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Quasiconformal geometry and uniformization of metric surfaces

THESIS

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Abstract

The classical uniformization theorem states that every simply connected Riemann surface is conformally equivalent to the unit disc, the complex plane, or the Riemann sphere. Over the past few decades, there has been an increasing interest in extending this statement to non-smooth metric surfaces: Bonk and Kleiner [BK02] formulate conditions for a metric sphere that imply the existence of a quasisymmetric parametrization, and Rajala [Raj17] characterizes planar metric surfaces that allow parametrizations by geometrically quasiconformal maps. The goal of this thesis is to further extend these results within the context of metric surfaces of locally finite Hausdorff 2-measure. Furthermore, we will apply the newly established uniformization theorems to address questions in the field of geometric mapping theory.

The central idea behind the proofs of the uniformization theorems presented in this thesis is to construct parametrizations with good geometric and analytic properties as energy minimizers of certain classes of Sobolev mappings. This strategy provides a novel proof of the theorem of Bonk and Kleiner, as shown by the work of Lytchak and Wenger [LW20]. In a joint collaboration with Martin Fitzi [A], we prove a version of the theorem of Bonk and Kleiner [BK02] for geodesic metric surfaces of higher topology. One of the main goals of this dissertation is to generalize the above mentioned statements of Bonk–Kleiner and Rajala to general metric surfaces X of locally finite Hausdorff 2-measure. By only assuming that X is furthermore locally geodesic, we show that Jordan domains in X having finite boundary length admit a weakly quasiconformal parametrization; this is based on joint work with Stefan Wenger [B]. The result was extended in [C] to metric surfaces with possibly higher genus and non-empty boundary.

The results presented in the second part of this thesis highly depend on the existence of weakly quasiconformal parametrizations. We first establish a coarea inequality for Sobolev mappings on metric surfaces with locally finite Hausdorff 2-measure. This is based on joint work with Dimitrios Ntalampekos [D] and builds on the article [EIR23]. In [D], the coarea inequality serves as a tool to generate new Lipschitz-volume rigidity results for mappings between metric surfaces. As a corollary of the main theorem of [D], we obtain that every 1-Lipschitz mapping from a closed metric surface to a closed Riemannian surface of the same area is an isometry. Another powerful application of the above mentioned uniformization results is presented in joint work with Kai Rajala [E, F], where we introduce a novel approach to study the distortion of mappings between general metric spaces. In [E], we investigate the main properties of mappings of finite distortion from a metric surface X of locally finite Hausdorff 2-measure to $\mathbb{R}^2$. The article [F] is devoted to the question of how different definitions of distortion relate for such mappings from X to $\mathbb{R}^2$, resulting in yet another uniformization theorem for metric surfaces.

Zusammenfassung

Das klassische Uniformisierungstheorem besagt, dass jede einfach zusammenhängende Riemannsche Fläche konform äquivalent zur Einheitskreisscheibe, der komplexen Ebene oder der Riemannschen Sphäre ist. In den letzten Jahrzehnten nahm das Interesse an der Erweiterung dieser Aussage auf nicht-glatte metrische Flächen stark zu: Bonk und Kleiner [BK02] formulieren Bedingungen für eine metrische Sphäre, die die Existenz einer quasisymmetrischen Parametrisierung implizieren, und Rajala [Raj17] charakterisiert ebene metrische Flächen, die geometrisch quasikonforme Parametrisierungen erlauben. Das Ziel dieser Dissertation ist es, diese Resultate im Kontext metrischer Flächen mit lokal endlichem Hausdorff 2-Mass zu generalisieren. Ausserdem werden wir die neu etablierten Uniformisierungstheoreme anwenden, um Fragen aus dem Gebiet der geometrischen Abbildungstheorie zu bearbeiten.

Die Kernidee hinter den Beweisen der in dieser Dissertation vorgestellten Uniformisierungstheoreme besteht darin, Parametrisierungen mit guten geometrischen und analytischen Eigenschaften als Energieminimierer bestimmter Klassen von Sobolev-Abbildungen zu konstruieren. Diese Strategie bietet einen neuartigen Beweis für das Theorem von Bonk und Kleiner, wie die Arbeit von Lytchak und Wenger zeigt [LW20]. In Zusammenarbeit mit Martin Fitzi [A] beweisen wir eine Version des Theorems von Bonk und Kleiner [BK02] für geodätische metrische Flächen höherer Topologie. Eines der Hauptziele dieser Dissertation ist die Verallgemeinerung der oben genannten Sätze von Bonk–Kleiner und Rajala auf allgemeine metrische Flächen X mit lokal endlichem Hausdorff 2-Mass. Unter der alleinigen zusätzlichen Annahme, dass X lokal geodätisch ist, zeigen wir, dass Jordangebiete in X mit endlicher Randlänge eine schwach quasikonforme Parametrisierung erlauben; dies basiert auf gemeinsamer Arbeit mit Stefan Wenger [B]. Das Ergebnis wurde in [C] auf metrische Flächen mit möglicherweise höherem Genus und nicht-leerem Rand erweitert.

Die Ergebnisse, die im zweiten Teil dieser Dissertation vorgestellt werden, basieren wesentlich auf der Existenz von schwach quasikonformen Parametrisierungen. Zunächst wird eine Koflächenungleichung für Sobolev-Abbildungen auf metrischen Flächen mit lokal endlichem Hausdorff 2-Mass aufgestellt. Dies basiert auf einer Zusammenarbeit mit Dimitrios Ntalampekos [D] und baut auf dem Artikel [EIR23] auf. In [D] dient die Koflächenungleichung als Werkzeug, um neue Lipschitz-Volumen-Starrheitsresultate für Abbildungen zwischen metrischen Flächen zu erzeugen. Als Korollar des Hauptsatzes von [D] erhalten wir, dass jede 1-Lipschitz-Abbildung von einer geschlossenen metrischen Fläche auf eine geschlossene Riemannsche Fläche desselben Flächeninhalts eine Isometrie ist. Eine weitere wichtige Anwendung der oben erwähnten Uniformierungsresultate wird in gemeinsamer Arbeit mit Kai Rajala in [E, F] vorgestellt, worin ein neuer Ansatz zur Untersuchung von Verzerrungen von Abbildungen zwischen allgemeinen metrischen Räumen eingeführt wird. In [E] untersuchen wir die wichtigsten Eigenschaften von Abbildungen endlicher Verzerrung von einer metrischen Fläche X mit lokal endlichem Hausdorff 2-Mass nach $\mathbb{R}^2$. Der Artikel [F] ist der Frage gewidmet, in welchem Zusammenhang verschiedene Definitionen von Verzerrung für solche Abbildungen von X nach $\mathbb{R}^2$ stehen, woraus ein weiteres Uniformisierungstheorem für metrische Flächen resultiert.

Acknowledgements

First, I want to sincerely thank my advisor Stefan Wenger for continuously supporting and encouraging me. This work would not have been possible without his guidance, knowledge and valuable advice - not only on a mathematical level but also in navigating academia. Thank you for all the time and energy you invested in me and for always believing in my potential.

Over the past few years, I have had the chance to work with and learn from wonderful coauthors. I am grateful to Martin Fitzi, Kai Rajala, Dimitrios Ntalampekos and Noa Vikman their inspiring collaborations and many helpful discussions.

I would also like the referees of this thesis, Kai Rajala and Stephan Stadler, for being part of the committee, for their interest in this work, and for their helpful comments.

Beyond research, I have been fortunate to engage in outreach and teaching. The curiosity and joy of my students consistently motivated me. I am especially thankful to Matthieu Jacquemet for sharing his expertise in didactics and for all the meaningful discussions. I am also grateful to the members of the Department of Mathematics for their kindness and support.

Nicht zuletzt möchte ich mich zutiefst bei meiner Familie und meinen engsten Freund*innen bedanken. Die letzten Jahre haben mir gezeigt, wie privilegiert ich bin, ein so wunderbares Umfeld zu haben - und wie sehr meine emotionale Gesundheit von euch abhängt. Diese Arbeit wäre ohne eure konstante Unterstützung, eure unermüdlichen Ermutigungen, eure stetige Bereitschaft zuzuhören, euer geduldiges Ausharren an meiner Seite und eure Fähigkeit, mich immer wieder auf andere Gedanken zu bringen, nicht möglich gewesen.

Organization

This cumulative thesis consists of the following scientific articles.

[A] Martin Fitzi and Damaris Meier. *Canonical parametrizations of metric surfaces of higher topology.* Ann. Fenn. Math. **48** (2023), no.1, 67–80.

[B] Damaris Meier and Stefan Wenger. *Quasiconformal almost parametrizations of metric surfaces.* J. Eur. Math. Soc. (2024), published online first.

[C] Damaris Meier. *Quasiconformal uniformization of metric surfaces of higher topology.* Indiana Univ. Math. J. **73** (2024), no. 5, 1689–1713.

[D] Damaris Meier and Dimitrios Ntalampekos. *Lipschitz-volume rigidity and Sobolev coarea inequality for metric surfaces.* J. Geom. Anal. **34** (2024), no.5, Paper No. 128.

[E] Damaris Meier and Kai Rajala. *Mappings of finite distortion on metric surfaces.* Math. Ann. **391** (2025), 2479–2507.

[F] Damaris Meier and Kai Rajala. *Definitions of quasiconformality on metric surfaces.* Preprint arXiv:2405.07476 (2024).

All of the articles evolve around the central theme of uniformization of metric surfaces in the following sense. Articles [A, B, C] provide new uniformization results for metric surfaces with certain additional properties: depending on the topological and geometric characteristics of a metric surface X, the main results in [A, B, C] show the existence of an uniformization map, i.e. a map from a smooth surface M to X with good geometric and analytic properties. Articles [D, E, F] contain new results for mappings between metric surfaces and can be seen as applications of uniformization theorems for metric surfaces. This is due to the fact that the techniques used to prove the statements in [D, E, F] crucially depend on the existence and regularity of the uniformization map.

The following work also appeared throughout the time of my doctoral studies. However, as it is only tangentially related to the topic of metric surfaces, it will not be included in this thesis.

[G] Damaris Meier, Noa Vikman and Stefan Wenger. *Energy minimizing harmonic* 2*-spheres in metric spaces.* Preprint arXiv: 2503.08553 (2025).

Contents

1. Introduction

1.1. The uniformization problem for metric surfaces

A *metric surface* is a metric space homeomorphic to a *smooth surface*, i.e. a smooth compact oriented and connected Riemannian 2-manifold with possibly non-empty boundary. Non-smooth metric surfaces naturally arise for example as limits, deformations or boundaries of classical smooth objects. In recent years, the interest in understanding these objects has gained enormous interest and led to the development of a new area called *analysis on metric spaces*. The goal of this new field is to find ways to do first-order calculus on very general and a priori not smooth metric measure spaces, see e.g. [Hei01, HKST15, BCH+20]. The development of the area was partly driven by the need to make sense of quasiconformal mappings in non-smooth spaces, playing a key role for example in the proof of Mostow's rigidity theorem.

In order to be able to perform first-order calculus on a metric surface X one wishes to understand the regularity and geometric properties of X. One step in this direction is to find uniform structures on metric surfaces under certain assumptions, resulting in classifications of these objects. The central question around which this thesis revolves can be stated as follows.

Question 1.1.1 (Uniformization Problem for metric surfaces)**.** *Under which conditions on a metric space X homeomorphic to a model surface M does there exist a map $u\colon M \to X$ with (certain given) good geometric and analytic properties?*

The map u in Question 1.1.1 will usually be referred to as *uniformization map* or *parametrization* of X. The existence and regularity of such uniformization maps are of great significance within the field of analysis on metric spaces and moreover have important implications to related areas such as e.g. geometric group theory, see e.g. [Bon06]. This thesis will provide positive answers to Question 1.1.1 for very general classes of metric surfaces. Moreover, we will show how the existence of such uniformization maps can be used to provide further results within analysis on metric spaces and adjacent fields.

1.2. History of the two-dimensional uniformization problem

Uniformization in a two-dimensional setting has a long history. Questions and proof strategies related to the existence and regularity of uniformization maps already appeared in early 19th century. In 1822, the Royal Danish Academy of Sciences and Letters in Copenhagen posed as a prize question the problem of finding a map between arbitrary regions of smooth surfaces embedded in Euclidean $\mathbb{R}^3$ such that the image is "similar to the domain in infinitesimally small regions". In modern mathematical terms, this problem asks to show that any two smooth surfaces embedded in Euclidean $\mathbb{R}^3$ are locally conformally equivalent. Gauss [Gau25] solved this local uniformization problem in 1825. Recall that a map $u\colon M \to N$ between smooth surfaces $M, N \subset \mathbb{R}^3$ is *conformal* if its differential D_u is orientation and angle preserving. In particular, a conformal map sends infinitesimally small balls to infinitesimally small balls and is locally bi-Lipschitz, see Definition 2.1.1.

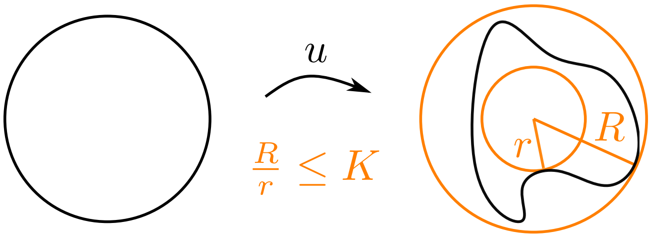


Figure 1.1.: Quasisymmetric and quasiconformal distortion of a (infinitesimal) ball.

In 1851, Riemann [Rie51] mentions in his thesis ideas on how to globally uniformize simply connected domains in the complex plane. Nowadays this is known as the Riemann mapping theorem: *every non-empty simply connected open set $U \subset \mathbb{C}$, $U \neq \mathbb{C}$, can be conformally mapped onto the open unit disk D.* The significance of the existence of these conformal maps is undisputed and finds applications in many areas of mathematics. Before the first rigorous mathematical proof of the Riemann mapping theorem was provided by Osgood [Osg00] in 1900, Poincaré [Poi82] and Klein [Kle83] independently conjectured the same generalization of the Riemann mapping theorem in 1882 and 1883, respectively. The conjecture was proven by Koebe [Koe07b, Koe07c, Koe07a] and Poincaré [Poi08] in 1907 and 1908, respectively. Today, this generalization is known as the classical uniformization theorem and can be phrased in modern terms as follows.

Theorem 1.2.1 (Classical uniformization theorem)**.** *Every simply connected Riemann surface is conformally diffeomorphic to the unit disc, the complex plane or the Riemann sphere.*

Every closed orientable Riemannian manifold has a simply connected universal cover. This fact together with Theorem 1.2.1 provide a classification of closed orientable Riemannian manifolds into hyperbolic, flat or spherical surfaces in the following sense. Every closed orientable Riemannian manifold can be equipped with a conformally equivalent Riemannian metric of constant sectional curvature -1, 0 or 1. For a more detailed historical survey on the Riemann mapping theorem and the classical uniformization theorem, see [Gra94, dSG16].

With the growth of the above mentioned field of analysis on metric spaces, the desire of possessing uniformization theorems for certain classes of metric surfaces has increased. The existence of conformal maps requires a high degree of regularity on domain and target. This is a priori not given in a general, non-smooth setting. Thus, one has to look for more flexible classes of mappings such as quasisymmetric or quasiconformal mappings. Quasisymmetric and quasiconformal maps distort relative shapes of sets in a controlled manner on a global and infinitesimal scale, respectively, as illustrated as in Figure 1.1. For precise definitions of quasisymmetry and geometric quasiconformality, we refer to Definition 3.1.1 and Definition 3.2.1, respectively. Uniformization of non-smooth metric surfaces will be discussed in more detail in Chapter 3.

1.3. Uniformization by minimizing energy

It is a standard approach to prove existence of conformal parametrizations of smooth surfaces through minimization of the energy of mappings onto the surface under consideration, see e.g. [Jos91, Chapter 3]. This proof strategy generalizes to a metric space setting in the following way.

Strategy 1.3.1. Let X be a metric space homeomorphic to some model surface M and with certain additional properties. Let E be a suitable notion of energy that is lower semicontinuous.

1. Define a class Λ of mappings from M onto X possessing desired regularity properties and show that the set Λ is not empty.
2. Take an energy *minimizing sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \Lambda$, i.e.
$$E(u_n) \to \inf\{E(v) : v \in \Lambda\} \quad \text{as } n \to \infty,$$
and show that, up to precomposition of suitable energy-invariant mappings $\varphi_n \colon M \to M$, there exists a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ converging to a map $u\colon M \to X$.
3. Show that the limiting map $u\colon M \to X$ is again an element of Λ.
4. By the lower semicontinuity of E, the map u is an energy minimizer in Λ.
5. The energy minimizing property of u can then be used to show that (a representative of) u possesses a high degree of regularity and thus fulfills additional geometric and analytic properties.

In the articles [A, B, C], Strategy 1.3.1 is applied to produce parametrizations of metric surfaces X with good geometric and analytic properties. We emphasize that in this general non-smooth setting, showing that the set Λ is non-empty, or equivalently, constructing a map from M to X of desired regularity, is often a very delicate and difficult part. In order to make Strategy 1.3.1 successful, we build on work of Lytchak and Wenger surrounding solving Plateau's problem in metric spaces admitting a local quadratic isoperimetric inequality, see Definition 4.0.1 below, and studying the regularity of these solutions. In the classical setting, Plateau's problem asks whether a given Jordan curve γ in $\mathbb{R}^n$ of finite length may be spanned by a minimal disc, i.e. a surface homeomorphic to the closed unit disc $\overline{D}$ whose boundary agrees with the image of γ and is of minimal area among all such surfaces. Following the initial contributions of Lytchak and Wenger [LW16, LW17a, LW17b, LW18a], the theory of energy and area minimizing Sobolev mappings in proper metric spaces admitting a local quadratic isoperimetric inequality has rapidly progressed and is still being developed, driven by advanced tools from geometric analysis and geometric measure theory in the framework of metric spaces, see [LW18b, FW20, FW21, LWY20, SW22, WY25, SW25]. We note here that a general metric surface X does not satisfy a local quadratic isoperimetric inequality. Nevertheless, by making use of the two-dimensional structure of X, we will show in Chapter 4 that Strategy 1.3.1 can be made successful in this generality.

1.4. Applications of uniformization of metric surfaces

Given a non-smooth metric space X homeomorphic to a smooth surface M and a map f of certain regularity from X into some metric space Y, one often wishes to make use of differentiability properties of f to derive further results. A priori, it is not possible to find a notion of derivative of f, even if f is Sobolev, due to lack of regularity of X. But the existence of a "good" parametrization $u\colon M \to X$, provided by the uniformization results derived in Chapters 3 and 4, allows to compare notions of derivatives of $f \circ u$ and u (see Section 2.4 for definition of derivative in this generality) as well as to apply more standard results holding for mappings defined on smooth domains to $f \circ u$ and u. Thus, one can establish novel results in the area of geometric mapping theory as applications of uniformization of metric surfaces. There already exist several results crucially depending on this fact, see e.g. [EIR23, Nta25, Raj24, D, E, F]. We will introduce some of these works in Part II of this thesis. All of the results mentioned in Part II highly depend on the existence of "good" parametrizations of a metric surface X as provided by Theorem 3.3.4 below.

In [EIR23], weakly quasiconformal parametrizations are used to show the existence of a coarea inequality for monotone Sobolev functions on metric surfaces; a result that has been extended to general Sobolev functions on metric surfaces in [D]. These results will be explored in more detail in Chapter 5.

In Chapter 6, we investigate Lipschitz volume rigidity results for metric surfaces, based on the work [D]. As a corollary of the main result in [D], we obtain the following statement.

Theorem 1.4.1 (Theorem D.1.1)**.** *If X is a closed metric surface and Y a closed Riemannian surface with $\mathcal{H}^2_X(X) = \mathcal{H}^2_Y(Y)$, then every* 1*-Lipschitz map from X onto Y is an isometry.*

Here, a map $f\colon X \to Y$ is called an *isometry* (or *isometric*) if it preserves distances, i.e. $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. Theorem 1.4.1 was previously mostly known for the case of X and Y being closed Riemannian n-manifolds, see Theorem 6.1.1.

Finally, in Chapter 7, we will discuss how to extend the definition of finite distortion to mappings defined on metric surfaces and develop their core properties. *Mappings of finite distortion* are natural non-homeomorphic generalizations of quasiconformal maps, allowing the distortion to vary from point to point while remaining finite almost everywhere. Chapter 7 is based on the works [E, F]: in [E], we show that a map of finite distortion from a metric surface X to $\mathbb{R}^2$ with integrable distortion is continuous, open and discrete; the core properties of planar analytic mappings. Moreover, in [F], we prove equivalence of different notions of finite distortion, implying a novel uniformization result for metric surfaces.

2. Preliminaries

This chapter is devoted to establishing definitions, theory and central results frequently used throughout this thesis.

2.1. Basic notation and terminology

Let (X, d) be a metric space. We denote the *open* and *closed ball* in X of radius $r > 0$ centered at a point $x \in X$ by $B(x, r)$ and $\overline{B}(x, r)$, respectively. For $r > 0$ the *open r-neighborhood* of a set $A \subset X$ is defined by $N_r(A) := \{x \in X : d(a, x) < r \text{ for some } a \in A\}$.

A *smooth surface* M is a smooth oriented and connected Riemannian 2-manifold M with possibly non-empty boundary. If we want to put emphasis on the Riemannian metric g chosen on M, we write (M, g) instead of M. By ∂M we denote the boundary of the smooth surface M, which is homeomorphic to a finite disjoint union of unit 1-spheres S^1. A *metric surface* X is a metric space homeomorphic to a smooth surface M. The boundary of X, denoted ∂X, is the subset of X that is homeomorphic to ∂M. We call a metric surface X *planar*, if X is homeomorphic to a domain $U \subset \mathbb{R}^2$.

A set $\Omega \subset X$ homeomorphic to the open unit disc D is a *Jordan domain* in X if its boundary $\partial\Omega \subset X$ is a *Jordan curve* in X, i.e. a subset of X homeomorphic to S^1. In particular, the completion $\overline{\Omega}$ of Ω is homeomorphic to the closed unit disc $\overline{D}$. The *image* of a curve γ in X is indicated by $|\gamma|$ and the *length* by $\ell(\gamma)$. A curve γ is *rectifiable* if $\ell(\gamma) < \infty$ and *locally rectifiable* if each of its compact subcurves is rectifiable. Moreover, a curve $\gamma\colon [a, b] \to X$ is *geodesic* if $\ell(\gamma) = d(\gamma(a), \gamma(b))$. A metric space X is called *geodesic* if any pair of points in X can be joined by a geodesic. It is called *locally geodesic* if every point $x \in X$ has a neighborhood U such that any two points in U can be joined by a geodesic in X.

For $s > 0$, the *Hausdorff s-measure* of a set $A \subset X$ is defined by

$$\mathcal{H}^s_X(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A), \text{ where } \mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{j=1}^{\infty} \frac{\omega_s}{2^s} \operatorname{diam}(A_j)^s \right\}$$

and the infimum is taken over all collections of sets $\{A_j\}_{j=1}^\infty$ such that $A \subset \bigcup_{j=1}^\infty A_j$ and $\operatorname{diam}(A_j) < \delta$ for each j. Here ω_s is a positive normalization constant, chosen so that for open subsets $A \subset \mathbb{R}^n$ the Hausdorff n-measure $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the *Lebesgue n-measure* $|\cdot|_n$. In particular, $\omega_1 = 2$ and $\omega_2 = \pi$. Moreover, if (M, g) is a smooth surface, then the Hausdorff 2-measure $\mathcal{H}^2_g$ on (M, g) coincides with the Riemannian area. We say that a metric surface X has *locally finite Hausdorff 2-measure* if $\mathcal{H}^2(A) < \infty$ for every $A \subset X$ compact. Throughout this paper, $\mathcal{H}^2_X$ will always be the reference measure on a metric surface X. If it is clear from the context to which metric space $\mathcal{H}^2_X$ refers to, we write $\mathcal{H}^2$ instead of $\mathcal{H}^2_X$.

Definition 2.1.1. A map $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called *L-Lipschitz*, $L > 0$, if

$$d_Y(f(x), f(y)) \le L \cdot d_X(x, y)$$

for all $x, y \in X$. Moreover, the map f is *L-bi-Lipschitz* if f is a homeomorphism and both f and its inverse $f^{-1}\colon Y \to X$ are L-Lipschitz. We say that f is *(bi-)Lipschitz* if f is L-(bi-)Lipschitz for some $L > 0$.

In analogy to rectifiability of curves, we define rectifiability of higher dimensional sets as follows. A metric space X is called *countably n-rectifiable*, $n \in \mathbb{N}$, if there exist $E_i \subset \mathbb{R}^n$ and Lipschitz mappings $f_i\colon E_i \to X$, $i \in \mathbb{N}$, with

$$\mathcal{H}^n_X\Big(X \setminus \bigcup_{i\in\mathbb{N}} f_i(E_i)\Big) = 0.$$

2.2. Modulus of curve families

Consider a metric space X equipped with the Hausdorff s-measure $\mathcal{H}^s_X$ for some $s > 0$. Let Γ be a family of curves in X. A Borel function $\rho\colon X \to [0,\infty]$ is *admissible* for Γ if $\int_\gamma \rho \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$. Here, the *path integral* of ρ over γ is defined as

$$\int_\gamma \rho := \int_0^{\ell(\gamma)} \rho(\gamma_s(t))\, dt,$$

where γ_s is the parametrization by arclength of γ. The *p-modulus* ($p \geq 1$) of Γ is given by

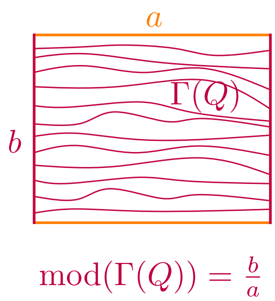
$$\operatorname{mod}_p(\Gamma) := \inf_\rho \int_X \rho^p \, d\mathcal{H}^s_X,$$

where the infimum is taken over all admissible functions for Γ.

By definition, $\operatorname{mod}_p(\Gamma) = \infty$ if Γ contains a constant curve. Note that p-modulus is an outer measure on the set of all curves in X. A property is said to hold for *p-almost every curve* in Γ if it holds for every curve in Γ_0 for some family $\Gamma_0 \subset \Gamma$ with $\operatorname{mod}_p(\Gamma \setminus \Gamma_0) = 0$. In the definition of $\operatorname{mod}_p(\Gamma)$, the infimum can equivalently be taken over all *weakly admissible functions*, that is, all Borel functions $\rho\colon X \to [0,\infty]$ with $\int_\gamma \rho \geq 1$ holding for p-almost every locally rectifiable curve $\gamma \in \Gamma$. Within this thesis we usually consider $p = 2$. In this case, we omit p in the definitions above. If we want to put emphasis on the ambient space X, we write mod_X instead of mod.

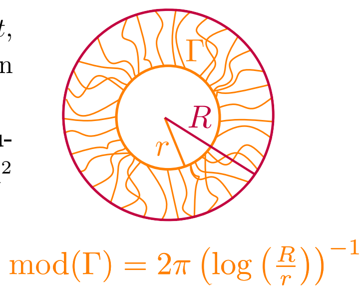
Example 2.2.1 (Modulus in the plane)**.** Consider a rectangle $Q = [0,a] \times [0,b] \subset \mathbb{R}^2$, $a, b > 0$, and denote by $\Gamma(Q)$ the family of curves joining the vertical sides of Q. Note that every curve $\gamma \in \Gamma(Q)$ has length at least a. In particular, the constant function $\varrho = \frac{1}{a}$ is admissible for $\Gamma(Q)$, implying that

$$\operatorname{mod}(\Gamma(Q)) \leq \frac{|Q|_2}{a^2} = \frac{ab}{a^2} = \frac{b}{a}. \tag{2.1}$$



By considering the subfamily $\{\gamma_t : t \in [0,b]\}$ of $\Gamma(Q)$ of straight line segments γ_t parallel to the x-Axis at height t, and applying Fubini as well as Hölder's inequality, one can show that equality holds in (2.1).
Let $0 < r < R$ and choose a point $x \in \mathbb{R}^2$. Similar computations as above show that if Γ is the family of curves in $\mathbb{R}^2$ joining $B(x,r)$ with $\mathbb{R}^2 \setminus B(x,R)$, then

$$\operatorname{mod}(\Gamma) = 2\pi\Big(\log\big(\frac{R}{r}\big)\Big)^{-1}.$$



For a more detailed introduction to modulus of curve families, we refer to [HKST15, Section 5].

2.3. Metric Sobolev spaces

We state some definitions from the theory of metric space valued Sobolev maps based on upper gradients. We refer to [Sha00, HKST01, HKST15] for a more detailed background on these so called Newton-Sobolev spaces.

Let $f\colon X \to Y$ be a map between metric spaces, where X is equipped with the Hausdorff s-measure $\mathcal{H}^s_X$ for some $s > 0$. A Borel function $\rho^u\colon X \to [0, \infty]$ is an *upper gradient* of f if

$$d_Y(f(x), f(y)) \leq \int_\gamma \rho^u \, ds \tag{2.2}$$

holds for all $x, y \in X$ and every rectifiable curve γ in X joining x and y. If, instead the *upper gradient inequality* (2.2) holds for all curves γ outside a curve family of 2-modulus zero, then we say that ρ^u is a *weak upper gradient* of f.

For $p \geq 1$ denote by $L^p(X, Y)$ the family of measurable essentially separably valued maps $f\colon X \to Y$ such that the function $x \mapsto d_Y(y, f(x))$ is in $L^p(X)$ for some $y \in Y$. A sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^p(X, Y)$ is said to *converge in* $L^p(X, Y)$ to a map $f \in L^p(X, Y)$ if

$$\int_X d^p(f_n(z), f(z)) \, d\mathcal{H}^s_X(z) \to 0$$

as n tends to infinity.

Definition 2.3.1. The *(Newton-)Sobolev space* $N^{1,p}(X, Y)$ is the space of maps $f \in L^p(X, Y)$ such that f has a weak upper gradient $\rho^u \in L^p(X)$.

If $Y = \mathbb{R}$, we simply write $N^{1,p}(X)$. Moreover, if $U \subset \mathbb{R}^n$ is open, then every element of the classical space of Sobolev functions $W^{1,p}(U)$ has a representative contained in $N^{1,p}(U)$, and vice versa, see [HKST15, Theorem 7.4.5]. The spaces $N^{1,p}_{\mathrm{loc}}(X, Y)$ and $L^{1,p}_{\mathrm{loc}}(X, Y)$ as well as L^p_{loc}-convergence are defined in obvious manner.

Every map $f \in N^{1,p}_{\mathrm{loc}}(X, Y)$ is absolutely continuous along almost every curve γ in X, see [HKST15, Proposition 6.3.2]. Moreover, each $f \in N^{1,p}_{\mathrm{loc}}(X, Y)$ has a *minimal weak upper gradient* ρ^u_f, which is unique up to sets of measure zero, see [HKST15, Theorem 6.3.20]. Here, the function ρ^u_f is *minimal* in the sense that for any other weak upper gradient ρ^u we have $\rho^u_f \leq \rho^u$ almost everywhere. If the domain is regular enough, the definition of Newton-Sobolev spaces agrees with other notions of Sobolev spaces, see [HKST15, Chapter 10].

2.4. Metric differentiability

The goal of this section is to study differentiability of Sobolev maps from a Euclidean or Riemannian domain into a complete metric space X. Let U be a domain in $\mathbb{R}^2$. A map $u\colon U \to X$ is said to be *approximately metrically differentiable at* $z \in U$ if there exists a seminorm s on $\mathbb{R}^2$ such that

$$\operatorname{ap}\lim_{y\to z} \frac{d(u(y), u(z)) - s(y - z)}{|y - z|} = 0,$$

where ap lim denotes the approximate limit, see e.g. [EG92, Section 1.7.2]. If such a seminorm exists, it is unique and is called *approximate metric derivative of* u *at* z, denoted $\operatorname{ap}\operatorname{md} u_z$.

Consider an open set $V \subset \mathbb{R}^2$, a point $v \in V$ and a diffeomorphism $\varphi\colon V \to U$. If the map $u\colon U \to X$ is approximately metrically differentiable at $\varphi(v)$ then the composition $u \circ \varphi$ is approximately metrically differentiable at v with

$$\operatorname{ap}\operatorname{md}(u \circ \varphi)_v = \operatorname{ap}\operatorname{md} u_{\varphi(v)} \circ d\varphi_v.$$

For the rest of this section we let M be a smooth surface equipped with a Riemannian metric g and $U \subset M$ a domain. The above made observation allows us to define the following: a map $u\colon U \to X$ is *approximately metrically differentiable at* $z \in U$ if the composition $u \circ \psi^{-1}$ is approximately metrically differentiable at $\psi(z)$ for an arbitrary chart (U_z, ψ) around z. We define the seminorm $\operatorname{ap}\operatorname{md} u_z$ on $(T_zM, g(z))$ by

$$\operatorname{ap}\operatorname{md} u_z := \operatorname{ap}\operatorname{md}(u \circ \psi^{-1})_{\psi(z)} \circ d\psi_z.$$

Note that this definition is independent of the choice of chart and $\operatorname{ap}\operatorname{md} u_z$ is called *approximate metric derivative of* u *at* z. The existence of an abundance of points of approximate metric differentiability for Sobolev maps $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ is provided by the following proposition, which in particular shows that every map $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ is approximately metrically differentiable at almost every $z \in U$.

Proposition 2.4.1 ([LW17a, Proposition 4.3])**.** *If $u \in N^{1,2}_{\mathrm{loc}}(U, X)$, then there exist countably many pairwise disjoint compact sets $K_i \subset U$, $i \in \mathbb{N}$, such that $\mathcal{H}^2(U \setminus \bigcup_{i\in\mathbb{N}} K_i) = 0$ with the following property. For every $i \in \mathbb{N}$ and every $\varepsilon > 0$ there exists $r_i(\varepsilon) > 0$ such that u is approximately metrically differentiable at every $z \in K_i$ and*

$$|d(u(z), u(z + v)) - \operatorname{ap}\operatorname{md} u_z(v)| \leq \varepsilon |v|$$

for all $z \in K_i$ and all $v \in \mathbb{R}^2$ with $|v| \leq r_i(\varepsilon)$ and $z + v \in K_i$.

The approximate metric derivative is a useful tool and can for example be applied to compute lengths of curves postcomposed with a Sobolev map $u \in N^{1,2}_{\mathrm{loc}}(U, X)$.

Lemma 2.4.2 ([LW18a, Lemma 3.1])**.** *If $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ then*

$$\ell(u \circ \gamma) = \int_a^b \operatorname{ap}\operatorname{md} u_{\gamma(t)}(\dot\gamma(t))\, dt$$

for almost every curve $\gamma\colon [a, b] \to U$ parametrized by arclength.

Remark 2.4.3. An application of Lemma 2.4.2 shows that the *maximal stretch* $L_u\colon U \to [0, \infty]$ of a map $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ defined by

$$L_u(z) = \max\{\operatorname{ap}\operatorname{md} u_z(v) : |v| = 1\}$$

is a representative of the minimal weak upper gradient ρ^u_u of u, for a proof see Lemma D.2.16.

2.5. Area of Sobolev maps

Unless otherwise stated, we assume throughout this section that M is a smooth surface equipped with a Riemannian metric g and $U \subset M$ is a domain. Let X be a metric space and let $u \in N^{1,2}_{\mathrm{loc}}(U, X)$.

Before providing the definition of Jacobian and area of a Sobolev map, we recall the following version of John's Theorem [Joh48], see also [Bal97, Theorem 3.1].

Theorem 2.5.1 (John's Theorem)**.** *Each symmetric convex body $K \subset \mathbb{R}^2$ contains a unique ellipsoid E of maximal area called* John's ellipse *of K. It holds that $E \subset K \subset \sqrt{2}E$. Moreover, if E is a round ball, then*

$$\frac{|K|_2}{|E|_2} \leq \frac{4}{\pi}.$$

Note that the ratio $4/\pi$ is attained for E being the closed unit disc $\overline{D}$ and $K = [-1,1]^2$. If s is a norm on $\mathbb{R}^2$, let $B_s = \{y \in \mathbb{R}^2 : s(y) \leq 1\}$ be the closed unit ball in $(\mathbb{R}^2, s)$. The set B_s is a planar symmetric convex body and we denote by E_s the John's ellipse of B_s as in Theorem 2.5.1. We will also refer to E_s as *John's ellipse of s*. After identifying $(T_zM, g(z))$ with $(\mathbb{R}^2, |\cdot|)$ via a linear isometry, we are able to define $B_z := B_{\operatorname{ap}\operatorname{md} u_z}$ and $E_z := E_{\operatorname{ap}\operatorname{md} u_z}$ whenever $\operatorname{ap}\operatorname{md} u_z$ is a norm on T_zM. The *Jacobian* of $\operatorname{ap}\operatorname{md} u_z$ is defined by

$$J(\operatorname{ap}\operatorname{md} u_z) = \frac{\pi}{|B_z|_2},$$

whenever $\operatorname{ap}\operatorname{md} u_z$ is a norm on T_zM and $J(\operatorname{ap}\operatorname{md} u_z) = 0$ otherwise.

As a consequence of [HKST15, Theorem 8.1.49], we obtain that U may be covered up to a set $G_0 \subset U$ of measure zero by countably many disjoint measurable sets G_j, $j \in \mathbb{N}$, such that $u|_{G_j}$ is Lipschitz. In particular, outside the set G_0 of measure zero, u satisfies Lusin's condition (N). Here, a map $f\colon X \to Y$ satisfies *Lusin's condition (N)* if $E \subset X$ with $\mathcal{H}^2_X(E) = 0$ implies $\mathcal{H}^2_Y(f(E)) = 0$. Now [Kar07, Theorem 3.2] implies the following area formula.

Theorem 2.5.2 (Area formula)**.** *If $u \in N^{1,2}_{\mathrm{loc}}(U, X)$, then there exists $G_0 \subset U$ with $\mathcal{H}^2(G_0) = 0$ such that for every measurable set $A \subset U \setminus G_0$ we have*

$$\int_A J(\operatorname{ap}\operatorname{md} u_z)\, d\mathcal{H}^2_g = \int_X N(y, u, A)\, d\mathcal{H}^2_X. \tag{2.3}$$

By $N(y, u, A)$, we denote the number of preimages of y under u in A.

Integrating the Jacobian $J(\operatorname{ap}\operatorname{md} u_z)$ gives rise to the following notion of area.

Definition 2.5.3. The *parametrized (Hausdorff) area* of $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ is given by

$$\operatorname{Area}(u) := \int_U J(\operatorname{ap}\operatorname{md} u_z)\; d\mathcal{H}^2_g(z).$$

We emphasize that the parametrized area of a map $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ is invariant under precompositions with bi-Lipschitz homeomorphisms, and thus independent of the Riemannian metric g chosen on M. By the area formula (Theorem 2.5.2) it holds that $\operatorname{Area}(u) = \mathcal{H}^2(u(U))$ in case of u being injective and satisfying Lusin's condition (N).

Part I.

Uniformization of metric surfaces

3. Uniformization theorems for metric surfaces

This chapter focuses on providing answers to Question 1.1.1 whenever X is a metric surface of locally finite Hausdorff 2-measure. The tools used to study the geometry of these surfaces heavily rely on an abundance of locally rectifiable curves. The local finiteness of the Hausdorff 2-measure guarantees such an abundance of rectifiable curves, by an application of the coarea inequality for Lipschitz functions (see Chapter 5 below) and Hölder's inequality, see [Raj17, Proposition 3.5].

Sections 3.1, 3.2 and 3.3 provide answers to Question 1.1.1 whenever the uniformization map $u\colon M \to X$ lies in a certain class of mappings. Specifically, we explore cases where u is assumed to be quasisymmetric, quasiconformal, and weakly quasiconformal in Sections 3.1, 3.2, and 3.3, respectively. In particular, we revisit the following breakthrough results as well as their generalizations: the quasisymmetric uniformization theorem of Bonk and Kleiner [BK02], Rajala's quasiconformal uniformization theorem [Raj17], and the weakly quasiconformal uniformization theorem due to Ntalampekos and Romney [NR23, NR24] as well as Wenger and the author [B, C].

3.1. Quasisymmetric uniformization

Quasisymmetries are natural generalizations of conformal mappings and were first introduced by Tukia and Väisälä in [TV80]. Quasisymmetric homeomorphisms also appear naturally in the setting of Gromov hyperbolic spaces (e.g. Cayley graphs of hyperbolic groups), as quasi-isometries between Gromov hyperbolic spaces induce quasisymmetric boundary homeomorphisms. This shows that quasisymmetric uniformization is not only important within the area of analysis on metric spaces but also possesses strong connections to problems in fields such as geometric group theory, see e.g. [Bon06].

Definition 3.1.1. A homeomorphism $u\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is *quasisymmetric* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{d_Y(u(x), u(y))}{d_Y(u(x), u(z))} \leq \eta\left(\frac{d_X(x,y)}{d_X(x,z)}\right)$$

whenever $x, y, z \in X$ with $x \neq z$.

In particular, every bi-Lipschitz map is quasisymmetric. Note that a quasisymmetric map preserves approximate shapes of sets and other geometric properties, compare to Figure 1.1.

In their celebrated work Bonk and Kleiner [BK02] ask for necessary and sufficient conditions on a metric space X homeomorphic to the 2-dimensional unit sphere S^2 under which X can be mapped onto S^2 by a quasisymmetric homeomorphism. Under the assumption of Ahlfors 2-regularity, Bonk and Kleiner [BK02] prove that linear local contractibility is necessary and sufficient for the existence of a quasisymmetric map from S^2 to X. Here, a metric space X is *Ahlfors Q-regular* if there exists $K > 0$ such that

$$K^{-1} \cdot r^Q \leq \mathcal{H}^Q(B(x,r)) \leq K \cdot r^Q$$

for all $x \in X$ and $0 < r < \operatorname{diam} X$. Moreover, X is *linearly locally contractible* if there exists a constant $\lambda \geq 1$ such that every ball $B(x, r)$ is contractible in the ball $B(x, \lambda r)$.

In contrast to linear local contractibility, Ahlfors 2-regularity is not invariant under quasisymmetric homeomorphisms. In fact, for any $\alpha \in (0, 1)$ the space $\mathbb{R}^2$ equipped with the metric

$$d^\alpha(x, y) = |x - y|^\alpha$$

is Ahlfors $(2/\alpha)$-regular, and the identity mapping $\mathrm{id}\colon (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, d^\alpha)$ is quasisymmetric.

We now state the theorem of Bonk and Kleiner [BK02].

Theorem 3.1.2 (Bonk–Kleiner uniformization theorem)**.** *Let X be an Ahlfors 2-regular metric space homeomorphic to S^2. Then, there exists a quasisymmetry $u\colon S^2 \to X$ if and only if X is linearly locally contractible.*

The significance of the Bonk–Kleiner uniformization theorem is undisputed. In particular, Theorem 3.1.2 answers to the affirmative an important question of Heinonen and Semmes [HS97, Question 3]. Note that Theorem 3.1.2 does not generalize to higher dimensions, see [Sem96b, HW10, PW14]. It is natural to wonder whether the conclusion of X being quasisymmetrically equivalent to S^2 is best possible. A more restrictive class of mappings to look for are bi-Lipschitz homeomorphisms. It is easy to see that Ahlfors 2-regularity as well as linear local contractibility are necessary conditions on a metric surface X homeomorphic to S^2 for the existence of a bi-Lipschitz parametrization of X by S^2. However, a construction based on the work of Laakso [Laa02] produces an example of an Ahlfors 2-regular and linearly locally contractible metric space homeomorphic to S^2 that does not admit a bi-Lipschitz parametrization by S^2. Up to this day, the problem of finding minimal assumptions under which a metric surface admits a bi-Lipschitz parametrization remains widely open.

If X is a closed surface, then it follows by [BK02, Lemma 2.5] that linear local contractibility is equivalent to linear local connectedness. Here, a metric space X is *linearly locally connected* if there exists a constant $\lambda \geq 1$ such that for all $x \in X$ and $r > 0$, every pair of distinct points in $B(x, r)$ can be connected by a continuum in $B(x, \lambda r)$ and every pair of distinct points in $X \setminus B(x, r)$ can be connected by a continuum in $X \setminus B(x, r/\lambda)$.

Theorem 3.1.2 was extended to Ahlfors 2-regular and linearly locally connected metric surfaces X whenever $X \setminus \partial X$ is a domain in S^2, see [Wil08] and [MW13], and whenever X is compact and has no boundary, see [GW18] and [Iko22]. In [A], Theorem 3.1.2 was extended to Ahlfors 2-regular and linearly locally connected geodesic metric surfaces with possibly non-empty boundary and of higher genus by building on the work of Lytchak and Wenger [LW20], which provides an alternate proof of the Bonk–Kleiner uniformization theorem in terms of energy and area minimization. We will introduce the works [LW20] and [A] in Section 4.2.

3.2. Geometrically quasiconformal uniformization

Let X and Y be metric surfaces of locally finite Hausdorff 2-measure. Recall the notion of modulus of curve families (Section 2.2) and note that modulus is a conformal invariant, see e.g. [Hei01, Section 7.3], in the sense that if X and Y are smooth surfaces and $f\colon X \to Y$ is conformal, then $\operatorname{mod}(\Gamma) = \operatorname{mod}(f \circ \Gamma)$, where $f \circ \Gamma$ denotes the family of all curves $\gamma' = f \circ \gamma$ for some $\gamma \in \Gamma$. It is therefore natural to define geometric quasiconformality as follows.

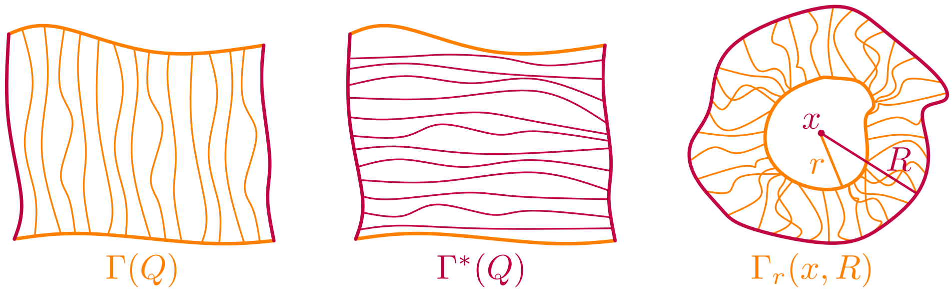


Figure 3.1.: Families of curves studied in this section.

Definition 3.2.1. A homeomorphism $f\colon X \to Y$ is *geometrically K-quasiconformal*, $K \geq 1$, if

$$K^{-1} \cdot \operatorname{mod}(\Gamma) \leq \operatorname{mod}(f \circ \Gamma) \leq K \cdot \operatorname{mod}(\Gamma) \tag{3.1}$$

holds for every family Γ of curves in X. We say that f is *geometrically quasiconformal* if (3.1) holds for some $K \geq 1$.

Let X be a metric surface homeomorphic to $\mathbb{R}^2$ and of locally finite Hausdorff 2-measure. Let $Q \subset X$ be a closed topological quadrilateral. We denote by $\Gamma(Q)$ and $\Gamma^*(Q)$ the two different families of curves joining opposite sides of Q as in Figure 3.1. Furthermore, for $x \in X$ and $0 < r < R < \infty$ with $X \setminus B(x,R) \neq \emptyset$ we define $\Gamma_r(x,R)$ to be the family of curves joining $B(x,r)$ and $X \setminus B(x,R)$ in X, compare to Figure 3.1.

Example 3.2.2 (Modulus in the plane)**.** Consider a domain $U \subset \mathbb{R}^2$ and let $Q \subset U$ be a rectangle. By making use of the computations from Example 2.2.1, we get

$$\operatorname{mod}(\Gamma(Q)) \cdot \operatorname{mod}(\Gamma^*(Q)) = \frac{a}{b} \cdot \frac{b}{a} = 1, \tag{3.2}$$

where $a, b > 0$ are the side lengths of Q. Moreover, if we choose a point $x \in U$ and let $R > 0$ be such that $U \setminus B(x,R) \neq \emptyset$, then we have by Example 2.2.1

$$\lim_{r \to 0} \operatorname{mod}(\Gamma_r(x,R)) = \lim_{r \to 0} 2\pi \Big(\log\big(\frac{R}{r}\big)\Big)^{-1} = 0. \tag{3.3}$$

Assume that X admits a geometrically quasiconformal parametrization $u\colon U \to X$, where $U \subset \mathbb{R}^2$ is a domain. The Riemann mapping theorem, along with Definition 3.2.1 and (3.2), provide the existence of a constant $\kappa \geq 1$ such that

$$\kappa^{-1} \leq \operatorname{mod}(\Gamma(Q)) \cdot \operatorname{mod}(\Gamma^*(Q)) \leq \kappa \tag{3.4}$$

holds for every closed topological quadrilateral $Q \subset X$. Moreover, by the Riemann mapping theorem, Definition 3.2.1 and (3.3) we obtain

$$\lim_{r \to 0} \operatorname{mod}(\Gamma_r(x,R)) = 0 \tag{3.5}$$

for every $x \in X$ and $R > 0$ with $X \setminus B(x,R) \neq \emptyset$.

Rajala [Raj17] calls a metric surface X satisfying properties (3.4) and (3.5) *reciprocal* and shows that these conditions are sufficient for characterizing planar metric surfaces admitting parametrizations by geometrically quasiconformal homeomorphisms.

Theorem 3.2.3 (Rajala's uniformization theorem)**.** *Let X be a metric space homeomorphic to $\mathbb{R}^2$ and of locally finite Hausdorff 2-measure. Then, there exists a geometrically quasiconformal map u from a domain $U \subset \mathbb{R}^2$ onto X, if and only if X is reciprocal.*

It was conjectured in [Raj17, Section 14] and proven by [Rom19a, Theorem 1.2] that if X is as in Theorem 3.2.3, then there exists a geometrically quasiconformal map $v\colon U \to X$ satisfying

$$\frac{\pi}{4} \cdot \operatorname{mod}(\Gamma) \leq \operatorname{mod}(v \circ \Gamma) \leq \frac{\pi}{2} \cdot \operatorname{mod}(\Gamma)$$

for every family Γ of curves in X. The constants are optimal as they are realized by the geometrically quasiconformal mapping $\mathrm{id}\colon (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, |\cdot|_\infty)$, see [Raj17, Example 2.2]. Here, $(\mathbb{R}^2, |\cdot|_\infty)$ denotes the space $\mathbb{R}^2$ equipped with the infinity-norm $|\cdot|_\infty$ defined by

$$|z|_\infty = |(z_1, z_2)|_\infty = \max\{|z_1|, |z_2|\}$$

for $z \in \mathbb{R}^2$. The lower bound of (3.4) is always satisfied for some $\kappa^{-1} > 0$ by [RR19, Theorem 1.3], and for $\kappa^{-1} = \pi^2/16$ by [EBPC22, Corollary 1.2]. The example of $(\mathbb{R}^2, |\cdot|_\infty)$ shows again that this constant is optimal. It follows by [NR24, Theorem 1.8] that the upper bound in (3.4) implies (3.5), the converse does not hold, see [NR24, Example 8.3]. In summary, we may define reciprocality as follows.

Definition 3.2.4. A metric surface X is *reciprocal* if there exists $\kappa > 0$ such that for every closed topological quadrilateral $Q \subset X$ and for the two different families $\Gamma(Q)$ and $\Gamma^*(Q)$ of curves joining opposite sides of Q we have

$$\operatorname{mod}(\Gamma(Q)) \cdot \operatorname{mod}(\Gamma^*(Q)) \leq \kappa. \tag{3.6}$$

Rajala's work has been extended to metric surfaces of higher topology in [Iko22] and [NR24]. In particular, [Raj17, Iko22, NR24] show that reciprocal surfaces are exactly the metric surfaces that admit quasiconformal parametrizations by smooth surfaces.

There are only few conditions known to imply reciprocality. One of them follows from the works [E, F] and will be discussed in Chapter 7. Another one is upper Ahlfors 2-regularity, see [Raj17, Theorem 1.6]. Here, a metric surface X is *upper Ahlfors 2-regular* if there exists a fixed constant $K > 0$ such that

$$\mathcal{H}^2(B(x, r)) \leq K\, r^2 \tag{3.7}$$

holds for every $x \in X$ and $0 < r < \operatorname{diam} X$. Moreover, we call X *locally upper Ahlfors 2-regular* if every point in X has a neighborhood U that is upper Ahlfors 2-regular. In particular, by [Raj17, Theorem 1.6], Theorem 3.2.3 applies to Ahlfors 2-regular metric surfaces. Under the assumptions of linear local contractibility and Ahlfors 2-regularity, a geometrically quasiconformal homeomorphism upgrades to a quasisymmetric mapping, see [Raj17, Corollary 1.7], and thus the Bonk–Kleiner uniformization theorem (Theorem 3.1.2) is recovered.

This equivalence of geometric quasiconformality and quasisymmetry does only hold under strong enough regularity of domain and target, see [Hei01, Section 11.13] and [Tys98]. Geometric quasiconformality provides local information, while quasisymmetry provides global. The following remark shows that in the context of general metric surfaces, neither of the conditions of quasisymmetry and geometric quasiconformality implies the other, even if domain or target are subsets of $\mathbb{R}^2$. The arguments presented in Remark 3.2.5 are illustrated in Table 3.1.

	$u\colon U \subset \mathbb{R}^2 \to X$	$u^{-1}\colon X \to U \subset \mathbb{R}^2$
u is quasisymmetric	u satisfies Lusin (N) [Tys00, Corollary 5.10]	Counterexample to Lusin (N) of u^{-1} [Rom19b]
u is geometrically quasiconformal	Counterexample to Lusin (N) of u [Raj17, Lemma 14.1]	u^{-1} satisfies Lusin (N) [Raj17, Remark 8.3]

Table 3.1.: Geometric quasiconformality vs. quasisymmetry in metric surface setting.

Remark 3.2.5. Let X be a metric surface of locally finite Hausdorff 2-measure and $u\colon U \to X$ a homeomorphism, where $U \subset \mathbb{R}^2$ is a domain. It follows from the works of Tyson [Tys00, Corollary 5.10] that if u is quasisymmetric, then u satisfies Lusin's condition (N).

Rajala [Raj17, Remark 8.3] proves that if u is geometrically quasiconformal then the inverse u^{-1} satisfies Lusin's condition (N). Furthermore, Rajala shows in [Raj17, Lemma 14.1] that if X is reciprocal and $u\colon U \to X$ the map that Rajala constructs in the proof of Theorem 3.2.3, then the space X has to be countably 2-rectifiable in case of u satisfying Lusin's condition (N). In [Raj17, Proposition 17.1], Rajala then provides an example of a reciprocal metric surface X of locally finite Hausdorff 2-measure that is not countably 2-rectifiable. This shows in particular that a geometrically quasiconformal map u as above does not need to satisfy Lusin's condition (N), and hence, by [Tys00, Corollary 5.10], can not be quasisymmetric.

A few years later Romney [Rom19b] provides an example of a metric surface X of locally finite Hausdorff 2-measure and a quasisymmetry $u\colon [0,1]^2 \to X$ with the property that u^{-1} does not satisfy Lusin's condition (N), establishing the existence of a quasisymmetry that is not geometrically quasiconformal. The result [Rom19b] also holds for higher dimensions and answers to the negative two famous questions of Heinonen and Semmes [HS97, Questions 15 and 16]. Moreover, the construction adapts to provide an example, where X is a hypersurface in $\mathbb{R}^3$, see [NR21].

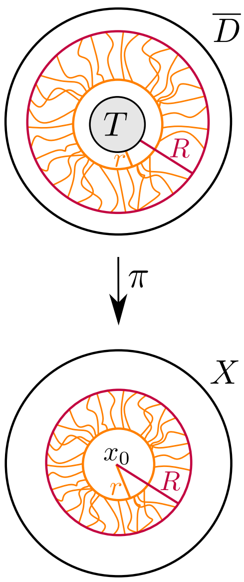
We end this section by remarking that there are plenty of metric surfaces that are not reciprocal. The following example appears in [LW18a, Example 11.3] and Example B.3.2. See [Raj17, IR22, NR23, NR24] for more examples along these lines.

Example 3.2.6 (Collapsed disc)**.** Let $T = \{z \in D : |z| \leq 1/2\}$ and let $X = \overline{D}/T$ be the quotient metric space equipped with the quotient metric, see e.g. [BBI01, Definition 3.1.12]. Then, the space X is geodesic, homeomorphic to $\overline{D}$, and has finite Hausdorff 2-measure.

Let $\pi\colon \overline{D} \to X$ be the natural projection map. The map π restricted to $\overline{D} \setminus T$ is a local isometry and leaves modulus invariant. In particular, if we set $x_0 = \pi(T)$, then

$$\begin{aligned}\lim_{r\to 0} \operatorname{mod}_X(\Gamma_r(x_0,R)) &= \lim_{r\to 0} \operatorname{mod}_{\mathbb{R}^2}(\Gamma_{r+1/2}(x_0,R+1/2)) \\ &= \lim_{r\to 0} 2\pi \left(\log\left(\frac{R+1/2}{r+1/2}\right)\right)^{-1} > 0.\end{aligned}$$

This shows that (3.5) is not satisfied at $x = x_0$ and hence, by [NR24, Theorem 1.8], the space X is not reciprocal.



The construction of the space X in Example 3.2.6 is very natural. Therefore, one wishes to show the existence of a uniformization map with good geometric and analytic properties for a more general class of metric surfaces. This will be the content of the next section.

3.3. Weakly quasiconformal uniformization

Within this section we aim to find answers to the following question, which is attributed to Rajala and Wenger, see [IR22, Question 1.1].

Question 3.3.1. *Let X be a metric space homeomorphic to a smooth surface M and with locally finite Hausdorff 2-measure. Does there exist a weakly quasiconformal map $u\colon M \to X$?*

Let X and Y be metric surfaces of locally finite Hausdorff 2-measure. A map $f\colon X \to Y$ is called *cell-like* if for every $y \in Y$ the set $f^{-1}(y)$ is a continuum that is contractible in each of its open neighborhoods in X. Assume that X and Y are compact and homeomorphic. If $f\colon X \to Y$ is surjective then cell-likeness is equivalent to monotonicity: a map $f\colon X \to Y$ is *monotone* if $f^{-1}(y)$ is a continuum for every $y \in Y$. If f is furthermore continuous, then f is a uniform limit of homeomorphisms, see [NR24, Theorem 6.3] or Proposition C.5.2 and the references therein. If X and Y have empty boundary, then every continuous, proper, and cell-like mapping $f\colon X \to Y$ is a uniform limit of homeomorphisms, see [Dav86, Corollary 25.1A].

Definition 3.3.2. A continuous, surjective, proper and cell-like map $f\colon X \to Y$ is *weakly K-quasiconformal*, $K \geq 1$, if

$$\operatorname{mod}\Gamma \leq K \cdot \operatorname{mod} f \circ \Gamma \tag{3.8}$$

holds for every family Γ of curves in X. If f is as above and (3.8) holds for some $K \geq 1$ we say that f is *weakly quasiconformal*.

It follows from [Wil12, Theorem 1.1] that every weakly quasiconformal map $f\colon X \to Y$ is an element of $N^{1,2}_{\text{loc}}(X, Y)$. See [NR23, Section 7] for more properties of weakly quasiconformal maps.

Together with Wenger [B], we apply Strategy 1.3.1 to produce the following uniformization result, providing a positive answer to Question 3.3.1 in case of X being locally geodesic.

Theorem 3.3.3. *Let X be a locally geodesic metric space homeomorphic to $\mathbb{R}^2$ and of locally finite Hausdorff 2-measure. If $\Omega \subset X$ is a Jordan domain of finite boundary length then there exists a weakly $(4/\pi)$-quasiconformal map $u\colon \overline{D} \to \overline{\Omega}$.*

The constant $4/\pi$ is optimal, by again considering the identity map $\mathrm{id}\colon (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, |\cdot|_\infty)$. The proof of Theorem 3.3.3 as well as the proof of the generalization of Theorem 3.3.3 to surfaces of higher topology (the main result of [C]) will be explained in detail in Section 4.3.

Simultaneously to the work of [B], Ntalampekos and Romney independently prove a variant of Theorem 3.3.3 for simply connected geodesic metric surfaces X of locally finite Hausdorff 2-measure in [NR23]. Their approach is based on approximating X by polyhedral surfaces and eventually allowed to drop the assumption of X being locally geodesic in [NR24].

Theorem 3.3.4 (Weakly quasiconformal uniformization, [NR24, Theorem 1.3])**.** *Let X be a metric surface of locally finite Hausdorff 2-measure. Then there exists a smooth surface M homeomorphic to X, a Riemannian metric g on M of constant curvature, and a weakly $(4/\pi)$-quasiconformal map $u\colon (M, g) \to X$.*

The smooth surface M in Theorem 3.3.4 might be non-compact. After additionally assuming that the metric surface X is reciprocal, the map u from Theorem 3.3.4 upgrades to being a geometrically quasiconformal homeomorphism. This follows from [NR24, Theorem 1.8] combined with Proposition B.3.3. Therefore, the theorems of Rajala (Theorem 3.2.3) as well as Bonk–Kleiner (Theorem 3.1.2) are recovered.

Theorem 3.3.4 completes the picture on uniformization of metric surfaces of locally finite Hausdorff 2-measure. The definitions of modulus of curve families and Sobolev maps as well as the notions of quasiconformality used in this thesis highly depend on an abundance of locally rectifiable paths. The existence of locally rectifiable paths is a priori not given on a metric surface that does not possess locally finite Hausdorff 2-measure. Hence, there is no hope of further extending Theorem 3.3.4 to surfaces of locally infinite Hausdorff 2-measure with the methods described in this dissertation.

4. Uniformization by minimizing energy

The goal of this chapter is to apply Strategy 1.3.1 to produce uniformization results for locally geodesic metric surfaces of locally finite Hausdorff 2-measure, possibly satisfying additional properties. In a series of papers [LW16, LW17a, LW17b, LW18a] Lytchak and Wenger develop a theory of existence and regularity of energy and area minimizing Sobolev maps in proper metric spaces admitting a local quadratic isoperimetric inequality.

Definition 4.0.1. A metric space X is said to satisfy a *local quadratic isoperimetric inequality* if there exist constants $C, l_0 > 0$ such that every Lipschitz curve $c\colon S^1 \to X$ of length $\ell(c) \leq l_0$ is the trace of a Sobolev map $u \in N^{1,2}(D, X)$ with

$$\operatorname{Area}(u) \leq C \cdot \ell(c)^2.$$

The theory of Lytchak and Wenger was extended to energy and area minimizing Sobolev maps from smooth surfaces of higher genus and/or with multiple boundary components into proper metric spaces admitting a local quadratic isoperimetric inequality by Fitzi and Wenger in [FW20, FW21]. Moreover, Soultanis and Wenger [SW22] provide a theory of energy and area minimizing surfaces in certain homotopy classes.

Note that a general metric surface X does not satisfy a local quadratic isoperimetric inequality. Nevertheless, in [B, C] we exploit the two-dimensional structure of a locally geodesic metric surface X of locally finite Hausdorff 2-measure to produce an energy minimizing parametrization of X, which possesses additional good geometric and analytic properties. In Section 4.1, we will introduce the the most important notions and results regarding existence and regularity of energy minimizing Sobolev maps into metric spaces. Section 4.2 is devoted to finding quasisymmetric parametrizations of Ahlfors 2-regular and linearly locally connected metric surfaces. In particular, we explain the alternate proof of the Bonk–Kleiner uniformization theorem (Theorem 3.1.2) due to Lytchak and Wenger [LW20] as well as the generalization to surfaces of higher topology by Fitzi and the author [A]. Section 4.3 is concerned with weakly quasiconformal parametrizations of geodesic metric surfaces of locally finite Hausdorff 2-measure (Theorem 3.3.3) and contains work of [B] and [C].

4.1. Existence and regularity of energy minimizing Sobolev maps

This section is mostly based on the works [LW17a, LW20, FW20, FW21] and aims at reviewing the main existence and regularity results of energy minimizing Sobolev maps into metric spaces. Throughout this section let M be a compact smooth surface, let $U \subset M$ be a domain and X a proper and complete metric space. Note that the articles [LW17a, FW20, FW21] make use of the definition of Sobolev mappings into metric spaces introduced by Reshetnyak [Res97, Res06]. In case of the domain U being a Lipschitz domain, every map in the Reshetnyak-Sobolev $W^{1,2}(U, X)$ has a representative in the Newton-Sobolev space $N^{1,2}(U, X)$, and vice versa, compare with [HKST15, Theorem 7.1.20]. Here, we say that $U \subset M \backslash \partial M$ is a *Lipschitz domain* if for every point $z \in \partial U$ there exists an open neighborhood $V \subset M$ and a bi-Lipschitz mapping $\psi\colon (0,1) \times [0,1) \to M$ such that $\psi((0,1) \times (0,1)) = U \cap V$ and $\psi((0,1) \times \{0\}) = V \cap \partial U$.

Definition 4.1.1. Let $u \in N^{1,2}(U, X)$ and assume that $U \subset M \setminus \partial M$ is a Lipschitz domain. As u is Sobolev, for almost every $s \in (0,1)$ the map $t \mapsto u \circ \psi(s,t)$ is absolutely continuous, see [HKST15, Proposition 6.3.2], which we denote by the same expression. The *trace* of u

$$\operatorname{tr}(u)(\psi(s,0)) := \lim_{t \searrow 0} (u \circ \psi)(s,t)$$

is defined for almost every $s \in (0,1)$.

It can be shown (see [KS93, Section 1.12]) that the trace is independent of the choice of the map ψ and lies in $L^2(\partial U, X)$. Moreover, if $u \colon \overline{U} \to X$ is continuous, then $\operatorname{tr}(u)$ agrees with the restriction of u to the boundary ∂U.

Assume that M has $k \geq 0$ boundary components and let Γ be a disjoint union of k Jordan curves in X. Denote by $[\Gamma]$ the set of all *weakly monotone parametrizations* of Γ, i.e. uniform limits of homeomorphisms from S to Γ, where S is any space homeomorphic to the disjoint union of k copies of S^1. We define the *set of admissible mappings* by

$$\Lambda(M, \Gamma, X) := \{u \in N^{1,2}(M, X) : \operatorname{tr}(u) \text{ has a continuous representative in } [\Gamma]\}.$$

Note that the family $\Lambda(M, \Gamma, X)$ might be empty. Showing the existence of a map $u \in \Lambda(M, \Gamma, X)$ is highly non-trivial and a major step in making Strategy 1.3.1 successful. Lytchak and Wenger [LW17a] use the local quadratic isoperimetric inequality to construct such a Sobolev map. The existence of non-trivial Sobolev maps into general metric surfaces will be considered in Section 4.3.2.

We use the minimal weak upper gradient to define the following notion of energy for Sobolev mappings in $N^{1,2}(U, X)$.

Definition 4.1.2. The *(Reshetnyak) energy* of a map $u \in N^{1,2}(U, X)$ with respect to a Riemannian metric g is defined by

$$E_+^2(u, g) := \int_U \rho_u^u(z)^2 \, d\mathcal{H}_g^2(z).$$

If it is clear from the context to which Riemannian metric g on M we are referring to, we write $E_+^2(u)$ instead of $E_+^2(u, g)$. Note that we always consider the standard Euclidean metric g_{Eucl} on D. Recall that the maximal stretch $L_u(z) = \max\{\operatorname{ap}\operatorname{md} u_z(v) : |v| = 1\}$ of $u \in N^{1,2}(U, X)$ is a representative of the minimal weak upper gradient ρ_u^u, see Remark 2.4.3. In particular, this definition of energy agrees with the one given in [FW21, Definition 2.2], and we obtain that E_+^2 is invariant under precompositions with conformal diffeomorphisms. Moreover, the unit ball $B_z = \{v \in \mathbb{R}^2 : \operatorname{ap}\operatorname{md} u_z(v) \leq 1\}$ contains a Euclidean ball of radius $1/L(z)$ and thus

$$J(\operatorname{ap}\operatorname{md} u_z) \leq L^2(z)$$

for almost every $z \in U$. In particular, $\operatorname{Area}(u) \leq E_+^2(u, g)$ for any Riemannian metric g on M. The Reshetnyak energy E_+^2 is lower semicontinuous, see [LW17a, Corollary 5.7] and [Res97, Theorem 4.2]. This is a crucial step in proving the existence of an energy minimizers.

Theorem 4.1.3 ([LW17a, Theorem 7.6])**.** *Let X be a proper metric space and $\Gamma \subset X$ a Jordan curve. If $\Lambda(\overline{D}, \Gamma, X)$ is not empty, then there exists $u \in \Lambda(\overline{D}, \Gamma, X)$ satisfying*

$$E_+^2(u) = \inf\{E_+^2(v) : v \in \Lambda(\overline{D}, \Gamma, X)\}.$$

The proof of Theorem 4.1.3 is provided via a direct variational method and, under certain additional assumptions on X or on mappings in $\Lambda(M, \Gamma, X)$, generalizes to the case where the domain is a smooth surface M of higher topology, see e.g. [FW21]. We now state the main steps of proof within this more general setting. If the family $\Lambda(M,\Gamma,X)$ is nonempty, we may choose a sequence $\{(u_n,g_n)\}_{n\in\mathbb{N}}$ with $u_n\in\Lambda(M,\Gamma,X)$ and g_n a Riemannian metric on M that is *energy minimizing* in the sense that

$$\lim_{n\to\infty} E_+^2(u_n,g_n)=\inf\{E_+^2(v,h): v\in\Lambda(M,\Gamma,X),\ h \text{ a Riemannian metric on } M\}.$$

We assume that $E_+^2(u_n,g_n)$ is uniformly bounded. The above mentioned additional assumptions should imply that the relative systoles of (M,g_n) (see [FW21, Definition 3.1]) are uniformly bounded from below. This ensures applicability of the Mumford compactness theorem (see [DHT10, Theorem 4.4.1] and [FW21, Theorem 3.3]). By furthermore utilizing the Rellich–Kondrachov compactness theorem (see [KS93, Theorem 1.13]), we obtain the existence of a Riemannian metric g on M and homeomorphisms $\varphi_n\colon (M,g)\to(M,g_n)$ such that the sequence $\{v_n:=u_n\circ\varphi_n\}_{n\in\mathbb{N}}$ is energy minimizing and a subsequence of $\{v_n\}_{n\in\mathbb{N}}$ converges in $L^2(U,X)$ to a map $u\in N^{1,2}(M,X)$. By lower semicontinuity of energy, see [LW17a, Corollary 5.7] and [Res97, Theorem 4.2], it hence follows that $E_+^2(u,g)\le\liminf_{n\to\infty}E_+^2(u_n,g_n)$.

The second property that has to be ensured by the above mentioned additional assumptions is equicontinuity of $\{tr(v_n)\}_{n\in\mathbb{N}}$, allowing the application of Arzelà–Ascoli to the sequence of traces. It follows that u is indeed an element of $\Lambda(M,\Gamma,X)$. In summary, we have established that (u,g) is an *energy minimizing pair in*

$$\Lambda_{\mathrm{metr}}(M,\Gamma,X)=\{(v,h): v\in\Lambda(M,\Gamma,X),\ h \text{ a Riemannian metric on } M\},$$

i.e. $(u,g)\in\Lambda_{\mathrm{metr}}(M,\Gamma,X)$ satisfies $E_+^2(u,g)=\inf\{E_+^2(v,h):(v,h)\in\Lambda_{\mathrm{metr}}(M,\Gamma,X)\}$.

Given the existence of an energy minimizing pair (u,g) in $\Lambda_{\mathrm{metr}}(M,\Gamma,X)$, we can study the regularity of u with respect to g. The first result we want to mention is of topological type.

Theorem 4.1.4 ([LW20, Theorem 1.2])**.** *Let X be a geodesic metric space homeomorphic to $\overline{D}$, and let $u\colon\overline{D}\to X$ be a continuous map. If $u\in\Lambda(\overline{D},\partial X,X)$ minimizes E_+^2 among all maps in $\Lambda(\overline{D},\partial X,X)$, then u is monotone.*

Recall that $u\colon\overline{D}\to X$ is monotone if the preimage of each point is a continuum. If u is furthermore continuous, then u is a uniform limit of homeomorphisms. Note that Theorem 4.1.4 generalizes to surfaces of higher topology by making use of the theory of energy minimizing Sobolev maps in certain homotopy classes [SW22] that will be introduced in Section 4.3.1, see Theorem C.1.4. We furthermore obtain the following regularity result for energy minimizers.

Theorem 4.1.5 ([LW17a, Theorem 6.2] and [FW20, Corollary 1.3])**.** *Let $u\in N^{1,2}(U,X)$ and let g be a Riemannian metric on M. If $E_+^2(u,g)\le E_+^2(u,h)$ holds for every Riemannian metric h on M, then u is infinitesimally isotropic with respect to g.*

Infinitesimal isotropy is the strongest metric variant of weak conformality: if X is a Riemannian manifold, or more generally a space with property (ET) (cf. [LW17a, Definition 11.1]), then infinitesimal isotropy is equivalent to weak conformality, see [LW17a, Theorem 11.3].

Definition 4.1.6. A map $u\in N^{1,2}_{\mathrm{loc}}(U,X)$ is called *infinitesimally isotropic with respect to a Riemannian metric g on M* if for almost every $z\in U$ the approximate metric derivative $\operatorname{ap}\operatorname{md}u_z$ is either zero or it is a norm and the John's ellipse E_z of $\operatorname{ap}\operatorname{md}u_z$ is a round ball with respect to g. Moreover, we call $u\in N^{1,2}(D,X)$ *infinitesimally isotropic* if u is infinitesimally isotropic with respect to g_{Eucl}.

By John's theorem (Theorem 2.5.1) it follows that if $u\in N^{1,2}_{\mathrm{loc}}(U,X)$ is infinitesimally isotropic with respect to g, then it is infinitesimally K-quasiconformal with respect to g for $K=4/\pi$.

Definition 4.1.7. A map $u \in N^{1,2}(U, x)$ is *infinitesimally K-quasiconformal with respect to g* if

$$\rho_u^u(z)^2 \leq K \cdot J(\operatorname{ap}\operatorname{md} u_z)$$

for almost every $z \in U$. Again, we say that $u \in N^{1,2}(D, X)$ is *infinitesimally K-quasiconformal* if u is infinitesimally K-quasiconformal with respect to g_{Eucl}.

If u is not only infinitesimally quasiconformal with respect to g but also continuous and monotone, then u satisfies the following modulus inequality.

Lemma 4.1.8 ([LW20, Proposition 3.5])**.** *Let X be a complete metric space, and let $u\colon M \to X$ be continuous and monotone. If $u \in N^{1,2}(M, X)$, and u is infinitesimally K-quasiconformal with respect to g then*

$$\operatorname{mod}(\Gamma) \leq K \cdot \operatorname{mod}(u \circ \Gamma) \tag{4.1}$$

for every family Γ of curves in U.

In particular, if in addition to the assumptions on u in Lemma 4.1.8, the map u is surjective, then u is weakly K-quasiconformal. Note that [LW20, Proposition 3.5] was established for the case of $M = \overline{D}$, but the same argument generalizes to surfaces of higher topology.

The last property of infinitesimally isotropic mappings we make use of is the following. If $u \in N^{1,2}(U, X)$ is infinitesimally isotropic with respect to a Riemannian metric g then, by [FW20, Proposition 1.1], the Reshetnyak energy agrees with the *inscribed Riemannian area*

$$\operatorname{Area}_{\mu^i}(u) := \int_M J_{\mu^i}(\operatorname{ap}\operatorname{md} u_z) d\mathcal{H}_g^2(z).$$

Here, the μ^i-Jacobian $\operatorname{Jac}_{\mu^i}(\operatorname{ap}\operatorname{md} u_z)$ of $\operatorname{ap}\operatorname{md} u_z$ is given by $\operatorname{Jac}_{\mu^i}(\operatorname{ap}\operatorname{md} u_z) = 0$ if $\operatorname{ap}\operatorname{md} u_z$ is degenerate and $\operatorname{Jac}_{\mu^i}(\operatorname{ap}\operatorname{md} u_z) = \pi/|E_z|_2$ if $\operatorname{ap}\operatorname{md} u_z$ is a norm, where E_z denotes the John's ellipse of $\operatorname{ap}\operatorname{md} u_z$. In particular, if $E_+^2(u, g) \leq E_+^2(v, g)$ for all $v \in \Lambda(M, \Gamma, X)$, then u minimizes the inscribed Riemannian area among all maps in $\Lambda(M, \Gamma, X)$.

4.2. Canonical quasisymmetric parametrizations of metric surfaces

The goal of this section is to use the results introduced in the previous section to provide an alternate proof following Strategy 1.3.1 of the Bonk–Kleiner uniformization theorem (Theorem 3.1.2) as well as its generalization to surfaces of higher topology, compare with [Wil08, MW13, GW18, Iko22]. This section explains strategies of proofs of the main statements of [LW20] and [A].

Let X be an Ahlfors 2-regular, geodesic metric surface. We first assume that $J \subset X$ is a Jordan domain with $\ell(\partial J) < \infty$ and such that $\overline{J}$ is linearly locally connected. This case was treated by Lytchak and Wenger in [LW20] and their main result may be stated as follows.

Theorem 4.2.1 ([LW20, Theorem 6.1])**.** *Let X be an Ahlfors 2-regular, geodesic metric surface. Let $J \subset X$ be a Jordan domain with $\ell(\partial J) < \infty$ and such that $\overline{J}$ is linearly locally connected. Then, there exist a continuous map $u \in \Lambda(\overline{D}, \partial J, \overline{J})$ satisfying*

$$E_+^2(u) = \inf\{E_+^2(v) : v \in \Lambda(\overline{D}, \partial J, \overline{J})\}.$$

Any such u is a quasisymmetric homeomorphism from $\overline{D}$ to $\overline{J}$ and is uniquely determined up to a conformal diffeomorphism of $\overline{D}$.

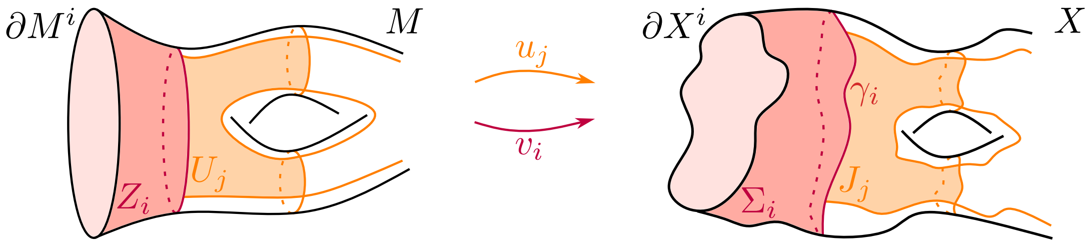


Figure 4.1.: Dissection of M and X in proof of Theorem 4.2.3.

We now describe the main steps in the proof of Theorem 4.2.1. After equipping $\overline{J}$ with its *intrinsic length metric*, i.e. the metric $d_{\overline{J}}$ given by

$$d_{\overline{J}}(x,y) := \inf\{\ell(\gamma) : \gamma\colon [0,1] \to \overline{J} \text{ locally rectifiable, } \gamma(0)=x,\, \gamma(1)=y\} \tag{4.2}$$

for $x, y \in X$, we may assume that $\overline{J}$ is geodesic. Note that this change of metric preserves lengths of curves as well as the Hausdorff 2-measure of Borel subsets, see [LW20, Lemma 2.1]. Hence, linear local connectivity and upper Ahlfors 2-regularity are preserved by this change of metric. By [LW20, Corollary 5.5], the space$(\overline{J}, d_{\overline{J}})$ admits a quadratic isoperimetric inequality, implying that $\Lambda(\overline{D}, \partial J, \overline{J})$ is not empty. The existence of a map $u \in \Lambda(\overline{D}, \partial J, \overline{J})$ minimizing the Reshetnyak energy $E_+^2(u)$ among all maps in $\Lambda(\partial J, \overline{J})$ follows from Theorem 4.1.3. By Theorem 4.1.5, the map u is infinitesimally isotropic and thus infinitesimally K-quasiconformal for $K = 4/\pi$. Moreover, u has a continuous representative, denoted again by u, which extends continuously to the boundary, see [LW17b, Theorem 4.4]. By Theorem 4.1.4, the map u is monotone and by [LW20, Theorem 3.6] a homeomorphism. Moreover, as $\overline{J}$ is upper Ahlfors 2-regular and linearly locally connected, the map u upgrades to being a quasisymmetry, compare to [LW20, Theorem 2.5] and [HK98, Theorem 4.7]. This finishes the sketch of proof of Theorem 4.2.1.

With Theorem 4.2.1 at hand, we may now construct the above mentioned quasisymmetric parametrization of an Ahlfors 2-regular, linearly locally connected and geodesic metric space X homeomorphic to a compact smooth surface M with $k \geq 0$ boundary components. The cases $M = \overline{D}$ and $M = S^2$ are the content of [LW20], where the other cases are treated in [A].

Define $\Lambda(M, X)$ to be the family of Newton-Sobolev maps $u \in N^{1,2}(M, X)$ such that u is a uniform limit of homeomorphisms from M to X. The boundary ∂X of X might consist of Jordan curves of unknown regularity. After cutting along bi-Lipschitz Jordan curves γ_i homotopic in X to a component ∂X^i of ∂X, we may assume that X has empty or bi-Lipschitz boundary; see Figure 4.1. The fact that γ_i can be chosen to be bi-Lipschitz follows from [LW20, Lemma 4.2]. The cylinders Σ_i bounded by γ_i and ∂X^i can be parametrized by quasisymmetries $v_i\colon Z_i \to \Sigma_i$, where $Z_i \subset M$ is homeomorphic to a cylinder and $\partial M^i \subset \partial Z_i$; see Proposition A.3.6. Here, ∂M^i denotes the i-th connected component of ∂M for $i \in \{1, ..., k\}$. After a suitable dissection of M and X into Jordan domains $U_j \subset M$, $J_j \subset X$, we may apply Theorem 4.2.1 to find quasisymmetric homeomorphisms $u_j\colon \overline{U_j} \to \overline{J_j}$. Note that the dissection is such that $\overline{U_j}$ is bi-Lipschitz equivalent to $\overline{D}$ and ∂J_j is a bi-Lipschitz curve. By using that every quasisymmetry $S^1 \to S^1$ extends to a quasisymmetry $\overline{D} \to \overline{D}$, see [BA56, Theorem 1], we may prescribe $u_j|_{\partial U_j}$ so that all u_j and all v_i align along common boundary. By the Sobolev gluing theorem, see [KS93, Theorem 12.1.3], the map $u\colon M \to X$ agreeing with u_j on U_j and with v_i on Σ_i is in $N^{1,2}(M, X)$. Moreover, by the quasisymmetric gluing theorem [AKT05, Theorem 3.1], the map u is a quasisymmetry. As quasisymmetries preserve linear local connectedness, this already establishes the following generalization of the Bonk–Kleiner uniformization theorem (Theorem 3.1.2).

Proposition 4.2.2. *Let X be a geodesic Ahlfors 2-regular metric space homeomorphic to a smooth surface M with possibly non-empty boundary. Then, X is quasisymmetrically equivalent to M if and only if X is linearly locally connected.*

Proposition 4.2.2 was previously only known for X being closed, see [GW18, Iko22], or for $X \setminus \partial X$ being a domain in S^2, see [Wil08, MW13, RR21].

The non-emptiness of $\Lambda(M, X)$ furthermore implies the existence of a *canonical* energy minimizing quasisymmetry $u \in \Lambda(M, X)$. Namely, assume that X has rectifiable boundary. After proving equicontinuity of a family of mappings in $\Lambda(M, X)$ of uniformly bounded energies (see Proposition A.4.1), we use a direct variational approach to find an energy minimizer in $\Lambda(M, X)$. This energy minimizer can be shown to be the canonical quasisymmetric parametrization of X, by using that X is linearly locally connected and Ahlfors 2-regular.

Theorem 4.2.3 (Theorem A.1.1)**.** *Let X be a geodesic metric space which is Ahlfors 2-regular, linearly locally connected, homeomorphic to a smooth surface M and has rectifiable boundary. Then, there exist a map $u \in \Lambda(M, X)$ and a Riemannian metric g on M with*

$$E_+^2(u, g) = \inf\{E_+^2(v, h) : v \in \Lambda(M, X),\ h \textit{ a smooth Riemannian metric on } M\}.$$

Any such u is a quasisymmetric homeomorphism from M to X and the pair (u, g) is uniquely determined up to a conformal diffeomorphism $\varphi\colon (M, g) \to (M, h)$.

Moreover, the metric g can be chosen to be of constant sectional curvature -1, 0 or 1 and such that ∂M is geodesic, if non-empty. The assumption of X being geodesic is natural and can be dropped if X is closed. Recall that in this case linear local connectedness is equivalent to linear local contractibility, see [BK02, Lemma 2.5]. Now, every Ahlfors 2-regular and linear local contractible metric surface is quasiconvex (see [Sem96a, Theorem B.6]) and thus geodesic up to a bi-Lipschitz change of metric. A natural question arising is the following.

Question 4.2.4. *Do Proposition 4.2.2 and Theorem A.1.1 still hold after removing the assumption of X being geodesic?*

4.3. Weakly quasiconformal parametrizations of metric surfaces

Within this section let M be a compact smooth surface and let X be a metric space homeomorphic to M and of finite Hausdorff 2-measure. By only assuming that X is locally geodesic and has rectifiable boundary, we want to show the existence a parametrization $u\colon M \to X$ that is weakly quasiconformal. The case where $M = \overline{D}$ was considered in [B], while the case for a general compact smooth surface M with non-empty boundary ∂M is contained in [C].

Let $\varphi\colon M \to X$ be a homeomorphism. In a first step we show that the family $\Lambda(M, \varphi, X)$ of all Sobolev maps $u \in \Lambda(M, \partial X, X)$ that are 1-homotopic to φ relative to the boundary ∂X is not empty. The theory of area minimizing mappings in relative 1-homotopy classes in metric spaces as introduced in [SW22] is reviewed in Section 4.3.1. A priori, the definition of relative 1-homotopy depends on the existence of a local quadratic isoperimetric inequality, which we circumvent by constructing the homotopy in a suitable ambient space that retracts onto X, see Section 4.3.1. In [SW22] the existence of a map in $\Lambda(M, \varphi, X)$ highly depends on the fact that X itself admits a local quadratic isoperimetric inequality. Instead, we make use of the 2-dimensional structure of X to prove the following statement.

Theorem 4.3.1 (Theorem B.1.4 and Theorem C.1.2)**.** *Let X be a locally geodesic metric space homeomorphic to a compact smooth surface M that is not a sphere and let $\varphi\colon M \to X$ be a homeomorphism. If $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$, then the family $\Lambda(M, \varphi, X)$ is not empty.*

In Section 4.3.2 we will describe how Theorem 4.3.1 is proven. By applying a direct variational method, the existence of $u \in \Lambda(M, \varphi, X)$ and a Riemannian metric g on M with

$$E_+^2(u, g) = \inf\{E_+^2(u', g') : u' \in \Lambda(M, \varphi, X), g' \text{ a Riemannian metric on } M\}$$

can be established, see Theorem C.3.4. Such a pair (u, g) is called *energy minimizing*. In a next step we show that energy minimizers possess continuous representatives.

Theorem 4.3.2 (Theorem B.1.3 and Theorem C.1.3)**.** *Let M be a smooth surface with non-empty boundary, let X be a locally geodesic metric space homeomorphic to M and let $\varphi\colon M \to X$ be a homeomorphism. If (u, g) is an energy minimizing pair, then u has a representative which is continuous and extends continuously to the boundary.*

The proof of Theorem 4.3.2 will be discussed in Section 4.3.3. By Theorem 4.3.2, without loss of generality, we may assume that the energy minimizing pair (u, g) is such that u is continuous. Any continuous map 1-homotopic to φ is homotopic to φ (see [SW22, Lemma 6.2]), as every surface not homeomorphic to S^2 has trivial second homotopy group. With the additional property that u is homotopic to φ, we may prove a generalization of Theorem 4.1.4 for surfaces of higher topology, see Theorem C.1.4. In particular, we show that u is monotone. By Theorem 4.1.5, the map u is infinitesimally isotropic with respect to g and thus infinitesimally $(4/\pi)$-quasiconformal with respect to g. By Lemma 4.1.8, we obtain that u satisfies the modulus inequality (3.8) with $K = 4/\pi$. This establishes the following main theorem of this section.

Theorem 4.3.3 (Theorem B.1.1 and Theorem C.1.1)**.** *Let X be a locally geodesic metric space homeomorphic to a compact smooth surface M with non-empty boundary. If X is of finite Hausdorff 2-measure and has rectifiable boundary, then there exists a Riemannian metric g on M and a weakly $(4/\pi)$-quasiconformal map $u\colon (M, g) \to X$.*

The Riemannian metric g can be chosen in such a way that it is of constant sectional curvature -1, 0 or 1 and the boundary of M is geodesic.

The conclusion of Theorem 4.3.3 holds in a more general setting, as already seen in Theorem 3.3.4. The only necessary assumption is that X is a metric surface of locally finite Hausdorff 2-measure. Our methods rely heavily on X being locally geodesic, compact and having rectifiable boundary. We are currently unable to eliminate these assumptions while using the energy minimization approach presented in this thesis.

Question 4.3.4. *Is it possible to drop the assumptions of X being locally geodesic, compact and having rectifiable boundary? In other words, can we construct a proof of Theorem 3.3.4 while following Strategy 1.3.1?*

In Theorem 4.3.1, we exclude the case of X being a metric sphere as, for spaces with non-contractible universal coverings, we do not have the same convergence results. A further challenge lies in the fact that, due to closedness and simply-connectedness of S^2, constant mappings are contained in the relative 1-homotopy class of any homeomorphism $\varphi\colon S^2 \to X$. It is natural to pose the following question.

Question 4.3.5. *Does Theorem 4.3.1 extend to the case of mappings defined on spheres?*

It is very plausible that the answer to Question 4.3.5 is "Yes". In [G], a theory of minimal 2-spheres in a metric space setting is developed, generalizing the famous work by Sacks and Uhlenbeck [SU81]. The results and tools established in [G] could potentially provide an analogue of Theorem 4.3.1 for metric spheres. The last question arising in this context is the following.

Question 4.3.6. *Does Theorem 4.3.2 hold for closed surfaces?*

As continuity is a local statement, Question 4.3.6 is expected to be answered affirmatively. Nevertheless, with the current approach of [C], we are not able to show the statement of Theorem 4.3.2 for closed surfaces, see Remark 4.3.10.

4.3.1. Relative 1-homotopy classes of Sobolev maps

In this section we present the theory of area minimizing surfaces in homotopy classes in metric spaces introduced by Soultanis and Wenger [SW22]. The definition of relative 1-homotopy classes in [SW22] highly depends on the existence of a local quadratic isoperimetric inequality. When the target X is a metric surface, we introduce ideas from [C] and define relative 1-homotopy classes using the existence of a local quadratic isoperimetric inequality in some ε-thickening X_ε of X and a suitable retraction $R : X_\varepsilon \to X$. Here, for $\varepsilon > 0$ a metric space Y is called *ε-thickening* of X, if there exists an isometric embedding $\iota\colon X \to Y$ such that the Hausdorff distance between $\iota(X)$ and Y is less than ε. It follows from [Wen08] that for any compact metric space X and any $\varepsilon > 0$ there exists a ε-thickening X_ε of X that is again compact and satisfies a local quadratic isoperimetric inequality, see also [LWY20, Lemma 3.3] and [CF23, Lemma 5.1].

Let $\varphi\colon M \to X$ be a homeomorphism, where M is a compact smooth surface and X a locally geodesic metric space that is of finite Hausdorff 2-measure. By Lemma C.2.4, there exists $\varepsilon > 0$ such that for any ε-thickening Y of X there is a continuous retraction from Y to X. In particular, we find a continuous retraction $R\colon X_\varepsilon \to X$ for all small enough $\varepsilon > 0$.

A finite collection K of compact convex polytopes (called *cells* of K) in some $\mathbb{R}^n$ is a *polyhedral complex* if each face of a cell is in K and the intersection of two cells of K is a face of each of them. We always equip K with the induced metric from $\mathbb{R}^n$, implying that a 2-cell Δ is isometric to a compact convex polygon in $\mathbb{R}^2$. A *triangulation* of M consists of a polyhedral complex K and a homeomorphism $h\colon K \to M$, where h restricted to any 2-cell Δ of K is a C^1-diffeomorphism onto its image. The *j-skeleton of K*, denoted K^j, is the union of all cells of K of dimension at most j and $\partial K \subset K^1$ is the preimage of ∂M under h.

Two continuous maps $\varrho, \varrho'\colon K^1 \to X$ with $\varrho|_{\partial K}, \varrho'|_{\partial K} \in [\partial X]$ are said to be *homotopic relative to ∂X* in some ambient space $Y \supset X$, denoted

$$\varrho \sim \varrho' \text{ rel } \Gamma \text{ in } Y,$$

if there exists a homotopy H in Y between ϱ and ϱ' with $H(\cdot, t)|_{\partial K} \in [\partial X]$ for all $t \in [0, 1]$. If Y is not mentioned, we assume $X = Y$. The relative homotopy class of ρ in X is denoted by $[\varrho]_{\partial X}$.

Definition 4.3.7. An *admissible deformation on a surface M* is a smooth map $\Phi\colon M \times \mathbb{R}^m \to M$, $m \in \mathbb{N}$, where $\Phi_\xi := \Phi(\cdot, \xi)$ is a diffeomorphism for every $\xi \in \mathbb{R}^m$ and $\Phi_0 = \mathrm{id}_M$, and such that the derivative of $\Phi^p := \Phi(p, \cdot)$ satisfies

$$D\Phi^p(0)(\mathbb{R}^m) = \begin{cases} T_pM & \text{if } p \in M \setminus \partial M \\ T_p(\partial M) & \text{if } p \in \partial M. \end{cases}$$

The existence of an admissible deformation $\Phi\colon M \times \mathbb{R}^m \to M$ on M follows from [SW22, Proposition 3.2], for related results see [Whi86, Whi88, HL03]. For a triangulation $h\colon K \to M$ of M and $\xi \in \mathbb{R}^m$ denote by $h_\xi\colon K \to M$ the triangulation given by $h_\xi := \Phi_\xi \circ h$. Furthermore, for $\xi \in \mathbb{R}^m$ and $u \in N^{1,2}(M, X)$ we denote by $u \circ h_\xi|_{K^1}$ the map agreeing with $u \circ h_\xi$ on $K^1 \setminus \partial K$ and with $\operatorname{tr}(u) \circ h_\xi$ on ∂K. Fix a Riemannian metric g on M. In [SW22, Section 3] it is shown that for every $u \in \Lambda(M, \Gamma, X)$ and every triangulation $h\colon K \to M$ of M there exists a ball $B_{\Phi,h} \subset \mathbb{R}^m$ centered at the origin such that for almost all $\xi, \zeta \in B_{\Phi,h}$ the maps $u \circ h_\xi|_{K^1}$ and $u \circ h_\zeta|_{K^1}$ are continuous and homotopic relative to Γ in X_ε. After postcomposition with R, the continuous representatives of $u \circ h_\xi|_{K^1}$ and $u \circ h_\zeta|_{K^1}$ are homotopic relative to Γ in X. We denote the common relative homotopy class by $u_{\#,1}[h]$. Note that $u_{\#,1}[h]$ is independent of the choice of deformation Φ and inducing the same relative homotopy class is independent of the triangulation h, see [SW22, Theorem 4.1]. Moreover, if u is continuous, then $u_{\#,1}[h] = [u \circ h|_{K^1}]_\Gamma$ for every triangulation h of M.

Definition 4.3.8. Two maps $u, v \in \Lambda(M, \Gamma, X)$ are 1-*homotopic relative to* Γ, denoted $u \sim_1 v$ rel Γ, if $u_{\#,1}[h] = v_{\#,1}[h]$ for one and thus any triangulation h of M.

4.3.2. Non-trivial Sobolev maps in geodesic metric surfaces

The goal of this section is to construct a non-trivial map in $\Lambda(M, \varphi, X)$, and thus, prove Theorem 4.3.1. The idea is to factorize through a simplicial complex Σ and use the 2-dimensional Euclidean structure of Σ to build for every $n \in \mathbb{N}$ a Lipschitz map v_n from M into a certain $(1/n)$-thickening $X_{1/n}$ of X with an upper bound on $\operatorname{Area}(v_n)$ not depending on n. Note that in general, the Lipschitz constant of v_n blows up as n tends to ∞, but the uniform bound on the area allows to apply compactness results. Similarly as in the proof of Theorem 4.1.3, we find that after precomposing each v_n with a certain diffeomorphism $M \to M$, a subsequence of $\{v_n\}_{n\in\mathbb{N}}$ converges in $L^2(M, X)$ to a map $u \in \Lambda(M, \varphi, X)$. For more details, we refer to the proofs of Theorem B.1.4 and Theorem C.1.2.

We first consider the case where $M = \overline{D}$, which is contained in Section B.5. Note that after equipping X with its intrinsic length metric as defined in (4.2), we may assume that X is geodesic. By the discussion above, we are left to prove the following proposition.

Proposition 4.3.9 (Proposition B.5.1)**.** *Suppose X is a geodesic metric space homeomorphic to $\overline{D}$. If $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$, then there exists $M > 0$ with the following property. For every $\varepsilon > 0$ there is a Lipschitz map $v\colon \overline{D} \to E(X)$ with* $\operatorname{Area}(v) \leq M$ *and such that $v|_{S^1}$ parametrizes ∂X and the image of v is contained in the ε-neighborhood of X in $E(X)$.*

A metric space E is called *injective* if for every metric space Z and any subset $Y \subset Z$, every 1-Lipschitz map $Y \to E$ extends to a 1-Lipschitz map $Z \to E$. By [Isb64], for every metric space X there exists an injective metric space $E(X)$ which contains X and is minimal in an appropriate sense among all injective metric spaces containing X. Such a space $E(X)$ is called the *injective hull* of X and is unique up to isometry. Moreover, if X is compact then so is $E(X)$. See [Isb64] for the proof of these properties.

Let X be as in Proposition 4.3.9. The Lipschitz extension properties of $E(X)$ can be used to prove the following Lipschitz factorization result, which is a consequence of [JL22, Theorem 2]

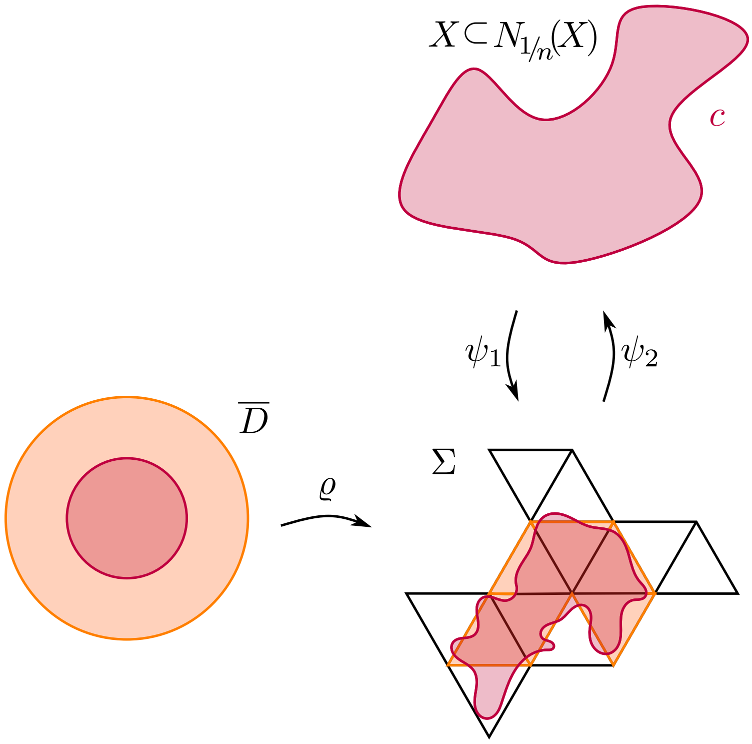


Figure 4.2.: Factorization in proof of Proposition 4.3.9.

and [BWY23, Theorem 1.6] (see also Lemma B.5.2). There exists a constant $C \geq 1$ such that for every $r > 0$ there is a finite 2-dimensional geodesic metric simplicial complex Σ, where every simplex σ of Σ is a Euclidean simplex of side length r, and there exist C-Lipschitz maps $\psi_1 \colon X \to \Sigma$ and $\psi_2 \colon \Sigma \to N_{Cr}(X) \subset E(X)$ with

$$d(x, \psi_2(\psi_1(x))) \leq Cr$$

for all $x \in X$. We illustrate the setup of the proof of Proposition 4.3.9 in Figure 4.2.

Let $c \colon S^1 \to X$ be a constant speed parametrization of ∂X and set $L := \ell(c)$. By Jordan-Schoenflies, there exists a homeomorphism $\eta \colon \overline{D} \to X$ extending c. Moreover, using the 2-dimensional Euclidean structure of Σ, we find the existence of a Lipschitz homotopy H between $\psi_1 \circ c$ and a Lipschitz curve $\gamma \colon S^1 \to \Sigma^1$, where Σ^1 denotes the 1-skeleton of Σ. Moreover, the map $\varrho \colon \overline{D} \to \Sigma$ obtained after gluing $\psi_1 \circ \eta$ and H and then reparametrizing satisfies

$$\int_{\Sigma} N(z, \varrho, \overline{D}) \, d\mathcal{H}^2(z) \leq C'(\mathcal{H}^2(X) + rL), \tag{4.3}$$

where C' only depends on C. In a next step we make use of a result of Radó [Rad38] to modify the map ϱ within D to create a map $\overline{\varrho}$ that is Lipschitz but still satisfies (4.3).

Let $f \colon U \to \mathbb{R}^2$ be continuous, where $U \subset \mathbb{R}^2$ is open. For $y \in \mathbb{R}^2$ with $N(y, f, U) < \infty$ and every $x \in f^{-1}(y)$ we denote by $\iota(f, x)$ the *winding number* of the curve $f \circ \gamma$ with respect to y, where $\gamma_r \colon S^1 \to \mathbb{R}^2$ is given by $\gamma_r(z) = x + rz$ and $r > 0$ is chosen so small that $\overline{B}(x, r) \subset U$ and $\overline{B}(x, r) \cap f^{-1}(y) = \{x\}$. Note that the winding number of $f \circ \gamma$ with respect to y is independent of the choice of such r. It follows from [Rad38, Lemma 5.2] that there exists an at most countable subset N of $A := \{y \in \mathbb{R}^2 : N(y, f, U) < \infty\}$ with $|\iota(f, x)| \leq 1$ for each $y \in A \setminus N$ and every $x \in f^{-1}(y)$.

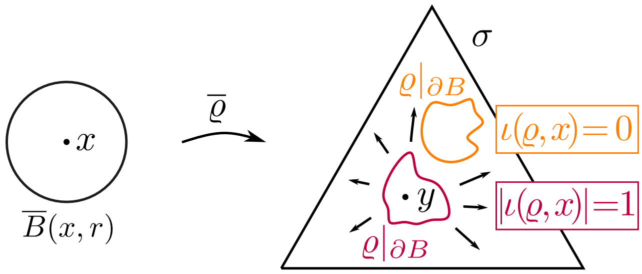


Figure 4.3.: Lipschitz modification of ϱ in proof of Proposition 4.3.9.

In particular, we may choose for every 2-cell σ of Σ a point $y \in \operatorname{int}(\sigma)$ with

$$N(y, \varrho, \overline{D}) \leq \frac{1}{|\sigma|} \int_{\sigma} N(\varrho, z)\, d\mathcal{H}^2(z)$$

and such that $|\iota(\varrho, x)| \leq 1$ for any $x \in \varrho^{-1}(y)$. We choose $r > 0$ so small that the balls $B(x,r)$, $x \in \varrho^{-1}(y)$, are disjoint and define $\overline{\varrho}$ on $\overline{B}(x,r)$ such that

- $\overline{\varrho}|_{\overline{B}(x,r)}$ is constant with image in $\partial\sigma$ if $\iota(\varrho, x) = 0$,
- $\overline{\varrho}|_{\overline{B}(x,r)}$ is a bi-Lipschitz homeomorphism and $\overline{\varrho}|_{\partial B(x,r)}$ is homotopic to $\pi \circ \varrho|_{\partial B(x,r)}$ in $\partial\sigma$ if $|\iota(\varrho, x)| = 1$.

The construction of the map $\overline{\varrho}|_{\overline{B}(x,r)}$ in these two cases is illustrated in Figure 4.3. The map $\overline{\varrho}$ defined on $\bigcup_{x\in\varrho^{-1}(y)} \overline{B}(x,r)$ extends to a Lipschitz map $\overline{\varrho}\colon \overline{D} \to \Sigma$ with $\overline{\varrho}|_{S^1} = \varrho|_{S^1}$ and

$$\overline{\varrho}\left(\overline{D} \backslash \bigcup B(x,r)\right) \subset \Sigma^{(1)}$$

since ϱ extends continuously to a map from $\overline{D}$ to Σ and $\Sigma^{(1)}$ is locally Lipschitz 1-connected, i.e. for some $d > 0$ there exists $\lambda > 0$ such that every L-Lipschitz map $f\colon S^1 \to \Sigma^{(1)}$ with $\operatorname{diam}(f(S^1)) < d$ extends to a λL-Lipschitz map $\overline{f}\colon \overline{D} \to \Sigma^{(1)}$. It can be checked that the Lipschitz map $\overline{\varrho}$ still satisfies (4.3). In particular, $\overline{\varrho}|_{S^1} = \varrho|_{S^1}$ has image in $\Sigma^{(1)}$, is $C'L$-Lipschitz and $C'r$-close to $\psi_1 \circ c$ for some constant C' only depending on C. The injectivity of $E(X)$ implies the existence of a Lipschitz homotopy G between c and $\psi_2 \circ \overline{\varrho}|_{S^1}$ with $\operatorname{Area}(G) \leq C''Lr$, where C'' only depends on C. The map $v\colon \overline{D} \to E(X)$ obtained by gluing $\psi_2 \circ \overline{\varrho}$ and G and then reparametrizing has the desired properties. A detailed proof can be found in Section B.5.

By applying gluing techniques similar as in Section 4.2, we may generalize Proposition 4.3.9 to the case where the domain is an arbitrary compact smooth surface M and the target X is a locally geodesic metric surface with $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$. Moreover, if $\varphi\colon M \to X$ is a given homeomorphism, we can construct the Lipschitz map $v\colon M \to E(X)$ in a way that φ and v are 1-homotopic relative to ∂X in some ε-thickening of X; see Section C.3.

4.3.3. Continuity of energy minimizers

The aim of this section is to describe the strategy of proof of Theorem 4.3.2. For a detailed proof we refer the reader to Section B.4 for the case where the domain is the disc D and Section C.4 for the general case.

We consider a smooth compact surface M with non-empty boundary and a homeomorphism $\varphi\colon M \to X$, where X is a locally geodesic metric space. Moreover, let (u, g) be an energy minimizing pair. It suffices to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\operatorname{osc}(u, z, \delta) < \varepsilon$

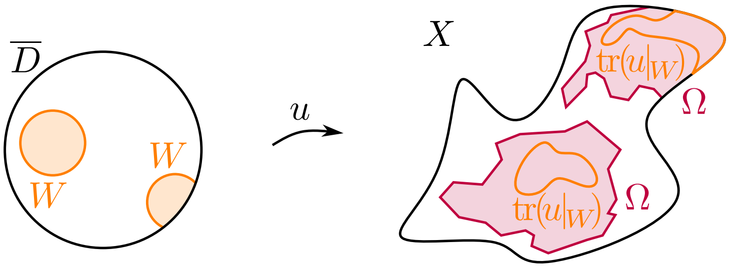


Figure 4.4.: Setup in proof of Theorem 4.3.2

for every $z \in M$ (see proof of Theorem B.1.3). Here, the *essential oscillation* of an arbitrary map $v\colon M \to X$ in the δ-ball, $\delta > 0$, around a point $z \in M$ is defined by

$$\operatorname{osc}(v, z, \delta) := \inf\{\operatorname{diam}(v(A)) : A \subset M \cap B(z, \delta) \text{ subset of full measure}\}.$$

As (u, g) is energy minimizing, it follows from Theorem 4.1.5 that the map u is infinitesimally isotropic with respect to g and thus minimizes the inscribed Riemannian area among all maps in $\Lambda(M, \partial X, X)$, recall the last paragraph of Section 4.1. Let $z \in M$ be an arbitrary point and let $\psi\colon \overline{D} \to M$ be a 2-bi-Lipschitz chart with $z \in \psi(D)$ (we choose $\psi = \mathrm{id}$ for $M = \overline{D}$). After applying the Courant-Lebesgue Lemma (see e.g. [LW17a, Lemma 7.3]) and some metric arguments, we find for every small enough $\varepsilon > 0$ a $\delta > 0$ such that for

$$W := \psi(B(z, \delta) \cap D)$$

the trace $\mathrm{tr}(u|_W)$ is contained in a Jordan domain $\Omega \subset X$ with $\operatorname{diam}(\Omega) < \varepsilon$ and $\partial\Omega \setminus \partial X$ being bi-Lipschitz, see Lemma B.4.2. The sets W and Ω are illustrated in Figure 4.4. We now consider the set

$$N := \{w \in W : u(w) \in X \setminus \overline{\Omega}\}.$$

For the moment, we assume that N is not negligible. In this case, a Fubini-type argument can be used to show that

$$\mathrm{Area}_{\mu^i}(u|_N) > 0,$$

see proof of Lemma B.4.3. Since Ω is bounded by a bi-Lipschitz curve and the boundary of M is non-empty, we find a Lipschitz retraction

$$\varrho\colon X \to \overline{\Omega} \quad \text{with} \quad \varrho(X \setminus \overline{\Omega}) \subset \partial\Omega, \tag{4.4}$$

see Lemma C.4.2. By using the general gluing theorem for Sobolev maps, see [KS93, Theorem 12.1.3], the map v agreeing with u on $M \setminus W$ and with $\varrho \circ u$ on W is contained in $\Lambda(M, \partial X, X)$. This contradicts the area minimization property of u as $\mathrm{Area}_{\mu^i}(v|_N) = 0$.

We conclude that the set N has to be negligible and thus, we obtain the desired bound on the essential oscillation.

Remark 4.3.10. Continuity is a local statement and thus the conclusion of Theorem 4.3.2 should hold for closed surfaces. However, the methods of proof described in this section do not apply to closed surfaces. Specifically, the existence of a Lipschitz retraction ϱ as in (4.4) is only guaranteed in case of X having non-empty boundary.

Part II.

Applications of uniformization of metric surfaces

5. Coarea inequality

5.1. Background

The following equality, known as the *coarea formula*, was originally established by Federer [Fed59] for Euclidean Lipschitz functions. Since then, it has become a fundamental tool in geometric measure theory and related fields. We assume that $1 \leq m < n$ and choose an open set $U \subset \mathbb{R}^n$. Let $f \colon U \to \mathbb{R}^m$ be Lipschitz, then

$$\int_{\mathbb{R}^m} \int_{f^{-1}(y)} g(x)\, d\mathcal{H}^{n-m}(x)\, dy = \int_U g(x)\, J_m(Df(x))\, dx, \tag{5.1}$$

where $g \colon U \to \mathbb{R}$ is a Borel function and $J_m(Df(x))$ denotes the *(m-dimensional) Jacobian determinant* of the differential $Df(x)$ of f at x. Here, the operator J_m is defined as follows. If $m \leq n$ and $L \colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear map with A_L being the corresponding $(n \times m)$-dimensional matrix, then

$$J_m(L) = \sqrt{\det(A_L^T A_L)}.$$

It is natural to wonder about generalizations of (5.1). For the purpose of this thesis, we restrict ourselves to the case where $n = 2$ and $m = 1$. The first extension we consider is mentioned in [MSZ03] and attributed to Federer [Fed69]: if $f \in W^{1,2}_{\mathrm{loc}}(U)$ is precisely represented, then $f^{-1}(y)$ is countably 1-rectifiable for almost every $y \in \mathbb{R}$ and the coarea formula (5.1) still holds after replacing $J_m(Df(x))$ with $|\nabla f(x)|$, where $\nabla f(x)$ denotes the weak derivative of f at a point $x \in U$. Note that every continuous mapping is precisely represented; for the exact definition we refer to [MSZ03, (2.5)].

The second extension worth noting applies to Lipschitz functions $f \colon X \to \mathbb{R}$ defined on an arbitrary metric space X. Instead of an equality as in (5.1), we derive the inequality

$$\int^* \int_{f^{-1}(t)} g\, d\mathcal{H}^1 dt \leq \frac{4}{\pi} \int_X g \cdot \operatorname{Lip}(f)\, d\mathcal{H}^2, \tag{5.2}$$

which was proven in [Fed69, Theorem 2.10.25] and further discussed and generalized in [EH21] and [EIR23, Lemma 5.2]. Here, $\operatorname{Lip}(f)$ represents the pointwise Lipschitz constant of f

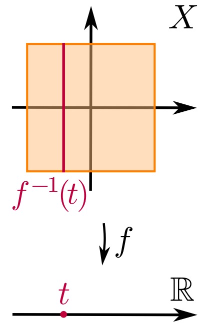
$$\operatorname{Lip}(f)(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x,y)},$$

and $\int^*$ denotes the upper integral, which is equivalent to the Lebesgue integral $\int$ whenever the integrand is a measurable function. Note that in case the right side of (5.2) is finite, the map $t \mapsto \int_{f^{-1}(t)} g\, d\mathcal{H}^1$ is measurable, and the upper integral $\int^*$ in (5.2) may be replaced by a Lebesgue integral $\int$, see [EIR23, Remark 2.12]. In the literature, inequality (5.2) is commonly referred to as the *coarea inequality* or *Eilenberg inequality*. It is important to note that achieving equality in (5.2) within this generality is not possible, and the constant $4/\pi$ is sharp, as demonstrated by the following basic example.

Example 5.1.1. Let $X = (\mathbb{R}^2, |\cdot|_\infty)$ and consider the indicator function $g = \chi_{[-1,1]^2}$ over the set $[-1,1]^2$. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := x$. It is easily seen that $\mathrm{Lip}(f) = 1$ and thus

$$\int_X g \cdot \mathrm{Lip}(f)\, d\mathcal{H}^2 = \mathcal{H}^2([-1,1]^2).$$

This equals π as $[-1,1]^2$ is the closed unit ball in X. On the other hand, the left side of (5.2) equals 4 as every $f^{-1}(t)$ is isometric to the interval $[-1,1]$.



5.2. Coarea inequality for Sobolev functions on metric surfaces

Extending the coarea inequality (5.2) to Sobolev functions with a metric surface domain is a delicate matter. According to [Che99, Proposition 1.11], the pointwise Lipschitz constant $\mathrm{Lip}(f)$ of a Lipschitz function $f\colon X \to \mathbb{R}$ serves as an upper gradient. However, it's important to note that the minimal weak upper gradient ρ_f^u may be significantly smaller than $\mathrm{Lip}(f)$. Cheeger [Che99] shows that if X satisfies strong geometric assumptions, such as the Hausdorff 2-measure on X being doubling and supporting a $(1,1)$-Poincaré inequality, then $\mathrm{Lip}(f)$ is comparable to ρ_f^u almost everywhere. For definitions of doubling and Poincaré inequality, see e.g. [HKST15]. If in addition X is geodesic, then $\mathrm{Lip}(f)(x) = \rho_f^u(x)$ for almost every $x \in X$, see [Che99, Theorem 5.1]. In this specific setting, we can replace $\mathrm{Lip}(f)$ in inequality (5.2) with any upper gradient of f. In the more general case of X being an arbitrary metric surface of locally finite Hausdorff 2-measure, this last statement does not hold, as illustrated by [EIR23, Theorem 1.7]. However, the following coarea inequality for continuous Sobolev functions on metric surfaces holds.

Theorem 5.2.1 (Theorem D.1.6)**.** *Let X be a metric surface of locally finite Hausdorff 2-measure and let $f\colon X \to \mathbb{R}$ be continuous with a 2-weak upper gradient $\rho_f \in L^2_{\mathrm{loc}}(X)$.*

(1) *If $\mathcal{A}_f$ denotes the union of all non-degenerate components of the level sets $f^{-1}(t)$, $t \in \mathbb{R}$, of f, then $\mathcal{A}_f$ is a Borel set.*

(2) *For every Borel function $g\colon X \to [0,\infty]$ we have*

$$\int^* \int_{f^{-1}(t)\cap\mathcal{A}_f} g\, d\mathcal{H}^1\, dt \le \frac{4}{\pi} \int g\rho_f\, d\mathcal{H}^2.$$

The strategy used to prove Theorem 5.2.1 highly depends on the existence of a weakly quasiconformal uniformization map $u\colon M \to X$, as guaranteed by Theorem 3.3.4. The central idea is to verify the applicability of the classical coarea formula (5.1) to the composition $v = f \circ u$, where $f\colon X \to \mathbb{R}$ is the map from Theorem 5.2.1. After some additional arguments and making use of properties of u, the statement follows. For a detailed proof we refer to Section D.5.

This proof strategy has recently been developed in [EIR23] for proving the coarea inequality (5.2) for monotone Sobolev functions defined on metric surfaces, see [EIR23, Section 4]. Here, a function $v\colon X \to \mathbb{R}$ is called a *weakly monotone function* if for every open Ω compactly contained in X

$$\sup_\Omega v \le \sup_{\partial\Omega} v < \infty \quad \text{and} \quad \inf_\Omega v \ge \inf_{\partial\Omega} v > -\infty.$$

Moreover, a continuous weakly monotone function is *monotone*.

As the level sets of monotone functions are always non-degenerate, see [Nta20, Corollary 2.8], it follows that $\mathcal{A}_f = X$ whenever f is monotone. Hence, Theorem 5.2.1 implies the main result of [EIR23] for $p \geq 2$. It is crucial to note that, without the monotonicity assumption, part (2) is optimal and does not hold for the complete level sets $f^{-1}(t)$ without restricting to $\mathcal{A}_f$, even in the case where f is Lipschitz. A relevant example illustrating this is provided in [EIR23, Section 5]. We refer the reader to [EH21] and [EIR23] for further background on the coarea inequality in metric spaces.

6. Lipschitz-volume rigidity

6.1. Background

The Lipschitz-volume rigidity problem, in its classical general formulation, poses the question of whether any surjective 1-Lipschitz map between metric spaces, which share the same volume (e.g. arising from Hausdorff measure), must be an isometry. It is well-known that the answer to the Lipschitz-volume rigidity problem is affirmative in a smooth manifold setting.

Theorem 6.1.1 (Lipschitz-volume rigidity for Riemannian manifolds)**.** *Let X and Y be closed Riemannian n-manifolds, where $n \geq 1$. If $\mathrm{Vol}(X) = \mathrm{Vol}(Y)$, then every 1-Lipschitz map from X onto Y is an isometric homeomorphism.*

See [BI10, Section 9] or [BCG95, Appendix C] for a proof of this fact. Theorem 6.1.1 has been generalized to singular settings of Alexandrov and limit RCD spaces by Storm [Sto06], Li [Li15], and Li–Wang [LW14]. See also [Li20] for an overview of the Lipschitz-volume rigidity problem in these settings. The problem in the context of integral current spaces has recently been studied by Basso–Creutz–Soultanis [BCS23], Del Nin–Perales [DNP23], and Züst [Züs24].

More generally, let $f\colon X \to Y$ be a surjective 1-Lipschitz map, where X and Y are metric spaces satisfying $\mathcal{H}^n_X(X) = \mathcal{H}^n_Y(Y) < \infty$. As f is 1-Lipschitz, we have $\mathcal{H}^n_Y(f(A)) \leq \mathcal{H}^n_X(A)$ for every measurable set $A \subset X$. After applying the same argument to the complement of A, we obtain that f is area-preserving. Here, a map $f\colon X \to Y$ is called *area-preserving* if $\mathcal{H}^n_X(A) = \mathcal{H}^n_Y(f(A))$ holds for every measurable set $A \subset X$.

We now restrict ourselves to the case $n = 2$ and pose the following question.

Question 6.1.2. *Let $f\colon X \to Y$ be an area-preserving surjective Lipschitz map between metric surfaces of locally finite Hausdorff 2-measure. Under which assumptions on X and/or Y does f upgrade to a map with better geometric and analytic properties?*

6.2. Lipschitz-Volume rigidity on metric surfaces

The goal of this section is to introduce the work [D], which provides answers to Question 6.1.2. As a corollary we obtain that Theorem 6.1.1 still holds for $n = 2$ after replacing X by a general metric surface (Theorem 1.4.1). The proofs within this section highly depend on the existence of a weakly quasiconformal uniformization map as provided by Theorem 3.3.4. Moreover, we will make use of the coarea inequality (Theorem 5.2.1) introduced in the previous section.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \subset \mathbb{R}^2 \arrow[u, "u"] \arrow[ur, "h"] & \end{tikzcd}$$

Let $f\colon X \to Y$ be an area-preserving surjective L-Lipschitz map, $L > 0$, between metric surfaces of locally finite Hausdorff 2-measure. For simplicity, we assume that X and Y are planar. By Theorem 3.3.4, there exists a weakly quasiconformal map $u\colon U \to X$, where $U \subset \mathbb{R}^2$ is a domain. We set $h := f \circ u$. As both u and h are defined on a Euclidean domain, we can compare their approximate metric derivatives to derive the following, see Section D.3.1.

Theorem 6.2.1 (Theorem D.1.4 (1))**.** *If X is reciprocal, then there exists a constant $K \geq 1$ depending only on L such that f is of K-bounded length distortion on almost every curve. Moreover, if f is 1-Lipschitz, then $K = 1$.*

Here, we call a map $f\colon X \to Y$ of *K-bounded length distortion* if

$$K^{-1} \cdot \ell(\gamma) \leq \ell(f \circ \gamma) \leq K \cdot \ell(\gamma) \tag{6.1}$$

for all curves γ in X; this includes curves of infinite length. If (6.1) holds for some constant $K \geq 1$, we say that f is of *bounded length distortion*. Moreover, if (6.1) holds for almost every curve γ in X, we say that f is of *(K-)bounded length distortion on almost every curve*.

It is an open question whether the reciprocality of X is necessary to obtain the statement of Theorem 6.2.1, see Question D.1.5. Note that a map as in Theorem 6.2.1 does not have to be a homeomorphism as illustrated by the following example.

Example 6.2.2 (Example D.4.1)**.** Let I be the interval $[0,1] \times \{0\}$ and $Y = \mathbb{R}^2/I$, equipped with the quotient metric. The natural projection map $f\colon \mathbb{R}^2 \to Y$ is area-preserving and 1-Lipschitz, but it is not a homeomorphism.

Instead of assuming that X is reciprocal, we now assume that Y is reciprocal. We may use planar topology to prove that f is a homeomorphism. Note that this step is one of the most technical parts of [D], for the arguments we refer the reader to Section D.3.2. The area-preservation and L-Lipschitz property of f imply that

$$\operatorname{mod}(\Gamma) \leq L^2 \cdot \operatorname{mod}(f \circ \Gamma) \tag{6.2}$$

holds for every family Γ of curves in X, see Lemma D.3.1. As f is a homeomorphism satisfying (6.2), the space X is reciprocal. We may apply Rajala's uniformization theorem (Theorem 3.2.3) and inequality (6.2) to show that f is geometrically quasiconformal. Here, we make use of the fact that a homeomorphism between Euclidean domains satisfying a modulus inequality as in (6.2) upgrades to being geometrically quasiconformal. Theorem 6.2.1 now implies the following.

Theorem 6.2.3 (Theorem D.1.4 (2))**.** *If Y is reciprocal, then there exists $K \geq 1$ depending only on L such that f is a geometrically K-quasiconformal homeomorphism that is of K-bounded length distortion on almost every curve. Moreover, if f is 1-Lipschitz, then we may choose $K = 1$.*

Theorem 6.2.3 is sharp as illustrated by the following example.

Example 6.2.4 (Example D.4.2)**.** Let I be the interval $(0,1] \times \{0\}$ and define a weight $\omega\colon \mathbb{R}^2 \to [0,1]$ by setting $\omega(x) = x_1$ if $x = (x_1, 0) \in I$ and $\omega(x) = 1$ otherwise. We let Y be the space $\mathbb{R}^2$ equipped with the metric

$$d(x,y) := \inf_{\gamma} \int_{\gamma} \omega \, ds, \tag{6.3}$$

where the infimum is taken over all rectifiable curves γ connecting the points $x, y \in \mathbb{R}^2$. Let $f\colon \mathbb{R}^2 \to Y$ be the identity map, which is 1-Lipschitz, since $\omega \leq 1$, and a local isometry on $\mathbb{R}^2 \setminus I$, hence area-preserving. It can be shown, see Example D.4.2, that f is a quasiconformal homeomorphism, and thus Y is reciprocal. For $t \in (0,1]$ denote by γ_t the straight line segment connecting $(0,0)$ and $(t,0)$. Then $\ell_{|\cdot|}(\gamma_t) = t$, whereas

$$\ell_d(\gamma_t) = \int_{\gamma_t} \omega \, ds = \frac{t^2}{2}.$$

This shows that the map f does not (quasi-)preserve the length of *every* curve.

We now assume that Y is locally upper Ahlfors 2-regular. Recall that every such space is reciprocal, see [Raj17, Theorem 1.6]. In particular, by Theorem 6.2.3, $f\colon X \to Y$ is a quasiconformal homeomorphism that is of bounded length distortion on almost every curve. Denote by $g\colon Y \to X$ the inverse of f, i.e. $g = f^{-1}$, and by Γ_0 the family of curves in Y on which g is not of bounded length distortion. We have that $\operatorname{mod}(\Gamma_0) = 0$. Let $\gamma \in \Gamma_0$. We want to find a nearby curve of comparable length, which is not contained in Γ_0. Note that the upper Ahlfors 2-regularity may be used to provide an upper bound on the Hausdorff 2-measure of small neighborhoods of γ, i.e. for all sufficiently small $r > 0$ we have

$$\mathcal{H}^2(N_r(|\gamma|)) \leq 2Cr\ell(\gamma) + 8Cr^2,$$

where C is the upper Ahlfors 2-regularity constant of Y, see Lemma D.3.9. After applying the coarea inequality for Lipschitz functions and planar topological arguments, we find for every $n \in \mathbb{N}$ a curve $\gamma_n \notin \Gamma_0$, $|\gamma_n| \subset N_{1/n}(|\gamma|)$ and $\ell(\gamma_n) < 4C\pi^{-1}\ell(\gamma) + n^{-1}$, see Lemma D.3.10. It follows that

$$\ell(g \circ \gamma) \leq \liminf_{n\to\infty} \ell(g \circ \gamma_n) \leq K \liminf_{n\to\infty} \ell(\gamma_n) \leq \frac{4KC}{\pi}\ell(\gamma),$$

where K is as in Theorem 6.2.3, compare to Lemma D.3.11. The fact that f is Lipschitz now implies the following.

Theorem 6.2.5 (Theorem D.1.4 (3))**.** *If Y is locally upper Ahlfors 2-regular with constant $C > 0$, then there exists a constant $K \geq 1$ depending only on L and C such that f is a homeomorphism of K-bounded length distortion.*

Note that in general, the constant K may not be improved to 1 in case of f being 1-Lipschitz.

Example 6.2.6. Let I be the interval $[0,1] \times \{0\}$ and define a weight $\omega\colon \mathbb{R}^2 \to [0,1]$ by $\omega = \chi_{\mathbb{R}^2\setminus I} + (1/2)\chi_I$. We let Y be the space $\mathbb{R}^2$ equipped with the metric d induced by ω as in (6.3). Then, Y is bi-Lipschitz equivalent to $\mathbb{R}^2$. Moreover, the identity map $f\colon \mathbb{R}^2 \to Y$ is 1-Lipschitz and area-preserving but not an isometry.

However, the obstructions of Example 6.2.6 do not appear in case of Y being smooth.

Theorem 6.2.7 (Theorem D.1.4 (4))**.** *If Y is smooth and f is 1-Lipschitz, then f is an isometric homeomorphism.*

As a corollary of Theorem 6.2.7, we obtain Theorem 1.4.1. For orientable surfaces this statement also follows from a combination of [BMW25, Theorems 1.1 and 1.3] and [Züs24], see [BMW25, Section 8]. Note that their proof also crucially depends on the existence of a weakly quasiconformal uniformization map as in Theorem 3.3.4. Recently, Marti [Mar25] showed a partial generalization of Theorem 1.4.1 to higher dimensional domains and targets.

7. Mappings of finite distortion on metric surfaces

Mappings of finite distortion are natural non-homeomorphic generalizations of quasiconformal maps, allowing the distortion to vary from point to point while remaining finite almost everywhere. Within the past few decades, a rich theory of mappings of finite distortion in a Euclidean setting has been developed (see e.g. [AIM09, HK14]), with applications to PDE, complex dynamics, inverse problems and non-linear elasticity theory, among other fields. In this chapter, we extend the definition of finite distortion to mappings on metric surfaces and develop their core properties; based on the articles [E] and [F]. Unless otherwise stated, we assume throughout this chapter that X and Y are planar metric surfaces of locally finite Hausdorff 2-measure. Our definitions and results are local and remain valid under the relaxed assumptions of X and Y being homeomorphic to smooth surfaces. The proofs of the main statements in this chapter are based on three main tools available for metric surfaces:

(a) weakly quasiconformal parametrizations of metric surfaces (Theorem 3.3.4),

(b) the coarea inequality for Sobolev functions on metric surfaces (Theorem 5.2.1), and

(c) the area inequalities on (the "good" part X' of) a metric surface X (Section 7.2 below).

Moreover, we will make use of estimates inspired by the value distribution theory of quasiregular mappings, see e.g. [Ric93].

7.1. Definitions of finite distortion

Let Ω be a domain in $\mathbb{R}^2$ and let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be non-constant. We say that the map f is of *finite distortion* if there exists a measurable function $K\colon \Omega \to [0, \infty)$ with

$$||Df(x)||^2 \leq K(x) \cdot J_2(Df(x)) \tag{7.1}$$

for almost every $x \in \Omega$, where $||Df(x)||$ and $J_2(Df(x))$ are the operator norm and Jacobian determinant of the differential $D_f(x)$ of f at x, respectively. If we can choose $K(x) = K$ for some fixed constant $K \geq 1$, the map f is called *quasiregular*. And if f is in addition a homeomorphism, then f is said to be *quasiconformal*.

A direct generalization of this definition to a metric space setting is not possible, due to lack of regularity, and thus, lack of differentiability. The goal of the current section is to provide new definitions of distortion applicable to non-homeomorphic mappings between general metric spaces. In the Euclidean context, every map of finite distortion is sense-preserving. This arises from inequality (7.1), after using the non-negativity of the Jacobian determinant and integration by parts. For a map $f : X \to Y$, we say that f is *sense-preserving* if for any domain Ω compactly contained in X so that $f|_{\partial\Omega}$ is continuous it follows that $\deg(y, f, \Omega) \geq 1$ for any $y \in f(\Omega) \setminus f(\partial\Omega)$. Here, deg is the local topological degree of f (see e.g. [Ric93, I.4]).

Remark 7.1.1 (Sense-preserving maps into $\mathbb{R}^2$). Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be sense-preserving, then f is continuous and satisfies Lusin's condition (N); see Remarks E.2.3 and E.2.8. Continuity follows from the proof of [EIR23, Theorem 1.4] after replacing weak monotonicity by sense preservation and the coarea inequality for monotone functions by Theorem 5.2.1. Moreover, an application of the weakly quasiconformal uniformization theorem (Theorem 3.3.4) in combination with the area formula (Theorem 2.5.2) shows that f satisfies Lusin's condition (N). Note that the converse is not true, as illustrated by [Raj17, Proposition 17.1]; compare to Remark 3.2.5.

Previous approaches to distortion of maps between metric spaces are often based on the analytic definition of quasiconformality, see e.g. [Wil12]. A homeomorphism $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ is analytically quasiconformal, if

$$\rho_f^u(x)^2 \leq K \cdot \mathrm{Jac}_f(x)$$

for some $K \geq 1$ and almost every $x \in X$. Here, $\mathrm{Jac}_f(x)$ denotes the Radon-Nikodym derivative of the measure $\nu(E) = \mathcal{H}^2(f(E))$ with respect to $\mathcal{H}^2$. Analytic quasiconformality provides a rich theory for homeomorphisms. But if f is not a homeomorphism, it is not always possible to make sense of Jac_f, as ν might not be a well-defined measure on X. We therefore propose to define the *Jacobian* of f at $x \in X$ by

$$J_f(x) = \limsup_{r \to 0} \frac{\mathcal{H}^2_Y(f(\overline{B}(x, r)))}{\pi r^2}.$$

Note that if f is a homeomorphism and X a domain in $\mathbb{R}^2$ or $Y = \mathbb{R}^2$, then J_f coincides with Jac_f almost everywhere, see Corollary F.3.4.

Definition 7.1.2. A sense-preserving map $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ has *finite analytic distortion* if there is a measurable function $C\colon X \to [1, \infty)$ such that

$$\rho_f^u(x)^2 \leq C(x) \cdot J_f(x) \quad \text{for almost every } x \in X. \tag{7.2}$$

The *analytic distortion* of f is defined as

$$C_f(x) := \begin{cases} \frac{\rho_f^u(x)^2}{J_f(x)}, & \text{if } J_f(x) \neq 0, \\ 1, & \text{if } J_f(x) = 0. \end{cases}$$

Within this chapter, we will call a homeomorphism $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ *analytically quasiconformal*, if there exists $C \geq 1$ such that $C_f \leq C$ almost everywhere.

For the second notion of distortion studied in this chapter, we introduce a notion of "minimal stretch factor" complimenting the "maximal stretch factor" represented by upper gradients. A Borel function $\rho^l\colon X \to [0, \infty]$ is a *lower gradient* of $f\colon X \to Y$ if $\rho^l \leq \rho_f^u$ almost everywhere and

$$\ell(f \circ \gamma) \geq \int_\gamma \rho^l \, ds \tag{7.3}$$

for every rectifiable curve γ in X with $f \circ \gamma$ being continuous. If the *lower gradient inequality* (7.3) holds for almost every rectifiable γ, we call ρ^l *weak lower gradient* of f.

This definition is motivated by the fact that if $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ is continuous, then the upper gradient inequality (2.2) is equivalent to the opposite inequality of (7.3). Note that 0 is always a lower gradient. Each $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ has a *maximal weak lower gradient* ρ_f^l, i.e. for any other weak lower gradient ρ^l we have $\rho_f^l \geq \rho^l$ almost everywhere. Moreover, ρ_f^l is unique up to a set of measure zero, see Section E.7. Similarly as in Remark 2.4.3, it follows that if $h \in N^{1,2}_{\mathrm{loc}}(U,Y)$, where U is a domain in $\mathbb{R}^2$, then the *minimal stretch factor*

$$l_h(z) = \min\{\operatorname{ap}\operatorname{md} h_z(v) : |v| = 1\}$$

is a representative of the maximal weak lower gradient ρ^l_h, see Lemma E.2.9.

Definition 7.1.3. A sense-preserving map $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ has *finite distortion along paths* if there is a measurable function $K\colon X \to [1,\infty)$ such that

$$\rho^u_f(x) \leq K(x) \cdot \rho^l_f(x) \tag{7.4}$$

for almost every $x \in X$. The *distortion along paths* K_f of f is given by

$$K_f(x) := \begin{cases} \frac{\rho^u_f(x)}{\rho^l_f(x)}, & \text{if } \rho^l_f(x) \neq 0, \\ 1, & \text{if } \rho^l_f(x) = 0. \end{cases}$$

Analogous to the Euclidean case, we call a map f *quasiregular* if the distortion K_f is uniformly bounded and *quasiconformal along paths* if f is in addition a homeomorphism.

Example 7.1.4. Let $u \in N^{1,2}_{\mathrm{loc}}(U,X)$ be a weakly $(4/\pi)$-quasiconformal parametrization of X as in Theorem 3.3.4. The proof of Theorem 3.3.4 implies that we may assume that the John's ellipse of $\operatorname{ap}\operatorname{md} u_z$ is a disc for almost every $z \in U$. By John's Theorem (Theorem 2.5.1) we know

$$L_u(z) \leq \sqrt{2} \cdot l_u(z).$$

As L_u and l_u are representatives of ρ^u_u and ρ^l_u, respectively, we obtain that u is $\sqrt{2}$-quasiregular; compare to Theorem F.2.5.

7.2. Area inequalities for Sobolev maps on metric surfaces

This section is devoted to establishing two area inequalities for Sobolev maps in $N^{1,2}_{\mathrm{loc}}(X,Y)$; both of which will play essential roles in the proofs of the main statements of this chapter. The two area inequalities highly depend on the existence of a weakly quasiconformal parametrization $u \in N^{1,2}_{\mathrm{loc}}(U,X)$ of X, where $U \subset \mathbb{R}^2$ is a domain, guaranteed by Theorem 3.3.4. Define $h\colon U \to Y$ by $h := f \circ u$. By making use of geometric and topological properties of u, we may conclude that $h \in N^{1,2}_{\mathrm{loc}}(U,Y)$, see Theorem F.2.5.

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{R}^2 \\ U \subset \mathbb{R}^2 \arrow[u, "u"] \arrow[ur, "h"'] & \end{tikzcd}$$

As described in the paragraph preceding Theorem 2.5.2, the set U may be exhausted by sets G_j, $j \geq 0$, such that $u|_{G_j}$ and $h|_{G_j}$ are j-Lipschitz for $j \geq 1$, and G_0 is of measure zero. We define

$$X_0 := u(G_0) \quad \text{and} \quad X' := X \setminus X_0.$$

As u is surjective, it follows that the set X' is countably 2-rectifiable. After applying the area formula (2.3) to both u and h and every measurable set $A \subset U \setminus G_0$, and comparing upper and lower gradients of the mappings f, u and h, we obtain the following area inequalities, see Section E.3.

Theorem 7.2.1 (Path area inequality). *Let $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$. If $g \colon Y \to [0, \infty]$ and $F \subset X'$ are Borel measurable, then*

$$\int_F g(f(x)) \cdot \rho_f^u(x) \rho_f^l(x) \, d\mathcal{H}^2_X \leq 4\sqrt{2} \int_Y g(y) \cdot N(y, f, F) \, d\mathcal{H}^2_Y.$$

If f additionally satisfies Lusin's condition (N), then

$$\int_F g(f(x)) \cdot \rho_f^u(x) \rho_f^l(x) \, d\mathcal{H}^2_X \geq \frac{1}{4\sqrt{2}} \int_Y g(y) \cdot N(y, f, F) \, d\mathcal{H}^2_Y.$$

For the second area inequality, we furthermore assume that f is sense-preserving. The same holds for the map $h = f \circ u$. By Remark 7.1.1, h satisfies Lusin's condition (N) and thus, $|f(X_0)|_2 = |h(G_0)|_2 = 0$. A covering argument now implies the following statement.

Lemma 7.2.2 (Lemma F.3.1). *If $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ is sense-preserving, then*

$$J_f(x) = 0 \quad \textit{for almost every } x \in X_0.$$

Lemma 7.2.2 and the fact that $|f(X_0)|_2 = 0$ allow to ignore subsets of X_0 when proving an area inequality involving J_f. Recall that the set $\widehat{X} = X \setminus X_0$ is countably 2-rectifiable. By a theorem of Kirchheim [Kir94, Theorem 9], there exists $E \subset X$, $\mathcal{H}^2(E) = 0$, so that

$$\lim_{r \to 0} \frac{\mathcal{H}^2(\overline{B}(x, r) \cap X')}{\pi r^2} = 1$$

for every $x \in X' \setminus E$. Together with Vitali's covering theorem, see e.g. [AT04, Theorem 2.2.2], and linear approximation of the Jacobian J_f, we may now prove the following statement, see Proposition F.3.2.

Theorem 7.2.3 (Analytic area inequality). *If $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ is sense-preserving and $F \subset X$ is Borel, then*

$$\int_F J_f(x) \, d\mathcal{H}^2_X \leq \int_{\mathbb{R}^2} N(y, f, F) \, dy. \tag{7.5}$$

If f is furthermore open and discrete, then equality holds in (7.5).

7.3. Openness and discreteness

Let Ω be a domain in $\mathbb{R}^2$. A map $f \colon \Omega \to \mathbb{R}^2$ is quasiregular of constant $K = 1$ if and only if f is complex analytic. The most important topological properties of complex analytic mappings are continuity, openness and discreteness. Here, a map is *discrete* if the preimage of every point is a discrete set. For general quasiregular mappings the same topological properties were established by Reshetnyak (see [Reš67]), and for mappings of finite distortion by Iwaniec and Šverák [IŠ93] under the additional assumption of locally integrable distortion, i.e. f satisfies (7.1) for some $K \in L^1_{\mathrm{loc}}(\Omega)$. This theorem extends to a metric surface setting, the main result of [E].

Theorem 7.3.1 (Theorem F.2.3). *Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be a non-constant mapping of finite distortion along paths with $K_f \in L^1_{\mathrm{loc}}(X)$. Then f is open and discrete.*

The following example shows that condition $K_f \in L^1_{\mathrm{loc}}(X)$ is sharp, even if $X = \mathbb{R}^2$, see [Bal81] and [HR02].

Example 7.3.2. Let $f_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_0(x,y) = (x, \eta(x,y))$, where

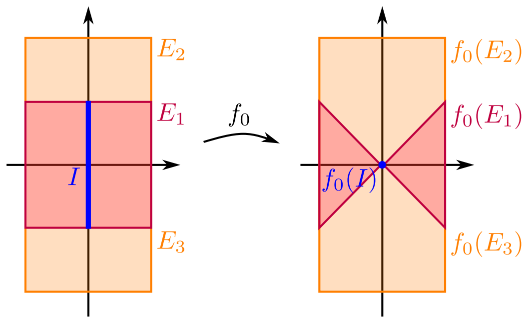
$$\eta(x,y) = \begin{cases} |x|y, & \text{if } (x,y) \in E_1, \\ (2(|y|-1) + |x|(2-|y|))\frac{y}{|y|}, & \text{if } (x,y) \in E_2 \cup E_3, \\ y, & \text{otherwise} \end{cases}$$

for $E_1 = [-1,1]^2$, $E_2 = [-1,1] \times [-2,-1]$ and $E_3 = [-1,1] \times [1,2]$. Note that f_0 is not open and discrete as f_0 maps the segment $\{0\} \times [-1,1]$ to the origin. The map f_0 is Lipschitz and the identity in the complement of $E = E_1 \cup E_2 \cup E_3$. Let $(x,y) \in E_1$, then

$$Df_0(x,y) = \begin{pmatrix} 1 & \frac{x}{|x|}y \\ 0 & |x| \end{pmatrix},$$

and thus $||Df_0(x,y)|| = 1$, $J_{f_0}(x,y) = |x|$ and $K_{f_0}(x,y) = 1/|x|$.
For $(x,y) \in E_2 \cup E_3$, we compute

$$Df_0(x,y) = \begin{pmatrix} 1 & \frac{xy(2-|y|)}{|x|\,|y|} \\ 0 & 2-|x| \end{pmatrix},$$



implying $||Df_0(x,y)|| \in [1,2]$, $J_{f_0}(x,y) \in [1,2]$, and $K_{f_0}(x,y) \leq 4$.
In conclusion, $K_{f_0} \notin L^1_{\mathrm{loc}}(\mathbb{R}^2)$, but $K_{f_0} \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for every $0 < p < 1$.

Let f be a map as in Theorem 7.3.1. Then, by Remark 7.1.1, f is continuous. To prove Theorem 7.3.1 it suffices to show that f is light. Indeed, if a map is continuous, sense-preserving and light, then it is open and discrete, see [TY62] and [Ric93, Lemma VI.5.6]. Here, $f\colon X \to \mathbb{R}^2$ is *light* if $f^{-1}(y)$ is totally disconnected for every $y \in \mathbb{R}^2$.

We know that f is not constant. In particular, for every $y_0 \in f(X)$ every component F of $f^{-1}(y_0)$ contains a point $x_0 \in X$ which is a boundary point of F. We find $s > 0$ so that $\overline{B}(x_0, 2s)$ is a compact subset of X. We will make use of the following two propositions.

Proposition 7.3.3 (Proposition E.4.1)**.** *Let $x_0 \in X$ and suppose that there are constants $s, r_0 > 0$ and $C > 0$ such that*

$$\int_0^{2\pi} N(f(x_0) + re^{i\theta}, f, B(x_0, s))\, d\theta \leq C \log \frac{1}{r} \tag{7.6}$$

for all $r < r_0$. Then, the component of $f^{-1}(f(x_0))$ containing x_0 either is $\{x_0\}$ or contains an open neighborhood of x_0.

Proposition 7.3.4 (Proposition E.4.2)**.** *Let $x_0 \in X$ and $s > 0$ so that $\overline{B}(x_0, 2s) \subset X$ is compact. Then Condition (7.6) holds with some $r_0, C > 0$.*

Proposition 7.3.4 allows us to apply Proposition 7.3.3, from which we know that $F = \{x_0\}$ or F contains an open neighborhood of x_0. The latter is not possible as x_0 is a boundary point of F. This shows that f is indeed a light map and hence open and discrete.

In the following we will introduce the main ideas in the proofs of Propositions 7.3.3 and 7.3.4; for a full proof we refer to Section E.4. Proposition 7.3.3 is established through a proof by contradiction, for a setup, see Figure 7.1. Denote by V_0 the component of $B(x_0, s)$ containing

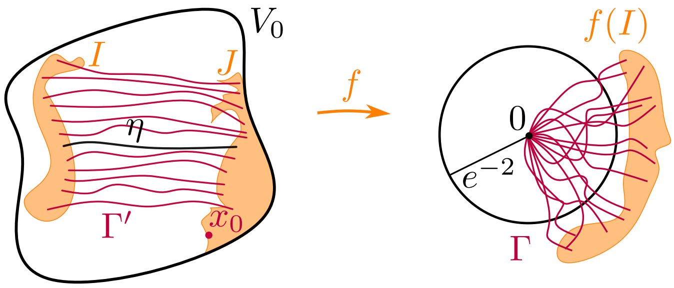


Figure 7.1.: Sketch of proof of Proposition 7.3.3.

x_0 and let J be the component of $f^{-1}(f(x_0))$ containing x_0 and intersected with V_0. Towards a contradiction, assume that J is a non-trivial continuum and there exists another non-trivial continuum $I \subset V_0 \setminus J$. By scaling and translating, we may assume that $f(x_0) = 0$, the set $f(I)$ does not intersect $B(0, e^{-2})$, and $r_0 \geq e^{-2}$. For a given family Γ of curves in X, we introduce the concept of *weighted modulus* by

$$\mathrm{mod}_{K^{-1}}(\Gamma) = \inf_{\rho} \int_X \frac{\rho(x)^2}{K_f(x)} d\mathcal{H}^2,$$

where the infimum is taken over all weakly admissible functions ρ for Γ. The underlying idea is to construct a set $E \subset \mathbb{R}$ with $|E|_1 > 0$ and a "good" family Γ' of curves γ_t, $t \in E$, connecting I and J, see Lemma E.4.5. Here, each γ_t is chosen to be a curve with image in the level set $\varphi^{-1}(t)$, where $\varphi(\cdot) := \mathrm{dist}(\cdot, |\eta|)$ for some fixed rectifiable curve η connecting I and J. The Lipschitz coarea inequality and Hölder's inequality imply $\mathrm{mod}_{K^{-1}}(\Gamma') > 0$. In Lemma E.4.4, we use (a), (b) and Hölder's inequality to show that $\mathcal{H}^1(|\gamma_t| \cap X_0) = 0$ for almost every $t \in E$. Recall that X_0 is the part of X on which the area formula (Theorem 2.5.2) can not be applied. By (c) and some additional arguments, we obtain

$$\mathrm{mod}_{K^{-1}}(\Gamma') \leq 4\sqrt{2} \int_{\mathbb{R}^2} g(y)^2 N(y, f, \Omega)\, dy, \tag{7.7}$$

whenever g is admissible for $\Gamma = f \circ \Gamma'$, see Lemma E.4.3. Every curve in Γ connects 0 with $\partial B(0, e^{-1})$ in $\mathbb{R}^2$. In Lemma E.4.6, we construct a family of functions g_ε, $\varepsilon > 0$, admissible for Γ and such that

$$\int_{\mathbb{R}^2} g_\varepsilon(y)^2 \log \frac{1}{|y|}\, dy \to 0,$$

as $\varepsilon \to 0$. The assumptions of Proposition 7.3.3 in combination with (7.7) now imply that $\mathrm{mod}_{K^{-1}}(\Gamma') = 0$, a contradiction.

We now sketch a proof of Proposition 7.3.4, see Figure 7.2. Let $x_0 \in X$ and $s > 0$ be so that $\overline{B}(x_0, 2s) \subset X$ is compact. After scaling and translating, we may assume $f(x_0) = 0$ and $f(\overline{B}(x_0, 2s)) \subset \overline{D}$. By using arguments as above and (b), we can show that $|\beta'|_1 = 0$ for

$$\beta' := \{\theta \in [0, 2\pi) \colon \exists R_\theta > 0 \text{ s.th. } f^{-1}(R_\theta e^{i\theta}) \text{ contains a non-degenerate continuum}\},$$

see Lemma E.4.7. In particular, $f^{-1}(y)$ is totally disconnected for most points $y \in f(X)$ around 0. Define $\varphi \colon X \to [0, 2\pi)$ by $\varphi(x) = \arg(f(x))$. Using sense-preservation of f, Proposition 7.3.3 and similar arguments as above, we find the existence of a set $\beta \supset \beta'$, $|\beta|_1 = 0$, and an open set $\Omega' \subset X$ with $\Omega' \supset X \setminus \varphi^{-1}(\beta)$ and such that $f|_{\Omega'}$ is a local homeomorphism; the content of Lemma E.4.8. We choose $m \in \mathbb{N}$ and $r \in (0, e^{-2})$, and define E_m to be the set of all

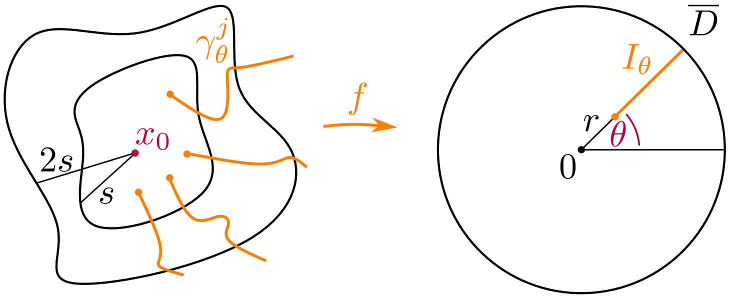


Figure 7.2.: Sketch of proof of Proposition 7.3.4.

$\theta \in [0, 2\pi)$ with $re^{i\theta}$ having m preimages in $B(x_0, s)$ under f. For a fixed $\theta \in E_m \setminus \beta$, we set $I_\theta = \{te^{i\theta} : t \in [r, 1]\}$ and obtain $f^{-1}(I_\theta) \subset \Omega'$. Hence, there exist maximal lifts $\{\gamma_\theta^1, ..., \gamma_\theta^m\}$ of I_θ, each starting at a distinct point of $f^{-1}(re^{i\theta}) \cap B(x_0, s)$. Note that γ_θ^j has image in the level set $\varphi^{-1}(\theta)$ and connects $B(x_0, s)$ with $X \setminus B(x_0, 2s)$. In particular, $\mathcal{H}^1(\varphi^{-1}(\theta)) \geq s \cdot m$. After combining with (b), Hölder's inequality and (c), and using polar coordinates, we obtain

$$m|E_m|_1 \leq C \int_{F_m} K_f \, d\mathcal{H} \cdot \log \frac{1}{r},$$

where $F_m = \varphi^{-1}(E_m)$ and $C > 0$ only depends on s, see Lemma E.4.9. After summing over m and possibly replacing X by a compactly contained subdomain to ensure integrability of K_f, Proposition 7.3.4 follows.

7.4. Regularity of the inverse

Another nontrivial question of interest is the following. Under which conditions on a map $f \in N^{1,2}(X, \mathbb{R}^2)$, that is a homeomorphism onto its image, can we say something about the regularity of its inverse $f^{-1} \colon f(X) \to X$? Hencl and Koskela [HK06] addressed this question in the case of X being a Euclidean domain, while assuming $W^{1,1}$-regularity. They showed that if f is a homeomorphism onto its image with locally integrable distortion, then the inverse f^{-1} is in $W^{1,2}(f(X), X)$ and a map of finite distortion. In [E] we provide the following partial generalization of this statement.

Theorem 7.4.1 (Theorem E.1.3)**.** *Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be an injective mapping of finite distortion with $K_f \in L^1_{\mathrm{loc}}(X)$. Then $f^{-1} \in N^{1,2}_{\mathrm{loc}}(f(X), X)$.*

Let f be as in Theorem 7.4.1 and set $\phi = f^{-1} \colon \Omega' \to X$. We define $\psi \colon \Omega' \to [0, \infty]$ by

$$\psi(y) = \frac{1}{\rho_f^l(\phi(y))}.$$

In the following, we want to make use of the characterization of Sobolev maps in terms of post-compositions with 1-Lipschitz functions, see [HKST15, Theorem 7.1.20], to show that $\phi \in N^{1,2}_{\mathrm{loc}}(\Omega', X)$. For this let $\alpha \colon X \to \mathbb{R}$ be 1-Lipschitz. By Lemma E.5.2, the map $v = \alpha \circ \phi$ is absolutely continuous on almost every line parallel to coordinate axes, and $|\partial_j v| \leq \frac{16\sqrt{2}}{\pi} \cdot \psi$ almost everywhere for $j = 1, 2$. It follows from Lemma E.5.1 that

$$\int_E \psi(y)^2 \, dy \leq 2 \int_{\phi(E)} K_f(x) \, d\mathcal{H}^2(x)$$

for every Borel set $E \subset \Omega'$. In particular, as $K_f \in L^1_{\mathrm{loc}}(X)$, we obtain $\psi \in L^2_{\mathrm{loc}}(\Omega')$ and $\partial_j v \in L^2_{\mathrm{loc}}(\Omega')$. The ACL-characterization of Sobolev functions, see [HKST15, Theorem 6.1.17], implies that $v \in W^{1,2}_{\mathrm{loc}}(\Omega')$. As $|\nabla v| \leq 32\psi\pi^{-1}$ almost everywhere, [HKST15, Theorem 7.1.20] shows that $\phi \in N^{1,2}_{\mathrm{loc}}(\Omega', X)$.

Comparing Theorem 7.4.1 with the above mentioned result of Hencl and Koskela [HK06] raises the following question.

Question 7.4.2. *Is it possible to construct a satisfactory theory of $N^{1,1}_{\mathrm{loc}}$-mappings of finite distortion between metric surfaces, potentially leading to improvements of Theorem 7.4.1?*

A challenge in this context is that the conclusions of Remark 7.1.1 do not hold in the $N^{1,1}$-setting without additional assumptions. There are examples of maps $f \in N^{1,1}_{\mathrm{loc}}(X, \mathbb{R}^2)$ of finite distortion that are not continuous nor do they satisfy Lusin's condition (N), see e.g. [HK14].

7.5. Equivalence of notions of finite distortion

The goal of this section is to give insight into the proof of the following theorem.

Theorem 7.5.1 (Theorem F.1.1)**.** *Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be sense-preserving.*

(i) *If f is of finite analytic distortion, then f is of finite distortion along paths and*

$$K_f(x) \leq 4\sqrt{2}\, C_f(x) \quad \text{for almost every } x \in X.$$

(ii) *If f is of finite distortion along paths with $K_f \in L^1_{\mathrm{loc}}(X)$, then f is of finite analytic distortion and*

$$C_f(x) \leq 4\sqrt{2}\, K_f(x) \quad \text{for almost every } x \in X.$$

It is an open question whether (ii) holds without the assumption of $K_f \in L^1_{\mathrm{loc}}(X)$. An application of Theorem 7.5.1 combined with a theorem of Williams [Wil12, Theorem 1.1] provides the following corollary.

Corollary 7.5.2 (Corollary F.1.2)**.** *If $f \colon X \to f(X) \subset \mathbb{R}^2$ is a homeomorphism, then the following conditions are quantitatively equivalent.*

1. *f is analytically quasiconformal,*
2. *f is geometrically quasiconformal,*
3. *f is quasiconformal along paths.*

Moreover, if f satisfies any of the three conditions, then so does f^{-1}.

The rest of this section is devoted to proving Theorem 7.5.1. Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be sense-preserving and let X_0 and $X' = X \setminus X_0$ be the sets from the paragraph preceding Theorem 7.2.1. The two area inequalities (Theorems 7.2.1 and 7.2.3) combined with Remark 7.1.1 give

$$J_f(x) \leq 4\sqrt{2}\, \rho^u_f(x)\rho^l_f(x) \tag{7.8}$$

for almost every $x \in X'$. If f is furthermore open and discrete, then also

$$\rho^u_f(x)\rho^l_f(x) \leq 4\sqrt{2}\, J_f(x) \tag{7.9}$$

for almost every $x \in X'$, see Corollary F.3.3.

If f is of finite analytic distortion, then, (7.8) implies that Part (i) holds on X'. Moreover, by definition of finite analytic distortion, ρ^u_f vanishes almost everywhere in the zero set of J_f. Hence, Lemma 7.2.2 implies that Theorem 7.5.1 (i) holds everywhere on X

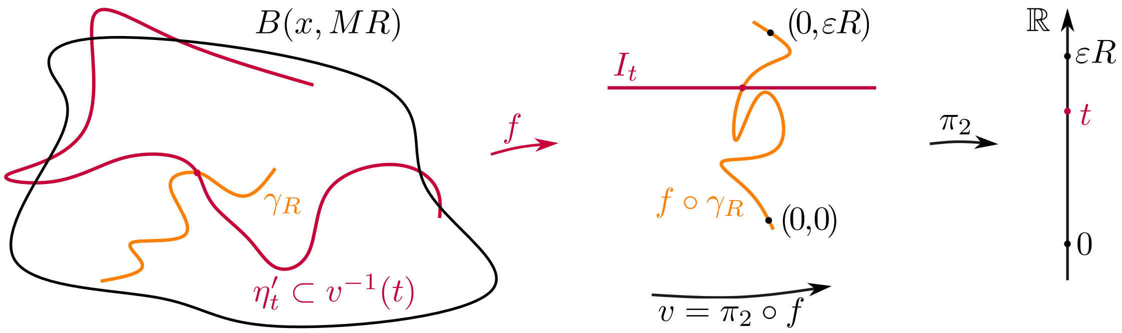


Figure 7.3.: Proof of Proposition 7.5.3.

If f is of finite distortion along paths with $K_f \in L^1_{\mathrm{loc}}(X)$, then, by Theorem 7.3.1, the map f is open and discrete. It follows from (7.9) that Theorem 7.5.1 (ii) holds on X'. We are left to prove the following proposition.

Proposition 7.5.3 (Proposition F.4.1)**.** *For almost every $x \in X_0$, we have that $\rho_f^l(x) = 0$ (and therefore $\rho_f^u(x) = 0$).*

Towards a contradiction, assume that the set $A := \{x \in X_0 : \rho_f^l(x) > 0\}$ has positive measure. As f is continuous, open and discrete, it is a local homeomorphism except on a discrete set of branch points $\mathcal{B}_f \subset X$. The following lemma summaries Lemma F.4.2 and Corollary F.4.3, and provides us with a positive measure set of points, each associated with a "good" curve.

Lemma 7.5.4 (Lemma F.4.2 and Corollary F.4.3)**.** *There exists $A' \subset A \setminus \mathcal{B}_f$ of positive measure such that for every $x \in A'$ the following conditions hold.*

1. *There exists a curve γ_x parametrized by arclength with $x = \gamma_x(t)$ for some $t \in (0, \ell(\gamma_x))$.*
2. *There exist $\delta_x, \varepsilon_x \in (0,1)$ such that for every $R \in (0, \delta_x)$ and $\gamma_R = \gamma_x|_{[t-R,t+R]}$ we have*

$$\mathrm{diam}(|f \circ \gamma_R|) \geq \varepsilon_x R.$$

We may choose $\varepsilon > 0$ such that the set $A_\varepsilon := \{x \in A' : \varepsilon_x \geq \varepsilon\}$ is of positive measure. We claim that $J_f(x) > 0$ for almost every $x \in A_\varepsilon$, contradicting Lemma 7.2.2, and thus, establishing Proposition 7.5.3.

We are left to prove the claim. By a density result, see e.g. [Fed69, Theorem 2.10.19(5)], there exists $E \subset X$, $\mathcal{H}^2(E) = 0$, and $r_x > 0$ so that

$$\mathcal{H}^2(\overline{B}(x,r)) \leq \pi r^2 \quad \text{for every } x \in X \setminus E \text{ and every } r < r_x. \tag{7.10}$$

Fix $x \in A_\varepsilon \setminus E$, let M be large enough, to be specified later, and let $R > 0$ be such that $5MR < \min\{\delta_x, r_x\}$. If γ_R is the curve from Lemma 7.5.4, then

$$x \in |\gamma_R| \subset B(x,R) \quad \text{and} \quad \mathrm{diam}(f(|\gamma_R|)) \geq \varepsilon R.$$

Without loss of generality we assume that $f(|\gamma_R|)$ contains $(0,0)$ and $(0,\varepsilon R)$. We have that $v = \pi_2 \circ f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R})$, where $\pi_2 \colon \mathbb{R}^2 \to \mathbb{R}$ is the projection to the second coordinate. The situation is sketched in Figure 7.3.

For every $0 < t < \varepsilon R$ we find a continuum $\eta_t' \subset v^{-1}(t)$ so that $\eta_t' \cap |\gamma_R| \neq \emptyset$ and $f(\eta_t')$ is an interval I_t of length $2R$ centered in $|f \circ \gamma_R|$. We define

$$Q_M(R) = \{0 < t < \varepsilon R : \eta_t' \subset B(x, MR)\}$$

and prove that for almost every $x \in A_\varepsilon$ we can choose M (depending on x) so that

$$|Q_M(R)|_1 \geq \frac{\varepsilon R}{2}$$

for all $R > 0$ satisfying $5MR < \min\{\delta_x, r_x\}$, see Lemma F.4.5. Note that the proof of this statement is not trivial and uses the tools (a)-(c), as well as (7.10).

If $t \in Q_M(R)$, then $f(\eta'_t) = I_t$, and in particular, $|f(\eta'_t)|_1 = 2R$. We set

$$G_M(R) = \bigcup_{t \in Q_M(R)} f(\eta'_t).$$

By definition, $G_M(R) \subset f(B(x, MR))$. Fubini's theorem now implies

$$\varepsilon R^2 \leq 2R \cdot |Q_M(R)|_1 = |G_M(R)|_2 \leq |f(B(x, MR))|_2.$$

The claim follows after letting $R \to 0$ and the proof of Theorem 7.5.1 is complete.

7.6. Quasiconformal uniformization

As mentioned in Section 3.2, it is difficult to find conditions on a metric surface X of locally finite Hausdorff 2-measure that imply reciprocality, and thus, by Theorem 3.2.3, existence of a geometrically quasiconformal parametrization. In this section we provide a uniformization result for metric surfaces in terms of existence of non-constant quasiregular mappings to $\mathbb{R}^2$. Note that in this setting quasiregularity in terms of distortion along paths is equivalent to quasiregularity in terms of analytic distortion, see Theorem 7.5.1, and thus. The following uniformization result is a consequence of Theorems 7.3.1 and 3.3.4 as well as Corollary 7.5.2.

Theorem 7.6.1 (Theorem F.1.3)**.** *If X admits a non-constant quasiregular map $f : X \to \mathbb{R}^2$, then X admits a quasiconformal homeomorphism $\phi : X \to U$ onto a domain $U \subset \mathbb{R}^2$.*

We emphasize that Theorem 7.6.1 is sharp in the following sense. There is no $p \geq 1$ for which the existence of a non-constant map $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ of finite distortion along paths and with $K_f \in L^p_{\mathrm{loc}}(X)$ implies the existence of a quasiconformal homeomorphism $\phi : X \to U$ onto a domain $U \subset \mathbb{R}^2$. Namely, if $f_0 \colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes the map from Example 7.3.2, it is possible to change the metric on $\mathbb{R}^2$, and thus the map f_0, such that the following holds: the domain X is a metric surface of locally finite Hausdorff 2-measure but not reciprocal, and $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ is a map of finite distortion with $K_f \in L^1_{\mathrm{loc}}(X)$; see Proposition E.6.1.

Theorem 7.6.1 in combination with the measurable Riemann mapping theorem (see e.g. [AIM09, Theorem 5.3.4]) allow the extension of the classical Stoïlow factorization theorem (see [AIM09, Chapter 5.5], [LP20]) to our setting.

Theorem 7.6.2 (Theorem F.1.4)**.** *Every non-constant quasiregular map $f : X \to \mathbb{R}^2$ admits a factorization $f = g \circ v$, where $v \colon X \to V$ is a quasiconformal homeomorphism onto a domain $V \subset \mathbb{R}^2$ and $g \colon V \to \mathbb{R}^2$ is complex analytic.*

Part III.

Appendix

A. Canonical parametrizations of metric surfaces of higher topology

with Martin Fitzi

Abstract. We give an alternate proof to the following generalization of the uniformization theorem by Bonk and Kleiner. Any linearly locally connected and Ahlfors 2-regular closed metric surface is quasisymmetrically equivalent to a model surface of the same topology. Moreover, we show that this is also true for surfaces as above with non-empty boundary and that the corresponding map can be chosen in a canonical way. Our proof is based on a local argument involving the existence of quasisymmetric parametrizations for metric discs as shown in a paper of Lytchak and Wenger.

A.1. Introduction and statement of main results

A.1.1. Introduction

The classical uniformization theorem states that any oriented Riemannian 2-manifold is conformally diffeomorphic to a model surface of constant curvature. The corresponding map provides a canonical parametrization of said Riemannian surface. An appropriate generalized notion of conformal diffeomorphisms in a non-smooth setting is given by quasisymmetric mappings. A homeomorphism $f\colon X \to Y$ between metric spaces is *quasisymmetric* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$d_Y(f(x), f(y)) \leq \eta(t) \cdot d_Y(f(x), f(z))$$

for all points $x, y, z \in X$ with $d_X(x, y) \leq t \cdot d_X(x, z)$. The quasisymmetric uniformization problem in the field of analysis on metric spaces then asks under which conditions on a metric space X topologically equivalent to some model space M one may identify X with M via a quasisymmetric homeomorphism.

A breakthrough result due to Bonk and Kleiner [BK02] asserts that if X is an Ahlfors 2-regular metric space homeomorphic to the 2-sphere S^2, then there exists a quasisymmetric homeomorphism between X and S^2 if and only if X is linearly locally connected. For definitions of Ahlfors 2-regularity and linear local connectedness we refer to Section A.2.1.

Lytchak and Wenger provide in [LW20] an alternate proof of the theorem of Bonk-Kleiner using a theory of energy and area minimizing discs in metric spaces admitting a quadratic isoperimetric inequality established in [LW17a] and [LW18a]. The aim of this paper is to use the existence result in [LW20] locally to obtain canonical parametrizations of metric surfaces of higher topology with possibly non-empty boundary.

Let X be a metric space homeomorphic to a smooth surface M. Here, a *smooth surface* refers to a smooth compact oriented and connected Riemannian 2-manifold with possibly non-empty boundary. Define $\Lambda(M, X)$ to be the family of Newton-Sobolev maps $u \in N^{1,2}(M, X)$ such that u is a uniform limit of homeomorphisms from M to X and let $E_+^2(u, g)$ be the *Reshetnyak energy* of a map $u \in N^{1,2}(M, X)$ with respect to the Riemannian metric g; for definitions see Section A.2.2. Our main result is the following version of [LW20, Theorem 1.1] for metric surfaces of higher topology. Note that the definition of $\Lambda(M, X)$ is different from [LW20].

Theorem A.1.1. *Let X be a geodesic metric space which is Ahlfors 2-regular, linearly locally connected and homeomorphic to a smooth surface M. Then, there exist a map $u \in \Lambda(M, X)$ and a Riemannian metric g on M such that*

$$E_+^2(u, g) = \inf\{E_+^2(v, h) : v \in \Lambda(M, X),\ h \text{ a smooth Riemannian metric on } M\}.$$

Any such u is a quasisymmetric homeomorphism from M to X and the pair (u, g) is uniquely determined up to a conformal diffeomorphism $\varphi\colon (M, g) \to (M, h)$.

Moreover, the metric g can be chosen to be of constant sectional curvature -1, 0 or 1 and such that ∂M is geodesic (if non-empty). Note that the assumption of X being geodesic is natural and can be dropped if X is closed, see Remark A.2.3.

The theorem of Bonk-Kleiner has been extended for example in [Raj17], [RRR21], [LW20], [B] and [NR24]. In the setting of X being an Ahlfors 2-regular and linearly locally connected metric surface, there exist quasisymmetric uniformization results if $X \setminus \partial X$ is a domain in S^2, see [Wil08], [MW13] and [RR21], and if X is closed, see [GW18] and [Iko22]. Theorem A.1.1 is a strengthening of these results in the sense that it states the existence of *canonical* quasisymmetric homeomorphism, regardless of X being closed or having non-empty boundary. A different canonical quasisymmetric homeomorphism was previously only provided by [Iko22] for X being closed. Note that the statement of [Iko22] also holds for non-orientable surfaces. Furthermore, in contrast to some results mentioned above, e.g. [GW18] and [Iko22], we do not obtain a quantitative statement in the sense that the quasisymmetric distortion function is not necessarily controlled by the Ahlfors 2-regularity and linear local connectedness constants of X.

As a corollary of Theorem A.1.1, we obtain the following generalization of the result of Bonk-Kleiner, which seems to be new for surfaces having non-empty boundary as well as higher genus.

Corollary A.1.2. *Let X be a geodesic Ahlfors 2-regular metric space homeomorphic to a smooth surface M with possibly non-empty boundary. Then, X is quasisymmetrically equivalent to M if and only if X is linearly locally connected.*

A.1.2. Elements of proof

We briefly sketch some of the arguments needed for proving Theorem A.1.1. For arbitrary M and X as in the paragraph before the theorem, the set $\Lambda(M, X)$ can be empty. A crucial step in this work is to show the existence of a map $u \in \Lambda(M, X)$ in the setting of Theorem A.1.1.

Proposition A.1.3. *Let M be a smooth surface and (X, d) a metric space which is geodesic, Ahlfors 2-regular, linearly locally connected and homeomorphic to M. Then the family $\Lambda(M, X)$ is non-empty and contains a quasisymmetric homeomorphism.*

The proposition follows by a dissection of M and X into appropriate disc-type subdomains, consequently applying [LW20, Theorem 6.1] yielding quasisymmetric parametrizations for each subdomain in X and finally gluing all these mappings together in order to obtain a global quasisymmetric homeomorphism $M \to X$. Note that Proposition A.1.3 already establishes Corollary A.1.2. Moreover, this procedure also works for non-orientable surfaces, see Remark A.3.9.

The map u provided by Proposition A.1.3 is not necessarily canonical, i.e. possibly not of minimal energy. In order to find an energy minimizer in $\Lambda(M, X)$, we will use similar arguments as in the proofs of [LW20, Theorem 6.1] and [FW21, Theorem 8.2]. In particular, we need to ensure that a family of mappings in $\Lambda(M, X)$ of uniformly bounded energies is equicontinuous.

The paper is structured as follows. In Section 2 we provide necessary definitions and some results on Newtonian Sobolev spaces that will be of use later on. Section 3 is devoted to the proof of Proposition A.1.3. In Section 4 we will show equicontinuity of energy bounded almost homeomorphisms. And finally, the proof of Theorem A.1.1 is given in Section 5.

A.2. Preliminaries

A.2.1. Basic definitions and notations

Let (X, d) be a metric space. The *open ball* in X of radius $r > 0$ centered at a point $x \in X$ is denoted by $B_X(x, r)$ or simply $B(x, r)$. Consider the Euclidean space $(\mathbb{R}^2, |\cdot|)$, where $|\cdot|$ is the Euclidean norm. The *open and closed unit discs* in $\mathbb{R}^2$ are given by

$$D := \{z \in \mathbb{R}^2 : |z| < 1\}, \qquad \overline{D} := \{z \in \mathbb{R}^2 : |z| \leq 1\}.$$

An open set $\Omega \subset X$ homeomorphic to the unit disc D is a *Jordan domain in* X if its completion $\overline{\Omega} \subset X$ is homeomorphic to $\overline{D}$. A *Jordan curve* in X is a subset of X homeomorphic to S^1 and it is called *chord-arc* if it is biLipschitz equivalent to S^1. The *image* of a curve c in X is denoted by $|c|$ and the *length* by $\ell_X(c)$ or $\ell(c)$. A curve $c\colon [a, b] \to X$ is called *geodesic* if $\ell(c) = d(c(a), c(b))$. A metric space (X, d) is *geodesic* if every pair of points in X can be joined by a geodesic.

A *metric surface* X is a metric space homeomorphic to a smooth surface M. We say that a metric surface X is of T*-type* if X is homeomorphic to a canonical topological surface T. By ∂M we denote the topological boundary of the smooth surface M, which is homeomorphic to a finite disjoint union of S^1. The boundary of X, denoted ∂X, is the subset of X that is homeomorphic to ∂M.

For $s \geq 0$, we denote the s*-dimensional Hausdorff measure* of a set $A \subset X$ by $\mathcal{H}^s_X(A)$ or simply $\mathcal{H}^s(A)$. The normalizing constant is chosen in such a way that if X is the Euclidean space $\mathbb{R}^n$, the Lebesgue measure agrees with $\mathcal{H}^n_X$. If (M, g) is a Riemannian manifold of dimension n then the n-dimensional Hausdorff measure $\mathcal{H}^n_g := \mathcal{H}^n_{(M,g)}$ on (M, g) coincides with the Riemannian volume.

Let g be a smooth Riemannian metric on a smooth surface M such that the boundary of M is geodesic with respect to g. We call the metric g *hyperbolic* if it is of constant sectional curvature -1, and *flat* if it has vanishing sectional curvature as well as an associated Riemannian 2-volume satisfying $\mathcal{H}^2_g(M) = 1$.

Definition A.2.1. A metric space X is said to be *Ahlfors* 2*-regular* if there exists $K > 0$ such that for all $x \in X$ and $0 < r < \operatorname{diam} X$, we have

$$K^{-1} \cdot r^2 \leq \mathcal{H}^2_X(B(x, r)) \leq K \cdot r^2.$$

Definition A.2.2. We say that a metric space X is *linearly locally connected (LLC)* if there exists a constant $\lambda \geq 1$ such that for all $x \in X$ and $r > 0$, every pair of distinct points in $B(x, r)$ can be connected by a continuum in $B(x, \lambda r)$ and every pair of distinct points in $X \setminus B(x, r)$ can be connected by a continuum in $X \setminus B(x, r/\lambda)$.

Here, *a continuum* refers to a non-empty compact connected subset of X.

Remark A.2.3. *If X is a closed surface, it follows by* [BK02, Lemma 2.5] *that linear local connectivity is equivalent to linear local contractibility, meaning that there exists $\lambda \geq 1$ such that every ball $B(x, r)$ of radius $0 < r < \lambda^{-1} \operatorname{diam}(X)$ is contractible in $B(x, \lambda r)$. Now, every Ahlfors 2-regular and linear local contractible metric surface is quasiconvex (see* [Sem96a, Theorem B.6]*) and thus geodesic up to a biLipschitz change of metric.*

A.2.2. Metric space valued Sobolev maps

In this subsection we give a brief overview over some basic concepts used in the theory of metric space valued Sobolev maps based on upper gradients. Note that several other equivalent definitions of Sobolev spaces exist. For more details consider e.g. [HKST15].

Let (X, d) be a complete metric space and M a smooth compact 2-dimensional manifold, possibly with non-empty boundary. Fix a Riemannian metric g on M and consider a domain $\Omega \subset M$. Let $u \colon \Omega \to X$ be a map and $\rho \colon \Omega \to [0, \infty]$ a Borel function. Then, ρ is called *(weak) upper gradient of u with respect to g* if

$$d(u(\gamma(a)), u(\gamma(b))) \leq \int_\gamma \rho(s) \, ds \tag{A.1}$$

for (almost) every rectifiable curve $\gamma \colon [a, b] \to \Omega$. A weak upper gradient ρ of u is said to be *minimal* if $\rho \in L^2(\Omega)$ and for every weak upper gradient ρ' of u in $L^2(\Omega)$ it holds that $\rho \leq \rho'$ almost everywhere on Ω. Denote by $L^2(\Omega, X)$ the family of measurable essentially separably valued maps $u \colon \Omega \to X$ such that the distance function $u_x(z) := d(u(z), x)$ is in the space $L^2(\Omega)$ of 2-integrable functions for some and hence any $x \in X$. A sequence $(u_k) \subset L^2(\Omega, X)$ is said to *converge in $L^2(\Omega, X)$* to a map $u \in L^2(\Omega, X)$ if

$$\int_\Omega d^2(u_k(z), u(z)) \, d\mathcal{H}^2_g(z) \to 0$$

as k tends to infinity. The *(Newton-)Sobolev space* $N^{1,2}(\Omega, X)$ is the collection of maps $u \in L^2(\Omega, X)$ such that u has a weak upper gradient in $L^2(\Omega)$. Every such u has a minimal weak upper gradient denoted by ρ_u, which is unique up to sets of measure zero (see e.g. [HKST15, Theorem 6.3.20]). Note also that the definition of $N^{1,2}(\Omega, X)$ is independant of the chosen metric g on M.

Definition A.2.4. The *Reshetnyak energy* of a map $u \in N^{1,2}(\Omega, X)$ with respect to g is defined by

$$E^2_+(u, g) := \int_\Omega |\rho_u(z)|^2 \, d\mathcal{H}^2_g(z).$$

This definition of energy agrees with the one given in [FW21, Definition 2.2]; in particular, E^2_+ is invariant under precompositions with conformal diffeomorphisms.

A.3. Noncanonical quasisymmetric parametrizations

The aim of this section is to prove Proposition A.1.3, which strongly depends on the following variant of [LW20, Theorem 6.1].

Theorem A.3.1. *Let X be an Ahlfors 2-regular geodesic metric space homeomorphic to a 2-dimensional manifold. Let $J \subset X$ be a Jordan domain with $\ell(\partial J) < \infty$ and such that $\overline{J}$*

is LLC. Then any quasisymmetric homeomorphism $f\colon S^1 \to \partial J$ *extends to a quasisymmetric homeomorphism* $\overline{f} \in \Lambda(\overline{D}, \overline{J})$.

Note that the conclusion of Theorem A.3.1 also holds if X is not geodesic and the boundary of J is not rectifiable, see [Wil08, Theorem 1.2] and [Iko22, Theorem 1.4].

The proof of [LW20, Theorem 6.1] depends on the existence and regularity of energy and area minimizing Sobolev discs in metric spaces developed by Lytchak and Wenger in [LW17a], [LW17b] and [LW18a]. In the following we describe the main steps in the proof of Theorem A.3.1. Let X and J be as in the hypotheses of Theorem A.3.1. We denote by $\Lambda(\partial J, \overline{J})$ the family of maps $v \in N^{1,2}(D, \overline{J})$ whose trace has a continuous representative which is a uniform limit of homeomorphisms $S^1 \to \partial J$, where the trace of $v \in N^{1,2}(D, \overline{J})$ is defined by $\operatorname{tr}(v)(s) := \lim_{t \nearrow 1} v(ts)$ for almost every $s \in S^1$. It can be shown that $\overline{J}$ admits a quadratic isoperimetric inequality, which implies that $\Lambda(\partial J, \overline{J})$ is not empty. The existence of a map $u \in \Lambda(\partial J, \overline{J})$ which minimizes the Reshetnyak energy $E_+^2(u, g_{\mathrm{Eucl}})$ among all maps in $\Lambda(\partial J, \overline{J})$ follows from [LW17a, Theorem 7.6]. By [LW17b, Theorem 4.4], u has a continuous representative, denoted again by u, which extends continuously to the boundary and by [LW17b, Lemmas 3.2 and 4.1], the map u is infinitesimally isotropic and thus infinitesimally quasiconformal (see [LW20, Definition 3.3] and the comment thereafter). After equipping $\overline{J}$ with the intrinsic length metric, it can be shown that u is a homeomorphism, see [LW20, Theorems 1.2 and 3.6]. Moreover, using the Ahlfors 2-regularity and LLC-condition on X, it follows that the map u is a quasisymmetry, compare to [LW20, Proposition 3.5 and Theorem 2.5].

The quasisymmetry $f^{-1} \circ u|_S^1\colon S^1 \to S^1$ extends to a quasisymmetry $g\colon \overline{D} \to \overline{D}$ after applying the extension result [BA56, Theorem 1]. The map $\overline{f} := u \circ g^{-1}$ then satisfies all desired properties.

A *cylinder* and *Y-piece* are connected topological surfaces of genus 0 with two and three boundary components, respectively. Furthermore, we refer to a metric space homeomorphic to a cylinder or a Y-piece as a *metric cylinder* or a *metric Y-piece*, respectively. In order to prove Proposition A.1.3, we will first decompose M and X into cylinders and Y-pieces, each of which can be further decomposed into suitable Jordan domains. This will be the content of Subsection A.3.1. Note that the Jordan domains in X should in particular satisfy the hypotheses of Theorem A.3.1. For a Jordan domain J adjacent to the boundary of X, we do not know how to ensure that $\overline{J}$ is LLC. Hence, we will prove a version of Theorem A.3.1 for boundary cylinders in Subsection A.3.2. In a last step we apply a quasisymmetric gluing theorem of Aseev, Kuzin and Tetenov [AKT05, Theorem 3.1] to construct the desired quasisymmetry from M to X. A rigorous proof of Proposition A.1.3 can be found in Subsection A.3.3.

A.3.1. Decompositions of metric Y-pieces and cylinders into Jordan domains

A crucial ingredient in our decomposition of a metric surface is [LW20, Lemma 4.2], stated next.

Lemma A.3.2. *Let X be a geodesic metric space, and let $\Gamma \subset X$ be a topological arc connecting two points $a, b \in X$. Then for every $\varepsilon > 0$ there exists a bi-Lipschitz curve contained in the ε-neighbourhood of Γ and connecting a and b.*

A similar statement also holds for Jordan curves, compare to the proof of [B, Lemma 4.2]. One can prove Lemma A.3.2 by choosing a piecewise geodesic injective curve Γ' in a small neighbourhood of Γ and modifying Γ' in the vicinity of every vertex by applying the following claim [LW20, Claim 4.3].

Claim A.3.3. *Let $s > 0$ and $\eta\colon [-s, s] \to X$ be an injective curve such that the restrictions $\eta|_{[0,s]}$ and $\eta|_{[-s,0]}$ are geodesics parametrized by their arc-length. Then there exist arbitrarily* *small $t \in (0, s)$ such that after replacing $\eta|_{[}-t, t]$ by a geodesic from $\eta(-t)$ to $\eta(t)$ we obtain a biLipschitz curve.*

Lemma A.3.4. *Let X be a geodesic metric surface and $\Sigma \subset X$ a metric cylinder or metric Y-piece such that each connected component of $\partial\Sigma$ can be parametrized by a piecewise geodesic chord-arc curve. Then there exist Jordan domains $J_1, J_2 \subset \Sigma$ with*

1. $\Sigma = \overline{J_1} \cup \overline{J_2}$,
2. $J_1 \cap J_2 = \emptyset$,
3. J_1, J_2 *are both bounded by a biLipschitz curve.*

Proof. We give a proof for Σ being a metric Y-piece, the case of a metric cylinder only needing minor adaptations in the following arguments. Denote by $\eta_i \colon S^1 \to \partial\Sigma$ the piecewise geodesic biLipschitz curves parametrizing the three components of $\partial\Sigma$. Choose three disjoint injective curves γ_i in Σ, each one connecting two boundary components such that Σ is separated into two Jordan domains when cutting along these curves. By Lemma A.3.2 and its proof, we may assume that each γ_i is biLipschitz and piecewise geodesic. Denote the endpoints of γ_i by a_i^1, a_i^2.

Choose $\varepsilon > 0$ so small that the balls $B(a_i^j, 2\varepsilon)$ are disjoint. We modify γ_i within $B(a_i^j, 2\varepsilon)$ with the following procedure. Without loss of generality assume $a_1^1 \in |\eta_1|$. Choose a point $x_1 \in B(a_1^1, \varepsilon) \cap |\gamma_1|$ distinct from a_1^1 and let $y_1 \in |\eta_1|$ be such that

$$d(x_1, y_1) = d(x_1, |\eta_1|), \tag{A.2}$$

where d denotes the metric on X. Let $c_1 \colon I \to \Sigma$ be a geodesic segment connecting x_1 with y_1. Thus, $|c_1| \subset B(a_1^1, 2\varepsilon)$. Then consider the concatenation of c_1 with one of the subcurves of η_1 emanating from y_1. Let $s > 0$ be such that the following holds. Subcurves of η_1 and c_1 with common endpoint y_1 can be reparametrized by arc-length on $[-s, 0]$ and $[0, s]$, respectively, such that $\eta_1(0) = c_1(0) = y_1$. Denote this concatenation defined on $[-s, s]$ by η. Equality (A.2) implies that for $r \in [0, s]$

$$d(\eta_1(-r), c_1(r)) \geq r.$$

It follows from the proof of Claim A.3.3 that η is a biLipschitz curve. Redefine γ_1 by replacing the subcurve from x_1 to a_1^1 by c_1. Analogously, construct segments $c_2, \dots, c_6$ in the vicinities of the other a_i^j and modify every γ_i near its endpoints in this way. By choosing appropriate subcurves, we have that all redefined γ_i are still injective. Moreover, Claim A.3.3 shows that if γ_i is not biLipschitz at a vertex in the interior of the curve, we can change it in an arbitrarily small ball around this vertex to obtain a global biLipschitz curve.

Finally, Σ is separated into Jordan domains J_1 and J_2 by cutting along redefined γ_i. Moreover, the boundaries ∂J_1 and ∂J_2 are parametrized by biLipschitz concatenations of the redefined γ_i with respective subcurves of η_j. □

The following lemma will be useful in the proof of Proposition A.3.6. A *metric disc* is a metric space homeomorphic to the closed unit disc $\overline{D}$.

Lemma A.3.5. *Let X be an Ahlfors 2-regular and LLC metric surface. Consider a subset $\Sigma \subset X$ that is either a metric disc bounded by a chord-arc curve in X, or that is a metric cylinder such that one component of $\partial\Sigma$ is contained in ∂X and the other component of $\partial\Sigma$ can be parametrized by a chord-arc curve in X. Then Σ equipped with the subspace metric is Ahlfors 2-regular and LLC.*

Lemma A.3.5 can be shown readily by using the LLC-property of X and replacing parts of the continua which lie in $X \setminus \Sigma$ with appropriate subcurves of the biLipschitz boundary component in order to obtain desired continua in Σ. Compare also to the proof of [LW20, Proposition 6.4]. The quadratic upper bound on the Hausdorff 2-measure of a ball is inherited by any subspace, while the lower bound essentially follows from the LLC condition and the coarea inequality for Lipschitz maps, see e.g. [Raj17, p. 1369].

A.3.2. Parametrizations of boundary cylinders

The aim of this section is to establish the following extension result for cylindrical surfaces which is needed later in the proof of Proposition A.1.3.

Proposition A.3.6. *Let Z be a smooth cylinder and $\partial Z^1 \subset \partial Z$ a boundary component. Let Σ be a geodesic, Ahlfors 2-regular and LLC metric cylinder and $\partial\Sigma^1 \subset \partial\Sigma$ a boundary component. Assume furthermore that there exists a biLipschitz homeomorphism $f\colon \partial Z^1 \to \partial\Sigma^1$. Then f extends to a quasisymmetric homeomorphism $\overline{f} \in \Lambda(Z, \Sigma)$.*

As a first step in the proof of Proposition A.3.6, we will perform a gluing of the metric cylinder Σ with the closed unit disc $\overline{D}$ along corresponding boundary components. We now introduce some notation and needed results concerning this gluing method.

Let (X, d_X) and (Y, d_Y) be two compact metric surfaces with non-empty boundary and let $\partial X^j \subset \partial X$, $\partial Y^k \subset \partial Y$ be two boundary components. Assume $\gamma\colon \partial X^j \to \partial Y^k$ is a biLipschitz homeomorphism and define the quotient

$$\widehat{XY} := (X \sqcup Y)/\sim,$$

where $x \sim y$ for $x \in X$, $y \in Y$ if $y = \gamma(x)$. Equip $\widehat{XY}$ with the quotient metric $\widehat{d}$, which for $[x], [y] \in \widehat{XY}$ is defined by

$$\widehat{d}([x],[y]) := \inf\left\{\sum_{i=1}^{k} d(p_i, q_i) : [p_{i+1}] = [q_i], p_1 = x, q_k = y, k \in \mathbb{N}\right\}.$$

Consider X and Y as subsets of $\widehat{XY}$ and set $X \cap Y := \{[x] : x \in \partial X^j\}$. It follows immediately that the identity maps $(X, d_X) \to (X, \hat{d}|_{X\times X})$ and $(Y, d_Y) \to (Y, \hat{d}|_{Y\times Y})$ are 1-Lipschitz. The next lemma is a consequence of the compactness of $X \cap Y$ and the biLipschitz property of γ.

Lemma A.3.7. *The identity maps $\mathrm{id}_X\colon (X, \hat{d}|_{X\times X}) \to (X, d_X)$ and $\mathrm{id}_Y\colon (Y, \hat{d}|_{Y\times Y}) \to (Y, d_Y)$ are L-Lipschitz, where $L \geq 1$ denotes the biLipschitz constant of γ. In particular, the restrictions $\hat{d}|_{X\times X}$ and $\hat{d}|_{Y\times Y}$ are L-biLipschitz equivalent to d_X and d_Y.*

Moreover, we have the following geometric property of the space $(\widehat{XY}, \hat{d})$.

Lemma A.3.8. *If (X, d_X) and (Y, d_Y) are Ahlfors 2-regular and LLC, then so is $(\widehat{XY}, \hat{d})$.*

The proof of Lemma A.3.8 can be found in the appendix. A similar gluing procedure with quantitative versions of Lemmas A.3.7 and A.3.8 was studied in [MW13, Section 9].

We are now able to provide a proof of Proposition A.3.6.

Proof of Proposition A.3.6. Consider the quotient space

$$\widehat{\Sigma D} := (\Sigma \sqcup \overline{D})/\sim$$

defined as above for some biLipschitz homeomorphism $\partial\Sigma^1 \to \partial D = S^1$ and equipped again with the quotient metric $\widehat{d}$. By definition, the metric disc $(\widehat{\Sigma D}, \widehat{d})$ is geodesic and from Lemma A.3.8

it follows that $(\widehat{\Sigma D}, \widehat{d})$ is Ahlfors 2-regular and LLC. Theorem A.3.1 implies the existence of a quasisymmetric homeomorphism $v \in \Lambda(\overline{D}, \widehat{\Sigma D})$. Consider $\widehat{D} = D$ as a subset of $\widehat{\Sigma D}$ and define

$$\Omega := \overline{D} \backslash v^{-1}(\widehat{D}).$$

By the annulus conjecture (see [TV81, Theorem 3.12]) there exists a quasisymmetric homeomorphism $g\colon A \to \Omega$, where

$$A := \{p \in \mathbb{R}^2 : 1/2 \le |p| \le 1\} \subset \overline{D}$$

denotes the standard annulus equipped with the Euclidean metric. Without loss of generality, we may assume that g maps the unit circle onto $\partial(v^{-1}(\widehat{D}))$. Let $\varphi\colon Z \to A$ be a biLipschitz homeomorphism with $\varphi(\partial Z^1) = S^1$. Then, the mapping $u \in N^{1,2}(Z, \Sigma)$ defined by $u := \mathrm{id}_\Sigma \circ v \circ g \circ \varphi$ is a quasisymmetric homeomorphism with $u(\partial Z^1) = \partial \Sigma^1$. Moreover, the composition

$$h := \varphi \circ u^{-1} \circ f \circ \varphi^{-1}|_{S^1} : S^1 \to S^1$$

is a quasisymmetric homeomorphism, which we may assume to be orientation-preserving. By [TV81, Theorem 3.14], the map h extends to a quasisymmetric homeomorphism $\overline{h}\colon \overline{D} \to \overline{D}$ such that $\overline{h}$ restricted to the ball $B(0, 1/2)$ is the identity map. Hence

$$\overline{f} := u \circ \varphi^{-1} \circ \overline{h} \circ \varphi$$

is a desired quasisymmetric homeomorphism from Z to Σ with $\overline{f}|_{\partial Z^1} = f$. □

A.3.3. Noncanonical quasisymmetric parametrizations

Using the extension result established in the previous subsection, we may obtain Proposition A.1.3 mentioned in the introduction.

Proof of Proposition A.1.3. The cases where M is a disc or a sphere follow from [LW20, Theorem 6.1] and [LW20, Proposition 6.4].

Depending on its topology, endow M with a hyperbolic or flat Riemannian metric (for a smooth surface M with non-empty boundary, see e.g. [Jos06, Exercices for §4.4]). Let $h\colon M \to X$ be a homeomorphism.

We first give a proof in the special case when X has either empty boundary or else is bounded by piecewise geodesic chord-arc curves. Choose a collection of simple closed geodesics $\{\gamma_i\colon S^1 \to M\}$ decomposing M into smooth Y-pieces or cylinders M_k, respectively. Using [LW20, Lemma 4.2], we may partition X into Y-pieces/cylinders X_k such that each X_k is homotopic to $h(M_k) \subset X$ and bounded by piecewise geodesic chord-arc curves. We then further decompose M_k and X_k into Jordan domains: if M_k is a Y-piece, then it is a standard result from hyperbolic geometry that M_k is isometric to the partial gluing of the boundary of two copies $\Omega_{k,1}, \Omega_{k,2}$ of a right-angled hexagon in $\mathbb{H}$, see e.g. [Bus10, Proposition 3.1.5]. If M_k is of cylindrical type, then a similar decomposition into isometric rectangles in the Euclidean plane, again denoted $\Omega_{k,1}$ and $\Omega_{k,2}$, is possible. Note that in either case $\Omega_{k,1}$ and $\Omega_{k,2}$ are biLipschitz equivalent to the closed unit disc $\overline{D}$. In X we decompose each X_k into Jordan domains $J_{k,1}$, $J_{k,2}$ as in Lemma A.3.4. After possibly inverting the notation of $J_{k,1}$ and $J_{k,2}$, let

$$f\colon \bigcup_{\substack{j=1,2\\k}} \partial\Omega_{k,j} \to \bigcup_{\substack{j=1,2\\k}} \partial J_{k,j}$$

be a biLipschitz homeomorphism satisfying $f(\partial\Omega_{k,j}) = \partial J_{k,j}$ for each j, k. By Lemma A.3.5 and

Theorem A.3.1, there exists for each k a quasisymmetric homeomorphism $g_{k,j}\colon \overline{\Omega_{k,j}} \to \overline{J_{k,j}}$ with $g_{k,j}|_{\partial\Omega_{k,j}} = f|_{\partial\Omega_{k,j}}$. The map $u\colon M \to X$ agreeing with $g_{k,j}$ on $\Omega_{k,j}$ satisfies the hypotheses of the quasisymmetric gluing theorem [AKT05, Theorem 3.1] as each $\Omega_{k,j}$ is bounded and has biLipschitz boundary and every $g_{k,j}$ is a quasisymmetric homeomorphism. Therefore, the map u itself is a quasisymmetric homeomorphism. This shows the proposition in the special case.

We now turn to the general case, where X might be bounded by curves of unknown regularity. For each boundary component ∂X^i, define a piecewise geodesic biLipschitz curve $c_i\colon S^1 \to X$ which is homotopic to an oriented parametrization of ∂X^i, but disjoint from it. Furthermore, we may assume that the curves $\{c_i\}$ are all pairwise disjoint. Let $\Sigma_i \subset X$ be the metric cylinder bounded by $c_i(S^1)$ and ∂X^i, and let $\Sigma \subset X$ be the subsurface bounded by $\bigcup_i c_i(S^1)$. Note that Σ is homeomorphic to X. The first part of the proof then shows that there exists a quasisymmetric homeomorphism $u\colon M \to \Sigma$. Then embed M smoothly into a surface $\tilde{M}$ such that for each i, there exists exactly one boundary component $\partial\tilde{M}^i$ which together with $\partial Z_i^1 := u^{-1}(c_i(S^1)) \subset \partial M$ bounds a smooth cylinder $Z_i \subset \tilde{M}$. Finally, use Lemma A.3.5 and Proposition A.3.6 to obtain quasisymmetric extensions $u_i\colon Z_i \to \Sigma_i$ of $u|_{\partial Z_i^1}$. Once again, the gluing result [AKT05, Theorem 3.1] ensures that the map $u\colon \tilde{M} \to X$ agreeing with u on M and with u_i on Z_i is a quasisymmetric homeomorphism. The proof of the proposition is complete. □

Remark A.3.9. *The proof of Proposition A.1.3 can be adapted to show a version of the proposition for non-orientable and homeomorphic surfaces M and X. In particular, we obtain a generalization of Bonk-Kleiner to all compact surfaces, meaning surfaces of arbitrary genus that are not necessarily orientable and possibly possess non-empty boundary; compare to Corollary A.1.2.*

A.4. Equicontinuity of energy bounded almost homeomorphisms

The map provided by Proposition A.1.3 does not need to be canonical, i.e. of minimal energy. In order to obtain such a parametrization in Section A.5, we will apply a direct variational method for which we need to know equicontinuity of a given energy-minimizing sequence of parametrizations. More explicitly, we prove the following statement in this section.

Proposition A.4.1. *Let M be a smooth surface endowed with a Riemannian metric g and which is neither of disc- nor of sphere-type. Let X be a metric surface homeomorphic to M and such that ∂X is rectifiable. Then the family*

$$\mathcal{F} := \{v \in \Lambda(M, X) : E_+^2(v, g) \leq K\}$$

is equicontinuous.

In order to show Proposition A.4.1, we need the following elementary lemma. Its proof is left to the reader.

Lemma A.4.2. *Let X be a metric surface which is not of sphere-type. Then for every $\varepsilon > 0$ there exists $\rho > 0$ such that the following holds. Every embedding $u\colon \overline{D} \to X$ with $\operatorname{diam}(u(S^1)) < \rho$ satisfies $\operatorname{diam}(u(\overline{D})) < \varepsilon$.*

By continuity, the statement holds for any uniform limit of embeddings from $\overline{D}$ to X.

Proof of Proposition A.4.1. Let $\varepsilon > 0$ and define

$$\eta := \inf\{\ell(c) \mid c\colon S^1 \to X \text{ is a non-contractible curve in } X\} > 0.$$

By Lemma A.4.2, there exists $0 < \rho < \min\{\varepsilon, \eta\}$ such that for any uniform limit of embeddings $u\colon \overline{D} \to X$ with $\operatorname{diam}(u(S^1)) < \rho$ there holds $\operatorname{diam}(u(\overline{D})) < \varepsilon$. Similarly, there exists $0 < \rho' <$

$\rho/2$ such that the following is true. If $x, x' \in \partial X$ satisfy $d(x, x') < \rho'$, then they lie on the same component $\partial X^i \subset \partial X$ and the shorter of the two subcurves of ∂X^i connecting x and x' has length at most $\rho/2$. Since M is compact, there exists $0 < \delta < 1$ so small that

$$\pi \cdot \left(\frac{8K}{|\log(\delta)|} \right)^{1/2} < \rho'$$

and such that every point $p \in M$ is contained in a neighbourhood in M which is the image of the set $B := B_{\mathbb{R}^2}(q, \sqrt{\delta}) \cap \overline{D}$ under a map ψ that is 2-biLipschitz and takes the point $q \in [0, 1] \subset \overline{D}$ to p, where q is chosen to be 1 if $p \in \partial M$ and 0 whenever the distance between p and ∂M is big enough. In particular, if the set $B_{\mathbb{R}^2}(q, \sqrt{\delta}) \cap S^1$ is not empty, then it is mapped onto a subcurve of ∂M.

Fix $p \in M$ and $v \in \mathcal{F}$. By the Courant-Lebesgue Lemma (see e.g. [LW17a, Lemma 7.3]) there exists $r \in (\delta, \sqrt{\delta})$ such that

$$\ell(v \circ \psi \circ \gamma_r) \leq \pi \cdot \left(\frac{2E_+^2(v \circ \psi)}{|\log(\delta)|} \right)^{1/2} \leq \pi \cdot \left(\frac{8E_+^2(v)}{|\log(\delta)|} \right)^{1/2} < \rho',$$

where γ_r is an arc-length parametrization of $\{z \in B : |z - q| = r\}$.

Consider the set $A := \{z \in B : |z - q| < r\}$. It holds that $B_M(p, \delta/2) \subset \psi(A)$ and $\overline{A}$ is biLipschitz equivalent to $\overline{D}$ with constant only depending on r. If $\psi(A)$ does not intersect ∂M, by applying Lemma A.4.2, we can conclude $\operatorname{diam}(v(\psi(A))) < \varepsilon$ and therefore $v(B_M(p, \delta/2)) \subset B_X(v(p), \varepsilon)$.

If $\psi(A) \cap \partial M$ is not empty, then $\psi(A)$ is bounded by $\psi \circ \gamma_r$ and a subarc of ∂M^i, denoted α_r. The endpoints $a_r, b_r \in \partial M^i$ of $\psi \circ \gamma_r$ satisfy $d(v(a_r), v(b_r)) < \rho' < \rho/2$. Thus, $v(a_r)$ and $v(b_r)$ lie on the same boundary component $\partial X^i \subset \partial X$ and the shorter subcurve of ∂X^i connecting $v(a_r)$ and $v(b_r)$ has length at most $\rho/2 < \eta/2$. This segment corresponds to the curve $v \circ \alpha_r$. Indeed otherwise, the concatenation of $v \circ \psi \circ \gamma_r$ with $v|_{\partial M^i \setminus \alpha_r}$ would yield a non-contractible closed curve in X of length strictly less than η, which is impossible. Again by applying Lemma A.4.2 we obtain $v(B_M(p, \delta/2)) \subset v(\psi(A)) \subset B_X(v(p), \varepsilon)$. Since the choice of δ was independant of v and of p, this proves equicontinuity of $\mathcal{F}$. $\square$

A.5. Proof of Main Theorem

We finally turn to the proof of Theorem A.1.1. First however, we introduce some notation. Define the family

$$\Lambda_{\text{metr}}(M, X) := \{(v, h) : v \in \Lambda(M, X),\ h \text{ a smooth Riemannian metric on M}\}.$$

An *energy minimizing sequence* in $\Lambda_{\text{metr}}(M, X)$ is a sequence of pairs $(u_n, g_n) \in \Lambda_{\text{metr}}(M, X)$ satisfying

$$E_+^2(u_n, g_n) \to \inf\{E_+^2(v, h) : (v, h) \in \Lambda_{\text{metr}}(M, X)\}$$

as n tends to infinity.

Proof of Theorem A.1.1. The proofs in the cases where M is of disc- or sphere-type follow from [LW20], and we therefore only consider M being of higher topological type. Moreover, we assume that M is equipped with a hyperbolic metric. The case where M only admits flat metrics follows analogously. In a first step, we show the existence of an energy minimizing pair in Λ_{metr}. By Proposition A.1.3, the set $\Lambda(M, X)$ is not empty. Therefore, we are able to consider an energy minimizing sequence (u_n, g_n) in $\Lambda_{\text{metr}}(M, X)$. We lose no generality in assuming that the metrics

g_n are all hyperbolic. Observe that each u_n, being a uniform limit of homeomorphisms, satisfies the condition of cohesion for some $\eta > 0$ in the sense of [FW21, Definition 8.1]. Thus by [FW21, Proposition 8.4] there exists $\varepsilon > 0$ depending only on η and $K := \sup_{n\in\mathbb{N}} E_+^2(u_n, g_n)$ such that for every n the relative systole of (M, g_n) (see [FW21, Definition 3.1]) is bounded from below by ε. Then, there exist diffeomorphisms $\varphi_n \colon M \to M$ such that a subsequence of $(\varphi_n^* g_n)$ converges smoothly to a hyperbolic metric g on M (see [DHT10, Theorem 4.4.1] if M is a closed surface; and e.g. [FW21, Theorem 3.3] if M has non-empty boundary). Set $v_n := u_n \circ \varphi_n$. The convergence above implies that

$$\lim_{n\to\infty} E_+^2(v_n, g) = \lim_{n\to\infty} E_+^2(u_n, g_n).$$

Thus, the sequence (v_n, g) is energy minimizing in $\Lambda_{\mathrm{metr}}(M, X)$. Now by Proposition A.4.1, the sequence (v_n) is equicontinuous and the Arzelà-Ascoli theorem implies that a subsequence of (v_n) converges uniformly to some continuous map $u\colon M \to X$. It follows that u is in $N^{1,2}(M, X)$ (compare to [KS93, Theorem 1.6.1]) as well as a uniform limit of homeomorphisms, hence $u \in \Lambda(M, X)$. By the lower semicontinuity of $E_+^2(\cdot)$ it follows that the pair (u, g) is an energy minimizer in $\Lambda_{\mathrm{metr}}(M, X)$.

We now show that any energy minimizing pair (u, g) in $\Lambda_{\mathrm{metr}}(M, X)$ is a quasisymmetric homeomorphism. As a uniform limit of homeomorphisms, the map u is continuous, monotone and surjective. Furthermore, by [FW21, Theorem 4.2], the map u is infinitesimally isotropic and hence infinitesimally $\sqrt{2}$-quasiconformal with respect to g (see [FW21, Definition 4.1] and the explanation thereafter). It follows from [LW20, Theorem 3.6] that u is a local homeomorphism. Monotonicity of u implies then that u is injective. Hence, u is a homeomorphism as it is a continuous bijection on a compact set M. Using analogous statements to Theorem 2.5 and Proposition 3.5 in [LW20] for the domain (M, g) instead of $(\overline{D}, g_{\mathrm{Eucl}})$, one can argue as in the proof of [LW20, Theorem 6.1] to obtain that u is a quasisymmetric homeomorphism with respect to g. Note that the analogue to [LW20, Theorem 2.5] follows since M admits a $(1, 2)$-Poincaré inequality and is thus a Loewner space, see [Hei01, Theorem 9.10].

It remains to show uniqueness of (u, g) up to precomposition with conformal diffeomorphisms. Let (u, g), (v, h) be energy minimizing pairs in $\Lambda_{\mathrm{metr}}(M, X)$. We claim that the map $\varphi := v^{-1} \circ u\colon (M, g) \to (M, h)$ is then a conformal diffeomorphism. Indeed, for any choice of conformal charts $\psi\colon \overline{U} \to \overline{D}$ of (M, g) and $\phi\colon \overline{V} \to \overline{D}$ of (M, h), we can argue as in the last paragraph in the proof of [LW20, Theorem 6.1] that the transition maps

$$\phi \circ v^{-1} \circ u \circ \psi^{-1} \colon \overline{D} \to \overline{D}$$

are conformal diffeomorphisms, which implies the respective property for the mapping φ. The proof of the theorem is complete. □

A.6. Appendix

Proof of Lemma A.3.8. Let $z \in \widehat{XY}$ and $r > 0$ be arbitrary. By symmetry, we may assume $z \in X$. Observe that there exists $y \in Y$ such that $B_{\widehat{XY}}(z, r)$ is contained in $(B_{\widehat{XY}}(z, r) \cap X) \cup (B_{\widehat{XY}}(y, 2r) \cap Y)$. The Ahlfors 2-regularity of $(\widehat{XY}, \hat{d})$ now follows from Lemma A.3.7 and the Ahlfors 2-regularity of X and Y.

It remains to prove that $(\widehat{XY}, \hat{d})$ is LLC. Both X and Y are quasiconvex (see [Sem96a, Theorem B.6]) with constants C_X and C_Y depending only on the LLC and Ahlfors 2-regularity constants of X and Y, respectively. Hence, the space $(\widehat{XY}, \hat{d})$ is quasiconvex with constant $\widehat{C} := \max\{C_X, C_Y\}$ implying that the first LLC condition holds with constant $\widehat{C}$.

Denote by λ_X and λ_Y the LLC-constants of X and Y, respectively, and choose

$$\hat{\lambda} \geq \max\{2, \lambda_X, \lambda_Y\}$$

such that $2\mathrm{diam}_{\hat{d}}(\widehat{XY})/\hat{\lambda} < \mathrm{diam}_{\hat{d}}(X \cap Y)$. Let $x, y \in \widehat{XY} \setminus B_{\widehat{XY}}(z, r)$. We want to prove the existence of a uniform $\lambda \geq 1$ such that x, y can be joined by a continuum in $\widehat{XY} \setminus B_{\widehat{XY}}(z, r/\lambda)$. If $x, y \in X$ or $x, y \in Y$, the statement follows from the LLC-property of X or Y and Lemma A.3.7. Consider $x \in X$, $y \in Y \setminus X$ and assume for the moment that $B := B_{\widehat{XY}}\left(z, r/(2L\hat{\lambda}^2)\right) \subset X$. Choose any point $a \in (X \cap Y) \setminus B_{\widehat{XY}}(z, r/\hat{\lambda})$. Then there exists a continuum in

$$X \setminus B_X\left(z, r/\hat{\lambda}^2\right) \subset \widehat{XY} \setminus B_{\widehat{XY}}\left(z, r/(L\hat{\lambda}^2)\right) \subset \widehat{XY} \setminus B$$

connecting x with a, which can be concatenated with any continuum in Y connecting a with y to obtain a desired path between x and y in $\widehat{XY} \setminus B$. If the intersection of $B_{\widehat{XY}}\left(z, r/(2L\hat{\lambda}^2)\right)$ with Y is not empty, choose a point $b \in X \cap Y \cap B$ and define

$$d := \hat{d}(b, z) < \frac{r}{2L\hat{\lambda}^2} < \frac{r}{\hat{\lambda}}.$$

It then follows from the triangle inequality that

$$d_X(b, x) \geq \hat{d}(b, x) \geq r - d \geq r - \frac{r}{\hat{\lambda}} \geq \frac{r}{\hat{\lambda}}$$

and similarly, that $d_Y(b, y) \geq r/\hat{\lambda}$. After picking a point $a \in (X \cap Y) \setminus B_{\widehat{XY}}(b, r/\hat{\lambda})$, we have the existence of continua $E \subset X \setminus B_X(b, r/\hat{\lambda}^2)$ connecting x with a respectively $F \subset Y \setminus B_Y(b, r/\hat{\lambda}^2)$ joining a with y; and therefore a continuum in

$$\widehat{XY} \setminus B_{\widehat{XY}}\left(b, r/(L\hat{\lambda}^2)\right) \subset \widehat{XY} \setminus B_{\widehat{XY}}\left(z, r/(L\hat{\lambda}^2) - d\right) \subset \widehat{XY} \setminus B$$

between x and y. We thus have proven that the space $(\widehat{XY}, \hat{d})$ is LLC with constant $\lambda := \max\{2L\hat{\lambda}^2, \widehat{C}\}$. $\square$

B. Quasiconformal almost parametrizations of metric surfaces

with Stefan Wenger

Abstract. We look for minimal conditions on a two-dimensional metric surface X of locally finite Hausdorff 2-measure under which X admits an (almost) parametrization with good geometric and analytic properties. Only assuming that X is locally geodesic, we show that Jordan domains in X of finite boundary length admit a quasiconformal almost parametrization. If X satisfies some further conditions, then such an almost parametrization can be upgraded to a geometrically quasiconformal homeomorphism or a quasisymmetric homeomorphism. In particular, we recover Rajala's recent quasiconformal uniformization theorem in the special case that X is locally geodesic as well as Bonk–Kleiner's quasisymmetric uniformization theorem. On the way, we establish the existence of Sobolev discs spanning a given Jordan curve in X under nearly minimal assumptions on X and prove the continuity of energy minimizers.

B.1. Introduction and statement of main results

B.1.1. Background

Every smooth Riemann surface is conformally diffeomorphic to a surface of constant curvature by the classical uniformization theorem. The uniformization problem for metric spaces, widely studied in the field of analysis in metric spaces and of importance also in other areas, asks to find conditions on a given metric space X, homeomorphic to some model space M, under which there still exists a homeomorphism from X to M with good geometric and analytic properties.

In this paper, we consider the uniformization problem for metric spaces homeomorphic to a two-dimensional surface and of locally finite Hausdorff 2-measure. In this setting, two outstanding uniformization results were proved in [BK02] and [Raj17]. Bonk–Kleiner [BK02] showed that an Ahlfors 2-regular metric space X homeomorphic to the standard two-sphere S^2 admits a quasisymmetric homeomorphism to S^2 if and only if X is linearly locally connected. Ahlfors 2-regular means that the Hausdorff 2-measure of balls of radius r is comparable to r^2, and a quasisymmetric homeomorphism is a homeomorphism that distorts shapes in a controlled manner. We refer to Section B.3 for the definitions of quasisymmetric homeomorphism and linear local connectedness. More recently, Rajala [Raj17] gave a characterization of metric planes admitting a geometrically quasiconformal homeomorphism to a Euclidean domain. His characterization involves a condition called reciprocality. A geometrically quasiconformal map is a homeomorphism that leaves the (conformal) modulus of curve families invariant up to a multiplicative constant. Rajala's result in particular gives a new approach to the Bonk–Kleiner quasisymmetric uniformization theorem. The results in [BK02] and [Raj17] have been extended, for example, in [BK05], [Wil08], [Wil10], [MW13], [Iko22]. In [LW20], Lytchak and the second author provided a further approach to the Bonk–Kleiner theorem which relies on results about the existence and regularity of energy and area minimizing discs in metric spaces admitting a quadratic isoperimetric inequality developed in [LW17a] and [LW18a].

While we work with metric surfaces of locally finite Hausdorff 2-measure in this paper, the uniformization problem has also been studied for fractal spaces, see, for example, [Mey02], [Mey10], [LRR18], [RRR21], [RR21]. The aim of our paper is to establish the existence of parametrizations or almost parametrizations with good properties under nearly minimal conditions on X. The properties are such that they upgrade to geometrically quasiconformal parametrizations under Rajala's reciprocality condition and to quasisymmetric parametrizations under the condition of Ahlfors 2-regularity and linear local connectedness. On the way to prove our parametrization results, we establish the existence of Sobolev discs spanning a given Jordan curve under nearly minimal assumptions on X and regularity of energy minimizers.

B.1.2. Parametrization results

We now turn to a rigorous discussion of our results. Let X be a metric space homeomorphic to a two-dimensional surface and assume that X has locally finite Hausdorff 2-measure. The modulus of a family Γ of curves in X is defined by

$$\operatorname{mod}(\Gamma) := \inf_{\rho} \int_X \rho^2 \, d\mathcal{H}^2,$$

where the infimum is taken over all Borel functions $\rho\colon X \to [0,\infty]$ for which $\int_\gamma \rho \, ds \geq 1$ for every $\gamma \in \Gamma$, see Section B.2. A homeomorphism $u\colon D \to \Omega$ from the unit disc $D \subset \mathbb{R}^2$ to a domain $\Omega \subset X$ is called geometrically quasiconformal if u leaves the modulus of curve families quasi-invariant, thus there exists $K \geq 1$ such that

$$K^{-1} \cdot \operatorname{mod}(\Gamma) \leq \operatorname{mod}(u \circ \Gamma) \leq K \cdot \operatorname{mod}(\Gamma)$$

for every family Γ of curves in D. Here, $u \circ \Gamma$ denotes the family of curves $u \circ \gamma$ with $\gamma \in \Gamma$. By Rajala's recent uniformization result [Raj17], Ω admits a geometrically quasiconformal parametrization if and only if Ω satisfies a certain reciprocality condition described below. It is natural to wonder to what extent one can weaken this condition and still obtain an (almost) parametrization of Ω with suitable properties which can then be upgraded to a geometrically quasiconformal homeomorphism when X is reciprocal. Our main result shows that the reciprocality condition can be dropped completely, at least when the underlying metric space X is locally geodesic.

Theorem B.1.1. *Let X be a locally geodesic metric space homeomorphic to $\mathbb{R}^2$ and of locally finite Hausdorff 2-measure. If $\Omega \subset X$ is a Jordan domain of finite boundary length then there exists a continuous, monotone surjection $u\colon \overline{D} \to \overline{\Omega}$ such that*

$$\operatorname{mod}(\Gamma) \leq K \cdot \operatorname{mod}(u \circ \Gamma) \tag{B.1}$$

for every family Γ of curves in $\overline{D}$, where $K = \frac{4}{\pi}$.

The map u is called monotone if $u^{-1}(x)$ is connected for every $x \in X$; equivalently, u is the uniform limit of homeomorphisms $u_n\colon \overline{D} \to \overline{\Omega}$. Notice that in the generality of Theorem B.1.1, there need not exist a homeomorphism satisfying (B.1), see Example B.3.2. When u is a homeomorphism, then (B.1) is equivalent to the so-called analytic definition of quasiconformality by [Wil12]. Theorem B.1.1 answers a question of Rajala and the second author stated e.g. in [IR22, Question 1.1] in the special case of locally geodesic metric spaces. The factor $K = \frac{4}{\pi}$ appearing in Theorem B.1.1 is optimal in general, see [Rom19b], and the condition that X be locally geodesic can be weakened, see Remark B.6.1 below. Theorem B.1.1 also implies an analogue in which the Jordan domain Ω is replaced by any open simply connected subset of X with compact closure, see Corollary B.6.2.

We now describe conditions on X under which a map u as in the theorem can be upgraded to a homeomorphism, to a geometrically quasiconformal homeomorphism, or to a quasisymmetric homeomorphism. Given subsets E, F and G of the metric space X, we denote by $\operatorname{mod}(E, F; G)$ the modulus of the family of curves joining E and F in G. Let u be a map as in Theorem B.1.1. If for every $x \in X$ and every $R > 0$ with $X \setminus B(x, R) \neq \emptyset$ we have

$$\lim_{r \to 0} \operatorname{mod}(B(x, r), X \setminus B(x, R); \overline{B}(x, R)) = 0 \tag{B.2}$$

then u is a homeomorphism, see Proposition B.3.1. If, moreover, there exists $\kappa > 0$ such that every closed topological square $Q \subset X$ with boundary edges $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ in cyclic order satisfies

$$\operatorname{mod}(\zeta_1, \zeta_3; Q) \cdot \operatorname{mod}(\zeta_2, \zeta_4; Q) \leq \kappa \tag{B.3}$$

then the map u is geometrically quasiconformal, as follows from the arguments in [Raj17], see Proposition B.3.3 below. Theorem B.1.1 thus yields the following result, which is variant for locally geodesic metric surfaces of Rajala's uniformization theorem [Raj17, Theorem 12.1].

Corollary B.1.2. *Let X be a locally geodesic metric space homeomorphic to $\mathbb{R}^2$ and of locally finite Hausdorff 2-measure. If X satisfies conditions (B.2) and (B.3), then for every Jordan domain $\Omega \subset X$, there exists a homeomorphism $u\colon \overline{D} \to \overline{\Omega}$ which is geometrically quasiconformal.*

Rajala's reciprocality condition [Raj17] mentioned above consists of (B.2) and (B.3) as well as a lower bound on the product in (B.3). It has recently been shown in [RR19] that the lower bound is always satisfied, see also [EBPC22]. Notice that reciprocality is also a necessary condition for the existence of geometrically quasiconformal homeomorphisms. This is because $\mathbb{R}^2$ satisfies (B.2) and (B.3) and they are preserved under geometrically quasiconformal homeomorphisms. The optimal geometric quasiconformality constants were obtained in [Rom19b]. It was shown in [Raj17, Theorem 1.6] that if there exists $C > 0$ such that

$$\mathcal{H}^2(B(x, r)) \leq C r^2 \tag{B.4}$$

for every $x \in X$ and $r > 0$, then X is reciprocal. It follows from [LW20, Theorem 2.5] that if u is a map as in Theorem B.1.1 and if X satisfies (B.4), then u is a quasisymmetric homeomorphism if and only if $\overline{\Omega}$ is linearly locally connected. In particular, Theorem B.1.1 recovers the Bonk–Kleiner quasisymmetric uniformization theorem [BK02] for metric discs (see Corollary B.6.3 below) and, by a quasisymmetric gluing argument as in [LW20], also for metric spheres.

B.1.3. Methods of proof and other results

We now describe the ingredients in the proof of Theorem B.1.1. As in the classical existence proof of conformal parametrizations of smooth surfaces, we will obtain a quasiconformal almost parametrization u as an energy minimizing disc in X spanning $\partial\Omega$. This is similar to the approach taken in [LW20]. The proofs in [LW20] heavily use regularity properties and the intrinsic structure of energy minimizers in spaces with a quadratic isoperimetric inequality established in [LW17a] and [LW18a]. Such results are not available in our setting.

Let X be a complete metric space and let $N^{1,2}(D,X)$ be the space of Newton–Sobolev maps from D to X in the sense of [HKST15]. For a map $u \in N^{1,2}(D,X)$, we denote by $\operatorname{tr}(u)$ the trace of u and by $E_+^2(u)$ its (Reshetnyak) energy. If $\Gamma \subset X$ is a Jordan curve, we denote by $\Lambda(\Gamma, X)$ the possibly empty family of maps $u \in N^{1,2}(D,X)$ whose trace has a continuous representative which is a weakly monotone parametrization of Γ. See Section B.2 for the definitions of these concepts.

Theorem B.1.3. *Let X be a locally geodesic metric space homeomorphic to $\overline{D}$ and let $\Gamma \subset X$ be a Jordan curve. If $u \in \Lambda(\Gamma, X)$ minimizes the Reshetnyak energy E_+^2 among all maps in $\Lambda(\Gamma, X)$, then u has a representative which is continuous and extends continuously to the boundary.*

Notice that the regularity results for energy minimizers proved in [LW17a] cannot be applied here since metric spaces as in Theorem B.1.3 need not admit a quadratic isoperimetric inequality for curves. In general, the family $\Lambda(\Gamma, X)$ in Theorem B.1.3 can be empty. However, we can prove the following theorem.

Theorem B.1.4. *Let X be a locally geodesic metric space homeomorphic to $\mathbb{R}^2$, $\overline{D}$, or S^2. If X has locally finite Hausdorff 2-measure, then $\Lambda(\Gamma, X) \neq \emptyset$ for every rectifiable Jordan curve $\Gamma \subset X$.*

Theorem B.1.1 now easily follows from Theorems B.1.3 and B.1.4 together with the results on the existence and structure of area and energy minimizers established in [LW17a] and [LW20]. Indeed, one easily reduces the theorem to the special case that X is geodesic, homeomorphic to $\overline{D}$, and $\overline{\Omega} = X$. Since the boundary circle ∂X of X has finite length, Theorem B.1.4 shows that the family $\Lambda(\partial X, X)$ is not empty. By [LW17a], there exists an energy minimizer u in $\Lambda(\partial X, X)$. By Theorem B.1.3, any such u (has a representative which) is continuous up to the boundary and it thus follows from [LW20, Theorem 1.2] that u is monotone. Finally, by [LW17a] energy minimizers are infinitesimally quasiconformal in the sense that

$$(g_u(z))^2 \leq K \cdot \operatorname{Jac}(\operatorname{ap}\operatorname{md} u_z) \tag{B.5}$$

with $K = \frac{4}{\pi}$ for almost every $z \in D$, where g_u denotes the minimal weak upper gradient of u and $\operatorname{Jac}(\operatorname{ap}\operatorname{md} u_z)$ is the Jacobian of u, see Section B.2. This implies that u satisfies (B.1), see Section B.3, and thus the outline of the proof of Theorem B.1.1 is complete. Notice that a homeomorphism $u \colon \overline{D} \to X$ is quasiconformal according to the analytic definition if u belongs to $N^{1,2}(D,X)$ and satisfies (B.5), see Section B.3.

We mention the following consequence of Theorem B.1.4.

Corollary B.1.5. *Let X be a locally geodesic metric space homeomorphic to $\mathbb{R}^2$. If X has locally finite Hausdorff 2-measure, then X contains a 2-rectifiable subset of positive Hausdorff measure.*

An example showing that X need not be countably 2-rectifiable is given in [SW10, Theorem A.1]. Corollary B.1.5 also holds for compact metric spaces of any topological dimension n and finite Hausdorff n-measure with positive lower density almost everywhere. This was proved in [DLD20] using a deep result of Bate [Bat20] about purely unrectifiable metric spaces.

The paper is structured as follows. In Section B.2, we collect the necessary definitions and some results on Newton-Sobolev maps that will be used later. In Section B.3, we show how quasiconformal almost parametrizations can be upgraded under additional conditions on the underlying space. We prove Theorem B.1.3 in Section B.4; Theorem B.1.4 and Corollary B.1.5 are established in Section B.5. In the final Section B.6, we discuss the proofs of the parametrizations results as well as some consequences and generalizations.

Subsequent to the submission of our paper on the arXiv in June 2021, the paper [NR23] appeared on the arXiv and has been published in the mean time. In [NR23], Ntalampekos and Romney prove a variant of Theorem B.1.1 for simply connected geodesic metric surfaces X of locally finite Hausdorff 2-measure using an approach based on approximating X by polyhedral surfaces. In [NR24], they were able to extend the result to all metric surfaces of locally finite Hausdorff 2-measure. See also [C] for a generalization of Theorem B.1.1 to compact geodesic metric surfaces of finite Hausdorff 2-measure and of higher topology following the approach used in this work.

B.2. Preliminaries

B.2.1. Basic notation

We denote the open and closed unit discs in the Euclidean plane $\mathbb{R}^2$ by D and $\overline{D}$, respectively; that is,

$$D := \left\{z \in \mathbb{R}^2 \mid |z| < 1\right\}, \quad \overline{D} := \left\{z \in \mathbb{R}^2 \mid |z| \leq 1\right\},$$

where $|v|$ denotes the Euclidean norm of the vector $v \in \mathbb{R}^2$. Let (X, d) be a metric space. The open and closed balls in X centred at some point x_0 of radius $r > 0$ are

$$B(x_0, r) := \{x \in X \mid d(x, x_0) < r\}, \quad \overline{B}(x_0, r) := \{x \in X \mid d(x, x_0) \leq r\}.$$

Let $c\colon I \to X$ be a curve defined on some interval $I \subset \mathbb{R}$. The length of c is denoted by $\ell(c)$. If c is absolutely continuous, then c has a metric derivative almost everywhere, thus the limit

$$|c'(t)| := \lim_{s \to t} \frac{d(c(s), c(t))}{|s - t|}$$

exists for almost every $t \in I$, and moreover $\ell(c) = \int_I |c'(t)|\, dt$, see [Kir94]. A curve $c\colon [a, b] \to X$ is called geodesic if $\ell(c) = d(c(a), c(b))$. The metric space X is called geodesic if any pair of points in X can be joined by a geodesic. It is called locally geodesic if every point $x \in X$ has a neighbourhood U such that any two points in U can be joined by a geodesic in X.

Given $m \geq 0$, the m-dimensional Hausdorff measure on X is denoted by $\mathcal{H}^m$. The normalizing constant is chosen so that $\mathcal{H}^n$ agrees with the Lebesgue measure on Euclidean $\mathbb{R}^n$. We write $|A|$ for the Lebesgue measure of a subset $A \subset \mathbb{R}^n$.

B.2.2. Conformal modulus

Let X be a metric space and Γ a family of curves in X. A Borel function $\rho\colon X \to [0, \infty]$ is said to be admissible for Γ if $\int_\gamma \rho\, ds \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$. See [HKST15] for the definition of the path integral $\int_\gamma \rho$. The modulus of Γ is defined by

$$\operatorname{mod}(\Gamma) := \inf_\rho \int_X \rho^2\, d\mathcal{H}^2,$$

where the infimum is taken over all admissible functions for the family Γ. We emphasize that throughout this paper, the reference measure on X will always be the two-dimensional Hausdorff measure. By definition, $\operatorname{mod}(\Gamma) = \infty$ if Γ contains a constant curve. A property is said to hold for almost every curve in Γ if it holds for every curve in Γ_0 for some family $\Gamma_0 \subset \Gamma$ with $\operatorname{mod}(\Gamma \setminus \Gamma_0) = 0$. In the definition of $\operatorname{mod}(\Gamma)$, the infimum can equivalently be taken over all weakly admissible functions, that is, Borel functions $\rho\colon X \to [0, \infty]$ such that $\int_\gamma \rho\, ds \geq 1$ for almost every locally rectifiable curve $\gamma \in \Gamma$.

B.2.3. Metric space valued Sobolev maps

We recall some definitions from the theory of metric space valued Sobolev maps based on upper gradients [Sha00], [HKST01], [HKST15] as well as two results concerning the existence and structure of energy minimizing discs established in [LW17a], [LW20]. Note that the results in [LW17a] are stated using a different but equivalent definition of Sobolev mappings.

Let (X, d) be a complete metric space and $U \subset \mathbb{R}^2$ a bounded domain. A Borel function $g\colon U \to [0, \infty]$ is said to be an upper gradient of a map $u\colon U \to X$ if

$$d(u(\gamma(a)), u(\gamma(b))) \leq \int_\gamma g\, ds \tag{B.6}$$

for every rectifiable curve $\gamma\colon [a, b] \to U$. If (B.6) only holds for almost every curve γ, then g is called a weak upper gradient of u. A weak upper gradient g of u is called minimal weak upper gradient of u if $g \in L^2(U)$ and if for every weak upper gradient h of u in $L^2(U)$, we have $g \leq h$ almost everywhere on U.

Denote by $L^2(U, X)$ the collection of measurable and essentially separably valued maps $u\colon U \to X$ such that the function $u_x(z) := d(u(z), x)$ belongs to $L^2(U)$ for some and thus any $x \in X$. A map $u \in L^2(U, X)$ belongs to the Newton–Sobolev space $N^{1,2}(U, X)$ if u has a weak upper gradient in $L^2(U)$. Every such map u has a minimal weak upper gradient g_u, unique up to sets of measure zero, see [HKST15, Theorem 6.3.20]. The *Reshetnyak energy* of a map $u \in N^{1,2}(U, X)$ is defined by

$$E_+^2(u) := \|g_u\|^2_{L^2(U)}.$$

If $u \in N^{1,2}(U, X)$, then for almost every $z \in U$ there exists a unique semi-norm on $\mathbb{R}^2$, denoted by $\operatorname{ap\,md} u_z$ and called the approximate metric derivative of u, such that

$$\operatorname{ap}\lim_{y\to z} \frac{d(u(y), u(z)) - \operatorname{ap\,md} u_z(y - z)}{|y - z|} = 0,$$

see [Kar07] and [LW17a, Proposition 4.3]. For the definition of the approximate limit ap lim, see [EG92]. The following notion of parametrized area was introduced in [LW17a].

Definition B.2.1. The parametrized (Hausdorff) area of a map $u \in N^{1,2}(U, X)$ is defined by

$$\operatorname{Area}(u) = \int_U \operatorname{Jac}(\operatorname{ap\,md} u_z)\, dz,$$

where the Jacobian $\operatorname{Jac}(s)$ of a semi-norm s on $\mathbb{R}^2$ is the Hausdorff 2-measure on $(\mathbb{R}^2, s)$ of the unit square if s is a norm and zero otherwise.

The area of the restriction of u to a measurable set $B \subset U$ is defined analogously. It is not difficult to show that $\operatorname{Jac}(\operatorname{ap\,md} u_z) \leq (g_u(z))^2$ for almost every $z \in U$, see [LW17a, Lemma 7.2]. If u is a homeomorphism onto its image, then the Jacobian $\operatorname{Jac}(\operatorname{ap\,md} u_z)$ agrees with the Radon–Nikodym derivative (of the absolutely continuous part) of the measure $u^*\mathcal{H}^2(B) := \mathcal{H}^2(u(B))$ with respect to the Lebesgue measure at almost every point $z \in U$. We emphasize that u need not satisfy Lusin's property (N).

Definition B.2.2. A map $u \in N^{1,2}(U, X)$ is called infinitesimally K-quasiconformal if

$$(g_u(z))^2 \leq K \cdot \operatorname{Jac}(\operatorname{ap\,md} u_z) \tag{B.7}$$

for almost every $z \in U$.

If $u \in N^{1,2}(D,X)$, then for almost every $v \in S^1$ the curve $t \mapsto u(tv)$ with $t \in [1/2,1)$ is absolutely continuous. The trace of u is defined by $\operatorname{tr}(u)(v) := \lim_{t\nearrow 1} u(tv)$ for almost every $v \in S^1$. It follows from [KS93] that $\operatorname{tr}(u) \in L^2(S^1,X)$. If u is the restriction to D of a continuous map $\hat{u}$ on $\overline{D}$ then $\operatorname{tr}(u) = \hat{u}|_{S^1}$. Given a Jordan curve $\Gamma \subset X$, we denote by $\Lambda(\Gamma,X)$ the possibly empty family of maps $v \in N^{1,2}(D,X)$ whose trace has a continuous representative which weakly monotonically parametrizes Γ. Recall that a continuous map $c\colon S^1 \to \Gamma$ is called a weakly monotone parametrization of Γ if c is the uniform limit of homeomorphisms $c_i\colon S^1 \to \Gamma$.

Theorem B.2.3. *Let X be a proper metric space and $\Gamma \subset X$ be a Jordan curve. If $\Lambda(\Gamma,X)$ is not empty, then there exists $u \in \Lambda(\Gamma,X)$ satisfying*

$$E_+^2(u) = \inf\left\{E_+^2(v) \mid v \in \Lambda(\Gamma,X)\right\},$$

and any such u is infinitesimally K-quasiconformal with $K = \frac{4}{\pi}$.

Proof. The existence of an energy minimizer in $\Lambda(\Gamma,X)$ follows from [LW17a, Theorem 7.6]. Energy minimizers are infinitesimally K-quasiconformal with $K = \frac{4}{\pi}$ by [LW17b], see also [LW17a, Lemma 6.5]. □

We will also need the following theorem proved in [LW20].

Theorem B.2.4. *Let X be a geodesic metric space homeomorphic to $\overline{D}$ and let $u\colon \overline{D} \to X$ be a continuous map. If u belongs to $\Lambda(\partial X,X)$ and minimizes the Reshetnyak energy E_+^2 among all maps in $\Lambda(\partial X,X)$, then u is monotone.*

By definition, the boundary circle ∂X of X is the image of S^1 under a homeomorphism from $\overline{D}$ to X. Recall that a continuous map $u\colon \overline{D} \to X$ is monotone if $u^{-1}(x)$ is connected for every $x \in X$. If X is homeomorphic to $\overline{D}$ then u is monotone if and and only if u is the uniform limit of homeomorphisms $u_n\colon \overline{D} \to X$, see [You48].

B.3. Upgrading a quasiconformal almost parametrization

The aim of this short section is to summarize some results which show that maps as in Theorem B.1.1 can be upgraded under certain additional conditions on the underlying space.

We first recall the connection with infinitesimally quasiconformal maps. Let X be a complete metric space and $u\colon \overline{D} \to X$ continuous and monotone. If $u \in N^{1,2}(D,X)$ and u is infinitesimally K-quasiconformal, then

$$\operatorname{mod}(\Gamma) \leq K \cdot \operatorname{mod}(u \circ \Gamma) \tag{B.8}$$

for every family Γ of curves in $\overline{D}$. Conversely, if u is a homeomorphism onto its image and satisfies (B.8), then u belongs to $N^{1,2}(D,X)$ and is infinitesimally K-quasiconformal. See [LW20, Proposition 3.5] and [Wil12, Theorem 1.1] for a proof.

Proposition B.3.1. *Let X be a complete metric space satisfying (B.2). Let $u\colon \overline{D} \to X$ be continuous, monotone, and non-constant. If u satisfies (B.8), then u is a homeomorphism onto its image.*

Proof. This follows exactly as in the proof of [LW20, Theorem 3.6]. □

In the setting of Theorem B.1.1, there need not exist a homeomorphism satisfying (B.8). The following example illustrating this appears in [LW18a, Example 11.3], see [Raj17] for other examples.

Example B.3.2. Let $T = \{z \in D \mid |z| \leq 1/2\}$ and let $X = \overline{D}/T$ be the quotient metric space equipped with the quotient metric. Then X is geodesic, homeomorphic to $\overline{D}$, and has finite Hausdorff 2-measure. We claim that there does not exist a homeomorphism $u\colon \overline{D} \to X$ satisfying (B.8). Suppose to the contrary that such u exists. Then u is analytically quasiconformal by the discussion above, thus u is in $N^{1,2}(D,X)$ and is infinitesimally quasiconformal. Let $\pi\colon \overline{D} \to X$ be the quotient map and set $p := \pi(T)$. After possibly precomposing u with a bi-Lipschitz homeomorphism of $\overline{D}$, we may assume that $u(0) = p$. Consider the homeomorphism $v\colon D\setminus\{0\} \to D \setminus T$ satisfying $\pi(v(z)) = u(z)$ for all $z \in D \setminus \{0\}$. Since π is a local isometry on $D \setminus T$, it follows that v is (analytically) quasiconformal, which is impossible since the punctured disc is not quasiconformally equivalent to the annulus, see [Väi71, Theorem 39.1]. This contradiction finishes the proof of the claim.

The next proposition follows from the arguments in [Raj17, Section 11].

Proposition B.3.3. *Let X be a metric space homeomorphic to $\overline{D}$. Suppose $u\colon \overline{D} \to X$ is a homeomorphism satisfying* (B.8). *Then u is geometrically quasiconformal if and only if X satisfies* (B.3) *for some κ.*

Proof. Notice that $\overline{D}$ satisfies (B.3). Therefore, if u is geometrically quasiconformal, then also X satisfies (B.3). Suppose now that X satisfies (B.3) for some κ. By the discussion at the beginning of this section, the map u thus belongs to $N^{1,2}(D,X)$ and is infinitesimally K-quasiconformal. Identifying $\overline{D}$ with the unit square $R := [0,1]^2$ via a bi-Lipschitz homeomorphism, we may view u as an element of $N^{1,2}(R,X)$. There exists a Borel set $A \subset R$ of full measure such that $u|_A$ has Lusin's property (N), see e.g. [LW17a, Proposition 3.2]. Let g_u be the weak minimal upper gradient of u. We may assume that $g_u = \infty$ on $R \setminus A$. The Borel function $h\colon X \to [0,\infty]$ defined by $h := \frac{1}{g_u \circ u^{-1}}$ is L^2-integrable since

$$\int_X h^2 \, d\mathcal{H}^2 = \int_{u(A)} h^2 \, d\mathcal{H}^2 = \int_A \frac{1}{g_u^2(z)} \operatorname{Jac}(\operatorname{ap}\operatorname{md} u_z) \, dz \leq |R| < \infty$$

by the area formula, see [Kir94] and [Kar07]. Arguing exactly as in the proof of [Raj17, Proposition 11.1] one shows, that there exists C only depending on K and κ such that $C \cdot h$ is a weak upper gradient for u^{-1}. The proof of this relies on a lower bound of the form

$$\int_{Q(i,j,k)} h^2 \, d\mathcal{H}^2 \geq K^{-1} 2^{-2k}, \tag{B.9}$$

where $Q(i,j,k) = u([2^{-k}i, 2^{-k}(i+1)] \times [2^{-k}j, 2^{-k}(j+1)])$ as well as on an upper bound of the form

$$\operatorname{mod}(\Gamma_\ell(i,j,k)) \leq 3\kappa \tag{B.10}$$

for suitable path families $\Gamma_\ell(i,j,k)$ defined in the proof of [Raj17, Proposition 11.1]. In our case, (B.9) follows from the area formula, and (B.10) follows from (B.8) and (B.3). Now let Γ be a family of curves in R and let ϱ be an admissible function for Γ. Since for almost every curve $\beta = u \circ \gamma \in u \circ \Gamma$ we have

$$1 \leq \int_\gamma \varrho \, ds \leq C \cdot \int_\beta \varrho \circ u^{-1} h \, ds,$$

it follows that $C\varrho \circ u^{-1} h$ is weakly admissible for $u \circ \Gamma$. The area formula yields

$$\operatorname{mod}(u \circ \Gamma) \leq C^2 \cdot \int_X h^2 \varrho^2 \circ u^{-1} \, d\mathcal{H}^2 \leq C^2 \int_R \varrho^2(z) \, dz,$$

and taking the infimum over ϱ, we conclude that $\operatorname{mod}(u \circ \Gamma) \leq C^2 \cdot \operatorname{mod}(\Gamma)$. □

We finally describe conditions that imply that u is quasisymmetric. Recall that a homeomorphism $u\colon M \to X$ between metric spaces is said to be quasisymmetric if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$d(u(z), u(a)) \leq \eta(t) \cdot d(u(z), u(b))$$

for all $z, a, b \in M$ with $d(z,a) \leq t \cdot d(z,b)$.

Proposition B.3.4. *Let X be a metric space homeomorphic to $\overline{D}$ and let $u\colon \overline{D} \to X$ be a homeomorphism satisfying* (B.8)*. If there exists $L > 0$ such that*

$$\mathcal{H}^2(B(x,r)) \leq L \cdot r^2$$

for all $x \in X$ and $r > 0$, then u is quasisymmetric if and only if X is linearly locally connected.

We refer, for example, to the appendix of [LW20] for a proof of the proposition. Here, a metric space X is called linearly locally connected if there exists $\lambda \geq 1$ such that for every $x \in X$ and for all $r > 0$, every pair of points in $B(x,r)$ can be joined by a continuum in $B(x, \lambda r)$, and every pair of points in $X \setminus B(x,r)$ can be joined by a continuum in $X \setminus B(x, r/\lambda)$.

B.4. Continuity of energy minimizers in locally geodesic metric discs

In this section, we prove Theorem B.1.3. For an arbitrary map $v\colon D \to X$ to a metric space X and for $z \in D$ and $\delta > 0$, set

$$\operatorname{osc}(v, z, \delta) := \inf\{\operatorname{diam}(v(A)) \mid A \subset D \cap B(z,\delta) \text{ subset of full measure}\},$$

called the essential oscillation of v in the δ-ball around z.

Proposition B.4.1. *Let X be a locally geodesic metric space homeomorphic to $\overline{D}$ and let $\Gamma \subset X$ be a Jordan curve. Suppose $u \in \Lambda(\Gamma, X)$ minimizes the Reshetnyak energy among all maps in $\Lambda(\Gamma, X)$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\operatorname{osc}(u, z, \delta) < \varepsilon$ for every $z \in D$.*

The theorem easily follows from this proposition.

Proof of Theorem B.1.3. Let $A = \{z_n \mid n \in \mathbb{N}\} \subset D$ be a countable dense set. For each $k \in \mathbb{N}$, apply the proposition with $\varepsilon = \frac{1}{k}$ to obtain $\delta_k > 0$ and negligible subsets $N_{k,n} \subset D$ such that

$$\operatorname{diam}(u(D \cap B(z_n, \delta_k) \setminus N_{k,n})) < \frac{1}{k}$$

for all $n \in \mathbb{N}$. Then the set $N := \bigcup_{k,n \in \mathbb{N}} N_{k,n}$ is negligible. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. If $z, z' \in D \setminus N$ satisfy $|z - z'| < \delta_k$, then there exists n such that $z, z' \in B(z_n, \delta_k) \setminus N_{k,n}$ and hence

$$d(u(z), u(z')) \leq \operatorname{diam}(u(D \cap B(z_n, \delta_k) \setminus N_{k,n})) < \frac{1}{k} < \varepsilon.$$

This shows that $u|_{D \setminus N}$ is uniformly continuous and hence has a unique continuous extension $\overline{u}$ to $\overline{D}$. □

We need the following lemma in the proof of Proposition B.4.1.

Lemma B.4.2. *Let X be a locally geodesic metric space homeomorphic to $\overline{D}$ and let $\varepsilon > 0$. Then there is $\varepsilon' > 0$ such that for every $x \in X$, there exists a bi-Lipschitz curve $T \subset X$ with the following property. Either T is the boundary of a Jordan domain Ω containing $\overline{B}(x, \varepsilon')$ and with*

$\operatorname{diam}\Omega \leq \varepsilon$; *or T is a Jordan arc intersecting ∂X exactly at its endpoints, and a component Ω of $X \setminus T$ contains $\overline{B}(x, \varepsilon')$ and satisfies* $\operatorname{diam}\Omega \leq \varepsilon$.

Proof. Let $0 < \varepsilon < \operatorname{diam} X$ and let $\varrho \colon \overline{D} \to X$ be a homeomorphism. Choose $\varepsilon', \delta > 0$ such that

$$B(\varrho(z), 2\varepsilon') \subset \varrho(\overline{D} \cap B(z, \delta)) \subset B(\varrho(z), \varepsilon/3)$$

for every $z \in \overline{D}$. Let $x \in X$, set $z = \varrho^{-1}(x)$ and $S = \{w \in \overline{D} \mid |z - w| = \delta\}$. We can approximate the curve $\varrho(S)$ by a bi-Lipschitz curve $T \subset X$ with the desired properties as follows.

We distinguish two cases and first assume that S does not intersect the boundary of $\overline{D}$ and hence is a circle. Let $\alpha \colon S^1 \to S$ be a constant speed parametrization and let $r > 0$ be sufficiently small, to be determined later. Let $\{t_0, t_1, \ldots, t_n\}$ be a fine partition of S^1 and let $\gamma \colon S^1 \to X$ be a curve such that $\gamma(t_k) = \varrho(\alpha(t_k))$ and the restriction of γ to the short segment $I_k = \overline{t_k t_{k+1}} \subset S^1$ is a geodesic for every k. If the partition is chosen sufficiently fine, we have $\operatorname{diam}(\varrho^{-1}(\gamma(I_k))) < \frac{r}{8}$ for all k. In particular, if $s, t \in S^1$ are such that $\gamma(s) = \gamma(t)$, then the shorter segment $\overline{st} \subset S^1$ is such that $\varrho^{-1}(\gamma(\overline{st}))$ is contained in a ball of radius $r/2$ centered on S and thus homotopic relative to endpoints to the constant curve inside this ball. It is then not difficult to see that, after deleting a finite number of subcurves from γ, we obtain a piecewise geodesic Jordan curve $T \subset X$ such that $\varrho^{-1}(T)$ is homotopic to S in the r-neighbourhood of S. Moreover, by applying the claim in the proof of [LW20, Lemma 4.2], we may further modify T to be a bi-Lipschitz Jordan curve. The Jordan domain Ω' enclosed by $\varrho^{-1}(T)$ satisfies $B(z, \delta - r) \subset \Omega'$. Thus, if $r > 0$ was chosen small enough, then the Jordan domain $\Omega = \varrho(\Omega')$ enclosed by T satisfies $\overline{B}(x, \varepsilon') \subset \Omega \subset B(x, \varepsilon/2)$, as desired.

The case that S intersects the boundary of $\overline{D}$ is analogous and is left to the reader. □

We now prove Proposition B.4.1. For this, let $u \in \Lambda(\Gamma, X)$ be an energy minimizer, and denote by $\alpha \colon S^1 \to X$ the continuous representative of the trace of u. Let $\varepsilon > 0$ and let $\varepsilon' \in (0, \varepsilon)$ be as in Lemma B.4.2. Choose $\delta \in (0, 1/4)$ so small that $2\pi^2 E_+^2(u) < (\varepsilon')^2 |\log \delta|$ and such that α maps segments of diameter $2\sqrt{\delta}$ to sets of diameter at most ε'.

Fix $z \in D$ and for $r > 0$ let γ_r be a constant speed parametrization of the curve $\{p \in \overline{D} \mid |p - z| = r\}$. By the Courant–Lebesgue lemma (see e.g. [LW17a, Lemma 7.3]), there exists a set $A \subset (\delta, \sqrt{\delta})$ of strictly positive measure such that $u \circ \gamma_r$ has an absolutely continuous representative, denoted again by $u \circ \gamma_r$, of length

$$\ell(u \circ \gamma_r) \leq \pi \left(\frac{2E_+^2(u)}{|\log \delta|} \right)^{\frac{1}{2}} < \varepsilon'$$

for every $r \in A$. For almost every $r \in A$ for which γ_r intersects S^1, the endpoints of the absolutely continuous curve $u \circ \gamma_r$ coincide with $\alpha(a_r)$ and $\alpha(b_r)$, where a_r, b_r are the endpoints of γ_r. We furthermore have

$$\operatorname{tr}(u|_{D \cap B(z,r)}) \circ \gamma_r = u \circ \gamma_r$$

for almost every $r \in A$. Fix $r \in A$ such that all of the above hold and set $W := D \cap B(z, r)$. Since α maps segments of diameter $2\sqrt{\delta}$ to sets of diameter at most ε', it follows that the image of (the continuous representative of) the trace of $u|_W$ is contained in $\overline{B}(x, \varepsilon')$ for some $x \in X$. By Lemma B.4.2, there exist a bi-Lipschitz curve $T \subset X$ and a set Ω with $\operatorname{diam}(\Omega) \leq \varepsilon$ and $\overline{B}(x, \varepsilon') \subset \Omega$ such that Ω is either a Jordan domain and $T = \partial\Omega$ or T is a Jordan arc intersecting ∂X exactly at its endpoints and Ω is a component of $X \setminus T$. We now claim the following assertion.

Lemma B.4.3. *The set $N = \{w \in W \mid u(w) \in X \setminus \overline{\Omega}\}$ is negligible.*

With this lemma at hand, we can easily finish the proof of the proposition. Indeed, we have

$u(W \setminus N) \subset \overline{\Omega}$ and therefore

$$\operatorname{diam}(u(W \setminus N)) \leq \operatorname{diam}(\overline{\Omega}) \leq \varepsilon.$$

Hence, since $|N| = 0$, we obtain

$$\operatorname{osc}(u, z, \delta) \leq \operatorname{osc}(u, z, r) \leq \varepsilon.$$

Since $\delta > 0$ was independent of z, this proves the proposition.

We are left to prove the lemma above.

Proof of Lemma B.4.3. Since u is an energy minimizer, it follows that u is infinitesimally quasiconformal and, by [LW17b, Theorem 1.1], minimizes the inscribed Riemannian area

$$\operatorname{Area}_{\mu^i}(u) := \int_D \operatorname{Jac}_{\mu^i}(\operatorname{ap}\operatorname{md} u_z) dz$$

among all maps in $\Lambda(\Gamma, X)$. Here, the μ^i-Jacobian $\operatorname{Jac}_{\mu^i}(s)$ of a semi-norm s on $\mathbb{R}^2$ is given by $\operatorname{Jac}_{\mu^i}(s) = 0$ if s is degenerate and $\operatorname{Jac}_{\mu^i}(s) = \frac{\pi}{|L|}$ if s is a norm, where $|L|$ denotes the Lebesgue measure of John's ellipse of $\{v \in \mathbb{R}^2 \mid s(v) \leq 1\}$.

In order to prove that N is negligible, we suppose to the contrary that $|N| > 0$. We then claim that $\operatorname{Area}_{\mu^i}(u|_N) > 0$. In order to prove the claim, we argue as in the proof of [GHP19, Proposition 7] and decompose W into horizontal curves β_t. For almost every t, the composition $u \circ \beta_t$ has an absolutely continuous representative c_t with speed

$$|c_t'(s)| = \operatorname{ap}\operatorname{md} u_{\beta_t(s)}(\beta_t'(s))$$

for almost every s, see [LW17a, Lemma 4.9]. If $\mathcal{H}^1(\beta_t \cap N) > 0$, then the set $c_t^{-1}(X \setminus \overline{\Omega})$ is nonempty and open, and thus an at most countable disjoint union of open intervals. Almost every point in such an interval I is contained in N and, as $c_t|_I$ has endpoints in $\overline{\Omega}$, it follows that $\ell(c_t|_I) > 0$. We conclude

$$\int_I \operatorname{ap}\operatorname{md} u_{\beta_t(s)}(\beta_t'(s))\, ds = \int_I |c_t'(s)|\, ds = \ell(c_t|_I) > 0$$

and therefore $\operatorname{ap}\operatorname{md} u_{\beta_t(s)}(\beta_t'(s))$ can not vanish for almost every $s \in I$. By Fubini theorem, we thus obtain that N contains a set A of strictly positive measure such that $\operatorname{ap}\operatorname{md} u_w \neq 0$ for every $w \in A$. Since u is infinitesimally quasiconformal, it follows that $\operatorname{Jac}_{\mu^i}(\operatorname{ap}\operatorname{md} u_w) > 0$ for almost every $w \in A$ and hence $\operatorname{Area}_{\mu^i}(u|_N) > 0$. This proves the claim.

Next, let T and Ω be as above and notice that there exists a continuous retraction $\varrho_0\colon X \to \overline{\Omega}$ such that $\varrho_0(X \setminus \Omega) \subset T$. Since T is a bi-Lipschitz curve, it is locally Lipschitz n-connected for every n. Moreover, X has Nagata dimension at most 2 by [JL22]. Hence, by the Lipschitz extension results in [Hoh93] and [LS05], we can approximate ϱ_0 arbitrarily closely by a Lipschitz retraction $\varrho\colon X \to \overline{\Omega}$ satisfying $\varrho(X \setminus \Omega) \subset T$. (Alternatively, such a Lipschitz retraction can be constructed using McShane's extension theorem, see the proof of [C, Lemma 4.2].) Let $v\colon D \to X$ be the map which agrees with u on $D \setminus W$ and with $\varrho \circ u$ on W. Since the trace of $u|_W$ has image in Ω it follows from the gluing theorem [KS93, Theorem 1.12.3] that $v \in N^{1,2}(D, X)$ and $\operatorname{tr}(v) = \operatorname{tr}(u)$; in particular, we have $v \in \Lambda(\Gamma, X)$. Notice that v agrees with u on $D \setminus N$ and that $\operatorname{Area}_{\mu^i}(v|_N) = 0$ since $v(N) \subset T$. It thus follows that

$$\operatorname{Area}_{\mu^i}(u) = \operatorname{Area}_{\mu^i}(u|_{D \setminus N}) + \operatorname{Area}_{\mu^i}(u|_N) > \operatorname{Area}_{\mu^i}(u|_{D \setminus N}) = \operatorname{Area}_{\mu^i}(v),$$

which contradicts the area minimization property of u. This completes the proof of the lemma. $\square$

B.5. Non-trivial Sobolev maps in locally geodesic metric surfaces

In this section, we establish Theorem B.1.4. In its proof, we will use the fact that every compact metric space isometrically embeds into an injective metric space which is again compact. Recall that a metric space E is injective if for every metric space Z, any 1-Lipschitz map $Y \to E$, defined on a subset $Y \subset Z$, extends to a 1-Lipschitz map $Z \to E$. By [Isb64], for every metric space X there exists an injective metric space $E(X)$ which contains X and which is minimal in an appropriate sense among injective metric spaces containing X. Such a space $E(X)$ is called injective hull of X and is unique up to isometry. Moreover, if X is compact, then so is $E(X)$. See [Isb64] for the proof of these properties.

The following proposition is the main ingredient in the proof of Theorem B.1.4.

Proposition B.5.1. *Suppose that X is a geodesic metric space homeomorphic to $\overline{D}$. If $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$, then there exists $M > 0$ with the following property. For every $\varepsilon > 0$, there is a Lipschitz map $v\colon \overline{D} \to E(X)$ with $\operatorname{Area}(v) \leq M$ and such that $v|_{S^1}$ parametrizes ∂X and the image of v is contained in the ε-neighbourhood of X in $E(X)$.*

It is not difficult to prove Theorem B.1.4 using this proposition.

Proof of Theorem B.1.4. We first assume that X is geodesic, homeomorphic to $\overline{D}$, and $\Gamma = \partial X$. By Proposition B.5.1, there exist $M > 0$ and a sequence (v_n) of Lipschitz maps $v_n\colon \overline{D} \to E(X)$ with $\operatorname{Area}(v_n) \leq M$ and such that $v_n|_{S^1}$ parametrizes ∂X and the image of v_n is contained in the $1/n$-neighbourhood of X for each $n \in \mathbb{N}$.

By Morrey's ε-conformality Lemma [FW20], there exist diffeomorphisms ϱ_n of $\overline{D}$ such that $u_n := v_n \circ \varrho_n$ satisfies

$$E_+^2(u_n) \leq \frac{4}{\pi} \cdot \operatorname{Area}(u_n) + 1 \leq \frac{4M}{\pi} + 1$$

for every n. Let $p_1, p_2, p_3 \in S^1$ and $q_1, q_2, q_3 \in \partial X$ be distinct points. After precomposing u_n with a Möbius transformation (this leaves the energy invariant), we may assume that every u_n satisfies the 3-point condition $u_n(p_i) = q_i$ for $i = 1, 2, 3$. Thus, the sequence (α_n) of curves $\alpha_n := u_n|_{S^1}$ is equicontinuous by [LW17a, Proposition 7.4]. Therefore, after passing to a subsequence, we may assume by the Arzelà–Ascoli theorem that (α_n) converges uniformly to a curve α. As the uniform limit of parametrizations of ∂X, the curve α is a weakly monotone parametrization of ∂X. Finally, after passing to a further subsequence, we may assume by the Rellich–Kondrachov theorem [KS93, Theorem 1.13] that (u_n) converges in $L^2(D, E(X))$ to some $u \in N^{1,2}(D, E(X))$. Since the image of u_n is contained in the $1/n$-neighbourhood of X for every n it follows that the essential image of u is contained in X, so we may view u as an element of $N^{1,2}(D, X)$. Since the traces converge in $L^2(S^1, E(X))$ to $\operatorname{tr}(u)$ by [KS93, Theorem 1.12.2], it follows that $\operatorname{tr}(u) = \alpha$ and hence that $u \in \Lambda(\partial X, X)$. This shows that $\Lambda(\partial X, X)$ is not empty, which completes the proof of the special case.

Now, let X and Γ be as in the statement of the theorem. Then Γ encloses a Jordan domain $\Omega \subset X$. Denote by d the metric on X and consider the length metric $d_{\overline{\Omega}}$ on $\overline{\Omega}$. The identity map $\pi\colon (\overline{\Omega}, d_{\overline{\Omega}}) \to (\overline{\Omega}, d)$ is a homeomorphism, is 1-Lipschitz, and preserves the lengths of curves and the Hausdorff 2-measures of Borel subsets, see [LW20, Lemma 2.1]. In particular, the metric space $Y = (\overline{\Omega}, d_{\overline{\Omega}})$ is geodesic, homeomorphic to $\overline{D}$, and $\ell(\partial Y)$ and $\mathcal{H}^2(Y)$ are finite. It thus follows from the first part of the proof that $\Lambda(\partial Y, Y)$ is not empty. Let $v \in \Lambda(\partial Y, Y)$. Then $u := \pi \circ v$ is an element of $N^{1,2}(D, X)$ with image in the compact set $\overline{\Omega}$. Since $\operatorname{tr}(u) = \pi \circ \operatorname{tr}(v)$

and $\operatorname{tr}(v)$ has a continuous representative which is a weakly monotone parametrization of ∂Y, we see that $u \in \Lambda(\Gamma, X)$. This shows that $\Lambda(\Gamma, X)$ is not empty and completes the proof. $\square$

The remainder of this section is dedicated to the proof of Proposition B.5.1. We will need two lemmas.

Lemma B.5.2. *There is a constant $C \geq 1$ with the following property. Let X be a geodesic metric space homeomorphic to $\overline{D}$. Then for every $r > 0$, there exist a finite metric simplicial complex Σ and C-Lipschitz maps $\psi\colon X \to \Sigma$ and $\varphi\colon \Sigma \to E(X)$ subject to:*

(1) *Σ has dimension ≤ 2 and the metric on Σ is geodesic and such that every simplex is a Euclidean simplex of side length r;*

(2) *the image of φ is in the Cr-neighbourhood of X and $d(x, \varphi(\psi(x))) \leq Cr$ for all $x \in X$.*

The lemma is a consequence of [JL22, Theorem 2] and [BWY23, Theorem 1.6]. For the convenience of the reader, we sketch the argument.

Proof. The space X has Nagata dimension at most 2 with some universal constant c by [JL22, Theorem 2]. Thus, for a given $r > 0$, there exists a finite cover $\{B_1, \dots, B_k\}$ of X by sets $B_i \subset X$ of diameter at most cr and such that every subset of X of diameter at most r intersects at most three of the B_i's. Define 1-Lipschitz functions $\tau_i\colon X \to \mathbb{R}$ by $\tau_i(x) = \max\{\frac{r}{2} - d(x, B_i), 0\}$. Then for every x, we have $\overline{\tau}(x) := \tau_1(x) + \cdots + \tau_k(x) \geq \frac{r}{2}$ and $\tau_i(x) > 0$ for at most three indices i. Therefore, the map $\psi(x) = \overline{\tau}(x)^{-1}(\tau_1(x), \dots, \tau_k(x))$ has image in the 2-skeleton of the simplex $\Delta = \{(v_1, \dots, v_k) \in \mathbb{R}^k \mid v_i \geq 0, v_1 + \cdots + v_k = 1\}$. One calculates as in the proof of [LS05, Theorem 5.2] that

$$|\psi(x) - \psi(y)| \leq 24 r^{-1} d(x, y)$$

for all $x, y \in X$. Let Σ be the smallest subcomplex of Δ containing $\psi(X)$ and define a map $\varphi\colon \Sigma \to E(X)$ as follows. For each vertex $e_i \in \Sigma^{(0)}$, let $\varphi(e_i)$ be a point in B_i. If e_i, e_j are adjacent vertices in Σ, then $d(\varphi(e_i), \varphi(e_j)) \leq (2c+1) r |e_i - e_j|$. Using the Lipschitz connectedness of $E(X)$, we can extend $\varphi|_{\Sigma^{(0)}}$ to the 1-simplices and 2-simplices of Σ and obtain a map φ which is Cr-Lipschitz on each simplex and satisfies $d(x, \varphi(\psi(x))) \leq Cr$ for some C only depending on c. Let d_Σ be the length metric on Σ and scaled by the factor $r/\sqrt{2}$. Then (Σ, d_Σ), ψ, and φ satisfy the properties of the lemma; see [BWY23, Section 3] for details. $\square$

Given sets Y, Z, and a map $f\colon Z \to Y$, we set for each $y \in Y$

$$N(f, y) := \#\{z \in Z \mid f(z) = y\},$$

the multiplicity of f at y. If Z is an open subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^n$ and f is continuous, then the multiplicity function $N(f, \cdot)$ is (Lebesgue) measurable, see [RR55]. We need the following lemma. See [Fed55, Lemma 7.3], [Fed48], and [WY25, Lemma A.1] for related results.

Lemma B.5.3. *Let $r > 0$ and let Σ be a finite simplicial complex of dimension at most 2, equipped with a metric such that each simplex is a Euclidean simplex of side length r. Let furthermore $\varrho\colon \overline{D} \to \Sigma$ be a continuous map such that $\varrho(S^1)$ is contained in the 1-skeleton $\Sigma^{(1)}$ of Σ and*

$$L := 4 \cdot 3^{-\frac{1}{2}} r^{-2} \int_\Sigma N(\varrho, y)\, d\mathcal{H}^2(y) < \infty.$$

Then there exist disjoint compact balls $B_1, \dots, B_m \subset D$ for some $0 \leq m \leq L$, and a continuous map $\overline{\varrho}\colon \overline{D} \to \Sigma$ which agrees with ϱ on S^1 and has the following property. For each $i = 1, \dots, m$, there exists a 2-simplex $\sigma_i \subset \Sigma$ such that $\overline{\varrho}$ maps B_i bi-Lipschitz homeomorphically onto σ_i, and

$\overline{\varrho}(\overline{D} \setminus \bigcup_{i=1}^{m} \operatorname{int}(B_i)) \subset \Sigma^{(1)}$. *Moreover, if* $\varrho|_{S^1}$ *is Lipschitz then* $\overline{\varrho}$ *can be taken to be Lipschitz on* $\overline{D}$.

Let $U \subset \mathbb{R}^2$ be open and $f\colon U \to \mathbb{R}^2$ continuous. Let $A \subset \mathbb{R}^2$ be the subset of points $y \in \mathbb{R}^2$ such that $N(f, y) < \infty$. For $y \in A$ and $x \in f^{-1}(y)$, we denote by $\iota(f, x)$ the winding number of the curve $f \circ \gamma$ with respect to y, where $\gamma\colon S^1 \to \mathbb{R}^2$ is given by $\gamma(z) = x + rz$ and $r > 0$ is chosen so small that $\overline{B}(x, r) \subset U$ and $\overline{B}(x, r) \cap f^{-1}(y) = \{x\}$. Clearly, the winding number of $f \circ \gamma$ with respect to y is independent of the choice of such r. It follows from [Rad38, Lemma 5.2] that there exists an at most countable set $N \subset A$ such that $|\iota(f, x)| \leq 1$ for each $y \in A \setminus N$ and every $x \in f^{-1}(y)$.

Proof of Lemma B.5.3. Denote by $\sigma_1, \dots, \sigma_n$ the finitely many 2-simplices of Σ and notice that $\mathcal{H}^2(\sigma_i) = |\sigma_i| = \frac{\sqrt{3}}{4} r^2$. By the remark after the statement of the lemma, for each $i = 1, \dots, n$ there exists $y_i \in \operatorname{int}(\sigma_i)$ such that

$$|\sigma_i| \cdot N(\varrho, y_i) \leq \int_{\sigma_i} N(\varrho, y) \, d\mathcal{H}^2(y)$$

and $|\iota(\varrho, x)| \leq 1$ for every $x \in \varrho^{-1}(y_i)$. If $\varrho^{-1}(y_i)$ is not empty, then we will write $\varrho^{-1}(y_i) = \{x_{i,1}, \dots, x_{i,m_i}\}$, where $m_i = N(\varrho, y_i)$, and choose $r_i > 0$ so small that the balls $\overline{B}(x_{i,j}, 2r_i)$ are contained in $\varrho^{-1}(\operatorname{int}(\sigma_i))$ and are pairwise disjoint. Let $\pi\colon \Sigma \setminus \{y_1, \dots, y_n\} \to \Sigma^{(1)}$ be the continuous map which is the identity on $\Sigma^{(1)}$ and such that $\pi|_{\sigma_i \setminus \{y_i\}}$ is the radial projection onto $\partial\sigma_i$ with projection centre y_i.

We let $\overline{\varrho}\colon \overline{D} \to \Sigma$ be the continuous map which agrees with $\pi \circ \varrho$ on the complement of the balls $B(x_{i,j}, 2r_i)$ and such that $\overline{\varrho}|_{\overline{B}(x_{i,j}, 2r_i)}$ is defined as follows. If $\iota(\varrho, x_{i,j}) = 0$, then $\overline{\varrho}|_{\partial B(x_{i,j}, 2r_i)}$ is contractible in $\partial\sigma_i$ and it therefore has a continuous extension to $\overline{B}(x_{i,j}, 2r_i)$ with image inside $\partial\sigma_i$. In this case, we may in particular assume that $\overline{\varrho}|_{\overline{B}(x_{i,j}, r_i)}$ is constant. If $\iota(\varrho, x_{i,j}) = \pm 1$, then $\overline{\varrho}|_{\partial B(x_{i,j}, 2r_i)}$ is homotopic inside $\partial\sigma_i$ to a bi-Lipschitz parametrization of $\partial\sigma_i$. We define $\overline{\varrho}|_{\overline{B}(x_{i,j}, 2r_i) \setminus B(x_{i,j}, r_i)}$ to be such a homotopy, and we let $\overline{\varrho}|_{\overline{B}(x_{i,j}, r_i)}$ be a bi-Lipschitz homeomorphism onto σ_i which extends $\overline{\varrho}|_{\partial B(x_{i,j}, r_i)}$. It is clear that $\overline{\varrho}$ has all the properties listed in the statement of the proposition, except the last one.

In order to prove the last statement, suppose $\varrho|_{S^1}$ is Lipschitz continuous. Set $\Omega := D \setminus \bigcup_{i,j} B(x_{i,j}, r_i)$ and notice that $\overline{\varrho}(\overline{\Omega}) \subset \Sigma^{(1)}$ and the restriction of $\overline{\varrho}$ to $\partial\Omega$ is Lipschitz. Since $\Sigma^{(1)}$ is locally Lipschitz k-connected for every $k \geq 0$, it follows from the Lipschitz extension results [LS05] and [Hoh93] that we can approximate $\overline{\varrho}|_{\overline{\Omega}}$ arbitrarily closely by a Lipschitz map with image in $\Sigma^{(1)}$ which agrees with $\overline{\varrho}$ on $\partial\Omega$. This concludes the proof. ☐

Proof of Proposition B.5.1. Let $r > 0$ be sufficiently small and let Σ, ψ, φ and the constant C be as in Lemma B.5.2. Let $c\colon S^1 \to X$ be a constant speed parametrization of ∂X and set $L := \ell(\partial X)$. We first claim that there exist a constant C_1 only depending on C and a continuous map $\varrho\colon \overline{D} \to \Sigma$ such that

$$\int_{\Sigma} N(\varrho, y) \, d\mathcal{H}^2(y) \leq C_1(\mathcal{H}^2(X) + Lr)$$

and $\varrho|_{S^1}$ is $C_1 L$-Lipschitz with image in $\Sigma^{(1)}$ and $d(\varrho(t), \psi(c(t))) \leq C_1 r$ for all $t \in S^1$. Indeed, the CL-Lipschitz curve $\psi \circ c$ is homotopic to a $C'L$-Lipschitz curve $\gamma\colon S^1 \to \Sigma$ satisfying $\gamma(S^1) \subset \Sigma^{(1)}$ and

$$d(\psi(c(t)), \gamma(t)) \leq C'r$$

for all $t \in S^1$ via a Lipschitz homotopy h of area $\operatorname{Area}(h) \leq C'Lr$, where C' only depends on C. Such γ can be obtained by replacing $\psi \circ c$ on the closure of each component of $(\psi \circ c)^{-1}(\operatorname{int}(\sigma))$

by the constant speed shortest curve in $\partial\sigma$ for every 2-simplex σ. The homotopy h is the straight line homotopy in σ. In particular, we obtain from the area formula that

$$\int_\Sigma N(h,y)\, d\mathcal{H}^2(y) = \operatorname{Area}(h) \leq C' Lr.$$

By the Jordan–Schoenflies theorem, there is a homeomorphism $\eta\colon \overline{D} \to X$ which extends c. The coarea inequality for Lipschitz maps [Fed69, Theorem 2.10.25] implies

$$\int_\Sigma N(\psi\circ\eta,y)\, d\mathcal{H}^2(y) = \int_\Sigma N(\psi,y)\, d\mathcal{H}^2(y) \leq C^2\mathcal{H}^2(X).$$

The map ϱ given by $\varrho(z) = \psi(\eta(2z))$ when $|z| \leq \frac{1}{2}$ and by $h(z/|z|, 2|z|-1)$ when $\frac{1}{2} \leq |z| \leq 1$ satisfies the claim above for some C_1 only depending on C.

Next, let $\overline{\varrho}\colon \overline{D} \to \Sigma$ be a Lipschitz map as in Lemma B.5.3 associated with the map ϱ. We then have

$$\operatorname{Area}(\overline{\varrho}) \leq \int_\Sigma N(\varrho,y)\, d\mathcal{H}^2(y) \leq C_1(\mathcal{H}^2(X) + Lr).$$

Moreover, since $\alpha := \varphi\circ\overline{\varrho}|_{S^1}$ is (CC_1L)-Lipschitz and satisfies

$$d(\alpha(t), c(t)) \leq d(\alpha(t), \varphi(\psi(c(t)))) + d(\varphi(\psi(c(t))), c(t)) \leq C(C_1+1)r$$

for all $t \in S^1$, the Lipschitz extension property of $E(X)$ implies that there exists a Lipschitz homotopy g from α to c in $E(X)$ of area bounded by $C''Lr$ and with image in the $(C''r)$-neighbourhood of c for some C'' only depending on C. The Lipschitz map $v\colon \overline{D} \to E(X)$ given by $v(z) = \varphi(\overline{\varrho}(2z))$ when $|z| \leq \frac{1}{2}$ and by $v(z) = g(z/|z|, 2|z|-1)$ when $\frac{1}{2} \leq |z| \leq 1$ agrees with c on S^1; its image is contained in the C_2r-neighbourhood of X, and

$$\operatorname{Area}(v) \leq \operatorname{Area}(\varphi\circ\overline{\varrho}) + \operatorname{Area}(g) \leq C^2C_1(\mathcal{H}^2(X) + Lr) + C''Lr \leq C_2(\mathcal{H}^2(X) + Lr)$$

for some constant C_2 only depending on C. The proposition now follows. □

Proof of Corollary B.1.5. Let $\Omega \subset X$ be a Jordan domain with finite boundary length. Such Ω can be constructed as in the proof of Lemma B.4.2. By Theorem B.1.4 and its proof, there exists $u \in \Lambda(\partial\Omega, \overline{\Omega})$. We claim that $\operatorname{Area}(u) > 0$. Indeed, otherwise the infimum of energies over all maps in $\Lambda(\partial\Omega, \overline{\Omega})$ would be zero by Morrey's ε-conformality lemma [FW20]. Hence an energy minimizer, which exists by Theorem B.2.3, would have zero energy and would thus be constant, a contradiction. This proves the claim.

Let $A_1 \subset A_2 \subset \cdots \subset D$ be measurable sets with $|D \setminus \bigcup A_i| = 0$ and such that the restriction $u|_{A_i}$ is Lipschitz for every i, see e.g. [LW17a, Proposition 3.2]. Since $\operatorname{Area}(u) > 0$ there exists i such that $\operatorname{Area}(u|_{A_i}) > 0$. By the area formula, we have

$$\operatorname{Area}(u|_{A_i}) = \int_{u(A_i)} N(u|_{A_i}, x)\, d\mathcal{H}^2(x)$$

and hence $\mathcal{H}^2(u(A_i)) > 0$. This completes the proof. □

B.6. Finishing the proofs of the almost parametrization results

In this section, we finish the proofs of the almost parametrization results given in the introduction and discuss some additional consequences.

Proof of Theorem B.1.1. Denote by d the metric on X and consider the length metric $d_{\overline{\Omega}}$ on $\overline{\Omega}$. It follows as in the second part of the proof of Theorem B.1.4 that the metric space $Y = (\overline{\Omega}, d_{\overline{\Omega}})$ is geodesic, homeomorphic to $\overline{D}$, and $\ell(\partial Y)$ and $\mathcal{H}^2(Y)$ are finite. Hence, $\Lambda(\partial Y, Y)$ is not empty by Theorem B.1.4. Therefore, by Theorem B.2.3, there exists an energy minimizer v in $\Lambda(\partial Y, Y)$, and every such v is infinitesimally $\frac{4}{\pi}$-quasiconformal. Theorem B.1.3 further implies that v has a continuous representative which continuously extends to the boundary, denoted by v again. Finally, Theorem B.2.4 implies that v is monotone.

Since the identity $\pi\colon Y \to (\overline{\Omega}, d)$ is a homeomorphism, the map $u\colon \overline{D} \to \overline{\Omega} \subset X$ defined by $u := \pi \circ v$ is continuous, surjective, and monotone. Since π is 1-Lipschitz and its restriction to $Y \setminus \partial Y$ is a local isometry, it follows furthermore that u belongs to $N^{1,2}(D, X)$ and that u is infinitesimally $\frac{4}{\pi}$-quasiconformal. Thus, u satisfies (B.1) by the discussion at the beginning of Section B.3. This completes the proof. □

Remark B.6.1. *The condition in Theorem B.1.1 that the metric space X be locally geodesic can be relaxed. It suffices to assume that $X = (X, d)$ is rectifiably connected, the length metric d_i induces the same topology, and $X_i = (X, d_i)$ has locally finite Hausdorff 2-measure. To see that this suffices, we first observe that the family Λ of maps $u \in N^{1,2}(D, X)$ such that u is the uniform limit of homeomorphisms $\overline{D} \to \overline{\Omega}$ is not empty. Indeed, by Theorem B.1.1, there exists a quasiconformal almost parametrization v of $(\overline{\Omega}, d_i)$. Let $\pi\colon (\overline{\Omega}, d_i) \to (\overline{\Omega}, d)$ be the identity map, and notice that the map $u := \pi \circ v$ is the uniform limit of homeomorphisms $\overline{D} \to \overline{\Omega}$. Since π is 1-Lipschitz, it follows that u belongs to $N^{1,2}(D, \overline{\Omega})$, so Λ is not empty. Next, one shows that Λ contains an energy minimizer. For this, let $(u_n) \subset \Lambda$ be an energy minimizing sequence. After precomposing with Möbius transformations, we may assume that the u_n satisfy a 3-point condition and so, by* [LW17a, Proposition 7.4]*, the sequence $(u_n|_{S^1})$ is equicontinuous. Hence, the proof of Proposition B.4.1 shows that the sequence (u_n) is equicontinuous. Thus, after passing to a subsequence, we may assume that (u_n) converges uniformly to a map u. This map belongs to Λ and is an energy minimizer in Λ and thus is infinitesimally $\frac{4}{\pi}$-quasiconformal by* [LW17b]*. In particular, u satisfies* (B.1)*, see Section B.3.*

The following variant of Theorem B.1.1 is closer to the statement of the Riemann mapping theorem.

Corollary B.6.2. *Let X be a locally geodesic metric space homeomorphic to $\mathbb{R}^2$ and of locally finite Hausdorff 2-measure. If $U \subset X$ is an open and simply connected set with compact closure then there exists a continuous, monotone surjection $u\colon D \to U$ such that*

$$\operatorname{mod}(\Gamma) \leq K \cdot \operatorname{mod}(u \circ \Gamma) \tag{B.11}$$

for every family Γ of curves in D, where $K = \frac{4}{\pi}$.

Proof. Let $\Omega \subset X$ be a Jordan domain containing $\overline{U}$. We can approximate $\partial\Omega$ by a biLipschitz Jordan curve as in the proof of Lemma B.4.2 and may therefore assume that Ω is of finite boundary length. By Theorem B.1.1, there exists a continuous, surjective, monotone map $v\colon \overline{D} \to \overline{\Omega}$ such that $\operatorname{mod}(\Gamma) \leq \frac{4}{\pi} \cdot \operatorname{mod}(v \circ \Gamma)$ for every family Γ of curves in $\overline{D}$. Since v is monotone, it follows that $V := v^{-1}(U) \subset D$ is also simply connected, see [LW20, Section 2.3]. By the Riemann mapping theorem, there exists a conformal diffeomorphism $\varphi\colon D \to V$. Then the map $u\colon D \to U$ given by $u := v \circ \varphi$ has the desired properties. □

Proof of Corollary B.1.2. Let $U \subset X$ be a Jordan domain containing $\overline{\Omega}$. Arguing as in the proof of Corollary B.6.2, we may assume that U has finite boundary length. By Theorem B.1.1, there exists a continuous, surjective, monotone map $v\colon \overline{D} \to \overline{U}$ such that $\operatorname{mod}(\Gamma) \leq \frac{4}{\pi} \cdot \operatorname{mod}(v \circ$

Γ) for every family Γ of curves in $\overline{D}$. It follows from Propositions B.3.1 and B.3.3 that v is geometrically quasiconformal homeomorphism. Finally, by the Riemann mapping theorem, there exists a conformal diffeomorphism $D \to \Omega'$, where $\Omega' = v^{-1}(\Omega)$, which moreover extends to a homeomorphism $\varrho\colon \overline{D} \to \overline{\Omega'}$. The composition $u := v \circ \varrho\colon \overline{D} \to \overline{\Omega}$ is a geometrically quasiconformal homeomorphism. □

Another consequence of Theorem B.1.1 is the following variant for discs of the Bonk–Kleiner quasisymmetric uniformization theorem [BK02], see also [LW20].

Corollary B.6.3. *Let X be a geodesic metric space homeomorphic to $\overline{D}$ and with finite boundary length. If there exists $L > 0$ such that*

$$\mathcal{H}^2(B(x,r)) \leq Lr^2$$

for all $x \in X$ and $r > 0$ and if X is linearly locally connected, then there exists a quasisymmetric homeomorphism $u\colon \overline{D} \to X$.

Using a quasisymmetric gluing theorem exactly as in the proof of [LW20, Proposition 6.4], one obtains an analogous statement when X is homeomorphic to S^2 and thus the Bonk–Kleiner quasisymmetric uniformization theorem [BK02].

Proof. By Theorem B.1.1, there exists a continuous, monotone surjection $u\colon \overline{D} \to X$ satisfying (B.1). Since the quadratic upper bound for the Hausdorff measure of balls implies (B.2) by [Hei01, Lemma 7.18], it follows from Proposition B.3.1 that u is a homeomorphism. Finally, u is quasisymmetric by Proposition B.3.4. □

C. Quasiconformal uniformization of metric surfaces of higher topology

Abstract. We establish the following uniformization result for metric spaces X of finite Hausdorff 2-measure. If X is homeomorphic to a smooth 2-manifold M with non-empty boundary, then we show that X admits a quasiconformal almost parametrization $M \to X$, by only assuming that X is locally geodesic and has rectifiable boundary. In particular, we recover a corollary of Ntalampekos and Romney by using the solution of the Plateau problem. After putting additional assumptions on X, we show that the quasiconformal almost parametrization upgrades to a quasisymmetry or a geometrically quasiconformal map, implying statements analogous to the uniformization theorems of Bonk and Kleiner as well as Rajala for surfaces of higher topology.

C.1. Introduction

C.1.1. Background and statement of main result

The classical uniformization theorem states that any Riemann surface is conformally equivalent to a surface of constant curvature. One of the main questions in the field of analysis on metric spaces asks under which conditions on a metric space X homeomorphic to some model space M there exists a homeomorphism $u\colon M \to X$ with good geometric and analytic properties.

The first breakthrough result regarding uniformization of metric surfaces is due to Bonk and Kleiner [BK02] and asserts that if a metric space X is Ahlfors 2-regular and homeomorphic to S^2, then X is quasisymmetrically equivalent to S^2 if and only if X is linearly locally connected. For the definitions we refer to Section C.6. In this work, a *smooth surface* refers to a smooth compact oriented and connected Riemannian 2-dimensional manifold with possibly non-empty boundary and a *metric surface* is a metric space homeomorphic to a smooth surface.

Let X be a metric space of locally finite Hausdorff 2-measure. The *modulus of a curve family* Γ in X is defined by

$$\operatorname{mod}(\Gamma) := \inf_{\rho} \int_X \rho^2 \, d\mathcal{H}^2,$$

where the infimum is taken over all Borel functions $\rho\colon X \to [0,\infty]$ for which $\int_\gamma \rho \geq 1$ holds for every $\gamma \in \Gamma$, see Section C.2.2. A homeomorphism $u\colon M \to X$ is *geometrically quasiconformal* if it leaves the modulus of curve families quasi-invariant. Rajala [Raj17] showed that every metric space X homeomorphic to $\mathbb{R}^2$ of locally finite Hausdorff 2-measure admits a geometrically quasiconformal map u from a domain $\Omega \subset \mathbb{R}^2$ to X if and only if X satisfies a condition called reciprocality, see Section C.6.

In a next step, Ntalampekos and Romney [NR23] as well as Wenger and the author [B], showed independently and with two different approaches the following result. If X is locally geodesic and one relaxes the assumptions on u, then the condition of reciprocality can be dropped completely. The assumptions on u are such that u is a continuous, monotone surjection satisfying

$$\operatorname{mod}(\Gamma) \leq K \cdot \operatorname{mod}(u \circ \Gamma) \tag{C.1}$$

for $K \geq 1$ and every family Γ of curves in M, where we denote by $u \circ \Gamma$ the family of curves $u \circ \gamma$ with $\gamma \in \Gamma$. Here, a map $u\colon M \to X$ is *monotone* if the preimage of a point is connected and equivalently, if u is a uniform limit of homeomorphisms $M \to X$ (see [You48] and Proposition C.5.2 below). Whenever $u\colon M \to X$ is in addition a homeomorphism, then by [Wil12], condition (C.1) is equivalent to the so-called analytic definition of quasiconformality. A similar relation holds for u being continuous, monotone and satisfying (C.1), see [NR23, Theorem 7.1].

All results mentioned so far were stated for simply connected surfaces. It is very natural to wonder to what extent these statements can be extended to surfaces of higher topology. Rajala's uniformization theorem has been generalized in [Iko22], while generalizations of the theorem of Bonk-Kleiner can be found in [MW13], [GW18] and [A].

Very recently, Ntalampekos and Romney [NR24] showed that the assumption of being locally geodesic in [NR23] and [B] is not needed. In particular, their result holds for all metric surfaces X of locally finite Hausdorff 2-measure, even for surfaces of higher genus and with non-empty boundary.

In this work we focus on the approach from [B], which is closely related to the existence and regularity of energy and area minimizing discs in metric spaces admitting a quadratic isoperimetric inequality developed by Lytchak and Wenger in [LW17a]. Moreover, we will make use of the theory of area minimizing surfaces in homotopy classes in metric spaces [SW22]. Our main result is the following Corollary of [NR24, Theorem 1.3], using a completely different proof strategy than in [NR24].

Theorem C.1.1. *Let X be a locally geodesic metric space homeomorphic to a smooth surface M with non-empty boundary. If X is of finite Hausdorff 2-measure and has rectifiable boundary, then there exists a Riemannian metric g on M and a continuous, monotone surjection $u\colon M \to X$ such that*

$$\operatorname{mod}(\Gamma) \leq K \cdot \operatorname{mod}(u \circ \Gamma)$$

holds for every family Γ of curves in M with $K = \frac{4}{\pi}$.

The Riemannian metric g can be chosen in such a way that it is of constant sectional curvature -1, 0 or 1 and the boundary of M is geodesic. Note that the constant $\frac{4}{\pi}$ is optimal (see [Iko22, Theorem 1.3]) and that in this generality, there doesn't have to exist a homeomorphism satisfying (C.1), see e.g., [B, Example 3.2]. We remark here that the approach of [NR24] also covers the case of closed surfaces, in particular the sphere.

C.1.2. Proof of main theorem

Let X be a metric space homeomorphic to a smooth surface M. By ∂M we denote the topological boundary of the smooth surface M, which is homeomorphic to the disjoint union of k copies of S^1 for some $k \geq 0$. The boundary of X, denoted ∂X, is the subset of X that is homeomorphic to ∂M. Denote by $[\partial X]$ the set of all weakly monotone parametrizations of ∂X, i.e., uniform limits of homeomorphisms from S to ∂X, where S is any space homeomorphic to the disjoint union of k copies of S^1. Let $\Lambda(M, \partial X, X)$ be the possibly empty family of Sobolev maps $u \in N^{1,2}(M, X)$ such that the trace $\operatorname{tr}(u)$ has a continuous representative in $[\partial X]$. Consider a homeomorphism $\varphi\colon M \to X$ and let $h\colon K \to M$ be a triangulation of M, see Section C.2.5. Denote by K^1

the 1-skeleton of K and by $\partial K \subset K^1$ the preimage of ∂M under h. Two continuous maps $\varrho, \varrho' \colon K^1 \to X$ with $\varrho|_{\partial K}, \varrho'|_{\partial K} \in [\partial X]$ are said to be *homotopic relative to* ∂X if there exists a homotopy H between ϱ and ϱ' with $H(\cdot, t)|_{\partial K} \in [\partial X]$ for all $t \in [0,1]$. The common relative homotopy class is denoted by $[\varrho]_{\partial X}$. Note that if ∂X is empty, then $[\varrho]_{\partial X}$ corresponds to the usual homotopy class of ϱ. If $u \in \Lambda(M, \partial X, X)$ is continuous, we say that *u is 1-homotopic to φ relative to ∂X*, denoted $u \sim_1 \varphi$ rel ∂X, if

$$[u \circ h|_{K^1}]_{\partial X} = [\varphi \circ h|_{K^1}]_{\partial X}$$

for some and thus any triangulation h on M, see [SW22]. The 1-homotopy class $u_{\#,1}[h]$ of an arbitrary $u \in \Lambda(M, \partial X, X)$ will be defined in Section C.2.5 using the existence of a local quadratic isoperimetric inequality in the ε-thickening X_ε of X and a suitable retraction $X_\varepsilon \to X$. In a first step we show that the family

$$\Lambda(M, \varphi, X) := \{u \in \Lambda(M, \partial X, X) : u \sim_1 \varphi \text{ rel } \partial X\}$$

is not empty. Notice that in [SW22] the existence of a map in $\Lambda(M, \varphi, X)$ highly depends on the fact that X admits a local quadratic isoperimetric inequality. In this article we make use of the 2-dimensional structure of X to prove the following generalization of [B, Theorem 1.4].

Theorem C.1.2. *Let X be a locally geodesic metric space homeomorphic to a smooth surface M that is not a sphere and let $\varphi \colon M \to X$ be a homeomorphism. If $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$ then the family $\Lambda(M, \varphi, X)$ is not empty.*

For convenience, we denote by $\Lambda_{\mathrm{metr}}(M, \varphi, X)$ the family of pairs (u, g), where $u \in \Lambda(M, \varphi, X)$ and g is a Riemannian metric on M. Moreover, a pair $(u, g) \in \Lambda_{\mathrm{metr}}(M, \varphi, X)$ satisfying

$$E_+^2(u, g) = \inf\{E_+^2(v, h) : v \in \Lambda(M, \varphi, X),\ h \text{ is a Riemannian metric on } M\},$$

is called *energy minimizing*. Here, E_+^2 denotes the Reshetnyak energy, see Section C.2.3. Assuming that X is locally geodesic, has rectifiable boundary and is of finite Hausdorff 2-measure, then by Theorem C.1.2, the family $\Lambda(M, \varphi, X)$ is not empty. Theorem C.3.4 below implies the existence of an energy minimizing pair $(u, g) \in \Lambda_{\mathrm{metr}}(M, \varphi, X)$. The regularity of such an energy minimizer follows from the next theorem generalizing [B, Theorem 1.3].

Theorem C.1.3. *Let M be a smooth surface with non-empty boundary, X a locally geodesic metric space homeomorphic to M and $\varphi \colon M \to X$ a homeomorphism. If $(u, g) \in \Lambda_{metr}(M, \varphi, X)$ is an energy minimizing pair, then u has a representative which is continuous and extends continuously to the boundary.*

Hence, we obtain the existence of an energy minimizing pair $(u, g) \in \Lambda_{\mathrm{metr}}(M, \varphi, X)$ with u being continuous. In a next step we show that every such u is a uniform limit of homeomorphisms and therefore monotone, compare to [LW20, Theorem 1.2].

Theorem C.1.4. *Let X be a locally geodesic metric space homeomorphic to a smooth surface M with non-empty boundary and let $\varphi \colon M \to X$ be a homeomorphism. If a continuous map $u \in \Lambda(M, \varphi, X)$ and a Riemannian metric g on M satisfy*

$$E_+^2(u, g) = \inf\{E_+^2(v, h) : v \in \Lambda(M, \varphi, X),\ h \textit{ is a Riemannian metric on } M\},$$

then u is a uniform limit of homeomorphisms from M to X.

By [FW20, Corollary 1.3], the map u is infinitesimally isotropic with respect to g and thus infinitesimally K-quasiconformal with respect to g for $K = \frac{4}{\pi}$ (see [LW17b]). Consider Section C.2.3 for the definitions of infinitesimal isotropy and infinitesimal quasiconformality. By arguing exactly as in the proof of [LW20, Corollary 3.5], we obtain that u satisfies (C.1) with $K = \frac{4}{\pi}$. This establishes Theorem C.1.1.

The paper is structured as follows. In Section C.2 we assemble necessary definitions and results needed later on. Theorem C.1.2 is shown in Section C.3, while Sections C.4 and C.5 are devoted to the proofs of Theorem C.1.3 and Theorem C.1.4, respectively. In Section C.6 we show that the map u from Theorem C.1.1 upgrades to a quasisymmetric or a geometrically quasiconformal map after further assumptions on the underlying spaces, recovering the above mentioned generalizations of Bonk-Kleiner and Rajala.

C.2. Preliminaries

C.2.1. Basic definitions and notations

Let (X, d) be a metric space. We denote the *open ball* in X of radius $r > 0$ centered at a point $x \in X$ by $B(x, r)$. The *open and closed unit discs* in $\mathbb{R}^2$ are given by

$$D := \{z \in \mathbb{R}^2 : |z| < 1\}, \qquad \overline{D} := \{z \in \mathbb{R}^2 : |z| \leq 1\},$$

where $|\cdot|$ is the Euclidean norm. A set $\Omega \subset X$ homeomorphic to the unit disc D is a *Jordan domain* in X if its boundary $\partial\Omega \subset X$ is a *Jordan curve* in X, i.e., a subset of X homeomorphic to S^1. The *length* of a curve c in X is denoted by $\ell(c)$. A curve c is said to be *rectifiable* if $\ell(c) < \infty$ and *locally rectifiable* if each of its compact subcurves is rectifiable. Moreover, a curve $c\colon [a, b] \to X$ is called *geodesic* if $\ell(c) = d(c(a), c(b))$. A metric space (X, d) is *geodesic* if every pair of points in X can be joined by a geodesic in X and it is called *locally geodesic* if every point $x \in X$ has a neighborhood U such that any two points in U can be joined by a geodesic in X.

For $s \geq 0$, we denote the *s-dimensional Hausdorff measure* of a set $A \subset X$ by $\mathcal{H}^s(A)$. The normalizing constant is chosen in such a way that if X is the Euclidean space $\mathbb{R}^n$, the Lebesgue measure agrees with $\mathcal{H}^n$ on open subsets of $\mathbb{R}^n$. If (M, g) is a Riemannian manifold of dimension n then the n-dimensional Hausdorff measure $\mathcal{H}^n_g$ on (M, g) coincides with the Riemannian volume. We emphasize that throughout this paper, the reference measure on metric spaces will always be the 2-dimensional Hausdorff measure.

Let g be a smooth Riemannian metric on a smooth surface M such that the boundary of M is geodesic with respect to g. We call the metric g *hyperbolic* if it is of constant sectional curvature -1, and *flat* if it has vanishing sectional curvature as well as an associated Riemannian 2-volume satisfying $\mathcal{H}^2_g(M) = 1$.

C.2.2. Conformal modulus

Let X be a metric space of locally finite Hausdorff 2-measure and Γ a family of curves in X. A Borel function $\rho\colon X \to [0, \infty]$ is said to be *admissible for* Γ if $\int_\gamma \rho \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$. For the definition of the path integral $\int_\gamma \rho$ see [HKST15]. The *modulus* of the curve family Γ is now defined by

$$\operatorname{mod}(\Gamma) := \inf_\rho \int_X \rho^2 \, d\mathcal{H}^2,$$

where the infimum is taken over all admissible functions for Γ. If Γ contains a constant curve, then $\operatorname{mod}(\Gamma) = \infty$. We say that a property holds *for almost every curve in* Γ if it holds for every curve in Γ_0 for some $\Gamma_0 \subset \Gamma$ with $\operatorname{mod}(\Gamma \setminus \Gamma_0) = 0$.

A homeomorphism $u\colon M \to X$ is *geometrically quasiconformal* if there exists $K \geq 1$ such that

$$K^{-1} \cdot \operatorname{mod}(\Gamma) \leq \operatorname{mod}(u \circ \Gamma) \leq K \cdot \operatorname{mod}(\Gamma)$$

for every family Γ of curves in M.

C.2.3. Metric space valued Sobolev maps

We now give a brief overview over some basic concepts used in the theory of metric space valued Sobolev maps based on upper gradients. Note that several other equivalent definitions of Sobolev spaces exist. For more details consider e.g., [HKST15].

Let (X, d) be a complete metric space and M a smooth surface. Fix a Riemannian metric g on M and consider a domain $U \subset M$. Let $u\colon U \to X$ be a map and $\rho\colon U \to [0, \infty]$ a Borel function. Then, ρ is called *(weak) upper gradient of u with respect to g* if

$$d(u(\gamma(a)), u(\gamma(b))) \leq \int_\gamma \rho(s)\, ds$$

for (almost) every rectifiable curve $\gamma\colon [a, b] \to U$.

Denote by $L^2(U, X)$ the family of measurable essentially separably valued maps $u\colon U \to X$ such that the function $u_x(z) := d(u(z), x)$ is in the space $L^2(U)$ of 2-integrable functions for some and hence any $x \in X$. A sequence $(u_k) \subset L^2(U, X)$ is said to *converge in $L^2(U, X)$* to a map $u \in L^2(U, X)$ if

$$\int_U d^2(u_k(z), u(z))\, d\mathcal{H}^2_g(z) \to 0$$

as k tends to infinity. The *(Newton-)Sobolev space* $N^{1,2}(U, X)$ is the collection of maps $u \in L^2(U, X)$ such that u has a weak upper gradient in $L^2(U)$. Every such u has a minimal weak upper gradient denoted by ρ_u, meaning that $\rho_u \in L^2(U)$ and for every weak upper gradient ρ of u in $L^2(U)$ it holds that $\rho_u \leq \rho$ almost everywhere on U. Moreover, ρ_u is unique up to sets of measure zero (see e.g., [HKST15, Theorem 6.3.20]). We emphasize that the definition of $N^{1,2}(U, X)$ is independant of the chosen metric g on M.

The *Reshetnyak energy* of a map $u \in N^{1,2}(U, X)$ with respect to g is defined by

$$E^2_+(u, g) := \int_U \rho_u(z)^2\, d\mathcal{H}^2_g(z).$$

Note that this definition of energy agrees with the one given in [FW21, Definition 2.2]; in particular, E^2_+ is invariant under precompositions with conformal diffeomorphisms.

Consider a domain $V \subset \mathbb{R}^2$. A map $v\colon V \to X$ is said to be *approximately metrically differentiable at $z \in V$* if there is a necessarily unique seminorm s on $\mathbb{R}^2$ such that

$$\operatorname{ap}\lim_{y \to z} \frac{d(v(y), v(z)) - s(y - z)}{|y - z|} = 0,$$

where $\operatorname{ap}\lim$ denotes the approximate limit (see e.g., [EG92]). If such a seminorm exists, it is called *approximate metric derivative of v at z*, denoted $\operatorname{ap}\operatorname{md} v_z$. Consider an open set $W \subset \mathbb{R}^2$, a point $w \in W$ and a diffeomorphism $\varphi\colon W \to V$. If the map $v\colon V \to X$ is approximately metrically differentiable at $\varphi(w)$ then the composition $v \circ \varphi$ is approximately metrically differentiable at w with

$$\operatorname{ap}\operatorname{md}(v \circ \varphi)_w = \operatorname{ap}\operatorname{md} v_{\varphi(w)} \circ d\varphi_w.$$

Along with [LW17a, Proposition 4.3] this implies that if $u \in N^{1,2}(U, X)$ then for almost every $z \in U$ the composition $u \circ \psi^{-1}$ is approximately metrically differentiable at $\psi(z)$ for an arbitrary chart (U_0, ψ) around z. Define the seminorm $\operatorname{ap}\operatorname{md} u_z$ on T_zM by

$$\operatorname{ap}\operatorname{md} u_z := \operatorname{ap}\operatorname{md}(u \circ \psi^{-1})_{\psi(z)} \circ d\psi_z.$$

Note that this definition is independent of the choice of chart and $\operatorname{ap}\operatorname{md} u_z$ is called *approximate metric derivative of u at z*.

The *Jacobian* $\operatorname{Jac}(s)$ of a seminorm s on $\mathbb{R}^2$ is defined to be the Hausdorff 2-measure on $(\mathbb{R}^2, s)$ of the unit square if s is a norm and zero otherwise. By identifying $(T_zM, g(z))$ with $(\mathbb{R}^2, |\cdot|)$ via a linear isometry, we are able to define the Jacobian of a seminorm s on T_zM.

Definition C.2.1. The *parametrized (Hausdorff) area* of $u \in N^{1,2}(U, X)$ is given by

$$\operatorname{Area}(u) := \int_U \operatorname{Jac}(\operatorname{ap}\operatorname{md} u_z)\, d\mathcal{H}^2_g(z).$$

We emphasize that the parametrized area of $u \in N^{1,2}(U, X)$ is invariant under precompositions with biLipschitz homeomorphisms, and thus independent of the Riemannian metric g. Moreover, if u is a homeomorphism onto its image then the Jacobian $\operatorname{Jac}(\operatorname{ap}\operatorname{md} u_z)$ agrees with the Radon-Nikodym derivative of the measure $u^*\mathcal{H}^2(B) := \mathcal{H}^2(u(B))$ with respect to the Lebesgue measure at almost every point $z \in U$.

Recall that by John's theorem (see e.g., [APT04, Theorem 2.18]), the unit ball B of a 2-dimensional space equipped with a norm s contains a unique ellipse E of maximal area, called *John's ellipse of s*. The μ^i-Jacobian $\operatorname{Jac}_{\mu^i}(s)$ of a semi-norm s on $\mathbb{R}^2$ is given by $\operatorname{Jac}_{\mu^i}(s) = 0$ if s is degenerate and $\operatorname{Jac}_{\mu^i}(s) = \frac{\pi}{|E|}$ if s is a norm, where $|E|$ denotes the Lebesgue measure of John's ellipse of $\{v \in \mathbb{R}^2 : s(v) \leq 1\}$. Again, by identifying $(T_zM, g(z))$ with $(\mathbb{R}^2, |\cdot|)$ via a linear isometry, we are able to define the μ^i-Jacobian of a seminorm s on T_zM.

Definition C.2.2. The *inscribed Riemannian area* of $u \in N^{1,2}(U, X)$ is defined as

$$\operatorname{Area}_{\mu^i}(u) := \int_U \operatorname{Jac}_{\mu^i}(\operatorname{ap}\operatorname{md} u_z)\, d\mathcal{H}^2_g(z).$$

From [LW17b, Section 2.4] it follows that the inscribed Riemannian area and the parametrized Hausdorff area are comparable; more explicitly

$$\frac{\pi}{4} \operatorname{Area}_{\mu^i}(u) \leq \operatorname{Area}(u) \leq \operatorname{Area}_{\mu^i}(u). \tag{C.2}$$

Definition C.2.3. A map $u \in N^{1,2}(U, X)$ is called *infinitesimally isotropic with respect to a Riemannian metric g on M* if for almost every $z \in U$ the approximate metric derivative $\operatorname{ap}\operatorname{md} u_z$ is either zero or it is a norm and the John's ellipse of $\operatorname{ap}\operatorname{md} u_z$ is a round ball with respect to g.

It follows from [LW17b] that $\operatorname{Area}_{\mu^i}(u) \leq E^2_+(u, g)$ for all Riemannian metrics g on M, with equality if and only if u is infinitesimally isotropic. Moreover, by [LW17b], if $u \in N^{1,2}(U, X)$ is infinitesimally isotropic with respect to g, then it is *infinitesimally K-quasiconformal with respect to g* with $K = \frac{4}{\pi}$ in the sense that

$$(\rho_u(z))^2 \leq K \cdot \operatorname{Jac}(\operatorname{ap}\operatorname{md} u_z)$$

for almost every $z \in U$.

Assume that X is a complete metric space and $u\colon M \to X$ continuous and monotone. By arguing as in the proof of [LW20, Proposition 3.5], we obtain that if $u \in N^{1,2}(M, X)$ and u is infinitesimally K-quasiconformal with respect to g then

$$\operatorname{mod}(\Gamma) \leq K \cdot \operatorname{mod}(u \circ \Gamma) \tag{C.3}$$

for every family Γ of curves in M. Conversely, if u is a homeomorphism onto its image and satisfies (C.3) then u belongs to $N^{1,2}(M, X)$ and is infinitesimally K-quasiconformal (see [Wil12, Theorem 1.1]).

We define the *trace* of a Sobolev map $u \in N^{1,2}(U, X)$ in the following way. Assume $U \subset M \backslash \partial M$ is a Lipschitz domain. Then for every point $z \in \partial U$ there exists an open neighborhood $V \subset M$ and a biLipschitz mapping $\psi\colon (0,1) \times [0,1) \to M$ such that $\psi((0,1) \times (0,1)) = U \cap V$ and $\psi((0,1) \times \{0\}) = V \cap \partial U$. As u is Sobolev, for almost every $s \in (0,1)$ the map $t \mapsto u \circ \psi(s,t)$ has an absolutely continuous representative which we denote by the same expression. The trace of u

$$tr(u)(\psi(s,0)) := \lim_{t \searrow 0} (u \circ \psi)(s,t)$$

is defined for almost every $s \in (0,1)$. It can be shown (see [KS93, Section 1.12]) that the trace is independent of the choice of the map ψ and is in $L^2(\partial U, X)$.

Assume that M has $k \geq 1$ boundary components and let Γ be a disjoint union of k Jordan curves in X. Recall that $[\Gamma]$ denotes the set of all weakly monotone parametrizations of Γ. Let $\Lambda(M, \Gamma, X)$ be the possibly empty family of Sobolev maps $u \in N^{1,2}(M, X)$ such that the trace $\operatorname{tr}(u)$ has a continuous representative in $[\Gamma]$.

C.2.4. Injective metric spaces and ε-thickenings

In this work we will make use of the theory of area minimizing surfaces in homotopy classes in metric spaces [SW22], which we will introduce in the next section. The definition of relative 1-homotopy classes in [SW22] highly depends on the existence of a *local quadratic isoperimetric inequality*, i.e., the existence of constants $C, l_0 > 0$ such that every Lipschitz curve $c\colon S^1 \to X$ of length $\ell(c) \leq l_0$ is the trace of a Sobolev map $u \in N^{1,2}(D, X)$ with

$$\operatorname{Area}(u) \leq C \cdot \ell(c)^2.$$

For any compact metric space X and any $\varepsilon > 0$ there exists a ε-thickening X_ε of X that is again compact and satisfies a local quadratic isoperimetric inequality. This result follows from [Wen08] and [LWY20, Lemma 3.3] if X locally geodesic and from [CF23, Lemma 5.1] otherwise. Here, a metric space Y is called *ε-thickening* of X, $\varepsilon > 0$, if there exists an isometric embedding $\iota\colon X \to Y$ such that the Hausdorff distance between $\iota(X)$ and Y is less than ε. For our definition of relative 1-homotopy in Section C.2.5, we will apply the statements of [SW22] to the ε-thickening X_ε of X and use the following lemma to pass to a notion of relative 1-homotopy in X.

Lemma C.2.4. *Let X be a metric surface. Then there is some $\varepsilon > 0$ such that for any ε-thickening Y of X there exists a continuous retraction*

$$R\colon Y \to X.$$

In order to prove Lemma C.2.4 we need some more notation. A metric space X is an *absolute neighbourhood retract (ANR)* if for each closed subset A of a metric space Y, every continuous map $f\colon A \to X$ has a continuous extension $F\colon U \to X$ defined on some neighbourhood U of A in Y. By [Dav86, Corollary 14.8A], every finite dimensional, locally contractible, compact metric space is an ANR. Thus, whenever M is a smooth surface and X a metric space homeomorphic to M, then both M and X are ANRs.

Moreover, a metric space E is *injective* if for every metric space Z, any 1-Lipschitz map $A \to E$, defined on a subset $A \subset Z$, extends to a 1-Lipschitz map $Z \to E$. For any metric space X there exists an injective metric space $E(X)$, called *injective hull of* X, which contains X and which is minimal in an appropriate sense among injective metric spaces containing X, see [Isb64]. Note that $E(X)$ is unique up to isometry and if X is compact then so is $E(X)$. Moreover, if $X \subset Y$ then $E(X)$ can be considered as a subset of $E(Y)$. For $r > 0$, we denote by $N_r(X)$ the r-neighbourhood of X in $E(X)$.

Proof of Lemma C.2.4. Let $\iota\colon X \to E(X)$ be the inclusion map, which is in particular closed. By [Dav86, Proposition 14.2], there is an open neighbourhood U of X in $E(X)$ and a continuous retraction $R_0\colon U \to X$. Choose $\varepsilon > 0$ so small that $N_\varepsilon(X) \subset U$. By injectivity of $E(X)$, the map ι extends to a 1-Lipschitz map $\bar\iota\colon Y \to E(X)$ for any Y containing X. In particular, if Y is a ε-thickening of X we obtain $\bar\iota(Y) \subset N_\varepsilon(X)$. Hence, the composition of $\bar\iota$ and R_0 is the desired retraction. □

C.2.5. Relative 1-Homotopy classes of Sobolev maps

We now introduce a notion of relative 1-homotopy classes of Sobolev mappings; for more information we refer to [SW22].

A finite collection K of compact convex polytopes (called cells of K) in some $\mathbb{R}^n$ is a *polyhedral complex* if each face of a cell is in K and the intersection of two cells of K is a face of each of them. We always equip K with the induced metric from $\mathbb{R}^n$, implying that a 2-cell Δ is isometric to a compact convex polygon in $\mathbb{R}^2$.

In the following let M be a smooth surface with possibly non-empty boundary. A *triangulation* of M consists of a polyhedral complex K and a homeomorphism $h\colon K \to M$, where h restricted to any 2-cell Δ of K is a C^1-diffeomorphism onto its image. The j-skeleton of K, denoted K^j, is the union of all cells of K of dimension at most j. Let $\partial K \subset K^1$ be the preimage of ∂M under h and define

$$\widehat{K}^1 := (K^1 \setminus \partial K) \cup K^0.$$

Note that if M has empty boundary, then $\widehat{K}^1 = K^1$.

Consider a proper geodesic metric space X. Two continuous maps $\varrho, \varrho'\colon K^1 \to X$ are homotopic relative to a set $A \subset K^1$ if there exists a homotopy $H\colon K^1 \times [0,1] \to X$ between ϱ and ϱ' satisfying $H(s,\cdot) = \varrho(s) = \varrho'(s)$ for every $s \in A$. If A is empty, then relative homotopy agrees with the usual definition of homotopy. For $\varrho|_{\partial K}, \varrho'|_{\partial K} \in [\Gamma]$ we say that ϱ and ϱ' are *homotopic relative to* Γ in some ambient space $Y \supset X$, denoted

$$\varrho \sim \varrho' \text{ rel } \Gamma \text{ in } Y,$$

if there exists a homotopy $H\colon K^1 \times [0,1] \to Y$ between ϱ and ϱ' so that $H(\cdot,t)|_{\partial K} \in [\Gamma]$ for every $t \in [0,1]$. If Y is not mentioned, we assume $X = Y$. The *homotopy class of ϱ relative to* Γ is the family

$$[\varrho]_\Gamma := \{\varrho'\colon K^1 \to X : \varrho' \text{ continuous}, \varrho'|_{\partial K} \in [\Gamma], \varrho \sim \varrho' \text{ rel } \Gamma\}.$$

We will also use this notation if Γ is empty; then the set $[\rho]_\Gamma$ coincides with the usual homotopy class $[\rho]$.

Definition C.2.5. An *admissible deformation on a surface* M is a smooth map $\Phi\colon M \times \mathbb{R}^m \to M$, $m \in \mathbb{N}$, such that $\Phi_\xi := \Phi(\cdot,\xi)$ is a diffeomorphism for every $\xi \in \mathbb{R}^m$ and $\Phi_0 = \mathrm{id}_M$, and such that the derivative of $\Phi^p := \Phi(p, \cdot)$ satisfies

$$D\Phi^p(0)(\mathbb{R}^m) = \begin{cases} T_pM & \text{if } p \in \operatorname{int}(M) \\ T_p(\partial M) & \text{if } p \in \partial M. \end{cases}$$

Consider a metric surface X and let $\varepsilon > 0$ and $R\colon X_\varepsilon \to X$ be as in Lemma C.2.4. The ε-thickening X_ε satisfies a quadratic isoperimetric inequality and we can thus apply the results form [SW22]. Let $\Phi\colon M \times \mathbb{R}^m \to M$ an admissible deformation on M, whose existence follows from [SW22, Proposition 3.2]. For a triangulation $h\colon K \to M$ of M and $\xi \in \mathbb{R}^m$ denote by $h_\xi\colon K \to M$ the triangulation given by $h_\xi := \Phi_\xi \circ h$. Furthermore, for $\xi \in \mathbb{R}^m$ and $u \in N^{1,2}(M, X)$ we denote by $u \circ h_\xi|_{K^1}$ the map agreeing with $u \circ h_\xi$ on $K^1 \setminus \partial K$ and with $\operatorname{tr}(u) \circ h_\xi$ on ∂K. Fix a Riemannian metric g on M. In [SW22, Section 3] it is shown that for every $u \in \Lambda(M, \Gamma, X)$ and every triangulation $h\colon K \to M$ of M there exists a ball $B_{\Phi,h} \subset \mathbb{R}^m$ centered at the origin such that for almost all $\xi, \zeta \in B_{\Phi,h}$ the maps $u \circ h_\xi|_{K^1}$ and $u \circ h_\zeta|_{K^1}$ have continuous representatives which are homotopic relative to Γ in X_ε. After postcomposition with R, the continuous representatives of $u \circ h_\xi|_{K^1}$ and $u \circ h_\zeta|_{K^1}$ are homotopic relative to Γ in X. We denote the common relative homotopy class by $u_{\#,1}[h]$. Note that $u_{\#,1}[h]$ is independent of the choice of deformation Φ and inducing the same relative homotopy class is independent of the triangulation h (see [SW22, Theorem 4.1]). Moreover, if u is continuous, then $u_{\#,1}[h] = [u \circ h|_{K^1}]_\Gamma$ for every triangulation h of M.

Two maps $u, v \in \Lambda(M, \Gamma, X)$ are 1-*homotopic relative to* Γ, denoted $u \sim_1 v \operatorname{rel} \Gamma$, if

$$u_{\#,1}[h] = v_{\#,1}[h]$$

for one and thus any triangulation h of M.

C.3. Non-trivial Sobolev maps

The purpose of this section is to establish the existence of a Sobolev map in $\Lambda(M, \varphi, X)$. The next proposition is a variant of [B, Proposition 5.1] and will play a crucial role in the construction of such a Sobolev map. In what follows, if Y is homeomorphic to $\overline{D}$, we denote by ∂Y the subset of Y that is homeomorphic to S^1.

Proposition C.3.1. *Let X be a locally geodesic metric space homeomorphic to a smooth surface M. Suppose $\Omega \subset M$ is biLipschitz equivalent to $\overline{D}$ and $J \subset X$ homeomorphic to $\overline{D}$. Consider a biLipsichitz map $\chi\colon I \to \partial J$ for $I \subset \partial\Omega$ connected. If $\mathcal{H}^2(J) < \infty$ and $\ell(\partial J) < \infty$ then there exists a constant $c > 0$ with the following property. For every $r > 0$ there is a Lipschitz map $v\colon \Omega \to E(J)$ with $\operatorname{Area}(v) \leq c$ and such that $v|_{\partial\Omega}$ parametrizes ∂J, $v|_I = \chi$ and $\operatorname{im}(v) \subset N_r(J)$.*

Proof. In the special case that J equipped with the subspace metric is geodesic, the statement follows from [B, Proposition 5.1]. Note that the parametrization of $v|_I$ can be prescribed by gluing a Lipschitz homotopy of zero area along I and reparametrizing. For the existence of such a Lipschitz homotopy consider e.g., [LWY20, Proposition 3.6].

For the general case, let d be the metric on X and denote by d_J the length metric on J. The identity map $\pi\colon Y := (J, d_J) \to (J, d)$ is a 1-Lipschitz homeomorphism which preserves lengths of curves and the Hausdorff 2–measure of Borel subsets, compare to [LW20, Lemma 2.1]. In particular, the metric space Y is a geodesic space homeomorphic to $\overline{D}$ with rectifiable boundary and finite Hausdorff 2-measure. Moreover, as $\chi(I) \subset \partial J$ is a biLipschitz curve, the inverse of π restricted to $\chi(I)$ is Lipschitz. The special case implies the existence of a constant $c > 0$ with the following property. For every $r > 0$ there is a Lipschitz map $v\colon \Omega \to E(Y)$ with $\operatorname{Area}(v) \leq c$ and such that $v|_{\partial\Omega}$ parametrizes ∂Y, $v|_I = \pi^{-1} \circ \chi|_I$ and $\operatorname{im}(v) \subset N_r(Y)$. By injectivity of $E(J)$, the

1-Lipschitz map π extends to a 1-Lipschitz map $\overline{\pi}\colon E(Y) \to E(J)$. The Lipschitz map $\overline{v} := \overline{\pi} \circ v$ fulfills the desired properties. $\square$

In order to apply Proposition C.3.1, we decompose M and X into Jordan domains, using a similar strategy as in [A]. A surface of genus 0 is called *cylinder* if it has two boundary components and *Y-piece* if it has three boundary components.

Lemma C.3.2. *Let X be a locally geodesic metric space homeomorphic to a smooth surface M and let $\varphi\colon M \to X$ be a homeomorphism. Then there exists a triangulation $h\colon K \to M$ of M and a biLipschitz embedding $\chi\colon h(\widehat{K}^1) \to X$ such that $\varphi \circ h|_{\widehat{K}^1}$ and $\chi \circ h|_{\widehat{K}^1}$ are homotopic relative to $K^0 \cap \partial K$ and $h(\Delta)$ is biLipschitz equivalent to $\overline{D}$ and $h(\Delta) \cap \partial M$ is connected for any 2-cell Δ of K.*

Recall that $\widehat{K}^1 := (K^1 \setminus \partial K) \cup K^0$, which agrees with K^1 if ∂K is empty.

Proof. If M is a disc, there's nothing to show. If $M = S^2$, equipped with the standard metric on S^2, we can choose three simple closed geodesics decomposing M into eight domains $\Omega_{k,l} \subset M$, $k \in \{1, 2, 3, 4\}$, $l \in \{1, 2\}$, that are triangles on the sphere and biLipschitz equivalent to $\overline{D}$. If M is not of disc- or sphere-type, depending on its topology, endow M with a hyperbolic or flat Riemannian metric. Choose a collection of simple closed geodesics decomposing M into smooth Y-pieces and cylinders M_k that intersect at most one boundary component of M. It is a standard result from hyperbolic geometry that if M_k is a Y-piece, then it is isometric to the partial gluing of the boundary of two right-angled hexagons $\Omega_{k,1}, \Omega_{k,2} \subset \mathbb{H}$, see e.g., [Bus10, Proposition 3.1.5]. Whenever M_k is a cylinder, then a similar decomposition into isometric rectangles $\Omega_{k,1}, \Omega_{k,2} \subset \mathbb{R}^2$ is possible. In either case $\Omega_{k,1}$ and $\Omega_{k,2}$ are biLipschitz equivalent to the closed unit disc $\overline{D}$.

Fix a triangulation $h\colon K \to M$ of M such that every $\Omega_{k,l}$ is the image of a 2-cell $\Delta_{k,l}$ of K under h. In particular, it has to hold that $h|_{\Delta_{k,l}}\colon \Delta_{k,l} \to \Omega_{k,l}$ is a C^1-diffeomorphism, which is only possible if $\Delta_{k,l}$ is the same type of polytope as $\Omega_{k,l}$. For every edge $e_i \in \widehat{K}^1$ we set $\alpha_i := h(e_i)$ and denote by a_i^1, a_i^2 the endpoints of α_i. There are piecewise geodesic biLipschitz curves β_i in X arbitrary close to $\varphi(\alpha_i)$ and with endpoints $\varphi(a_i^1), \varphi(a_i^2)$, see [LW20, Lemma 4.2]. By arguing as in the proof of [A, Lemma 3.1], we can modify each β_i in an arbitrary small neighbourhood of β_i, while fixing an endpoint $\varphi(a_i^j)$ if $a_i^j \in \partial M$, such that $\bigcup_i \beta_i$ is biLipschitz equivalent to $\bigcup_i \alpha_i = h(\widehat{K}^1)$. In particular, there exists a biLipschitz map $\chi\colon \bigcup_i \alpha_i \to \bigcup_i \beta_i$ sending each α_i to β_i and a homotopy relative to $K^0 \cap \partial K$ between $\varphi \circ h|_{\widehat{K}^1}$ and $\chi \circ h|_{\widehat{K}^1}$. $\square$

With Proposition C.3.3 and Lemma C.3.2 at hand, we are able to prove the following generalization of [B, Proposition 5.1].

Proposition C.3.3. *Let X be a locally geodesic metric space homeomorphic to a smooth surface M and $\varphi\colon M \to X$ a homeomorphism. If $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$ then there exists a constant $C > 0$ with the following property. For every $r \in (0, r_0]$ there exists a Lipschitz map $v\colon M \to E(X)$ with $\operatorname{Area}(v) \leq C$ and such that*

(i) $v|_{\partial M}$ *parametrizes* ∂X,
(ii) $v|_{h(\widehat{K}^1)} = \chi|_{h(\widehat{K}^1)}$,
(iii) $\operatorname{im}(v) \subset N_r(X)$,
(iv) $v \sim_1 \varphi$ *rel* ∂X *in* $(N_{r_0}(X))_\varepsilon$,

where $h\colon K \to M$, $\chi\colon M \to X$ are as in Lemma C.3.2 and $\varepsilon, r_0 > 0$ are so small that their sum is less than the constant from Lemma C.2.4.

Proof. Denote by $\Delta_1, ..., \Delta_N$ the 2-cells of K and define $\Omega_i := h(\Delta_i)$ and $I_i := h(\widehat{K}^1) \cap \partial\Omega_i$. Let J_i be the closure of the Jordan domain obtained by cutting along $\chi(I_i)$. By Proposition C.3.1, for each i there exists a constant $c_i > 0$ such that the following holds. For every $r > 0$ there is a Lipschitz map $v_i \colon \Omega_i \to E(J_i)$ with $\mathrm{Area}(v_i) \leq c_i$ and such that $v_i|_{\partial\Omega}$ parametrizes ∂J_i, $v_i|_{I_i} = \chi|_{I_i}$ and $\mathrm{im}(v_i) \subset N_r(J)$. After gluing all v_i along corresponding boundaries we obtain a Lipschitz map $v \colon M \to E(X)$ of area less than $C := \sum_{i=1}^N c_i$ satisfying (i), (ii) and (iii), where the constant C is independent of r.

By Lemma C.3.2, the maps $\varphi \circ h|_{\widehat{K}^1}$ and $\chi \circ h|_{\widehat{K}^1} = v \circ h|_{\widehat{K}^1}$ are homotopic relative to $K^0 \cap \partial K$. Let $e \subset \partial K$ be a 1-cell of K. As $\varphi \circ h|_e$ and $v \circ h|_e$ both parametrize $\varphi(h(e)) \subset \partial X$, there exists a homotopy $F_e \colon e \times [0,1] \to X$ relative endpoints between $\varphi \circ h|_e$ and $v \circ h|_e$ such that $F_e(\cdot, t)$ parametrizes $\varphi(h(e))$ for every $t \in [0,1]$. After gluing the obtained homotopies along corresponding points in $K^0 \cap \partial X$, we obtain a homotopy $H \colon K^1 \times [0,1] \to X$ between $v \circ h|_{K^1}$ and $\varphi \circ h|_{K^1}$ such that $H(\cdot, t)|_{\partial K}$ is a parametrization of ∂X for every $t \in [0,1]$. □

The proof of Theorem C.1.2 uses similar arguments as in [FW21, Section 8] and the proofs of [SW22, Proposition 6.1] and [B, Theorem 1.4]. Let X be a complete metric space and M a smooth surface. A family $\mathcal{F}$ of continuous maps from M to X is said to *satisfy the condition of cohesion* if there exists $\eta > 0$ such that each $u \in \mathcal{F}$ satisfies $\ell(u \circ c) \geq \eta$ for every non-contractible closed curve c in M. Furthermore, two 1-cells in K are called *non-neighbouring* whenever they do not intersect.

Proof of Theorem C.1.2. Let $\varepsilon, r_0 > 0$ be such that there exists a retraction $R_0 \colon (N_{r_0}(X))_\varepsilon \to X$ as in Lemma C.2.4. Consider the triangulation $h \colon K \to M$ and the biLipschitz map $\chi \colon h(\widehat{K}^1) \to X$ from Lemma C.3.2. By Proposition C.3.3, there exists a constant $C > 0$ and a sequence of Lipschitz mappings (v_n) such that each $v_n \colon M \to E(X)$ has area less than C and satisfies properties (i) to (iv) from Proposition C.3.3 with $r := 1/n$.

As χ is biLipschitz, it holds that the constant

$$\eta := \inf\{\mathrm{dist}(\chi(h(e_1)), \chi(h(e_2))) : e_1, e_2 \text{ are non-neighbouring 1-cells of } K\}$$

is strictly positive. Observe that if c is a non-contractible closed curve in M, then $h^{-1} \circ c$ intersects at least two non-neighbouring 1-cells e_1, e_2 of K. Denote the intersection point of c and e_i by p_i. As X embeds isometrically into $E(X)$ it holds that

$$\ell(v_n \circ c) \geq d_{E(X)}(v_n(h(p_1)), v_n(h(p_2))) = d(\chi(h(p_1)), \chi(h(p_2))) \geq \eta.$$

Thus, (v_n) satisfies the condition of cohesion. By Morrey's ε-conformality lemma (see [FW20, Theorem 1.2]) and inequality (C.2), we obtain a sequence of hyperbolic (or flat) metrics (g_n) on M such that

$$E_+^2(v_n, g_n) \leq \frac{4}{\pi} \mathrm{Area}(v_n) + 1 \leq \frac{4}{\pi} C + 1.$$

By [FW21, Proposition 8.4], there exists a uniform constant $\hat{\varepsilon} > 0$ such that the relative systole of (M, g_n) is bounded from below by $\hat{\varepsilon}$. Hence, we can apply the Mumford compactness theorem to obtain orientation preserving diffeomorphisms $\phi_n \colon M \to M$ such that a subsequence of $(\phi_n^* g_n)$ converges to a hyperbolic (or flat) metric h on M (see [FW21, Theorem 3.3] and e.g., [DHT10, Theorem 4.4.1] for the fact that the diffeomorphisms may be chosen to be orientation preserving). This convergence implies that for maps $w_n := v_n \circ \phi_n \in \Lambda(M, \partial X, E(X))$ it holds that

$$E_+^2(w_n, h) \leq C_n \cdot E_+^2(v_n, g_n),$$

where $C_n \geq 1$ tends to 1 as $n \to \infty$. By the Rellich-Kondrachov compactness theorem (see [KS93, Theorem 1.13]), a further subsequence of (w_n) converges in $L^2(M, E(X))$ to a finite energy map w. As each v_n is 1-homotopic to φ relative to ∂X in $(N_{r_0}(X))_\varepsilon$ and every ϕ_n is orientation preserving, the maps w_n induce the same orientation on ∂X. From [SW22, Theorem 4.7] we know that there exists $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$ the map w_{n_j} is 1-homotopic to $w_{n_{j_0}}$ relative to ∂X in $(N_{r_0}(X))_\varepsilon$. Then, the maps $u_j := w_{n_j} \circ \phi_{n_{j_0}}^{-1} \in \Lambda(M, \partial X, E(X))$ satisfy for $j \geq j_0$

$$u_j \sim_1 w_{n_{j_0}} \circ \phi_{n_{j_0}}^{-1} = v_{n_{j_0}} \sim_1 \varphi \text{ rel } \partial X \text{ in } (N_{r_0}(X))_\varepsilon.$$

The sequence (u_j) converges in $L^2(M, X)$ to the map $u := w \circ \phi_{n_{j_0}}^{-1}$ and we set $g := (\phi_{n_{j_0}}^{-1})^* h$. Since the image of u_j is in $N_{1/n_j}(X)$ we have that the essential image of u is in X and thus u can be viewed as an element of $N^{1,2}(M, X)$. This already shows that $u \in \Lambda(M, \partial X, X)$ if $\partial X = \emptyset$. Assume now that ∂X is not empty. By [FW21, Proposition 8.3], the sequence $(\operatorname{tr}(u_j))$ is equicontinuous as (u_j) still satisfies the condition of cohesion. Hence, by Arzelà-Ascoli, a subsequence of $(\operatorname{tr}(u_j))$ converges uniformly to some continuous map $\gamma\colon \partial M \to X$, which is a weakly monotone parametrization of ∂X. The convergence of $(\operatorname{tr}(u_j))$ in $L^2(\partial M, X)$ to $\operatorname{tr}(u)$, see [KS93, Theorem 1.12.2], implies that $\operatorname{tr}(u)$ coincides with γ. This shows that $u \in \Lambda(M, \partial X, X)$. It follows from [SW22, Theorem 4.7] that the map u is 1-homotopic to φ relative to ∂X in $(N_{r_0}(X))_\varepsilon$. Since both u and φ have image in X and any homotopy in $(N_{r_0}(X))_\varepsilon$ can be projected by R_0 to a homotopy with image in X, u is in fact 1-homotopic to φ relative to ∂X in X. □

Note that we exclude the case of a sphere in Theorem C.1.2, since for spaces with non-contractible universal coverings we can not apply the same methods to find a converging subsequence of (v_n). The methods used to prove the existence of energy minimizing spheres in the smooth case (see [SU81]) have not yet been extended to this generality.

In a next step we apply a direct variational method to show the existence of an energy minimizing pair in $\Lambda_{\mathrm{metr}}(M, \varphi, X)$.

Theorem C.3.4. *Let M be a smooth surface that is not a sphere, X a metric space homeomorphic to M and $\varphi\colon M \to X$ a homeomorphism. If $\Lambda(M, \varphi, X)$ is not empty, then there exists an energy minimizing pair $(u, g) \in \Lambda_{metr}(M, \varphi, X)$.*

Proof. Take an energy minimizing sequence (v_n, g_n) in $\Lambda_{\mathrm{metr}}(M, \varphi, X)$, i.e., a sequence of pairs $(v_n, g_n) \in \Lambda_{\mathrm{metr}}(M, \varphi, X)$ satisfying

$$E_+^2(v_n, g_n) \to \inf\{E_+^2(v, g) : (v, g) \in \Lambda_{\mathrm{metr}}(M, \varphi, X)\}$$

as n tends to infinity. Every non-contractible closed curve c in X satisfies $\ell(c) \geq \eta$ for some $\eta > 0$ as X is homeomorphic to a smooth surface. Thus, by arguing as in the proofs of Propositions 8.4 and 8.3 in [FW21], we obtain that the relative systole of (M, g_n) is bounded away from zero independently of n and the sequence $(\operatorname{tr}(v_n))$ is equicontinuous. Note that the maps in [FW21, Section 8] were additionally assumed to be continuous, but the proofs of Propositions 8.4 and 8.3 in [FW21] can be adapted to the current setting. We proceed as in the proof of Theorem C.1.2 to obtain a map $u \in \Lambda(M, \varphi, X)$ and a hyperbolic metric g on M with the following property. The map u is 1-homotopic to φ relative to ∂X and after precomposing each v_n with a suitable diffeomorphism of M and passing to a subsequence, the maps v_n converge to u in $L^2(M, X)$. The statement follows from lower semicontinuity of energy. □

C.4. Continuity of energy minimizers

The goal of this section is to provide a proof of Theorem C.1.3, equipping us with the right regularity of energy minimizers. For an arbitrary map $v\colon M \to X$ from a smooth surface M to

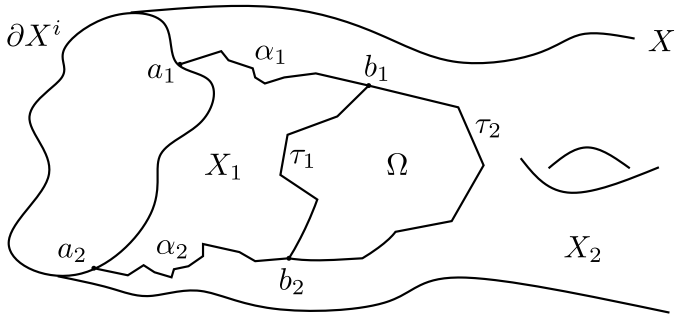


Figure C.1.: Construction of Lipschitz Retraction in proof of Lemma C.4.2.

a metric space X and for $z \in M$ and $\delta > 0$ the *essential oscillation of v in the δ-ball around z* is defined by

$$\operatorname{osc}(v, z, \delta) := \inf\{\operatorname{diam}(v(A)) : A \subset M \cap B(z, \delta) \text{ subset of full measure}\}.$$

We can show as in the proof of [B, Theorem 1.3] that Theorem C.1.3 is implied by the following generalization of [B, Proposition 4.1].

Proposition C.4.1. *Let X be a locally geodesic metric space homeomorphic to a smooth surface M with non-empty boundary and $\varphi\colon M \to X$ a homeomorphism. If $(u, g) \in \Lambda_{metr}(M, \varphi, X)$ is an energy minimizing pair, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that* $\operatorname{osc}(u, z, \delta) < \varepsilon$ *for every $z \in M$.*

The proof of Proposition C.4.1 is very similar to the proof of [B, Proposition 4.1]. We repeat the most important steps and adapt them to the current setting. For details we refer to [B, Section 4].

Consider M, X and $\varphi\colon M \to X$ as in Proposition C.4.1. Let $(u, g) \in \Lambda_{\mathrm{metr}}(M, \varphi, X)$ be an energy minimizing pair and let $\varepsilon > 0$. For an application of the Courant-Lebesgue Lemma, we consider a family $\mathcal{F}$ of 2-biLipschitz mappings from $\overline{D}$ to M such that every point in M is contained in the image of at least one map in $\mathcal{F}$; compare to the usage of Courant-Lebesgue in the proof of [A, Proposition 4.1]. Fix $z \in M$ and choose $\psi \in \mathcal{F}$ with $z \in \operatorname{im}(\psi)$. Let $\delta > 0$ be the constant from the Courant-Lebesgue Lemma applied to the map $u \circ \psi$. Note that we can choose $\delta > 0$ so small that for any $r \in (\delta, \sqrt{\delta})$ the set $\psi(\partial B(\psi^{-1}(z), r) \cap \overline{D})$ is contractible in M and intersects at most one boundary component of M. As in the proof of [B, Proposition 4.1], we find $r \in (\delta, \sqrt{\delta})$ such that for $W := \psi(D \cap B(\psi^{-1}(z), r))$ we have that the image of the trace of $u|_W$ is contained in a Jordan domain Ω of diameter less than ε that is bounded by a bi Lipschitz Jordan curve or the concatenation of a biLipschitz Jordan arc and a connected subcurve of ∂X, compare to [B, Lemma 4.2].

We are left to argue why the set $N := \{w \in W : u(w) \in X \setminus \overline{\Omega}\}$ is negligible (compare to [B, Lemma 4.3]). For this we need the following lemma.

Lemma C.4.2. *Let X be a locally geodesic metric space homeomorphic to a smooth surface M with non-empty boundary. If $\Omega \subset X$ is a Jordan domain that is either bounded by a biLipschitz Jordan curve or by the concatenation of a connected subcurve of ∂X and a biLipschitz Jordan arc, then there exists a Lipschitz retraction $\varrho\colon X \to \overline{\Omega}$ with $\varrho(X \setminus \Omega) \subset \partial\Omega$.*

Proof. We provide a proof for $\partial\Omega$ being a Jordan curve contained in the interior of X, the case of $\partial\Omega$ intersecting ∂X only needing minor adaptions in the following arguments. To get a better understanding of the construction described below, consider Figure C.1.

Choose a boundary component ∂X^i and two points $a_1, a_2 \in \partial X^i$. Decompose $\partial\Omega$ into two subcurves τ_1, τ_2 with endpoints $\tau_j(0) =: b_1$ and $\tau_j(1) =: b_2$. From [LW20, Lemma 4.2] we obtain the existence of two disjoint biLipschitz curves α_k connecting a_k and b_k in such a way that the concatenation of α_1, α_2, τ_1 and the corresponding component of $\partial X^i \setminus \{a_1, a_2\}$ bounds a Jordan domain X_1. Similarly, define X_2 to be the set bounded by the concatenation of α_1, α_2, τ_2 and the other component of $\partial X^i \setminus \{a_1, a_2\}$. After possibly changing α_k in the vicinity of b_k as in the proof of [A, Lemma 3.1] and redefining b_k and τ_j, we can assume that the concatenation of α_1, α_2 and τ_j is biLipschitz.

Let ϱ_j be the Lipschitz map agreeing with the identity on the image of τ_j and sending every point on the curve α_k to b_k. Since τ_j is a biLipschitz curve, we can apply McShane's theorem to obtain the existence of a Lipschitz map $\overline{\varrho}_j \colon X_j \to \tau_j$ extending ϱ_j. Then, the map $\varrho \colon X \to \overline{\Omega}$ agreeing with $\overline{\varrho}_j$ on X_j and with the identity on Ω is Lipschitz, as the intersections of respective domains are biLipschitz curves. □

Recall that $(u, g) \in \Lambda_{\mathrm{metr}}(M, \varphi, X)$ is an energy minimizing pair. By [FW20, Corollary 1.3], the map u is infinitesimally isotropic with respect to g and, by [FW20, Proposition 1.1], u minimizes the inscribed Riemannian area $\mathrm{Area}_{\mu^i}(u)$ among all maps in $\Lambda(M, \varphi, X)$.

We want to show that N is negligible and suppose to the contrary that N is not negligible. As in the proof of [B, Lemma 4.3], a Fubini-type argument implies that $\mathrm{Area}_{\mu^i}(u|_N) > 0$. Let $\varrho \colon X \to \overline{\Omega}$ be a Lipschitz retract as in Lemma C.4.2 and denote by v the map agreeing with u on $M \setminus W$ and with $\varrho \circ u$ on W. Since the image of the trace of $u|_W$ is contained in Ω, it follows from the Sobolev gluing theorem [KS93, Theorem 1.12.3] that $v \in N^{1,2}(M, X)$ and $\mathrm{tr}(v) = \mathrm{tr}(u)$. Moreover, it holds that $v \in \Lambda(M, \varphi, X)$. Indeed, if $h \colon K \to M$ is a triangulation of M with $W \subset h(\mathrm{int}(\Delta))$ for some 2-cell Δ of K and Φ is any admissible deformation on M. Then for every sufficiently small ξ the image of K^1 under h_ξ does not intersect W and therefore $u \circ h_\xi|_{K^1} = v \circ h_\xi|_{K^1}$.

Observe that $\mathrm{Area}_{\mu^i}(v|_N) = 0$ as $v(N) \subset \partial\Omega$ and hence,

$$\mathrm{Area}_{\mu^i}(u) = \mathrm{Area}_{\mu^i}(u|_{M \setminus N}) + \mathrm{Area}_{\mu^i}(u|_N) > \mathrm{Area}_{\mu^i}(u|_{M \setminus N}) = \mathrm{Area}_{\mu^i}(v),$$

contradicting the area minimization property of u. Hence, N is negligible and we finished the sketch of proof of Proposition C.4.1.

C.5. Almost homeomorphism

In this section we provide a proof of Theorem C.1.4 for surfaces that are not homeomorphic to a disc. If M is a disc, we refer to the proof of [LW20, Theorem 1.2].

A compact metric space is called *cell-like* if it admits an embedding into the Hilbert cube in which it is null-homotopic in every neighborhood of itself. A continuous surjection $v \colon Y \to Z$ between metric spaces is called *cell-like* if $v^{-1}(z)$ is cell-like, and in particular compact, for every $z \in Z$. Cell-like mappings are closely related to uniform limits of homeomorphisms as illustrated by the following theorem of Moore (see e.g., [Edw80, p. 116] or [Dav86, Theorem 25.1] for closed surfaces and [Sie72, Theorem A] for compact surfaces with non-empty boundary).

Theorem C.5.1. *Let M be a smooth surface and $v \colon M \to X$ a cell-like map such that $v|_{\partial M} \colon \partial M \to \partial X$ is cell-like if ∂M is non-empty. Then X is homeomorphic to M and v is a uniform limit of homeomorphisms.*

Moreover, by arguing exactly as in the proof of [LW20, Proposition 2.9] and using the fact that every metric surface is an ANR, we obtain the next proposition.

Proposition C.5.2. *Let M be a smooth surface with possibly non-empty boundary and let X be a metric surface homeomorphic to M. If $v \colon M \to X$ is a continuous surjection, then the following statements are equivalent:*

1. v is monotone,
2. v is cell-like,
3. v is a uniform limit of homeomorphisms $v_i\colon M \to X$.

From now on, we assume that X is a locally geodesic metric space homeomorphic to a smooth surface M with non-empty boundary. The next theorem generalizes [LW20, Theorem 4.1] and will play a crucial role in the proof of Theorem C.1.4.

Theorem C.5.3. *Let $v\colon M \to X$ be a continuous surjection satisfying the following properties:*
(i) The restriction of v to ∂M is a weakly monotone parameterization of ∂X.
(ii) Whenever $T \subset X$ is a single point or biLipschitz homeomorphic to a closed interval, every connected component of $v^{-1}(T)$ is cell-like.
Then v is a cell-like map.

Proof. We follow the same strategy as in the proof of [LW20, Theorem 4.1]. By the monotone-light factorization theorem due to Eilenberg and Whyburn (see e.g., [You51, Theorem 3.5]) there exists a compact metric space Z and continuous surjective maps $v_1\colon M \to Z$, $v_2\colon Z \to X$, where v_1 is monotone and v_2 is light, such that for every $z \in Z$ the fibers $v_1^{-1}(z)$ are the connected components of $v^{-1}(v_2(z))$. Recall that the map $v_2\colon Z \to X$ is light if $v_2^{-1}(x)$ is totally disconnected for every $x \in X$. The mappings v_1 and $v_1|_{\partial M}$ are cell-like as v satisfies (i) and (ii). Hence, it follows from Theorem [Sie72, Theorem A] that Z is homeomorphic to M and v_1 is a uniform limit of homeomorphisms. Identify Z with M. Since [Lac69, Theorem 1.4] holds for an arbitrary ANR, we can follow as in the proof of [LW20, Lemma 4.4] that the map v_2 satisfies properties (i) and (ii). If v_2 is cell-like, then so is v. Thus, it suffices to consider the case where v is in addition a light map.

As an application of Theorem C.5.1, Lemma 2.8 in [LW20] remains true for an arbitrary surface (M, g) instead of (S^2, g_{Eucl}). Therefore, Lemma 4.5 and Lemma 4.6 in [LW20] can be generalized to our setting. Proceeding as in the last paragraph of [LW20, Section 4] completes the proof of Theorem C.5.3. □

The rest of this section is devoted to the proof of Theorem C.1.4. Recall that M has non-empty boundary and we excluded the case of M being a disc. Let $\varphi\colon M \to X$ be a homeomorphism. Consider an energy minimizing pair $(u, g) \in \Lambda_{\text{metr}}(M, \varphi, X)$. As above, the map u minimizes the inscribed Riemannian area Area_{μ^i} among all maps in $\Lambda(M, \varphi, X)$ with respect to g. By Proposition C.5.2, it suffices to show that the hypotheses of Theorem C.5.3 hold. The map u is surjective for topological reasons and $u \in \Lambda(M, \partial X, X)$ implies that u satisfies (i). Towards a contradiction, assume there exists T as in (ii) such that some connected component K of $u^{-1}(T)$ is not cell-like. Let $\varepsilon > 0$ be so small that the ε-neighbourhood of T in X is contractible in X. By continuity of u and compactness of K, there is $r_0 > 0$ such that $u(N_{r_0}(K)) \subset N_\varepsilon(T)$.

Assume that for any $0 < r < r_0$ there exists a curve γ_r in $N_r(K)$ which is not contractible in M. Any continuous map 1-homotopic to φ is homotopic to φ (see [SW22, Lemma 6.2]), as every surface not of sphere-type has trivial second homotopy group. This implies that the curve $u \circ \gamma_r \subset N_\varepsilon(T)$ is not contractible in X, a contradiction. Thus, we can choose $0 < r < r_0$ such that any curve in $N_r(K)$ is contractible in M. We claim:

Lemma C.5.4. *Let $K \subset M$ be a compact and connected set and let $r > 0$ be such that every curve in $U := N_r(K)$ is contractible in M. Then, the set K is contained in the closure of a Jordan domain in M.*

With this lemma at hand we can easily finish the proof of the theorem. Indeed, from Lemma C.5.4 we can deduce the existence of a Jordan domain $\Omega \subset M$ such that $K \subset \overline{\Omega}$. As K is not cell-like, there exists a connected component of $\overline{\Omega} \setminus K$ that does not intersect $\partial\Omega$. In particular, there exists a connected component $U \subset \overline{\Omega}$ of $M \setminus u^{-1}(T)$ not intersecting ∂M. By arguing as in the proof of [B, Lemma 4.2], we may assume that Ω is bounded by a biLipschitz curve and thus is a Lipschitz domain. Moreover, T is an absolute Lipschitz retract as T is a single point or biLipschitz homeomorphic to a closed interval. In particular, there exists a Lipschitz retraction $P\colon X \to T$. Define $w := P \circ u$. By arguing as in the proof of [LW20, Theorem 1.2], there exists a map $u_1 \in N^{1,2}(\Omega, X)$ having the same trace as $u|_\Omega$ and agreeing with u on $\Omega \setminus U$ and with w on U. After applying the general gluing theorem for Sobolev maps, see [KS93, Theorem 12.1.3], we obtain a map $u_2 \in \Lambda(M, \partial X, X)$ agreeing with u on $M \setminus U$ and with w on U. As both U and T are contractible in their respective spaces, we obtain that u_2 is homotopic to φ. Thus, $u_2 \in \Lambda(M, \varphi, X)$.

The set T is a single point or biLipschitz homeomorphic to a closed interval implying that $\operatorname{ap}\operatorname{md} w_z$ is degenerate for almost every z. Hence, the inscribed Riemannian area of $w|_U$ is zero. Since u minimizes the inscribed Riemannian area among all maps in $\Lambda(M, \varphi, X)$, it follows that the inscribed Riemannian area of $u|_U$ is zero. By [FW20, Proposition 1.1], the Reshetnyak energy of $u|_U$ is zero as well and it follows that $u|_U$ is constant. Therefore $u(U)$ is contained in T, a contradiction. Thus every connected component of $u^{-1}(T)$ is cell-like and u satisfies (ii). This finishes the proof of Theorem C.1.4.

We are left to prove the lemma above. For a topological space Y, a subset $A \subset Y$ and two distinct points $x, y \in Y \setminus A$, we say that A *separates* x from y if every connected subset $B \subset Y$ containing x and y intersects A. Note that Lemma C.5.4 holds for all surfaces that are not of disc- or sphere-type. Hence, we provide a proof also for surfaces with empty boundary.

Proof of Lemma C.5.4. We first assume that M is a closed surface. Equip M with a Riemannian metric g and let $\phi\colon \widehat{M} \to M$ be the universal cover of M. Fix $x_0 \in U$. The fiber of U under ϕ is given by

$$\phi^{-1}(U) = \bigcup_{\alpha \in \pi_1(M, x_0)} U_\alpha,$$

where we define

$$U_\alpha := \{[\alpha + \beta] : \beta\colon [0,1] \to U \text{ continuous, } \beta(0) = x_0\}.$$

As every curve in U is contractible we have $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$ and $\phi|_{U_\alpha}\colon U_\alpha \to U$ is a homeomorphism for every $\alpha \in \pi_1(M, x_0)$. Fix $\alpha \in \pi_1(M, x_0)$. The set U_α is bounded. Indeed, if U_α is not bounded, there exists a curve τ in U winding infinitely many times around a handle of M. Since every point in the image of τ is contained in a ball of radius r in $U = N_r(K)$, we can find a non-contractible curve contained in U, a contradiction. Hence, there exists a set $Z \subset \widehat{M}$ homeomorphic to $\overline{D}$ with $U_\alpha \subset Z$. Observe that the set $K_\alpha := {\phi|_{U_\alpha}}^{-1}(K)$ is homeomorphic to K. The coarea inequality for Lipschitz maps (see e.g., [Fed69, Theorem 2.10.25]) implies that the set

$$S_t := \{y \in Z : \operatorname{dist}(y, K_\alpha) = t\}$$

has finite $\mathcal{H}^1$-measure for almost every $t > 0$. Let $t \in (0, r)$ be such that $\mathcal{H}^1(S_t) < \infty$. The set S_t separates any point $p \in K_\alpha$ from any $q \in \partial Z$ and, by [LW18a, Corollary 7.6], contains a Jordan curve γ_t still separating p from q. By the Jordan curve theorem, γ_t bounds a Jordan domain Ω_t in Z with $K_\alpha \subset \Omega_t$.

Consider the set $\gamma := \phi(\gamma_t) \subset U$, which is homeomorphic to γ_t and hence a Jordan curve. By assumption, every curve in U is contractible in M and therefore γ bounds a Jordan domain $\Omega \subset M$. As $\overline{\Omega}$ is contractible, the map $\phi|_{\overline{\Omega}_\alpha}\colon \overline{\Omega}_\alpha \to \overline{\Omega}$ is a homeomorphism, where $\overline{\Omega}_\alpha$ is defined analogously to U_α. Hence, $\overline{\Omega}_\alpha$ is homeomorphic to $\overline{D}$ and shares the same boundary as $\overline{\Omega_t}$. Assume that K is not contained in $\overline{\Omega}$. Since $\phi|_{\overline{\Omega}_\alpha}$ is a homeomorphism and $\partial\overline{\Omega}_\alpha = \gamma_t$ does not intersect K_α, it holds that $K_\alpha \subset \widehat{M} \setminus \Omega_\alpha$. This implies $\overline{\Omega}_\alpha \neq \overline{\Omega_t}$ and therefore $\overline{\Omega}_\alpha \cup \overline{\Omega_t}$ is a sphere, a contradiction.

If M has non-empty boundary we consider the Schottky double (M^*, g^*) of (M, g), obtained by gluing two copies of M along their boundaries and by doubling the metric g; compare to the proof of [FW21, Lemma 2.4]. By construction, (M^*, g^*) is a smooth closed surface containing an isometric copy of M, denoted again by M. We use the same strategy as above to obtain the existence of a Jordan domain Ω in M^* containing $K \subset M \subset M^*$. A connected component of the intersection of Ω with M is again a Jordan domain whose closure contains K. □

C.6. Applications

In this short section we briefly describe how a quasiconformal almost parametrization upgrades to a quasisymmetric map under the assumptions of Ahlfors 2-regularity and linear local connectedness and to a geometrically quasiconformal map after assuming reciprocality. In particular, we show that Theorem C.1.1 implies generalizations of the uniformization theorems of Bonk and Kleiner as well as Rajala

A homeomorphism $f\colon X \to Y$ between metric spaces is *quasisymmetric* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$d_Y(f(x), f(y)) \leq \eta(t) \cdot d_Y(f(x), f(z))$$

for all points $x, y, z \in X$ with $d_X(x, y) \leq t \cdot d_X(x, z)$. Moreover, a metric space X is said to be *Ahlfors 2-regular* if there exists $K > 0$ such that for all $x \in X$ and $0 < r < \operatorname{diam} X$, we have

$$K^{-1} \cdot r^2 \leq \mathcal{H}^2(B(x, r)) \leq K \cdot r^2.$$

We say that X is *linearly locally connected (LLC)* if there exists a constant $\lambda \geq 1$ such that for all $x \in X$ and $r > 0$, every pair of distinct points in $B(x, r)$ can be connected by a continuum in $B(x, \lambda r)$ and every pair of distinct points in $X \setminus B(x, r)$ can be connected by a continuum in $X \setminus B(x, r/\lambda)$.

Note that every compact Ahlfors 2-regular metric space is in particular of finite Hausdorff 2-measure. Denote by $\Lambda(M, X)$ the family of Newton-Sobolev maps $u \in N^{1,2}(M, X)$ such that u is a uniform limit of homeomorphisms from M to X. Theorem C.1.1 shows that $\Lambda(M, X)$ is not empty for M having non-empty boundary and X being geodesic, Ahlfors 2-regular and homeomorphic to M. By arguing exactly as in [A, Section 5], there exists a canonical quasisymmetric homeomorphism from M to X, if X is furthermore LLC. After applying gluing techniques as in [A], we receive that this also holds for closed surfaces. Hence, we obtain the following theorem which recovers [A, Theorem 1.1] and generalizes Bonk-Kleiner's theorem [BK02, Theorem 1.1].

Theorem C.6.1. *Let X be a geodesic metric space which is Ahlfors 2-regular, linearly locally connected and homeomorphic to a smooth surface M. Then, there exist a map $u \in \Lambda(M, X)$ and a Riemannian metric g on M such that*

$$E_+^2(u, g) = \inf\{E_+^2(v, h) : v \in \Lambda(M, X),\ h \textit{ a smooth Riemannian metric on } M\}.$$

Any such u is a quasisymmetric homeomorphism from M to X and the pair (u, g) is uniquely determined up to a conformal diffeomorphism $(M, g) \to (M, h)$.

The assumption of X being geodesic is not needed if X is closed, since every closed, LLC and Ahlfors 2-regular metric surface is geodesic up to a biLipschitz change of metric (see [Sem96a, Theorem B.6] and [BK02, Lemma 2.5]).

We now turn to the generalization of Rajala's uniformization theorem. Consider a metric space X homeomorphic to a smooth surface M and of finite Hausdorff 2-measure. A homeomorphism $u\colon M \to X$ is *geometrically quasiconformal* if there exists $K \geq 1$ such that

$$K^{-1} \cdot \operatorname{mod}(\Gamma) \leq \operatorname{mod}(u \circ \Gamma) \leq K \cdot \operatorname{mod}(\Gamma) \tag{C.4}$$

for every family Γ of curves in M with respect to a Riemannian metric g on M. We call X a *quasiconformal surface* if every point of X is contained in a geometrically quasiconformal image of $\overline{D}$. By Rajala's uniformization theorem [Raj17], this is equivalent to being *locally recirocal*, i.e., every point of X is contained in a reciprocal neighbourhood U that is homeomorphic to $\overline{D}$. We call U *reciprocal* if the following two conditions hold. For every $x \in U$ and $R > 0$ with $U \setminus B(x, R) \neq \emptyset$ we have

$$\lim_{r \to 0} \operatorname{mod}(B(x,r), X \setminus B(x,R); \overline{B}(x,R)) = 0, \tag{C.5}$$

where $\operatorname{mod}(E, F; G)$ denotes the modulus of the family of curves joining E and F in G for some subsets $E, F, G \subset U$. Moreover, there exists $\kappa > 0$ such that every closed topological square $Q \subset U$ with boundary edges $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ in cyclic order satisfies

$$\operatorname{mod}(\zeta_1, \zeta_3; Q) \cdot \operatorname{mod}(\zeta_2, \zeta_4; Q) \leq \kappa. \tag{C.6}$$

Notice that Rajala [Raj17] originally assumed an additional lower bound on the product in (C.6). It has been shown that this lower bound is always satisfied, see [RR19] and [EBPC22].

If X is locally geodesic and has non-empty, rectifiable boundary, then by Theorem C.1.1, there exists a continuous, monotone surjection $u\colon M \to X$ satisfying (C.1) with $K = \frac{4}{\pi}$. By compactness, there exists a constant $C \geq 1$ and finitely many C-biLipschitz maps $\psi_i\colon \overline{D} \to M$ such that the sets $U_i := \operatorname{im}(u \circ \psi_i)$ are homeomorphic to $\overline{D}$ and cover X. If each U_i satisfies (C.5), then the maps $u \circ \psi_i\colon \overline{D} \to U_i$ are homeomorphisms (see [B, Proposition 3.1]) and, by [B, Proposition 3.3], upgrade to geometrically quasiconformal maps if each U_i satisfies (C.6). Since ψ_i are biLipschitz and U_i cover X, the map u itself is geometrically quasiconformal. We have thus established the following version of [Iko22, Theorem 1.2] for locally geodesic surfaces with non-empty boundary.

Theorem C.6.2. *Every locally geodesic quasiconformal surface with $k \geq 1$ rectifiable boundary components is quasiconformally equivalent to a Riemannian surface with k boundary components.*

D. Lipschitz-Volume rigidity and Sobolev coarea inequality for metric surfaces

with Dimitrios Ntalampekos

Abstract. We prove that every 1-Lipschitz map from a closed metric surface onto a closed Riemannian surface that has the same area is an isometry. If we replace the target space with a non-smooth surface, then the statement is not true and we study the regularity properties of such a map under different geometric assumptions. Our proof relies on a coarea inequality for continuous Sobolev functions on metric surfaces that we establish, and which generalizes a recent result of Esmayli–Ikonen–Rajala.

D.1. Introduction

The Lipschitz-volume rigidity problem in its general formulation asks whether every 1-Lipschitz and surjective map between metric spaces that have the same volume (e.g. arising from Hausdorff measure) is necessarily an isometry. It is well-known that the answer to this problem is affirmative for maps between smooth manifolds.

Let X, Y be closed Riemannian n-manifolds, where $n \geq 1$. If $\mathrm{Vol}(X) = \mathrm{Vol}(Y)$, then every 1-Lipschitz map from X onto Y is an isometric homeomorphism.

See [BI10, Section 9] or [BCG95, Appendix C] for a proof of this fact. Moreover, this statement has been generalized to singular settings of Alexandrov and limit RCD spaces by Storm [Sto06], Li [Li15], and Li–Wang [LW14]. See also [Li20] for an overview of the Lipschitz-volume rigidity problem. The problem in the setting of integral current spaces has been recently studied by Basso–Creutz–Soultanis [BCS23], Del Nin–Perales [DNP23], and Züst [Züs24].

The recent developments in the uniformization of non-smooth metric surfaces by Rajala, Romney, Wenger, and the current authors [Raj17, NR23, B, NR24], allow us to establish the above rigidity statement in the two-dimensional setting under no geometric, smoothness, or curvature assumptions on X.

Theorem D.1.1. *Let X be a closed metric surface and Y be a closed Riemannian surface. If $\mathcal{H}^2(X) = \mathcal{H}^2(Y)$, then every 1-Lipschitz map from X onto Y is an isometric homeomorphism.*

Here a closed metric surface is a compact topological 2-manifold without boundary, equipped with a metric that induces its topology. Also, an isometric map is a distance-preserving map. We state an immediate corollary.

Corollary D.1.2. *Among all metrics d on $\mathbb{S}^2$ that are at least as large as the spherical metric, the map $d \mapsto \mathcal{H}^2_d(\mathbb{S}^2)$ has a unique minimum attained by the spherical metric.*

We note in the next example that the conclusion is not true in general if we replace the spherical metric with a non-smooth metric.

Example D.1.3. Consider a non-constant rectifiable curve E in $\mathbb{S}^2$ and let d_0 be the length metric $\chi_{\mathbb{S}^2\setminus E}\, ds + (1/2)\chi_E\, ds$. Then there exist infinitely many distinct metrics $d \geq d_0$ having the same area as d_0. Namely, for each $\delta \in (1/2, 1]$, the metric $\chi_{\mathbb{S}^2\setminus E}\, ds + \delta\chi_E\, ds$ has this property.

One of the most technical difficulties of Theorem D.1.1 is establishing the injectivity of the map in question; see Lemma D.3.7. Since this issue is not present in Corollary D.1.2, it is conceivable that the result can be obtained in higher dimensions as well by a modification of our argument.

D.1.1. Area-preserving and Lipschitz maps between surfaces

A map as in Theorem D.1.1 preserves the Hausdorff 2-measure, or else area measure, of every measurable set. Theorem D.1.1 is a consequence of Theorem D.1.4 below, which provides several topological and regularity results for area-preserving and Lipschitz maps between surfaces of locally finite Hausdorff 2-measure.

We provide the necessary definitions. Let X and Y be metric surfaces of locally finite Hausdorff 2-measure. A map $f\colon X \to Y$ is *area-preserving* if $\mathcal{H}(A) = \mathcal{H}(f(A))$ for every measurable set $A \subset X$. A map $f\colon X \to Y$ is *Lipschitz* if there exists $L > 0$ such that for all $x_1, x_2 \in X$ we have

$$d(f(x_1), f(x_2)) \leq L\, d(x_1, x_2).$$

In this case, we say that f is L-Lipschitz. A homeomorphism $f\colon X \to Y$ is *quasiconformal* (abbr. QC) if there exists $K \geq 1$ such that

$$K^{-1} \operatorname{mod} f(\Gamma) \leq \operatorname{mod} \Gamma \leq K \operatorname{mod} f(\Gamma)$$

for each path family Γ in X; here mod refers to 2-modulus and the precise definition is given in Section D.2.3. In this case we say that f is K-quasiconformal. A map $f\colon X \to Y$ is a map of *bounded length distortion* (abbr. BLD) if there exists a constant $K \geq 1$ such that

$$K^{-1} \cdot \ell(\gamma) \leq \ell(f \circ \gamma) \leq K \cdot \ell(\gamma)$$

for all curves γ in X; this includes curves of infinite length. In this case we say that f is a map of K-bounded length distortion.

We say that the surface X is *reciprocal* if there exists a constant $\kappa > 0$ such that for every quadrilateral $Q \subset X$ and for the families $\Gamma(Q)$ and $\Gamma^*(Q)$ of curves joining opposite sides of Q we have

$$\operatorname{mod} \Gamma(Q) \cdot \operatorname{mod} \Gamma^*(Q) \leq \kappa.$$

By a result of Rajala [Raj17, Section 14], if a surface is reciprocal then the above holds for some $\kappa \leq (\pi/2)^2$. Reciprocal surfaces are important because they are precisely the metric surfaces that admit quasiconformal parametrizations by Riemannian surfaces [Raj17, Iko22, NR24]. We say that X is *upper Ahlfors 2-regular* if there exists $K > 0$ such that

$$\mathcal{H}^2(B(x,r)) \leq Kr^2$$

for every ball $B(x,r) \subset X$. If X is (locally) upper Ahlfors 2-regular, then it is also reciprocal [Raj17]. See Section D.2.5 for further details. We state our main theorem, which is also concisely

Reference	X	Y	f	Conclusion about f
Question D.1.5	-	-	Lip.	BLD on a.e. curve?
Thm. D.1.4 (1)	Reciprocal	-	(1-)Lip.	(1-)BLD on a.e. curve
Example D.4.1	Riemannian	-	1-Lip.	*Not* homeomorphic
Thm. D.1.4 (2)	-	Reciprocal	(1-)Lip.	(1-)QC homeom., (1-)BLD on a.e. curve
Example D.4.2	Riemannian	Reciprocal	1-Lip.	*Not* BLD
Thm. D.1.4(3)	-	Upper regular	Lip.	QC homeom., BLD
Example D.1.3	Riemannian	Upper regular	1-Lip.	*Not* isometric
Thm. D.1.4(4)	-	Riemannian	1-Lip.	Isometry

Table D.1.: The conclusions of Theorem D.1.4. In all cases f is assumed to be area-preserving.

presented in Table D.1.

Theorem D.1.4. *Let X, Y be metric surfaces without boundary and with locally finite Hausdorff 2-measure, and let $f\colon X \to Y$ be an area-preserving surjective map.*

(1) *If X is reciprocal and f is Lipschitz, then there exists a constant $K \geq 1$ such that*

$$K^{-1} \cdot \ell(\gamma) \leq \ell(f \circ \gamma) \leq K \cdot \ell(\gamma)$$

for all curves γ in X outside a curve family Γ_0 with $\operatorname{mod}\Gamma_0 = 0$. Moreover, if f is 1-Lipschitz, then $K = 1$.

(2) *If Y is reciprocal and f is Lipschitz, then there exists a constant $K \geq 1$ such that f is a K-quasiconformal homeomorphism and*

$$K^{-1} \cdot \ell(\gamma) \leq \ell(f \circ \gamma) \leq K \cdot \ell(\gamma)$$

for all curves γ in X outside a curve family Γ_0 with $\operatorname{mod}\Gamma_0 = 0$. Moreover, if f is 1-Lipschitz, then $K = 1$.

(3) *If Y is upper Ahlfors 2-regular and f is Lipschitz, then there exists a constant $K \geq 1$ such that f is a homeomorphism of K-bounded length distortion.*

The constant K in (1)–(3) depends quantitatively on the assumptions.

(4) *If Y is Riemannian and f is 1-Lipschitz, then f is an isometric homeomorphism.*

We were neither able to show that part (1) holds without the assumption that X is reciprocal, nor were we able to find a counterexample. This raises the following question.

Question D.1.5. *Suppose that X, Y are metric surfaces of locally finite Hausdorff 2-measure. If $f\colon X \to Y$ is an area-preserving and Lipschitz map, does it quasi-preserve the length of a.e. path in X?*

We note that an affirmative answer to the question has been provided by Creutz–Soultanis [CS20, Proposition 4.1] with the additional assumptions that X is 2-rectifiable and f is 1-Lipschitz. This result does not imply Theorem D.1.4 (1) or vice versa.

In Section D.4 we present examples illustrating the optimality of Theorem D.1.4. We first note that area-preserving and 1-Lipschitz maps are not injective in general without any assumptions on Y; a sufficient condition is the reciprocity of Y in part (2). Moreover, one cannot expect in part (2) that the length of *all* curves (rather than a.e. curve) is quasi-preserved; a sufficient condition is upper Ahlfors 2-regularity of Y as in (3). Finally, in part (3) one cannot expect a 1-Lipschitz map f to be an isometry without further assumptions on Y, such as smoothness, as in (4); this has already been illustrated in Example D.1.3.

D.1.2. Coarea inequality

The proof of Theorem D.1.4 relies on a coarea inequality for continuous Sobolev functions on metric surfaces. The following result is an improvement of the coarea inequality for *monotone* Sobolev functions that was established recently in [EIR23]; here monotonicity means that the maximum and minimum of a function on a precompact open set are attained at the boundary. We direct the reader to [EIR23] for further background on the coarea inequality in metric spaces.

Theorem D.1.6. *Let X be a metric surface of locally finite Hausdorff 2-measure and $u\colon X \to \mathbb{R}$ be a continuous function with a 2-weak upper gradient $\rho_u \in L^2_{\mathrm{loc}}(X)$.*

(1) *If $\mathcal{A}_u$ denotes the union of all non-degenerate components of the level sets $u^{-1}(t)$, $t \in \mathbb{R}$, of u, then $\mathcal{A}_u$ is a Borel set.*

(2) *For every Borel function $g\colon X \to [0,\infty]$ we have*

$$\int^* \int_{u^{-1}(t)\cap\mathcal{A}_u} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int g\rho_u \, d\mathcal{H}^2.$$

(3) *If, in addition, u is Lipschitz, then for every Borel function $g\colon X \to [0,\infty]$ we have*

$$\int^* \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int g \cdot (\rho_u \chi_{\mathcal{A}_u} + \mathrm{Lip}(u)\chi_{X\setminus\mathcal{A}_u}) \, d\mathcal{H}^2.$$

Here $\mathrm{Lip}(u)$ denotes the pointwise Lipschitz constant of a Lipschitz function $u\colon X \to \mathbb{R}$, defined by

$$\mathrm{Lip}(u)(x) = \limsup_{y\to x} \frac{|u(y)-u(x)|}{d(x,y)}.$$

Also, $\int^*$ denotes the upper integral, which is equal to the Lebesgue integral for measurable functions. The main result of [EIR23] (for $p \geq 2$) states that (2) holds with the additional assumption that u is monotone and with $u^{-1}(t)$ in place of $u^{-1}(t) \cap \mathcal{A}_u$. Since the level sets of monotone functions are always non-degenerate (see e.g. [Nta20, Corollary 2.8]), we see that $\mathcal{A}_u = X$ when u is monotone; hence our theorem implies the main result of [EIR23] for $p \geq 2$. Moreover, without the monotonicity assumption, we note that part (2) is optimal and does not hold for the full level sets $u^{-1}(t)$ if we do not restrict to $\mathcal{A}_u$, even if u is Lipschitz. A relevant example is provided in [EIR23, Section 5].

The proof of Theorem D.1.6 relies on recent developments in the theory of uniformization of metric surfaces. Specifically, we use a result of Romney and the second-named author [NR24], which states that every metric surface of locally finite Hausdorff 2-measure admits a weakly quasiconformal parametrization by a Riemannian surface of the same topological type.

D.2. Preliminaries

D.2.1. Hausdorff measures

For a metric space X and $s > 0$, the *Hausdorff s-measure* of a set $A \subset X$ is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A), \text{ where } \mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{j=1}^{\infty} \frac{\omega_s}{2^s} \operatorname{diam}(A_j)^s \right\}$$

and the infimum is taken over all collections of sets $\{A_j\}_{j=1}^{\infty}$ such that $A \subset \bigcup_{j=1}^{\infty} A_j$ and $\operatorname{diam}(A_j) < \delta$ for each j. Here ω_s is a positive normalization constant, chosen so that the Hausdorff n-measure coincides with Lebesgue measure in $\mathbb{R}^n$. Note that $\omega_1 = 2$ and $\omega_2 = \pi$. If we need to emphasize the metric d being used for the Hausdorff s-measure, we write $\mathcal{H}^s_d$ instead of $\mathcal{H}^s$.

We state the coarea inequality for Lipschitz functions and the classical coarea formula for Sobolev functions.

Theorem D.2.1 (Coarea inequality and formula)**.** *Let X be a metric space, $u\colon X \to \mathbb{R}$ be a continuous function, and $g\colon X \to [0, \infty]$ be a Borel function.*

(1) *If u is Lipschitz, then for $K = 4/\pi$ we have*

$$\int^* \int_{u^{-1}(t)} g \, d\mathcal{H}^1 dt \leq K \int_X g \cdot \operatorname{Lip}(u) \, d\mathcal{H}^2.$$

If X is a Riemannian surface, we may take $K = 1$.

(2) *If X is an open subset of $\mathbb{R}^2$ and $u \in W^{1,1}_{\mathrm{loc}}(X)$, then*

$$\int \int_{u^{-1}(t)} g \, d\mathcal{H}^1 dt = \int_X g \cdot |\nabla u| \, d\mathcal{H}^2.$$

Part (1) is a consequence of [Fed69, Theorem 2.10.25] for general metric spaces X and of [Fed59, Theorem 3.1] for Riemannian manifolds with $K = 1$. Part (2) is stated in [MSZ03] and attributed to Federer. See also [EH21] for a more general statement than (1) and [EIR23, Lemma 5.2].

We will make use of the following area formula. Below, $N(f, y)$ denotes the number of preimages of a point y under a map f.

Theorem D.2.2 ([Fed69, Theorem 2.10.10])**.** *Let X, Y be metric spaces such that X is separable. Consider a map $f\colon X \to Y$ such that for every Borel set $A \subset X$ the set $f(A)$ is $\mathcal{H}^2$-measurable. For $S \subset X$ define $\zeta(S) = \mathcal{H}^2(f(S))$ and denote by ψ the measure on X resulting by Carathéodory's construction from ζ on the family of all Borel subsets of X. Then, for each Borel set $A \subset X$ we have*

$$\psi(A) = \int_Y N(f|_A, y) \, d\mathcal{H}^2.$$

D.2.2. Topological preliminaries

Let X be a metric space. A *path* or *curve* is a continuous map $\gamma\colon [a, b] \to X$. The *trace* of γ is the set $|\gamma| = \gamma([a, b])$. The *length* of γ is its total variation and is denoted by $\ell(\gamma)$. The following theorem is a consequence of Theorem D.2.2 and provides an area formula for length.

Theorem D.2.3 ([Fed69, Theorem 2.10.13])**.** *Let X be a metric space and $\gamma\colon [a,b] \to X$ be a curve. Then*

$$\ell(\gamma) = \int_X N(\gamma, x)\, d\mathcal{H}^1.$$

We say that a curve $\gamma\colon [a,b] \to X$ is a *Jordan arc* if γ is injective. Here we allow the possibility $a = b$, in which case γ is a *degenerate* Jordan arc. We say that γ is a *Jordan curve* if $\gamma|_{[a,b)}$ is injective and $\gamma(a) = \gamma(b)$. We also say that a set $K \subset X$ is a Jordan arc (resp. Jordan curve) if there exists a Jordan arc (resp. Jordan curve) γ with $|\gamma| = K$. A *continuum* is a compact and connected metric space. A *Peano continuum* is a locally connected continuum.

Lemma D.2.4. *Let $\{K_i\}_{i\in I}$ be a collection of pairwise disjoint Peano continua in $\mathbb{R}^2$. Then, with the exception of countably many $i \in I$, each K_i is a Jordan arc or a Jordan curve.*

Proof. A *triod* is the union of three non-degenerate Jordan arcs that have a common endpoint, the *junction point*, but are otherwise disjoint. A theorem of Moore [Moo28] (see also [Pom92, Proposition 2.18]) states that there is no uncountable collection of pairwise disjoint triods in the plane. On the other hand, if a Peano continuum is not a Jordan arc or Jordan curve, then it contains a triod [Nta20, Lemma 2.4]. This completes the proof. □

Lemma D.2.5. *Let K be a continuum with $\mathcal{H}^1(K) < \infty$. Then K is a Peano continuum.*

Proof. If $\mathcal{H}^1(K) < \infty$, a result of Eilenberg–Harrold [EH43, Theorem 2] states that there exists a continuous and surjective mapping $\gamma\colon [0,1] \to K$ (with $\ell(\gamma) \leq 2\mathcal{H}^1(K) - \operatorname{diam}(K)$). By the Hahn–Mazurkiewicz theorem [Wil70, Theorem 31.5], Peano continua are characterized as continuous images of the unit interval. □

Lemma D.2.6 ([BBI01, Theorem 2.6.2])**.** *Let X be a metric space and let $\gamma\colon [a,b] \to X$ be a curve. Then $\ell(\gamma) \geq \mathcal{H}^1(|\gamma|)$. Moreover, if γ is a Jordan arc or Jordan curve, then $\ell(\gamma) = \mathcal{H}^1(|\gamma|)$.*

We state a consequence of Lemmas D.2.4, D.2.5, and D.2.6, and of the existence of arclength parametrizations of rectifiable curves [HKST15, Section 5.1].

Corollary D.2.7. *Let X be a metric space homeomorphic to a subset of $\mathbb{R}^2$. Let $\{K_i\}_{i\in I}$ be a collection of pairwise disjoint continua in X with $\mathcal{H}^1(K_i) < \infty$ for each $i \in I$. Then, with the exception of countably many $i \in I$, each K_i is a Jordan arc or a Jordan curve and there exists a Lipschitz parametrization $\gamma\colon [a_i, b_i] \to K_i$ that is injective in $[a_i, b_i)$.*

Lemma D.2.8. *Let X be a topological space homeomorphic to $\mathbb{S}^2$ or to a closed disk. Let $K \subset X$ be a compact set separating two points $a, b \in X$. Then there exists a connected component of K that also separates a and b.*

In $\mathbb{S}^2$ this is a consequence of [Wil63, Lemma II.5.20, p. 61]. For topological disks the conclusion follows from [LW18a, Lemma 7.1].

Throughout the paper $\operatorname{int}(X)$ denotes the manifold interior of a surface X. The topological interior of a set A in a topological space is denoted by $\operatorname{int}_\top(A)$. Similar notation is adopted for the notion of boundary.

D.2.3. Metric Sobolev spaces

Let X be a metric space and Γ be a family of curves in X. A Borel function $\rho\colon X \to [0,\infty]$ is *admissible* for Γ if $\int_\gamma \rho\, ds \geq 1$ for all rectifiable paths $\gamma \in \Gamma$. We define the 2-*modulus* of Γ as

$$\operatorname{mod}\Gamma = \inf_\rho \int_X \rho^2\, d\mathcal{H}^2,$$

where the infimum is taken over all admissible functions ρ for Γ. By convention, $\operatorname{mod}\Gamma = \infty$ if there are no admissible functions for Γ. Observe that we consider X to be equipped with the Hausdorff 2-measure. This definition may be generalized by allowing for an exponent different from 2 or a different measure, though this generality is not needed for this paper.

Let $h\colon X \to Y$ be a map between metric spaces. We say that a Borel function $g\colon X \to [0,\infty]$ is an *upper gradient* of h if

$$d_Y(h(x), h(y)) \leq \int_\gamma g\, ds \tag{D.1}$$

for all $x, y \in X$ and every rectifiable path γ in X joining x and y. This is called the *upper gradient inequality.* If, instead the above inequality holds for all curves γ outside a curve family of 2-modulus zero, then we say that g is a *(*2*-)weak upper gradient* of h. In this case, there exists a curve family Γ_0 with $\operatorname{mod}\Gamma_0 = 0$ such that all paths outside Γ_0 and all subpaths of such paths satisfy the upper gradient inequality.

We equip the space X with the Hausdorff 2-measure $\mathcal{H}^2$. Let $L^2(X)$ denote the space of 2-integrable Borel functions from X to the extended real line $\widehat{\mathbb{R}}$, where two functions are identified if they agree $\mathcal{H}^2$-almost everywhere. The Sobolev space $N^{1,2}(X, Y)$ is defined as the space of Borel maps $h\colon X \to Y$ with a 2-weak upper gradient g in $L^2(X)$ such that the function $x \mapsto d_Y(y, h(x))$ is in $L^2(X)$ for some $y \in Y$. If $Y = \mathbb{R}$, we simply write $N^{1,2}(X)$. The spaces $L^2_{\text{loc}}(X)$ and $N^{1,2}_{\text{loc}}(X, Y)$ are defined in the obvious manner. Each map $h \in N^{1,2}_{\text{loc}}(X, Y)$ has a *minimal* 2-weak upper gradient g_h, in the sense that for any other 2-weak upper gradient g we have $g_h \leq g$ a.e. See the monograph [HKST15] for background on metric Sobolev spaces.

We state a consequence of the coarea inequality for Lipschitz functions.

Lemma D.2.9 ([EIR23, Lemma 2.13])**.** *Let X be a metric surface of finite Hausdorff* 2*-measure and $u\colon X \to \mathbb{R}$ be a Lipschitz function. If Γ_0 is a curve family in X with $\operatorname{mod}\Gamma_0 = 0$, then for a.e. $t \in \mathbb{R}$, every Lipschitz curve $\gamma\colon [a,b] \to u^{-1}(t)$ that is injective on $[a,b)$ lies outside Γ_0.*

D.2.4. Quasiconformal maps

Let X, Y be metric surfaces of locally finite Hausdorff 2-measure. Recall that a homeomorphism $h\colon X \to Y$ is quasiconformal if there exists $K \geq 1$ such that

$$K^{-1} \operatorname{mod}\Gamma \leq \operatorname{mod} h(\Gamma) \leq K \operatorname{mod}\Gamma$$

for every curve family Γ in X. A continuous map between topological spaces is *cell-like* if the preimage of each point is a continuum that is contractible in each of its open neighborhoods. A continuous, surjective, proper, and cell-like map $h\colon X \to Y$ is *weakly quasiconformal* if there exists $K > 0$ such that for every curve family Γ in X we have

$$\operatorname{mod}\Gamma \leq K \operatorname{mod} h(\Gamma).$$

In this case, we say that h is weakly K-quasiconformal.

If X and Y are compact surfaces that are homeomorphic to each other, then we may replace cell-likeness with the weaker requirement that h is monotone; that is, the preimage of every point is a continuum. In that case, continuous, surjective, and monotone maps from X to Y coincide with uniform limits of homeomorphisms; see [NR24, Theorem 6.3] and the references therein. Alternatively, if X, Y have empty boundary, then continuous, proper, and cell-like maps from X to Y also coincide with uniform limits of homeomorphisms, see [Dav86, Corollary 25.1A].

We note that a weakly K-quasiconformal map between planar domains is a K-quasi-conformal homeomorphism. Indeed, by [NR23, Theorem 7.4], such a map is a homeomorphism. Also, note that a quasiconformal homeomorphism between planar domains is a priori required to satisfy only one modulus inequality, as in the definition of a weakly quasiconformal map; see [LV73, Section I.3].

The next theorem of Williams ([Wil12, Theorem 1.1 and Corollary 3.9]) relates the above definitions of quasiconformality with the "analytic" definition that relies on upper gradients; see also [NR23, Section 2.4].

Theorem D.2.10 (Definitions of quasiconformality)**.** *Let X, Y be metric surfaces of locally finite Hausdorff 2-measure, $h\colon X \to Y$ be a continuous map, and $K > 0$. The following are equivalent.*

(i) *$h \in N^{1,2}_{\mathrm{loc}}(X, Y)$ and there exists a 2-weak upper gradient g of h such that for every Borel set $E \subset Y$ we have*
$$\int_{h^{-1}(E)} g^2\, d\mathcal{H}^2 \leq K\mathcal{H}^2(E).$$

(ii) *Each point of X has a neighborhood U such that $h|_U \in N^{1,2}(U, Y)$ and there exists a 2-weak upper gradient g_U of $h|_U$ such that for every Borel set $E \subset Y$ we have*
$$\int_{(h|_U)^{-1}(E)} g_U^2\, d\mathcal{H}^2 \leq K\mathcal{H}^2(E).$$

(iii) *For every curve family Γ in X we have*
$$\operatorname{mod}\Gamma \leq K \operatorname{mod} h(\Gamma).$$

Theorem D.2.11 ([NR23, Theorem 7.1 and Remark 7.2])**.** *Let X, Y be metric surfaces of locally finite Hausdorff 2-measure and $h\colon X \to Y$ be a weakly K-quasiconformal map for some $K > 0$.*

(1) *The set function $\nu(E) = \mathcal{H}^2(h(E))$ is a locally finite Borel measure on X. Moreover, for a.e. $x \in X$ we have*
$$g_h(x)^2 \leq KJ_h(x), \quad \textit{where } J_h = \frac{d\nu}{d\mathcal{H}^2}.$$

(2) *$N(h, y) = 1$ for a.e. $y \in Y$.*

Recall that $N(h, y)$ denotes the number of preimages of y under h.

D.2.5. Reciprocal surfaces

Let X be a metric surface of locally finite Hausdorff 2-measure. For a set $G \subset X$ and disjoint sets $E, F \subset G$ we define $\Gamma(E, F; G)$ to be the family of curves in G joining E and F. A *quadrilateral* in X is a closed Jordan region Q together with a partition of ∂Q into four non-overlapping edges $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \subset \partial Q$ in cyclic order. When we refer to a quadrilateral Q, it will be implicitly understood that there exists such a marking on its boundary. We define $\Gamma(Q) = \Gamma(\zeta_1, \zeta_3; Q)$ and $\Gamma^*(Q) = \Gamma(\zeta_2, \zeta_4; Q)$. According to the definition of Rajala [Raj17], the metric surface X is reciprocal if there exist constants $\kappa, \kappa' \geq 1$ such that

$$\kappa^{-1} \leq \operatorname{mod}\Gamma(Q) \cdot \operatorname{mod}\Gamma^*(Q) \leq \kappa' \quad \text{for each quadrilateral } Q \subset X \tag{D.2}$$

and

$$\lim_{r\to 0} \operatorname{mod}\Gamma(\overline{B}(a, r), X \setminus B(a, R); X) = 0 \quad \text{for each ball } B(a, R). \tag{D.3}$$

By work of Rajala and Romney [RR19] it is now known that the lower bound in (D.2) is always satisfied for some uniform constant κ. In fact, the optimal constant was shown to be $\kappa = (4/\pi)^2$

[EBPC22]. Moreover, (D.3) follows from the upper bound in (D.2), as was shown by Romney and the second-named author [NR24]. Therefore, we may only require the upper inequality of (D.2) in the definition of a reciprocal surface.

Rajala [Raj17] proved that a metric surface X of locally finite Hausdorff 2-measure that is homeomorphic to $\mathbb{R}^2$ is 2-quasiconformally equivalent to an open subset of $\mathbb{R}^2$ if and only if X is reciprocal. This result was generalized to all metric surfaces (with or without boundary) of locally finite Hausdorff 2-measure, where $\mathbb{R}^2$ is replaced with a Riemannian surface [Iko22, NR24].

More generally, it was shown in [NR24] that any metric surface of locally finite Hausdorff 2-measure admits a weakly quasiconformal parametrization by a Riemannian surface of the same topological type. The following special case is sufficient for our purposes.

Theorem D.2.12 ([NR24, Theorem 1.2])**.** *Let X be a metric surface of finite Hausdorff 2-measure that is homeomorphic to a topological closed disk. Then there exists a weakly $(4/\pi)$-quasiconformal map from $\overline{\mathbb{D}}$ onto X.*

Here $\mathbb{D}$ denotes the open unit disk in the plane. We show that weakly quasiconformal maps can be upgraded to quasiconformal homeomorphisms under certain conditions.

Lemma D.2.13. *Let X, Y be metric surfaces without boundary and with locally finite Hausdorff 2-measure such that Y is reciprocal. Then every weakly quasiconformal map $f\colon X \to Y$ is a quasiconformal homeomorphism, quantitatively.*

Proof. Let $f\colon X \to Y$ be a weakly K-quasiconformal map for some $K > 0$. Since Y is reciprocal, condition (D.3) implies that the modulus of the family of non-constant curves passing through any point of Y is zero. By [NR23, Theorem 7.4] we conclude that f is a homeomorphism. Now, the reciprocity of Y implies that the upper bound in (D.2) is satisfied for X as well. Therefore, X is reciprocal.

Consider a domain $V' \subset Y$ that is homeomorphic to $\mathbb{R}^2$. By Rajala's theorem, there exists a 2-quasiconformal homeomorphism ϕ from V' onto a domain $V \subset \mathbb{R}^2$. The set $U' = f^{-1}(V')$ is homeomorphic to $\mathbb{R}^2$, so by Rajala's theorem there exists a 2-quasiconformal homeomorphism ψ from U' onto a domain $U \subset \mathbb{R}^2$. The composition $g = \phi \circ f \circ \psi^{-1}$ is a weakly $4K$-quasiconformal map from U onto V. Since the domains are planar, g is a $4K$-quasiconformal homeomorphism. Therefore, f is a $16K$-quasiconformal homeomorphism from U' onto V'. By Theorem D.2.10, quasiconformality is a local condition, so $f\colon X \to Y$ is $16K$-quasiconformal. □

D.2.6. Metric differentiability

Throughout the section we let $U \subset \mathbb{R}^2$ be a domain and Y be a metric space. We say that a map $h\colon U \to Y$ is *approximately metrically differentiable* at a point $x \in U$ if there exists a seminorm N_x on $\mathbb{R}^2$ for which

$$\operatorname{ap}\lim_{y \to x} \frac{d(h(y), h(x)) - N_x(y - x)}{y - x} = 0.$$

Here, $\operatorname{ap}\lim$ denotes the approximate limit as defined in [EG92, Section 1.7.2]. In this case, the seminorm N_x is unique, is denoted by $\operatorname{ap}\operatorname{md} h_x$, and we call it the *approximate metric derivative* of h at x.

Proposition D.2.14 ([LW17a, Proposition 4.3])**.** *If $h \in N^{1,2}(U, Y)$ then there exist countably many pairwise disjoint compact sets $K_i \subset U$, $i \in \mathbb{N}$, such that $\mathcal{H}^2(U \setminus \bigcup_{i \in \mathbb{N}} K_i) = 0$ with the following property. For every $i \in \mathbb{N}$ and every $\varepsilon > 0$ there exists $r_i(\varepsilon) > 0$ such that h is approximately metrically differentiable at every $x \in K_i$ and*

$$|d(h(x), h(x + v)) - \operatorname{ap}\operatorname{md} h_x(v)| \leq \varepsilon |v|$$

for all $x \in K_i$ *and all* $v \in \mathbb{R}^2$ *with* $|v| \leq r_i(\varepsilon)$ *and* $x + v \in K_i$.

In particular, every map $h \in N^{1,2}(U, Y)$ is approximately metrically differentiable at a.e. $x \in U$.

Lemma D.2.15 ([LW18a, Lemma 3.1])**.** *If* $h \in N^{1,2}(U, Y)$ *then*

$$\ell(h \circ \gamma) = \int_a^b \operatorname{ap}\operatorname{md} h_{\gamma(t)}(\dot{\gamma}(t))\, dt$$

for every curve $\gamma \colon [a, b] \to U$ *parametrized by arclength outside a family* Γ_0 *with* $\operatorname{mod}\Gamma_0 = 0$.

Lemma D.2.16. *If* $h \in N^{1,2}(U, X)$ *then the function* $L \colon U \to [0, \infty]$ *defined by*

$$L(x) = \max\{\operatorname{ap}\operatorname{md} h_x(v) : |v| = 1\}$$

is a representative of the minimal 2-weak upper gradient of h.

Proof. It is an immediate consequence of Lemma D.2.15 that L is a 2-weak upper gradient of h. It remains to show that if g is an upper gradient of h in $L^2(U)$, then $L(x) \leq g(x)$ for a.e. $x \in U$; this will imply that the same conclusion is true for the minimal 2-weak upper gradient. Let $g \in L^2(U)$ be an upper gradient of h. It can be deduced from Fubini's theorem that for each $v \in \mathbb{S}^1$ and for a.e. $x \in U$ we have

$$g(x) = \lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta g(x + tv)\, dt = \lim_{\delta \to 0} \frac{1}{\delta} \int_{\gamma_v|_{[0,\delta]}} g\, ds, \tag{D.4}$$

where $\gamma_v \colon [0, 1] \to \mathbb{R}^2$ is the curve $\gamma_v(t) = x + tv$. Consider a set K_i as in Proposition D.2.14. An application of Fubini's theorem shows that for each $v \in \mathbb{S}^1$ and for a.e. $x \in K_i$ we have $x + \delta v \in K_i$ for arbitrarily small values of $\delta > 0$. Let $\varepsilon > 0$, $v \in \mathbb{S}^1$, and $x \in K_i$ such that (D.4) is true and $x + \delta_n v \in K_i$ for a sequence $\delta_n \to 0$. By Proposition D.2.14, whenever $|\delta_n v| \leq r_i(\varepsilon)$, we have

$$\operatorname{ap}\operatorname{md} h_x(v) \leq \frac{1}{\delta_n} d(h(x), h(x + \delta_n v)) + \varepsilon|v| \leq \frac{1}{\delta_n} \int_{\gamma_v|_{[0,\delta_n]}} g + \varepsilon.$$

We let $n \to \infty$ and then $\varepsilon \to 0$ to obtain $\operatorname{ap}\operatorname{md} h_x(v) \leq g(x)$. Since this is true for every $v \in \mathbb{S}^1$, we obtain $L(x) \leq g(x)$ for a.e. $x \in K_i$. The sets K_i, $i \in \mathbb{N}$, cover U up to a set of measure zero, so the conclusion follows. □

Before providing the definition of the Jacobian of a Sobolev map, we state the following version of John's Theorem; see [Bal97, Theorem 3.1].

Theorem D.2.17 (John's Theorem)**.** *Each symmetric convex body* $K \subset \mathbb{R}^2$ *contains a unique ellipse* E *of maximal area, called the John ellipse of* K*. Moreover,*

$$E \subset K \subset \sqrt{2}E.$$

If $\operatorname{ap}\operatorname{md} h_x$ is a norm, let $B_x = \{y \in \mathbb{R}^2 : \operatorname{ap}\operatorname{md} h_x(y) \leq 1\}$ be the closed unit ball in $(\mathbb{R}^2, \operatorname{ap}\operatorname{md} h_x)$. The *Jacobian* of $\operatorname{ap}\operatorname{md} h_x$ is defined to be $J(\operatorname{ap}\operatorname{md} h_x) = \pi/|B_x|$, where $|B_x|$ is the Lebesgue measure of B_x. Since B_x is a symmetric convex body, by John's theorem there exists a unique ellipse $E_x \subset B_x$ of maximal area. When $\operatorname{ap}\operatorname{md} h_x$ is not a norm, the closed unit ball B_x has infinite area and we define $J(\operatorname{ap}\operatorname{md} h_x) = 0$.

Theorem D.2.18 (Area formula). *If $h \in N^{1,2}(U, Y)$, then there exists a set $G_0 \subset U$ with $\mathcal{H}^2(G_0) = 0$ such that for every measurable set $A \subset U \setminus G_0$ we have*

$$\int_A J(\operatorname{ap}\operatorname{md} h_x)\, d\mathcal{H}^2 = \int_Y N(h|_A, y)\, d\mathcal{H}^2.$$

Proof. It is a consequence of [HKST15, Theorem 8.1.49] that U can be covered up to a set of measure zero by countably many disjoint measurable sets G_j, $j \in \mathbb{N}$, such that $h|_{G_j}$ is Lipschitz. This implies that outside a set of measure zero $G_0 \subset U$, h satisfies Lusin's condition (N). The statement now follows from [Kar07, Theorem 3.2]. □

Lemma D.2.19. *Let Y be a metric surface of locally finite Hausdorff 2-measure and $h \colon U \to Y$ be a weakly K-quasiconformal map for some $K > 0$. Then*

$$J(\operatorname{ap}\operatorname{md} h_x) \leq \max\{(\operatorname{ap}\operatorname{md} h_x(v))^2 : |v| = 1\} \leq K J(\operatorname{ap}\operatorname{md} h_x)$$

for a.e. $x \in U$. In particular, for a.e. $x \in U$ we have $J(\operatorname{ap}\operatorname{md} h_x) = 0$ if and only if $\operatorname{ap}\operatorname{md} h_x \equiv 0$.

Proof. By Theorem D.2.10, $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$, so h is approximately metrically differentiable at a.e. $x \in U$. We set $N_x = \operatorname{ap}\operatorname{md} h_x$ and $J_x = J(\operatorname{ap}\operatorname{md} h_x)$ for a.e. $x \in U$. By Lemma D.2.16, the quantity $L_x = \max\{N_x(v) : |v| = 1\}$ is a representative of the minimal 2-weak upper gradient of h, so $L_x = g_h(x)$ for a.e. $x \in U$. By the area formula of Theorem D.2.18, there exists a set $G_0 \subset U$ of measure zero such that for each measurable set $A \subset U \setminus G_0$ we have

$$\int_A J_x = \int_Y N(h|_A, y)\, d\mathcal{H}^2 = \mathcal{H}^2(h(A)),$$

where the latter equality follows from Theorem D.2.11. This implies that J_x is the Radon–Nikodym derivative of the measure $A \mapsto \mathcal{H}^2(h(A))$, so $J_x = J_h(x)$ for a.e. $x \in U$, again by Theorem D.2.11. Finally, since $g_h(x)^2 \leq K J_h(x)$, we conclude that $L_x^2 \leq K J_x$ for a.e. $x \in U$. The inequality $J_x \leq L_x^2$ follows by the fact that the unit ball $B_x = \{y \in \mathbb{R}^2 : N_x(y) \leq 1\}$ contains a Euclidean ball of radius $1/L_x$. □

Remark D.2.20. It is a consequence of Lemma D.2.19 that if f is a weakly K-quasi-conformal map from a planar (or Riemannian) domain U onto a metric surface Y, then we necessarily have $K \geq 1$. It is unclear how to show this for maps between arbitrary metric surfaces.

D.3. Proof of main theorem

This section is devoted to the proof of Theorem D.1.4. Throughout the section we assume that X, Y are metric surfaces without boundary and with locally finite Hausdorff 2-measure.

D.3.1. Preservation of length

In this section we establish Theorem D.1.4 (1).

Lemma D.3.1. *Let $f \colon X \to Y$ be a map that is area-preserving and L-Lipschitz for some $L > 0$. Then $\operatorname{mod}\Gamma \leq L^2 \operatorname{mod} f(\Gamma)$ for each curve family Γ in X.*

Proof. Since f is L-Lipschitz, the constant function L is an upper gradient of f. Moreover, for every Borel set $A \subset Y$ we have

$$\int_{f^{-1}(A)} L^2\, d\mathcal{H}^2 = L^2 \mathcal{H}^2(f^{-1}(A)) = L^2 \mathcal{H}^2(f(f^{-1}(A))) \leq L^2 \mathcal{H}^2(A).$$

The conclusion now follows from Theorem D.2.10. □

Lemma D.3.2. *Let $f\colon X \to Y$ be a map that is area-preserving and continuous. Then $N(f,y) = 1$ for a.e. $y \in f(X)$.*

Proof. For each Borel set $A \subset X$ the set $f(A)$ is analytic [Kec95, Proposition 14.4] and thus $\mathcal{H}^2$-measurable [Kec95, Theorem 29.7]. Define $\zeta(S) = \mathcal{H}^2(f(S))$, where $S \subset X$ is a Borel set. By assumption, $\zeta(S) = \mathcal{H}^2(S)$. The measure on X resulting by Carathéodory's construction from ζ is precisely $\mathcal{H}^2$. By Theorem D.2.2, for each Borel set $A \subset X$ we have

$$\mathcal{H}^2(A) = \int_Y N(f|_A, y)\, d\mathcal{H}^2.$$

In particular, since f is area-preserving we have

$$\mathcal{H}^2(A) = \int_{f(A)} N(f|_A, y)\, d\mathcal{H}^2 \geq \mathcal{H}^2(f(A)) = \mathcal{H}^2(A).$$

If $\mathcal{H}^2(A) < \infty$, we conclude that $N(f|_A, y) = 1$ for a.e. $y \in f(A)$. Since X has σ-finite Hausdorff 2-measure, we have $N(f,y) = 1$ for a.e. $y \in f(X)$. □

Lemma D.3.3. *Let $U \subset \mathbb{R}^2$ be a domain and $\phi\colon U \to X$ be a weakly quasiconformal map. Let $f\colon X \to Y$ be a map that is area-preserving and L-Lipschitz for some $L > 0$. Then there exists a constant $C(L) > 0$ such that*

$$C(L)\ell(\phi \circ \beta) \leq \ell(f \circ \phi \circ \beta) \leq L\ell(\phi \circ \beta)$$

for all curves β in U outside a curve family Γ_0 with $\operatorname{mod} \Gamma_0 = 0$. Moreover, if $L = 1$, then we can choose $C(1) = 1$.

Proof. By Lemma D.3.1 and the weak quasiconformality of ϕ, there exists a constant $K \geq 1$ such that for each curve family Γ in U we have

$$\operatorname{mod} \Gamma \leq K \operatorname{mod} f(\phi(\Gamma)).$$

By Theorem D.2.10, $f \circ \phi \in N^{1,2}_{\mathrm{loc}}(U, Y)$ and $\phi \in N^{1,2}_{\mathrm{loc}}(U, X)$. In particular, both maps are approximately metrically differentiable almost everywhere.

Set $N_x = \operatorname{ap\,md} \phi_x$ and $\widetilde{N}_x = \operatorname{ap\,md}(f \circ \phi)_x$ for a.e. $x \in U$. We use the notation B_x, $\widetilde{B}_x$ for the corresponding unit balls, and J_x, $\widetilde{J}_x$ for the corresponding Jacobians. By Lemma D.2.15 we have

$$\ell(\phi \circ \beta) = \int_a^b N_{\beta(t)}(\dot\beta(t))\, dt \tag{D.5}$$

for every curve $\beta\colon [a,b] \to U$ parametrized by arclength outside a family Γ_1 with $\operatorname{mod} \Gamma_1 = 0$. Analogously, we get

$$\ell((f \circ \phi) \circ \beta) = \int_a^b \widetilde{N}_{\beta(t)}(\dot\beta(t))\, dt \tag{D.6}$$

for every curve $\beta\colon [a,b] \to U$ parametrized by arclength outside a family Γ_2 with $\operatorname{mod} \Gamma_2 = 0$.

Next, we claim that for a.e. $x \in U$ and all $v \in \mathbb{R}^2$ we have,

$$C(L)N_x(v) \leq \widetilde{N}_x(v) \leq LN_x(v)$$

for some constant $C(L) > 0$ with $C(1) = 1$. This implies that there exists a curve family Γ_3 in U with $\operatorname{mod}\Gamma_3 = 0$ such that for all curves $\beta\colon [a,b] \to U$ parametrized by arclength that are outside Γ_3 we have

$$C(L)\int_a^b N_{\beta(t)}(\dot\beta(t))\,dt \le \int_a^b \widetilde{N}_{\beta(t)}(\dot\beta(t))\,dt \le L\int_a^b N_{\beta(t)}(\dot\beta(t))\,dt. \tag{D.7}$$

Let Γ_0 be the family of curves that have a reparametrization in $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Then $\operatorname{mod}\Gamma_0 = 0$. By combining (D.5), (D.6), and (D.7), we see that the conclusions of the lemma are true for the family Γ_0.

Now, we prove the claim. Theorem D.2.18 applied to ϕ provides a set of measure zero $G_1 \subset U$ such that for any measurable set $A \subset U \setminus G_1$ we have

$$\int_A J_x = \int_X N(\phi|_A, x)\,d\mathcal{H}^2 = \mathcal{H}(\phi(A)), \tag{D.8}$$

where the last equality follows from Theorem D.2.11. Similarly, there exists a set $G_2 \subset U$ of measure zero such that for any measurable set $A \subset U \setminus G_2$,

$$\int_A \widetilde{J}_x = \int_Y N(f\circ\phi|_A, y)\,d\mathcal{H}^2.$$

From Lemma D.3.2 we know that $N(f,y) = 1$ for a.e. $y \in f(X)$. By Theorem D.2.11, for a.e. $x \in X$, $\phi^{-1}(x)$ is a singleton. Since f is area-preserving and in particular has the Lusin (N) property, we conclude that for a.e. $y \in f(X)$ the set $\phi^{-1}(x)$ is a singleton whenever $f(x) = y$. In summary, $N(f\circ\phi|_A, y) = 1$ for a.e. $y \in f(\phi(A))$. In particular, for any measurable set $A \subset U \setminus G_2$,

$$\int_A \widetilde{J}_x = \int_Y N(f\circ\phi|_A, y)\,d\mathcal{H}^2 = \mathcal{H}(f(\phi(A))).$$

The area-preserving property of f and (D.8) now imply that $J_x = \widetilde{J}_x$ for a.e. $x \in U$ and hence

$$|B_x| = |\widetilde{B}_x| \tag{D.9}$$

for a.e. $x \in U$. This equality implies that N_x is not a norm if and only if $\widetilde{N}_x$ is also not a norm. By Lemma D.2.19, if N_x is not a norm, then $N_x \equiv 0$.

Let $K_i, \widetilde{K}_j \subset U$, $i,j \in \mathbb{N}$, be the sets from Proposition D.2.14 applied to $\phi, f\circ\phi$, respectively. Let $\varepsilon > 0$. The Lipschitz property of f implies that

$$\widetilde{N}_x(v) \le LN_x(v) + (1+L)\varepsilon|v|$$

for every $x \in K_{i,j} = K_i \cap \widetilde{K}_j$ and every $v \in \mathbb{R}^2$ with $|v| \le \min\{r_i(\varepsilon), \widetilde{r}_j(\varepsilon)\}$ and $x + v \in K_{i,j}$. This shows that

$$\widetilde{N}_x(v) \le LN_x(v) \quad \text{and thus} \quad B_x \subset L\widetilde{B}_x \tag{D.10}$$

for a.e. $x \in U$ and all $v \in \mathbb{R}^2$. Here $L\widetilde{B}_x$ denotes the closed ball $\{y \in \mathbb{R}^2 : \widetilde{N}_x(y) \le L\}$. In particular, if N_x is not a norm, then $\widetilde{N}_x \equiv N_x \equiv 0$.

If $L = 1$, then (D.10) implies that $B_x \subset \widetilde{B}_x$ for a.e. $x \in U$. By (D.9), we have $B_x = \widetilde{B}_x$ for a.e. $x \in U$, since N_x and $\widetilde{N}_x$ are either both norms or vanish identically. Hence, $N_x(v) = \widetilde{N}_x(v)$ for a.e. $x \in U$ and all $v \in \mathbb{R}^2$.

Denote by $E_x, \widetilde{E}_x$ the John ellipse of $B_x, \widetilde{B}_x$, respectively, whenever N_x and $\widetilde{N}_x$ are norms.

John's Theorem (Theorem D.2.17) implies that

$$E_x \subset B_x \subset \sqrt{2}E_x \quad \text{and} \quad \widetilde{E}_x \subset \widetilde{B}_x \subset \sqrt{2}\widetilde{E}_x. \tag{D.11}$$

Denote by $a_x, \widetilde{a}_x$ (resp. $b_x, \widetilde{b}_x$) the length of the major (resp. minor) axis of $E_x, \widetilde{E}_x$, respectively. By (D.10) and (D.11) we have that

$$L^{-1}E_x \subset L^{-1}B_x \subset \widetilde{B}_x \subset \sqrt{2}\widetilde{E}_x,$$

which implies that $b_x \leq \sqrt{2}L\widetilde{b}_x$. Moreover, combining (D.9) and (D.11) gives

$$|\widetilde{E}_x| \leq |\widetilde{B}_x| = |B_x| \leq 2|E_x|.$$

Since $|E_x| = \pi a_x b_x$ and $|\widetilde{E}_x| = \pi \widetilde{a}_x \widetilde{b}_x$, we get

$$\widetilde{a}_x \leq 2\,\frac{a_x b_x}{\widetilde{b}_x} \leq 2\sqrt{2}La_x.$$

In particular, if we assume in addition that E_x is a geometric ball, then $\widetilde{E}_x \subset 2\sqrt{2}LE_x$. All in all we obtain that

$$L^{-1}B_x \subset \widetilde{B}_x \subset \sqrt{2}\widetilde{E}_x \subset 4LE_x \subset 4LB_x, \tag{D.12}$$

with the additional assumption that E_x is a geometric ball. Note that (D.12) shows that the claim holds for $C(L) = (4L)^{-1}$.

For the general case that E_x is not a geometric ball, we consider a linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(E_x)$ is a round ball. Note that (D.9) remains true for the images of $B_x, \widetilde{B}_x$ under T. Since the John ellipse is preserved under linear maps, the above calculations are true for the images of the corresponding sets under T, and thus one obtains the inclusions (D.12) for the images. Therefore, the inclusions also hold for the original sets. □

Proof of Theorem D.1.4 (1). We cover X with a countable collection of open sets $\{X_n\}_{n\in\mathbb{N}}$, each homeomorphic to $\mathbb{R}^2$. Every X_n is reciprocal and, by Rajala's theorem [Raj17], there exists a quasiconformal homeomorphism $\phi_n\colon U_n \to X_n$, where $U_n \subset \mathbb{R}^2$ is an open set. By Lemma D.3.3,

$$C(L)\ell(\phi_n \circ \beta) \leq \ell(f \circ \phi_n \circ \beta) \leq L\ell(\phi_n \circ \beta)$$

holds for every curve β in U_n outside a curve family Γ_n with $\operatorname{mod}\Gamma_n = 0$, where $C(L) > 0$ is some constant with $C(1) = 1$. Since ϕ_n is quasiconformal, $\operatorname{mod}\phi_n(\Gamma_n) = 0$ for each $n \in \mathbb{N}$. Note that if γ is a curve in X_n outside $\phi_n(\Gamma_n)$, then after setting $\beta = \phi_n^{-1} \circ \gamma$ we see that the statement of Theorem D.1.4 (1) holds for γ. We define Γ_0 to be the family of curves in X that have a subcurve in some $\phi_n(\Gamma_n)$, $n \in \mathbb{N}$. Then $\operatorname{mod}\Gamma_0 = 0$ and the conclusions of Theorem D.1.4 (1) hold for all curves γ in X outside Γ_0. □

D.3.2. Injectivity

In this section we establish Theorem D.1.4 (2). The main difficulty is to establish the injectivity of f. A map $f\colon X \to Y$ is *light* if $f^{-1}(y)$ is totally disconnected for each $y \in Y$.

Lemma D.3.4. *Suppose that Y is reciprocal. Let $f\colon X \to Y$ be a non-constant continuous map such that there exists $K > 0$ with the property that $\operatorname{mod}\Gamma \leq K \operatorname{mod} f(\Gamma)$ for each curve family Γ in X. Then f is a light map.*

Proof. Let $y \in Y$ and suppose that $f^{-1}(y)$ contains a non-degenerate continuum E. Consider a non-degenerate continuum $F \subset X \setminus f^{-1}(y)$; note that the latter set is non-empty because f is non-constant. The family Γ of curves joining E and F has positive modulus [Raj17, Proposition 3.5]. On the other hand, each curve of $f(\Gamma)$ joins the continuum $f(F)$ to y. Since Y is reciprocal, we have $\operatorname{mod} f(\Gamma) = 0$ (see (D.3)). This is a contradiction. $\square$

For $y_0 \in Y$ and $r > 0$ we denote by $S(y_0, r)$ the set $\{y \in Y : d(y, y_0) = r\}$.

Lemma D.3.5. *Let $y_0 \in Y$ and $K \subset Y \setminus \{y_0\}$ be a closed set. There exists $\delta > 0$ such that for a.e. $r \in (0, \delta)$ there exists a component $E \subset S(y_0, r)$ that is a rectifiable Jordan curve separating y_0 and K.*

Proof. Let $U \subset Y$ be the interior of a topological closed disk $\overline{U} \subset Y$ such that $Y \setminus U$ is connected, $y_0 \in U$, and $K \subset Y \setminus U$. Note that $\mathcal{H}^2(\overline{U}) < \infty$. Let $\delta > 0$ be sufficiently small such that $\overline{B}(y_0, \delta) \subset U$. Then for all $r \in (0, \delta)$ the set $S(y_0, r)$ is compact. By the coarea inequality for Lipschitz functions (Theorem D.2.1), $\mathcal{H}^1(S(y_0, r)) < \infty$ for a.e. $r \in (0, \delta)$. By Corollary D.2.7 (see also [Nta20, Theorem 1.5]), for a.e. $r \in (0, \delta)$, each component of $S(y_0, r)$ is a rectifiable Jordan arc or Jordan curve. Fix such a parameter r. Since $S(y_0, r)$ separates y_0 from all points of ∂U, by Lemma D.2.8 there exists a component E of $S(y_0, r)$ that separates y_0 from ∂U. In particular, E must be a Jordan curve and separates y_0 from K. $\square$

Lemma D.3.6. *Let $Z \subset X$ be homeomorphic to a topological closed disk and let $f : Z \to Y$ be a continuous map in $N^{1,2}(Z, Y)$. For every $y_0 \in Y$ and for a.e. $r \in (0, \infty)$, each component of $f^{-1}(S(y_0, r))$ is a Jordan arc or a Jordan curve.*

Proof. Define $u(x) = d(f(x), y_0)$ on Z, which is continuous and lies in $N^{1,2}(Z)$. Observe that $u^{-1}(r) = f^{-1}(S(y_0, r))$ for every $r > 0$. By the coarea inequality of Theorem D.1.6 we see that $\mathcal{H}^1(u^{-1}(r) \cap \mathcal{A}_u) < \infty$ for a.e. $r > 0$. In particular, for such values r, if E is a non-degenerate component of $u^{-1}(r)$, then $E \subset \mathcal{A}_u$, so $\mathcal{H}^1(E) < \infty$. By Corollary D.2.7, for a.e. $r > 0$, every non-degenerate component of $u^{-1}(r)$ is a Jordan arc or a Jordan curve. $\square$

Lemma D.3.7. *Let $f : X \to Y$ be a continuous light map in $N^{1,2}(X, Y)$ such that $N(f, y) \leq 1$ for a.e. $y \in Y$. Then $N(f, y) \leq 1$ for every $y \in Y$. In particular, f is injective.*

Proof. Let $y \in f(X)$ and $x \in f^{-1}(y)$. For the moment, we consider the restriction $g = f|_Z$ to a compact neighborhood Z of x that is homeomorphic to a closed disk and contains x in its interior. Since g is light, it is non-constant on $\operatorname{int}(Z)$ and there exists a point $z \in \operatorname{int}(Z) \setminus g^{-1}(y)$. Note that for each $r \in (0, d(y, g(z)))$ the set $S(y, r)$ separates y from $g(z)$. Therefore, the compact set $g^{-1}(S(y, r))$ separates x from z. By Lemma D.3.6, for a.e. $r > 0$, each component of $g^{-1}(S(y, r))$ is a Jordan arc or a Jordan curve. Combining these facts with Lemma D.2.8, we see that there exists a full measure subset I of $(0, d(y, g(z)))$ such that for each $r \in I$, there exists a component of $g^{-1}(S(y, r))$ that separates x from z and is a Jordan arc or a Jordan curve.

We claim for all sufficiently small $r \in I$, each such component must be a Jordan curve. To prove this, suppose that there exists a sequence of positive numbers $r_n \to 0$ and components F_{r_n} of $g^{-1}(S(y, r_n))$ that are Jordan arcs and separate x from z. Fix a continuum $K \subset \operatorname{int}(Z)$ connecting x and z. Since F_{r_n} separates x from z, it intersects K. Moreover, since F_{r_n} is a Jordan arc, it cannot be contained in $\operatorname{int}(Z)$, as $\operatorname{int}(Z) \setminus F_{r_n}$ would then be connected. Therefore, F_{r_n} intersects ∂Z and

$$\operatorname{diam}(F_{r_n}) \geq \operatorname{dist}(K, \partial Z) > 0$$

for all $n \in \mathbb{N}$. After passing to a subsequence, F_{r_n} converges in the Hausdorff sense to a non-degenerate continuum F. Since $r_n \to 0$, we have that $F \subset g^{-1}(y)$. This contradicts the lightness of g. The claim is proved.

By the assumption that $N(f,w) \leq 1$ for a.e. $w \in Y$ and the coarea inequality for Lipschitz functions (Theorem D.2.1), we see that for a.e. $r > 0$, $\mathcal{H}^1$-a.e. point of $S(y,r)$ has at most one preimage under f. Also, given a closed set $K \subset Y \setminus \{y\}$, by Lemma D.3.5, for a.e. sufficiently small $r > 0$ there exists a Jordan curve $E \subset S(y,r)$ separating y from K. Altogether, there exists $\delta' > 0$ and a set $I' \subset (0,\delta')$ of full measure so that for every $r \in I'$ the following statements are true.

1. $\mathcal{H}^1$-a.e. point of $S(y,r)$ has at most one preimage under f.
2. There exists a component of $S(y,r)$ that is a Jordan curve separating y and K.
3. Each component of $g^{-1}(S(y,r))$ that separates x and z is a Jordan curve.

Let E be a component of $S(y,r)$, $r \in I'$, that is a Jordan curve and let $F \subset g^{-1}(E)$ be a Jordan curve. We claim that $g(F) = E$. By (1), $\mathcal{H}^1$-a.e. point of E has at most one preimage under g. The map $g|_F$ is conjugate to a continuous map $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ with the property that a dense set of points of $\mathbb{S}^1$ have at most one preimage. Suppose that $g(F)$ is a strict subarc of E. Note that $g(F)$ cannot be a point since g is light. Then $\phi(\mathbb{S}^1)$ is a non-degenerate strict subarc of $\mathbb{S}^1$. This contradicts the fact that a dense set of points of $\mathbb{S}^1$ have at most one preimage. We have shown the following.

4. If E is a component of $S(y,r)$ that is a Jordan curve and $F \subset g^{-1}(E)$ is a Jordan curve, then $g(F) = E$.

We have completed our preparation to show the injectivity of f. Suppose that $f^{-1}(y)$ contains two points x_1, x_2 for some $y \in f(X)$. We consider disjoint topological closed disks $Z_1, Z_2 \subset X$ such that $x_i \in \operatorname{int}(Z_i)$, $i = 1,2$. We also fix $z_i \in \operatorname{int}(Z_i) \setminus f^{-1}(y)$. Consider the restrictions $g_i = f|_{Z_i}$, $i = 1,2$. By the previous, for $i = 1,2$, there exists a set I_i' of full measure in an interval $(0,\delta_i')$, such that (1)–(4) are true for the map g_i; specifically, in (2) we use the set $K = \{f(z_1), f(z_2)\}$. Let $I' = I_1' \cap I_2'$, which has full measure in $(0,\delta')$, where $\delta' = \min\{\delta_1', \delta_2'\}$. By (2), for $r \in I'$ there exists a component E of $S(y,r)$ that is a Jordan curve separating each of the pairs $(y, f(z_1))$ and $(y, f(z_2))$. Let F_i be a component of $g_i^{-1}(E)$ that separates x_i and z_i, $i = 1,2$; such components exist by Lemma D.2.8. Note that F_i is also a component of $g_i^{-1}(S(y,r))$. By (3), F_i is a Jordan curve for $i = 1,2$. By (4), we conclude that $g_i(F_i) = E$, $i = 1,2$. Thus, each point of E has at least two preimages under f. This contradicts (1). ☐

Lemma D.3.8. *Let $f\colon X \to Y$ be an area-preserving map that is a quasiconformal homeomorphism. Suppose that there exists $K \geq 1$ such that*

$$K^{-1/2}\ell(\gamma) \leq \ell(f \circ \gamma) \leq K^{1/2}\ell(\gamma)$$

for all curves γ in X outside a curve family Γ_0 with $\operatorname{mod}\Gamma_0 = 0$. Then f is K-quasiconformal.

Proof. The constant function $K^{1/2}$ is a 2-weak upper gradient of f and lies in $N^{1,2}_{\mathrm{loc}}(X)$. Moreover, by the preservation of area, for each Borel set $E \subset Y$ we have

$$\int_{f^{-1}(E)} K \, d\mathcal{H}^2 = K\mathcal{H}^2(f^{-1}(E)) = K\mathcal{H}^2(E).$$

In view of Theorem D.2.10, we derive that f is weakly K-quasiconformal. Since f is quasiconformal, we have

$$\ell(f^{-1} \circ \gamma) \leq K^{1/2}\ell(\gamma)$$

for all curves γ in Y outside a curve family Γ_0' with $\operatorname{mod}\Gamma_0' = 0$. Thus, the same argument applies to f^{-1} and shows that it is weakly K-quasiconformal. Altogether, f is K-quasiconformal. ☐

Proof of Theorem D.1.4 (2). Suppose that f is L-Lipschitz and area-preserving. By Lemma D.3.2, $N(f,y)=1$ for a.e. $y \in f(X)$. Also, Lemmas D.3.1 and D.3.4 imply that f is a light map. Now, Lemma D.3.7 implies that the restriction of f to any precompact open subset U of X (so that $f|_U \in N^{1,2}(U,Y)$) is injective. This implies that f is injective in all of X. The invariance of domain theorem implies that f is an embedding. Since f is surjective by assumption, we conclude that f is a homeomorphism. By Lemma D.3.1, we see that f is a weakly L^2-quasiconformal homeomorphism. Since Y is reciprocal, Lemma D.2.13 yields that f is K-quasiconformal for some $K=K(L)\geq 1$. In particular, this implies that X is also reciprocal.

The final inequality in Theorem D.1.4 (2) involving the lengths follows from Theorem D.1.4 (1). In the case that f is 1-Lipschitz, we obtain $\ell(\gamma)=\ell(f\circ\gamma)$ for all curves γ in X outside a curve family Γ_0 with $\operatorname{mod}\Gamma_0=0$. By Lemma D.3.8, we conclude that f is 1-quasiconformal. ☐

D.3.3. Bounded length distortion and isometry

Here we prove Theorem D.1.4 (3). Our goal is to upgrade the conclusion of Theorem D.1.4 (2) so that the length of every path, rather than almost every path, is quasi-preserved. This is achieved with the aid of upper Ahlfors 2-regularity. We say that a space is locally upper Ahlfors 2-regular with constant $K>0$ if each point has a neighborhood U such that $\mathcal{H}^2(B(x,r))\leq Kr^2$ for all $x\in U$ and $r<\operatorname{diam}(U)$. We denote by $N_r(E)$ the open r-neighborhood of a set E.

Lemma D.3.9. *Suppose that Y is locally upper Ahlfors 2-regular with constant $K>0$ and γ is a curve in Y. Then for all sufficiently small $r>0$ we have*

$$\mathcal{H}(N_r(|\gamma|))\leq 2Kr\ell(\gamma)+8Kr^2.$$

Proof. Without loss of generality, $\gamma\colon[0,\ell(\gamma)]\to X$ is non-constant, rectifiable and parametrized by arclength. Assume that $0<r<\ell(\gamma)/2$ and that for every $x\in|\gamma|$ we have

$$\mathcal{H}(B(x,2r))\leq 4Kr^2.$$

Consider a partition $\{t_0,\dots,t_n\}$ of $[0,\ell(\gamma)]$ such that $|t_i-t_{i-1}|\leq 2r$, $i\in\{1,\dots,n\}$, and $(n-1)2r<\ell(\gamma)\leq 2nr$. Then $\{B(\gamma(t_i),2r)\}_{i=0}^n$ covers $N_r(|\gamma|)$ and we can compute

$$\mathcal{H}(N_r(|\gamma|))\leq\sum_{i=0}^{n}\mathcal{H}(B(\gamma(t_i),2r))\leq(n+1)4Kr^2\leq 2Kr\ell(\gamma)+8Kr^2. \qquad \square$$

Lemma D.3.10. *Suppose that Y is locally upper Ahlfors 2-regular with constant $K>0$. Let Γ_0 be a curve family in Y with $\operatorname{mod}\Gamma_0=0$. Then for each curve $\gamma\colon[a,b]\to Y$ and for each $\varepsilon>0$ there exists a curve $\gamma_\varepsilon\colon[a,b]\to Y$ with the following properties.*

(1) $\gamma_\varepsilon\notin\Gamma_0$.

(2) $|\gamma(a)-\gamma_\varepsilon(a)|<\varepsilon$, $|\gamma(b)-\gamma_\varepsilon(b)|<\varepsilon$, *and* $|\gamma_\varepsilon|\subset N_\varepsilon(|\gamma|)$.

(3) $\ell(\gamma_\varepsilon)\leq 4\pi^{-1}K\ell(\gamma)+\varepsilon$.

Moreover, if Y is Riemannian, then

(3') $\ell(\gamma_\varepsilon)\leq\ell(\gamma)+\varepsilon$.

Proof. Assume that γ is simple, otherwise we consider a simple curve with trace in $|\gamma|$ connecting $\gamma(a)$ and $\gamma(b)$. Let $\varepsilon>0$. Consider the distance function $g(x)=d(x,|\gamma|)$. By the coarea inequality for Lipschitz functions (Theorem D.2.1) and Lemma D.3.9, there exists $r_1>0$ such that for all $0 < r < r_1$ we have

$$\int^{*} \chi_{(0,r)}(t)\mathcal{H}^1(g^{-1}(t))\,dt \leq \frac{4}{\pi}\mathcal{H}^2(N_r(|\gamma|)) < \frac{8}{\pi}Kr\ell(\gamma) + \varepsilon r. \tag{D.13}$$

Therefore, for all $0 < r < r_1$ we have

$$\operatorname*{essinf}_{t\in(0,r)} \mathcal{H}^1(g^{-1}(t)) < \frac{8}{\pi}K\ell(\gamma) + \varepsilon. \tag{D.14}$$

By Lemma D.2.9, for a.e. $t \in (0, r_1)$, every Lipschitz and injective curve $\alpha\colon [a, b] \to g^{-1}(t)$ does not lie in Γ_0.

Let $U \subset Y$ be a neighborhood of $|\gamma|$ homeomorphic to $\mathbb{D}$. Since γ is simple, the space $Z := U/|\gamma|$ equipped with the quotient metric is homeomorphic to $\mathbb{D}$. The quotient map $\pi\colon U \to Z$ is a local isometry on $U \setminus |\gamma|$. This together with Lemma D.3.5 provides the existence of $r_2 > 0$ such that for a.e. $t \in (0, r_2)$, the level set $g^{-1}(t)$ contains a rectifiable Jordan curve γ_t in U separating $|\gamma|$ from ∂U. Note that $|\gamma_t|$ converges to $|\gamma|$ in the Hausdorff sense as $t \to 0$. Thus, there exists $r_3 \in (0, r_2)$ such that for a.e. $t \in (0, r_3)$ we can find distinct points $a_t \in \overline{B}(\gamma(a), \varepsilon) \cap |\gamma_t|$ and $b_t \in \overline{B}(\gamma(b), \varepsilon) \cap |\gamma_t|$. Let γ_t' be a Lipschitz and injective parametrization of the closure of the shorter component of $|\gamma_t| \setminus \{a_t, b_t\}$. For $0 < r < \min\{r_1, r_2, r_3, \varepsilon\}$ we have

$$\operatorname*{essinf}_{t\in(0,r)} \ell(\gamma_t') < \frac{4}{\pi}K\ell(\gamma) + \frac{\varepsilon}{2}.$$

By the previous, $\gamma_t' \notin \Gamma_0$ for a.e. $t \in (0, r)$. Moreover, $|\gamma_t'| \subset g^{-1}(t) \subset N_\varepsilon(|\gamma|)$. Therefore, there exists $t \in (0, r)$ so that γ_t' satisfies (1)–(3).

If Y is Riemannian we have a local upper area bound of the form

$$\mathcal{H}(N_r(|\gamma|)) \leq 2r\ell(\gamma) + O(r^2);$$

see [Gra04, Corollary 9.24]. By arguing as in (D.13) while applying the coarea inequality for Riemannian manifolds (Theorem D.2.1), we obtain

$$\operatorname*{essinf}_{t\in(0,r)} \ell(\gamma_t') \leq \ell(\gamma) + \varepsilon,$$

for all sufficiently small $r > 0$. Hence, (3') follows. □

Lemma D.3.11. *Suppose that Y is locally upper Ahlfors 2-regular with constant $K > 0$. Let $g\colon Y \to X$ be continuous map such that there exists $L > 0$ with the property that $\ell(g\circ\gamma) \leq L\ell(\gamma)$ for all curves γ in Y outside a curve family Γ_0 with $\operatorname{mod}\Gamma_0 = 0$. Then*

$$\ell(g \circ \gamma) \leq \frac{4}{\pi}KL\ell(\gamma)$$

for every rectifiable curve γ in Y. Moreover, if Y is Riemannian then

$$\ell(g \circ \gamma) \leq L\ell(\gamma)$$

for every rectifiable curve γ in Y.

Proof. Let γ be a rectifiable Jordan arc in Y. By Lemma D.3.10, for each $n \in \mathbb{N}$ we can find a curve $\gamma_n \subset N_{1/n}(|\gamma|)$ whose endpoints are $(1/n)$-close to the endpoints of γ, $\gamma_n \notin \Gamma_0$, and

$$\ell(\gamma_n) \leq 4\pi^{-1}K\ell(\gamma) + n^{-1}.$$

Suppose that γ_n is parametrized by $[0,1]$ with constant speed. After passing to a subsequence, we may assume that γ_n converges uniformly to a path $\widetilde{\gamma}\colon [0,1] \to |\gamma|$ with the same endpoints as γ. It follows that $\widetilde{\gamma}$ is surjective, but it is possibly not injective. Moreover, $g \circ \gamma_n$ converges uniformly to $g \circ \widetilde{\gamma}$. Since γ is a Jordan arc, we have $N(g \circ \widetilde{\gamma}, y) \geq N(g \circ \gamma, y)$ for each $y \in g(|\gamma|)$. The area formula for length (Theorem D.2.3) and the lower semi-continuity of length imply that

$$\ell(g \circ \gamma) \leq \ell(g \circ \widetilde{\gamma}) \leq \liminf_{n \to \infty} \ell(g \circ \gamma_n).$$

Since $\gamma_n \notin \Gamma_0$, the latter is bounded by

$$L \liminf_{n \to \infty} \ell(\gamma_n) \leq 4\pi^{-1} K L \ell(\gamma).$$

This completes the proof in the case of Jordan arcs.

Now, suppose that $\gamma\colon [a,b] \to Y$ is an arbitrary path. Let $\{t_0, \ldots, t_n\}$ be a partition of $[a,b]$. For $i \in \{1, \ldots, n\}$, let $\gamma_i\colon [t_{i-1}, t_i] \to \gamma([t_{i-1}, t_i])$ be a Jordan arc with endpoints $\gamma(t_{i-1}), \gamma(t_i)$. Then

$$\begin{aligned}\sum_{i=1}^{n} d(g(\gamma(t_{i-1})), g(\gamma(t_i))) &\leq \sum_{i=1}^{n} \ell(g \circ \gamma_i) \leq 4\pi^{-1} K L \sum_{i=1}^{n} \ell(\gamma_i) \\ &\leq 4\pi^{-1} K L \sum_{i=1}^{n} \ell(\gamma|_{[t_{i-1}, t_i]}) = 4\pi^{-1} K L \ell(\gamma).\end{aligned}$$

This yields $\ell(g \circ \gamma) \leq 4\pi^{-1} K L \ell(\gamma)$.

If Y is Riemannian, the statement follows after applying (3') from Lemma D.3.10 instead of (3). □

Proof of Theorem D.1.4 (3). The upper Ahlfors 2-regularity implies that Y is reciprocal [Raj17, Theorem 1.6]. By Theorem D.1.4 (2), we have that f is a quasiconformal homeomorphism and the length of a.e. path is quasi-preserved. We now apply Lemma D.3.11 to $g = f^{-1}$, together with the fact that f is Lipschitz, and conclude that the length of every rectifiable path is quasi-preserved. It also follows that $\ell(\gamma) < \infty$ if and only if $\ell(f \circ \gamma) < \infty$. Therefore, f is a map of bounded length distortion. □

Proof of Theorem D.1.4 (4). Since Y is reciprocal, by Theorem D.1.4 (2), f is a 1-quasiconformal homeomorphism and preserves the length of all curves in X outside a curve family Γ_0 with $\operatorname{mod} \Gamma_0 = 0$. It follows from Lemma D.3.11 that $\ell(f^{-1} \circ \gamma) \leq \ell(\gamma)$ for every rectifiable curve γ in Y. If $x, y \in X$ and γ is a rectifiable curve in Y joining $f(x)$ and $f(y)$ then

$$d(x, y) \leq \ell(f^{-1} \circ \gamma) \leq \ell(\gamma).$$

Infimizing over γ gives $d(x, y) \leq d(f(x), f(y))$. Equality follows from f being 1-Lipschitz. □

D.4. Examples

We present examples that show the optimality of Theorem D.1.4. In all examples X, Y are metric surfaces of locally finite Hausdorff 2-measure and $f\colon X \to Y$ is an area-preserving and 1-Lipschitz map.

Example D.4.1. This example shows that f is not a homeomorphism in general, even if X is Euclidean. Let I be the interval $[0,1] \times \{0\}$ and $Y = \mathbb{R}^2 / I$, equipped with the quotient metric. The natural projection map $f\colon \mathbb{R}^2 \to Y$ is area-preserving and 1-Lipschitz, but it is not a homeomorphism.

Example D.4.2. This example shows that if Y is reciprocal as in Theorem D.1.4 (2), then f is not BLD in general, even if X is Euclidean. Define the weight $\omega\colon \mathbb{R}^2 \to [0,1]$ by $\omega(x) = x_1$ if $x = (x_1, 0) \in I := (0,1] \times \{0\}$ and $\omega(x) = 1$ otherwise. We define a metric d on $\mathbb{R}^2$ by

$$d(x,y) := \inf_{\gamma} \int_{\gamma} \omega \, ds,$$

where the infimum is taken over all rectifiable curves γ connecting $x, y \in \mathbb{R}^2$. Let $f\colon \mathbb{R}^2 \to Y := (\mathbb{R}^2, d)$ be the identity map, which is 1-Lipschitz, since $\omega \leq 1$, and a local isometry on $\mathbb{R}^2 \setminus I$, hence area-preserving. Moreover, f is a homeomorphism, and thus Y is a metric space homeomorphic to $\mathbb{R}^2$.

One can show that for each Borel set $E \subset \mathbb{R}^2$ we have $\mathcal{H}^1_d(E) = \int_E \omega \, d\mathcal{H}^1$; in fact, it suffices to show this for sets $E \subset I$. This fact and the area formula for length (Theorem D.2.3) imply that if γ is a rectifiable curve with respect to the Euclidean metric, then $\ell_d(\gamma) = \int_\gamma \omega \, ds$. This implies that

$$\int_\gamma \rho \, ds_d = \int_\gamma \rho\omega \, ds$$

for every Borel function $\rho\colon \mathbb{R}^2 \to [0,\infty]$.

Let Γ be a family of curves in $\mathbb{R}^2$. Since f is 1-Lipschitz and area-preserving, by Lemma D.3.1 we have $\operatorname{mod} \Gamma \leq \operatorname{mod} f(\Gamma)$; here the latter modulus is with respect to the metric d. We now show the reverse inequality. Let $\rho\colon \mathbb{R}^2 \to [0,\infty]$ be admissible for Γ. We set $\rho' = \rho\omega^{-1}$. If $\gamma \in \Gamma$, then

$$\int_{f\circ\gamma} \rho' \, ds_d = \int_\gamma \rho\omega^{-1}\omega \, ds = \int_\gamma \rho \, ds \geq 1.$$

Thus, ρ' is admissible for $f(\Gamma)$. Since $\mathcal{H}^2_d(I) = 0$, we conclude that

$$\operatorname{mod} f(\Gamma) \leq \int \rho^2 \, d\mathcal{H}^2$$

and thus $\operatorname{mod} f(\Gamma) \leq \operatorname{mod} \Gamma$. This shows that f is 1-quasiconformal and that Y is reciprocal.

By Theorem D.1.4 (1), f preserves the length of a.e. curve with respect to 2-modulus; this can also be seen immediately here, since a.e. curve intersects I at a set of length zero. However, f does not preserve the length of *every* curve and is not BLD. Indeed, for $t \in (0,1]$ denote by γ_t the straight line segment connecting $(0,0)$ and $(t,0)$. Then $\ell(\gamma_t) = t$, whereas

$$\ell_d(\gamma_t) = \int_{\gamma_t} \omega \, ds = t^2/2.$$

D.5. Coarea inequality

In this section we establish the general coarea inequality of Theorem D.1.6. First we prove the statement in the case that X is a topological closed disk. The proof follows the same strategy as in [EIR23, Theorem 4.8].

Theorem D.5.1. *Let X be a metric surface of finite Hausdorff 2-measure that is homeomorphic to a topological closed disk and suppose that there exists a weakly K-quasiconformal map from $\overline{\mathbb{D}}$ onto X for some $K \geq 1$. Let $u\colon X \to \mathbb{R}$ be a continuous function with a 2-weak upper gradient $\rho_u \in L^2(X)$.*

(1) *If $\mathcal{A}_u$ denotes the union of all non-degenerate components of the level sets $u^{-1}(t)$, $t \in \mathbb{R}$, of u, then $\mathcal{A}_u$ is a Borel set.*

(2) *For every Borel function $g\colon X \to [0,\infty]$ we have*

$$\int^* \int_{u^{-1}(t)\cap \mathcal{A}_u} g\, d\mathcal{H}^1\, dt \leq K \int g\rho_u\, d\mathcal{H}^2.$$

Proof. First we show that $\mathcal{A}_u$ is a Borel set. We can write

$$\mathcal{A}_u = \bigcup_{k=1}^{\infty} A_k,$$

where A_k is the union of the components E of $u^{-1}(t)$, $t \in \mathbb{R}$, with $\operatorname{diam}(E) \geq 1/k$. We will show that A_k is closed for each $k \in \mathbb{N}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in A_k. If $x_n \in E_n \subset u^{-1}(t_n)$, $n \in \mathbb{N}$, then after passing to a subsequence, the continua E_n converge in the Hausdorff sense to a continuum E with $\operatorname{diam}(E) \geq 1/k$. Moreover, after passing to a further subsequence, t_n converges to some $t \in \mathbb{R}$, so $E \subset u^{-1}(t)$. This shows that $E \subset A_k$. Therefore, all limit points of $\{x_n\}_{n\in\mathbb{N}}$ lie in A_k, as desired.

Let $f : \overline{\mathbb{D}} \to X$ be a weakly K-quasiconformal map. By Theorem D.2.10 there exists a 2-weak upper gradient $\rho_f \in L^2(\overline{\mathbb{D}})$ such that

$$\int_{f^{-1}(E)} \rho_f^2\, d\mathcal{H}^2 \leq K\mathcal{H}^2(E)$$

for each Borel set $E \subset X$. This implies that

$$\int (g \circ f) \cdot \rho_f^2\, d\mathcal{H}^2 \leq K \int g\, d\mathcal{H}^2 \tag{D.15}$$

for each Borel function $g\colon X \to [0,\infty]$. Moreover, for all curves γ in $\overline{\mathbb{D}}$ outside a curve family Γ_0 with $\operatorname{mod}\Gamma_0 = 0$ we have (see [HKST15, Prop. 6.3.3])

$$\int_{f\circ\gamma} g\, ds \leq \int_\gamma (g \circ f) \cdot \rho_f\, ds. \tag{D.16}$$

Consider the function $v = u \circ f$ on $\overline{\mathbb{D}}$. Then by [EIR23, Lemma 4.5], v has a 2-weak upper gradient ρ_v such that for a.e. $x \in \overline{\mathbb{D}}$ we have

$$\rho_v(x) \leq (\rho_u \circ f)(x) \cdot \rho_f(x).$$

In conjunction with (D.15), this implies that $\rho_v \in L^2(\overline{\mathbb{D}})$, so $v \in W^{1,2}(\mathbb{D})$, and

$$|\nabla v(x)| \leq (\rho_u \circ f)(x) \cdot \rho_f(x) \tag{D.17}$$

for a.e. $x \in \mathbb{D}$, because $|\nabla v|$ is the minimal 2-weak upper gradient of v (see [HKST15, Theorem 7.4.5]). We can extend v by reflection to a continuous function $\widetilde{v} \in W^{1,2}(U)$ for some neighborhood U of $\overline{\mathbb{D}}$. By the classical coarea formula (Theorem D.2.1), the set $v^{-1}(t) = \widetilde{v}^{-1}(t) \cap \overline{\mathbb{D}}$ has finite Hausdorff 1-measure for a.e. $t \in \mathbb{R}$. Corollary D.2.7 implies that for a.e. $t \in \mathbb{R}$ each component E of $v^{-1}(t)$ is a Jordan arc or a Jordan curve and can be parametrized by a Lipschitz function $\gamma\colon [a,b] \to E$ that is injective on $[a,b)$. Moreover, using the classical coarea formula for $\widetilde{v}$ one can show that for a.e. $t \in \mathbb{R}$, each Lipschitz curve $\gamma\colon [a,b] \to v^{-1}(t)$ that is injective on $[a,b)$ lies outside the given curve family Γ_0 of 2-modulus zero (cf. Lemma D.2.9); hence γ satisfies (D.16). Therefore, the following statements are true for a.e. $t \in \mathbb{R}$.

1. $\mathcal{H}^1(v^{-1}(t)) < \infty$. (Consequence of classical coarea formula.)
2. Each non-degenerate component E of $v^{-1}(t)$ is a Jordan arc or a Jordan curve and there exists a Lipschitz parametrization $\gamma\colon [a,b] \to E$ that is injective in $[a,b)$. (Consequence of Corollary D.2.7.)
3. For each Lipschitz curve $\gamma\colon [a,b] \to v^{-1}(t)$ that is injective on $[a,b)$ and for each Borel function $g\colon X \to [0,\infty]$, we have
$$\int_{f\circ\gamma} g\, ds \leq \int_\gamma (g\circ f)\cdot \rho_f\, ds.$$
(Consequence of classical coarea formula and (D.16).)

We fix a Borel function $g\colon X \to [0,\infty]$, a value $t \in \mathbb{R}$ satisfying the above statements, a non-degenerate component E of $v^{-1}(t)$, and a Lipschitz parametrization $\gamma\colon [a,b] \to E$ that is injective in $[a,b)$. We have

$$\int_{f(E)} g\, d\mathcal{H}^1 = \int_{f(|\gamma|)} g\, d\mathcal{H}^1 \leq \int_{f\circ\gamma} g\, ds \leq \int_\gamma (g\circ f)\cdot\rho_f\, ds = \int_E (g\circ f)\cdot \rho_f\, d\mathcal{H}^1.$$

Note that if G is a non-degenerate component of $u^{-1}(t)$, then by the monotonicity of f, $f^{-1}(G)$ is a non-degenerate component of $v^{-1}(t)$. Hence,

$$\int_G g\, d\mathcal{H}^1 \leq \int_{f^{-1}(G)} (g\circ f)\cdot \rho_f\, d\mathcal{H}^1.$$

The finiteness of the Hausdorff 1-measure of $v^{-1}(t)$ implies that it can have at most countably many non-degenerate components. Summing over all the non-degenerate components gives

$$\int_{u^{-1}(t)\cap \mathcal{A}_u} g\, d\mathcal{H}^1 \leq \int_{v^{-1}(t)} (g\circ f)\cdot\rho_f\, d\mathcal{H}^1.$$

We now integrate over $t \in \mathbb{R}$, use the classical coarea formula for $\widetilde{v}$, and inequalities (D.17) and (D.15), to obtain

$$\begin{aligned}
\int^* \int_{u^{-1}(t)\cap\mathcal{A}_u} g\, d\mathcal{H}^1 dt &\leq \int\int_{v^{-1}(t)} (g\circ f)\cdot\rho_f\, d\mathcal{H}^1 dt \\
&= \int_{\overline{\mathbb{D}}} (g\circ f)\cdot \rho_f \cdot |\nabla \widetilde{v}|\, d\mathcal{H}^2 \\
&= \int_{\mathbb{D}} (g\circ f)\cdot\rho_f\cdot|\nabla v|\, d\mathcal{H}^2 \\
&\leq \int (g\circ f)\cdot(\rho_u\circ f)\cdot \rho_f^2\, d\mathcal{H}^2 \leq K\int g\rho_u\, d\mathcal{H}^2.
\end{aligned}$$

This completes the proof. □

Proof of Theorem D.1.6. We write X as the countable union of topological closed disks X_n with $\mathcal{H}^2(X_n) < \infty$, $n \in \mathbb{N}$. We also consider topological closed disks $Z_n \supset X_n$, so that the topological interior $\mathrm{int}_\top(Z_n)$ contains X_n. We have $\mathrm{int}(Z_n) \subset \mathrm{int}_\top(Z_n) \subset Z_n$, where $\mathrm{int}(Z_n)$ refers to the manifold interior. Therefore the topological closure of $\mathrm{int}_\top(Z_n)$ is precisely the closed disk Z_n.

Let $u_n = u|_{Z_n}$. We claim that

$$\mathcal{A}_u = \bigcup_{n=1}^{\infty} \mathcal{A}_{u_n}. \tag{D.18}$$

For this, it suffices to show that

$$\mathcal{A}_u \cap X_n \subset \mathcal{A}_{u_n} \tag{D.19}$$

for each $n \in \mathbb{N}$. Let $x \in \mathcal{A}_u \cap X_n$ and consider a non-degenerate component E of $u^{-1}(t)$ for some $t \in \mathbb{R}$ such that $x \in E$. Note that x lies in $\mathrm{int}_\top(Z_n)$. If $E \subset Z_n$, then $E \subset \mathcal{A}_{u_n}$ and $x \in \mathcal{A}_{u_n}$. Suppose that E is not contained in Z_n; in this case $E \cap \partial_\top Z_n \neq \emptyset$ by the connectedness of E. Since E is a generalized continuum (i.e., a locally compact connected set), by [Why42, (10.1), p. 16], we conclude that each component of $E \cap Z_n$ intersects $\partial_\top Z_n$. In particular, the component E_x of $E \cap Z_n$ that contains x must intersect $\partial_\top Z_n$, and thus E_x is non-degenerate. We conclude that $E_x \subset \mathcal{A}_{u_n}$, so $x \in \mathcal{A}_{u_n}$. The claim is proved. Now, each $\mathcal{A}_{u_n}$ is a Borel set by Theorem D.5.1, so $\mathcal{A}_u$ is Borel measurable by (D.18) and we have established (1).

Let $g\colon X \to [0, \infty]$ be a Borel function. For $n \in \mathbb{N}$, let $g_n = g \cdot \chi_{X_n \setminus \bigcup_{i=1}^{n-1} X_i}$. Let $x \in \mathcal{A}_u \cap (X_n \setminus \bigcup_{i=1}^{n-1} X_i)$. Then $x \in \mathcal{A}_{u_n}$ by (D.19), so

$$g(x)\chi_{\mathcal{A}_u}(x) = g_n(x)\chi_{\mathcal{A}_u}(x) = g_n(x)\chi_{\mathcal{A}_{u_n}}(x).$$

We conclude that

$$g\chi_{\mathcal{A}_u} = \sum_{n\in\mathbb{N}} g_n \chi_{\mathcal{A}_{u_n}}.$$

By Theorem D.5.1, applied to $u_n\colon Z_n \to \mathbb{R}$, and the existence of weakly $(4/\pi)$-quasiconformal parametrizations (Theorem D.2.12), we have

$$\int^* \int_{u^{-1}(t)} g_n \chi_{\mathcal{A}_{u_n}} \, d\mathcal{H}^1 dt \leq \frac{4}{\pi} \int g_n \rho_u \, d\mathcal{H}^2$$

for each $n \in \mathbb{N}$. Thus, upon summing we obtain the claimed inequality (2).

Finally, part (3) follows from part (2) and the coarea inequality for Lipschitz functions. Namely, one applies (2) to the Borel function $g\chi_{\mathcal{A}_u}$ and Theorem D.2.1 to $g\chi_{X\setminus\mathcal{A}_u}$. $\square$

E. Mappings of finite distortion on metric surfaces

with Kai Rajala

Abstract. We investigate basic properties of *mappings of finite distortion* $f : X \to \mathbb{R}^2$, where X is any *metric surface*, i.e., metric space homeomorphic to a planar domain with locally finite 2-dimensional Hausdorff measure. We introduce *lower gradients*, which complement the upper gradients of Heinonen and Koskela, to study the distortion of non-homeomorphic maps on metric spaces.
We extend the Iwaniec-Šverák theorem to metric surfaces: a non-constant $f : X \to \mathbb{R}^2$ with locally square integrable upper gradient and locally integrable distortion is continuous, open and discrete. We also extend the Hencl-Koskela theorem by showing that if f is moreover injective then f^{-1} is a Sobolev map.

E.1. Introduction

E.1.1. Background

Let $\Omega \subset \mathbb{R}^2$ be a domain. We say that map $f : \Omega \to \mathbb{R}^2$ in the Sobolev space $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ has *finite distortion* if there is a measurable function $K : \Omega \to [1, \infty)$ so that

$$||Df(x)||^2 \leq K(x) J_f(x) \quad \text{for a.e. } x \in \Omega. \tag{E.1}$$

Here $||Df(x)||$ and $J_f(x)$ are the operator norm and determinant of $Df(x)$, respectively.

If $K(x) = 1$ for almost every $x \in \Omega$, then (E.1) is valid if and only if f is complex analytic. The basic topological properties of non-constant analytic functions are *continuity*, *openness* and *discreteness* (the preimage of every point is discrete in Ω).

By *Stoïlow factorization* (see [AIM09, Chapter 5.5], [LP20]) non-constant *quasiregular maps*, i.e., maps f satisfying (E.1) with constant function $K(x) = K \geq 1$, admit a factorization $f = g \circ h$, where h is a quasiconformal homeomorphism and g is analytic. In particular, every such f is also continuous, open and discrete.

In [IŠ93] Iwaniec and Šverák showed that boundedness of $K(x)$ may be replaced with local integrability.

Theorem E.1.1 (Iwaniec-Šverák theorem)**.** *Suppose $f \in W^{1,2}_{loc}(\Omega, \mathbb{R}^2)$ is non-constant and satisfies (E.1) for some locally integrable $K(x)$. Then f is continuous, open and discrete.*

The assumption on $K(x)$ is essentially best possible (see [Bal81] and [HR13]). Since the work of Iwaniec and Šverák [IŠ93], a rich theory of mappings of finite distortion has been developed (see [AIM09], [HK14]), with applications to PDE, complex dynamics, inverse problems and non-linear elasticity theory, among other fields.

The theory extends to $W^{1,1}_{\mathrm{loc}}$-maps with exponentially integrable distortion and also to higher dimensions, where continuity, openness and discreteness of quasiregular maps was proved by Reshetnyak already in the 1960s (see [Reš67]). Reshetnyak's theorem has been extended to spatial mappings of finite distortion by several authors (see [VG76], [VM98], [KKM01], [IKO01], [IM01], [KKM$^+$03], [OZ08], [Raj10], [HR13]).

Partially motivated by works of Heinonen-Rickman [HR02], Heinonen-Sullivan [HS02] and Heinonen-Keith [HK11] on BLD- and bi-Lipschitz parametrizations of metric spaces, Kirsilä [Kir16] furthermore extended Reshetnyak's theorem to maps $f : X \to \mathbb{R}^n$, where X is a *generalized n-manifold* satisfying assumptions such as Ahlfors n-regularity and Poincaré inequality.

In this paper we extend the Iwaniec-Šverák theorem to maps $f : X \to \mathbb{R}^2$, where X is a *metric surface*, i.e., a metric space homeomorphic to a domain in $\mathbb{R}^2$ with locally finite 2-dimensional Hausdorff measure. The novelty of our results is that we do not impose any additional conditions on X.

Our research is partially inspired by recent advances on the uniformization problem on metric surfaces (see [BK02], [Raj17], [Iko22], [B], [C], [NR23], [NR24]) and the properties of the associated homeomorphisms, such as quasiconformal maps $f : X \to \mathbb{R}^2$. It is desirable to explore the properties of non-homeomorphic maps on metric surfaces. The aim of our paper is to provide the first results in this direction.

E.1.2. Mappings of finite distortion on metric surfaces

A (euclidean) *metric surface* X is a metric space homeomorphic to a domain $U \subset \mathbb{R}^2$ and with locally finite 2-dimensional Hausdorff measure. Below, $\mathcal{H}^2$ will always be the reference measure on X.

Let X and Y be metric surfaces. We want to establish what it means for a map $f\colon X \to Y$ to have finite distortion. We first observe that in the euclidean case every mapping of finite distortion is sense-preserving. This follows from inequality (E.1) by applying non-negativity of the Jacobian determinant and integration by parts, a method which is not available in our generality. We call $f : X \to Y$ *sense-preserving* if for any domain Ω compactly contained in X so that $f|_{\partial\Omega}$ is continuous it follows that $\deg(y, f, \Omega) \geq 1$ for any $y \in f(\Omega) \setminus f(\partial\Omega)$. Here deg is the local topological degree of f, see [Ric93, I.4] for a definition in the euclidean setting and note that the concept transfers to our setting as every metric surface is homeomorphic to a domain in $\mathbb{R}^2$.

We apply the theory of Sobolev spaces based on *upper gradients* ([HKST15]). A Borel function $\rho^u : X \to [0, \infty]$ is an *upper gradient* of $f : X \to Y$, if

$$d_Y(f(x), f(y)) \leq \int_\gamma \rho^u \, ds \tag{E.2}$$

for all $x, y \in X$ and every rectifiable curve γ in X joining x and y. We say that f belongs to the Sobolev space $N^{1,2}_{\mathrm{loc}}(X, Y)$ if f has an upper gradient $\rho^u \in L^2_{\mathrm{loc}}(X)$ and if $d_Y(y, f(\cdot)) \in L^2_{\mathrm{loc}}(X)$ for some $y \in Y$ (see Section E.2.3).

It follows from the proof of [EIR23, Theorem 1.4] that a sense-preserving map $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ is continuous (see Remark E.2.3). Such an f also satisfies Lusin's Condition (N): if $E \subset X$ and $\mathcal{H}^2(E) = 0$, then $|f(E)|_2 = 0$ (see Remark E.2.8). The converse implication does not hold ([Raj17, Section 17]).

In order to define the distortion of f, we introduce *lower gradients*: a Borel function $\rho^l : X \to [0,\infty]$ is a *lower gradient* of $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$, if $\rho^l \leq \rho^u_f$ almost everywhere and

$$\ell(f \circ \gamma) \geq \int_\gamma \rho^l \, ds \tag{E.3}$$

for every rectifiable curve γ in X with $f \circ \gamma$ being continuous. Our definition is motivated by the observation that the upper gradient inequality (E.2) is equivalent to the reverse inequality of (E.3) for ρ^u (see Section E.2.3). Every $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ has an essentially unique *minimal weak upper gradient* ρ^u_f (see Section E.2.3). Similarly, we prove in Section E.7 that every such f has an essentially unique *maximal weak lower gradient* ρ^l_f.

We say that a sense-preserving $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ has *finite distortion (along paths) and denote* $f \in \mathrm{FDP}(X,Y)$, if there is a measurable $K\colon X \to [1,\infty)$ such that

$$\rho^u_f(x) \leq K(x) \cdot \rho^l_f(x) \quad \text{for almost every } x \in X. \tag{E.4}$$

The *distortion* K_f of f is

$$K_f(x) := \begin{cases} \frac{\rho^u_f(x)}{\rho^l_f(x)}, & \text{if } \rho^l_f(x) \neq 0, \\ 1, & \text{if } \rho^l_f(x) = 0. \end{cases}$$

Our main result is the following extension of the Iwaniec-Šverák theorem. Here X is any metric surface.

Theorem E.1.2. *Let $f \in \mathrm{FDP}(X,\mathbb{R}^2)$ be non-constant with $K_f \in L^1_{\mathrm{loc}}(X)$. Then f is open and discrete.*

Generalizing the euclidean result by Hencl-Koskela (who assumed $W^{1,1}$-regularity, see [HK06]), we show that if f is furthermore a homeomorphism, then the inverse is also a Sobolev map. For a related result see [BC23].

Theorem E.1.3. *Let $f \in \mathrm{FDP}(X,\mathbb{R}^2)$ be injective with $K_f \in L^1_{\mathrm{loc}}(X)$. Then*

$$f^{-1} \in N^{1,2}_{\mathrm{loc}}(f(X),X).$$

Examples in [Bal81] (f_0 in Proposition E.6.1 below, see also [HR13]) and [HK06, Example 1.4], respectively, show that condition $K_f \in L^1_{\mathrm{loc}}(X)$ is sharp both in Theorem E.1.2 and in Theorem E.1.3, even if $X = \mathbb{R}^2$.

We show in Section E.6 that there are metric surfaces X which do not admit any quasiconformal maps $h : X \to \mathbb{R}^2$ but do admit maps $f : X \to \mathbb{R}^2$ satisfying the assumptions of Theorem E.1.2. By [F, Theorem 1.3], such surfaces do not exist if we require K_f to be bounded instead of integrable.

Previous approaches to distortion of maps between metric spaces are mostly based on the *analytic definition*: We say that a sense-preserving $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ has *finite analytic distortion* and denote $f \in \mathrm{FDA}(X,Y)$, if there is a measurable $C\colon X \to [1,\infty)$ such that

$$\rho^u_f(x)^2 \leq C(x) \cdot J_f(x) \quad \text{for almost every } x \in X, \tag{E.5}$$

where

$$J_f(x) = \limsup_{r \to 0} \frac{\mathcal{H}^2_Y(f(\overline{B}(x,r)))}{\pi r^2}.$$

Inequality (E.5) is equivalent to (E.4) in the euclidean setting, and also provides a rich theory for homeomorphisms between metric spaces. However, unlike our approach based on lower gradients, the analytic approach is not convenient for treating non-homeomorphic maps between metric surfaces. We nevertheless prove the following in [F].

Theorem E.1.4 ([F, Theorem 1.1])**.** *If $f \in \mathrm{FDA}(X, \mathbb{R}^2)$, then $f \in \mathrm{FDP}(X, \mathbb{R}^2)$. Moreover, for every $C(x)$ in (E.5) we have*

$$K_f(x) \leq 4\sqrt{2}\, C(x) \quad \textit{for almost every } x \in X.$$

Theorem E.1.2 can be applied to prove the converse of Theorem E.1.4 assuming $K_f \in L^1_{\mathrm{loc}}(X, \mathbb{R}^2)$, see [F]. Combining Theorems E.1.2, E.1.3 and E.1.4 shows that our main results hold under the analytic assumption.

Corollary E.1.5. *Let $f \in \mathrm{FDA}(X, \mathbb{R}^2)$ be non-constant with $C(x) \in L^1_{\mathrm{loc}}(X)$. Then f is open and discrete. If f is injective, then $f^{-1} \in N^{1,2}_{\mathrm{loc}}(f(X), X)$.*

The definition of a metric surface can be relaxed by requiring X to be homeomorphic to an oriented topological surface M instead of a domain in $\mathbb{R}^2$. Our definitions and results are local and remain valid under the relaxed definition. We state them only for euclidean metric surfaces to simplify the presentation.

This paper is organized as follows. In Section E.2 we recall the background on Analysis in metric spaces needed to prove our main results. In Section E.3 we prove an area inequality for maps on the rectifiable part of a metric surface which involves lower gradients and may be of independent interest. We prove Theorems E.1.2 and E.1.3 in Sections E.4 and E.5, respectively.

The proofs are based on three main tools: the coarea inequality for Sobolev functions on metric surfaces by Meier-Ntalampekos [D] and Esmayli-Ikonen-Rajala [EIR23], weakly quasiconformal parametrizations of metric surfaces by Ntalampekos-Romney [NR24], [NR23] and Meier-Wenger [B], and the area inequality proved in Section E.3. In addition, to prove Theorem E.1.2 we apply estimates inspired by the value distribution theory of quasiregular mappings (see [Ric93]).

In Section E.6, we discuss connections between our results and the uniformization problem on metric surfaces, as well as different definitions of mappings with controlled distortion. Finally, in Section E.7 we prove the existence of maximal weak lower gradients.

E.2. Preliminaries

E.2.1. Basic definitions and notations

Let (X, d) be a metric space. We denote the *open* and *closed ball* in X of radius $r > 0$ centered at a point $x \in X$ by $B(x, r)$ and $\overline{B}(x, r)$, respectively. When $X = \mathbb{R}^2$ we use notation $\mathbb{D}(x, r)$ instead of $B(x, r)$.

A set $\Omega \subset X$ homeomorphic to the unit disc $\mathbb{D}(0, 1)$ is a *Jordan domain* in X if its boundary $\partial\Omega \subset X$ is a *Jordan curve* in X, i.e., a subset of X homeomorphic to $\mathbb{S}^1$. The *image* of a curve γ in X is indicated by $|\gamma|$ and the *length* by $\ell(\gamma)$.

A curve γ is *rectifiable* if $\ell(\gamma) < \infty$ and *locally rectifiable* if each of its compact subcurves is rectifiable. Moreover, a curve $\gamma\colon [a, b] \to X$ is *geodesic* if $\ell(\gamma) = d(\gamma(a), \gamma(b))$. A curve $\gamma\colon [0, \ell(\gamma)] \to X$ is *parametrized by arclength* if $\ell(\gamma|_I) = |I|_1$ for every interval $I \subset [0, \ell(\gamma)]$. Here, $|\cdot|_n$ denotes the *n-dimensional Lebesgue measure.*

For $s \geq 0$, we denote the *s-dimensional Hausdorff measure* of $A \subset X$ by $\mathcal{H}^s(A)$. The normalizing constant is chosen so that $|V|_n = \mathcal{H}^n(V)$ for open subsets V of $\mathbb{R}^n$.

We equip X with $\mathcal{H}^2$. Let $L^p(X)$ ($L^p_{\mathrm{loc}}(X)$) denote the space of p-integrable (locally p-integrable) Borel functions from X to $\mathbb{R} \cup \{-\infty, \infty\}$. Here locally p-integrable means p-integrable on compact subsets. We say that a subdomain G of X is *compactly contained* in X if the closure $\overline{G}$ is compact.

E.2.2. Modulus

Let X be a metric space and Γ be a family of curves in X. A Borel function $g\colon X \to [0,\infty]$ is *admissible* for Γ if $\int_\gamma g\, ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$. We define the (2-)*modulus* of Γ as

$$\operatorname{mod}\Gamma = \inf_g \int_X g^2\, d\mathcal{H}^2,$$

where the infimum is taken over all admissible functions g for Γ. If there are no admissible functions for Γ we set $\operatorname{mod}\Gamma = \infty$. A property is said to hold for *almost every* curve in Γ if it holds for every curve in $\Gamma \setminus \Gamma_0$ for some family $\Gamma_0 \subset \Gamma$ with $\operatorname{mod}(\Gamma_0) = 0$. In the definition of $\operatorname{mod}(\Gamma)$, the infimum can equivalently be taken over all *weakly admissible* functions, i.e., Borel functions $g\colon X \to [0,\infty]$ such that $\int_\gamma g \geq 1$ for almost every locally rectifiable curve $\gamma \in \Gamma$.

E.2.3. Metric Sobolev spaces

Let $f\colon X \to Y$ be a map between metric spaces. A Borel function $\rho^u\colon X \to [0,\infty]$ is an *upper gradient* of f if

$$d_Y(f(x), f(y)) \leq \int_\gamma \rho^u\, ds \tag{E.6}$$

for all $x, y \in X$ and every rectifiable curve γ in X joining x and y. If the *upper gradient inequality* (E.6) holds for almost every rectifiable curve γ in X joining x and y we call ρ^u *weak upper gradient* of f.

The Sobolev space $N^{1,2}(X,Y)$ is the space of Borel maps $f\colon X \to Y$ with upper gradient $\rho^u \in L^2(X)$ such that $x \mapsto d_Y(y, f(x))$ is in $L^2(X)$ for some and thus any $y \in Y$. The space $N^{1,2}_{\text{loc}}(X,Y)$ is defined in the obvious manner.

Each $f \in N^{1,2}_{\text{loc}}(X,Y)$ has a *minimal* weak upper gradient ρ^u_f, i.e., for any other weak upper gradient ρ^u we have $\rho^u_f \leq \rho^u$ almost everywhere. Moreover, ρ^u_f is unique up to a set of measure zero. See monograph [HKST15] for more background on metric Sobolev spaces.

We apply a notion of "minimal stretching" which compliments the "maximal stretching" represented by upper gradients. To motivate the definition, notice that for continuous maps $f \in N^{1,2}_{\text{loc}}(X,Y)$ the upper gradient inequality (E.6) is equivalent to

$$\ell(f \circ \gamma) \leq \int_\gamma \rho^u\, ds$$

for almost every rectifiable curve γ in X. We call a Borel function $\rho^l\colon X \to [0,\infty]$ a *lower gradient* of $f \in N^{1,2}_{\text{loc}}(X,Y)$, if $\rho^l \leq \rho^u_f$ almost everywhere and

$$\ell(f \circ \gamma) \geq \int_\gamma \rho^l\, ds \tag{E.7}$$

for every rectifiable curve γ in X with $f \circ \gamma$ being continuous. If the *lower gradient inequality* (E.7) holds for almost every rectifiable γ, we call ρ^l *weak lower gradient* of f. Note that 0 is always a lower gradient.

Each $f \in N^{1,2}_{\text{loc}}(X,Y)$ has a *maximal weak lower gradient* ρ^l_f, i.e., for any other weak lower gradient ρ^l we have $\rho^l_f \geq \rho^l$ almost everywhere. Moreover, ρ^l_f is unique up to a set of measure zero. The proof is analogous to the existence of minimal weak upper gradients, see [HKST15, Theorem 6.3.20]. For completeness, we provide a proof in Section E.7.

E.2.4. Coarea inequality on metric surfaces

We state the following coarea inequality for Lipschitz functions, which is a consequence of [EHa21, Theorem 1.1] (see [EIR23, Section 5]). Here, $\mathrm{Lip}(u)$ denotes the pointwise Lipschitz constant of a Lipschitz function $u\colon X \to \mathbb{R}$, defined by

$$\mathrm{Lip}(u)(x) = \limsup_{x \neq y \to x} \frac{|u(y) - u(x)|}{d(x, y)}.$$

Theorem E.2.1 (Lipschitz coarea inequality)**.** *Let X be a metric space and $u\colon X \to \mathbb{R}$ a Lipschitz function. Then*

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 dt \leq \frac{4}{\pi} \int_X g \cdot \mathrm{Lip}(u) \, d\mathcal{H}^2$$

for every Borel measurable $g : X \to [0, \infty]$.

Here $\int^*$ denotes the upper integral, which is equal to Lebesgue integral for measurable functions. An important tool throughout this work will be the following coarea inequality for continuous Sobolev functions on metric surfaces.

Theorem E.2.2 (Sobolev coarea inequality, [D, Theorem 1.6])**.** *Let X be a metric surface and $v\colon X \to \mathbb{R}$ be a continuous function in $N^{1,2}_{\mathrm{loc}}(X)$.*

(1) *If $\mathcal{A}_v$ denotes the union of all non-degenerate components of the level sets $v^{-1}(t)$, $t \in \mathbb{R}$, of v, then $\mathcal{A}_v$ is a Borel set.*

(2) *For every Borel function $g\colon X \to [0, \infty]$ we have*

$$\int^{*} \int_{v^{-1}(t) \cap \mathcal{A}_v} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int g \cdot \rho_v^u \, d\mathcal{H}^2.$$

Theorem E.2.2 generalizes the coarea inequality for monotone Sobolev functions established in [EIR23]. Here $v\colon X \to \mathbb{R}$ is called a *weakly monotone function* if for every open Ω compactly contained in X

$$\sup_{\Omega} v \leq \sup_{\partial\Omega} v < \infty \quad \text{and} \quad \inf_{\Omega} v \geq \inf_{\partial\Omega} v > -\infty.$$

A continuous weakly monotone function is *monotone.*

Remark E.2.3. In the proof of [EIR23, Theorem 1.4] the coarea inequality for monotone Sobolev functions is used to show that every weakly monotone function $v \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R})$ is continuous and hence monotone. Continuity of a sense-preserving map $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ now follows by applying the exact same proof strategy while replacing weak monotonicity with sense-preservation and the coarea inequality for monotone Sobolev maps with Theorem E.2.2.

E.2.5. Metric differentiability

Let (Y, d) be a complete metric space and $U \subset \mathbb{R}^n$, $n \geq 1$, a domain. We say that $h\colon U \to Y$ is *approximately metrically differentiable* at $z \in U$ if there exists a seminorm N_z on $\mathbb{R}^2$ for which

$$\operatorname{ap}\lim_{y \to z} \frac{d(h(y), h(z)) - N_z(y - z)}{|y - z|} = 0.$$

Here, $\operatorname{ap}\lim$ denotes the approximate limit (see [EG92, Section 1.7.2]). If such a seminorm exists, it is unique and is called *approximate metric derivative* of h at z, denoted $\operatorname{ap}\operatorname{md} h_z$. The following result follows from [LW18a, Lemma 3.1].

Lemma E.2.4. *Let X and Y be metric surfaces and $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$. Almost every curve $\gamma\colon [a,b] \to X$ parametrized by arclength satisfies*

$$\int_{f\circ\gamma} g\, ds = \int_a^b g(f(\gamma(t))) \cdot \operatorname{ap}\operatorname{md}(f\circ\gamma)_t\, dt$$

for all Borel measurable $g : Y \to [0,\infty]$.

Lemma E.2.4 leads to the following properties of upper and lower gradients (see [HKST15, Proposition 6.3.3] for a proof involving upper gradients).

Corollary E.2.5. *Let X and Y be metric surfaces and $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$. Almost every curve $\gamma\colon [a,b] \to X$ parametrized by arclength satisfies the following properties.*

1. *f is absolutely continuous on γ,*
2. *$\rho^l_f(\gamma(t)) \le \operatorname{ap}\operatorname{md}(f\circ\gamma)_t \le \rho^u_f(\gamma(t))$ for almost every $a < t < b$,*
3. *if $g : Y \to [0,\infty]$ is Borel measurable, then*

$$\int_\gamma \rho^l_f \cdot (g\circ f)\, ds \le \int_{f\circ\gamma} g\, ds \le \int_\gamma \rho^u_f \cdot (g\circ f)\, ds.$$

E.2.6. Area formula on euclidean domains

Suppose $U \subset \mathbb{R}^2$ is a domain and $h \in N^{1,2}_{\mathrm{loc}}(U,Y)$. Then U can be covered up to a set of measure zero by countably many disjoint measurable sets G_j, $j \in \mathbb{N}$, such that $h|_{G_j}$ is Lipschitz. In particular, outside a set of measure zero $G_0 \subset U$, h satisfies Lusin's condition (N) (see [HKST15, Theorem 8.1.49]).

By [LW17a, Proposition 4.3], every $h \in N^{1,2}_{\mathrm{loc}}(U,Y)$ is approximately metrically differentiable at a.e. $z \in U$. The following area formula follows from [Kar07, Theorem 3.2]. Here, the *Jacobian* $J(N_z)$ of a seminorm N_z on $\mathbb{R}^2$ is zero if N_z is not a norm and $J(N_z) = \pi/|\{y \in \mathbb{R}^2 : N_z(y) \le 1\}|_2$ otherwise.

Theorem E.2.6 (Area formula)**.** *If $h \in N^{1,2}_{\mathrm{loc}}(U,Y)$, then there exists $G_0 \subset U$ with $\mathcal{H}^2(G_0) = 0$ such that for every measurable set $A \subset U \setminus G_0$ we have*

$$\int_A J(\operatorname{ap}\operatorname{md} h_z)\, d\mathcal{H}^2 = \int_Y N(y,h,A)\, d\mathcal{H}^2. \tag{E.8}$$

Here, $N(y,h,A)$ denotes the *multiplicity* of $y \in Y$ with respect to h in A:

$$N(y,h,A) := \#\{z \in A : h(z) = y\}. \tag{E.9}$$

E.2.7. Weakly quasiconformal parametrizations

A map $h\colon X \to Y$ between metric surfaces is *cell-like* if the preimage of each point is a continuum that is contractible in each of its open neighborhoods. A continuous, surjective, proper and cell-like map $h\colon X \to Y$ is *weakly C-quasiconformal* if

$$\operatorname{mod}\Gamma \le C \operatorname{mod} h(\Gamma)$$

holds for every family of curves Γ in X. It follows from [Wil12, Theorem 1.1] that every weakly quasiconformal map $h\colon X \to Y$ is contained in $N^{1,2}_{\mathrm{loc}}(X,Y)$.

It was shown in [NR24] that any metric surface admits a weakly quasiconformal parametrization, see also [NR23], [B], [C].

Theorem E.2.7 ([NR24, Theorem 1.2])**.** *Let X be any metric surface. There is a weakly $(4/\pi)$-quasiconformal $u\colon U \to X$, where $U \subset \mathbb{R}^2$ is a domain.*

Remark E.2.8. Condition (N) for sense-preserving maps $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ can be proved using the area formula and Theorem E.2.7 as follows: suppose $E \subset X$ and $\mathcal{H}^2(E) = 0$, and let $u\colon U \to X$ be a (sense-preserving) weakly $(4/\pi)$-quasiconformal parametrization of X provided by Theorem E.2.7. Define $h\colon U \to \mathbb{R}^2$ by $h := f \circ u$. Then $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ and $h \in N^{1,2}_{\mathrm{loc}}(U, \mathbb{R}^2)$, see [F, Theorem 2.5].

By Theorem E.2.6 there exists $G_0 \subset U$ with $|G_0|_2 = 0$ and such that (E.8) holds for u and h and every measurable set $A \subset U \setminus G_0$. We set $X_0 := u(G_0)$. Now h is sense-preserving and thus monotone. Therefore, h satisfies Condition (N) by [MM95]. In particular, with the above notation,

$$|f(E)|_2 \leq \int_{u^{-1}(E)} J(\operatorname{ap} \operatorname{md} h_z)\, dz.$$

On the other hand, applying Theorem E.2.6 to u shows that

$$\int_{u^{-1}(E)} J(\operatorname{ap} \operatorname{md} u_z)\, dz \leq \mathcal{H}^2(E) = 0,$$

and so $J(\operatorname{ap} \operatorname{md} u_z) = 0$ almost everywhere in $u^{-1}(E)$. Since u is weakly quasiconformal, it moreover follows that $\operatorname{ap} \operatorname{md} u_z = 0$. Then, by Lemmas E.2.9 and E.2.10 below, $J(\operatorname{ap} \operatorname{md} h_z) = 0$ almost everywhere in $u^{-1}(E)$ as well. We conclude that $|f(E)|_2 = 0$.

E.2.8. Distortion of Sobolev maps

Let $U \subset \mathbb{R}^2$ be a domain. We define the *maximal and minimal stretches* of $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$ at points of approximate differentiability by

$$L_h(z) = \max\{\operatorname{ap} \operatorname{md} h_z(v) : |v| = 1\}, \quad l_h(z) = \min\{\operatorname{ap} \operatorname{md} h_z(v) : |v| = 1\}.$$

Recall that maps $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$ are approximately differentiable almost everywhere.

Lemma E.2.9. *Let $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$. Then L_h and l_h are representatives of the minimal weak upper gradient and the maximal weak lower gradient of h, respectively. Moreover,*

$$2^{-1} L_h(z) l_h(z) \leq J(\operatorname{ap} \operatorname{md} h_z) \leq 2 L_h(z) l_h(z) \tag{E.10}$$

at points of approximate differentiability.

Proof. The first claim concerning upper gradients is [D, Lemma 2.14]. A slight modification of the proof gives the claim concerning lower gradients.

Towards (E.10), we may assume that $\operatorname{ap} \operatorname{md} h_z$ is a norm. Then the unit ball B_z of $\operatorname{ap} \operatorname{md} h_z(v)$ contains a unique ellipse of maximal area E_z, called the *John ellipse* of B_z, which satisfies

$$E_z \subset B_z \subset \sqrt{2} E_z, \tag{E.11}$$

see [Bal97, Theorem 3.1]. Let N_z be the norm whose unit ball is E_z, let $M_z = \max\{N_z(v) : |v| = 1$ and $m_z = \min\{N_z(v) : |v| = 1\}$. Then $J(N_z) = \pi/|E_z|_2 = M_z m_z$, and (E.11) gives

$$L_h(z) l_h(z) \leq M_z m_z = J(N_z) = 2\pi/|\sqrt{2} E_z|_2 \leq 2\pi/|B_z|_2 = 2J(\operatorname{ap} \operatorname{md} h_z).$$

On the other hand, (E.11) also gives

$$J(\operatorname{ap}\operatorname{md} h_z) \leq J(N_z) = M_z m_z \leq 2L_h(z) l_h(z).$$

The proof is complete. □

We will apply distortion estimates on composed mappings.

Lemma E.2.10. *Let X and Y be metric surfaces and $U \subset \mathbb{R}^2$ a domain, $u : U \to X$ weakly quasiconformal, and $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$. Then*

$$l_{f\circ u}(z) \geq \rho^l_f(u(z)) \cdot l_u(z) \quad \text{and} \quad L_{f\circ u}(z) \leq \rho^u_f(u(z)) \cdot L_u(z)$$

for almost every $z \in U$.

Proof. Let Γ_0 be the family of paths γ in U so that l_u does not satisfy the lower gradient inequality (E.7) for u on some subcurve of γ or ρ^l_f does not satisfy the lower gradient inequality for f on some subcurve of $u \circ \gamma$. Then, since u is weakly quasiconformal and l_u, ρ^l_f are weak lower gradients (Lemma E.2.9), we conclude that $\operatorname{mod}(\Gamma_0) = 0$. Applying Corollary E.2.5, we have

$$\ell(f \circ u \circ \gamma) \geq \int_{u\circ\gamma} \rho^l_f \, ds \geq \int_\gamma (\rho^l_f \circ u) \cdot l_u \, ds$$

for every $\gamma \notin \Gamma_0$ parametrized by arclength. We conclude that $(\rho^l_f \circ u) \cdot l_u$ is a weak lower gradient of $f \circ u$. But $l_{f\circ u}$ is a maximal weak lower gradient of $f \circ u$ by Lemma E.2.9. The first inequality follows. The second inequality is proved in a similar way. □

E.3. Area inequality on metric surfaces

Let X and Y be metric surfaces. In this section we establish Theorem E.3.1, an area inequality for Sobolev maps in $N^{1,2}_{\mathrm{loc}}(X, Y)$ on measurable subsets of the rectifiable part of X. We apply Theorem E.3.1 in Sections E.4 and E.5 below to prove our main results, Theorems E.1.2 and E.1.3.

As in Remark E.2.8, let $u \colon U \to X$ be a weakly $(4/\pi)$-quasiconformal parametrization of X provided by Theorem E.2.7, and $h \colon U \to Y$, $h := f \circ u$. Then $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ and $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$. By Theorem E.2.6, there exists $G_0 \subset U$ with $|G_0|_2 = 0$ and such that (E.8) holds for both u and h and every measurable set $A \subset U \setminus G_0$. We set $X_0 := u(G_0)$.

Theorem E.3.1 (Area inequality)**.** *If $g \colon Y \to [0, \infty]$ and $E \subset X \setminus X_0$ are Borel measurable, then*

$$\int_E g(f(x)) \cdot \rho^u_f(x) \rho^l_f(x) \, d\mathcal{H}^2 \leq 4\sqrt{2} \int_Y g(y) \cdot N(y, f, E) \, dy.$$

If in addition, the map f satisfies Lusin's condition (N), then

$$\int_E g(f(x)) \cdot \rho^u_f(x) \rho^l_f(x) \, d\mathcal{H}^2 \geq \frac{1}{4\sqrt{2}} \int_Y g(y) \cdot N(y, f, E) \, dy.$$

In order to establish Theorem E.3.1, we make use of the following proposition which can be seen as a counterpart to Lemma E.2.10.

Proposition E.3.2. *Let f, u and $h = f \circ u$ be as above. Then*

$$\rho^u_f(u(z)) \cdot l_u(z) \leq L_h(z) \quad \text{and} \quad l_h(z) \leq \rho^l_f(u(z)) \cdot L_u(z) \tag{E.12}$$

for almost every $z \in U \setminus G_0$.

Proof. Fix Borel representatives of the maps $z \mapsto \operatorname{ap}\operatorname{md} u_z$ and $z \mapsto \operatorname{ap}\operatorname{md} h_z$. Towards the first inequality in (E.12), we denote

$$G_0' = G_0 \cup \{z \in U : \, l_u(z) = 0\},$$

and notice that it suffices to prove the inequality for almost every $z \in U \setminus G_0'$. By [LW17a, Proposition 4.3], there are pairwise disjoint Borel sets $K_i \subset U \setminus G_0'$, $i \in \mathbb{N}$, so that

$$|U \setminus (G_0' \cup (\cup_i K_i))|_2 = 0 \tag{E.13}$$

and so that for every $i \in \mathbb{N}$ we have

(i) $\operatorname{ap}\operatorname{md} u_z$ and $\operatorname{ap}\operatorname{md} h_z$ exist for every $z \in K_i$ and

(ii) for every $\varepsilon > 0$ there is $r_i(\varepsilon) > 0$ so that

$$|d_X(u(z+v), u(z+w)) - \operatorname{ap}\operatorname{md} u_z(v-w)| \le \varepsilon|v-w| \quad \text{and}$$
$$|d_Y(h(z+v), h(z+w)) - \operatorname{ap}\operatorname{md} h_z(v-w)| \le \varepsilon|v-w|$$

for every $z \in K_i$ and all $v, w \in \mathbb{R}^2$ with $|v|, |w| \le r_i(\varepsilon)$ and such that $z+v, z+w \in K_i$.

We will show that if $i \in \mathbb{N}$ then almost every curve γ in X parametrized by arclength has the following property: almost every $t \in \gamma^{-1}(u(K_i))$ satisfies

$$\operatorname{ap}\operatorname{md}(f \circ \gamma)_t \le \frac{L_h(z)}{l_u(z)} \quad \text{for all } z \in u^{-1}(\gamma(t)) \cap K_i. \tag{E.14}$$

We show how to conclude the first inequality in (E.12) from (E.14). By Lemma E.2.4, Corollary E.2.5 and (E.14), $\rho\colon X \to [0,\infty]$ is a weak upper gradient of f, where $\rho(x) = \rho_f^u(x)$ for $x \in X \setminus u(K_i)$ and

$$\rho(x) = \inf_{z \in K_i,\, u(z) = x} \frac{L_h(z)}{l_u(z)}$$

when $x \in u(K_i)$. By the definition of minimal weak upper gradients, we then have that

$$\rho_f^u(x) \le \rho(x) \quad \text{for almost every } x \in u(K_i). \tag{E.15}$$

Since $K_i \subset U \setminus G_0'$, we have $l_u > 0$ and thus $J(\operatorname{ap}\operatorname{md} u_z) > 0$ in K_i. Combining (E.15) with the Area formula (Theorem E.2.6) for u now yields

$$\rho_f^u(u(z)) \cdot l_u(z) \le L_h(z)$$

for almost every $z \in K_i$. The first inequality in (E.12) follows from (E.13).

We now prove (E.14). Denote by $\widehat{X} \subset X$ the set of points x for which $N(x, u, U) = 1$. By [NR23, Remark 7.2], $\mathcal{H}^2(X \setminus \widehat{X}) = 0$. In particular, almost every rectifiable curve $\gamma : [0, \ell(\gamma)] \to X$ parametrized by arclength satisfies $\gamma(t) \in \widehat{X}$ for $\mathcal{H}^1$-almost every $0 < t < \ell(\gamma)$.

We fix such a γ and a density point $t_0 \in \gamma^{-1}(u(K_i) \cap \widehat{X}) =: T$ of T. By Corollary E.2.5, we may moreover assume that $f \circ \gamma$ is approximately metrically differentiable at t_0. It suffices to show that (E.14) holds for t_0 and the unique $z_0 = u^{-1}(\gamma(t_0)) \in K_i$.

Fix a sequence (t_j) of points in T converging to t. Then $x_j := \gamma(t_j) \to \gamma(t_0) =: x_0$. Moreover, since $x_0 \in \widehat{X}$, we have $z_j := u^{-1}(x_j) \to z_0$. We are now in position to apply Property (ii) above.

Denoting $y_j = f(x_j)$ for $j = 0, 1, \ldots$, (ii) and triangle inequality yield

$$\begin{aligned}\frac{d_X(x_j, x_0)}{|z_j - z_0|} &\geq \operatorname{ap\,md} u_{z_0}\left(\frac{z_j - z_0}{|z_j - z_0|}\right) - o(|z_j - z_0|) \geq l_u(z_0) - o(|z_j - z_0|),\\ \frac{d_Y(y_j, y_0)}{|z_j - z_0|} &\leq \operatorname{ap\,md} h_{z_0}\left(\frac{z_j - z_0}{|z_j - z_0|}\right) + o(|z_j - z_0|) \leq L_h(z_0) + o(|z_j - z_0|).\end{aligned}$$

Combining the inequalities, we have

$$\frac{d_Y(y_j, y_0)}{d_X(x_j, x_0)} = \frac{d_Y(y_j, y_0) \cdot |z_j - z_0|}{|z_j - z_0| \cdot d_X(x_j, x_0)} \leq \frac{L_h(z_0)}{l_u(z_0)} + o(|z_j - z_0|). \tag{E.16}$$

Since γ is parametrized by arclength, (E.16) gives (E.14). The first inequality in (E.12) follows. The second inequality follows in a similar way, namely showing that instead of (E.14) we have

$$\operatorname{ap\,md}(f \circ \gamma)_t \geq \frac{l_h(z)}{L_u(z)}$$

outside suitable exceptional sets. We leave the details to the reader. □

Proof of Theorem E.3.1. We may approximate g with simple functions and replace E with appropriate subsets to see that it suffices to show the claim for $g \equiv 1$. We set $E' = E \cap \widehat{X}$, where $\widehat{X}$ is as in the proof of Proposition E.3.2, and obtain

$$N(y, h, u^{-1}(E')) = \sum_{x \in f^{-1}(y)} N(x, u, u^{-1}(E')) = N(y, f, E') \tag{E.17}$$

for every $y \in f(E')$.

The area formula (Theorem E.2.6) implies

$$\begin{aligned}\int_E \rho_f^u(x)\rho_f^l(x)\, d\mathcal{H}^2 &= \int_{E'} \rho_f^u(x)\rho_f^l(x) N(x, u, u^{-1}(E'))\, d\mathcal{H}^2\\ &= \int_{u^{-1}(E')} \rho_f^u(u(z))\rho_f^l(u(z)) J(\operatorname{ap\,md} u_z)\, dz.\end{aligned}$$

By Lemma E.2.9, $J(\operatorname{ap\,md} u_z) \leq 2L_u(z) \cdot l_u(z)$ for almost every $z \in u^{-1}(E')$. Moreover, it follows from the proof of Theorem E.2.7 given in [NR24] that we can choose u so that the John ellipse of $\operatorname{ap\,md} u_z$ (see (E.11)) is a disk. Then $L_u(z) \leq \sqrt{2} l_u(z)$, which leads to

$$J(\operatorname{ap\,md} u_z) \leq 2L_u(z) \cdot l_u(z) \leq 2\sqrt{2} \cdot l_u(z)^2 \quad \text{for almost every } z \in u^{-1}(E').$$

Combining with Lemma E.2.10 and Proposition E.3.2, we conclude that

$$\int_E \rho_f^u(x)\rho_f^l(x)\, d\mathcal{H}^2 \leq 2\sqrt{2} \int_{u^{-1}(E')} L_h(z) l_h(z)\, dz.$$

Applying Lemma E.2.9 and the area formula (Theorem E.2.6) to h, we finally obtain

$$\int_E \rho_f^u(x)\rho_f^l(x)\, d\mathcal{H}^2 \leq 4\sqrt{2} \int_{u^{-1}(E')} J(\operatorname{ap\,md} h_z)\, dz = 4\sqrt{2} \int_{f(E')} N(y, h, u^{-1}(E'))\, dy.$$

The theorem follows by combining with (E.17).

For the second statement we note that f satisfying Lusin's condition (N) implies $\mathcal{H}^2(f(E \setminus E')) = 0$ as, by [NR23, Remark 7.2], $\mathcal{H}^2(E \setminus E') = 0$. The rest of the proof is analogous to the arguments above. $\square$

E.4. Openness and discreteness

Throughout this section let f be as in Theorem E.1.2, i.e., $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ is non-constant, sense-preserving and satisfies $K_f \in L^1_{\mathrm{loc}}(X)$. Recall that f is continuous by Remark E.2.3.

A map $f\colon X \to \mathbb{R}^2$ is *light* if $f^{-1}(y)$ is totally disconnected for every $y \in \mathbb{R}^2$. It is well-known that if f is continuous, sense-preserving and light, then f is open and discrete [TY62], [Ric93, Lemma VI.5.6]. Thus, in order to prove Theorem E.1.2 it suffices to show that f is in fact light. The proof of this fact relies on the following two propositions involving estimates on the multiplicity of f (recall notation $N(y, h, A)$ for multiplicity in (E.9)).

Proposition E.4.1. *Suppose that there are $s, r_0 > 0$ and $C > 0$ such that*

$$\int_0^{2\pi} N(f(x_0) + re^{i\theta}, f, B(x_0, s))\, d\theta \leq C \log \frac{1}{r} \tag{E.18}$$

for all $r < r_0$. Then the x_0-component of $f^{-1}(f(x_0))$ either is $\{x_0\}$ or contains an open neighborhood of x_0.

Recall that X is homeomorphic to a planar domain. In particular, for every $x_0 \in X$ there is $s > 0$ so that $\overline{B}(x_0, 2s)$ is a compact subset of X.

Proposition E.4.2. *Let $x_0 \in X$ and $s > 0$ so that $\overline{B}(x_0, 2s) \subset X$ is compact. Then Condition (E.18) holds with some $r_0, C > 0$.*

Theorem E.1.2 follows by combining Propositions E.4.1 and E.4.2: since f is not constant, for every $y_0 \in f(X)$ every component F of $f^{-1}(y_0)$ contains a point $x_0 \in X$ which is a boundary point of F. Combining Propositions E.4.1 and E.4.2, we see that $F = \{x_0\}$. We conclude that f is light and therefore open and discrete.

E.4.1. Proof of Proposition E.4.1

Let $f\colon X \to \mathbb{R}^2$ be a map of finite distortion and Γ a curve family in X. We define the weighted modulus

$$\operatorname{mod}_{K^{-1}} \Gamma = \inf_g \int_X \frac{g(x)^2}{K_f(x)}\, d\mathcal{H}^2,$$

where the infimum is taken over all weakly admissible functions g for Γ.

Let $u\colon U \to X$ be a weakly $(4/\pi)$-quasiconformal parametrization of X as in Theorem E.2.7. Let $G_0 \subset U$ and $X_0 = u(G_0) \subset X$ be as in the paragraph preceding Theorem E.3.1. Recall that $|G_0|_2 = 0$. We set $X' := X \setminus X_0$.

Lemma E.4.3. *Let Γ' be a family of curves in $\Omega \subset X$ with $\mathcal{H}^1(|\gamma| \cap X_0) = 0$ for every $\gamma \in \Gamma'$. Then*

$$\operatorname{mod}_{K^{-1}} \Gamma' \leq 4\sqrt{2} \int_{\mathbb{R}^2} g(y)^2 N(y, f, \Omega)\, dy,$$

whenever g is admissible for $\Gamma = f(\Gamma')$.

Proof. Fix an admissible g for Γ, and let $g'\colon X \to \mathbb{R}$,

$$g'(x) := g(f(x)) \cdot \rho^u_f(x) \cdot \chi_{\Omega \cap X'}(x).$$

Here, χ_E denotes the indicator function on a set $E \subset X$, i.e., $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ else. For almost every $\gamma \in \Gamma'$ we have that f is absolutely continuous on γ, $\mathcal{H}^1(|\gamma| \cap X_0) = 0$, and

$$\int_\gamma g'\,ds = \int_\gamma (g\circ f)\cdot \rho_f^u\,ds \geq \int_{f\circ\gamma} g\,ds,$$

see Corollary E.2.5. Since g is admissible for $\Gamma = f(\Gamma')$, it follows that g' is weakly admissible for Γ'. Moreover,

$$\begin{aligned}\operatorname{mod}_{K^{-1}}\Gamma' &\leq \int_X \frac{g'(x)^2}{K_f(x)}\,d\mathcal{H}^2 = \int_{\Omega\cap X'} g(f(x))^2\cdot \rho_f^u(x)\rho_f^l(x)\,d\mathcal{H}^2\\ &\leq 4\sqrt{2}\int_{\mathbb{R}^2} g(y)^2\cdot N(y,f,\Omega)\,dy,\end{aligned}$$

where the last inequality follows from the area inequality, Theorem E.3.1. □

Lemma E.4.4. *Let $\varphi \in N^{1,2}_{\mathrm{loc}}(X,\mathbb{R})$, and consider $E\subset\mathbb{R}$ with $|E|_1 > 0$ and so that each level set $\varphi^{-1}(t)$, $t\in E$, contains a non-degenerate continuum η_t. Then $\mathcal{H}^1(\eta_t\cap X_0) = 0$ for almost every $t\in E$.*

Proof. Note that $\widehat{\varphi} = \varphi\circ u$ is in $N^{1,2}_{\mathrm{loc}}(U,\mathbb{R})$. For every $t\in E$, let $\widehat{\eta}_t = u^{-1}(\eta_t)$. Then, since u is continuous and proper, $\widehat{\eta}_t$ is a non-degenerate continuum for every $t\in E$. Moreover, the coarea inequality for Sobolev functions (Theorem E.2.2) shows that $\mathcal{H}^1(\widehat{\eta}_t) < \infty$ for almost every $t\in E$. For every such t, there is a surjective two-to-one 1-Lipschitz curve

$$\widehat{\gamma}_t : [0, 2\mathcal{H}^1(\widehat{\eta}_t)] \to \widehat{\eta}_t,$$

cf. [RR19, Proposition 5.1]. Let $\widehat{\Gamma}$ be the family of the curves $\widehat{\gamma}_t$, and let $g\colon U\to[0,\infty]$ be admissible for $\widehat{\Gamma}$. We apply the coarea inequality for Sobolev functions (Theorem E.2.2) and Hölder's inequality to obtain

$$\begin{aligned}|E|_1 &\leq \int_E^* \int_{\widehat{\gamma}_t} g\,ds\,dt \leq 2\int_E^*\int_{\widehat{\eta}_t} g\,d\mathcal{H}^1\,dt \leq \frac{8}{\pi}\int_{\widehat{\varphi}^{-1}(E)} g\cdot\rho^u_{\widehat{\varphi}}\,d\mathcal{H}^2\\ &\leq \frac{8}{\pi}\left(\int_{\widehat{\varphi}^{-1}(E)} g^2\,d\mathcal{H}^2\right)^{1/2}\left(\int_{\widehat{\varphi}^{-1}(E)} (\rho^u_{\widehat{\varphi}})^2\,d\mathcal{H}^2\right)^{1/2}.\end{aligned}$$

Since $\rho^u_{\widehat{\varphi}} \in L^2_{\mathrm{loc}}(U)$ and $|E|_1 > 0$ it follows that $\operatorname{mod}(\widehat{\Gamma}) > 0$. As a Sobolev function, u is therefore absolutely continuous along $\widehat{\gamma}_t$ for almost every $t\in E$, see e.g. [HKST15, Lemma 6.3.1]. Moreover, for almost every $t\in E$ we have that $\mathcal{H}^1(\widehat{\eta}_t\cap G_0) = 0$, since $|G_0|_2 = 0$. Combining these two facts shows that $\mathcal{H}^1(\eta_t\cap X_0) = 0$ for almost every $t\in E$. □

Lemma E.4.5. *Let $V\subset X$ be open and connected, and $I, J\subset V$ disjoint non-trivial continua. There are $E\subset\mathbb{R}$, $|E|_1 > 0$, and a family $\Gamma' = \{\gamma_t : t\in E\}$ satisfying*

1. *every $\gamma_t\in\Gamma'$ is a non-degenerate curve connecting I and J in V,*
2. *there exists $\varphi\in N^{1,2}_{\mathrm{loc}}(V,\mathbb{R})$ such that for every $t\in E$ the curve $\gamma_t\in\Gamma'$ has image in the level set $\varphi^{-1}(t)$, and*
3. $\operatorname{mod}_{K^{-1}}\Gamma' > 0$.

Proof. Replacing V with a compactly connected subdomain if necessary, we may assume that

$$\int_V K_f(x)\,d\mathcal{H}^2(x) = K < \infty. \tag{E.19}$$

Fix points $a \in I$ and $b \in J$ and a continuous curve η joining a and b in V. Define $\varphi\colon X \to \mathbb{R}$ by $\varphi(x) = \operatorname{dist}(x, |\eta|)$. As described in the proof of [Raj17, Proposition 3.5], we find $\varepsilon' > 0$, a set $E_0 \subset (0, \varepsilon')$ with $\mathcal{H}^1(E_0) = 0$, and for every $t \in E = (0, \varepsilon') \setminus E_0$ a rectifiable injective curve γ_t joining I and J in V, with image in the level set $\varphi^{-1}(t)$. We set $\Gamma' = \{\gamma_t : t \in E\}$.

Let $g\colon V \to [0, \infty]$ be admissible for Γ'. We apply the coarea inequality for Lipschitz maps (Theorem E.2.1) and Hölder's inequality to obtain

$$\begin{aligned}\varepsilon' &\leq \int_0^{\varepsilon'} \int_{\gamma_t} g \, ds \, dt \leq \frac{4}{\pi} \int_V g(x) K_f(x)^{-1/2} K_f(x)^{1/2} \, d\mathcal{H}^2(x) \\ &\leq \frac{4}{\pi} \left(\int_V K_f(x) \, d\mathcal{H}^2(x) \right)^{1/2} \left(\int_V \frac{g(x)^2}{K_f(x)} \, d\mathcal{H}^2(x) \right)^{1/2}.\end{aligned}$$

Combining with (E.19) gives

$$\operatorname{mod}_{K^{-1}} \Gamma' \geq \left(\frac{\pi \varepsilon'}{4K} \right)^2 > 0,$$

where we used that the estimate above holds for all admissible functions. □

If Z is a metric surface, $G \subset Z$ a domain, and $E, F \subset \overline{G}$ disjoint sets, we denote by $\Gamma(E, F; G)$ the family of curves joining E and F in $\overline{G}$.

Lemma E.4.6. *For any $\varepsilon > 0$ the function $g_\varepsilon\colon \mathbb{R}^2 \to [0, \infty)$ defined by*

$$g_\varepsilon(y) = \varepsilon \left(|y| \log \frac{1}{|y|} \log\log \frac{1}{|y|} \right)^{-1} \chi_{\mathbb{D}(0, e^{-2})}$$

is admissible for $\Gamma(\{0\}, \partial\mathbb{D}(0, e^{-2}); \mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} g_\varepsilon(y)^2 \log \frac{1}{|y|} \, dy \to 0 \qquad \textit{as } \varepsilon \to 0.$$

Proof. Fix $\gamma \in \Gamma(\{0\}, \partial\mathbb{D}(0, e^{-2}); \mathbb{R}^2)$. We may assume that $\gamma\colon [0, \ell(\gamma)] \to \mathbb{R}^2$ is parametrized by arclength and $\gamma(0) = 0$. Then $\ell(\gamma) \geq e^{-2}$ and $|\gamma(t)| \leq t$ for every $0 \leq t \leq \ell(\gamma)$. We compute

$$\begin{aligned}\int_\gamma g_1 \, ds &= \int_0^{\ell(\gamma)} g_1(\gamma(t)) \, dt = \int_0^{\ell(\gamma)} \left(|\gamma(t)| \log \frac{1}{|\gamma(t)|} \log\log \frac{1}{|\gamma(t)|} \right)^{-1} dt \\ &\geq \int_0^{e^{-2}} \left(t \log \frac{1}{t} \log\log \frac{1}{t} \right)^{-1} dt = \infty,\end{aligned}$$

where the last equality follows since

$$\frac{d}{ds} \log\log\log \frac{1}{s} = -\left(s \log \frac{1}{s} \log\log \frac{1}{s} \right)^{-1}.$$

Thus, $g_\varepsilon = \varepsilon \cdot g_1$ is admissible for $\Gamma(\{0\}, \partial\mathbb{D}(0, e^{-2}); \mathbb{R}^2)$ for any $\varepsilon > 0$.

In order to prove the second claim we use polar coordinates and compute

$$\begin{aligned}\int_{\mathbb{R}^2} g_\varepsilon(y)^2 \log \frac{1}{|y|} \, dy &= \varepsilon^2 \int_{\mathbb{R}^2} \left(|y|^2 \log \frac{1}{|y|} \left(\log\log \frac{1}{|y|} \right)^2 \right)^{-1} \chi_{\mathbb{D}(0, e^{-2})} \, dy \\ &= \varepsilon^2 \int_0^{2\pi} \int_0^{e^{-2}} \left(r \log \frac{1}{r} \left(\log\log \frac{1}{r} \right)^2 \right)^{-1} dr \, d\varphi.\end{aligned}$$

The last term converges to 0 as $\varepsilon \to 0$ since

$$\frac{d}{ds}\left(\log\log\frac{1}{s}\right)^{-1} = \left(s\log\frac{1}{s}\left(\log\log\frac{1}{s}\right)^2\right)^{-1}.$$

The second claim follows. □

We are now able to prove Proposition E.4.1. Let V_0 be the x_0-component of $B(x_0, s)$. Denote the x_0-component of $f^{-1}(f(x_0)) \cap V_0$ by J. We may assume that $V_0 \setminus f^{-1}(f(x_0)) \neq \emptyset$, since otherwise there is nothing to prove. Towards contradiction, assume that J is a non-trivial continuum. Fix another non-trivial continuum $I \subset V_0 \setminus f^{-1}(f(x_0))$.

By scaling and translating the target we may assume that $f(x_0) = 0$, $f(I) \cap \mathbb{D}(0, e^{-2}) = \emptyset$, and that the constant r_0 in Condition (E.18) satisfies $r_0 \geq e^{-2}$. Let Γ' be the curve family from Lemma E.4.5. Note that $\Gamma = f(\Gamma')$ is a subfamily of $\Gamma(\{0\}, \partial\mathbb{D}(0, e^{-2}); \mathbb{R}^2)$. Hence, we know from Lemma E.4.6 that for any $\varepsilon > 0$ the function g_ε is admissible for Γ. Lemma E.4.4 implies that Lemma E.4.3 can be applied to our setting and thus

$$\operatorname{mod}_{K^{-1}} \Gamma' \leq 4\sqrt{2}\int_{\mathbb{R}^2} g_\varepsilon(y)^2 N(y, f, B(x_0, s))\, dy.$$

Since g_ε is symmetric with respect to the origin, combining Assumption (E.18) with polar coordinates yields

$$\begin{aligned}\int_{\mathbb{R}^2} g_\varepsilon(y)^2 N(y, f, B(x_0, s))\, dy &= \int_0^{e^{-2}} r g_\varepsilon(r)^2 \int_0^{2\pi} N(re^{i\theta}, f, B(x_0, s))\, d\theta\, dr \\ &\leq C\int_0^{e^{-2}} r g_\varepsilon(r)^2 \log\frac{1}{r}\, dr = C\int_{\mathbb{R}^2} g_\varepsilon(y)^2 \log\frac{1}{|y|}\, dy.\end{aligned}$$

By the second part of Lemma E.4.6, the right hand integral converges to 0 as ε goes to 0. Thus, $\operatorname{mod}_{K^{-1}} \Gamma' = 0$, contradicting Lemma E.4.5. The proof is complete.

E.4.2. Proof of Proposition E.4.2

Let x_0 and s be as in the statement. We may assume that $f(x_0) = 0$. We first show that $f^{-1}(y)$ is totally disconnected for most points $y \in f(X)$ around 0.

Lemma E.4.7. *Let β' be the set of those $0 \leq \theta < 2\pi$ for which there is $R_\theta > 0$ so that $f^{-1}(R_\theta e^{i\theta})$ contains a non-degenerate continuum. Then $|\beta'|_1 = 0$.*

Proof. We define

$$\varphi\colon X \setminus f^{-1}(0) \to \mathbb{S}^1, \quad \varphi(x) = \frac{f(x)}{|f(x)|},$$

and note that $\rho_f^u/|f|$ is a weak upper gradient of φ. Towards a contradiction we assume that $|\beta'|_1 > 0$. Then there are $\delta, \varepsilon > 0$ and a set $\beta'_\delta \subset \beta'$, $|\beta'_\delta|_1 > 0$, such that for every $\theta \in \beta'_\delta$ there exists $R_\theta \in [\varepsilon, 1]$ for which $f^{-1}(R_\theta e^{i\theta})$ contains a continuum E_θ with $\mathcal{H}^1(E_\theta) \geq \delta$. As in the proof of Lemma E.4.4, we see that almost every $\theta \in \beta'_\delta$ the continuum E_θ is the image of a rectifiable curve γ_θ, and the modulus of the family of such curves is positive. By the definition of lower gradients and since $f \circ \gamma_\theta$ is constant by construction, we then have that $\rho_f^l = 0$ almost everywhere in

$$E = \bigcup_{\theta \in \beta'_\delta} E_\theta.$$

Furthermore, since f has finite distortion, also $\rho_f^u = 0$ almost everywhere in E. Let

$$F = \{x \in X : |f(x)| \geq \varepsilon,\, \rho_f^u(x) = 0\} \supset E.$$

We apply the Sobolev coarea inequality (Theorem E.2.2) to compute

$$0 < \delta |\beta'_\delta|_1 \leq \int_{\beta'_\delta}^{*} \mathcal{H}^1(E_\theta)\, d\theta \leq \frac{4}{\pi} \int_F \frac{\rho_f^u}{|f|}\, d\mathcal{H}^2 = 0,$$

a contradiction. The proof is complete. □

Lemma E.4.8. *Let β' be the set in Lemma E.4.7. There exists $\beta \supset \beta'$ with $|\beta|_1 = 0$, and an open $\Omega' \subset X$, such that*

1. *$f|_{\Omega'}$ is a local homeomorphism, and*
2. *if $V = \{te^{i\theta} : t > 0, \theta \in \beta\}$, then $\Omega' \supset X \setminus f^{-1}(V)$.*

Proof. Set $V' = \{te^{i\theta} : \theta \in \beta', t > 0\}$. Let $y \in f(X) \setminus V'$ and $x \in f^{-1}(y)$. Then, since $\{x\}$ is a component of $f^{-1}(y)$, there is a Jordan domain $\widetilde{U}_x$ in X such that $x \in \widetilde{U}_x$ and $y \notin f(\partial \widetilde{U}_x)$. Let W_x be the y-component of $\mathbb{R}^2 \setminus f(\partial \widetilde{U}_x)$ and U_x the x-component of $f^{-1}(W_x)$. It follows that $f(\partial U_x) \subset \partial W_x$. Indeed, otherwise there is a point $a \in \partial U_x$ with $f(a) \in W_x$ and therefore there exists a neighbourhood Y of $f(a)$ in W_x, but the a-component of $f^{-1}(Y)$ is not contained in U_x, which is a contradiction.

The assumption that f is sense-preserving now implies $f(\partial U_x) = \partial W_x$. Using basic degree theory, we conclude that $f^{-1}(z)$ has at most $\deg(y, f, U_x)$ components in U_x for every $z \in W_x$. Furthermore, arguing as in the proof of Lemma E.4.7 we see that for almost every such z all of these components are points. In other words,

$$N(z, f, U_x) \leq \deg(y, f, U_x) < \infty$$

for almost every $z \in W_x$. In particular, every $x \in U_x$ satisfies the conditions in Proposition E.4.1, and therefore $f|_{U_x}$ is open and discrete.

We have established the following.

(i) If $y \in f(X) \setminus V'$ and $x \in f^{-1}(y)$, then x has a neighbourhood U_x such that $f|_{U_x}$ is open and discrete.

We define

$$\widehat{\Omega} = \{x \in X : x\text{-component of } f^{-1}(f(x)) \text{ is } \{x\}\}.$$

Note that if $x \in \widehat{\Omega}$, then there exists a neighbourhood Y of $f(x)$ such that the closure of the x-component of $f^{-1}(Y)$ is compact. As above, we find a neighbourhood U_x of x such that $f|_{U_x}$ is open and discrete. In particular, $\widehat{\Omega}$ is open. Moreover, it follows from (i) that $\widehat{\Omega} \supset X \setminus f^{-1}(V')$. We have shown that

(ii) $\widehat{\Omega}$ is open, $f|_{\widehat{\Omega}}$ is open and discrete, and $\widehat{\Omega} \supset X \setminus f^{-1}(V')$.

Denote by $\mathcal{B}_f$ the branch set of $f|_{\widehat{\Omega}}$, i.e., the set of points where $f|_{\widehat{\Omega}}$ fails to be locally invertible, and define

$$\beta'' = \{0 \leq \theta < 2\pi : Re^{i\theta} \in f(\mathcal{B}_f) \text{ for some } R > 0\}.$$

Recall that $\mathcal{B}_f$ is closed and countable, see [Čer64], [Čer65] and [Väi66], thus β'' is countable. It follows from Lemma E.4.7 and (ii) that the sets $\Omega' = \widehat{\Omega} \setminus \mathcal{B}_f$ and $\beta = \beta' \cup \beta''$ possess the desired properties. □

Lemma E.4.9. *Let $m \in \mathbb{N}$, $0 < r < e^{-2}$, and assume that $\overline{B}(x_0, 2s)$ is compact and satisfies $f(\overline{B}(x_0, 2s)) \subset \mathbb{D}(0,1)$. If*

$$E_m = \{0 \le \theta < 2\pi : N(re^{i\theta}, f, B(x_0, s)) = m\},$$

then

$$m|E_m|_1 \le \frac{64\sqrt{2}}{\pi s^2} \int_{F_m} K_f \, d\mathcal{H}^2 \cdot \log \frac{1}{r},$$

where $F_m = \{x \in X : \arg(f(x)) \in E_m\}$.

Proof. We assume $|E_m|_1 > 0$, otherwise there is nothing to show. Let β and Ω' be as in Lemma E.4.8. We set $E'_m = E_m \setminus \beta$ and note that $|E'_m|_1 = |E_m|_1$ since $|\beta|_1 = 0$. We also denote

$$F'_m = \{x \in X : \arg(f(x)) \in E'_m\} \subset F_m.$$

Fix $\theta \in E'_m$, then

$$f^{-1}(\{te^{i\theta} : t \ge r\}) \subset \Omega'.$$

We can therefore apply path lifting of local homeomorphisms to curves $I_\theta = \{te^{i\theta} : r \le t \le 1\}$ as follows: if $\{x_1, ..., x_m\} = f^{-1}(re^{i\theta}) \cap B(x, s)$ then for every $j \in \{1, ..., m\}$ there exists a maximal lift γ_θ^j of I_θ starting at x_j, see [Ric93, Theorem II.3.2]. Note that if $\varphi \colon X \to [0, 2\pi)$ is defined by $\varphi(x) = \arg(f(x))$, then the image of each γ_θ^j is contained in the level set $\varphi^{-1}(\theta)$.

Since $\overline{B}(x, 2s)$ is compact and $f(\overline{B}(x, 2s)) \subset \mathbb{D}(0,1)$, every curve γ_θ^j connects $B(x, s)$ and $X \setminus B(x, 2s)$, and so $\mathcal{H}^1(|\gamma_\theta^j|) \ge s$. Moreover, $f|_{|\gamma_\theta^j|}$ is injective. It follows that

$$s \cdot m \le \sum_{j=1}^{m} \mathcal{H}^1(|\gamma_\theta^j|) \le \mathcal{H}^1(\{x \in X : \arg(f(x)) = \theta\}) \tag{E.20}$$

for every $\theta \in E'_m$.

We combine (E.20) with the Sobolev coarea inequality (Theorem E.2.2) and Hölder's inequality to compute

$$\begin{aligned} s \cdot m \cdot |E_m|_1 = s \cdot m \cdot |E'_m|_1 &\le \int_{E'_m} \mathcal{H}^1(\{x \in X : \arg(f(x)) = \theta\}) \, d\theta \\ &\le \frac{4}{\pi} \int_{F_m} \frac{\rho_f^u}{|f|} \, d\mathcal{H}^2 \le \frac{4}{\pi} \int_{F_m} K_f^{1/2} \cdot \frac{(\rho_f^u \cdot \rho_f^l)^{1/2}}{|f|} \, d\mathcal{H}^2 \\ &\le \frac{4}{\pi} \left(\int_{F_m} K_f \, d\mathcal{H}^2 \right)^{1/2} \underbrace{\left(\int_{F'_m} \frac{\rho_f^u \cdot \rho_f^l}{|f|^2} \, d\mathcal{H}^2 \right)^{1/2}}_{=:I}. \end{aligned}$$

For each $j \in \{1, ..., m\}$ we define the curve family

$$\Gamma'_j = \{\gamma_\theta^j : t \in E'_m\}.$$

Lemma E.4.4 applied to Γ'_j shows that $\mathcal{H}^1(|\gamma_\theta^j| \cap X_0) = 0$ for almost every $\theta \in E'_m$ and every $j \in \{1, ..., m\}$, where X_0 is as in Theorem E.3.1. Hence, if

$$F''_m = \{x \in X : \, x \in |\gamma_\theta^j| \text{ for some } \theta \in E'_m \text{ and } 1 \le j \le m\} \supset F'_m,$$

then $\mathcal{H}^2(F''_m \cap X_0) = 0$ and $N(y, f, F''_m) \le m$ for every $y \in \mathbb{R}^2$. By the area inequality (Theorem

E.3.1) and polar coordinates,

$$I \leq 4\sqrt{2}\int_{E_m}\int_r^1 \frac{N(se^{i\theta}, f, F_m'')}{s}\, ds\, d\theta \leq 4\sqrt{2}\cdot |E_m|_1 \cdot m \cdot \log\frac{1}{r}.$$

The lemma follows by combining the estimates. □

Proposition E.4.2 follows from Lemma E.4.9: notice that by scaling we may assume that $f(\overline{B}(x_0, 2s)) \subset \mathbb{D}(0,1)$, so that the conditions of Lemma E.4.9 are satisfied. Recall that the sets F_m are pairwise disjoint. Therefore, summing the estimate in Lemma E.4.9 over m gives

$$\begin{aligned}\int_0^{2\pi} N(re^{i\theta}, f, B(x_0,s))\, d\theta = \sum_{m=1}^{\infty} m|E_m|_1 &\leq C\log\frac{1}{r}\sum_{m=1}^{\infty}\int_{F_m} K_f(x)\, d\mathcal{H}^2 \\ &\leq C\log\frac{1}{r}\int_X K_f(x)\, d\mathcal{H}^2.\end{aligned}$$

We may replace X with a compactly contained subdomain if necessary to guarantee that K_f is integrable. Proposition E.4.2 follows.

E.5. Regularity of the inverse

In this section we study the regularity of the inverse of a mapping of finite distortion and prove Theorem E.1.3. Let $f \in N^{1,2}_{\mathrm{loc}}(X, \Omega')$ be a homeomorphism with $K_f \in L^1_{\mathrm{loc}}(X)$, where $\Omega' \subset \mathbb{R}^2$. We set $\phi = f^{-1}\colon \Omega' \to X$ and define $\psi\colon \Omega' \to [0,\infty]$ by

$$\psi(y) = \frac{1}{\rho_f^l(\phi(y))}.$$

Lemma E.5.1. *We have*

$$\int_E \psi(y)^2\, dy \leq 2\int_{\phi(E)} K_f(x)\, d\mathcal{H}^2(x)$$

for every Borel set $E \subset \Omega'$. In particular, $\psi \in L^2_{\mathrm{loc}}(\Omega')$.

Proof. Again, let $u : U \to X$, $U \subset \mathbb{R}^2$, be a weakly $(4/\pi)$-quasiconformal parametrization and $h = f \circ u$. Then h is locally in $N^{1,2}(U, \mathbb{R}^2)$ and monotone. Therefore, h satisfies Condition (N) and consequently the euclidean area formula, see [MM95]. Combining the area formula with distortion estimates established in previous sections, we have

$$\begin{aligned}\int_E \psi(y)^2\, dy &= \int_{h^{-1}(E)} \frac{J(\operatorname{ap}\operatorname{md} h_z)}{\rho_f^l(u(z))^2}\, dz = \int_{h^{-1}(E)} \frac{L_h(z)\cdot l_h(z)}{\rho_f^l(u(z))^2}\, dz \\ &\leq \int_{h^{-1}(E)} \frac{\rho_f^u(u(z))\cdot\rho_f^l(u(z))}{\rho_f^l(u(z))^2} L_u(z)^2\, dz \\ &\leq 2\int_{h^{-1}(E)} K_f(u(z))\cdot J(\operatorname{ap}\operatorname{md} u_z)\, dz.\end{aligned}$$

Here the second equality holds since both the domain and target of h are euclidean domains and the first inequality holds by Lemma E.2.10 and Proposition E.3.2. The second inequality holds by (E.11) and recalling that we can choose u so that the John ellipses of $\operatorname{ap}\operatorname{md} u_z$ are disks for almost every z. The claim now follows from the area formula for u (Theorem E.2.6). □

Lemma E.5.2. *Suppose $\alpha : X \to \mathbb{R}$ is 1-Lipschitz. Then $v = \alpha \circ \phi$ is absolutely continuous on almost every line parallel to coordinate axes, and $|\partial_j v| \leq \frac{16\sqrt{2}}{\pi} \cdot \psi$ almost everywhere for $j = 1, 2$.*

Proof. It suffices to consider horizontal lines. Fix a square $Q \subset \Omega'$ with sides parallel to coordinate axes. By scaling and translating, we may assume that $Q = [0,1]^2$.

By Lebesgue's theorem, there exists a set $\Phi \subset (0,1)$ of full measure so that if $s_0 \in \Phi$ then

$$\frac{1}{2\varepsilon} \int_{F_\varepsilon} \psi(y)\, dy = \frac{1}{2\varepsilon} \int_{s_0-\varepsilon}^{s_0+\varepsilon} \int_{t_1}^{t_2} \psi(t,s)\, dt\, ds \to \int_{t_1}^{t_2} \psi(t, s_0)\, dt \tag{E.21}$$

as $\varepsilon \to 0$ for every $0 \leq t_1 < t_2 \leq 1$, where $F_\varepsilon = [t_1, t_2] \times [s_0 - \varepsilon, s_0 + \varepsilon]$.

Fix $s_0 \in \Phi$. The claim now follows from Lemma E.5.1 if we can show that

$$|\phi(t_2, s_0) - \phi(t_1, s_0)| \leq \frac{16\sqrt{2}}{\pi} \int_{t_1}^{t_2} \psi(t, s_0)\, dt \tag{E.22}$$

for every $0 \leq t_1 < t_2 \leq 1$.

Given $0 < \varepsilon < \min\{s_0, 1 - s_0\}$ we set $E_\varepsilon = \phi(F_\varepsilon)$. Let $\varphi = \pi_2 \circ f|_{E_\varepsilon}$, where π_2 denotes projection to the s-axis on the (t,s)-plane. By continuity of φ, Lemma E.4.4, and the Sobolev coarea inequality (Theorem E.2.2) applied to φ, we have

$$\begin{aligned} |\phi(t_2, s_0) - \phi(t_1, s_0)| &\leq \delta(\varepsilon) + \frac{1}{2\varepsilon} \int_{s_0-\varepsilon}^{s_0+\varepsilon} \mathcal{H}^1(\varphi^{-1}(s) \setminus X_0)\, ds \\ &\leq \delta(\varepsilon) + \frac{2}{\pi\varepsilon} \int_{E_\varepsilon \setminus X_0} \frac{\rho_f^u \cdot \rho_f^l}{\rho_f^l} \chi_{\rho_f^l \neq 0}\, d\mathcal{H}^2, \end{aligned}$$

where X_0 is the set in the Area inequality (Theorem E.3.1) and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Combining with Theorem E.3.1, we obtain

$$|\phi(t_2, s_0) - \phi(t_1, s_0)| \leq \delta(\varepsilon) + \frac{8\sqrt{2}}{\pi\varepsilon} \int_{F_\varepsilon} \psi(y)\, dy. \tag{E.23}$$

Now (E.22) follows by combining (E.23) and (E.21). ☐

We are ready to prove Theorem E.1.3. Since ϕ is continuous, $d_X(\phi(\cdot), x_0) \in L^2_{\mathrm{loc}}(\Omega')$ for every $x_0 \in X$. By Lemma E.5.1 and the ACL-characterization of Sobolev functions (see [HKST15, Theorem 6.1.17]), we see that every v in Lemma E.5.2 belongs to $W^{1,2}_{\mathrm{loc}}(\Omega')$ and satisfies $|\nabla v| \leq \frac{32\psi}{\pi}$ almost everywhere. Furthermore, the characterization of Sobolev maps in terms of post-compositions with 1-Lipschitz functions, i.e., in terms of the functions v above (see [HKST15, Theorem 7.1.20 and Proposition 7.1.36]), shows that $\phi \in N^{1,2}_{\mathrm{loc}}(\Omega', X)$. The proof is complete.

Remark E.5.3. When $X \subset \mathbb{R}^2$, the $N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$-regularity assumption in Theorem E.1.3 may be replaced with $f \in N^{1,1}_{\mathrm{loc}}(X, \mathbb{R}^2)$. Moreover, the conclusion on the regularity of f^{-1} is more precise, see [HK06]. While our results only concern $N^{1,2}_{\mathrm{loc}}$-maps, it would be interesting to extend the definition of finite distortion to $N^{1,1}_{\mathrm{loc}}$-maps between metric surfaces and develop basic properties including improvements of Theorem E.1.3. One cannot expect the conclusions of Remarks E.2.3 and E.2.8 to hold in the $N^{1,1}$-setting without additional assumptions; maps $f \in N^{1,1}_{\mathrm{loc}}(X, \mathbb{R}^2)$ of finite distortion need not be continuous or satisfy Condition (N) even when $X \subset \mathbb{R}^2$ (see e.g. [HK14]).

E.6. Reciprocal surfaces

Recall the *geometric definition of quasiconformality*: a homeomorphism $f\colon X \to Y$ is *quasiconformal* if there exists $C \geq 1$ such that

$$C^{-1} \operatorname{mod} f(\Gamma) \leq \operatorname{mod} \Gamma \leq C \operatorname{mod} f(\Gamma) \tag{E.24}$$

for each curve family Γ in X.

We say that metric surface X is *reciprocal* if there exists $\kappa > 0$ such that for every topological quadrilateral $Q \subset X$ and for the families $\Gamma(Q)$ and $\Gamma^*(Q)$ of curves joining opposite sides of Q we have

$$\operatorname{mod} \Gamma(Q) \cdot \operatorname{mod} \Gamma^*(Q) \leq \kappa.$$

If X is reciprocal, $x \in X$ and $R > 0$ so that $X \setminus B(x, R) \neq \emptyset$, then by [NR24, Theorem 1.8] we have

$$\lim_{r \to 0} \operatorname{mod} \Gamma(B(x, r), X \setminus B(x, R); X) = 0. \tag{E.25}$$

Recall that $\Gamma(E, F; G)$ is the family of curves joining E and F in $\overline{G}$.

Reciprocal surfaces are the metric surfaces that admit quasiconformal parametrizations by euclidean domains, see [Raj17], [Iko22], [NR24]. See [Raj17], [RR19], [EBPC22], [B], [NR23] and [NR24] for further properties of reciprocal surfaces.

It is desirable to find non-trivial conditions which imply reciprocality. For instance, one could hope that the existence of maps satisfying the conditions of Theorem E.1.2 forces X to be reciprocal. However, this is not the case.

Proposition E.6.1. *Given an increasing $\phi : [1, \infty) \to [1, \infty)$ so that $\phi(t) \to \infty$ as $t \to \infty$, there is a non-reciprocal metric surface X and a homeomorphism $f\colon X \to \mathbb{R}^2$ so that $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ and $\phi(K_f)$ is locally integrable.*

The map f_0 defined in the proof below is known as Ball's map ([Bal81]) and illustrates that the integrability condition in Theorem E.1.2 is sharp.

Proof. Let $f_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_0(x, y) = (x, \eta(x, y))$, where

$$\eta(x, y) = \begin{cases} |x|y, & 0 \leq |x| \leq 1,\ 0 \leq |y| \leq 1, \\ (2(|y| - 1) + |x|(2 - |y|))\frac{y}{|y|}, & 0 \leq |x| \leq 1,\ 1 \leq |y| \leq 2, \\ y, & \text{otherwise.} \end{cases}$$

Note that f_0 is not open and discrete since it maps the segment $I = \{0\} \times [-1, 1]$ to the origin. Also, f_0 is the identity outside $(-1, 1) \times (-2, 2)$. Calculating the Jacobian matrix shows that f_0 is sense-preserving and Lipschitz, K_{f_0} is bounded outside $(-1, 1) \times (-1, 1)$, and

$$K_{f_0}(x, y) = \frac{1}{|x|} \quad \text{for all } (x, y) \in (-1, 1) \times (-1, 1). \tag{E.26}$$

It follows that K_{f_0} is not in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ but $K_{f_0} \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for every $0 < p < 1$.

We change the metric on $\mathbb{R}^2$ to obtain the desired metric surface X and $f\colon X \to \mathbb{R}^2$. Define $\omega : \mathbb{R}^2 \to [0, 1]$ by $\omega(z) = 1$ when $\operatorname{dist}(z, I) \geq 1$ and by

$$\omega(z) = \frac{1}{\phi(\operatorname{dist}(z, I)^{-1})} \tag{E.27}$$

otherwise, where $I = \{0\} \times [-1,1]$. Moreover, let

$$d_\omega(x,y) := \inf_\gamma \int_\gamma \omega\, ds,$$

where the infimum is taken over all rectifiable curves γ connecting $x, y \in \mathbb{R}^2$.

Now $X = (\mathbb{R}^2/I, d_\omega)$ is homeomorphic to $\mathbb{R}^2$ and has locally finite $\mathcal{H}^2$-measure. Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/I$ be the projection map, $\mathrm{id}_\omega\colon \mathbb{R}^2/I \to X$ the identity, and $\pi_\omega\colon \mathbb{R}^2 \to X$, $\pi_\omega = \mathrm{id}_\omega \circ \pi$.

Since modulus is conformally invariant and ω is a conformal change of metric outside I, the family of curves joining any non-trivial continuum and the point $p := \pi_\omega(I)$ in X has positive modulus. By (E.25), it follows that X is non-reciprocal.

We define $f : X \to \mathbb{R}^2$ by $f := f_0 \circ \pi_\omega^{-1}$. Then f is absolutely continuous on almost every rectifiable curve in X, and $\rho_f^u(z) \leq (\omega(z))^{-1} \cdot L$ for almost every $z \in X$, where L is the Lipschitz constant of f_0. Therefore,

$$\int_E (\rho_f^u)^2 \, d\mathcal{H}^2 \leq L^2 |\pi_\omega^{-1}(E)|_2$$

for every Borel set $E \subset X$. We conclude that $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$.

It remains to estimate the integral of $\phi(K_f)$. To this end, notice that since ω is a conformal change of metric, we have

$$K_f(z) = K_{f_0}(\pi_\omega^{-1}(z))$$

for almost every $z \in X$. Therefore, it suffices to check that $\phi(K_f)$ is integrable over $E = \pi_\omega((-1,1) \times (-1,1))$. By (E.26) and (E.27), we have

$$\int_E \phi(K_f(z))\, d\mathcal{H}^2 = \int_{(-1,1)^2} \phi(K_{f_0}) \cdot \omega^2 \, dx\, dy \leq \int_{(-1,1)^2} \frac{1}{\phi(|x|^{-1})} \, dx\, dy < \infty.$$

The proof is complete. $\square$

We prove in [F, Theorem 1.3] that if there is a non-constant $f \in \mathrm{FDP}(X, \mathbb{R}^2)$ (not necessarily a homeomorphism) with *bounded* distortion, then X is reciprocal. We also show (see [F, Corollary 1.2]) that the geometric definition (E.24) is quantitatively equivalent with the path definition (requiring K_f to be bounded) of quasiconformality for homeomorphisms $f : X \to \mathbb{R}^2$. By Williams' theorem [Wil12], the equivalence between the analytic (requiring $C(x)$ to be bounded in (E.5)) and geometric definitions of quasiconformality for homeomorphisms holds in even greater generality.

E.7. Existence of maximal weak lower gradients

Let X and Y be metric surfaces. We now complete the discussion in Section E.2.3 by proving that each $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ has a *maximal weak lower gradient*. Precisely, we claim that there is a weak lower gradient ρ_f^l of f so that if ρ^l is another weak lower gradient of f then

$$\rho_f^l(x) \geq \rho^l(x) \quad \text{for almost every } x \in X.$$

Moreover, ρ_f^l is unique up to a set of measure zero. The proof of these facts is analogous to the existence of minimal weak upper gradients, see [HKST15, Theorem 6.3.20].

First, recall that f is absolutely continuous along almost every curve [HKST15, Lemma 6.3.1]. It follows from [HKST15, Lemma 5.2.16] that if ρ is a weak lower gradient of f and $\sigma\colon X \to [0,\infty]$ is a Borel function such that $\sigma = \rho$ almost everywhere in X, then σ is a weak lower gradient of f. In particular, if $E \subset X$ is Borel and satisfies $\mathcal{H}^2(E) = 0$ then $\rho\chi_{X \setminus E}$ is a weak lower gradient of

u, compare to [HKST15, Lemma 6.2.8]. We conclude that if there exists a maximal weak lower gradient ρ_f^l of f, it has to be unique up to sets of measure zero.

To prove existence of ρ_f^l, we may assume without loss of generality that $\mathcal{H}^2(X) < \infty$. Arguing exactly as in the proof of [HKST15, Lemma 6.3.8], we can show that if $\sigma, \tau \in L^2(X)$ are weak lower gradients of a map $f\colon X \to Y$ that is absolutely continuous along almost every curve in X and if E is a measurable subset of X then the function

$$\rho = \sigma \cdot \chi_E + \tau \cdot \chi_{X \setminus E}$$

is a weak lower gradient of f. Now, by choosing $E = \{x \in X : \sigma > \tau\}$, it follows that $\rho\colon X \to [0, \infty]$ defined by

$$\rho(x) = \max\{\sigma(x), \tau(x)\}$$

is a 2-integrable weak lower gradient of f. After applying Fuglede's lemma, see e.g. [HKST15, Section 5.1], we established the following lemma.

Lemma E.7.1. *If $f\colon X \to Y$ is absolutely continuous along almost every curve, then the collection $\mathcal{L}$ of 2-integrable weak lower gradients of f is closed under pointwise maximum operations.*

Let $(\rho_i) \subset \mathcal{L}$ be a sequence such that

$$\lim_{i \to \infty} ||\rho_i||_{L^2} = \sup\{||\rho||_{L^2} : \rho \in \mathcal{L}\}.$$

By Lemma E.7.1, the sequence (ρ_i') given by $\rho_i'(x) = \max_{1 \le j \le i} \rho_j(x)$ is in $\mathcal{L}$. Note that (ρ_i') is pointwise increasing. The limit function

$$\rho_f^l := \lim_{i \to \infty} \rho_i'$$

is Borel by [HKST15, Proposition 3.3.22]. The monotone convergence theorem implies that $\rho_i' \to \rho_f^l$ in $L^2(X)$ and by Fuglede's lemma $\rho_f^l \in \mathcal{L}$, see e.g. [HKST15, Section 5.1]. By construction, ρ_f^l is a maximal weak lower gradient of f. The proof is complete.

F. Definitions of quasiconformality on metric surfaces

with Kai Rajala

Abstract. We explore the interplay between different definitions of distortion for mappings $f\colon X \to \mathbb{R}^2$, where X is any metric surface, meaning that X is homeomorphic to a domain in $\mathbb{R}^2$ and has locally finite 2-dimensional Hausdorff measure. We establish that finite distortion in terms of the familiar analytic definition always implies finite distortion in terms of *maximal and minimal stretchings along paths.* The converse holds for maps with locally integrable distortion. In particular, we prove the equivalence of various notions of quasiconformality, implying a novel uniformization result for metric surfaces.

F.1. Introduction

Within this note we study the relation between different notions of distortion for mappings on metric surfaces. Here, a *metric surface* X is a metric space homeomorphic to a domain in $\mathbb{R}^2$ with locally finite 2-dimensional Hausdorff measure. Most importantly, we show that locally integrable *distortion along paths* introduced by the authors in [E] is comparable to the analytic distortion for mappings $f : X \to \mathbb{R}^2$.

Before stating the main theorem, we provide the relevant definitions. Let X and Y be metric surfaces and consider the Newton-Sobolev space $N^{1,2}_{\mathrm{loc}}(X,Y)$, see Section F.2.3. We call a map $f : X \to Y$ *sense-preserving* if for any domain Ω compactly contained in X so that $f|_{\partial\Omega}$ is continuous it follows that $\deg(y, f, \Omega) \geq 1$ for any $y \in f(\Omega) \setminus f(\partial\Omega)$. Here, deg is the local topological degree of f (see [Ric93, I.4]).

Let $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ be sense-preserving. We say that f has *finite distortion along paths* and denote $f \in \mathrm{FDP}(X,Y)$ if there is a measurable $K\colon X \to [1,\infty)$ such that

$$\rho_f^u(x) \leq K(x) \cdot \rho_f^l(x) \quad \text{for almost every } x \in X, \tag{F.1}$$

where ρ_f^u and ρ_f^l denote the minimal weak upper and maximal weak lower gradient of f, respectively; for definitions see Section F.2.3 and Section F.2.5. The *distortion along paths* K_f of f is

$$K_f(x) := \begin{cases} \frac{\rho_f^u(x)}{\rho_f^l(x)}, & \text{if } \rho_f^l(x) \neq 0, \\ 1, & \text{if } \rho_f^l(x) = 0. \end{cases}$$

We say that f has *finite analytic distortion*, denoted $f \in \mathrm{FDA}(X,Y)$, if there is a measurable $C\colon X \to [1,\infty)$ such that

$$\rho_f^u(x)^2 \leq C(x) \cdot J_f(x) \quad \text{for almost every } x \in X, \tag{F.2}$$

where

$$J_f(x) = \limsup_{r\to 0} \frac{\mathcal{H}_Y^2(f(\overline{B}(x,r)))}{\pi r^2}. \tag{F.3}$$

If f is a homeomorphism and X is a domain in $\mathbb{R}^2$ or $Y = \mathbb{R}^2$, then J_f coincides with the Radon-Nikodym derivative of the corresponding pull-back measure with respect to $\mathcal{H}_X^2$, see Corollary F.3.4. Notice, however, that such a pull-back is not defined for non-homeomorphic maps.

The *analytic distortion* C_f of f is

$$C_f(x) := \begin{cases} \frac{\rho_f^u(x)^2}{J_f(x)}, & \text{if } J_f(x) \neq 0, \\ 1, & \text{if } J_f(x) = 0. \end{cases}$$

Inequality (F.2) is equivalent to (F.1) whenever f is a map between euclidean domains. However, in the generality of metric spaces, it is unclear how the two definitions relate. The following main theorem of this work shows equivalence for mappings from a metric surface into $\mathbb{R}^2$.

Theorem F.1.1. *Let $f \in N^{1,2}_{\mathrm{loc}}(X,\mathbb{R}^2)$ be sense-preserving.*

1. *If $f \in \mathrm{FDA}(X,\mathbb{R}^2)$, then $f \in \mathrm{FDP}(X,\mathbb{R}^2)$ and*

$$K_f(x) \leq 4\sqrt{2}\, C_f(x) \quad \textit{for almost every } x \in X.$$

2. *If $f \in \mathrm{FDP}(X,\mathbb{R}^2)$ and $K_f \in L^1_{\mathrm{loc}}(X)$, then $f \in \mathrm{FDA}(X,\mathbb{R}^2)$ and*

$$C_f(x) \leq 4\sqrt{2}\, K_f(x) \quad \textit{for almost every } x \in X.$$

We do not know if the second part holds without assumption $K_f \in L^1_{\mathrm{loc}}(X)$. As a consequence of Theorem F.1.1 we obtain the following characterization of quasiconformal homeomorphisms.

Corollary F.1.2. *If $f\colon X \to f(X) \subset \mathbb{R}^2$ is a homeomorphism, then the following conditions are quantitatively equivalent.*

1. *f is analytically quasiconformal,*
2. *f is geometrically quasiconformal,*
3. *f is quasiconformal along paths.*

Moreover, if f satisfies any of the three conditions, then so does f^{-1}.

Here a homeomorphism $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ is *analytically quasiconformal* (resp., *quasiconformal along paths*), if C_f (resp., K_f) is uniformly bounded. Moreover, f is *geometrically quasiconformal* if there is $C \geq 1$ such that

$$C^{-1} \cdot \operatorname{mod}\Gamma \leq \operatorname{mod} f(\Gamma) \leq C \cdot \operatorname{mod}\Gamma \tag{F.4}$$

for each curve family Γ in X, where mod refers to 2-modulus, see Section F.2.2, and $f(\Gamma)$ denotes the family of curves $f \circ \gamma$ for $\gamma \in \Gamma$.

There is a large body of literature on different definitions of quasiconformality in metric spaces, showing in particular the equivalence of the *metric definition* with the analytic and geometric definitions for homeomorphisms between metric spaces with *controlled geometry*, see [BKR07], [HK95], [HK98], [HKST01], [Tys98], [Tys01]. However, in the generality of metric surfaces the metric definition is not equivalent with the other definitions (see [RRR21, Section 5]), and does not lead to a satisfactory theory.

To prove the equivalence of Conditions (1) and (2) in Corollary F.1.2 we apply Williams' theorem [Wil12]. The target $\mathbb{R}^2$ cannot be replaced with a general metric surface Y in Corollary F.1.2 (see e.g. [Raj17, Example 2.1]), even if $X = \mathbb{R}^2$ and f, f^{-1} are both metrically quasiconformal (see [Rom19b], [NR21]). We do not know if Theorem F.1.1 holds for such general targets Y.

Lower gradients and the class $\mathrm{FDP}(X, Y)$ were introduced in [E] as a tool for developing the fundamental properties of non-homeomorphic maps under minimal assumptions. In particular, we proved in [E] that a non-constant $f \in \mathrm{FDP}(X, \mathbb{R}^2)$ with $K_f \in L^1_{\mathrm{loc}}(X)$ is continuous, discrete and open. Non-homeomorphic maps with controlled distortion in metric spaces have previously been considered e.g. in [Cri06], [Guo15], [Kir14], [OR09].

Theorem F.1.1 can be applied to the *uniformization problem* of metric surfaces (see e.g. [BK02, Raj17, LW17a, LW18a, Iko22, B, NR23, C]) as follows. Here $f \in \mathrm{FDP}(X, Y)$ is *quasiregular*, or has *bounded distortion* (along paths), if K_f is uniformly bounded.

Theorem F.1.3. *If X admits a non-constant quasiregular map $f : X \to \mathbb{R}^2$, then X admits a quasiconformal homeomorphism $\phi : X \to U$ onto a domain $U \subset \mathbb{R}^2$.*

Non-homeomorphic maps are easier to construct than homeomorphisms, so Theorem F.1.3 offers flexibility for finding quasiconformal parametrizations of a given surface. Theorem F.1.3 is sharp in the following sense: There is no $p \geq 1$ for which the existence of a non-constant $f \in \mathrm{FDP}(X, \mathbb{R}^2)$ with $K_f \in L^p_{\mathrm{loc}}(X)$ implies the existence of a quasiconformal homeomorphism $\phi : X \to U$ onto a domain $U \subset \mathbb{R}^2$, see [E, Proposition 6.1].

Theorem F.1.1 also allows the extension of the classical *Stoïlow factorization theorem* (see [AIM09, Chapter 5.5], [LP20]) to our setting.

Theorem F.1.4. *Every non-constant quasiregular map $f : X \to \mathbb{R}^2$ admits a factorization $f = g \circ v$, where $v\colon X \to V$ is a quasiconformal homeomorphism onto a domain $V \subset \mathbb{R}^2$ and $g\colon V \to \mathbb{R}^2$ is complex analytic.*

The proof of Theorem F.1.1 depends on plane topology and recent results on the uniformization problem, see Section F.2. Such methods fail in higher dimensions. It would be interesting to find higher-dimensional versions of Theorem F.1.1 under minimal assumptions on a metric n-manifold X.

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F.2. Preliminaries

F.2.1. Basic definitions and notations

Let (X, d) be a metric space. We denote the *open ball* in X of radius $r > 0$ centered at a point $x \in X$ by $B(x, r)$. If $B = B(x, r)$ is a ball and $k > 0$, we denote by kB the ball $B(x, kr)$. We say that a subdomain Ω of X is *compactly contained* in X if the closure $\overline{\Omega}$ is compact. Given a set $A \subset X$ and $\delta > 0$, we denote the closed δ-neighborhood of A in X by $N_\delta(A)$.

The *image* of a curve γ in X is indicated by $|\gamma|$ and the *length* by $\ell(\gamma)$. A curve γ is *rectifiable* if $\ell(\gamma) < \infty$ and *locally rectifiable* if each of its compact subcurves is rectifiable. If $\gamma\colon [a, b] \to X$ is rectifiable, then for almost every $t \in [a, b]$ we can define the *metric differential* of γ at t by

$$\gamma'(t) := \lim_{s \to t,\, s \neq t} \frac{d(\gamma(t), \gamma(s))}{|t - s|}.$$

A curve $\gamma\colon [0,\ell(\gamma)] \to X$ is *parametrized by arclength* if $\ell(\gamma|_I) = |I|_1$ for every interval $I \subset [0,\ell(\gamma)]$. Here and later on, $|\cdot|_n$ denotes the *n-dimensional Lebesgue measure*.

For $s \geq 0$, we denote the *s-dimensional Hausdorff measure* of a set $A \subset X$ by $\mathcal{H}^s(A)$ or $\mathcal{H}^s_X(A)$ if we want to emphasize that A is a subset of X. The normalizing constant is chosen so that $|U|_n = \mathcal{H}^n(U)$ for open subsets U of $\mathbb{R}^n$.

If X is a metric surface, we equip X with $\mathcal{H}^2$. Let $L^p(X)$ ($L^p_{\mathrm{loc}}(X)$) denote the *space of p-integrable (locally p-integrable) Borel functions* from X to $\mathbb{R}\cup\{-\infty,\infty\}$. Here locally p-integrable means p-integrable on compact subsets.

F.2.2. Modulus

Let X be a metric surface and Γ a family of curves in X. A Borel function $g\colon X \to [0,\infty]$ is *admissible* for Γ if $\int_\gamma g\,ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$. We define the (2-)*modulus* of Γ as

$$\operatorname{mod}\Gamma = \inf_g \int_X g^2\,d\mathcal{H}^2,$$

where the infimum is taken over all admissible functions g for Γ. If there are no admissible functions for Γ we set $\operatorname{mod}\Gamma = \infty$. A property is said to hold for *almost every* curve in Γ if it holds for every curve in $\Gamma \setminus \Gamma_0$ for some family $\Gamma_0 \subset \Gamma$ with $\operatorname{mod}(\Gamma_0) = 0$.

F.2.3. Metric Sobolev spaces

Let $f\colon X \to Y$ be a map from metric surface X to a metric space Y. A Borel function $\rho^u\colon X \to [0,\infty]$ is an *upper gradient* of f if

$$d_Y(f(x), f(y)) \leq \int_\gamma \rho^u\,ds \tag{F.5}$$

for all $x, y \in X$ and every rectifiable curve γ in X joining x and y. If the *upper gradient inequality* (F.5) holds for almost every rectifiable curve γ in X joining x and y we call ρ^u *weak upper gradient* of f.

The Sobolev space $N^{1,2}(X,Y)$ is the space of Borel maps $f\colon X \to Y$ with upper gradient $\rho^u \in L^2(X)$ such that $x \mapsto d_Y(y, f(x))$ is in $L^2(X)$ for some $y \in Y$. The space $N^{1,2}_{\mathrm{loc}}(X,Y)$ is defined in the obvious manner. Each $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ has a *minimal weak upper gradient* ρ^u_f, i.e., for any other weak upper gradient ρ^u we have $\rho^u_f \leq \rho^u$ almost everywhere. Moreover, ρ^u_f is unique up to a set of measure zero, see [HKST15, Theorem 6.3.20]. We refer to the monograph [HKST15] for more background on metric Sobolev spaces.

F.2.4. Metric differentiability

Let X be a metric surface and $U \subset \mathbb{R}^2$ a domain. We say that $u\colon U \to X$ is *approximately metrically differentiable* at $z \in U$ if there exists a seminorm N_z on $\mathbb{R}^2$ for which

$$\operatorname{ap}\lim_{y\to z} \frac{d_X(u(y), u(z)) - N_z(y-z)}{|y-z|} = 0.$$

Here $\operatorname{ap}\lim$ denotes the approximate limit (see [EG92, Section 1.7.2]). If such a seminorm exists, it is unique and is called *approximate metric derivative* of u at z, denoted $\operatorname{ap}\operatorname{md} u_z$. The *Jacobian* of $\operatorname{ap}\operatorname{md} u_z$ is

$$J(\operatorname{ap}\operatorname{md} u_z) = \frac{\pi}{|B_z|_2},$$

whenever $\operatorname{ap}\operatorname{md} u_z$ is a norm and $J(\operatorname{ap}\operatorname{md} u_z) = 0$ otherwise. Here B_z refers to the closed unit ball in $(\mathbb{R}^2, \operatorname{ap}\operatorname{md} u_z)$. Every map $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ is approximately metrically differentiable at almost every $z \in U$, see [LW17a, Proposition 4.3].

Let $U \subset \mathbb{R}^2$ be a domain and $u \in N^{1,2}_{\mathrm{loc}}(U, X)$. By [HKST15, Theorem 8.1.49], U is the union of pairwise disjoint Borel sets G^u_j, $j = 0, 1, \ldots$, so that $|G^u_0|_2 = 0$ and $u|_{G^u_j}$ is j-Lipschitz continuous for every $j \geq 1$. Recall the classical area formula following from [Kar07, Theorem 3.2].

Theorem F.2.1 (Classical area formula)**.** *For $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ and every measurable set $A \subset U \setminus G^u_0$ we have*

$$\int_A J(\operatorname{ap}\operatorname{md} u_z)\, dz = \int_X N(x, u, A)\, d\mathcal{H}^2.$$

Here $N(x, u, A)$ denotes the number of preimages of x under u in A. If $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ is a homeomorphism, Theorem F.2.1 implies $J_u(z) = J(\operatorname{ap}\operatorname{md} u_z)$ for almost every $z \in U$; recall the definition of $J_u(z)$ in (F.3).

F.2.5. Lower gradients and distortion along paths

Let X and Y be metric surfaces. We call a Borel function $\rho^l \colon X \to [0, \infty]$ a *lower gradient* of $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ if $\rho^l \leq \rho^u_f$ almost everywhere and

$$\ell(f \circ \gamma) \geq \int_\gamma \rho^l\, ds \tag{F.6}$$

for every rectifiable curve γ in X such that f is continuous along γ. If the *lower gradient inequality* (F.6) holds for almost every rectifiable γ on which f is continuous, we call ρ^l *weak lower gradient* of f. Note that 0 is always a lower gradient. Up to exceptional curve families of zero modulus, the upper gradient inequality (F.5) is equivalent to the converse inequality in (F.6). Each $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ has a *maximal weak lower gradient* ρ^l_f, i.e., for any other weak lower gradient ρ^l we have $\rho^l_f \geq \rho^l$ almost everywhere, that is uniquely defined up to sets of measure zero, see [E, Section 7].

Mappings of finite distortion along paths, i.e., class $\mathrm{FDP}(X, Y)$ (defined in the introduction), were introduced in [E]. We now state the most important results from [E] that will be repeatedly used throughout this work.

Proposition F.2.2 ([E, Remarks 2.3 and 2.8])**.** *Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be sense-preserving. Then, f is continuous and satisfies Lusin's condition (N).*

Here a map $f \colon X \to Y$ satisfies *Lusin's condition (N)* if $\mathcal{H}^2_Y(f(E)) = 0$ for every $E \subset X$ with $\mathcal{H}^2_X(E) = 0$. Recall that f is *discrete*, if $f^{-1}(y)$ is a discrete set in X for every $y \in Y$.

Theorem F.2.3 ([E, Theorem 1.2])**.** *Let $f \in \mathrm{FDP}(X, \mathbb{R}^2)$ be non-constant with $K_f \in L^1_{\mathrm{loc}}(X)$. Then f is open and discrete.*

Let $U \subset \mathbb{R}^2$ be a domain. The *maximal and minimal stretches* of a map $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$ at points of approximate differentiability are defined by

$$L_h(z) = \max\{\operatorname{ap}\operatorname{md} h_z(v) : |v| = 1\}, \quad l_h(z) = \min\{\operatorname{ap}\operatorname{md} h_z(v) : |v| = 1\}.$$

Lemma F.2.4 ([E, Lemma 2.9])**.** *Let $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$. Then L_h and l_h are representatives of the minimal weak upper gradient and the maximal weak lower gradient of h, respectively. Moreover,*

$$2^{-1} L_h(z) l_h(z) \leq J(\operatorname{ap}\operatorname{md} h_z) \leq 2 L_h(z) l_h(z)$$

at points $z \in U$ of approximate differentiability.

F.2.6. Weakly quasiconformal parametrizations of metric surfaces

The proof of Theorem F.1.1 depends on the existence of a weakly quasiconformal parametrization of X provided by [NR24]. See also [NR23] and [B]. The following theorem summarizes the main properties of a weakly quasiconformal parametrization and will be repeatedly applied within this work. A map $u\colon U \to X$ is called *monotone*, if $u^{-1}(x)$ is connected for every $x \in X$.

Theorem F.2.5. *If X is a metric surface then there exists a continuous, surjective, sense-preserving and monotone map $u \in N^{1,2}_{\mathrm{loc}}(U, X)$, where U is a domain in $\mathbb{R}^2$, such that*
(i) u is $\sqrt{2}$-quasiregular.
Let Y be a metric surface, let $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ and $h := f \circ u$. Then
(ii) $h \in N^{1,2}_{\mathrm{loc}}(U, Y)$, and if f is sense-preserving then so is h.
Moreover, if $f \in \mathrm{FDP}(X, Y)$, then
(iii) $h \in \mathrm{FDP}(U, Y)$ with $K_h(z) \leq \sqrt{2}\, K_f(u(z))$ for almost every $z \in U$.

Proof. The existence of a sense-preserving weakly $(4/\pi)$-quasiconformal parametrization $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ follows from [NR24, Theorem 1.3]. We refrain from defining weak quasiconformality here, but note that such a map u is continuous, surjective and monotone, and satisfies

$$\operatorname{mod}\Gamma \leq \frac{4}{\pi} \operatorname{mod} u(\Gamma) \tag{F.7}$$

for every family Γ of curves in X.

If $\operatorname{ap}\operatorname{md} u_z$ is a norm, then the closed unit ball B_z of $(\mathbb{R}^2, \operatorname{ap}\operatorname{md} u_z)$ contains a unique ellipse of maximal area E_z, called *John's ellipse* of $\operatorname{ap}\operatorname{md} u_z$. The proof of [NR24, Theorem 1.3] implies that we may assume E_z to be a disc for almost every $z \in U$. By John's Theorem (see e.g. [Bal97, Theorem 3.1]) and Lemma F.2.4 we know that

$$\rho^u_u(z) \leq \sqrt{2} \cdot \rho^l_u(z)$$

holds for almost every $z \in U$. Thus u is $\sqrt{2}$-quasiregular, which proves (i).

It follows from (F.7) that u maps curve families of positive modulus to curve families of positive modulus. Therefore,

$$\rho = \rho^u_u \cdot (\rho^u_f \circ u)$$

is a weak upper gradient of h, see Lemma F.2.4 and [E, Lemma 2.10]. By [NR23, Remark 7.2], we know that $N(x, u, U) = 1$ for almost every $x \in u(U)$. Let $E \subset U$ be compact and $G^u_0 \subset U$ the exceptional set in the classical area formula, Theorem F.2.1. Combining the formula with (i) and Lemma F.2.4, we have

$$\begin{aligned}\int_E \rho^2\, dz &= \int_{E \setminus G^u_0} (\rho^u_u)^2 \cdot (\rho^u_f \circ u)^2\, dz \leq \sqrt{2} \int_{E \setminus G^u_0} \rho^u_u \rho^l_u \cdot (\rho^u_f \circ u)^2\, dz \\ &\leq 2\sqrt{2} \int_X (\rho^u_f)^2 \cdot N(x, u, E)\, d\mathcal{H}^2_X = 2\sqrt{2} \int_{u(E)} (\rho^u_f)^2\, d\mathcal{H}^2_X < \infty.\end{aligned}$$

In particular, $h \in N^{1,2}_{\mathrm{loc}}(X, Y)$. The second claim in (ii) follows from the basic properties of topological degree.

Finally by [E, Lemma 2.10] and Lemma F.2.4 we have

$$\rho^l_h(z) \geq \rho^l_f(u(z)) \cdot \rho^l_u(z) \quad \text{and} \quad \rho^u_h(z) \leq \rho^u_f(u(z)) \cdot \rho^u_u(z) \tag{F.8}$$

for almost every $z \in U$. Combining (F.8) with (i) and (ii) gives (iii). $\square$

F.2.7. Area inequality

Another important ingredient in the proof of Theorem F.1.1 is the following area inequality for Sobolev maps on metric surfaces.

Let X, Y be metric surfaces and $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ a weakly quasiconformal parametrization as in Theorem F.2.5. Given $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$, let $G_0 := G_0^u \cup G_0^h$, where $G_0^u \subset U$ and $G_0^h \subset U$ are the exceptional sets in the classical area formula (Theorem F.2.1) associated with u and $h = f \circ u$, respectively. We denote

$$u(G_0) =: X_0 \quad \text{and} \quad X \setminus X_0 =: X'. \tag{F.9}$$

Theorem F.2.6 (Area inequality, [E, Theorem 3.1])**.** *Let $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$. If $g\colon Y \to [0, \infty]$ and $F \subset X'$ are Borel measurable, then*

$$\int_F g(f(x)) \cdot \rho_f^u(x)\rho_f^l(x)\, d\mathcal{H}^2_X \le 4\sqrt{2} \int_Y g(y) \cdot N(y, f, F)\, d\mathcal{H}^2_Y.$$

If f additionally satisfies Lusin's condition (N), then

$$\int_F g(f(x)) \cdot \rho_f^u(x)\rho_f^l(x)\, d\mathcal{H}^2_X \ge \frac{1}{4\sqrt{2}} \int_Y g(y) \cdot N(y, f, F)\, d\mathcal{H}^2_Y.$$

F.2.8. Covering theorems

We recall the basic $5r$-covering lemma. For a proof see e.g. [Hei01, Theorem 1.2].

Lemma F.2.7 ($5r$-covering lemma)**.** *Every family $\mathcal{F}$ of balls in X of uniformly bounded diameter contains a subfamily $\mathcal{G}$ such that every two distinct balls in $\mathcal{G}$ are disjoint and*

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B.$$

For a Borel function $g\colon X \to \mathbb{R}$, we define the maximal function

$$\mathcal{M}g(x) = \sup_{r>0} \frac{1}{\mathcal{H}^2(B(x, 5r))} \int_{B(x,r)} g\, d\mathcal{H}^2.$$

The proof of the following lemma is a standard application of the $5r$-covering theorem, see e.g. [Hei01, Theorem 2.2]

Lemma F.2.8. *If $g \in L^1_{\mathrm{loc}}(X)$ and $A \subset X$ with $\mathcal{H}^2(A) > 0$, then there are $E' \subset A$ with $\mathcal{H}^2(A \setminus E') > 0$ and $L \ge 1$ such that*

$$\mathcal{M}g(x) \le L \quad \textit{for every } x \in A \setminus E'.$$

We will also apply the Vitali covering theorem for Hausdorff measures, see e.g. [AT04, Theorem 2.2.2].

Theorem F.2.9. *Let $G \subset X$, and let $\mathcal{F}$ be a* fine covering *of G by closed sets. Then there exists a countable disjoint subfamily $\{V_j\} \subset \mathcal{F}$ such that one of the following holds:*

(i) $\sum \mathrm{diam}(V_j)^2 = \infty$.

(ii) $\mathcal{H}^2(G \setminus \cup_j V_j) = 0$.

Here a covering $\mathcal{F}$ of G by closed sets is *fine* if for every $x \in G$ and every $\varepsilon > 0$ there exists $V \in \mathcal{F}$ such that $x \in V$ and $\mathrm{diam}(V) < \varepsilon$.

Lemma F.2.7 and Theorem F.2.9 hold in arbitrary metric spaces, and the latter holds with exponent 2 replaced by any $\alpha \geq 0$.

F.2.9. Regularity of the Hausdorff measure

Let X be a metric surface and $u \in N^{1,2}_{\mathrm{loc}}(U, X)$ a weakly quasiconformal parametrization as in Theorem F.2.5. Moreover, let G_0, X_0 and X' be as in (F.9). As described in the paragraph preceding Theorem F.2.1, $U \setminus G_0$ may be covered with pairwise disjoint Borel sets $G^u_j \subset U$, $j = 1, 2, ...$, so that $u|_{G^u_j}$ is j-Lipschitz. In particular, $X' = u(U \setminus G_0)$ is countably 2-rectifiable. The following density result follows by combining [Fed69, Theorem 2.10.19(5)] and [Kir94, Theorem 9].

Theorem F.2.10. *There exists $E \subset X$, $\mathcal{H}^2(E) = 0$, so that*

$$\limsup_{r \to 0} \frac{\mathcal{H}^2(\overline{B}(x, r))}{\pi r^2} \leq 1 \quad \textit{for every } x \in X \setminus E, \quad \textit{and}$$
$$\lim_{r \to 0} \frac{\mathcal{H}^2(\overline{B}(x, r) \cap X')}{\pi r^2} = 1 \quad \textit{for every } x \in X' \setminus E.$$

F.3. Differentiation of Hausdorff measures

Metric surfaces do not need to be doubling or even Vitali spaces, so standard results on differentiation of measures do not hold automatically. In this section we prove such results for sense-preserving Sobolev maps.

Let X be a metric surface. We fix a weakly quasiconformal parametrization $u : U \to X$ as above. Given $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$, we denote $h = f \circ u$ and let G_0 and $X_0 = u(G_0)$ be the exceptional sets in (F.9). Recall notations $X' = X \setminus X_0$ and

$$J_f(x) = \limsup_{r \to 0} \frac{|f(\overline{B}(x, r))|_2}{\pi r^2}.$$

Lemma F.3.1. *If $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ is sense-preserving, then*

$$J_f(x) = 0 \quad \textit{for almost every } x \in X_0.$$

Proof. Suppose towards contradiction that there are $\varepsilon > 0$ and $W \subset X_0$ with $\mathcal{H}^2_X(W) > 0$ and such that $J_f(x) \geq 2\varepsilon$ for every $x \in W$. By choosing a subset if necessary, we may assume that W is compact. We fix $\delta > 0$. Then the collection of balls $B(x, r) \subset N_\delta(W)$ satisfying $x \in W$, $0 < 10r < \delta$, and

$$|f(\overline{B}(x, r))|_2 \geq \varepsilon \pi r^2$$

covers W. By the $5r$-covering lemma (Lemma F.2.7) there is a subcollection $\{B_j = \overline{B}(x_j, r_j)\}$ of disjoint closed balls so that collection $\{5B_j\}$ covers W. Then

$$\mathcal{H}^2_{\delta,X}(W) \leq \sum_j 25\pi r_j^2 \leq 25\varepsilon^{-1} \sum_j |f(B_j)|_2. \tag{F.10}$$

As before, we denote $h = f \circ u$ and recall that h satisfies Condition (N) by Proposition F.2.2 and Theorem F.2.5. In particular, if we denote $F_j = u^{-1}(B_j)$ then the classical area formula (Theorem F.2.1) shows that

$$|f(B_j)|_2 \leq \int_{F_j} J(\operatorname{ap} \operatorname{md} h_z)\, dz \quad \text{for all } j. \tag{F.11}$$

Since sets F_j are pairwise disjoint, combining (F.10) and (F.11) shows that

$$\mathcal{H}^2_{\delta,X}(W) \leq 25\varepsilon^{-1} \int_{u^{-1}(N_\delta(W))} J(\operatorname{ap}\operatorname{md} h_z)\, dz.$$

But $u^{-1}(W) \subset G_0$ has zero area, so the right hand term tends to zero when $\delta \to 0$. We conclude that $\mathcal{H}^2_X(W) = 0$, which is a contradiction. The proof is complete. □

Proposition F.3.2. *If $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ is sense-preserving, then*

$$\int_F J_f(x)\, d\mathcal{H}^2_X \leq \int_{\mathbb{R}^2} N(y, f, F)\, dy \tag{F.12}$$

for every Borel set $F \subset X$. If f is furthermore open and discrete, then equality holds in (F.12).

Proof. We start with the proof of (F.12). By Lemma F.3.1 we may assume that $F \subset X' \setminus E$, where E is the exceptional set in Theorem F.2.10. Given $1 < t < 2$ and $k \in \mathbb{Z}$, denote

$$A^k_t = \{x \in F : t^{k-1} \leq J_f(x) < t^k\}.$$

Then, once we prove

$$\int_{A^k_t} J_f(x)\, d\mathcal{H}^2_X \leq t \int_{\mathbb{R}^2} N(y, f, A^k_t)\, dy, \tag{F.13}$$

inequality (F.12) follows by summing both sides of (F.13) over k and letting $t \to 1$.

To prove (F.13), we fix t and k and notice that it suffices to prove (F.13) for all compact subsets $A \subset A^k_t$. We fix such an A, and $\varepsilon > 0$.

Then, since $A \subset X' \setminus E$, Theorem F.2.10 and the definition of A^k_t show that the collection

$$\mathcal{F} = \{\overline{B}(x, r) : x \in A,\ 0 < r < \varepsilon,\ \text{(F.14) and (F.15) hold}\ \}$$

is a fine covering of A; here we apply conditions

$$(1+\varepsilon)^{-1}\pi r^2 \leq \mathcal{H}^2_X(\overline{B}(x,r) \cap X') \leq \mathcal{H}^2_X(\overline{B}(x,r)) \leq (1+\varepsilon)\pi r^2, \tag{F.14}$$

and

$$(1+\varepsilon)^{-1} t^{k-1} \mathcal{H}^2_X(\overline{B}(x,r)) \leq |f(\overline{B}(x,r))|_2 \leq (1+\varepsilon) t^k \mathcal{H}^2_X(\overline{B}(x,r)). \tag{F.15}$$

Since X is homeomorphic to $\mathbb{R}^2$ and A is compact, we may choose ε to be small enough so that $\mathcal{H}^2_X(N_\varepsilon(A)) < \infty$. Then (F.14) shows that if $\mathcal{G}$ is a subcollection of $\mathcal{F}$ consisting of pairwise disjoint balls $B_1, B_2, \ldots, B_j = \overline{B}(x_j, r_j)$, then

$$(1+\varepsilon)^{-1}\pi \sum_j r_j^2 \leq \sum_j \mathcal{H}^2_X(B_j) \leq \mathcal{H}^2_X(N_\varepsilon(A)) < \infty.$$

Thus, by the Vitali covering theorem (Theorem F.2.9), the pairwise disjoint balls B_j can be chosen so that

$$\mathcal{H}^2_X(A \setminus \cup_j B_j) = 0. \tag{F.16}$$

Using (F.15) and (F.16), we have

$$\begin{aligned}\int_A J_f(x)\,d\mathcal{H}^2_X &\leq t^k\mathcal{H}^2_X(A) \leq t^k\sum_j \mathcal{H}^2_X(B_j) \leq t(1+\varepsilon)\sum_j |f(B_j)|_2 \\ &\leq t(1+\varepsilon)\sum_j \int_{\mathbb{R}^2} N(y,f,B_j)\,dy \\ &= t(1+\varepsilon)\int_{\mathbb{R}^2} N(y,f,\cup_j B_j)\,dy \\ &\leq t(1+\varepsilon)\int_{\mathbb{R}^2} N(y,f,N_\varepsilon(A))\,dy.\end{aligned}$$

By compactness of A, letting $\varepsilon \to 0$ yields (F.13) for A^k_t replaced with A. Inequality (F.12) follows.

We now assume that f is open and discrete and claim that

$$\int_F J_f(x)\,d\mathcal{H}^2_X \geq \int_{\mathbb{R}^2} N(y,f,F)\,dy \tag{F.17}$$

for every Borel set $F \subset X$. Recall that $|f(X_0)|_2 = 0$ by Proposition F.2.2 and Theorem F.2.5. Therefore, we may again assume that $F \subset X' \setminus E$.

Also, recall that an open and discrete map is locally invertible outside a discrete set $\mathcal{B}_f$. Therefore, we may replace F with $F \setminus \mathcal{B}_f$ if needed and assume without loss of generality that f is locally invertible at every $x \in F$.

As in the proof of (F.12), we see that (F.17) follows if we can show that

$$\int_{\mathbb{R}^2} N(y,f,A)\,dy \leq t\int_A J_f(x)\,d\mathcal{H}^2_X \tag{F.18}$$

for every $1 < t < 2$, $k \in \mathbb{Z}$, and every compact set $A \subset A^k_t$. We can choose a family of pairwise disjoint balls $B_1, B_2, \ldots$ satisfying the conditions of collection $\mathcal{F}$ above, and require the additional property that $f_{|B_j}$ is invertible for each j. In particular, (F.16) holds and as f satisfies Lusin's condition (N), by Proposition F.2.2, also $|f(A \setminus \cup_j B_j)|_2 = 0$. Combining with (F.15), we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} N(y,f,A)\,dy &\leq \int_{\mathbb{R}^2} N(y,f,\cup_j B_j)\,dy = \sum_j \int_{\mathbb{R}^2} N(y,f,B_j)\,dy \\ &= \sum_j |f(B_j)|_2 \leq (1+\varepsilon)t^k \sum_j \mathcal{H}^2_X(B_j) \\ &\leq (1+\varepsilon)t^k\mathcal{H}^2_X(N_\varepsilon(A)).\end{aligned}$$

Letting $\varepsilon \to 0$, the last term converges to

$$t^k\mathcal{H}^2_X(A) \leq t\int_A J_f(x)\,d\mathcal{H}^2_X.$$

Combining the estimates gives (F.18). The proof is complete. □

Proposition F.3.2 together with Theorem F.2.6 and Proposition F.2.2 now imply the following corollary.

Corollary F.3.3. *If $f \in N^{1,2}_{\mathrm{loc}}(X,\mathbb{R}^2)$ is sense-preserving, then*

$$J_f(x) \leq 4\sqrt{2}\,\rho^u_f(x)\rho^l_f(x) \quad \textit{for almost every } x \in X'.$$

If f is furthermore open and discrete, then

$$\rho_f^u(x)\rho_f^l(x) \leq 4\sqrt{2}\, J_f(x) \quad \text{for almost every } x \in X'.$$

Corollary F.3.4. *Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be a homeomorphism and μ the corresponding pull-back measure, i.e., $\mu(A) = |f(A)|_2$ for all Borel sets $A \subset X$. Then J_f is the Radon-Nikodym derivative of μ with respect to $\mathcal{H}^2_X$.*

Proof. Recalling that $|f(X_0)|_2 = 0$, the claim follows from Proposition F.3.2 and the definition of Radon-Nikodym derivative. □

F.4. Proof of the main theorem

The goal of this section is to prove Theorem F.1.1. Part 1 (from analytic distortion to distortion along paths) follows by combining Lemma F.3.1 and Corollary F.3.3, and recalling that if $f \in \mathrm{FDA}(X, \mathbb{R}^2)$ then $\rho_f^u = 0$ almost everywhere in the zero set of J_f.

It remains to prove Part 2 (from distortion along paths to analytic distortion). We know from Theorem F.2.3 that f is open and discrete. By Corollary F.3.3, the analytic distortion is controlled by the distortion along the paths in X' (recall that $X' = X \setminus X_0$, where the exceptional set X_0 is defined in (F.9)). Therefore, the proof of Theorem F.1.1 is complete after we have established the following result.

Proposition F.4.1. *If $f \in \mathrm{FDP}(X, \mathbb{R}^2)$ and $K_f \in L^1_{\mathrm{loc}}(X)$, then $\rho_f^l(x) = 0$ (and therefore $\rho_f^u(x) = 0$) for almost every $x \in X_0$.*

F.4.1. Vanishing lower gradient

This section is devoted to proving Proposition F.4.1. Let $f \in \mathrm{FDP}(X, \mathbb{R}^2)$ with $K_f \in L^1_{\mathrm{loc}}(X)$. Towards a contradiction we assume that there exists a set $A \subset X_0$ of positive measure such that $\rho_f^l(x) > 0$ for every $x \in A$.

Lemma F.4.2. *There exists a set $A'' \subset A$, $\mathcal{H}^2_X(A \setminus A'') = 0$, such that for every $x \in A''$ we find a rectifiable curve $\gamma_x \colon [0, \ell(\gamma_x)] \to X$ parametrized by arclength and such that*

(i) the lower gradient inequality (F.6) holds for the pair (f, ρ_f^l) on γ_x,

(ii) f is absolutely continuous along γ_x, and

(iii) there is $0 < t < \ell(\gamma_x)$ such that $\gamma_x(t) = x$ and $f \circ \gamma_x$ is differentiable at t with $(f \circ \gamma_x)'(t) > 0$.

Proof. Denote by Γ the family of all compact rectifiable curves in X and by Γ_0 the family of curves $\gamma \in \Gamma$ such that either ρ_f^u is not integrable on γ or the upper gradient inequality (F.5) does not hold for the pair (f, ρ_f^u) along γ. Note that f is absolutely continuous along every $\gamma \in \Gamma \setminus \Gamma_0$ and $\mathrm{mod}\, \Gamma_0 = 0$, see [HKST15, Propositions 6.3.1 and 6.3.2]. As $0 \leq \rho_f^l(x) \leq \rho_f^u(x)$ for almost every $x \in X$, we have that ρ_f^l is integrable on every $\gamma \in \Gamma \setminus \Gamma_0$. Let Γ_1 be the family of curves $\gamma \in \Gamma \setminus \Gamma_0$ such that the lower gradient inequality (F.6) does not hold for the pair (f, ρ_f^l) along γ. As f is absolutely continuous along every $\gamma \in \Gamma_1$, we have by definition of ρ_f^l that $\mathrm{mod}(\Gamma_1) = 0$.

Now the claim follows if for almost every $x \in A$ there is a $\gamma_x \colon [0, \ell(\gamma_x)] \to X$ in $\Gamma' = \Gamma \setminus (\Gamma_0 \cup \Gamma_1)$ parametrized by arclength and satisfying (iii).

Suppose towards contradiction that there is a set $A_0 \subset A$ with positive measure so that, given $x \in A_0$, no $\gamma = \gamma_x \in \Gamma'$ satisfies (iii). Recall that if $\gamma \in \Gamma'$ then $f \circ \gamma$ is differentiable at almost every $0 < t < \ell(\gamma)$ (see e.g. [HKST15, Remark 4.4.10]). But then, by the definition of the line integral, every $\gamma \in \Gamma'$ satisfies

$$\int_{f\circ\gamma} \chi_{f(A_0)}\, ds = \int_0^{\ell(\gamma)} \chi_{A_0}(\gamma(t)) \cdot (f\circ\gamma)'(t)\, dt = 0$$

and therefore

$$\ell(f\circ\gamma) = \int_{f\circ\gamma} \chi_{f(X\setminus A_0)}\, ds.$$

Moreover, the upper gradient inequality (F.5) implies

$$\int_{f\circ\gamma} \chi_{f(X\setminus A_0)}\, ds \leq \int_{\gamma} \chi_{X\setminus A_0} \cdot \rho_f^u\, ds.$$

In particular, $\chi_{X\setminus A_0} \cdot \rho_f^u$ is a weak upper gradient of f. From minimality of ρ_f^u we conclude that $\rho_f^u(x) = 0$ for almost every $x \in A_0$, which contradicts the definition of A. The proof is complete. □

Recall that f is discrete and open by Theorem F.2.3, so that f is locally invertible outside a countable *branch set* $\mathcal{B}_f$. We denote $A' = A'' \setminus \mathcal{B}_f$.

Corollary F.4.3. *Fix $x \in A'$, γ_x, and $0 < t < \ell(\gamma_x)$ as in Lemma F.4.2. There are $0 < \delta_x, \varepsilon_x < 1$ such that if $0 < R \leq \delta_x$ and $\gamma_R := \gamma_x|_{[t-R,t+R]}$, then $\ell(\gamma_R) = 2R$ and*

$$diam(|f\circ\gamma_R|) \geq \varepsilon_x R.$$

Moreover, $|\gamma_R|$ has a neighborhood W so that the restriction of f to W is a homeomorphism onto $B(f(x), 10(R + \operatorname{diam}(|f\circ\gamma_R|))$.

Proof. We set

$$\varepsilon_x := \frac{(f\circ\gamma_x)'(t)}{2} > 0.$$

By definition of the metric derivative, we find $0 < \delta < t$ such that

$$\frac{d(f(\gamma_x(t-R)), f(\gamma_x(t+R)))}{2R} \geq (f\circ\gamma_x)'(t) - \varepsilon_x = \varepsilon_x$$

for every $0 < R \leq \delta$. In particular,

$$\operatorname{diam}(|f\circ\gamma_R|) \geq d(f(\gamma_x(t-R)), f(\gamma_x(t+R))) \geq 2\varepsilon_x R$$

and, as γ_x is parametrized by arclength, $\ell(\gamma_R) = 2R$. By local invertibility of f at x, we may choose δ to be smaller if necessary so that the last claim also holds. □

As $\rho_f^l(x) > 0$ for almost every $x \in A'$ and $\mathcal{H}^2(A') > 0$, there is $\varepsilon > 0$ such that $\mathcal{H}^2(A_\varepsilon) > 0$ for

$$A_\varepsilon = \{x \in A' : \varepsilon_x \geq \varepsilon\}.$$

Proposition F.4.4. *We have $J_f(x) > 0$ for almost every $x \in A_\varepsilon$.*

Proposition F.4.4 contradicts Lemma F.3.1, so Proposition F.4.1, and thus Theorem F.1.1, follow once we have proved Proposition F.4.4.

To start the proof of Proposition F.4.4 we fix $x \in A_\varepsilon \setminus E$, where E is the null set in Theorem F.2.10. Then there is $r_x > 0$ so that

$$\mathcal{H}^2(B(x,r)) \leq 4r^2 \quad \text{for all } 0 < r < r_x. \tag{F.19}$$

Let $\delta_x, \varepsilon_x > 0$ be as in Corollary F.4.3. We fix a large number M to be specified later, and let $R > 0$ be small enough so that

$$5MR < \min\{r_x, \delta_x\}. \tag{F.20}$$

Consider the curve γ_R in Corollary F.4.3. We have

$$x \in |\gamma_R| \subset B(x, R) \quad \text{and} \quad \operatorname{diam}(f(|\gamma_R|)) \geq \varepsilon R.$$

Without loss of generality we assume that the points $(0,0)$ and $(0, \varepsilon R)$ are contained in $f(|\gamma_R|)$. Let $\pi_2 \colon \mathbb{R}^2 \to \mathbb{R}$ be the projection to the second coordinate. Then $v = \pi_2 \circ f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R})$.

Recall that f is invertible in a neighborhood W of x with image

$$f(W) = B(f(x), 10(R + \operatorname{diam}(|f \circ \gamma_R|)).$$

In particular, for every $0 < t < \varepsilon R$ there are s_t and a continuum $\eta_t' \subset v^{-1}(t)$ so that $(s_t, t) \in f(|\gamma_R|)$,

$$\eta_t' \cap |\gamma_R| \neq \emptyset \quad \text{and} \quad f(\eta_t') = I_t := [s_t - R, s_t + R] \times \{t\}.$$

We fix $a_t \in \eta_t' \cap |\gamma_R| \subset B(x, R)$ and define

$$E_M(R) = \{0 < t < \varepsilon R : \eta_t' \not\subset B(x, MR)\}.$$

We may choose continua η_t' so that $E_M(R)$ is a Borel set.

Lemma F.4.5. *For almost every $x \in A_\varepsilon$ we can choose M (depending on x) so that*

$$|E_M(R)|_1 \leq \frac{\varepsilon R}{2}$$

for all $R > 0$ satisfying (F.20).

Proof. We may assume that $|E_M(R)|_1 > 0$ since otherwise there is nothing to show. We may also assume that $M = 2^l$ for some $l \in \mathbb{N}$. Define $\phi \colon X \to \mathbb{R}$ by

$$\phi(y) = \frac{\chi_{B(x,MR) \setminus B(x,R)}(y)}{l \cdot d(y, x)}.$$

Let η_t be the a_t-component of $\eta_t' \cap B(x, MR)$, and

$$\eta_t^j = \eta_t \cap B(x, 2^j R) \setminus B(x, 2^{j-1} R).$$

If $t \in E_M(R)$, then

$$\begin{aligned}
\int_{\eta_t} \phi \, d\mathcal{H}^1 &= \sum_{j=1}^{l} \int_{\eta_t^j} \phi \, d\mathcal{H}^1 \geq \frac{1}{l} \sum_{j=1}^{l} \mathcal{H}^1(\eta_t^j) \min_{y \in \eta_t^j} \frac{1}{d(y,x)} \\
&\geq \frac{1}{l} \sum_{j=1}^{l} (2^{j-1} R) \cdot \frac{1}{2^j R} \geq \frac{1}{2}.
\end{aligned} \tag{F.21}$$

Note that each η_t is a non-degenerate continuum contained in the level set $v^{-1}(t)$. Hence, by [E, Lemma 4.4], $\mathcal{H}^1(\eta_t \cap X_0) = 0$ for almost every $t \in E_M(R)$. Let

$$F_M = \bigcup_{t \in E_M(R)} \eta_t \subset B(x, MR).$$

We apply (F.21), the coarea inequality for Sobolev mappings [D, Theorem 1.6], and Hölder's inequality to obtain

$$\begin{aligned}\frac{|E_M(R)|_1}{2} &\leq \int_{E_M(R)} \int_{\eta_t} \phi \, d\mathcal{H}^1 \, ds = \int_{E_M(R)} \int_{\eta_t \cap X'} \phi \, d\mathcal{H}^1 \, ds \\ &\leq \frac{4}{\pi} \int_{F_M \cap X'} \phi \, \rho_f^u \, d\mathcal{H}^2 \leq \frac{4}{\pi} \int_{F_M \cap X'} \phi \, K_f^{1/2} (\rho_f^u \rho_f^l)^{1/2} \, d\mathcal{H}^2 \\ &\leq \frac{4}{\pi} \underbrace{\left(\int_{B(x,MR) \setminus B(x,R)} \phi^2 K_f \, d\mathcal{H}^2 \right)^{1/2}}_{=:\mathcal{I}_1} \underbrace{\left(\int_{F_M \cap X'} \rho_f^u \rho_f^l \, d\mathcal{H}^2 \right)^{1/2}}_{=:\mathcal{I}_2}.\end{aligned} \tag{F.22}$$

Recall that $K_f \in L^1_{\mathrm{loc}}(X)$ and therefore, by Lemma F.2.8, for almost every $x \in A_\varepsilon \setminus E$ there exists $L \geq 1$ such that the maximal function satisfies $\mathcal{M}K_f(x) \leq L$. Combining with (F.19), we obtain

$$\begin{aligned}\mathcal{I}_1 &\leq \frac{4}{l^2 R^2} \sum_{j=1}^{l} 2^{-2j} \int_{B(x,2^j R) \setminus B(x,2^{j-1}R)} K_f \, d\mathcal{H}^2 \\ &\leq \frac{4}{l^2 R^2} \sum_{j=1}^{l} 2^{-2j} \frac{\mathcal{H}^2(B(x, 5 \cdot 2^j R))}{\mathcal{H}^2(B(x, 5 \cdot 2^j R))} \int_{B(x,2^j R)} K_f \, d\mathcal{H}^2 \\ &\leq \frac{400}{l^2} \sum_{j=1}^{l} \frac{1}{\mathcal{H}^2(B(x, 5 \cdot 2^j R))} \int_{B(x,2^j R)} K_f \, d\mathcal{H}^2 \\ &\leq \frac{400 L l}{l^2} = \frac{400 L}{l}.\end{aligned} \tag{F.23}$$

We may apply the area inequality (Theorem F.2.6) and Fubini's theorem to estimate $\mathcal{I}_2$ as follows:

$$\mathcal{I}_2 \leq 4\sqrt{2} \int_{f(F_M)} 1 \, dA \leq 8\sqrt{2} R \, |E_M(R)|_1, \tag{F.24}$$

where the last inequality holds as $f(\eta_t) \subset I_t$, I_t is a segment of length $2R$ for every $t \in E_M(R)$, and $f(F_M) = \bigcup_{t \in E_M(R)} f(\eta_t)$. Combining (F.22), (F.23) and (F.24) gives

$$\frac{|E_M(R)|_1}{2} \leq \left(3200\sqrt{2} L \, \frac{R}{\ell} \, |E_M(R)|_1 \right)^{1/2}.$$

After setting $\kappa = 50000L$ we obtain

$$|E_M(R)|_1 \leq \kappa \frac{R}{l},$$

and thus $|E_M(R)|_1 \leq \frac{\varepsilon R}{2}$ for l large enough. □

We now finish the proof of Proposition F.4.4. Choose $x \in A_\varepsilon$, M and R so that Lemma F.4.5 holds. We denote $Q_M(R) = (0, \varepsilon R) \setminus E_M(R)$. By Lemma F.4.5,

$$|Q_M(R)|_1 \geq \frac{\varepsilon R}{2}.$$

Moreover, local invertibility of f around x and the definition of $Q_M(R)$ show that if $t \in Q_M(R)$

then $f(\eta_t') = I_t$, so that $|f(\eta_t')|_1 = 2R$. We denote

$$G_M(R) = \bigcup_{t \in Q_M(R)} f(\eta_t').$$

Note that by definition, $G_M(R) \subset f(B(x, MR))$. Fubini's theorem now gives

$$\varepsilon R^2 \leq 2R \cdot |Q_M(R)|_1 = |G_M(R)|_2 \leq |f(B(x, MR))|_2.$$

Proposition F.4.4 follows by letting $R \to 0$. The proof of Theorem F.1.1 is complete.

F.5. Quasiconformal uniformization

This section is devoted to proving Corollary F.1.2 and Theorems F.1.3 and F.1.4. Before proving Corollary F.1.2, we recall the following theorem of Williams ([Wil12, Theorem 1.1]).

Theorem F.5.1. *Let $f\colon X \to Y$ a homeomorphism between metric surfaces. Then the following conditions are equivalent with the same constant $C \geq 1$.*

(i) *$f \in N^{1,2}_{\mathrm{loc}}(X, Y)$ and*

$$\rho_f^u(x)^2 \leq C \cdot \mathrm{Jac}_f(x) \quad \textit{for almost every } x \in X,$$

where Jac_f denotes the Radon-Nikodym derivative of the corresponding pull-back measure with respect to $\mathcal{H}^2_X$.

(ii) *For every family Γ of curves in X we have*

$$\operatorname{mod}\Gamma \leq C \cdot \operatorname{mod} f(\Gamma). \tag{F.25}$$

Proof of Corollary F.1.2. We first note that Corollary F.3.4 implies the equivalence of analytic quasiconformality as defined in our work and Condition (i) in Theorem F.5.1. In particular, the constant may be chosen to be the same.

In the next step we show that within this setting f satisfying (F.25) for some $C \geq 1$ implies that f is geometrically $4C$-quasiconformal. Namely, as f maps into $\mathbb{R}^2$ and satisfies (F.25), it follows from [Raj17] (see also [NR24], [RR19]) that there exists a geometrically 2-quasiconformal homeomorphism $u\colon U \to X$, where $U \subset \mathbb{R}^2$ is a domain. Now the map $h := f \circ u\colon U \to \mathbb{R}^2$ is a homeomorphism satisfying $\operatorname{mod}\Gamma \leq 2C \cdot \operatorname{mod} h(\Gamma)$ for every family Γ of curves in X. As the domains are planar, h is geometrically $2C$-quasiconformal. Thus, f is geometrically $4C$-quasiconformal. This shows that (F.25) is quantitatively equivalent to geometric quasiconformality.

Theorem F.5.1 now implies the equivalence between Conditions (1) and (2) in Corollary F.1.2, i.e., between analytic and geometric quasiconformality. Moreover, it follows from our main theorem (Theorem F.1.1) that (1) is quantitatively equivalent with (3). More explicitly, if f is analytically C-quasiconformal then f is K-quasiconformal along paths with $K = 4\sqrt{2}C$ and vice versa. We have proven the equivalence of Conditions (1)-(3).

It remains to show that if f is quasiconformal according to any of the conditions above, then so is f^{-1}. First, it follows from the definition of geometric quasiconformality that f^{-1} is geometrically quasiconformal. Moreover, applying Theorem F.5.1 to f^{-1} shows that f^{-1} is also analytically quasiconformal. In particular, $f^{-1} \in N^{1,2}_{\mathrm{loc}}(f(X), X)$. Quasiconformality of f^{-1} along paths now follows from analytic quasiconformality by combining Theorem F.2.1 and Lemma F.2.4. The proof is complete. □

Proof of Theorem F.1.3. Let $f \in \mathrm{FDP}(X, \mathbb{R}^2)$ be K-quasiregular. By Theorem F.2.5 (iii), there is a weakly quasiconformal parametrization $u : U \to X$, where $U \subset \mathbb{R}^2$, so that $h = f \circ u$ is in $\mathrm{FDP}(U, \mathbb{R}^2)$ and $\sqrt{2}K$-quasiregular. By Theorem F.2.3, h is discrete and open. We conclude that u is both discrete and monotone and therefore a homeomorphism. We will show that $\phi := u^{-1} \colon X \to U$ is (geometrically) quasiconformal.

Denote by $\mathcal{B}_f$ the set of branch points of f, i.e., the set of points at which f is not locally invertible, and recall that $\mathcal{B}_f$ is a discrete set. For every $x \in X \setminus \mathcal{B}_f$ we find a neighbourhood $V_x \subset X$ of x such that $f|_{V_x}$ is a homeomorphism onto its image. It follows from the proof of Corollary F.1.2 that $f|_{V_x}$ is geometrically $16\sqrt{2}K$-quasiconformal and $h|_{u^{-1}(V_x)}$ is geometrically $32K$-quasiconformal. In particular, $\phi|_{V_x}$ is geometrically C-quasiconformal with $C = 512\sqrt{2}K$.

The proof of Corollary F.1.2 implies that the restriction of ϕ to $\tilde{X} = X \setminus \mathcal{B}_f$ is analytically C-quasiconformal. In particular, there is a family Γ_0 of curves in $\tilde{X}$ with zero modulus so that ϕ is absolutely continuous on all paths $\tilde{\gamma} \notin \Gamma_0$ in $\tilde{X}$. Theorem F.1.3 follows if we can show that ϕ is absolutely continuous on almost every rectifiable curve γ in X.

We fix such a γ. Then, since $\mathcal{B}_f$ is discrete, $|\gamma| \cap \mathcal{B}_f$ is finite. Since ϕ is continuous, we conclude that if ϕ is not absolutely continuous on γ then there is a subpath $\tilde{\gamma}$ with image in $\tilde{X}$ where ϕ is not absolutely continuous, that is, $\tilde{\gamma} \in \Gamma_0$. The family of paths in X which contain a subpath in Γ_0 has zero modulus. The proof is complete. □

Proof of Theorem F.1.4. By Theorem F.1.3 there is a geometrically quasiconformal homeomorphism $\phi \colon X \to U$, where U is a domain in $\mathbb{R}^2$. The map $h = f \circ \phi^{-1} : U \to \mathbb{R}^2$ is quasiregular. By the measurable Riemann mapping theorem (see e.g. [AIM09, Theorem 5.3.4]) there exists a quasiconformal map $\psi \colon U \to V$, $V \subset \mathbb{R}^2$, such that $g := h \circ \psi^{-1} : V \to \mathbb{R}^2$ is analytic. The statement follows after setting $v := \psi \circ \phi$. □

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My research interests lie at the intersection of several active areas of mathematics, including geometric analysis, metric and differential geometry, geometric measure theory, and analysis on metric spaces.

Education

09/2024 - today	**Certificate of Advanced Studies in Higher Education and Educational Technology**, University of Fribourg, Switzerland
10/2020 - today	**PhD in Mathematics** under the supervision of Stefan Wenger, University of Fribourg, Switzerland
09/2019 - 09/2020	**Master of Science in Mathematics**, University of Bern, Switzerland
09/2015 - 09/2019	**Bachelor of Science in Mathematics**, Minor in Computer Science and Physics, University of Bern, Switzerland

Employment

10/2020 - today	**Diploma assistant**, Department of Mathematics, University of Fribourg, Switzerland
02/2021 - 12/2022	**ICT technician**, Department of Mathematics, University of Fribourg, Switzerland
09/2017 - 05/2020	**Teaching assistant**, Mathematical Institute, University of Bern, Switzerland

Funding and awards

01/2023 - 07/2023	**Doc.Mobility** grant of the University of Fribourg (7 months research stay at Stony Brook University and travel funds for conferences)

Extended research stays

01/2025 - 04/2025	**Trimester Program: "Metric Analysis"**, Hausdorff Research Institute for Mathematics, Bonn, Germany
01/2023 - 07/2023	**Doc.Mobility research stay**, Stony Brook University, New York, USA

Publications and preprints

[7] with Noa Vikman and Stefan Wenger: *Energy minimizing harmonic 2-spheres in metric spaces*, Preprint arXiv: 2503.08553.

[6] with Kai Rajala: *Definitions of quasiconformality on metric surfaces*, Preprint arXiv: 2405.07476.

[5] with Stefan Wenger: *Quasiconformal almost parametrizations of metric surfaces*, J. Eur. Math. Soc., published online first.

[4] with Kai Rajala: *Mappings of finite distortion on metric surfaces*, Math. Ann. 391 (2025), 2479–2507.

[3] *Quasiconformal uniformization of metric surfaces of higher topology*, Indiana Univ. Math. J. 73 (2024), no. 5, 1689–1713.

[2] with Dimitrios Ntalampekos: *Lipschitz-Volume rigidity and Sobolev coarea inequality for metric surfaces*, J. Geom. Anal. 34 (2024), No. 5, Paper No. 128.

[1] with Martin Fitzi: *Canonical parametrizations of metric surfaces of higher topology*, Ann. Fenn. Math. 48 (2023), No. 1, 67–80.

Invited Conference and Workshop talks

06/2025 *Quasiweekend III*, University of Helsinki

03/2025 *Differential geometry beyond Riemannian manifolds*, Hausdorff Research Institute for Mathematics Bonn

01/2025 *Geometric structures and infinite-dimensional manifolds*, Erwin Schrödinger International Institute for Mathematics and Physics (ESI) Vienna

08/2024 *Metric Geometry and Geometric Measure Theory*, University of Fribourg

07/2024 *Geometry and Analysis on metric surfaces*, IMPAN, Warsaw

04/2024 *AMS Spring Eastern Sectional Meeting*, Howard University, Washington DC

04/2023 *AMS Spring Eastern Sectional Meeting*, online

Invited Seminar talks

02/2025 *Trimester Seminar Series*, Hausdorff Research Institute for Mathematics Bonn

11/2024 *Geometrie Seminar*, Karlsruher Institut für Technologie

04/2024 *Analysis Seminar*, Stony Brook University

04/2024 *Geometric Analysis and Topology Seminar*, New York University

02/2024 *Geometric analysis seminar*, University of Jyväskylä

03/2023 *Differential Geometry and Geometric Analysis Seminar*, Princeton University

03/2023	*Analysis Seminar*, University of Tennessee, Knoxville
02/2023	*Analysis Seminar*, Stony Brook University
09/2022	*Geometry Graduate Colloquium*, ETH Zürich
05/2022	*Oberseminar Differentialgeometrie*, Max Planck Institute Bonn
12/2021	*Bernoullis Tafelrunde*, University of Basel
11/2021	*Quasiworld Seminar*, online

Organized research activities

02/2022 - 01/2025	Co-founder and co-organizer of the *Bern–Fribourg graduate seminar*, University of Bern and University of Fribourg
03/2025	Co-organizer of the *CUSO career day*, University of Fribourg
11/2023	Co-organizer of the *22th CUSO Graduate Colloquium*, University of Fribourg

Teaching

Fribourg	Teaching assistant for analysis, introduction to numerical analysis, propedeutic analysis, analysis on metric spaces
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Outreach activities

09/2024	*Preparatory course in mathematics*, University of Fribourg
10/2023	*Leonardo Math Camp - A camp for highschool students*, Les Diablerets
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Personal skills

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