PANTS DECOMPOSITIONS OF RANDOM SURFACES

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Abstract. Our goal is to show, in two different contexts, that "random" surfaces have large pants decompositions. First we show that there are hyperbolic surfaces of genus g for which any pants decomposition requires curves of total length at least $g^{7/6-\varepsilon}$. Moreover, we prove that this bound holds for most metrics in the moduli space of hyperbolic metrics equipped with the Weil-Petersson volume form. We then consider surfaces obtained by randomly gluing euclidean triangles (with unit side length) together and show that these surfaces have the same property.

Any surface of genus g, $g \geq 2$, can be decomposed into three-holed spheres (colloquially, pairs of pants). We say that a surface has pants length $\leq l$ if it can be divided into pairs of pants by curves each of length $\leq l$. We say that a surface has total pants length $\leq L$ if it can be divided into pairs of pants by curves with the sum of the lengths $\leq L$. The pants length and total pants length measure the size and complexity of a surface. In particular, they describe how hard it is to divide the surface into simpler parts. One of the main open problems in this area is to understand how big the pants length of a genus g hyperbolic surface can be. It would also be interesting to understand how big the total pants length of a genus g hyperbolic surface can be. In this paper, we use a random construction to find hyperbolic surfaces with surprisingly large total pants length.

To put the paper in context, we review the known results about pants length and total pants length. In [Ber74, Ber85], Bers proved that for each genus g, the supremal pants length of a genus g hyperbolic surface is finite. This result is non-trivial because the moduli space of hyperbolic surfaces is not compact, and Bers's result gives information about the geometry of surfaces near the ends of moduli space. The supremal pants length of a genus g hyperbolic surface is called the Bers constant, $\mathcal{B}_g$. Later work gave explicit estimates for the Bers constant. In [BS92], Buser and Seppälä proved that every genus g hyperbolic surface has pants length at most Cg (where C is a constant independent of g). On the other hand, Buser [Bus81] gave examples of hyperbolic surfaces with pants length at least $cg^{1/2}$ for arbitrarily large g. Buser conjectured that the Bers constant of a hyperbolic surface is bounded by $Cg^{1/2}$ [Bus92].

The total pants length has not been studied as much as the pants length, but it also seems like a natural invariant. Since a pants decomposition has $3g-3$ curves in it, the estimate of Buser and Seppälä implies that every genus g hyperbolic surface has total pants length at most Cg^2. This is the best known general upper bound. In the other direction, it is easy to construct hyperbolic surfaces with total pants length at least cg for every g by taking covers of a genus 2 surface. The only previous non-trivial estimate comes from work of Buser and Sarnak on the geometry of certain arithmetic surfaces [BS94]. They proved that there exist families of surfaces, one in each genus g, with the property that every topologically non-trivial curve has length at least $\sim \log g$. Since each curve in a pants decomposition is non-trivial, the total pants length of these arithmetic hyperbolic surfaces is at least $\sim g \log g$.

Now we have the background to state our main theorem.

Theorem 1. For any $\varepsilon > 0$, a "random" hyperbolic surface of genus g has total pants length at least $g^{7/6-\varepsilon}$ with probability tending to 1 as $g \to \infty$. In particular, for all sufficiently large g, there are hyperbolic surfaces with total pants length at least $g^{7/6-\varepsilon}$.

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(To define a "random" hyperbolic surface we need a probability measure on the moduli space of hyperbolic metrics. We use the renormalized Weil-Petersson volume form. We discuss this notion of randomness more below.)

As another piece of context for our result, we mention the analogous questions for hyperbolic surfaces with small genus but many cusps. For simplicity, let's consider complete hyperbolic surfaces with genus 0 and n cusps for $n \geq 3$. These surfaces also have pants decompositions, and pants length is defined in the same way. The arguments of Buser and Seppälä show that such a surface has pants length at most Cn, and Buser's conjecture was that the pants length should be at most $Cn^{1/2}$. Balacheff and Parlier proved this conjecture in [BP09]. In a more general context in [BPS10], it is shown how to recuperate these results via Balacheff and Sabourau's diastolic inequality [BS10]. There are also very good estimates for the total pants length in this context. Balacheff, Parlier and Sabourau showed that the total pants length of a hyperbolic surface with genus 0 and n cusps is at most $Cn \log n$. It's easy to find examples where the total pants length is at least cn, so their bound is sharp up to logarithmic factors. The same authors went on to show that hyperelliptic genus g hyperbolic surfaces have total pants length at most $\sim g \log g$. The hyperbolic surfaces in Theorem 1 are very different from hyperelliptic surfaces or from surfaces of genus 0 with many cusps.

Our lower bound is a lot stronger than the one coming from the Buser-Sarnak estimate. Instead of improving the trivial bound by a factor of $\log g$, we improve it by a polynomial factor $g^{1/6-\varepsilon}$. Let's take a moment to explain why it is difficult to prove such a lower bound. Almost all the random hyperbolic surfaces we construct have diameter around $\log g$. Therefore, they have lots of non-trivial curves with length around $\log g$. For example, we can make a basis for the first homology of the surface using curves of length around $\log g$. So there are lots of short curves that look like good candidates to include in a pants decomposition. The key issue seems to be that the curves in a pants decomposition need to be disjoint. After we pick a first curve, the second curve needs to avoid it. This cuts down our options for the second curve. Then the third curve needs to avoid the first two curves, so we have even fewer options. If the pants length of the surface is near $g^{7/6}$, then the curves "get in each other's way" so much that we have to pick dramatically longer curves to finish building a pants decomposition. This kind of effect looks difficult to prove because there are so many choices about how to choose the early curves. At the present time, for any particular hyperbolic surface, we cannot prove that the total pants length is any larger than $g \log g$. But for a random hyperbolic surface, the pants length is usually much larger than $g \log g$.

Our proof is essentially a counting argument. If a surface has total pants length at most L, we can construct it by gluing some hyperbolic pairs of pants with total boundary length $\leq L$. A hyperbolic pair of pants is determined by its boundary lengths, so the number (really, Weil-Petersson volume) of possible surfaces with total pants length $\leq L$ is governed by the number of possible ways of choosing these hyperbolic pairs of pants and gluing them together. We estimate this volume and show that if $L \leq g^{7/6-\varepsilon}$, then it is much smaller than the total volume of moduli space.

To define the random hyperbolic surfaces above, we use the Weil-Petersson volume form on moduli space. Moduli space and the Weil-Petersson volume form are both fairly abstract, making it particularly hard to visualize what is going on. To help make things more down-to-earth, we will also consider an elementary random construction: randomly gluing together equilateral triangles to form a closed surface. Suppose that we take N Euclidean triangles of side length 1, where N is even. These triangles can be glued together to make a surface, and Gamburd and Makover [GM02] showed that this surface usually has genus close to $(N+2)/4$. We consider the set of triangulated surfaces obtained in this way. We identify surfaces that are simplicially isomorphic, and the equivalence classes form a kind of combinatorial "moduli space". We put the uniform measure on this finite set, so we can speak of a random combinatorial surface. (This definition goes back to Brooks and Makover [BM04], who studied the first eigenvalue of the Laplacian and the systole of random combinatorial surfaces.)

Theorem 2. For any $\varepsilon > 0$, a random combinatorial surface with N triangles has total pants length at least $N^{7/6-\varepsilon}$ with probability tending to 1 as $N \to \infty$.

The proof of Theorem 2 is closely parallel to the proof of Theorem 1. In some ways, the proof is not as clean, but on the other hand the proof is completely elementary and self-contained.

In a similar spirit, Brooks and Makover suggested that for large N, the counting measure on the combinatorial moduli space above may be close to the Weil-Petersson measure on the moduli space of hyperbolic metrics. We're not sure how to phrase this in a precise way, but there does seem to be a strong analogy between combinatorial surfaces with N triangles and hyperbolic surfaces of genus g when $N = g/4$. For example, there are roughly $N^{N/2} \approx g^{2g}$ different surfaces with N triangles, comparable to the volume of moduli space, which is roughly g^{2g}.

The combinatorial metrics we study in Theorem 2 are not hyperbolic, so let us mention what is known about the pants lengths of arbitrary metrics on a surface of genus g. Here the natural question is how the pants length relates to the area. Generalizing his work with Seppälä, Buser proved that any (not necessarily hyperbolic) genus g surface with area A has pants length at most $Cg^{1/2}A^{1/2}$ [Bus92]. It's easy to give examples of surfaces with area A and pants length at least $CA^{1/2}$. Generalizing his conjecture for hyperbolic metrics, Buser conjectured that any metric on a surface with area A should have pants length at most $CA^{1/2}$. Recall that the systolic inequality for surfaces says that any surface of genus $g \geq 1$ and area A has a non-trivial curve of length at most $CA^{1/2}$. If it's true, Buser's conjecture about pants lengths would greatly strengthen the systolic inequality. For an introduction to systolic geometry, see Chapter 4 of [Gro07].

In both cases, hyperbolic and combinatorial, our results seem to be based primarily on genus. Just as spheres with cusps have much smaller pants length than high-genus hyperbolic surfaces with the same area, small-genus random surfaces, e.g. random combinatorial surfaces conditioned to have genus 0, might have simpler geometry than high-genus random surfaces.

The study of small-genus random surfaces is an active area in probability theory especially in recent years, and it has been an active area in physics for a long time. A lot of study has been paid to random metrics on S^2, and one of the key ideas is that there is essentially only one really natural random metric on S^2. Following this idea, one conjectures that different procedures for producing a random metric on S^2 should give the same result. In addition to the example above, where one glues N triangles together and rejects any result which isn't a sphere, one can begin with the standard metric on S^2 and change it by a randomly chosen conformal factor (the conformal factor being a Gaussian free field). Conjecturally, for large N, the probability distribution coming from gluing triangles converges to the probability distribution for the random conformal factor. For an introduction to these issues, see the survey article [LG07].

There are many open questions related to the work in this paper. The central problem, understanding the maximal pants length of a genus g hyperbolic surface, remains wide open. We found random surfaces to be useful in studying the total pants length, and there are also many questions about pants length for random surfaces. For instance, what is the average pants length of a random genus g surface, and is the average pants length close to the Bers constant? Do the pants lengths of random genus g surfaces concentrate near a single value? One also has the analogous questions for total pants length.

In several ways, arithmetic surfaces have properties similar to random surfaces. How does the pants length of an arithmetic surface compare with the pants length of a random surface? In particular, is it true that an arithmetic surface has pants length at least $g^{1/6-\varepsilon}$?

Many of the same questions arise for random combinatorial surfaces along with some new ones. Most random combinatorial surfaces have large genus, but we can also consider the set of combinatorial surfaces where the genus g is fixed, but the number of triangles N increases. If g is small, say $g = 0$, and we look at large values of N, then we get something quite similar to a "random planar map" studied in probability and physics - see [LG07]. If we take g large and N to be roughly

$4g$, it seems that we get some kind of approximation to the moduli space of hyperbolic metrics. There are many questions about the geometry of such a random surface - what is its systole, first eigenvalue of Laplacian, or its Uryson width? What are its isoperimetric properties? How many balls of various radii are needed to cover it? What are its pants length and total pants length? And so on.

In the first section of the paper, we review the topology of pants decompositions. In the second section, we review some key background theorems about the Weil-Petersson volume form, and we use them to prove Theorem 1. In the third section, we introduce the combinatorial moduli space and prove Theorem 2. In the last section, we mention some open problems.

Notation: Many of the numbers we will be concerned with are super-exponential. For two numbers $A(x)$ and $B(x)$ that depend on a variable x, $A(x) \approx B(x)$ (resp. $A(x) \gtrsim B(x)$) will mean that they are equal (resp. the inequality holds) up to an exponential factor in x. For example, by Stirling's inequality, $g! \approx g^g$.

1. Topological types of pants decompositions

Pants decompositions come in different topological types. Let us fix a surface Σ. A pants decomposition determines a trivalent graph where each pair of pants corresponds to a vertex and two vertices are joined by an edge if the corresponding pants share a boundary. (This trivalent graph may have multiple edges or loops.) We call this graph the *topological type* of the pants decomposition. We say that two pants decompositions are topologically equivalent if their topological types are isomorphic graphs. It's straightforward to check that if two pants decompositions are topologically equivalent, then there is a diffeomorphism of Σ taking one to the other.

For example, if Σ is a surface of genus 2, then it has two topological types of pants decomposition. The two types each correspond to trivalent graphs with two vertices. In the first case, there are three edges that go between the two vertices. In the second case, each vertex has one edge connecting it to the other vertex and one loop connecting it to itself.

The first result that we need is an estimate for the number of different topological types of pants decomposition on a surface of genus g. In [Bol82], Bollobas gave a precise asymptotic formula for the number of trivalent graphs. We need only the following cruder version of Bollobas's formula.

Lemma 1. *(Bollobas) There are $\approx n^n$ trivalent graphs with $2n$ vertices.*

This lemma is cruder than Bollobas's result, and it also has a simpler proof. For reference, we include a proof here.

Proof. We start with $2n$ labeled tripods and consider all the ways to glue them together to produce a trivalent graph. The tripods have total degree $6n$, and there are

$$\frac{(6n)!}{(3n)!2^{3n}} \approx n^{3n}$$

ways of dividing the $6n$ half-edges into pairs, each of which corresponds to a trivalent graph with vertices numbered $v_1, \ldots, v_{2n}$. An unlabeled trivalent graph occurs many times in this collection. The permutation group S_{2n} acts on the set of labeled graphs by permuting the labels, and each orbit of the permutation group consists of isomorphic graphs. The number of equivalence classes is thus at most the number of orbits. If G is a labeled graph, recall that its orbit has $\operatorname{Card} S_{2n} / \operatorname{Card} S_{2n}^G$ elements, where S_{2n}^G is the stabilizer of G. The stabilizer of G consists of permutations of the vertices which lead to an isomorphic graph. We can describe such a permutation in terms of the image of a basepoint in G and a permutation of the neighbors of each vertex, so

$$1 \leq \operatorname{Card} S_{2n}^G \leq 2n6^{2n} \lesssim 1.$$

Hence each orbit has $\approx n^{2n}$ elements, and the number of orbits is $\approx n^n$. □

Consequently, a pants decomposition of a genus g surface has one of $\approx g^g$ possible topological types.

2. The moduli space of hyperbolic metrics

In this section we show that a random hyperbolic metric on a genus g surface has total pants length at least roughly $g^{7/6}$ with very high probability. To begin, let us define what we mean by a random metric and make a precise statement.

We denote the moduli space of closed hyperbolic surfaces of genus g by $\mathcal{M}_g$. The Weil-Petersson metric is a Riemannian metric on $\mathcal{M}_g$. We use the volume form of the Weil-Petersson metric to define volumes on moduli space. By renormalizing the Weil-Petersson volume form, we get a probability measure on moduli space. We take random metrics with respect to this probability measure.

Theorem 1. *For any $\varepsilon > 0$, a random metric in $\mathcal{M}_g$ has total pants length at least $g^{\frac{7}{6}-\varepsilon}$ with high probability: the probability tends to 1 as $g \to \infty$.*

Let us indicate the plan of our proof.

First, we observe that without loss of generality, we may assume that the curves in the pants decomposition are closed geodesics. Suppose we begin with a pants decomposition of a hyperbolic surface Σ. By definition, the pants decomposition consists of disjoint embedded curves $\gamma_1, ..., \gamma_{3g-3}$ so that each component of the complement is a three-holed sphere. This is equivalent to just requiring that the curves γ_i are disjoint and non-parallel: no two curves among them bound an annulus. Following a standard trick, we 'tighten' the curves γ_i: we replace each curve γ_i with a closed geodesic $\bar{\gamma}_i$ in its free homotopy class. By standard arguments in hyperbolic geometry, the closed geodesics will be embedded and disjoint, and no two of them will bound an annulus. Hence the curves $\bar{\gamma}_i$ give a pants decomposition of Σ also, and they have smaller lengths than the curves γ_i.

From now on we assume that the curves in the pants decomposition are closed geodesics. So each three-holed sphere in the pants decomposition has a hyperbolic metric with geodesic boundary. We call such a pair of pants with such a metric a *hyperbolic pair of pants*. The classification of hyperbolic pairs of pants is easy to descibe: the boundary curves may have any positive lengths, and for each choice of lengths there is a unique metric.

Now our plan consists of describing all the ways to glue together hyperbolic pairs of pants to make a closed hyperbolic surface, and then estimating the volume in moduli space covered by these surfaces.

To prove our theorem, we use two fundamental facts about the Weil-Petersson volume form. Recall that the Teichmüller space $\mathcal{T}_g$ denotes the space of hyperbolic metrics on a fixed surface of genus g, where two metrics are equivalent if they are related by an isometry *isotopic to the identity*. The moduli space $\mathcal{M}_g$ is the quotient of Teichmüller space $\mathcal{T}_g$ by the action of the mapping class group. The Weil-Petersson metric on Teichmüller space is a non-complete Kähler metric with negative sectional curvature, and is very much related to the hyperbolic geometry of surfaces. Although it is a very natural metric to consider, it is quite technical to define, so we refer the reader to [Wol03] for details. The Weil-Petersson metric is invariant under the action of the mapping class group, so it descends to a metric on moduli space.

We will need the following result of Schumacher and Trapani [ST01].

Background Theorem 1 (Asymptotic volume growth)**.** The volume of moduli space $\mathcal{M}_g$ grows (up to an exponential factor) like g^{2g}; i.e.,

$$\operatorname{Vol} \mathcal{M}_g \approx g^{2g}.$$

This result was an improvement of previous lower [Pen92] and upper bounds [Gru01].

The second background theorem expresses the Weil-Petersson volume form in a set of Fenchel-Nielsen coordinates. Before stating the result, we quickly recall Fenchel-Nielsen coordinates.

Fix a pants decomposition of the genus-g surface. We denote the curves in the pants decomposition by $\gamma_1, ..., \gamma_{3g-3}$. Recall that l_i and τ_i, the length and twist parameters, define coordinates on the Teichmüller space $\mathcal{T}_g$. The length parameter l_i measures the length of the shortest curve homotopic to γ_i in the given metric; this is a positive real number. The twist parameter τ_i measures the twist in the gluing across this geodesic; this is a real number measured in units of length. Different length and twist parameters may correspond to the same point in $\mathcal{M}_g$; for instance, replacing τ_i by $\tau_i \pm l_i$ yields a metric isometric to the original one by a Dehn twist around γ_i.

The volume form for the Weil-Petersson metric has a very simple form in terms of these coordinates [Wol03].

Background Theorem 2 (Wolpert)**.** In Fenchel-Nielsen coordinates, the volume form of the Weil-Petersson metric is simply the standard volume form $dl_1 \wedge ... \wedge dl_{3g-3} \wedge d\tau_1 \wedge ... \wedge d\tau_{3g-3}$.

The region in moduli space with total pants length less than L *in our fixed pants decomposition* is covered by the following region of Teichmüller space:

$$S = \{(l_i, \tau_i) \in T_g | \sum_i l_i \leq L, 0 \leq \tau_i \leq l_i\}$$

The Weil-Petersson volume of this set is the same as its volume in the standard Euclidean metric on $\mathbb{R}^{6g-6}$ and it is not hard to estimate.

Lemma 2. *If $1 \leq L \leq exp(g)$, then $\operatorname{Vol} S \lesssim (L/g)^{6g}$, where $\lesssim$ is taken with respect to g.*

Proof. First consider the $(3g-3)$-dimensional simplex defined by the inequalities $0 < l_i$, $\sum_i l_i \leq L$. We denote this simplex by Δ_L. By Fubini's theorem,

$$\operatorname{Vol} S = \int_{\Delta_L} \prod_{i=1}^{3g-3} l_i.$$

By the arithmetic-geometric mean inequality,

$$\prod l_i \leq (\frac{L}{3g-3})^{3g-3} \lesssim (\frac{L}{g})^{3g}.$$

Hence $\operatorname{Vol} S \lesssim \operatorname{Vol}(\Delta_L)(L/3g)^{3g}$.

The volume of the simplex Δ_L may be calculated inductively using the formula for the volume of a pyramid. It is equal to $\frac{L^{3g-3}}{(3g-3)!} \lesssim (L/g)^{3g}$. □

Remark. The calculation in this lemma is basically sharp: the region of Teichmüller space above has volume $\approx (L/g)^{6g}$. The region of moduli space covered by this region of Teichmüller space has volume $\lesssim (L/g)^{6g}$. (The volume in moduli space may be much smaller if the covering map is highly non-injective. We don't know how to estimate this effect.)

Now let $\mathcal{M}_g(\leq L) \subset \mathcal{M}_g$ denote the subset of hyperbolic metrics that admit pants decompositions of total length $\leq L$.

If E denotes a topological type of pants decomposition, we let $\mathcal{M}_g(\leq L, E) \subset \mathcal{M}_g(\leq L)$ denote the the subset of hyperbolic metrics that admit a pants decomposition of type E and total length $\leq L$. For each E, the calculation in Fenchel-Nielsen coordinates shows that the volume of $\mathcal{M}_g(\leq L, E)$ is $\lesssim (L/g)^{6g}$. There are $\approx g^g$ different topological types E. Every pants decomposition belongs to one of these $\approx g^g$ types, and so the volume of $\mathcal{M}_g(\leq L)$ is $\lesssim L^{6g} g^{-5g}$.

As we saw above, the volume of $\mathcal{M}_g$ is $\approx g^{2g}$. If we set $L = g^{\frac{7}{6}-\varepsilon}$, then we see that the volume of $\mathcal{M}_g(\leq L) \lesssim g^{(2-6\varepsilon)g}$. Recalling the definition of $\lesssim$ and $\gtrsim$, we see that the volume of $\mathcal{M}_g$ is at least $ce^{-cg}g^{2g}$, while the volume of $\mathcal{M}_g(\leq g^{(7/6)-\varepsilon})$ is at most $Ce^{Cg}g^{(2-6\varepsilon)g}$. So for g sufficiently

large, the volume of $\mathcal{M}_g$ is much larger than the volume of $\mathcal{M}_g(\leq g^{(7/6)-\varepsilon})$. This proves Theorem 1.

3. The combinatorial viewpoint

If N is even, one can construct a oriented surface by gluing together N triangles (we allow edges of the same triangle to be glued together and allow edges to form loops). We call the corresponding CW-complex a combinatorial surface. We declare two combinatorial surfaces to be equivalent if there is a homeomorphism which sends edges to edges and faces to faces and we define Comb_N to be the set of equivalence classes of such surfaces with N triangles. Gamburd and Makover [GM02] showed that as $N \to \infty$, a random element of Comb_N has genus at least $(1/4 - \varepsilon)N$ with high probability, so Comb_N is a rough combinatorial equivalent of $\mathcal{M}_{N/4}$.

We think of each triangle in a combinatorial surface as a Euclidean equilateral triangle with side length 1. In this way, each combinatorial surface in Comb_N has a metric on it. In particular, we can define its pants length and total pants length. We also have a good notion of a random combinatorial surface given by the counting measure on Comb_N. Using these definitions, we get the following combinatorial version of our main result.

Theorem 2. *For any $\varepsilon > 0$, a random combinatorial surface in Comb_N has total pants length at least $N^{7/6-\varepsilon}$ with probability tending to 1 as $N \to \infty$.*

The proof of Theorem 2 is morally analogous to the proof of Theorem 1. It is more elementary, because it does not rely on the Weil-Petersson metric, but there are also some additional subtleties.

One of the key observations in the proof of Theorem 1 was that a hyperbolic surface of total pants length L can be cut into *hyperbolic pairs of pants* whose boundaries have total length at most $2L$. The main subtlety in the proof of Theorem 2 is to find the right combinatorial analogue for this step.

Suppose we start with an arbitrary pants decomposition of a combinatorial surface Σ. The curves γ_i are arbitrary curves, and so the pairs of pants in the decomposition do not have combinatorial structures. To get a combinatorial decomposition, we need the γ_i to be combinatorial curves. For each i, we can approximate γ_i by a combinatorial curve $\bar{\gamma}_i$ which is homotopic to γ_i and has comparable length. At this point, some problems appear: the curves $\bar{\gamma}_i$ need not be embedded and need not be disjoint. If the $\bar{\gamma}_i$ are chosen carefully, however, we can still express Σ as the union of combinatorial pairs of pants, glued along the $\bar{\gamma}_i$. A combinatorial pair of pants will consist of triangles and edges, but some of the edges may not border any of the triangles. These isolated edges can be thought of as 'infinitely thin' pieces of surface. We will show that we can choose the $\bar{\gamma}_i$ to be combinatorial geodesics, so that each one has minimal length compared to all combinatorial curves in its homotopy class.

At this point, there is a further problem. A hyperbolic pair of pants is determined by the lengths of its boundary curves. But there are many different combinatorial pairs of pants with the same boundary lengths. When we count how many ways we can glue together combinatorial pairs of pants with given boundary lengths, we get a large unwanted factor coming from the different choices for a pair of pants with fixed boundary curves. The underlying cause of this problem is that combinatorial geodesics — unlike hyperbolic ones — are not unique. The solution to this problem is to consider only special combinatorial pants decompositions which we call 'tight pants decompositions'. We define these below. Roughly speaking, they are pants decompositions of minimal complexity in an appropriate sense.

With this well-chosen definition, the analogy runs smoothly. In Section 3.1, we prove that Comb_N has cardinality $\approx N^{N/2}$, but that for any fixed g, the number of N-triangle combinatorial surfaces of genus g grows only exponentially. In Section 3.2, we introduce combinatorial pants decompositions and tight combinatorial pants decompositions. We show that any pants decomposition can be

improved to make a tight combinatorial pants decomposition. In Section 3.3, we count the number of tight combinatorial pants decompositions of total length $\leq L$.

3.1. **Counting combinatorial surfaces.** The goal of this part is to count surfaces that lie in our combinatorial moduli space. Our main goal is to prove that the cardinality of Comb_N is $\approx N^{N/2}$. Since most surfaces in Comb_N have genus close to $N/4$, this is analogous to the fact that the Weil-Petersson volume of moduli space of surfaces of genus g is $\approx g^{2g}$.

Over the course of our argument, we will need to consider the set of combinatorial surfaces with a particular genus. Let $\mathrm{Comb}_N\mathrm{Mod}_{g,k}$ denote all combinatorial surfaces of genus g with k boundary components made from N triangles. Again, we will allow two edges of the same triangle to be glued together, and again we consider surfaces up to homeomorphisms preserving edges and faces.

For fixed g and k, we will see that the cardinality of $\mathrm{Comb}_N\mathrm{Mod}_{g,k}$ grows only exponentially with N.

We begin by studying the cardinality of Comb_N.

Lemma 3 (Combinatorial volume growth)**.**

$$\mathrm{Card}(\mathrm{Comb}_N) \approx N^{N/2}.$$

Proof. If $\Sigma \in \mathrm{Comb}_N$, we can construct a trivalent graph with N vertices by letting the vertices of the graph be the faces of Σ and connecting vertices whose corresponding faces share an edge. Given a trivalent graph, we can construct a surface with the corresponding pattern of gluings, so by Lemma 1, $\mathrm{Card}(\mathrm{Comb}_N) \gtrsim N^{N/2}$.

On the other hand, many surfaces might correspond to the same graph, since the graphs do not record which of the three edges of each triangle is glued to which other edge. With this information, a graph uniquely identifies a surface, but there are only 6^N ways to add this information to the graph, so $\mathrm{Card}(\mathrm{Comb}_N) \lesssim N^{N/2}$ as well. □

In contrast, the number of ways to triangulate a surface of a fixed genus with many triangles grows only exponentially:

Lemma 4. *For any $g \geq 0$, $k \geq 0$,*

$$\mathrm{Card}(\mathrm{Com}_n\mathrm{Mod}_{g,k}) \precsim e^n.$$

In other words,

$$\mathrm{Card}(\mathrm{Com}_n\mathrm{Mod}_{g,k}) \leq C(g,k)e^n.$$

Proof. In the special case that $(g,k) = (0,1)$ (i.e., for triangulations of a disk), this follows from a result of Brown [Bro64]; we will reduce the general case to the case of a disk.

Brown counted the number of rooted simplicial triangulations of the disk, that is, triangulations with a marked oriented boundary edge such that the endpoints of each edge are distinct and no face is glued to itself, and showed that the number of such with $j+3$ boundary vertices and k interior vertices is

$$\frac{2(2j+3)!(4k+2j+1)!}{(j+2)!j!k!(3k+2j+3)!}.$$

Restating this in terms of the number $n = j+2k+1$ of triangles, we find that

$$\frac{2(2j+3)!(2n-1)!}{(j+2)!j!\left(\frac{n-j-1}{2}\right)!\left(\frac{3n+j-3}{2}\right)!},$$

where we require that $0 \leq j \leq n-1$ and $j \equiv n-1 \mod 2$. Call this number $\theta(n,j)$. Our combinatorial surfaces differ from Brown's in that the two endpoints of an edge may be identified, but any

element of $\mathrm{Com}_n\mathrm{Mod}_{0,1}$ can be barycentrically subdivided twice to get a simplicial triangulation. We thus find that

$$\mathrm{Card}(\mathrm{Com}_n\mathrm{Mod}_{0,1}) \leq \sum_{\substack{j=0 \\ j\equiv 6n-1 \pmod 2}}^{6n-1} \theta(6n,j).$$

Since $j \leq n-1$, there is a c such that

$$\theta(n,j) \leq e^{cn}\frac{(2n-1)!}{(\frac{n-j-1}{2})!(\frac{3n+j-3}{2})!} = e^{cn}(2n-1)\binom{2n-2}{\frac{n-j-1}{2}} \leq e^{cn}2^{2n-2}(2n-1),$$

so there is a c' such that for all $n > 0$,

$$\sum_{\substack{j=0 \\ j\equiv n-1 \pmod 2}}^{n-1} \theta(6n,j) \leq e^{c'n}.$$

Now consider $\mathrm{Comb}_N\mathrm{Mod}_{g,k}$. If Σ is a triangulated genus-g surface with k holes and $(g,k) \neq (0,0)$, we can cut it along non-separating simple closed curves or simple arcs between boundary components until we get a disk; this takes at most $2g+k$ cuts. We can thus obtain any element of $\mathrm{Comb}_N\mathrm{Mod}_{g,k}$ by performing at most $2g+k$ gluings on an element of $\mathrm{Com}_n\mathrm{Mod}_{0,1}$. Each gluing identifies two edge paths on the boundary, so a gluing is determined by the endpoints of the paths that are glued. There are at most $(3n)^4$ ways to perform each gluing, so as long as $(g,k) \neq (0,0)$,

$$\mathrm{Card}(\mathrm{Com}_n\mathrm{Mod}_{g,k}) \leq e^{c'n}(3n)^{2g+k} \precsim e^n.$$

For the case that $(g,k) = (0,0)$, note that if Σ is a triangulation of a sphere, then we can obtain a triangulation of the disc by cutting along an edge of Σ, so

$$\mathrm{Card}(\mathrm{Com}_n\mathrm{Mod}_{0,0}) \leq e^{c'n} \precsim e^n$$

as desired. □

3.2. **Combinatorial pants decompositions.** In this part, we define pants decompositions of combinatorial surfaces and their lengths. We then focus our interest on pants decompositions of minimal length, and we show that in the isotopy class of such a pants decomposition there is always a pants decomposition of a particular type, called a tight pants decomposition.

A pants decomposition of a surface of genus g is a maximal set of disjoint and freely homotopically distinct non-trivial simple closed curves. A pants decomposition always contains $3g-3$ curves, and its complementary region consists of a set of $2g-2$ three holed spheres, or pairs of pants. To define a pants decomposition in the combinatorial setting, we focus on these pairs of pants.

A combinatorial pair of pants will consist of a simplicial complex equipped with some boundary curves. Let Δ be a simplicial complex which is a deformation retract of a three-holed sphere $M_{0,3}$. If we consider Δ as a subset of $M_{0,3}$, this implies that a regular neighborhood of Δ is a three-holed sphere. The boundary curves of this three-holed sphere correspond to simplicial curves in the boundary of Δ, and when Δ is equipped with these boundary curves, we call it a *combinatorial pair of pants*. These curves inherit an orientation from $M_{0,3}$. Note that Δ need not be a manifold; for instance, it could be two vertices connected by three edges. In general, Δ may contain edges that are not boundaries of triangles. Such edges we call stranded.

We can glue pairs of pants to get surfaces. If $P_1, \ldots, P_{2g}$ are combinatorial pairs of pants, each one has three boundary components, and we can identify pairs of boundary components which have the same length. Like its geometric analogue, there are many ways to identify the same pair of boundary components, and these ways differ by a shift; by fixing a basepoint on each boundary curve, we can define twist parameters for the gluing. If we glue all of the boundary curves in pairs, we obtain a complex $\bigcup P_i/\sim$; this may not be a surface, because there may still be edges which are

not part of a triangle. If $\bigcup P_i/\sim$ is isomorphic to a triangulated surface Σ, we call the collection of the P_i, the gluing instructions, and the isomorphism a *combinatorial pants decomposition* of Σ. The boundary curves of the P_i project to simplicial curves on Σ; we call these the boundary curves of the pants decomposition. If the boundary curves have minimal (combinatorial) length in their free homotopy classes, we say that the pants decomposition is minimal.

We will show that a geometric pants decomposition gives rise to a minimal combinatorial pants decomposition. Recall that a Lipschitz closed curve on a triangulated surface is homotopic to a simplicial curve whose length is bounded by a constant times the length of the original curve:

Lemma 5. *Let $\alpha : S^1 \to \Sigma$ be a Lipschitz curve on a triangulated surface Σ. There is a simplicial curve λ on Σ which is homotopic to α and whose length satisfies*

$$\ell(\lambda) \leq 2\ell(\alpha).$$

Proof. For any $\varepsilon > 0$, we can perturb α to get a smooth curve α' of length $\ell(\alpha') \leq \ell(\alpha) + \varepsilon$ which avoids vertices of Σ and intersects its edges transversely. The edges of the triangulation cut α' into finitely many arcs, each contained in a single triangle. Each arc a cuts the boundary of its triangle into two arcs. We homotope the arc a to the shortest of these two boundary arcs (say b). By elementary euclidean geometry, we have $\ell(b) \leq 2\ell(a)$. As such, the resulting curve β satisfies $\ell(\beta) \leq 2\ell(\alpha)$.

The resulting curve is not necessarily a simplicial curve as it may go partway along an edge and then backtrack. A further homotopy removes this backtracking and decreases the length. The resulting curve λ is now a simplicial curve and has length at most the length of β. This proves the lemma. □

We can now focus our attention on the equivalent statement for full pants decompositions.

Proposition 1. *Let Σ be a triangulated surface and let $\alpha_1, \ldots, \alpha_k : S^1 \to \Sigma$ be the boundary curves of a pants decomposition for Σ. There is a minimal combinatorial pants decomposition of Σ with boundary curves $\lambda_1, \ldots, \lambda_k : S^1 \to \Sigma$ such that for all i, λ_i is homotopic to α_i and $\ell(\lambda_i) \leq 2\ell(\alpha_i)$.*

Proof. The first step is to approximate the α_i by simplicial curves. For all i, let β_i be a simplicial closed curve of minimal length which is homotopic to α_i. In general, the β_i may share edges; we will subdivide Σ and make them disjoint.

We first duplicate the edges of Σ so that the β_i do not share edges. If $e = (x, y)$ is an edge of Σ which occurs $n \geq 2$ times in the β_i, we replace e with $n - 1$ bigons glued edge-to-edge. We make no changes to edges which do not occur or occur only once. Now we replace the vertices of Σ; if a vertex has degree d, we replace it with a d-gon which we call a *vertex polygon*. Each incoming edge connects to one vertex of this d-gon; this makes the bigons created in the previous step into rectangles which we call *edge rectangles*. Each edge rectangle has two edges which are shared by vertex polygons and two edges which connect vertex polygons; we call the edges which connect vertex polygons *long edges*. Call the resulting complex Σ'. This complex is homeomorphic to Σ, and there is a natural map $p : \Sigma' \to \Sigma$ which collapses the edge rectangles to edges and the vertex polygons to vertices; this map sends long edges to edges of Σ homeomorphically.

We call a curve *semi-simplicial* if it consists of alternating long edges and curves in vertex polygons; the image of a semi-simplicial curve under p is thus a simplicial curve. If γ is a semi-simplicial curve, we define its length by $\ell(\gamma) := \ell(p(\gamma))$. We can lift each of the β_i to semi-simplicial curves in Σ' by replacing edges of β_i with corresponding long edges and replacing vertices of β_i with curves in the vertex polygons. There are enough long edges in Σ' that we can ensure that no long edge is used twice, so this gives us curves β_i' in Σ' which intersect only in the vertex polygons. We may assume that at most two curves intersect at a point and that all intersections are transverse.

A standard argument allows us to perform surgeries to make these curves disjoint. If two curves β_i' and β_j' intersect (where possibly $i = j$), the fact that the two curves are homotopic to disjoint

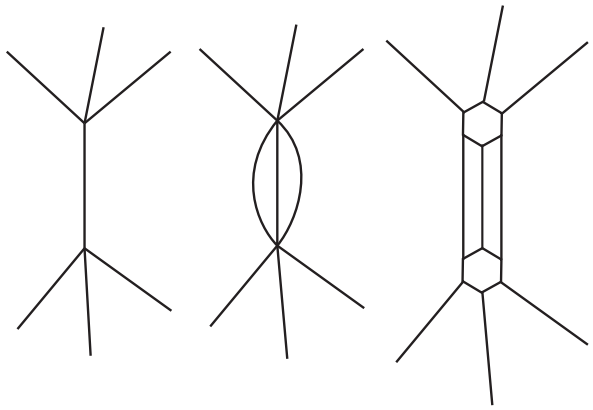

FIGURE 1. From Σ to Σ'

curves (or, in the case that $i = j$, to a simple curve) implies that there is a pair of intersection points x, y, a segment γ_1 of β'_i, and a segment γ_2 of β'_j, such that γ_1 and γ_2 connect x and y and are homotopic relative to their endpoints. Because β'_i and β'_j have minimal length, we have $\ell(\gamma_1) = \ell(\gamma_2)$. We can modify β'_i and β'_j by swapping γ_1 and γ_2 and deforming the resulting curves near x and y to eliminate those two intersection points. This operation reduces the number of intersection points by two, so we can repeat it to eliminate all intersection points. Since we have only swapped subsegments of curves and performed homotopies inside vertex polygons, the resulting curves are still semi-simplicial and still have minimal length; call them $\beta''_1, \dots, \beta''_k$.

We will get a combinatorial pants decomposition of Σ by cutting Σ' along these curves and collapsing edge rectangles and vertex polygons. The curves $\beta''_1, \dots, \beta''_k$ are the boundary curves of a geometric pants decomposition of Σ' into subsurfaces $P_1, \dots, P_{2k/3}$. Each of these subsurfaces is homeomorphic to a pair of pants (i.e., a three-holed sphere), and is the union of edge rectangles, subsets of the vertex polygons, and faces of Σ' which come from triangles of Σ. If we collapse each edge rectangle in P_i to an edge and each connected component of a vertex polygon to a vertex, we obtain a complex P'_i which comes with a map $p_i : P'_i \to \Sigma$. The boundary curves of P_i correspond to simplicial curves in P'_i, and with these boundary curves, P'_i forms a combinatorial pair of pants. Furthermore, gluing the P'_i along these boundary curves reconstructs the original surface Σ, making them a combinatorial pants decomposition of Σ. The boundary curves of this pants decomposition are $\lambda_i := p(\beta''_i)$; as required, λ_i is homotopic to α_i, and since it has minimal combinatorial length, its length is no more than a constant factor larger than that of α_i. □

Let Σ be a genus g surface and let $P_1, \dots, P_{2g-2}$ be the pants in a combinatorial pants decomposition of Σ. We can view a combinatorial pair of pants P as a collection of *clusters* connected by *strands*; indeed, we will construct a graph G whose vertices correspond to clusters of P and whose edges correspond to strands. We construct this graph as follows (see Fig. 2):

(1) For all vertices $v \in P$, if the link of v has d connected components and more than one is an interval, replace v by a star with d edges.
(2) Shrink paths of edges to single edges.
(3) Shrink groups of triangles which are glued along edges to single vertices.

Each vertex of G corresponds to a group of triangles (indeed, a submanifold of P with boundary) or a single point; we call these clusters. We will call a single-point cluster *degenerate*. Each edge corresponds to a path of stranded edges of P (possibly a path of length zero); we call these strands. Since each cluster corresponds to a vertex of G, we can define the *degree* of a cluster to be the degree of the corresponding vertex.

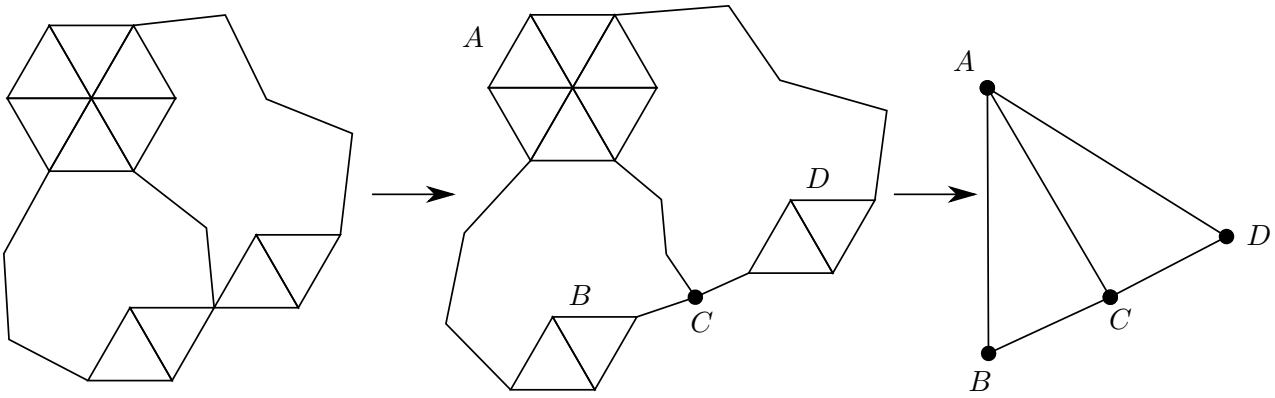

FIGURE 2. The cluster graph. This pair of pants has four disk-type clusters, one of which (C) is degenerate and two of which (B, D) are loose disks.

The interior of a cluster can be homeomorphic to a disk, a cylinder or a three holed sphere, and we call it disk-type, cylinder-type, or three-holed-sphere-type accordingly. If a cluster is a single point, we say it is disk-type. If the pants decomposition was minimal, then a disk-type cluster can have degree two, three, or four. A cylinder-type cluster has degree one or two, and a three-holed-sphere-type cluster has degree zero. A pair of pants P is called tight if none of its disk-type clusters have degree 2. Such a cluster will be called a loose disk. A minimal length pants decomposition is called tight if all of its pants are tight, i.e., do not contain any loose disks.

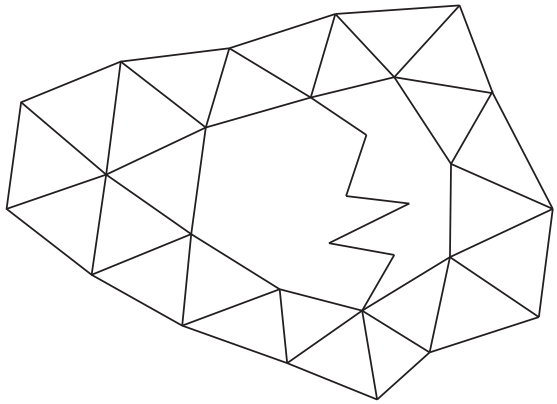

FIGURE 3. A pair of pants with a annulus-type cluster

We are now able to introduce the main result of this section.

Lemma 6. *Any combinatorial surface Σ admits a minimal length tight pants decomposition.*

Proof. We need to show that if a minimal pants decomposition has pants with loose disks we can isotope the curves to remove the disk from the pair of pants without increasing length. We further need to make sure that by doing so we are not just moving a loose disk somewhere else, and that in fact we will have reduced the number of loose disks.

The key technique in this proof is sliding a loose disk from one pair of pants to another. Let D be a loose disk which is part of a pair of pants P in a minimal pants decomposition, and say that P has boundary curves $\gamma_a : S_a \to P$, $a = 1, 2, 3$. Since D is a loose disk, there are two strands which enter D, say at vertices x_1 and x_2. These two vertices divide the boundary of D into two paths, and since the boundary curves of P have minimal length, the two paths have equal length.

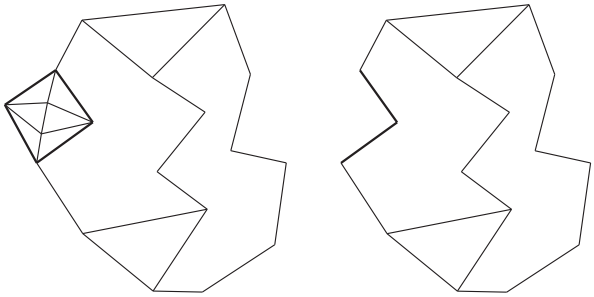

FIGURE 4. Removing a loose disk to get a tight pair of pants

We denote them $c, c' : [0, \ell] \to P$, and we can choose n and n' so that c is a subsegment of γ_n and c' is a subsegment of $\gamma_{n'}$. If P is glued to Q along γ_n, then we can slide that common boundary curve over D to transfer D from P to Q. This has the effect of cutting D out of P (leaving c') and gluing it on to Q along c, and it produces a new pants decomposition of Σ. We call this process *sliding c to c'*. Furthermore, since c and c' must have equal lengths, the lengths of the boundary curves are unchanged and the new pants decomposition is also minimal.

After such a slide, the pants decomposition differs from the original only around D. All the clusters of the original decomposition have counterparts in the new one, except possibly for clusters in P and Q; the move deleted one cluster from P and added at most one to Q, depending on whether D was glued to a cluster or to a strand. If D was glued to a cluster, then the move reduced the number of loose disks by one. Otherwise, the slide glues D to a strand of Q. In this case, after the slide D becomes a loose disc in Q bounded by segments c_1 and c_1'.

We will inductively define a sequence $X_0, X_1, \ldots, X_k$ of pants decompositions which all differ by slides. Each X_i except X_k will have a loose disk D_i in a pair of pants P_i. This disk will be isomorphic to D and which is bounded by two curves $c_i : [0, \ell] \to P_i$ and $c_i' : [0, \ell] \to P_i$. Recall that Σ is a quotient of the pairs of pants in X_i; let $\mu_i : P_i \to \Sigma$ be the restriction of the quotient map to P_i. We will require that $\mu_i \circ c_i : [0, \ell] \to \Sigma$ be the same curve for all $i < k$ and likewise for $\mu_i \circ c_i' : [0, \ell] \to \Sigma$, and we will define $f := \mu_i \circ c_i$ and $f' := \mu_i \circ c_i'$.

Let X_0 be the original decomposition of Σ; this has a loose disk $D_0 \cong D$ bounded by $c_0 := c$ and $c_0' := c'$. We construct X_{i+1} from X_i by sliding c_i to c_i'. If this reduces the number of loose disks, we stop, letting $k = i + 1$; otherwise, D_i corresponds to a loose disk of X_{i+1}, bounded by c_{i+1} and c_{i+1}', and we continue. We claim that this process eventually stops.

By way of contradiction, say that the process does not stop. Sliding c_i to c_i' affects the boundary curves of X_i by replacing an occurrence of f by f'. If the process does not stop, then we can replace f by f' infinitely many times. In particular, each edge of Σ occurs as many times in f as it does in f', so each edge of Σ occurs an even number of times in $\mu(\partial D)$ (indeed, either 0 or 2). Consequently, $\mu(D) = \Sigma$. Since μ is injective on the interior of D, this means that we can obtain Σ by gluing the edges of D together.

If $w = c(0)$ or $w = c(\ell)$, we call w an endpoint of D_0. We claim that for all $v \in \Sigma$, $\mu^{-1}(v)$ contains at most 2 non-endpoint vertices of D. Say v is such that $\{w_1, w_2, w_3\} \subset \mu^{-1}(v)$ for some 3 distinct non-endpoint vertices $w_1, w_2, w_3 \in D$. The link of v is a circle, and it contains 3 intervals corresponding to the links of the w_i. Let $S \subset \operatorname{link} v$ be the complement of the interiors of these intervals; this consists of three connected components, S_1, S_2, and S_3. If a length-2 segment of a boundary curve of X passes through v, there are j and k such that it approaches v from the direction of S_j and leaves v in the direction of S_k. We call this a jk-segment. Note that the only jk-segments of ∂D with $j \neq k$ are those centered at the w_i. Since D and the S_a's are unaffected by repeatedly sliding c to c', we can discuss jk-segments of the boundary curves of the X_i too.

One of the w_i is in the image of c; say $w_1 = c(x)$, so that f passes through v at x, and number the S_i so that this is a 12-segment. Replacing f by f' deletes this segment, and we claim that replacing f by f' decreases the number of 12-segments by one. The path f' has no 12-segments, so it only remains to check that replacing f by f' can't introduce new 12-segments centered at the endpoints of D. But if $f(0) = f'(0) = v$ or $f(\ell) = f'(\ell) = v$, then f and f' both leave v in the direction of the same S_i, so a jk-segment centered at an endpoint remains a jk-segment when f is replaced by f'. Since the number of 12-segments in X is finite, the process terminates after a finite number of slides.

Thus, Σ can be obtained by gluing a disc to itself along its edges; the resulting gluing has one face, namely D, ℓ edges, and at least $(\ell - 1)$ vertices. Thus, if the process does not terminate, then Σ has genus at most 1, which is a contradiction. □

3.3. **Counting tight pants decompositions.** The goal of this part is to count the number of tight pants decompositions of bounded length.

Main Estimate. There is a $c > 0$ such that the number of triangulated surfaces in Comb_N with genus g and (tight) pants decompositions of total length at most L is $\leq exp(cN)g^g(L/g)^{6g}$.

First we will count the number of different tight pairs of pants with boundary curves of controlled length. Next we will count the number of ways of gluing these pants together into a surface.

Lemma 7. *There is a $c_0 > 0$ such that the number of tight pairs of pants with boundary curves of lengths l_1, l_2, and l_3 and with A triangles is $\leq c_0 e^{c_0 A}$.*

Proof. A tight pair of pants consists of some clusters (which may be collections of triangles or may be single points) joined by some strands. There are several combinatorial types, including:

(1) One annulus-type cluster, joined to itself with one strand.
(2) Two disk-type clusters, joined by three strands running between them.
(3) One disk-type cluster, joined to itself by two strands.

What matters to us is that there are only a finite number of combinatorial types. To see this, we begin by observing that each cluster of triangles has to be a subsurface with genus 0 and at most 3 boundary components, so there are finitely many types of cluster. If c is the number of clusters, s is the number of strands, and b is the total of the first Betti numbers of the clusters in a pair of pants P, then the first Betti number of P, $\beta_1(P)$ is given by $\beta_1(P) = b + s - c + 1$. If P is tight, each disk-type cluster has degree at least 3, so each cluster contributes at least $1/2$ to $\beta_1(P)$. As such, there can be at most 2 clusters in a tight pair of pants and at most 3 strands, so there are at most, say, 100 types of pairs of pants.

We can now prove that for each combinatorial type, there are $\precsim e^A$ tight pairs of pants with A triangles. Since there are a finite number of combinatorial types, this will imply the lemma.

By Lemma 4, there is a $c > 0$ such that there are at most e^{cA} discs or annuli or three-holed spheres with A triangles. Now to build a tight pair of pants, we have to add strands to the triangulated surfaces. We have to choose the attaching points. There are at most six attaching points, and each attaching point has at most $3A$ choices, and so there are $\leq (3A)^6$ choices of attaching points. Once we have chosen where to attach each strand, the lengths of the strands are determined by the lengths l_1, l_2, and l_3 of the three boundary circles. Thus the total number of tight pants with fixed boundary lengths, a fixed combinatorial type, and A triangles is bounded by $100e^{2cA}(3A)^6 \precsim e^A$. □

Remark. In hyperbolic geometry, there is a unique hyperbolic pair of pants for every choice of boundary lengths l_1, l_2, and l_3. The closest combinatorial analogue of this phenomenon is the fact that tight pairs of pants with no triangles are determined by their boundary lengths. For every triple of lengths, $l_1, l_2, l_3 \in \mathbb{Z}$, there is a unique triangle-less pair of pants with the given lengths. If the lengths obey the triangle inequality, the graph looks like θ and if not the graph looks like a

pair of glasses, that is, two circles connected by a line. If we are not careful when we add triangles, the number of pairs of pants explodes: if we add one triangle, we have $\approx l_1 + l_2 + l_3$ different edges where we can put it, so we get many different pairs of pants. For this reason, we introduced tight pairs of pants; tightness restricts the possible places that a triangle can go. The number of tight pairs of pants with fixed boundary lengths is bounded by $\exp(A)$, with constant independent of the chosen boundary lengths. The $\exp(A)$ factor is fairly harmless, so tight pairs of pants are a good analogue of hyperbolic pairs of pants.

Recall that a combinatorial pants decomposition consisted of a collection of pants and some gluing information. Next we consider combinatorial analogues of length and twist parameters (i.e., Fenchel-Nielsen coordinates) and bound the number of possible ways to glue pants.

Lemma 8. *There is a C such that the number of genus g combinatorial pants decompositions with total pants length $\leq L$ and total area $\leq N$ is bounded by $\leq Cg^g(L/g)^{6g}exp(CN)$.*

Proof. In this proof, we will write $f(g, L, N) \lesssim h(g, L, N)$ to mean that there is some c such that $f(g, L, N) \leq ce^{cN}h(g, L, N)$ for all applicable g, L, N. Note that we may assume that $g \leq N/4$, and so $e^g \lesssim 1$ as usual.

As with hyperbolic pants decompositions, a genus-g combinatorial pants decomposition has a topological type. By Lemma 1, the number of topological types of genus g is $\approx g^g$.

Now we count tight pants decompositions with a fixed topological type. We first have to choose the lengths of the $3g-3$ boundary curves in the pants decomposition. How many ways can we choose positive integers $l_1, ..., l_{3g-3}$ so that $\sum l_i \leq L$? This number is less than the volume of the set $x_i \geq 0, \sum x_i \leq L$. The volume of that simplex can be computed by induction on the dimension; it has volume $\frac{1}{(3g-3)!}L^{3g-3} \lesssim (L/g)^{3g}$.

For each choice of lengths, we next have to choose how many triangles to put in each pair of pants. Here we have to choose $A_1, ..., A_{2g-2} \geq 0$ with $\sum A_i = N$. The number of ways to choose A_i is exactly $\binom{N+2g-3}{2g-3} \leq 2^{N+2g-3} \leq 4^N$. (Since $g \leq N/4$.) So the number of ways of choosing A_i is $\lesssim 1$.

Next we have to choose a tight pants structure for each pair of pants with the given area A_i and the given boundary lengths. If c_0 is the constant from Lemma 4, then the number of ways to do this is at most $\prod_i c_0 exp(c_0 A_i) \lesssim 1$.

Now we count the number of possible gluings. Since we already chose the topological type, a gluing is determined by its twist parameters. For each of the $3g-3$ curves, the twist parameter is an integer in the range $0 \leq t_i \leq l_i - 1$. The number of choices for the twist parameters is

$$\prod_{j=1}^{3g-3} l_j \leq (\frac{L}{3g-3})^{3g-3} \lesssim (L/g)^{3g}.$$

Multiplying all of these together, we find that the number of possible pants decompositions is

$$\lesssim g^g \left(\frac{L}{g}\right)^{3g} \left(\frac{L}{g}\right)^{3g} \lesssim g^g \left(\frac{L}{g}\right)^{6g},$$

as desired. □

In particular, the number of underlying surfaces (up to simplicial isomorphism) is $\lesssim (L/g)^{6g}g^g$. This proves the main estimate.

The total number of combinatorial surfaces in Comb_N is $\approx N^{N/2}$. If N is sufficiently large and $L = N^{7/6-\varepsilon}$, then the number of surfaces in Comb_N with genus ≥ 2 and total pants length at most L is

$$\lesssim \sum_{i=2}^{(N+2)/4} e^{cN}(L/i)^{6i}i^i \lesssim e^{cN}N^{(\frac{1}{6}-\varepsilon)\cdot\frac{6N}{4}}(N/4)^{\frac{N}{4}} \approx N^{N/2-3\varepsilon/2}.$$

The number of surfaces in Comb_N with genus < 2 is $\lesssim 1$ by Lemma 4, so for large N, most surfaces in Comb_N have no pants decomposition of total length $\leq L$. This finishes the proof of Theorem 2.

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