A note on collars of simple closed geodesics

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Abstract

For compact hyperbolic Riemann surfaces, the collar theorem gives a lower bound on the distance between a simple closed geodesic and all other simple closed geodesics that do not intersect the initial geodesic. Here it is shown that there are two possible configurations, and in each configuration there is a natural collar width associated to a simple closed geodesic. If one extends the natural collar of a simple closed geodesic α by $\varepsilon > 0$, then the extended collar contains an infinity of simple closed geodesics that do not intersect α.

A *surface* will always be a compact Riemann surface equipped with a metric of constant curvature -1. We consider surfaces to be without boundary and denote their genus by g. Such a surface is always locally isometric to the hyperbolic plane $\mathbb{H}$. A surface will be represented by S and distance on S (between points, curves or other subsets) by $d_S(\cdot,\cdot)$. A curve is considered non-oriented and primitive. In this respect, the length of the curve and the curve itself will not be denoted differently. A curve γ on S is called dividing if $S \setminus \gamma$ is not connected. Let the set of all simple closed geodesics on S be denoted by $\mathbb{G}(S)$. S can be decomposed into $2g-2$ Y-pieces (surfaces of signature $(0,3)$) by cutting along $3g-3$ pairwise disjoint closed geodesics. Such a collection of curves is called a partition. A partition can be viewed as either a collection of $3g-3$ simple closed geodesics or a collection of $2g-2$ Y-pieces. A Y-piece (α,β,γ) can be decomposed into two isometric right-angled hyperbolic hexagons with non-adjacent edges of lengths $\alpha/2, \beta/2$ and $\gamma/2$. These lengths determine the hexagon up to isometry (i.e. [2], [3]). If H is a right-angled hexagon with a, b and c non-adjacent sides, and $\tilde{c}$ the remaining edge adjacent to a and b, the following formula holds:

$$\cosh c = \sinh a \sinh b \cosh \tilde{c} - \cosh a \cosh b. \tag{1}$$

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The collar theorem (i.e. [1], [4] and [6]) states essentially the following.

Theorem 1 *Let γ_1 and γ_2 be non-intersecting simple closed geodesics on S. Then the collars*

$$\mathcal{C}(\gamma_i) = \{p \in S \mid d_S(p, \gamma_i) \leq w(\gamma_i)\}$$

of widths

$$w(\gamma_i) = \operatorname{arcsinh}(1/\sinh \frac{\gamma_i}{2})$$

are pairwise disjoint for $i = 1, 2$. Furthermore, each $\mathcal{C}(\gamma_i)$ is isometric to the cylinder $[-w(\gamma_i), w(\gamma_i)] \times \mathbb{S}^1$ with the metric $ds^2 = d\rho^2 + \gamma_i^2 \cosh^2\!\rho \, dt^2$.

Notice that α divides its collar into two connected spaces which we will call *half-collars.*

An immediate consequence of this theorem is a lower bound on the minimal distance between two non-intersecting geodesics.

Corollary 2 *Let γ_1 and γ_2 be geodesics as above. Then*

$$d_S(\gamma_1, \gamma_2) > \operatorname{arcsinh}(1/\sinh \frac{\gamma_1}{2}) + \operatorname{arcsinh}(1/\sinh \frac{\gamma_2}{2}).$$

Let us return to S with a given partition $\mathcal{P}$ and suppose that $\gamma \in \mathcal{P}$. The geodesic γ is thus the boundary of either one (case 1) or two (case 2) distinct Y-pieces in $\mathcal{P}$. Thus γ is the interior geodesic of a surface of signature $(1, 1)$ in the first case, and is the interior geodesic of a surface of signature $(0, 4)$ in the second case. Let $\delta \in \mathbb{G}(S)$ such that γ is the only geodesic in $\mathcal{P}$ that intersects δ. Furthermore, δ can be chosen such that $\operatorname{int}(\gamma, \delta) = 1$ in case 1 and $\operatorname{int}(\gamma, \delta) = 2$ in case 2. Let $k \in \mathbb{Z}$ and let the result of k Dehn twists around δ on γ be denoted by $\mathcal{D}_{k,\delta}(\gamma)$. Notice that for any k, $\mathcal{P}' = \{\mathcal{P} \setminus \gamma\} \cup \mathcal{D}_{k,\delta}(\gamma)$ is a partition on S. The convexity of geodesic length functions along earthquake paths [5] implies that k can be chosen so that $\ell(\mathcal{D}_{k,\delta}(\gamma))$ is arbitrarily large. Define the minimal distance $\delta(\alpha)$ between a geodesic α and all other geodesics on S that do not intersect α:

$$\delta(\alpha) = \inf\{d_S(\alpha, \beta) \mid \beta \text{ is a simple closed geodesic on } S \text{ such that } \alpha \cap \beta = \emptyset\}.$$

Corollary 2 implies that $\delta(\alpha) \geq \operatorname{arcsinh}(1/\sinh \frac{\alpha}{2})$ [7]. We now show:

Proposition 3 *Let $\alpha \in \mathbb{G}(S)$. Then if α is a dividing geodesic on a surface of genus 2 then*

$$\delta(\alpha) = \operatorname{arcsinh}(1/\sinh \frac{\alpha}{4}).$$

Otherwise

$$\delta(\alpha) = \operatorname{arcsinh}(1/\sinh\frac{\alpha}{2}).$$

Proof: Suppose that α is not a dividing geodesic of a surface of genus 2. Then there exists β and γ distinct geodesics in $\mathbb{G}(S)$ with (α,β,γ) a Y-piece. Complete the three into a partition. The distance between α and β (denoted $\ell_{\alpha\beta}$ here) is, using formula 1, given by the following equation:

$$\cosh \ell_{\alpha\beta} = \frac{\cosh\frac{\gamma}{2} + \cosh\frac{\alpha}{2}\cosh\frac{\beta}{2}}{\sinh\frac{\alpha}{2}\sinh\frac{\beta}{2}}.$$

The above considerations (applying Dehn twists to elements of $\mathcal{P}$) imply that β can be replaced by a geodesic of length as large as wanted. Call the length of the geodesic obtained x. It is easy to see in this case that

$$\delta(\alpha) \leq \lim_{x\longrightarrow\infty} \operatorname{arccosh}\frac{\cosh\frac{\gamma}{2} + \cosh\frac{\alpha}{2}\cosh\frac{x}{2}}{\sinh\frac{\alpha}{2}\sinh\frac{x}{2}} = \operatorname{arcsinh}(1/\sinh\frac{\alpha}{2}).$$

Corollary 2 completes the equality in this case.

Now if α is a dividing geodesic of a genus 2 surface, α divides S into two pieces of signature $(1,1)$. Thus $\beta \in \mathbb{G}(S)$ with $\alpha \cap \beta = \emptyset$ is the interior geodesic of one of the two Q-pieces and the distance between α and β is given by

$$\cosh \ell_{\alpha\beta} = \frac{\cosh\frac{\beta}{2} + \cosh\frac{\alpha}{2}\cosh\frac{\beta}{2}}{\sinh\frac{\alpha}{2}\sinh\frac{\beta}{2}}.$$

A simple calculation shows that $\ell_{\alpha\beta} > \operatorname{arccosh}(\frac{1+\cosh\frac{\alpha}{2}}{\sinh\frac{\alpha}{2}}) = \operatorname{arcsinh}(1/\sinh\frac{\alpha}{4})$. Using the same technique and notations as above we have

$$\delta(\alpha) \leq \lim_{x\longrightarrow\infty} \operatorname{arccosh}\frac{\cosh\frac{x}{2} + \cosh\frac{\alpha}{2}\cosh\frac{x}{2}}{\sinh\frac{\alpha}{2}\sinh\frac{x}{2}} = \operatorname{arcsinh}(1/\sinh\frac{\alpha}{4}),$$

and this proves the first equality. □

When one considers half-collars around α instead of collars, what we have effectively shown is the following. If a half-collar is included in the interior of a surface of signature $(1,1)$, then the optimal width is $\operatorname{arcsinh}(1/\sinh\frac{\alpha}{4})$. By optimal it is meant that for $\varepsilon > 0$, if one widens the half-collar by ε, then an infinity of simple closed geodesics that do not intersect α intersect the half-collar. If the half-collar is not included in a surface of signature $(1,1)$, then the optimal value is $\operatorname{arcsinh}(1/\sinh\frac{\alpha}{2})$. In a nutshell, there are two natural half-collar widths, and if one increases the width of these collars, then an infinite number of simple closed geodesics enter the half-collar without crossing it.

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