

# Planar lattices do not recover from forest fires

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Joint work with Demeter Kiss and Vladas Sidoravicius

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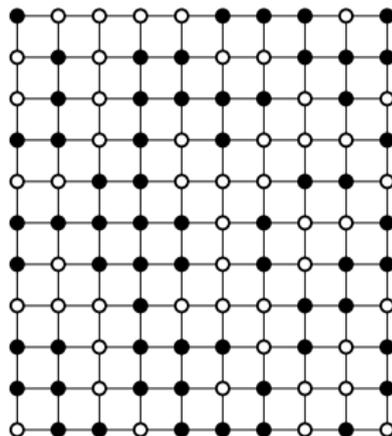
31 March 2014

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Let  $p, \delta \in [0, 1]$ .

Two percolation configurations:

- $\omega$  - intensity  $p$  (measure  $\mathbb{P}_p$ ).

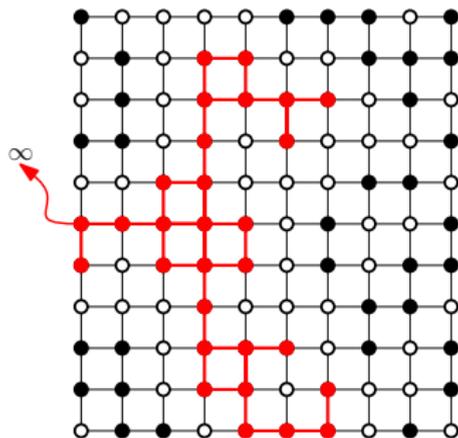


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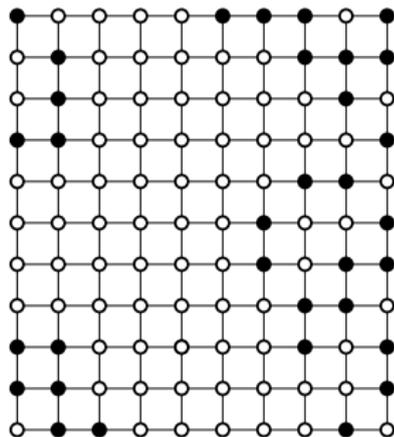
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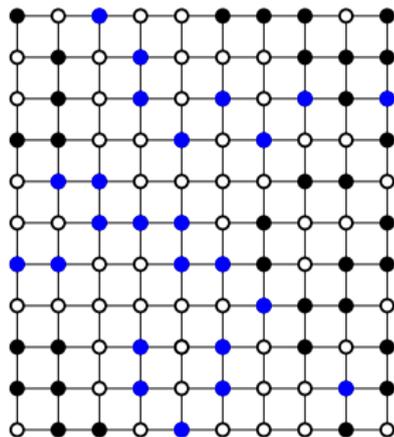
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$\bar{\omega}^\delta = \bar{\omega} \vee \sigma$  is an enhancement of  $\bar{\omega}$ .

Call  $\mathbb{P}_{p,\delta}$  the measure for  $\omega, \sigma, \bar{\omega}^\delta, \dots$



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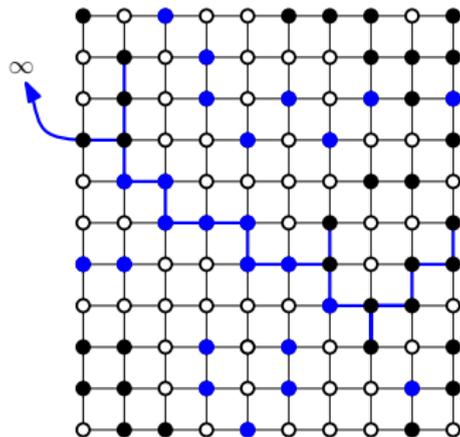
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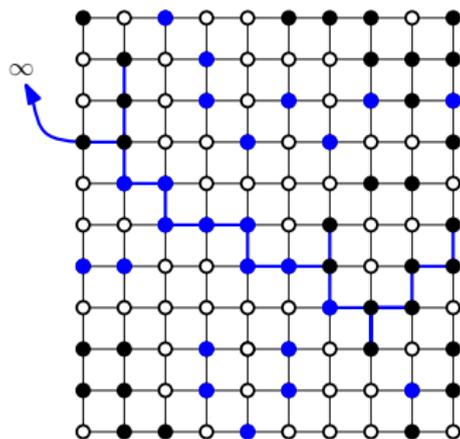
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**Question:**  $\delta_c(p) \rightarrow 0$  as  $p \searrow p_c$ ?

# Result

Theorem (Kiss, M., Sidoravicius)

There exists  $\delta > 0$  such that, for all  $p > p_c$ ,

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See Van den Berg and Brouwer 2004 for proof.

There exists  $p_c \in [0, 1]$  such that

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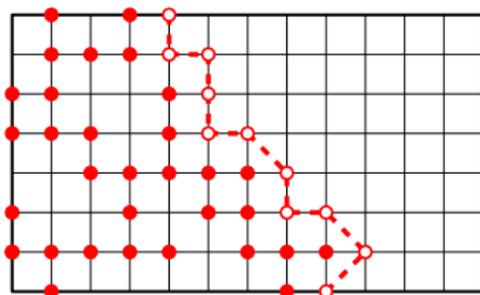
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At  $p_c..$

$$\forall n, \mathbf{P}_{p_c} \left[ \begin{array}{c} \text{[Diagram: Solid red path of length } n \text{ in a } 2n \text{ box]} \\ 2n \end{array} \right] \geq \epsilon$$

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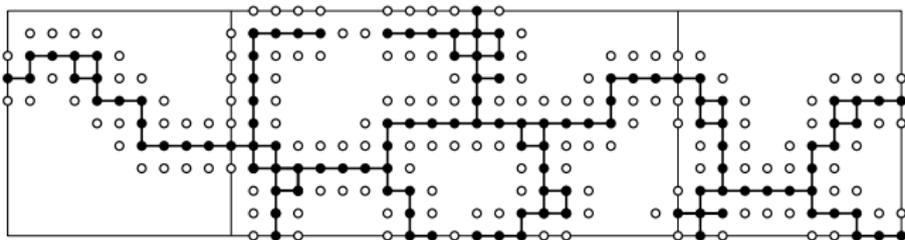
At  $p_c..$

$$\forall n, \mathbf{P}_{p_c} \left[ \begin{array}{c} \text{[Diagram: Solid red path in } 2n \times n \text{ box]} \\ 2n \end{array} \right] \geq \epsilon$$

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$$\mathbf{P}_{p_c} \left[ \begin{array}{c} \text{[Diagram: Solid red path in } n \times n \text{ box with a dot]} \\ n \end{array} \right] \leq n^{-\alpha_1}$$

$$\mathbf{P}_{p_c} \left[ \begin{array}{c} \text{[Diagram: Mixed solid and dashed red paths in } n \times n \text{ box with a dot]} \\ n \end{array} \right] \leq n^{-\alpha_5}$$



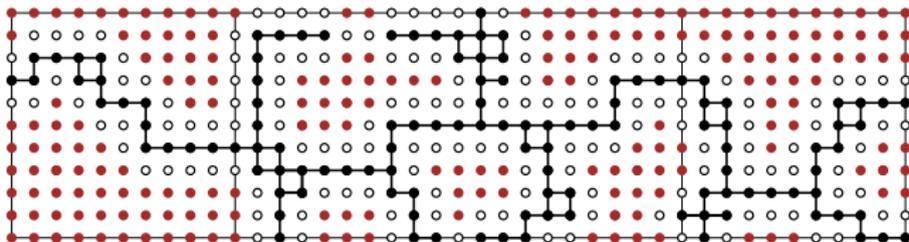
$\omega$  containing crossing

## Proposition

For  $\delta > 0$  small enough, as  $n \rightarrow \infty$ ,

$$\mathbb{P}_{p_c, \delta} \left( \left( \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) \right) \rightarrow 0$$

The diagram shows two rectangular domains. The first domain has a width of  $6n$  and a height of  $n$ . It is divided into three vertical sections of width  $n$ ,  $4n$ , and  $n$ . A black path  $\omega$  is shown crossing the domain. The second domain has a width of  $6n$  and a height of  $n$ , also divided into three vertical sections of width  $n$ ,  $4n$ , and  $n$ . A blue path  $\tilde{\omega}^\delta$  is shown crossing the domain.



$\omega$  containing crossing  $\xrightarrow{\text{delete crossing cluster}} \tilde{\omega}$

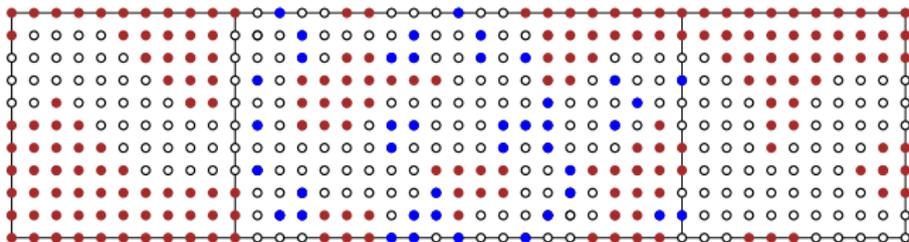
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$\leftarrow n \quad 4n \quad n \rightarrow$





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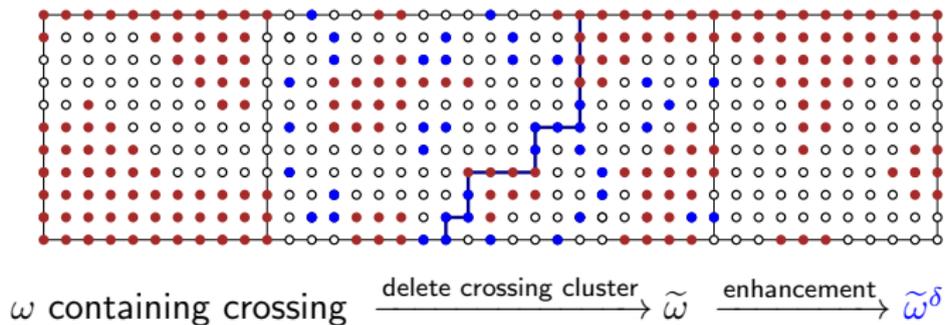
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Diagram 1: A rectangle with width  $6n$  and height  $n$ . It is divided into three vertical sections of width  $n$ . A black curve labeled  $\omega$  starts in the left section, crosses the first boundary, crosses the second boundary, and ends in the right section.

Diagram 2: A rectangle with width  $6n$  and height  $n$ . It is divided into three vertical sections of width  $n$ . A blue curve labeled  $\tilde{\omega}^\delta$  starts in the left section, crosses the first boundary, crosses the second boundary, and ends in the right section.

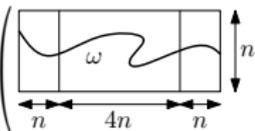


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$$\mathbb{P}_{p_c, \delta} \left( \begin{array}{c} \text{[Diagram of } \omega \text{ crossing a rectangle of width } 6n \text{ and height } n \text{]} \\ \text{and} \\ \text{[Diagram of } \tilde{\omega}^\delta \text{ crossing a rectangle of width } 6n \text{ and height } n \text{]} \end{array} \right) \rightarrow 0$$

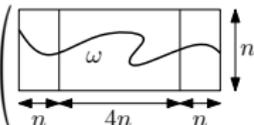
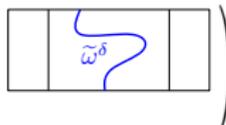
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For  $p_1 \geq p_2$  and  $\delta_1, \delta_2$  such that  $p_1 + (1 - p_1)\delta_1 \leq p_2 + (1 - p_2)\delta_2$ ,

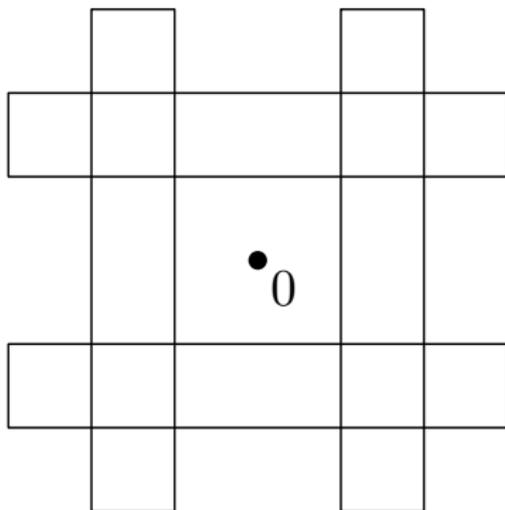
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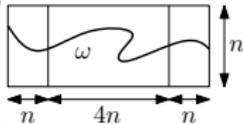
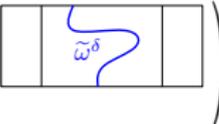
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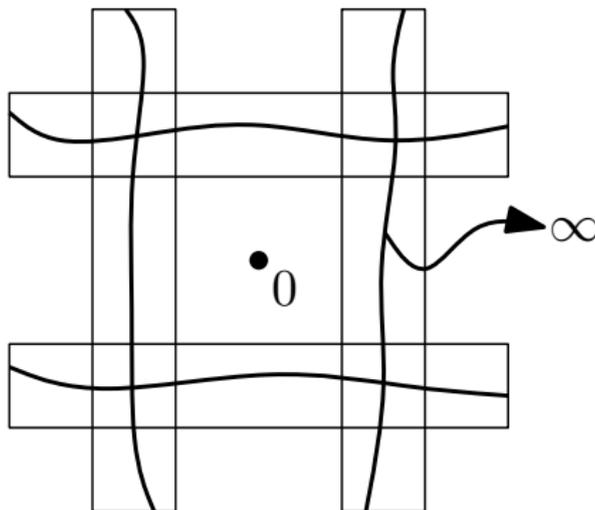
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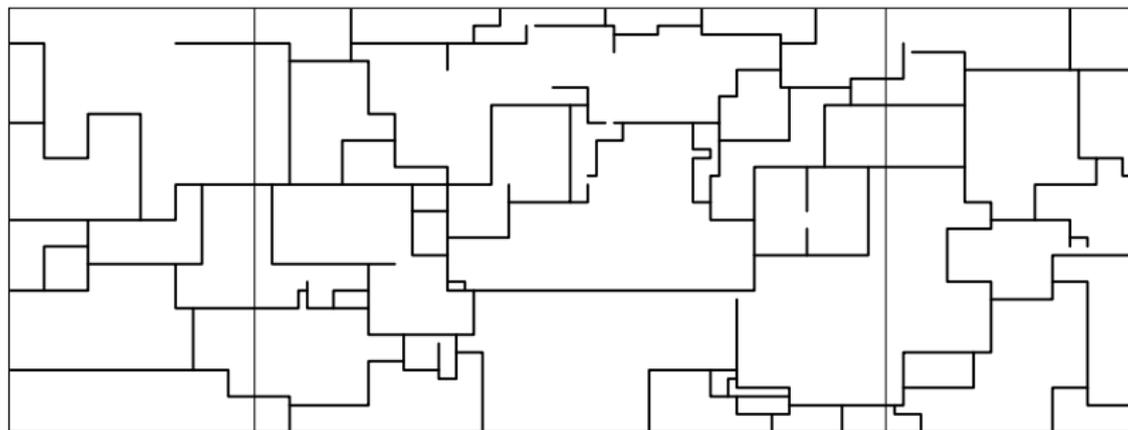
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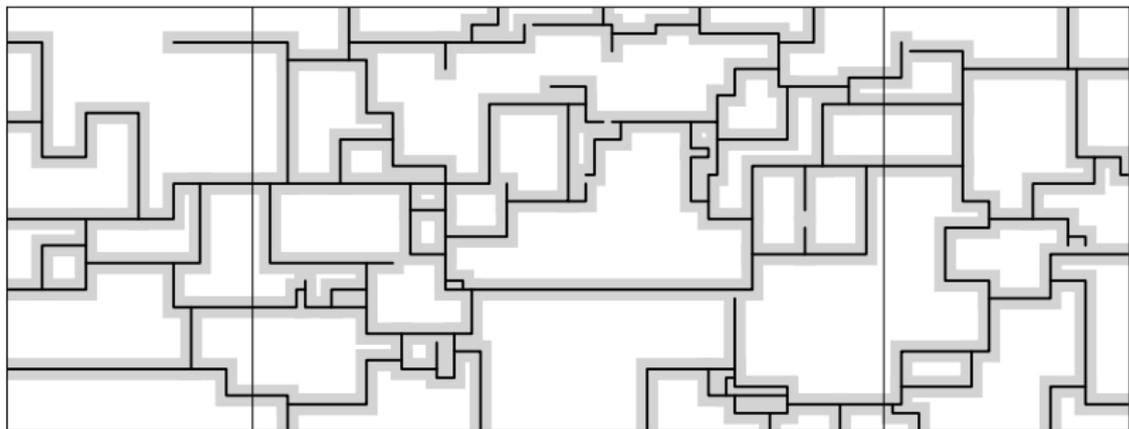


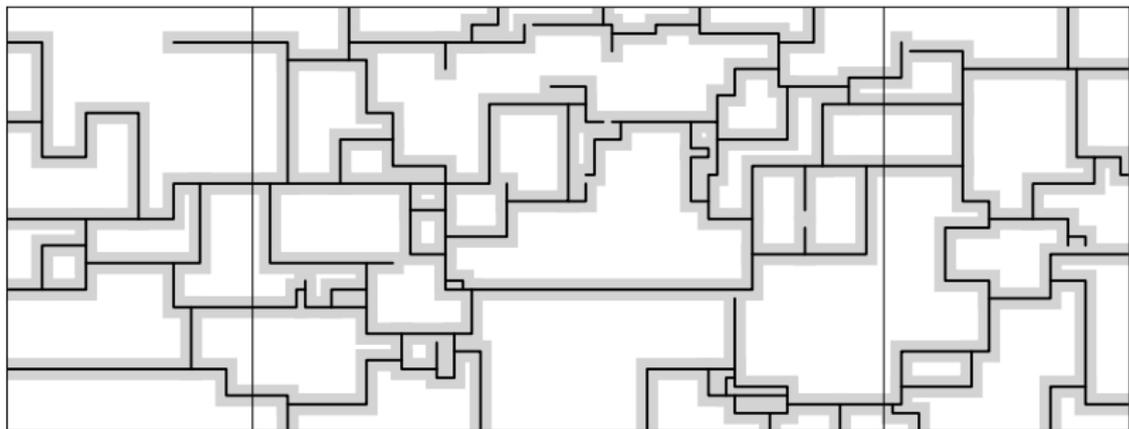
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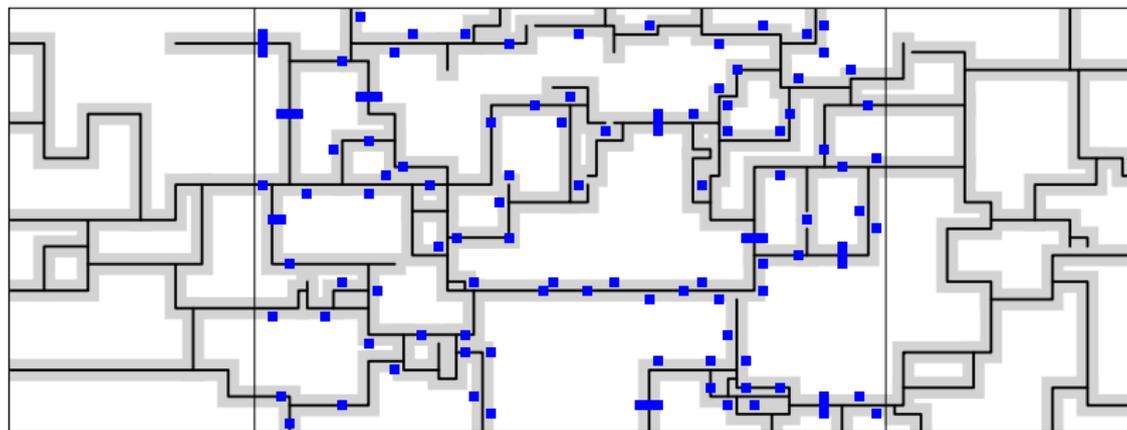


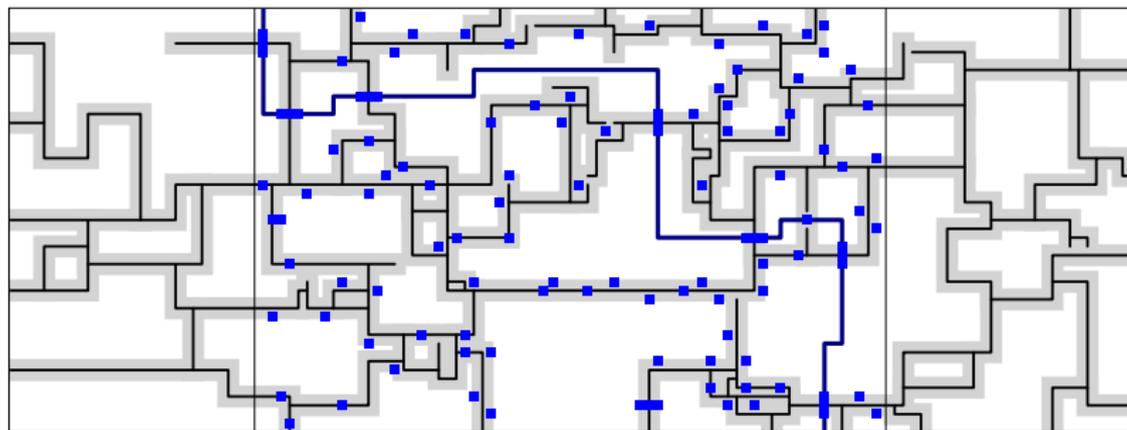




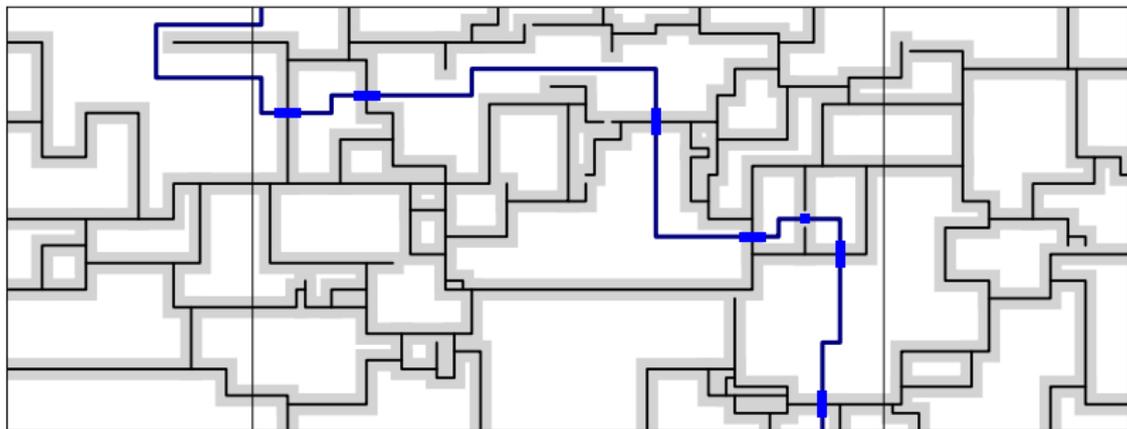








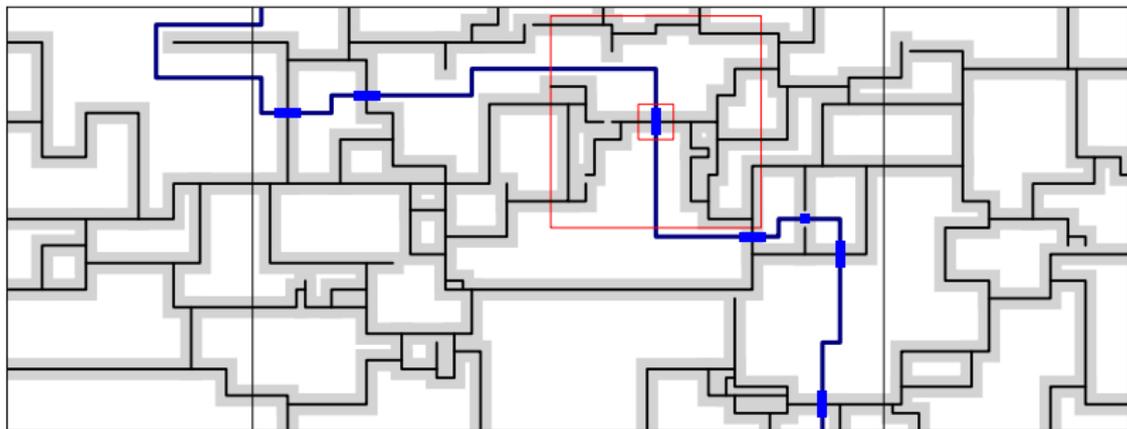
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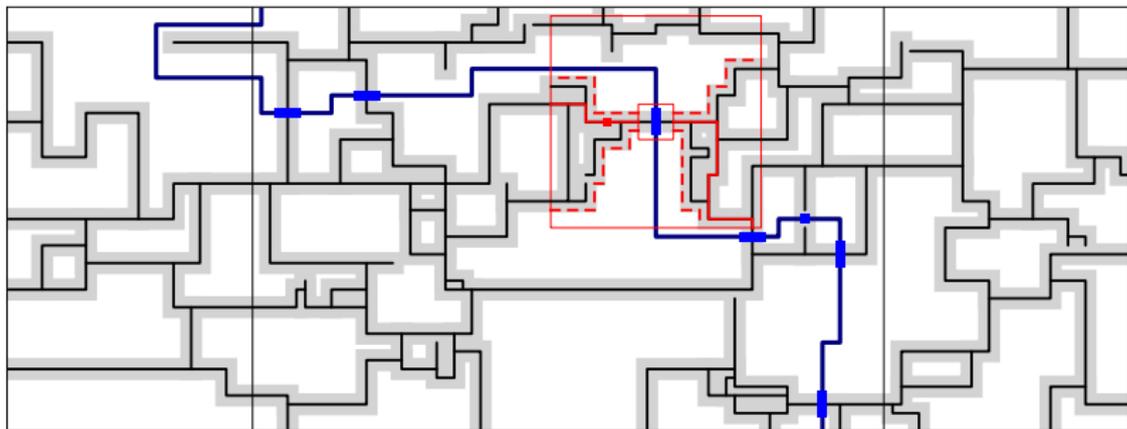
$\mathcal{X} = \{\text{enhanced points used by } \gamma\}$ . If no crossing  $\mathcal{X} = \emptyset$ .

$$\mathbb{P}_{\rho_c, \delta}(\text{vertical crossing in } \tilde{\omega}^\delta) = \sum_{\mathcal{X} \neq \emptyset} \mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X).$$

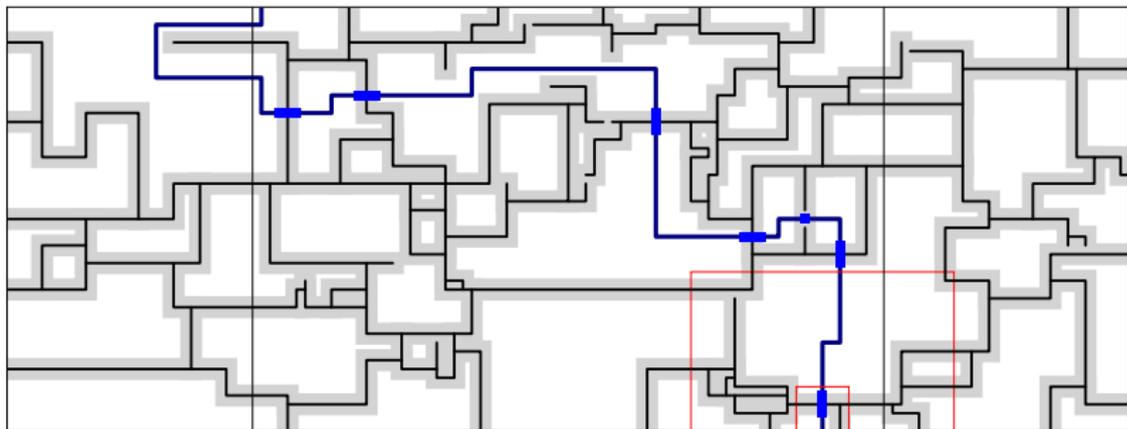


Annulus surrounding passage points but not containing passage points:  
 6 arms or 4 half-plane arms in  $\omega$  (possibly with one defect).

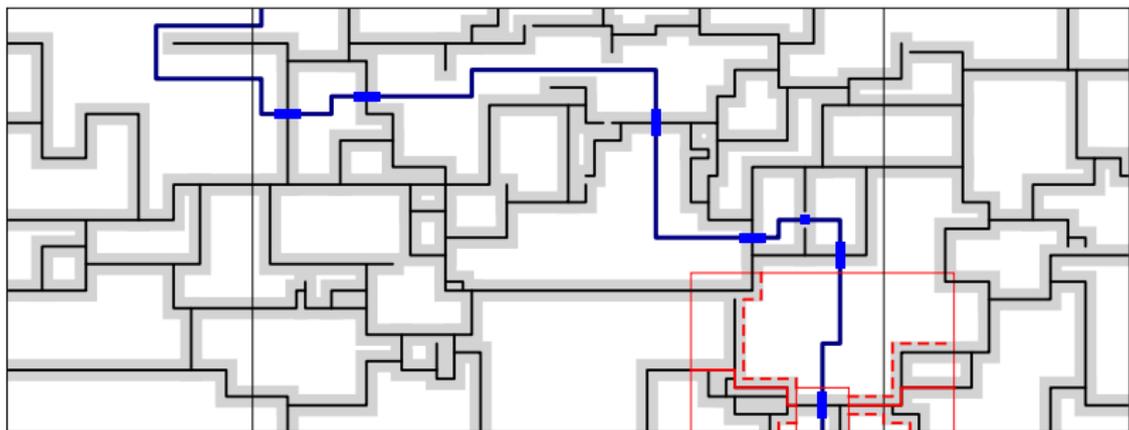




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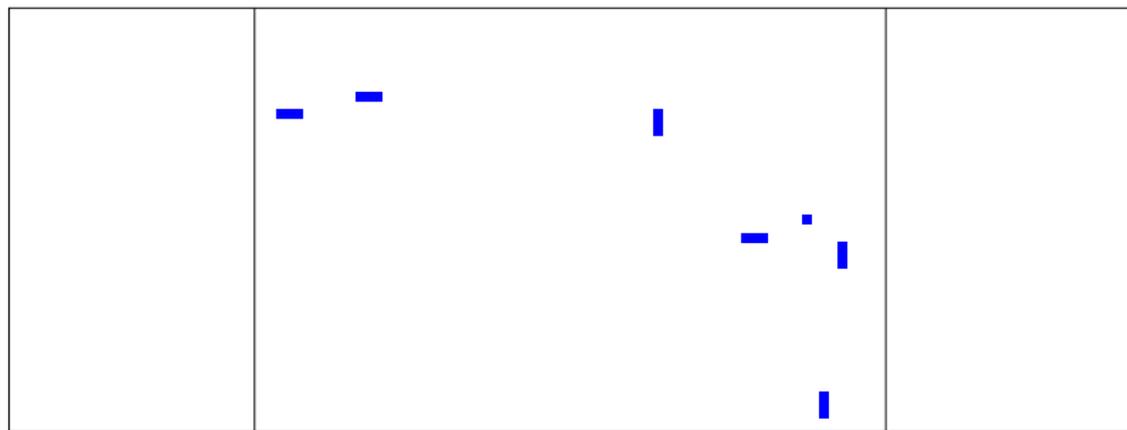


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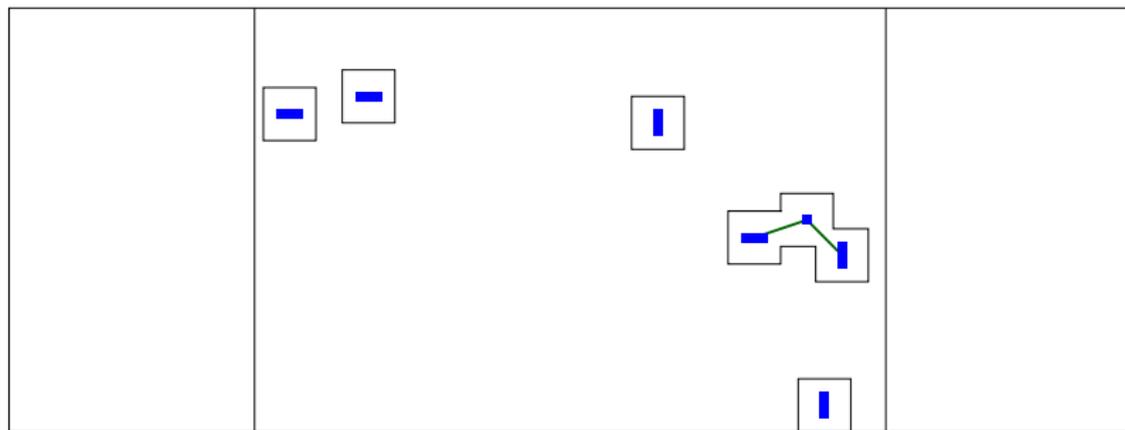


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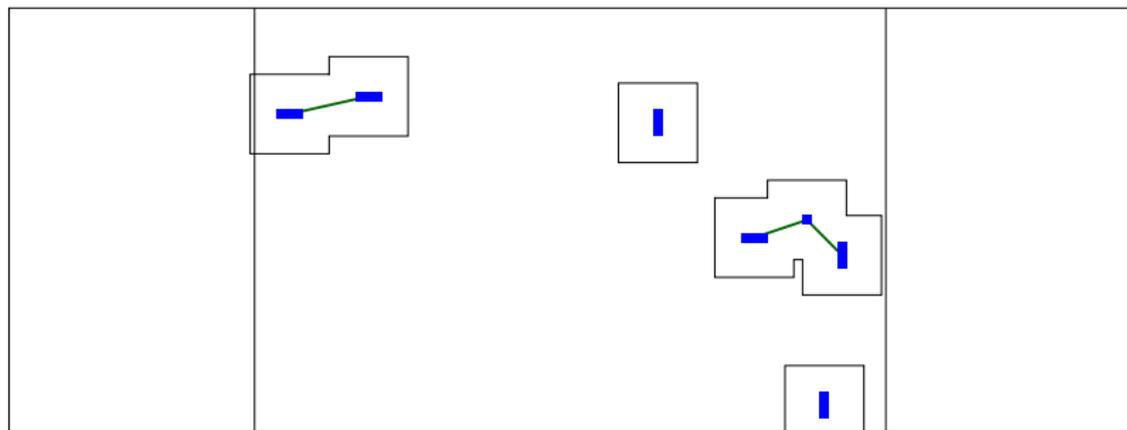
$$\mathbb{P}_{p_c} \left( \begin{array}{c} \text{Diagram 1: A square with side length } R \text{ and } r. \text{ A solid red path winds through the square, crossing the boundary multiple times.} \\ \text{Diagram 2: A square with side length } R \text{ and } r. \text{ A solid red path winds through the square, crossing the boundary multiple times.} \end{array} \right) \leq \left( \frac{r}{R} \right)^{2+\lambda}$$



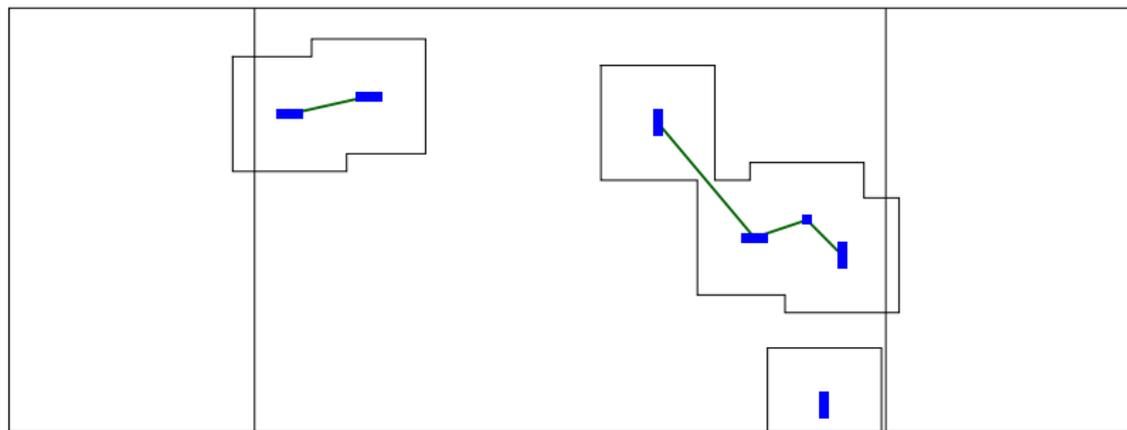
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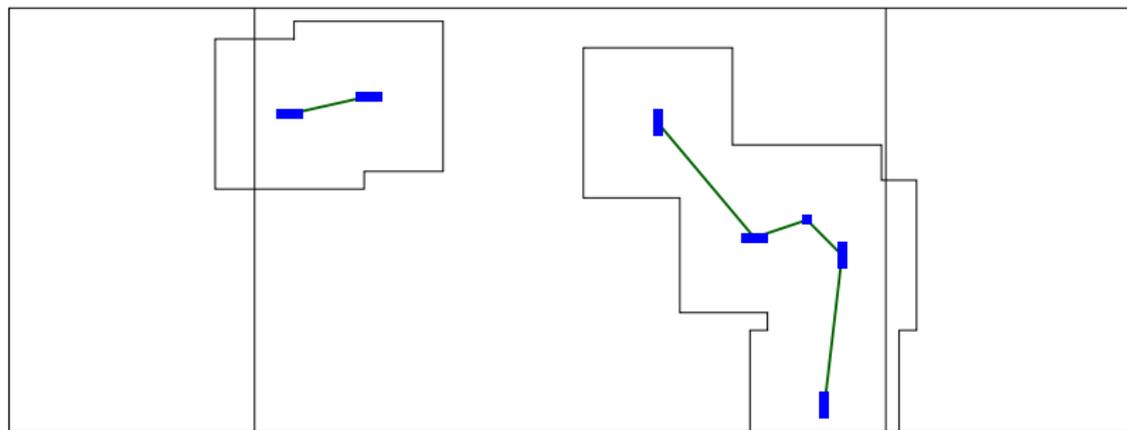
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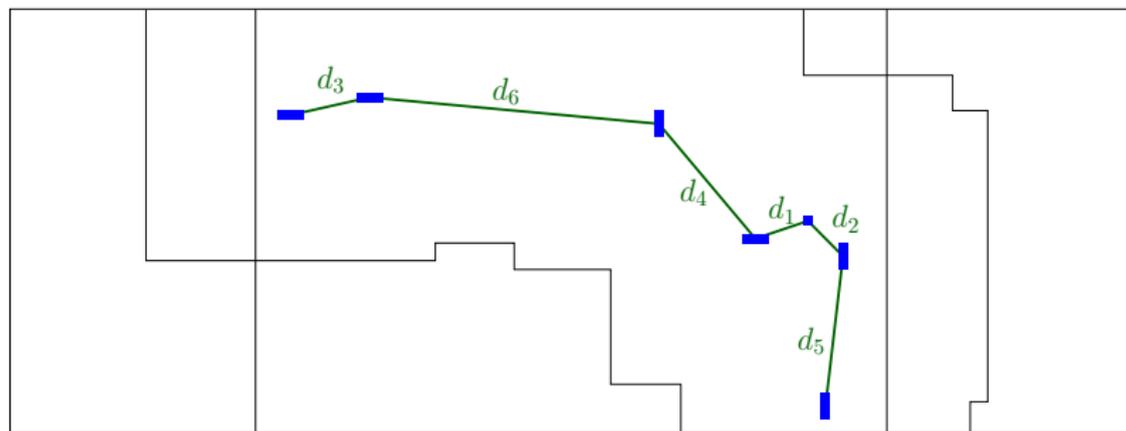
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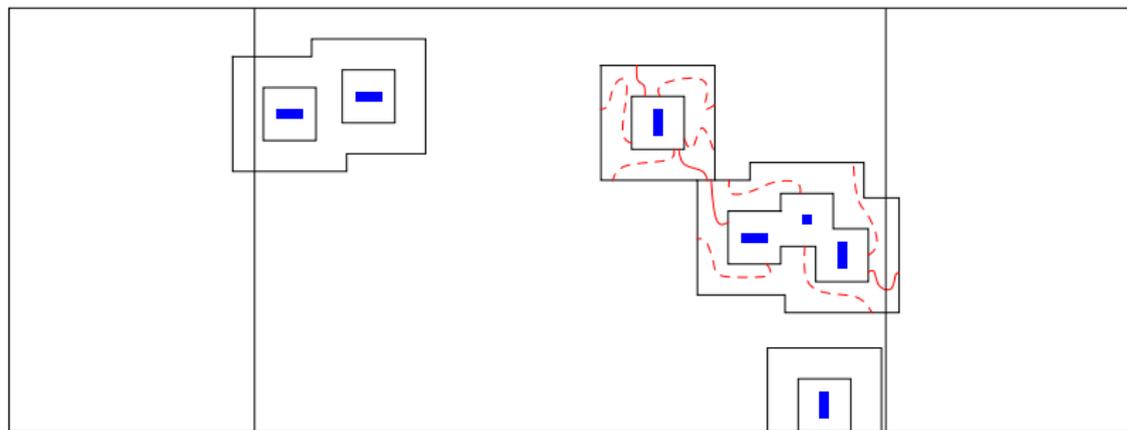
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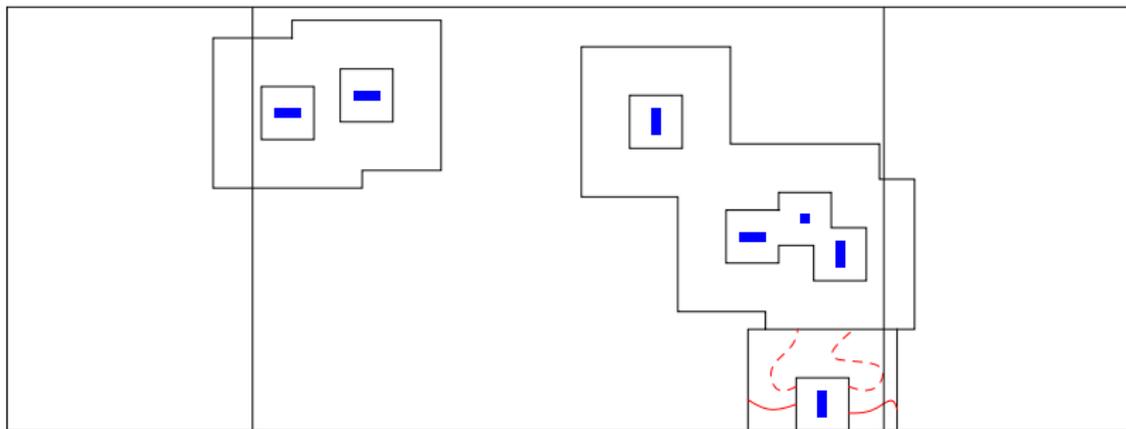
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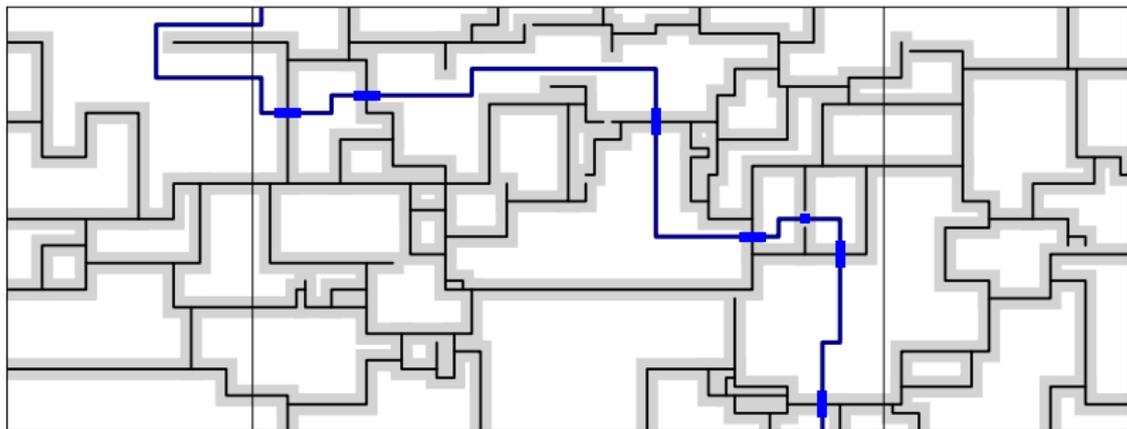
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$$\mathbb{P}_{p,\delta}(\mathcal{X} = X) \leq c^k n^{-2-\lambda} \prod_j d_j^{-2-\lambda} \times \delta^k,$$

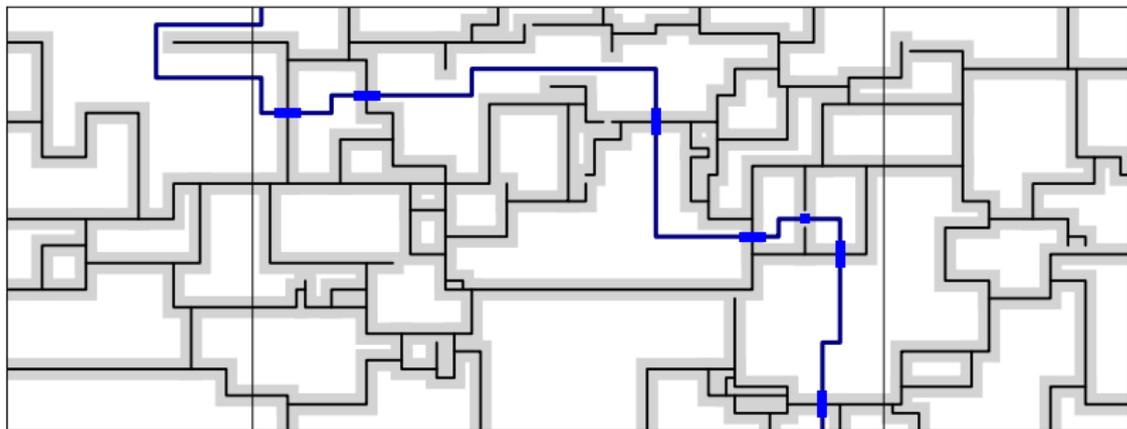
where  $d_1, \dots, d_k$  are the merger times of  $X$ .

$$\#\{X \text{ with merger times } d_1, \dots, d_k\} \leq C^k n^2 \prod_j d_j.$$



$$\begin{aligned}
 \mathbb{P}(\text{vertical crossing in } \tilde{\omega}^\delta) &\leq n^{-\lambda} \sum_{\substack{k \geq 1 \\ d_1, \dots, d_k}} \left( \delta^k c^k \prod_k d_k^{-1-\lambda} \right) \\
 &= n^{-\lambda} \sum_{k \geq 1} \left( \delta c \sum_{d \geq 1} d^{-1-\lambda} \right)^k \rightarrow 0,
 \end{aligned}$$

for  $\delta > 0$  small.



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