Quasiconformal almost parametrizations of metric surfaces

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Uniformization of metric surfaces

Under which conditions on a metric surface X, homeomorphic to some model surface M, does there exist

$$u\colon M \to X$$

with good geometric and analytic properties?

- *Classical uniformization theorem:* Every simply connected Riemann surface is conformally diffeomorphic to D, $\mathbb{C}$ or S^2.
- *Uniformization theorem of Bonk-Kleiner [1]:* Let $X \simeq S^2$ be Ahlfors 2-regular. Then there exists a quasisymmetry $u\colon S^2 \to X$ if and only if X is linearly locally connected.
- *Uniformization theorem of Rajala [6]:* Let $X \simeq \mathbb{R}^2$ be of locally finite $\mathscr{H}^2$-measure. Then there exists a geometrically quasiconformal map $u\colon U \to X$, $U \subset \mathbb{R}^2$, if and only if X is reciprocal.

Goal: Generalization to a larger class of metric surfaces.

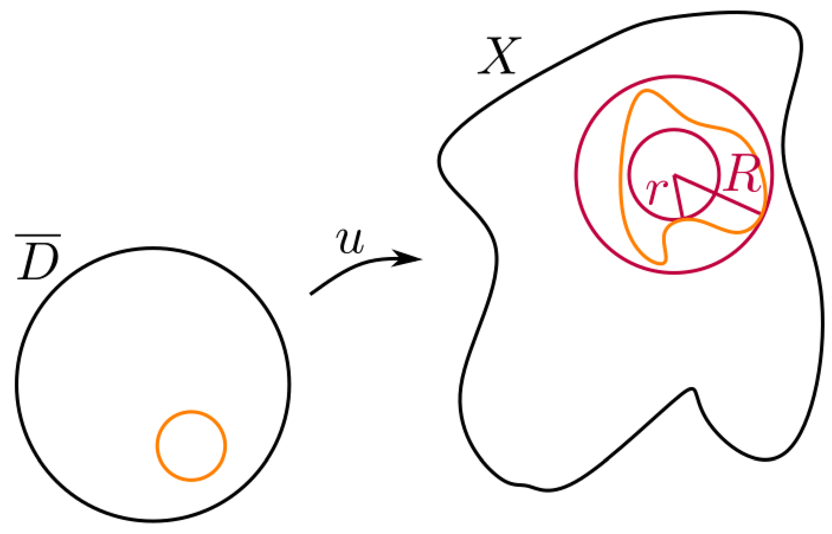
Main result

Let $X \simeq \overline{D}$ be geodesic, $\mathscr{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$. Then there exists a continuous, monotone surjection $u\colon \overline{D} \to X$ such that

$$\operatorname{mod}(\Gamma) \leq \frac{4}{\pi} \cdot \operatorname{mod}(u \circ \Gamma) \qquad (1)$$

for every family Γ of curves in $\overline{D}$.

- The factor $\frac{4}{\pi}$ is optimal.
- If u is a homeo then (1) is equivalent to the analytic definition of quasiconformality.
- u upgrades to a geometrically quasiconformal homeomorphism if X is reciprocal.
- u upgrades to a quasisymmetry if X is Ahlfors 2–regular and linearly locally connected.
- Similar result by Ntalampekos and Romney [4].



$(1) \Rightarrow \frac{R}{r} \leq Q$ on infinitesimal scales

Sobolev maps into metric spaces

A map $u\colon D \to X$ is in the Sobolev space $W^{1,2}(D,X)$ if there is a non-negative function $g \in L^2(D)$ such that for every Lipschitz function $f : X \to \mathbb{R}$ we have

$$f \circ u \in W^{1,2}(D) \text{ and } |\nabla(f \circ u)| \leq \operatorname{Lip}(f)g \text{ a.e.}$$

- $u \in W^{1,2}(D,X)$ has a minimal weak upper gradient $g_u \in L^2(D)$. Define the energy of u by
$$E_+^2(u) := \|g_u\|_{L^2(D)}^2.$$
- $u \in W^{1,2}(D,X)$ admits an approximate metric derivative a.e., allowing us to make sense to notions of quasiconformality and area.
- $u \in W^{1,2}(D,X)$ extends to S^1 a.e. by means of a well-defined trace operator, denoted by $\operatorname{tr}(\cdot)$.
- $\Lambda(\partial X, X)$ is the family of maps $u \in W^{1,2}(D,X)$ such that $\operatorname{tr}(u)$ almost parametrizes ∂X.

Modulus of curve families

- Modulus $\operatorname{mod}(\cdot)$ is an outer measure on the class of curves and a conformal invariant.
- $\operatorname{mod}(\Gamma)$ measures how many locally rectifiable curves are contained in the curve family Γ.

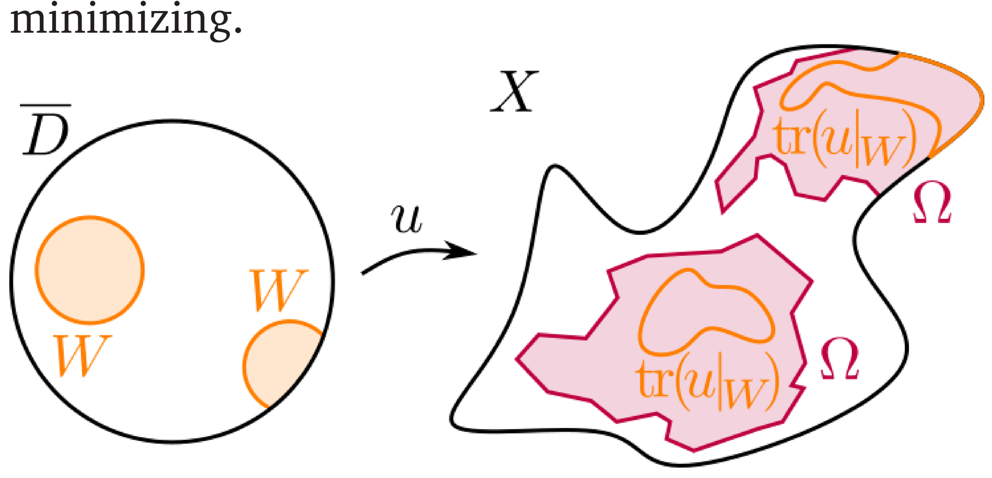
Strategy of proof

- Show that $\Lambda(\partial X, X) \neq \emptyset$.
- Use existence of an energy minimizing map $u \in \Lambda(\partial X, X)$, see [2].
- Prove continuity of energy minimizers.
- Use results from [2] and [3] to show that u is monotone and the modulus inequality (1) is fulfilled.

Continuity of energy minimizers

In this setting, an energy minimizing map $u \in \Lambda(\partial X, X)$ has a representative which is continuous and extends continuously to S^1:

- There is a notion of area such that u is area minimizing.

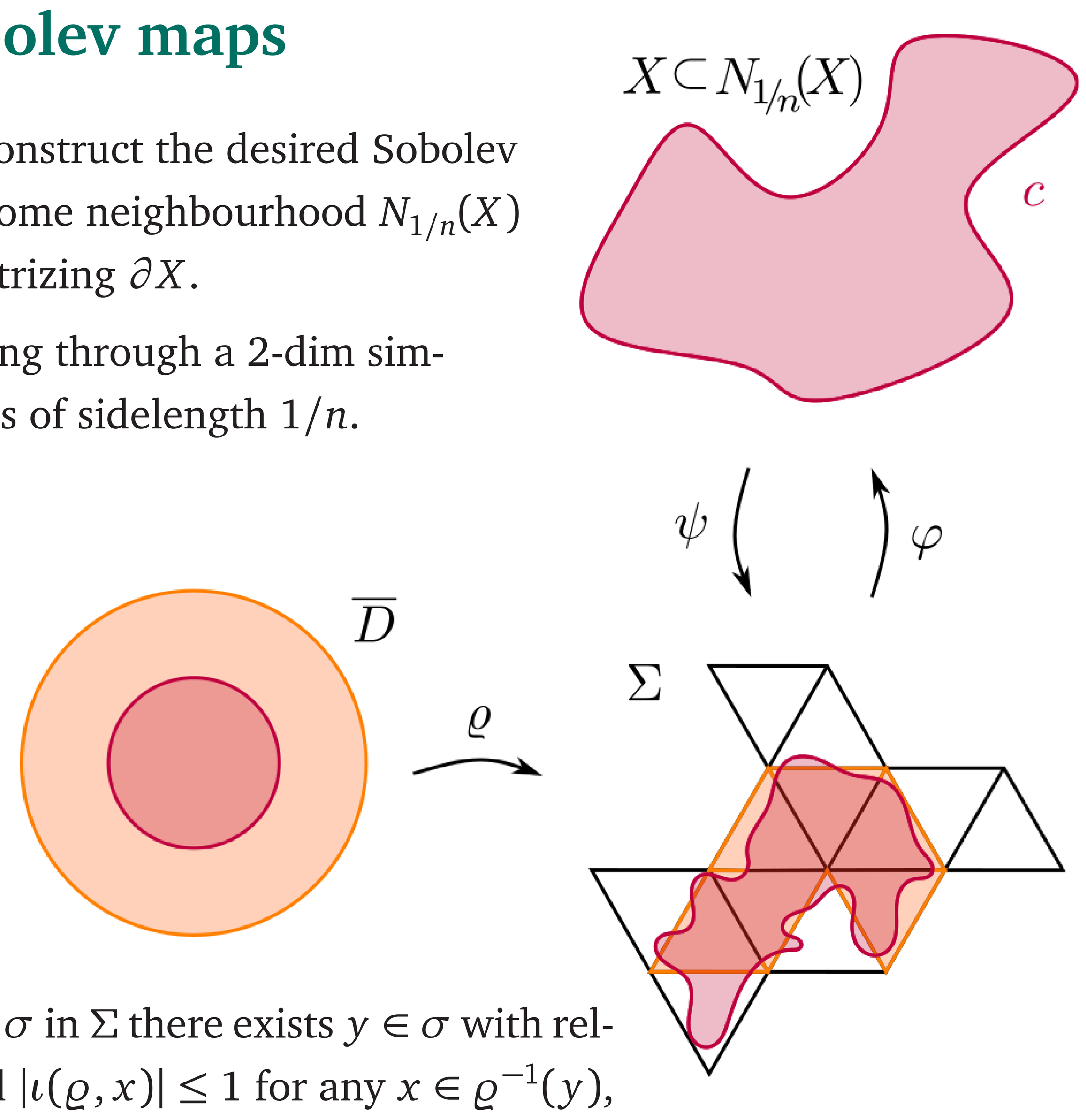


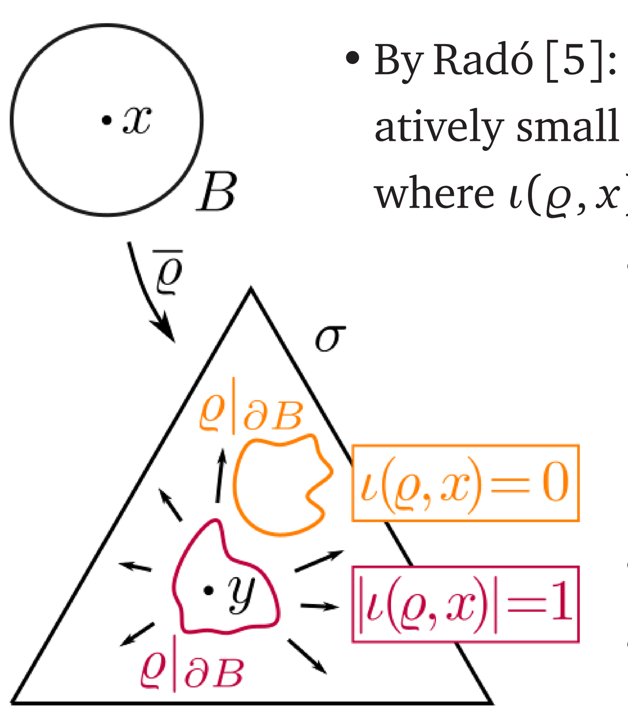
- After applying the Courant-Lebesgue Lemma and some metric arguments, we find for every small enough $\varepsilon > 0$ a $\delta > 0$ such that for
$$W := B(z,\delta) \cap D$$
the trace $\operatorname{tr}(u|_W)$ is contained in a Jordan domain $\Omega \subset X$ bounded by a biLipschitz curve and with $\operatorname{diam}(\Omega) < \varepsilon$.
- Consider the set
$$N := \{w \in W : u(w) \in X \setminus \overline{\Omega}\}.$$
If N is not negligible, one can use a Fubini-type argument to show that
$$\operatorname{Area}(u|_N) > 0.$$
- Since Ω is bounded by a biLipschitz curve, we find a Lipschitz retraction
$$R\colon X \to \overline{\Omega} \quad \text{with} \quad R(X \setminus \overline{\Omega}) \subset \partial\Omega.$$
- Then the map v agreeing with u on $D \setminus W$ and with $R \circ u$ on W is also contained in $\Lambda(\partial X, X)$ and contradicts the area minimizing property of u, since $\operatorname{Area}(v|_N) = 0$.

Existence of Sobolev maps

We will show that $\Lambda(\partial X, X) \neq \emptyset$. For this, we construct the desired Sobolev map as a limit of Lipschitz maps v_n from $\overline{D}$ to some neighbourhood $N_{1/n}(X)$ with uniformly bounded area and $v_n|_{S^1}$ parametrizing ∂X.

- The Lipschitz map v_n is obtained via factorizing through a 2-dim simplicial complex Σ consisting of Euclidean cells of sidelength $1/n$.
- There exist Lipschitz maps
$$\psi: X \to \Sigma \quad \text{and} \quad \varphi: \Sigma \to N_{1/n}(X)$$
that are almost inverse to each other.
- Construct a continuous map $\varrho: \overline{D} \to \Sigma$, where $\varrho|_{S^1}: S^1 \to \Sigma^{(1)}$ is Lipschitz and close to $\psi(\partial X)$ and the integral over the multiplicity function of ϱ is bounded.





- By Radó [5]: For every 2-cell σ in Σ there exists $y \in \sigma$ with relatively small multiplicity and $|\iota(\varrho,x)| \leq 1$ for any $x \in \varrho^{-1}(y)$, where $\iota(\varrho,x)$ is the winding number.
- Define $\overline{\varrho}$ on small balls B such that
  - $\overline{\varrho}|_B$ is constant with image in $\partial\sigma$ if $\iota(\varrho,x) = 0$,
  - $\overline{\varrho}|_B$ is a biLipschitz homeomorphism and $\overline{\varrho}|_{\partial B}$ is homotopic to the projection of $\varrho|_{\partial B}$ to $\partial\sigma$ if $|\iota(\varrho,x)| = 1$.
- Extend $\overline{\varrho}|_{\bigcup B \cup S^1}$ to a Lipschitz map $\overline{\varrho}: \overline{D} \to \Sigma$ with bounded area.
- Use properties of $N_{1/n}(X)$ to change $\varphi \circ \overline{\varrho}$ into the desired Lipschitz map.

References

[1] Mario Bonk and Bruce Kleiner. "Quasisymmetric parametrizations of two-dimensional metric spheres". In: *Invent. Math.* 150.1 (2002), pp. 127–183.

[2] Alexander Lytchak and Stefan Wenger. "Area Minimizing Discs in Metric Spaces". In: *Arch. Ration. Mech. Anal.* 223.3 (2017), pp. 1123–1182.

[3] Alexander Lytchak and Stefan Wenger. "Canonical parameterizations of metric disks". In: *Duke Math. J.* 169.4 (2020), pp. 761–797.

[4] Dimitrios Ntalampekos and Matthew Romney. "Polyhedral approximation of metric surfaces and applications to uniformization". In: *Duke Math. J.* (to appear).

[5] Tibor Radó. "On absolutely continuous transformations in the plane". In: *Duke Math. J.* 4.1 (1938), pp. 189–221.

[6] Kai Rajala. "Uniformization of two-dimensional metric surfaces". In: *Invent. Math.* 207.3 (2017), pp. 1301–1375.

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