface of infinite type admits an ideal triangulation.*

Before outlining the proof we make note of the following restriction for such ideal triangulations.

Theorem 1.2. *An ideal multiarc is a subset of an ideal triangulation if and only if it intersects any simple closed curve a finite number of times.*

Hence, an ideal triangulation is a multiarc maximal with respect to inclusion among all multiarcs which intersect any given simple closed curve a finite number of times. And as a consequence:

Corollary 1.3. *Any ideal triangulation of an orientable surface must contain only finitely many arcs between any distinct pair of ends.*

Following this observation, the first step in proving Theorem 1.1 is to show that we can triangulate a one-ended surface with countably many boundary components. We then show that any infinite type surface can by decomposed into such subsurfaces.

2. Setup

By an orientable surface X of infinite type, we mean an orientable topological surface with infinitely generated fundamental group. The homeomorphism type of a such a surface is determined by its genus and space of ends.

An end is a nested sequence $\{U_i\}_{i\in\mathbb{N}}$ of subsurfaces of X (so $U_i \supset U_{i+1}$). Two ends $\{U_i\}_{i\in\mathbb{N}}$, $\{V_j\}_{j\in\mathbb{N}}$ are equivalent if for any i_0, there exists a j_0 such that $U_{i_0} \subset V_{j_0}$ and vice versa.

If each U_i contains genus then we say the end is *nonplanar*, and *planar* otherwise. An end $e = \{U_i\}_{i\in\mathbb{N}}$ is *isolated* if there exists some subsurface U_{i_0} such that every nested sequence containing U_{i_0} is equivalent to e. Usually, we call an isolated planar end a puncture or cusp.

We adopt the standard definition of an arc between ends (see for instance [5], and note this is more general than the definition in [7] which is adapted for flip graphs).

Definition 2.1. An arc of X is a proper embedding $\alpha : \mathbb{R} \to X$ such that the each of its ends correspond to an end of X. Note that since X is orientable, each arc has exactly two sides.

Arcs play an important role in dynamical and topological considerations on infinite type surfaces, see for instance [4, 6]. The definition above means that there exist ends $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_j\}_{j\in\mathbb{N}}$ (not necessarily distinct) such that for all i_0, there exists $t_0 \in \mathbb{R}$ such that $\alpha(t) \in U_{i_0}$ for all $t > t_0$, and there exists $\{V_j\}_{j\in\mathbb{N}}$ such that for all j_0, there exists s_0 with $\alpha(t) \in V_{j_0}$ for all $t < s_0$. An arc is non-trivial if its complementary region does not contain a topological disk. This is equivalent to there not existing an end $\{U_i\}_{i\in\mathbb{N}}$ such that for any $i_0 \in \mathbb{N}$, the arc can be freely homotoped to an arc lying entirely inside U_{i_0}.

Curves (or simple closed curves) are much easier to define: they are just embeddings of S^1. They are non-trivial if they are not freely homotopic to a point or a puncture.

We are only interested in curves and arcs up to (free) homotopy.

A pants decomposition is a collection of dijoint curves whose complementary regions are homeomorphic to thrice punctured spheres. A corollary of Richards' classification theorem for infinite type surfaces [10] is the following:

Corollary 2.2. *Any surface X can be obtained by pasting together pants. Equivalently, any surface can be decomposed into a pants decomposition by cutting along disjoint simple closed curves.*

Observe that dual to a pants decomposition is a (possibly infinite) trivalent (or cubic) graph.

Definition 2.3. An ideal triangulation of X is a collection of arcs $\mu = \{\mu_i\}_{i \in \mathbb{N}}$ such that each component of $X \setminus \mu$ is an open disk and each disk is bounded by at most three sides of arcs.

We end this section with a triangulation of the flute surface (see [2]).

Proposition 2.4. *There exists an ideal triangulation of the flute surface.*

Proof. First we take the standard Farey triangulation T of an open disk D. By removing a single point from each triangle of T we arrive at a surface $F \subset D$ homeomorphic to the flute surface (see Figure 1).
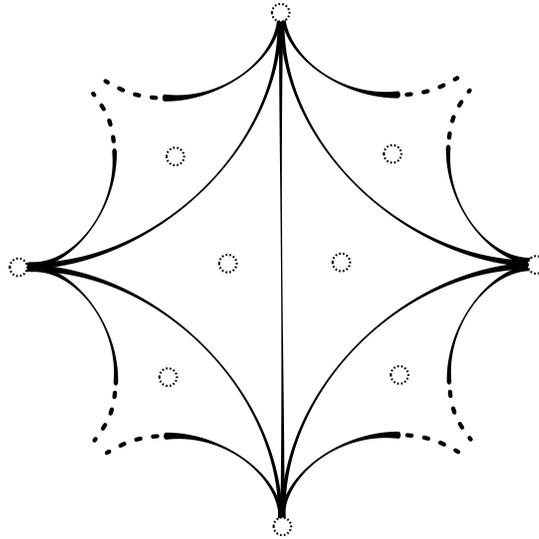
Figure 1: The punctured Farey triangulation

While the projection of T onto F is not a triangulation, it can be extended to one by triangulating each connected component of $F \setminus T$. Each such component is homeomorphic and can be easily triangulated as in Figure 2. □
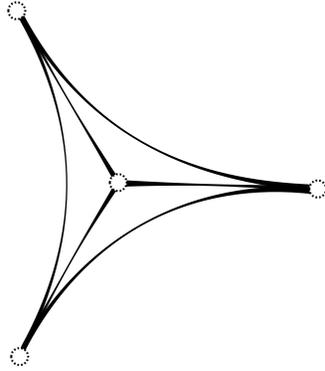
Figure 2: A triangulation of a punctured triangle

3. Triangulations galore

Our construction for more general triangulations draws inspiration from the example above.

3.1. Perforated Farey Triangulation

To obtain a perforated Farey triangulation, you start with the Farey triangulation of the hyperbolic plane and you add a puncture or a boundary curve to a collection of triangles (see Figures 3 and 4), and so that each arc leaves a puncture or a boundary curve on either side.
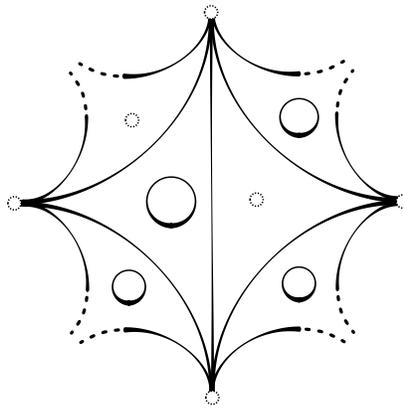
Figure 3: A perforated Farey triangulation

Note that with this condition, no two arcs are isotopic and any arc leaves infinitely many punctures or boundary curves on either of its sides.

From now on we will use the term flute surface to refer to any surface obtained in this way. That is, our flute surfaces may have compact boundary components as opposed to just punctures.

The following lemma is illustrated by Figure 5.

Lemma 3.1. *Any two one holed triangles pasted together along their holes can be triangulated using 6 arcs.*
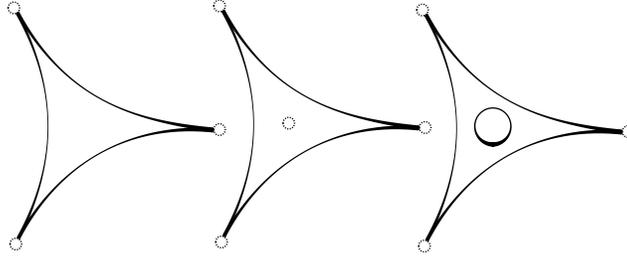
Figure 4: Triangles may be left as triangles, have a puncture, or a boundary curve

Figure 5: Triangulating a cylinder with three ideal vertices on each boundary

A key observation for our purposes is that any surface can be decomposed into subsurfaces homeomorphic to flute surfaces.

Lemma 3.2. *Any pants decomposition contains a collection of curves Γ such that $X \setminus \Gamma$ is a disjoint union of flute surfaces.*

Proof. Consider P, a pants decomposition of X. Let G_P be its dual graph. The edges of G_P are the curves of P, and the vertices are the pants. The degree of every vertex is 3, except for vertices which correspond to pants with one or two boundary curves which are isolated planar ends, and whose degree is either two or one.

Now choose a spanning tree T_0 of G_P. Every vertex of G_P belongs to T_0, but certain edges do not. Let E_0 be the set of edges of G_P that do not belong to T and let Γ_0 be the corresponding curves of P. We set $X_0 = X \setminus \Gamma_0$. Note that by construction X_0 is of genus 0, and we have a subset of P that is a pants decomposition of X_0 and whose dual graph is T_0. Vertices of T_0 are either of degree 1, 2 or 3: those of degree 1 and 2 correspond to pants bounding either isolated planar ends, or curves in Γ_0.

The next step in the construction will be iterated.

If T_0 is one ended, then X_0 is a flute surface. If not, it contains an infinite geodesic c_1 (for the graph metric). As T_0 is a tree, each of its edges is separating. If an edge does not separate two infinite trees,

we call it irrelevant. Otherwise it is relevant. Note that for instance if T_0 is an infinite trivalent tree, every edge is relevant. On the contrary, if T_0 is one ended, all edges are irrelevant.

We consider the set of all edges of T_0 that are both adjacent to c_1 and relevant and denote by $\Gamma_1 \subset \Gamma_0$ the corresponding multicurve. Note that $X_0 \setminus \gamma_1 = U \sqcup V$ where U is homeomorphic to a two ended flute surface. We set $X_1 = V$ and repeat the above procedure, possibly infinitely many times.

The end result of the procedure is a multicurve $\Gamma' = \cup_{i \in I} \Gamma_i \subset P$ where I is an index set (possibly empty, possibly infinite) such that $X \setminus \Gamma'$ is a collection of one or two ended flute surfaces. To obtain a collection of one ended flute surfaces, it suffices to add to Γ' a single relevant curve for each of the two end ended flute surfaces, which results in the multicurve Γ as desired. □

3.2. Triangulating any surface

Here we prove the following, which is a slight generalization of Theorem 1.1.

Theorem 3.3. *Let X be an infinite type surface with boundary consisting in a collection of simple closed curves. Then there exists an ideal multiarc T such that $X \setminus T$ is a collection of triangles or one holed monogons.*

Proof. Let Γ be the collection of curves described by Lemma 3.2 and let F be one of the flute components of $X \setminus \Gamma$. We then take a perforated Farey triangulation of F and extend it to F by adding three arcs for each isolated planar end, and a single arc for each boundary component of X. Denote by T' the resulting multiarc.

By construction each non-triangular component of $X \setminus T'$ is either a one holed monogon or two triangles pasted together along a simple closed curve. By Lemma 3.1 we can extend T' to a multiarc T satisfying the properties of the theorem. □

4. Completing ideal multiarcs into triangulations

We can now prove Theorem 1.2 which we restate for convenience.

Theorem 4.1. *A collection of arcs μ_0 can be completed into a triangulation if and only if $i(\mu_0, \alpha) < \infty$ for any simple closed curve α.*

Proof. First, assume α and μ_0 have infinite intersection. Let z be a point in α accumulated by intersection points with μ_0. Now, if μ is a triangulation of X containing μ_0 then z cannot belong to any disk in $X \setminus \mu$, as this would imply infinitely many arcs in μ_0 intersect this disk. Similarly, z cannot belong an arc δ in μ, as this would imply that infinitely many arcs of μ_0 intersect the union of the two disks bounded by δ.

To prove the other direction, we construct a triangulation containing μ_0. We begin by looking at $X \setminus \mu_0$ to see if it contains any finite type pieces (hence bounded by arcs in μ_0). If so, we triangulate

the pieces that are not already triangles, and we set X_0 to be this collection of triangles. If not, X_0 is the empty set.

Let P be a pants decomposition of X. The main idea is to add arcs to μ_0 such that they contain triangles that contain every curve in P. To do so we give curves in P an order (so $P = \cup_{i \in \mathbb{N}} \alpha_i$). Now we proceed inductively to construct a multiarc μ_i, and a subsurface X_i, which will be triangulated by arcs of μ_i and which contains all curves $\alpha_1, \ldots, \alpha_i$. By construction μ_i will contain μ_{i-1}. The induction starts with X_0 and μ_0.

We now describe the inductive step. Let $i > 0$. There are two cases for α_i, depending on whether it intersects μ_{i-1}.

Case 1: α_i and μ_{i-1} are disjoint.

First, suppose Y is a component of $X \setminus \alpha_i$ such that Y and μ_{i-1} do not intersect. (For this to happen, note that α_i must be a separating curve.) Then from Theorem 3.3 we can construct a multiarc in Y whose complement is a collection of triangles and single one holed monogon containing ∂Y. We can add this multiarc to μ_{i-1} and look at another component.

Note that unless $\mu_{i-1} = \varnothing$, at least one component of $X \setminus \alpha_i$ must intersect μ_{i-1}.

We now consider Y, a component of $X \setminus \alpha_i$ which intersects μ_{i-1}. Suppose it has a connected boundary (hence α_i is again separating). Let a be an embedding of $[0, 1]$ such that $a \cap \mu_{i-1} = a(0)$, and $a(1) \in \partial Y$. Now, let γ be the arc of μ_{i-1} intersecting a and let b be a component of $\gamma \setminus a$. Finally, define γ_1 be the boundary of a regular neighborhood of $a \cup b \cup \partial Y$. Note that, up to homotopy, γ_1 is an arc disjoint from μ_{i-1} and $Y \setminus \gamma_1$ contains a one holed monogon.

If Y contains arcs in μ_{i-1} and does not have connected boundary (and in particular α_i is nonseparating) then add γ_1 to μ_{i-1} and repeat the above step in order to define at an arc γ_2 associated to the second boundary component.

We therefore arrive at two arcs γ_1, γ_2 that are disjoint from μ_{i-1} such that the component Z of $X \setminus \{\gamma_1, \gamma_2\}$ containing α_i is of finite type. Let ζ_{i-1} be a triangulation of Z. We then define μ_i to be the union of $\mu_{i-1}, \gamma_1, \gamma_2$, and ζ_{i-1}. We appropriately define X_i to be the subsurface spanned by μ_i, that is, $X_i = X_{i-1} \cup Z \cup \{\gamma_1, \gamma_2\}$.

Case 2: α_i and μ_{i-1} intersect.

Let $\Gamma = \alpha_i \cup \mu_{i-1}$ and let $N(\Gamma)$ be a regular neighborhood of Γ. Define $A(\Gamma)$ to be the set of essential arcs in $\partial N(\Gamma)$ and define $C(\Gamma)$ to be the set of essential curves in $\partial N(\Gamma)$. Note that since $i(\mu_{i-1}, \alpha_i) < \infty$ both $C(\Gamma)$ and $A(\Gamma)$ are finite. For each curve in $C(\Gamma)$, repeat the argument from case 1 to arrive at a multiarc μ'_{i-1} and a subsurface X'_{i-1}. Now, there exists a subsurface $X_i \subset X$ such that $X'_{i-1} \cup A(\Gamma) \subset X_i$ and $X_i \setminus X'_{i-1}$ is finite type. We can therefore extend μ'_{i-1} to a triangulation μ_i of X_i. Note that X_i containts α_i as desired.

The surfaces $\cup_{i \in \mathbb{N}} X_i$ clearly form an exhaustion of X. We set $T = \cup_{i \in \mathbb{N}} \mu_i$, which by construction is

thus a triangulation of X. □

References

[1] Alessandrini, Daniele, Liu, Lixin, Papadopoulos, Athanase, Su, Weixu and Sun, Zongliang. On Fenchel-Nielsen coordinates on TeichmÃijller spaces of surfaces of infinite type. Ann. Acad. Sci. Fenn. Math. 36 (2011), no. 2, 621–659.

[2] Basmajian, Ara. Hyperbolic structures for surfaces of infinite type. Trans. Amer. Math. Soc. 336 (1993), no. 1, 421–444.

[3] Basmajian, Ara and Šarić, Dragomir. Geodesically complete hyperbolic structures. Math. Proc. Cambridge Philos. Soc. 166 (2019), no. 2, 219–242.

[4] Bavard, Juliette. Hyperbolicité du graphe des rayons et quasi-morphismes sur un gros groupe modulaire. Geom. Topol. 20 (2016), no. 1, 491–535.

[5] Durham, Matthew Gentry, Fanoni, Federica and Vlamis, Nicholas. Graphs of curves on infinite-type surfaces with mapping class group actions. Ann. Inst. Fourier (Grenoble) 68 (2018), no. 6, 2581–2612.

[6] Fanoni, Federica, Ghaswala, Tyrone and McLeay, Alan. Homeomorphic subsurfaces and the omnipresent arcs, Ann. H. Lebesgue, to appear.

[7] Fossas, Ariadna and Parlier, Hugo. Flip graphs for infinite type surfaces. Groups Geom. Dyn., to appear.

[8] Korkmaz, Mustafa and Papadopoulos, Athanase. On the ideal triangulation graph of a punctured surface. Ann. Inst. Fourier (Grenoble) 62 (2012), no. 4, 1367–1382.

[9] Penner, Robert. The decorated Teichmüller space of punctured surfaces. Comm. Math. Phys. 113 (1987), no. 2, 299–339.

[10] Richards, Ian. On the classification of noncompact surfaces. Trans. Amer. Math. Soc. 106 (1963), 259–269.

[11] Šarić, Dragomir. Fenchel-Nielsen coordinates on upper bounded pants decompositions. Math. Proc. Cambridge Philos. Soc. 158 (2015), no. 3, 385–397.

Address:

Department of Mathematics, University of Luxembourg, Luxembourg

Email: alan.mcleay@uni.lu, hugo.parlier@uni.lu