Percolation on isoradial graphs

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Percolation

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Homogeneous percolation on $\mathbb{Z}^2$: all edges have intensity $p \in [0, 1]$.

**Question:** is there an infinite connected component?
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1
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**Question:** is there an infinite connected component?

- **Subcriticality:**
  - No infinite cluster
  - Exponential tail for cluster size
  - Trivial large scale behaviour!

- **Supercriticality:**
  - Existence of infinite cluster
  - Unique infinite cluster
  - Exponential tail for distance to infinite cluster
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Homogeneous percolation on $\mathbb{Z}^2$: all edges have intensity $p \in [0, 1]$.

**Question:** is there an infinite connected component?

- **Subcriticality:**
  - No infinite cluster
  - Exponential tail for cluster size
  - Trivial large scale behaviour!

- **Criticality:**
  - Scale invariance
  - Large scale limit
  - Universality

- **Supercriticality:**
  - Existence of infinite cluster
  - Unique infinite cluster
  - Exponential tail for distance to infinite cluster.
  - Trivial large scale behaviour!
Isoradial percolation

Each face of $G$ is inscribed in a circle of radius 1.

$\mathbb{P}_G$ percolation with $p_e$:

$$\frac{p_e}{1 - p_e} = \frac{\sin\left(\frac{\pi - \theta(e)}{3}\right)}{\sin\left(\frac{\theta(e)}{3}\right)}.$$

![Graph](image-url)
Isoradial Percolation

Bond Percolation on $\mathbb{Z}^2$

Isoradiality: $p = \frac{1}{2}$

Theorem (Kesten 80)

$p \leq \frac{1}{2}$, a.s. no infinite cluster;

$p > \frac{1}{2}$, a.s. existence of an infinite cluster.

Method:

self-duality $+$ RSW $+$ sharp-threshold

$$\mathbb{P}^{\frac{1}{2}}(\square) = \frac{1}{2} \Rightarrow \mathbb{P}^{\frac{1}{2}}(\text{rectangle}) \geq c \Rightarrow \mathbb{P}^{\frac{1}{2} + \epsilon}(0 \leftrightarrow \infty) > 0$$
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The box-crossing property (RSW)

A model satisfies the box-crossing property if for all rectangles $ABCD$ there exists $c(BC/AB) = c(\rho) > 0$ s. t. for all $N$ large enough:

$$\in [c, 1 - c]$$

Equivalent for the primal and dual model.

Theorem

If $\mathbb{P}_p$ satisfies the box-crossing property, then it is critical.
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$$\mathbb{P} \begin{bmatrix} A \\ B \\ \rho N \\ C \end{bmatrix} N \in [c, 1 - c]$$

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*If $P_p$ satisfies the box-crossing property, then it is critical.*
Results I: the box-crossing property

For an isoradial graph $G$ with the percolation measure $\mathbb{P}_G$, subject to conditions:

Theorem

$\mathbb{P}_G$ satisfies the box-crossing property.

Corollary

$\mathbb{P}_G$ is critical.

- $\mathbb{P}_p(\text{infinite cluster}) = 0$,
- $\mathbb{P}_{p+\epsilon}(\text{infinite cluster}) = 1$. 
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Arm exponents

For a critical percolation measure \( P \), as \( n \to \infty \), we expect:

- one-arm exponent \( \frac{5}{48} \):
  \[
  P(\text{rad}(C_0) \geq n) = P(A_1(n)) \approx n^{-\rho_1},
  \]

- \( 2j \)-alternating-arms exponents \( \frac{4j^2 - 1}{12} \):
  \[
  P[A_{2j}(n)] \approx n^{-\rho_{2j}}.
  \]

Moreover, \( \rho_i \) does not depend on the underlying model.

Power-law bounds are given by the box-crossing property.
Critical exponents

For $\mathbb{P}_p$ critical we expect:

Exponents at criticality.

Volume exponent $\delta = \frac{91}{5}$:
$$\mathbb{P}_p(|C_0| = n) \approx n^{-\frac{1}{\delta} - 1}.$$

Connectivity exponent $\eta = \frac{5}{24}$:
$$\mathbb{P}_p(0 \leftrightarrow x) \approx |x|^{-\eta}.$$

Radius exponent $\rho = \frac{48}{5}$:
$$\mathbb{P}_p(\text{rad}(C_0) = n) \approx n^{-1 - \frac{1}{\rho}}.$$

(\rho = \frac{1}{\rho_1})

Exponents near criticality.

Percolation probability $\beta = \frac{5}{36}$:
$$\mathbb{P}_{p+\epsilon}(|C_0| = \infty) \approx \epsilon^\beta \text{ as } \epsilon \downarrow 0.$$

Correlation length $\nu = \frac{4}{3}$:
$$\xi(p - \epsilon) \approx \epsilon^{-\nu} \text{ as } \epsilon \downarrow 0, \text{ were }$$
$$-\frac{1}{n} \log \mathbb{P}_{p-\epsilon}(\text{rad}(C_0) \geq n) \to_{n \to \infty} \frac{1}{\xi(p-\epsilon)}.$$

Mean cluster-size $\gamma = \frac{43}{18}$:
$$\mathbb{P}_{p+\epsilon}(|C_0|; |C_0| < \infty) \approx |\epsilon|^{-\gamma} \text{ as } \epsilon \to 0.$$

Gap exponent $\Delta = \frac{91}{36}$:
$$\frac{\mathbb{P}_{p+\epsilon}(|C_0|^{k+1}; |C_0| < \infty)}{\mathbb{P}_{p+\epsilon}(|C_0|^k; |C_0| < \infty)} \approx |\epsilon|^{-\Delta} \text{ for } k \geq 1, \text{ as } \epsilon \to 0.$$
Results II: arm exponents

For an isoradial graph $G$ with the percolation measure $\mathbb{P}_G$, subject to conditions

**Theorem**

For $k \in \{1, 2, 4, \ldots \}$ there exist constants $c_1, c_2 > 0$ such that:

$$c_1 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)] \leq \mathbb{P}_G[A_k(n)] \leq c_2 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)],$$

for $n \in \mathbb{N}$. 
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**Corollary**

The one arm exponent and the $2j$ alternating arm exponents are universal for percolation on isoradial graphs.
Isoradial Graphs

G isoradial graph
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G isoradial graph
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$G$ isoradial graph

$G^*$ dual isoradial graph
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Track system
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Track system
Conditions for isoradial graphs.

**Bounded angles condition:**
There exist \( \epsilon_0 > 0 \) such that for any edge \( e \), \( \theta_e \in [\epsilon_0, \pi - \epsilon_0] \).
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**Bounded angles condition:**
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**Square grid property:**
Families of "parallel" tracks $(s_i)_{i \in \mathbb{Z}}$ and $(t_j)_{j \in \mathbb{Z}}$.
The number of intersections on $s_i$ between $t_j$ and $t_{j+1}$ is uniformly bounded by a constant $I$. (same for $t$).
Examples: Penrose tilings and no square grid
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Star–triangle transformation

\[ \kappa_\triangle(p) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1. \]

Take \( \omega \), respectively \( \omega' \), according to the measure on the left, respectively right. The families of random variables

\[ \left( x \xrightarrow{\omega} y : x, y = A, B, C \right), \quad \left( x \xrightarrow{\omega'} y : x, y = A, B, C \right), \]

have the same joint law.
Coupling

\[ P = (1 - p_0)(1 - p_1)(1 - p_2). \]
Path transformation
Track exchange

Two parallel tracks $s_1$ and $s_2$ with no intersection between them. We may exchange $s_1$ and $s_2$ using star–triangle transformations.
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Proof for box-crossing property
From $\mathbb{Z}^2$ to isoradial square lattice.

- **Initial configuration**
- **Principal outcome**
- **Secondary outcome**
- **Probability of secondary outcome**

Open paths are preserved (unless the deleted edge was part of the path).
Strategy

Proposition

If two isoradial square lattices have same transverse angles for the vertical/horizontal tracks, and one has the box-crossing property, then so does the other.
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If two isoradial square lattices have same transverse angles for the vertical/horizontal tracks, and one has the box-crossing property, then so does the other.
Transport of horizontal crossings

Construct a mixed isoradial square lattice: 
"regular" in the gray part, "irregular" in the rest.
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\[ \mathbb{P}_{\text{gen}}(C_h[B(\rho N, N)]) \geq \mathbb{P}_{\text{sq}}(C_h[B(I\rho N, N)] \mathbb{P}_{\text{sq}}(C_v[B(N, N)])^2 \]
Transport of the arm exponents . . .

. . . using the same strategy as for the box-crossing property.
Square lattices
Square lattices
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Square lattices

\[ c_1 P_{\text{reg}}(A_k(n)) \leq P_{\text{irreg}}(A_k(n)) \]
Square lattices

\[ c_1 \mathbb{P}_{\text{reg}}(A_k(n)) \leq \mathbb{P}_{\text{irreg}}(A_k(n)) \]
Proof for box-crossing property

Arm exponents

Square lattices

\[ c_1 P_{\text{reg}}(A_k(n)) \leq P_{\text{irreg}}(A_k(n)) \leq c_2 P_{\text{reg}}(A_k(n)). \]
From square lattices to general graphs
From square lattices to general graphs
From square lattices to general graphs
From square lattices to general graphs
From square lattices to general graphs
From square lattices to general graphs

\[ c_1 \mathbb{P}_{sq}(A_k(n)) \leq \mathbb{P}_{gen}(A_k(n)) \leq c_2 \mathbb{P}_{sq}(A_k(n)). \]
Thank you!