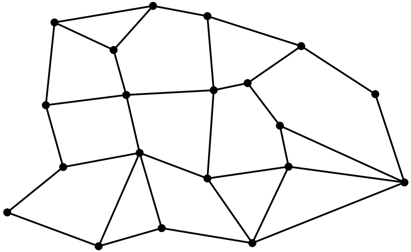
Percolation on isoradial graphs

Ioan Manolescu
Joint work with Geoffrey Grimmett

University of Geneva

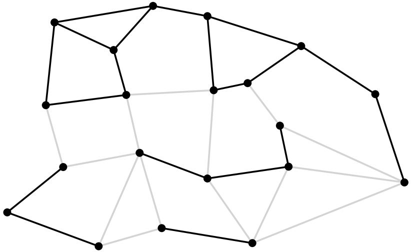
15 August 2013

Percolation



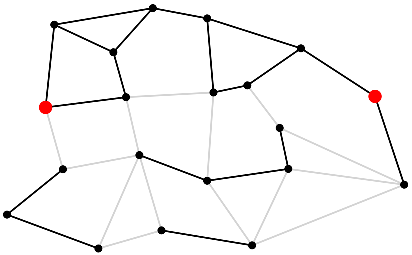
Under $\mathbb{P}_{\mathbf{p}}$, an edge e is $\begin{cases} \text{open with probability } p_e \\ \text{closed with probability } 1 - p_e \end{cases}$

Percolation



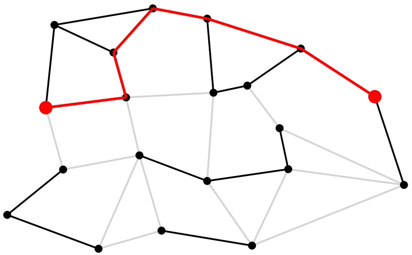
Under $\mathbb{P}_{\mathbf{p}}$, an edge e is $\begin{cases} \text{open with probability } p_e \\ \text{closed with probability } 1 - p_e \end{cases}$

Percolation



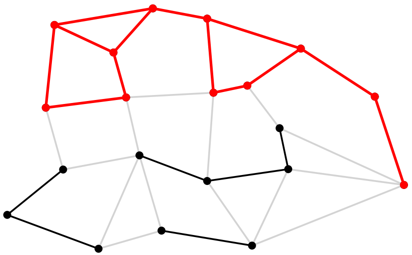
Under $\mathbb{P}_{\mathbf{p}}$, an edge e is $\begin{cases} \text{open with probability } p_e \\ \text{closed with probability } 1 - p_e \end{cases}$

Percolation



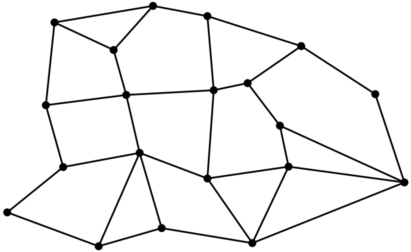
Under $\mathbb{P}_{\mathbf{p}}$, an edge e is $\begin{cases} \text{open with probability } p_e \\ \text{closed with probability } 1 - p_e \end{cases}$

Percolation



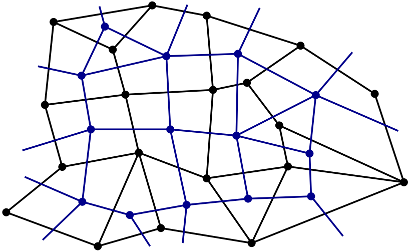
Under $\mathbb{P}_{\mathbf{p}}$, an edge e is $\begin{cases} \text{open with probability } p_e \\ \text{closed with probability } 1 - p_e \end{cases}$

Percolation



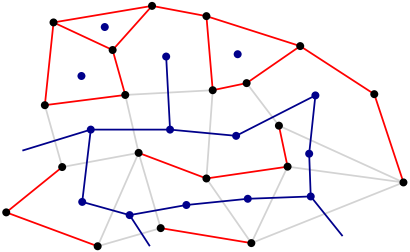
Under $\mathbb{P}_{\mathbf{p}}$, an edge e is $\begin{cases} \text{open with probability } p_e \\ \text{closed with probability } 1 - p_e \end{cases}$

Percolation



Under $\mathbb{P}_{\mathbf{p}}$, an edge e is $\begin{cases} \text{open with probability } p_e \\ \text{closed with probability } 1 - p_e \end{cases}$

Percolation



Under $\mathbb{P}_{\mathbf{p}}$, an edge e is $\begin{cases} \text{open with probability } p_e \\ \text{closed with probability } 1 - p_e \end{cases}$

Homogeneous percolation on $\mathbb{Z}^2$: all edges have intensity $p \in [0,1]$.

Question: is there an infinite connected component?

Homogeneous percolation on $\mathbb{Z}^2$: all edges have intensity $p \in [0, 1]$.

Question: is there an infinite connected component?

0 [———————— • ————————] 1

Subcriticality: No infinite cluster

Supercriticality: Existence of infinite cluster

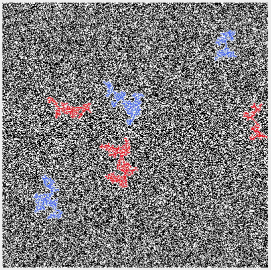
Homogeneous percolation on $\mathbb{Z}^2$: all edges have intensity $p \in [0,1]$.

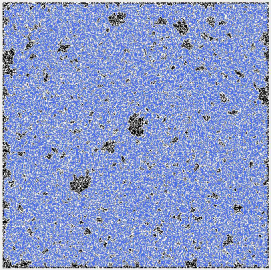
Question: is there an infinite connected component?

0 — 1

Subcriticality:
No infinite cluster
Exponential tail for cluster size
Trivial large scale behaviour!

Supercriticality:
Existence of infinite cluster
Unique infinite cluster
Exponential tail for distance to infinite cluster.
Trivial large scale behaviour!





Homogeneous percolation on $\mathbb{Z}^2$: all edges have intensity $p \in [0,1]$.

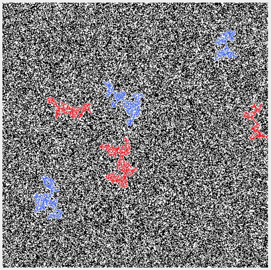
Question: is there an infinite connected component?

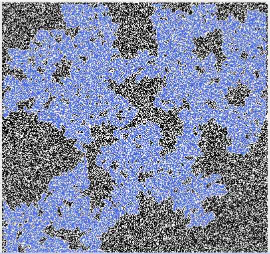
0 $\qquad$ p_c $\qquad$ 1

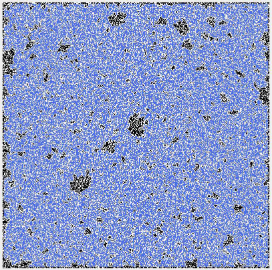
Subcriticality:
No infinite cluster
Exponential tail for cluster size
Trivial large scale behaviour!

Supercriticality:
Existence of infinite cluster
Unique infinite cluster
Exponential tail for distance to infinite cluster.
Trivial large scale behaviour!

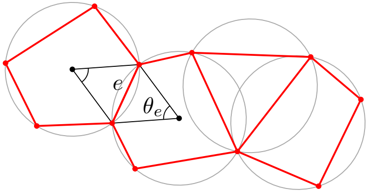
Criticality
Scale invariance
Large scale limit.
Universality.





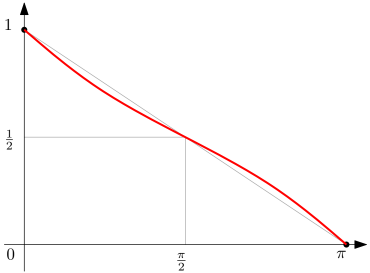


Isoradial percolation

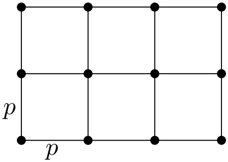


Each face of G is inscribed in a circle of radius 1.
$\mathbb{P}_G$ percolation with p_e:

$$\frac{p_e}{1-p_e} = \frac{\sin\left(\frac{\pi-\theta(e)}{3}\right)}{\sin\left(\frac{\theta(e)}{3}\right)}.$$



Bond Percolation on $\mathbb{Z}^2$



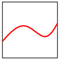
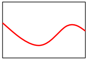
Isoradiality: $p = \frac{1}{2}$

Theorem (Kesten 80)

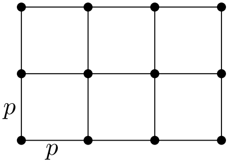
$p \leq \frac{1}{2}$, a.s. no infinite cluster;
$p > \frac{1}{2}$, a.s. existence of an infinite cluster.

Method:

self-duality $+$ RSW $+$ sharp-threshold

$\mathbb{P}_{\frac{1}{2}}\Bigg($  $\Bigg) = \frac{1}{2} \Rightarrow \mathbb{P}_{\frac{1}{2}}\Bigg($  $\Bigg) \geq c \Rightarrow \mathbb{P}_{\frac{1}{2}+\epsilon}(0 \leftrightarrow \infty) > 0$

Bond Percolation on $\mathbb{Z}^2$



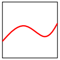
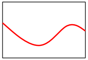
Isoradiality: $p = \frac{1}{2}$

Theorem (Kesten 80)

$p \leq \frac{1}{2}$, a.s. no infinite cluster;
$p > \frac{1}{2}$, a.s. existence of an infinite cluster.

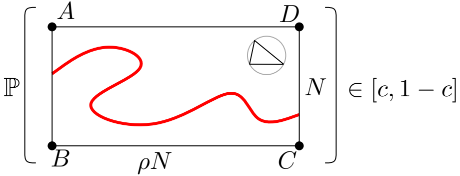
Method:

self-duality $+$ RSW $+$ sharp-threshold

$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) = \frac{1}{2} \Rightarrow \mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \geq c \Rightarrow \mathbb{P}_{\frac{1}{2}+\epsilon}(0 \leftrightarrow \infty) > 0$

The box-crossing property (RSW)

A model satisfies the box-crossing property if for all rectangles *ABCD* there exists $c(BC/AB) = c(\rho) > 0$ s. t. for all N large enough:



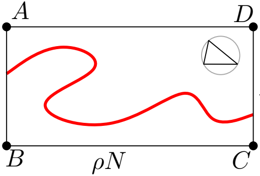
Equivalent for the primal and dual model.

Theorem

If $\mathbb{P}_{\mathbf{p}}$ satisfies the box-crossing property, then it is critical.

The box-crossing property (RSW)

A model satisfies the box-crossing property if for all rectangles $ABCD$ there exists $c(BC/AB) = c(\rho) > 0$ s. t. for all N large enough:



$$\mathbb{P}\left[\ \cdots\ \right] \in [c, 1-c]$$

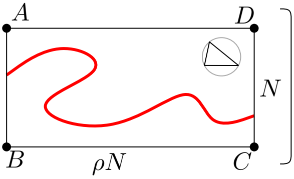
Equivalent for the primal and dual model.

Theorem

If $\mathbb{P}_{\mathbf{p}}$ satisfies the box-crossing property, then it is critical.

The box-crossing property (RSW)

A model satisfies the box-crossing property if for all rectangles $ABCD$ there exists $c(BC/AB) = c(\rho) > 0$ s. t. for all N large enough:

$\mathbb{P}\Big[$ 

$\Big] \in [c, 1-c]$

Equivalent for the primal and dual model.

Theorem

If $\mathbb{P}_{\mathbf{p}}$ satisfies the box-crossing property, then it is critical.

Results I: the box-crossing property

For an isoradial graph G with the percolation measure $\mathbb{P}_G$, subject to conditions:

Theorem

$\mathbb{P}_G$ satisfies the box-crossing property.

Corollary

$\mathbb{P}_G$ is critical.

- $\mathbb{P}_{\mathbf{p}}(\textit{infinite cluster}) = 0$,
- $\mathbb{P}_{\mathbf{p}+\epsilon}(\textit{infinite cluster}) = 1$.

Results I: the box-crossing property

For an isoradial graph G with the percolation measure $\mathbb{P}_G$, subject to conditions:

Theorem

$\mathbb{P}_G$ satisfies the box-crossing property.

Corollary

$\mathbb{P}_G$ is critical.

- $\mathbb{P}_{\mathbf{p}}(\textit{infinite cluster}) = 0$,
- $\mathbb{P}_{\mathbf{p}+\epsilon}(\textit{infinite cluster}) = 1$.

Arm exponents

For a critical percolation measure $\mathbb{P}$, as $n \to \infty$, we expect:

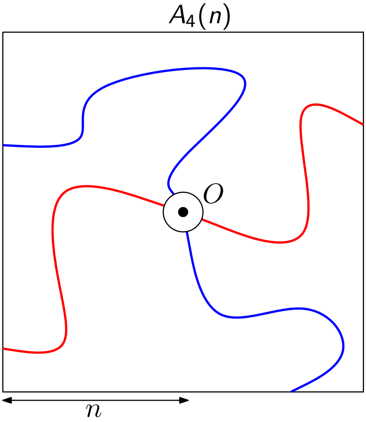
- one-arm exponent $\frac{5}{48}$:

$$\mathbb{P}(\mathrm{rad}(C_0) \geq n) = \mathbb{P}(A_1(n)) \approx n^{-\rho_1},$$

- $2j$-alternating-arms exponents $\frac{4j^2-1}{12}$:

$$\mathbb{P}[A_{2j}(n)] \approx n^{-\rho_{2j}}.$$

Moreover ρ_i **does not depend on the underlying model**.



Power-law bounds are given by the box-crossing property.

Critical exponents

For $\mathbb{P}_{\mathbf{p}}$ critical we expect:

Exponents at criticality.

Volume exponent $\delta = \frac{91}{5}$:
$\mathbb{P}_{\mathbf{p}}(|C_0| = n) \approx n^{-1-1/\delta}$.

Connectivity exponent $\eta = \frac{5}{24}$:
$\mathbb{P}_{\mathbf{p}}(0 \leftrightarrow x) \approx |x|^{-\eta}$.

Radius exponent $\rho = \frac{48}{5}$:
$\mathbb{P}_{\mathbf{p}}(\mathrm{rad}(C_0) = n) \approx n^{-1-1/\rho}$.

$(\rho = \frac{1}{\rho_1})$

Exponents near criticality.

Percolation probability $\beta = \frac{5}{36}$:
$\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0| = \infty) \approx \epsilon^{\beta}$ as $\epsilon \downarrow 0$.

Correlation length $\nu = \frac{4}{3}$:
$\xi(\mathbf{p} - \epsilon) \approx \epsilon^{-\nu}$ as $\epsilon \downarrow 0$, were
$-\frac{1}{n} \log \mathbb{P}_{\mathbf{p}-\epsilon}(\mathrm{rad}(C_0) \geq n) \to_{n\to\infty} \frac{1}{\xi(\mathbf{p}-\epsilon)}$.

Mean cluster-size $\gamma = \frac{43}{18}$:
$\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|; |C_0| < \infty) \approx |\epsilon|^{-\gamma}$ as $\epsilon \to 0$.

Gap exponent $\Delta = \frac{91}{36}$:
$\frac{\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|^{k+1}; |C_0| < \infty)}{\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|^{k}; |C_0| < \infty)} \approx |\epsilon|^{-\Delta}$. for $k \geq 1$, as $\epsilon \to 0$.

Results II: arm exponents

For an isoradial graph G with the percolation measure $\mathbb{P}_G$, subject to conditions

Theorem

For $k \in \{1, 2, 4, \ldots\}$ there exist constants $c_1, c_2 > 0$ such that:

$$c_1 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)] \le \mathbb{P}_G[A_k(n)] \le c_2 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)],$$

for $n \in \mathbb{N}$.

Results II: arm exponents

For an isoradial graph G with the percolation measure $\mathbb{P}_G$, subject to conditions

Theorem

For $k \in \{1, 2, 4, \dots\}$ *there exist constants* $c_1, c_2 > 0$ *such that:*

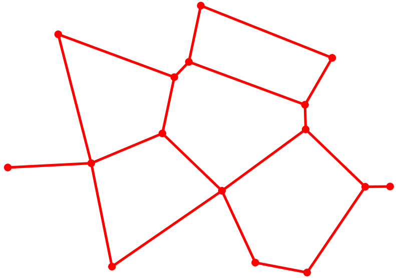
$$c_1 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)] \leq \mathbb{P}_G[A_k(n)] \leq c_2 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)],$$

for $n \in \mathbb{N}$.

Corollary

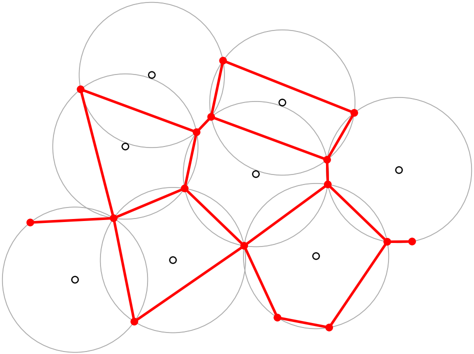
The one arm exponent and the $2j$ *alternating arm exponents are universal for percolation on isoradial graphs.*

Isoradial Graphs



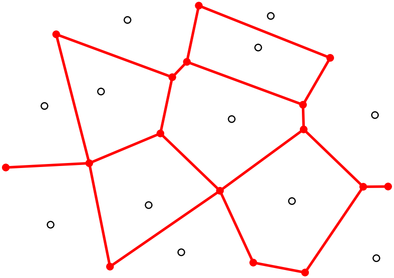
G isoradial graph

Isoradial Graphs



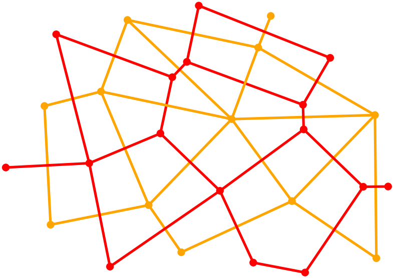
G isoradial graph

Isoradial Graphs



G isoradial graph

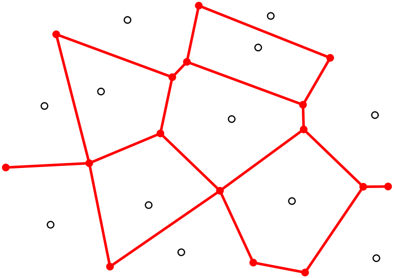
Isoradial Graphs



G isoradial graph

G^* dual isoradial graph

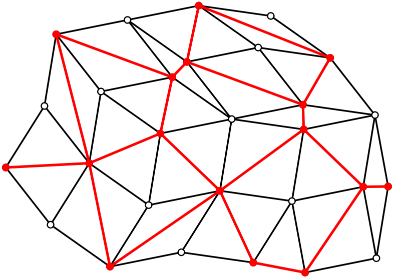
Isoradial Graphs



G isoradial graph

G^* dual isoradial graph

Isoradial Graphs

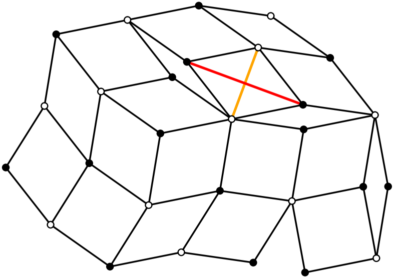


G isoradial graph

G^* dual isoradial graph

$G^\diamond$ diamond graph

Isoradial Graphs

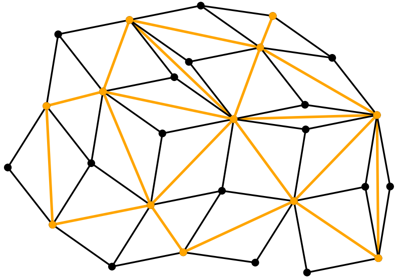


G isoradial graph

G^* dual isoradial graph

$G^\diamond$ diamond graph

Isoradial Graphs

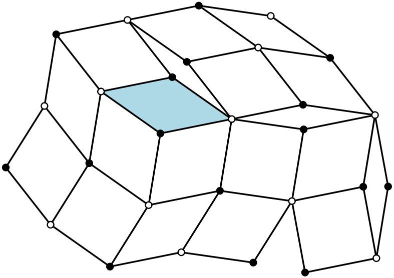


G isoradial graph

G^* dual isoradial graph

$G^\diamond$ diamond graph

Isoradial Graphs

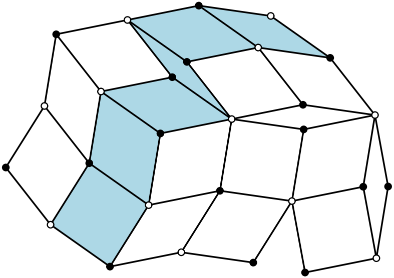


G isoradial graph

G^* dual isoradial graph

$G^\diamond$ diamond graph

Isoradial Graphs

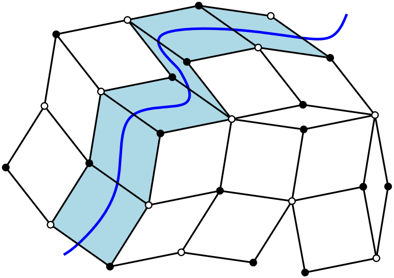


G isoradial graph

G^* dual isoradial graph

$G^\diamond$ diamond graph

Isoradial Graphs

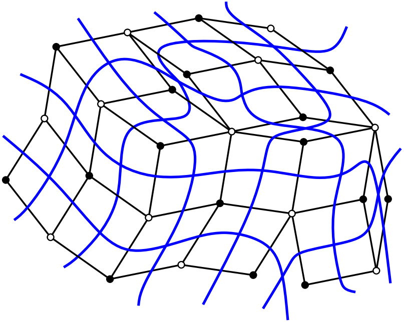


G isoradial graph

G^* dual isoradial graph

$G^\diamond$ diamond graph

Isoradial Graphs



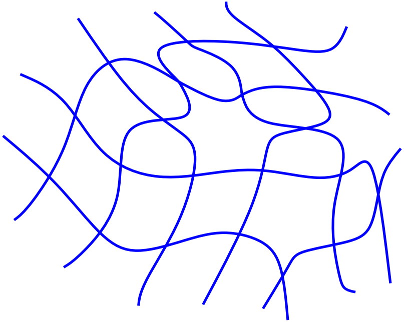
G isoradial graph

G^* dual isoradial graph

$G^\diamond$ diamond graph

Track system

Isoradial Graphs



G isoradial graph

G^* dual isoradial graph

$G^\diamond$ diamond graph

Track system

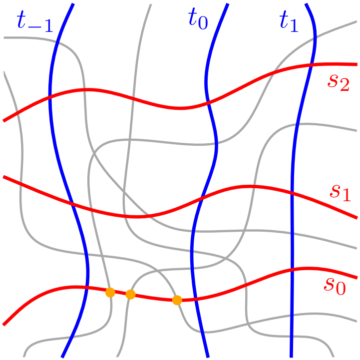
Conditions for isoradial graphs.

Bounded angles condition:
There exist $\epsilon_0 > 0$ such that for any edge e, $\theta_e \in [\epsilon_0, \pi - \epsilon_0]$.

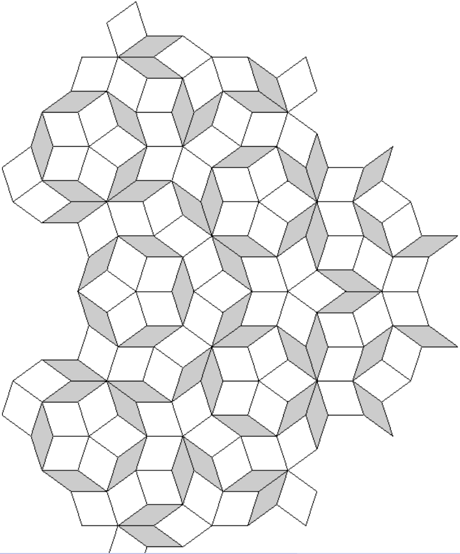
Conditions for isoradial graphs.

Bounded angles condition:
There exist $\epsilon_0 > 0$ such that for any edge e, $\theta_e \in [\epsilon_0, \pi - \epsilon_0]$.

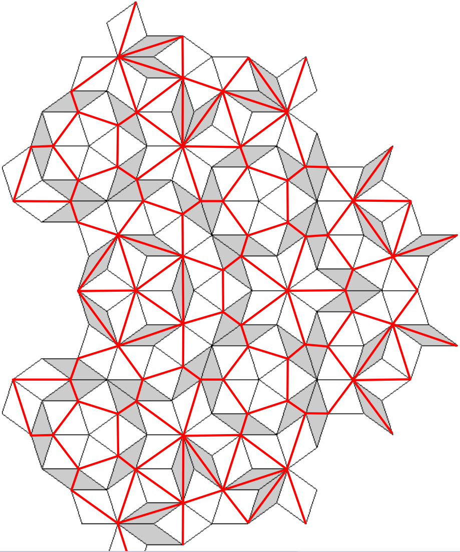
Square grid property:
Families of "parallel" tracks $(s_i)_{i\in\mathbb{Z}}$ and $(t_j)_{j\in\mathbb{Z}}$.
The number of intersections on s_i between t_j and t_{j+1} is uniformly bounded by a constant I. (same for t).



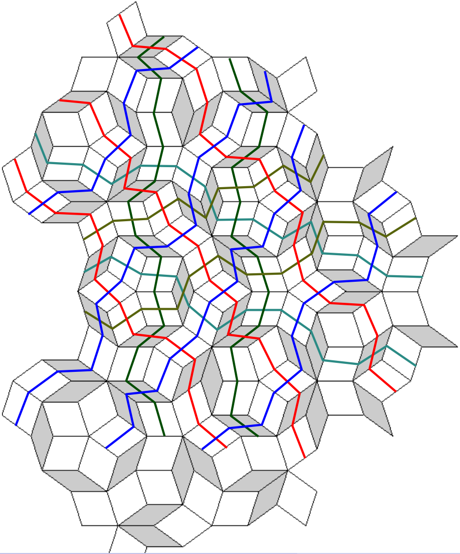
Examples: Penrose tilings and no square grid



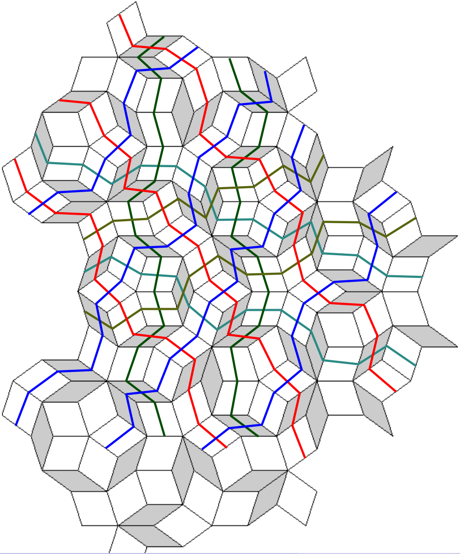
Examples: Penrose tilings and no square grid

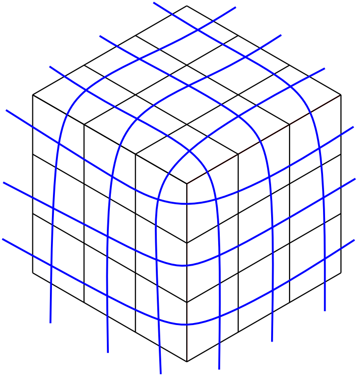


Examples: Penrose tilings and no square grid

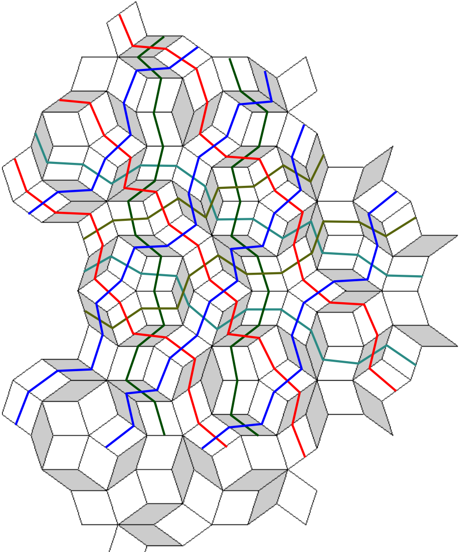


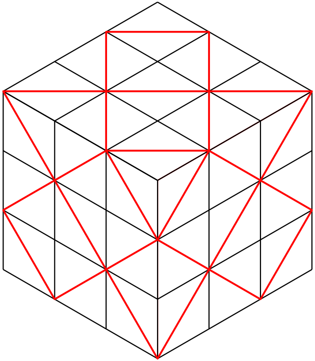
Examples: Penrose tilings and no square grid



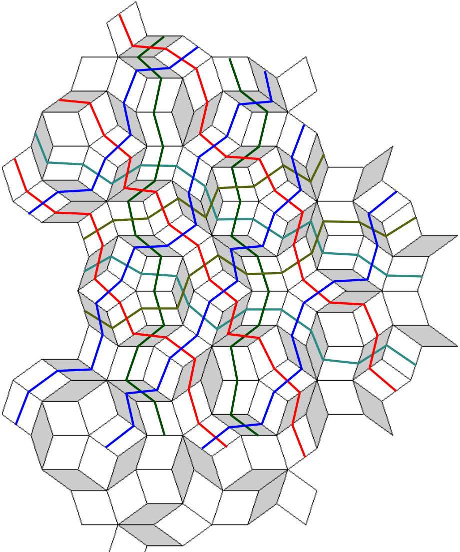


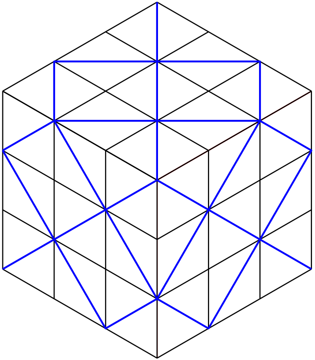
Examples: Penrose tilings and no square grid



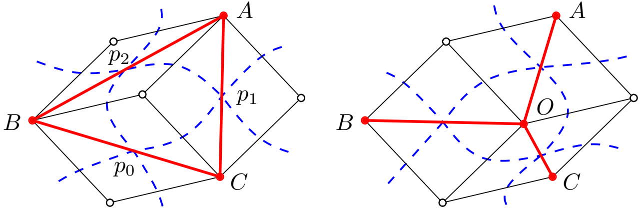


Examples: Penrose tilings and no square grid





Star–triangle transformation



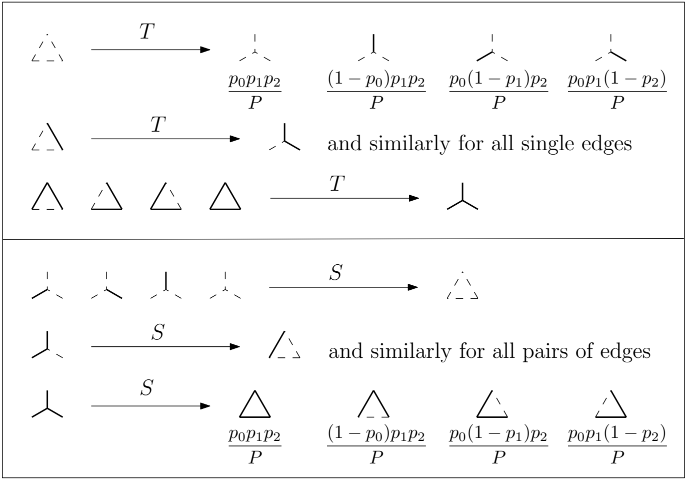
$$\kappa_{\triangle}(\mathbf{p}) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1.$$

Take ω, respectively ω', according to the measure on the left, respectively right. The families of random variables

$$\left(x \overset{\omega}{\longleftrightarrow} y : x, y = A, B, C\right), \quad \left(x \overset{\omega'}{\longleftrightarrow} y : x, y = A, B, C\right),$$

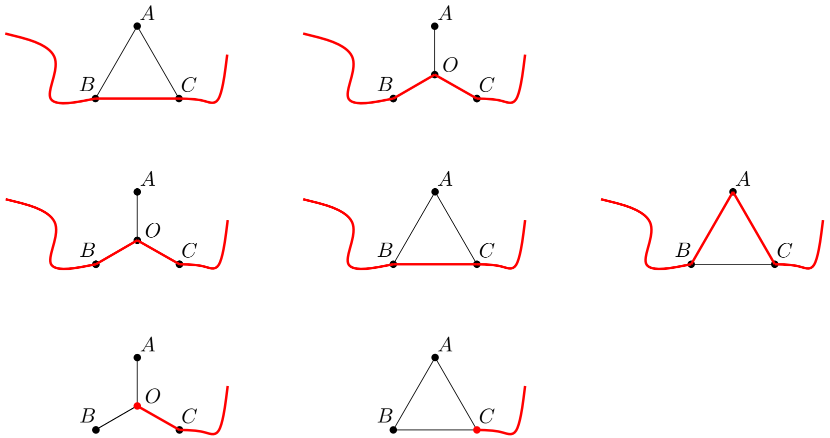
have the same joint law.

Coupling



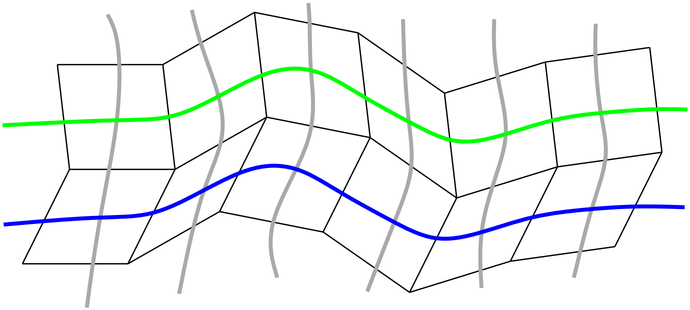
where $P = (1 - p_0)(1 - p_1)(1 - p_2)$.

Path transformation



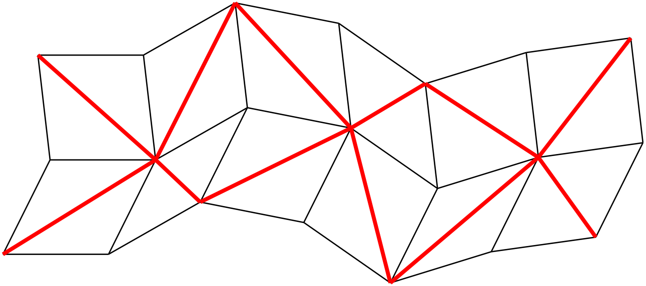
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



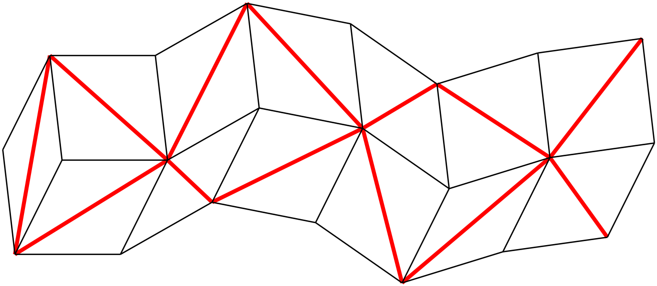
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



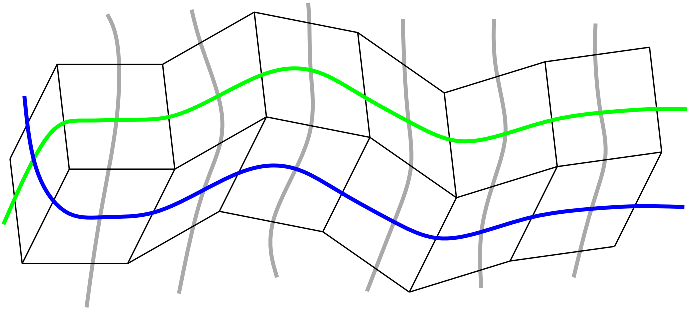
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



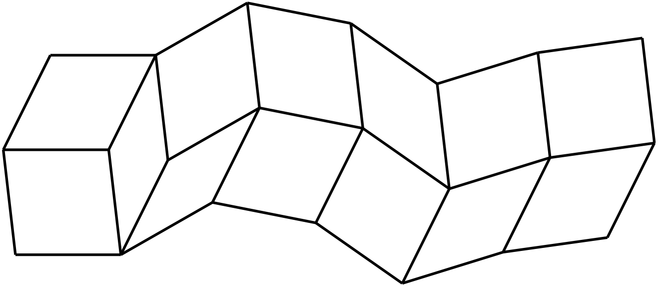
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



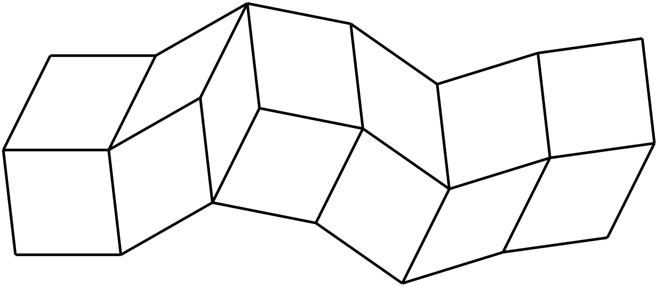
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



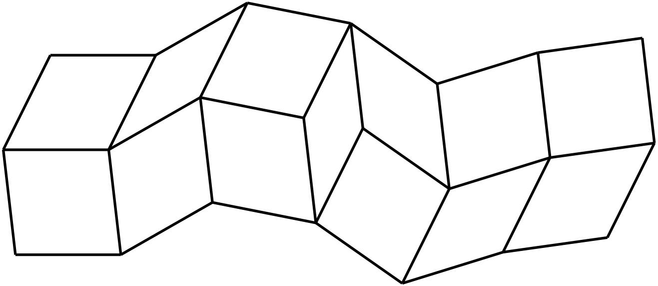
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



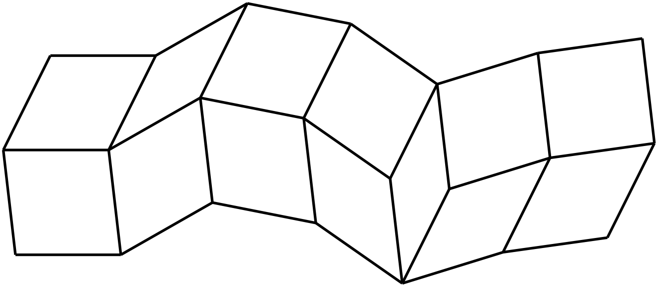
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



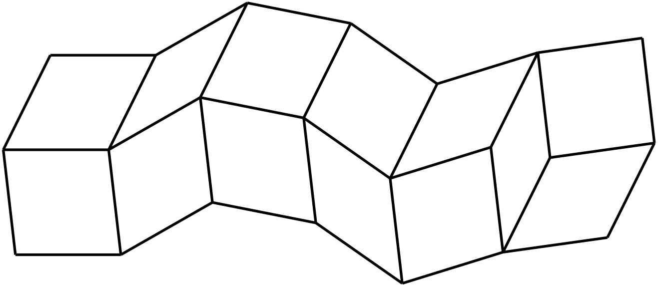
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



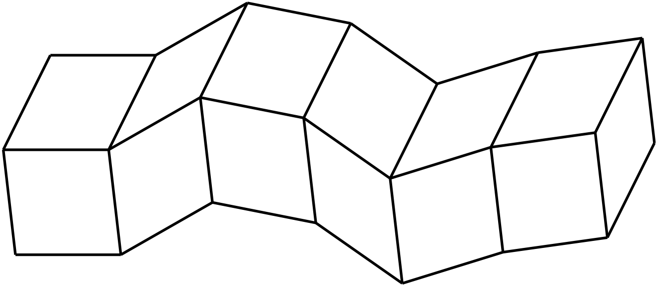
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



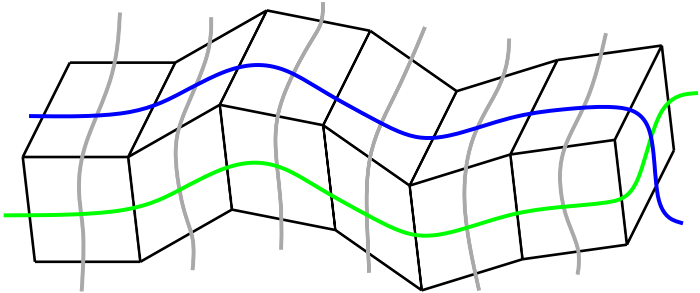
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



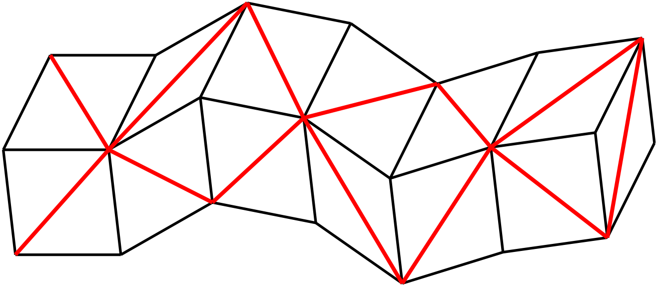
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



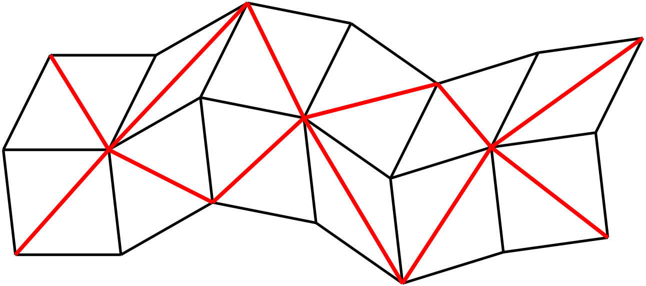
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



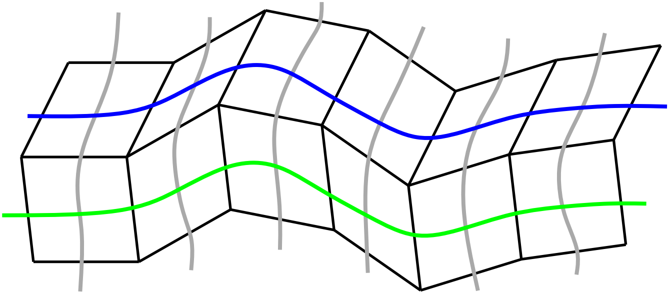
Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.

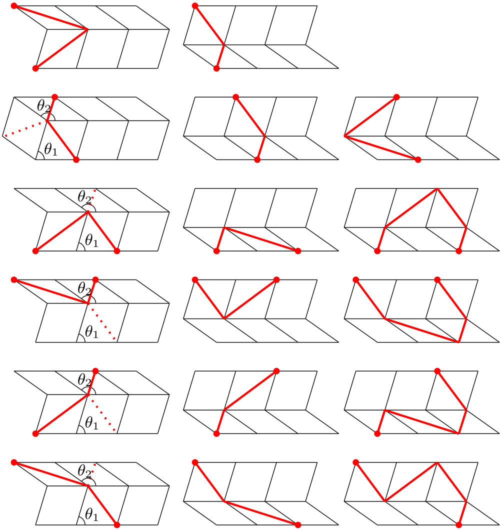


Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



Initial configuration	Principal outcome	Secondary outcome	Probability of secondary outcome



$$\frac{p_{\pi-\theta_1}p_{\theta_2}}{p_{\theta_1}p_{\pi-\theta_2}}$$

$$\frac{p_{\pi-\theta_1}p_{\pi-\theta_2+\theta_1}}{p_{\theta_1}p_{\theta_2-\theta_1}}$$

$$\frac{p_{\theta_2}p_{\pi-\theta_2+\theta_1}}{p_{\pi-\theta_2}p_{\theta_2-\theta_1}}$$

$$\frac{p_{\theta_2}p_{\pi-\theta_2+\theta_1}}{p_{\pi-\theta_2}p_{\theta_2-\theta_1}}$$

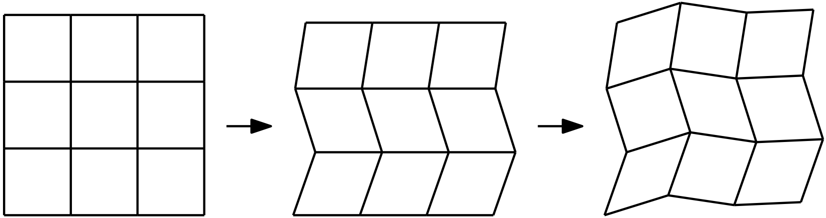
$$\frac{p_{\pi-\theta_1}p_{\pi-\theta_2+\theta_1}}{p_{\theta_1}p_{\theta_2-\theta_1}}$$

Open paths are preserved (unless the deleted edge was part of the path).

Strategy

Proposition

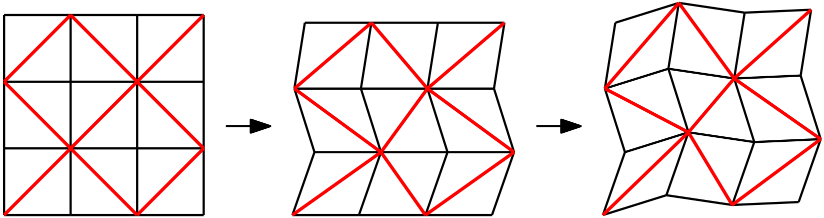
If two isoradial square lattices have same transverse angles for the vertical/horizontal tracks, and one has the box-crossing property, then so does the other.



Strategy

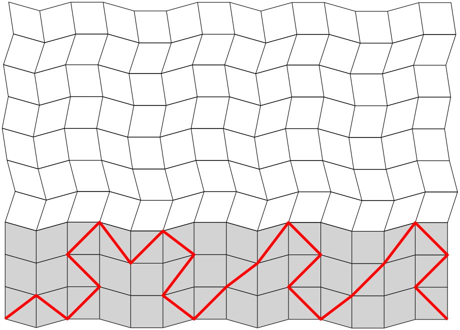
Proposition

If two isoradial square lattices have same transverse angles for the vertical/horizontal tracks, and one has the box-crossing property, then so does the other.



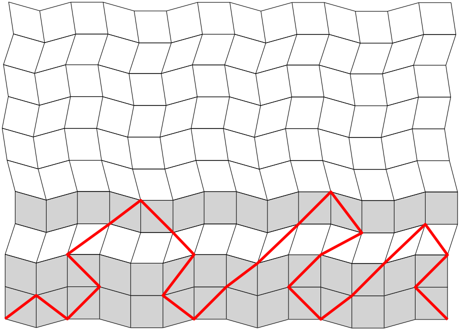
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



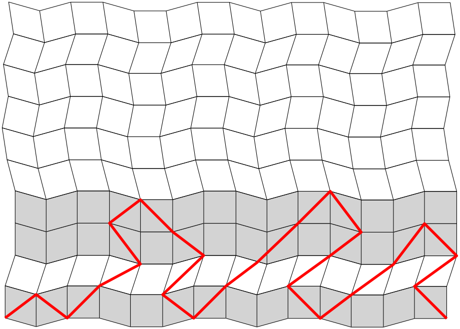
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



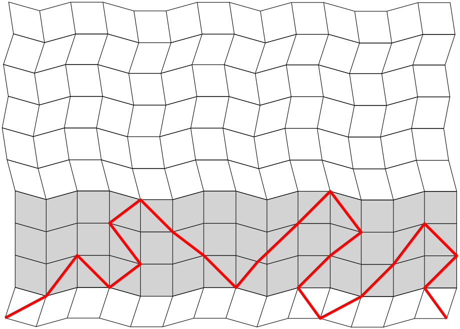
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



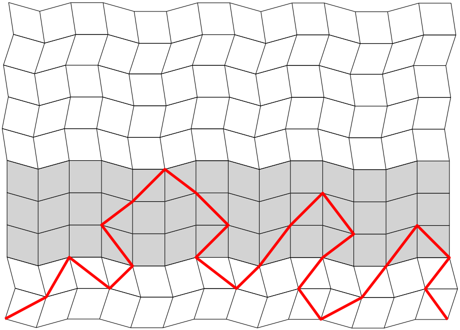
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



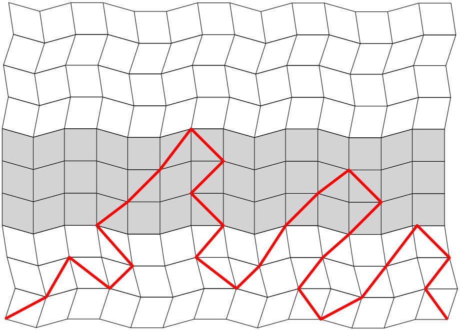
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



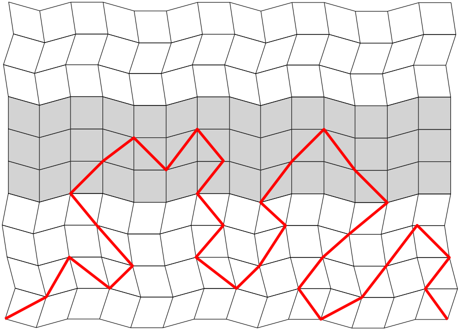
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



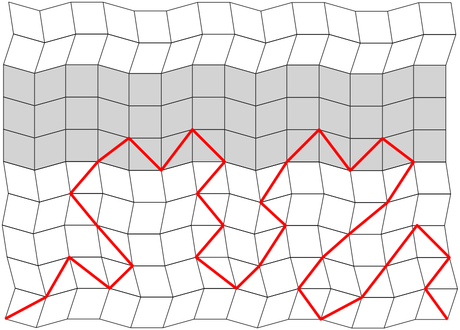
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



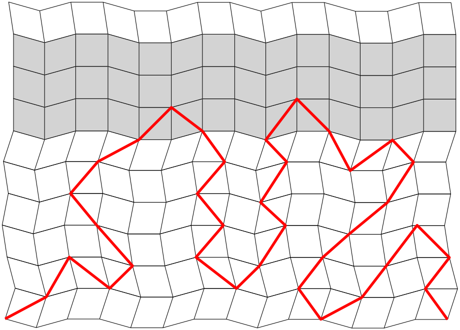
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



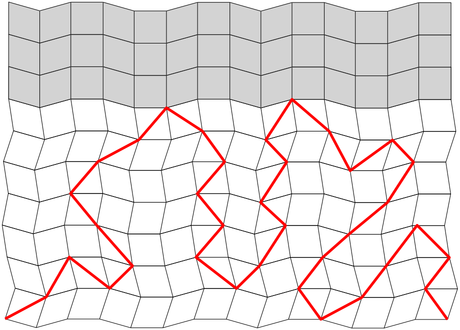
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



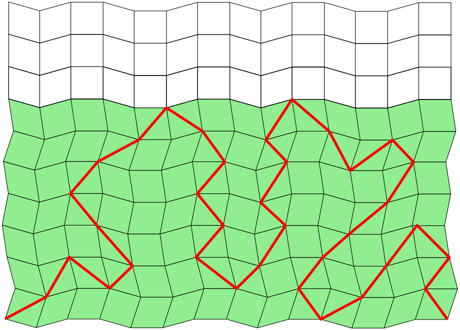
Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.

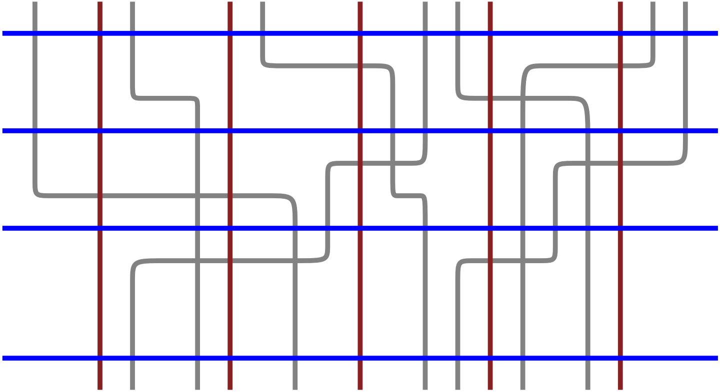


Transport of horizontal crossings

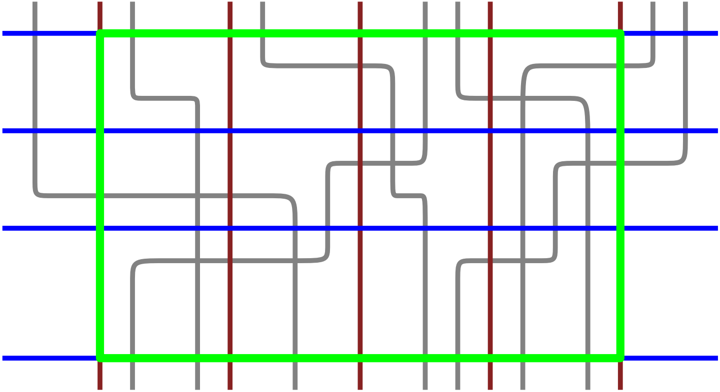
Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



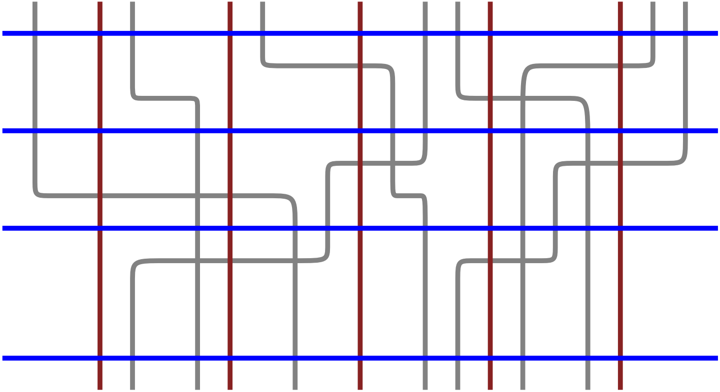
Track stacking



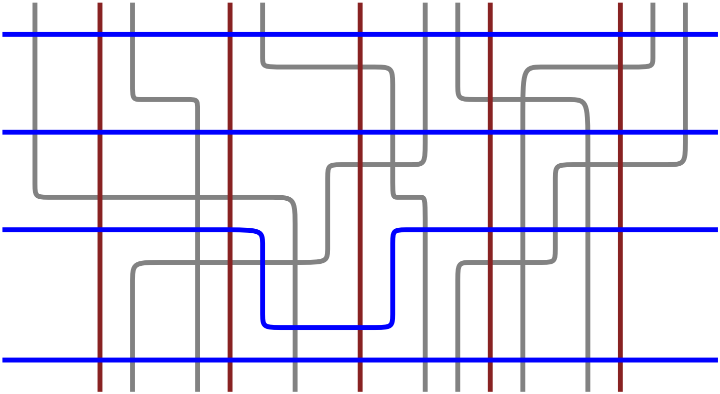
Track stacking



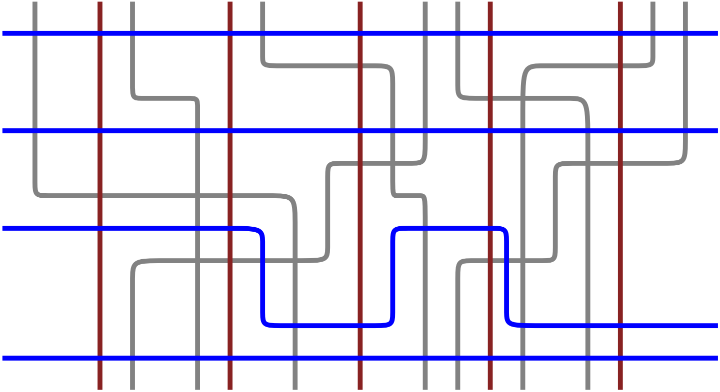
Track stacking



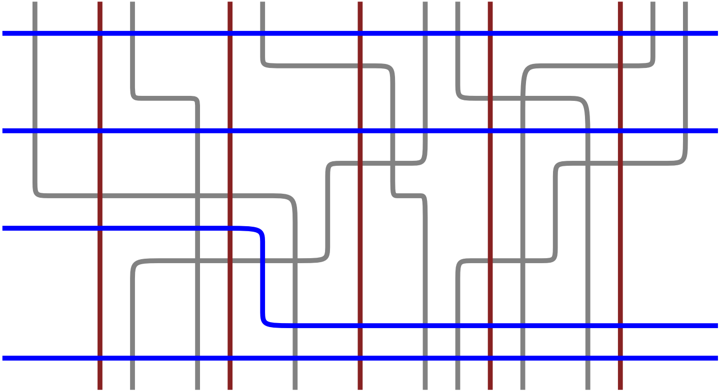
Track stacking



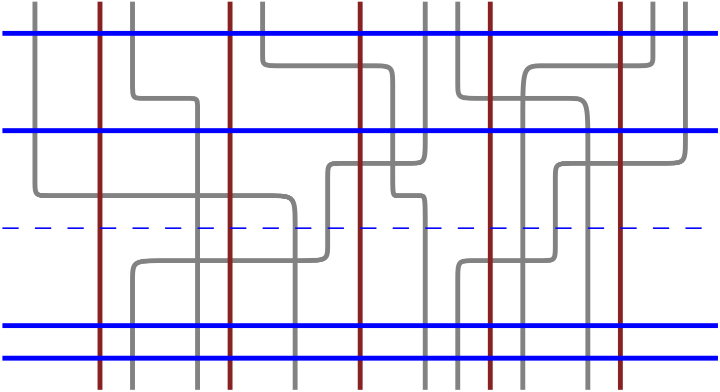
Track stacking



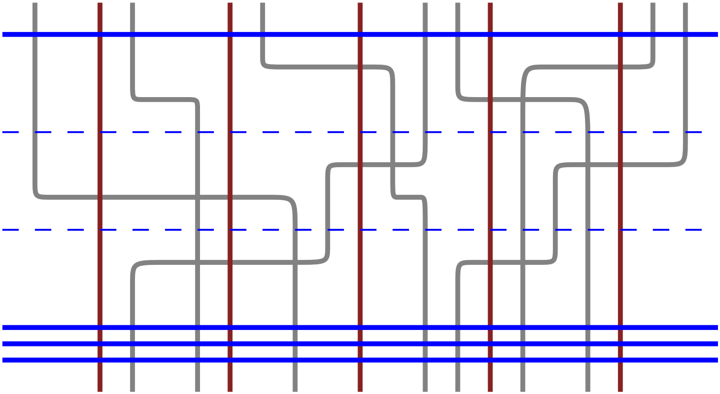
Track stacking



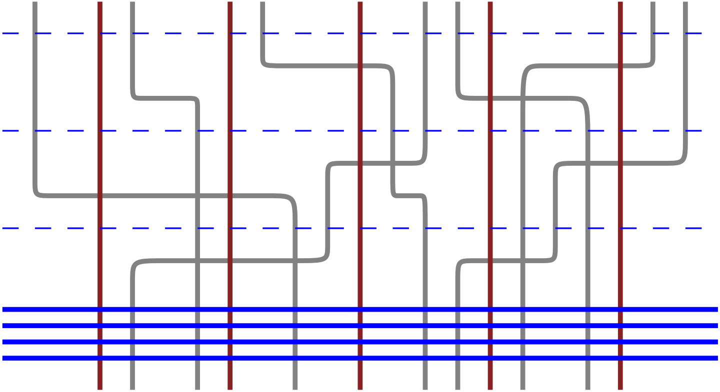
Track stacking



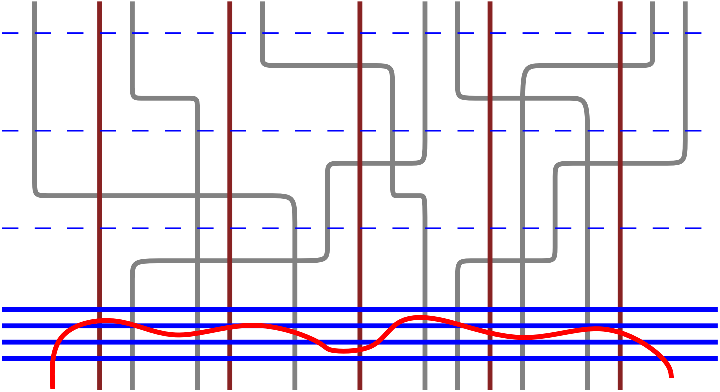
Track stacking



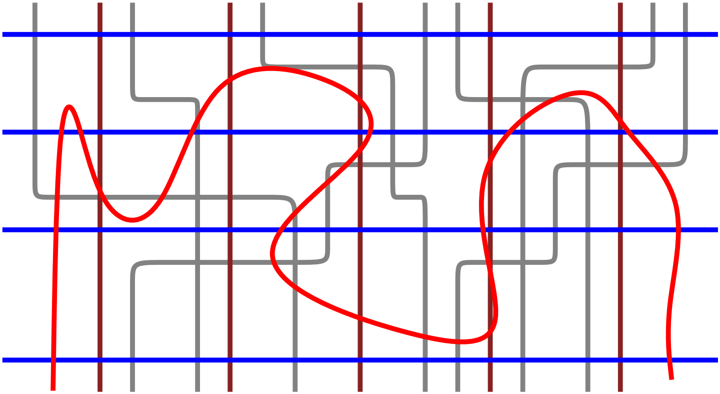
Track stacking



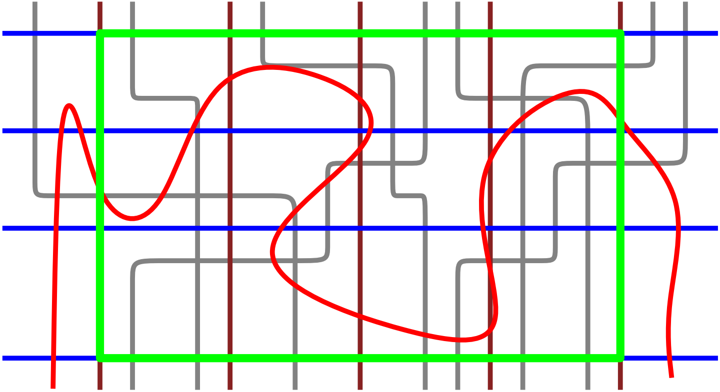
Track stacking



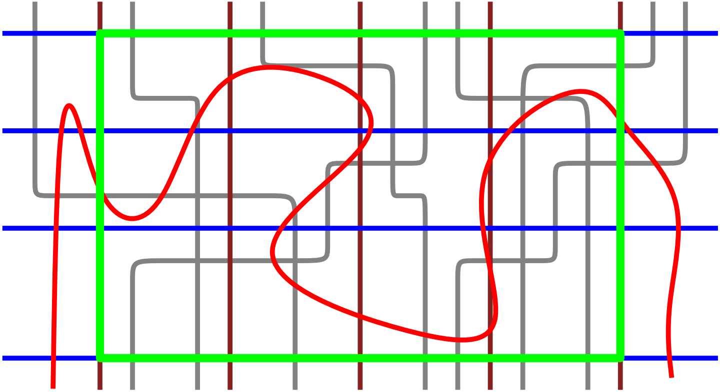
Track stacking



Track stacking



Track stacking

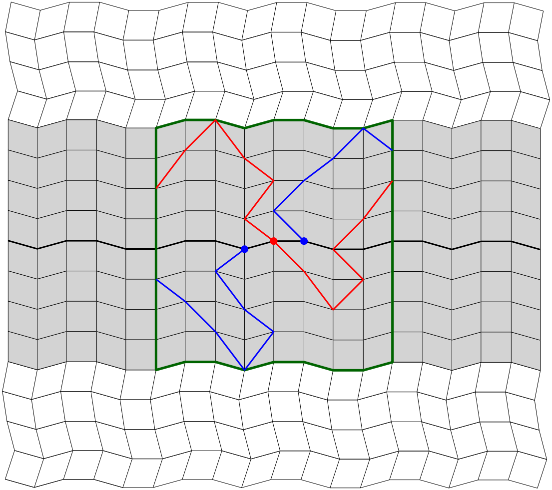


$$\mathbb{P}_{gen}(C_\mathrm{h}[B(\rho N, N)]) \geq \mathbb{P}_{sq}(C_\mathrm{h}[B(I\rho N, N)])\mathbb{P}_{sq}(C_\mathrm{v}[B(N, N)])^2$$

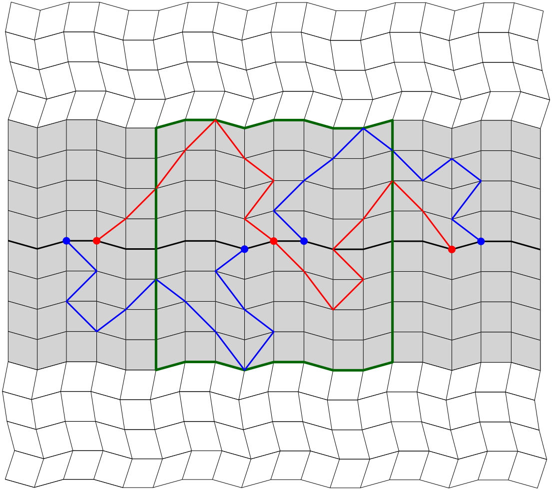
Transport of the arm exponents . . .

. . . using the same strategy as for the box-crossing property.

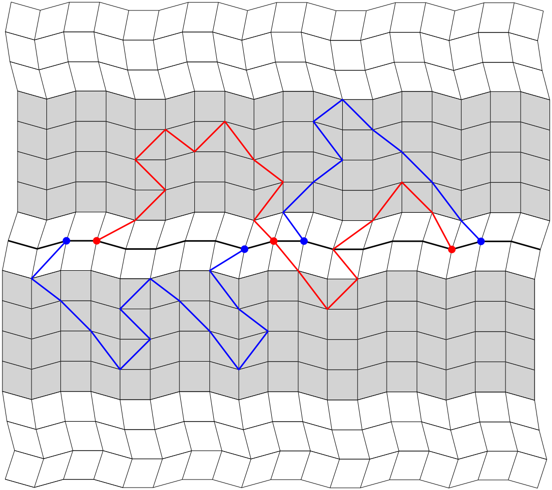
Square lattices



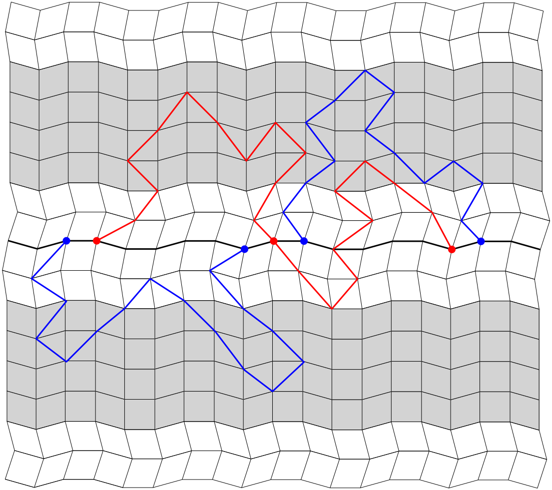
Square lattices



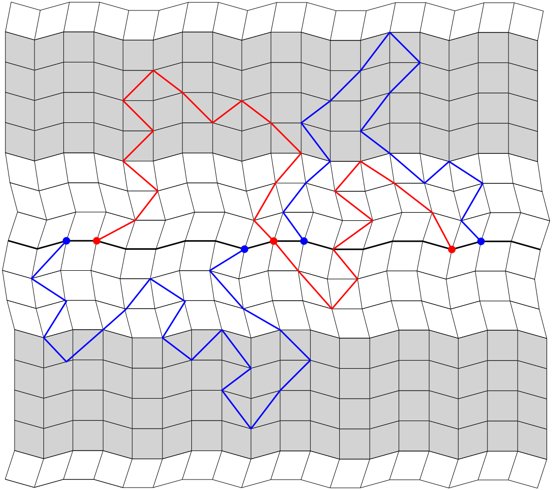
Square lattices



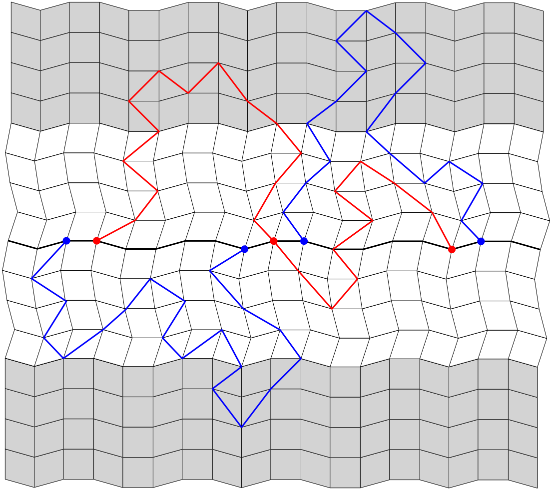
Square lattices



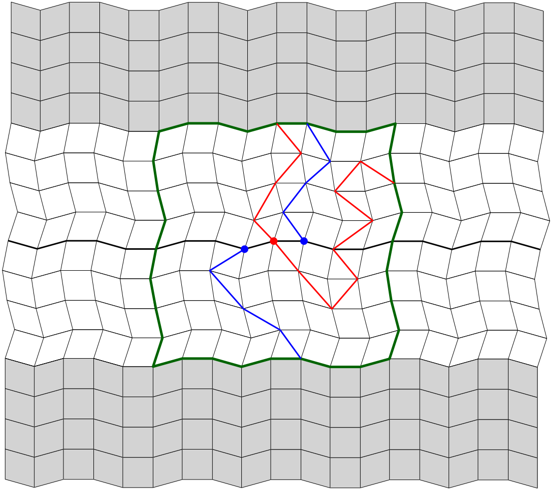
Square lattices



Square lattices

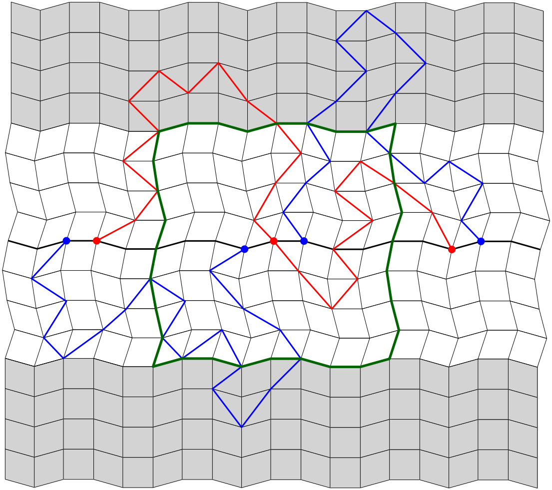


Square lattices



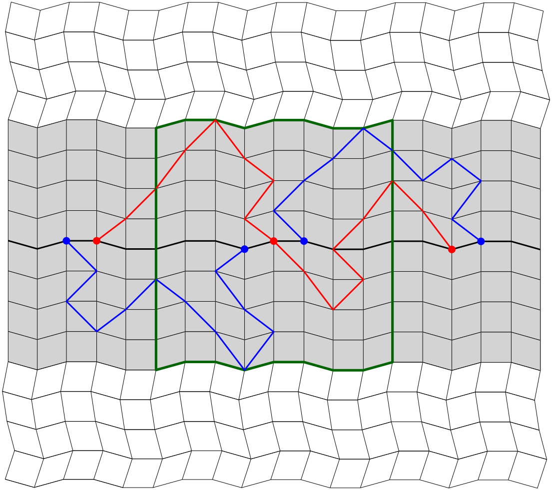
$$c_1 \mathbb{P}_{reg}(A_k(n)) \leq \mathbb{P}_{irreg}(A_k(n))$$

Square lattices



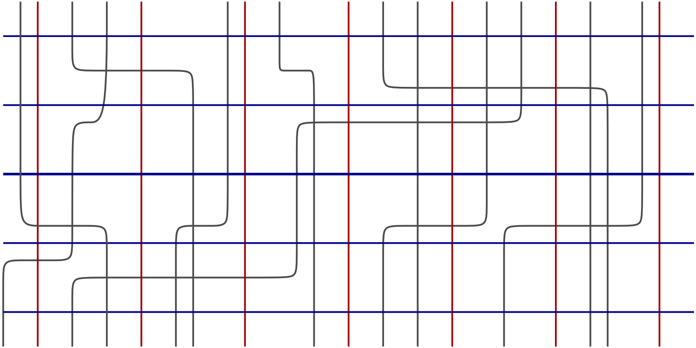
$$c_1 \mathbb{P}_{reg}(A_k(n)) \leq \mathbb{P}_{irreg}(A_k(n))$$

Square lattices

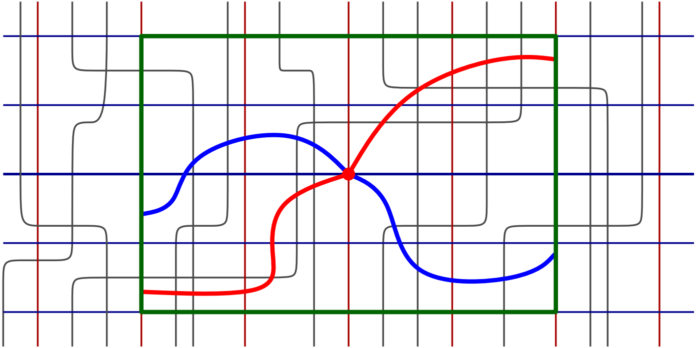


$$c_1 \mathbb{P}_{reg}(A_k(n)) \leq \mathbb{P}_{irreg}(A_k(n)) \leq c_2 \mathbb{P}_{reg}(A_k(n)).$$

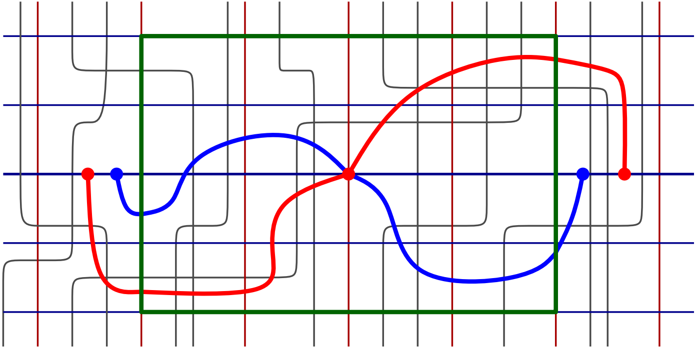
From square lattices to general graphs



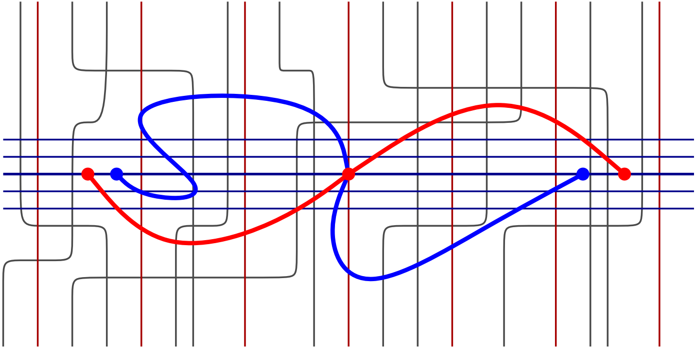
From square lattices to general graphs



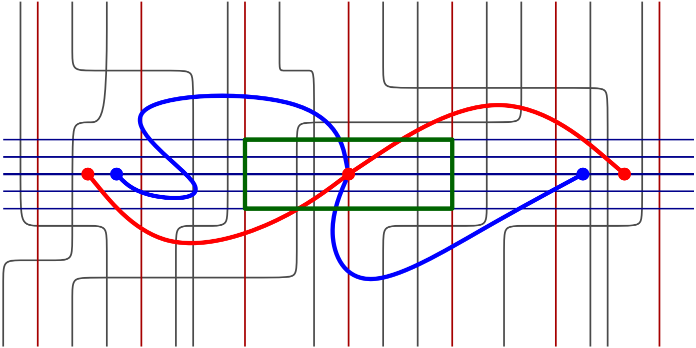
From square lattices to general graphs



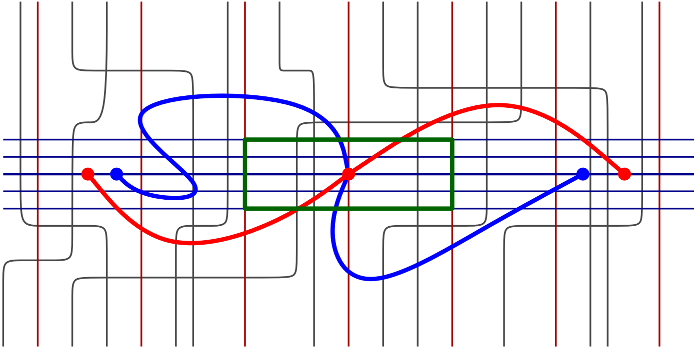
From square lattices to general graphs



From square lattices to general graphs



From square lattices to general graphs



$$c_1 \mathbb{P}_{sq}(A_k(n)) \leq \mathbb{P}_{gen}(A_k(n)) \leq c_2 \mathbb{P}_{sq}(A_k(n)).$$

Thank you!