Scaling limits and influence of the seed graph in preferential attachment trees

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joint work with Nicolas Curien, Thomas Duquesne and Igor Kortchemski

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9th December 2014
Journée cartes aléatoires - IHES

Linear preferential attachment: the model

- Initial tree: $T_{n_0}^{(S)} = S$ (where $n_0 = |S|$)
- $T_{n+1}^{(S)}$ is obtained from $T_n^{(S)}$ by adding an edge to a random vertex $v \in T_n^{(S)}$, chosen proportionally to its degree.

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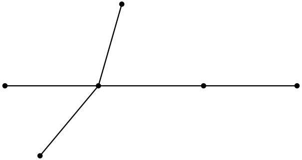

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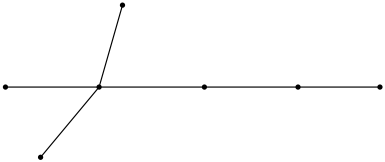

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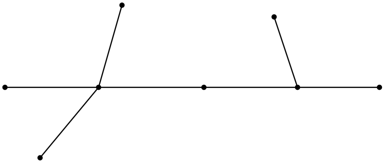

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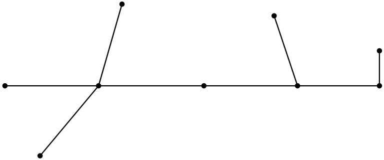

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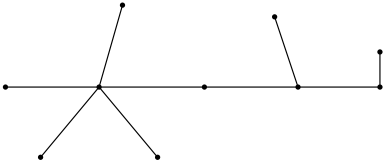

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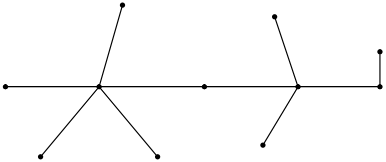

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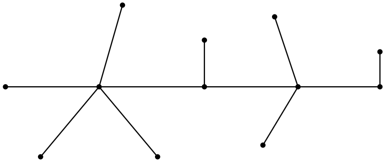

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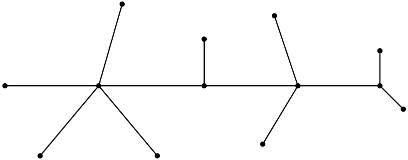

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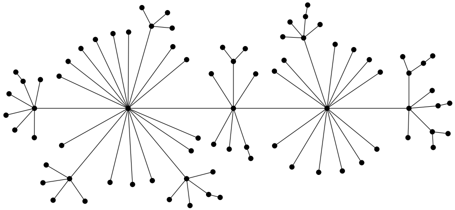

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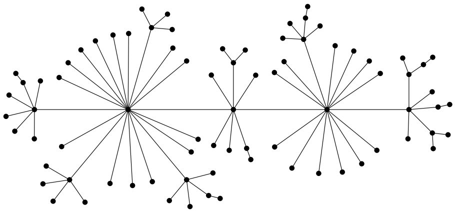

Questions:

- Does the process mix?
  For $S_1 \neq S_2$, does $\mathrm{d}_{\mathrm{TV}}(T_n^{(S_1)}; T_n^{(S_2)}) \to 0$ as $n \to \infty$?
- How does $T_n^{(S)}$ look like when n is very large? Scaling limit?

Recognise the seed

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Example:

$\mathcal{T}_2^{(S_1)} = S_1$

$\mathcal{T}_3^{(S_2)} = S_2$

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Study the degree sequence of T_n: $Deg(T_n) = \{\deg(v) : v \in T_n\}$.

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Theorem (Bubeck, Mossel, Rácz)

For seeds $S_1 \neq S_2$ with $|S_1| = |S_2|$ but $Deg(S_1) \neq Deg(S_2)$,

$$\mathrm{d}_{\mathrm{TV}}(Deg(T_n^{(S_1)}); Deg(T_n^{(S_2)})) \not\to 0.$$

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Proof: The degree sequence is given by a Pólya urn.
The tail of $\max Deg(T_n^{(S)})$ depends on $Deg(S)$.

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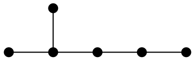
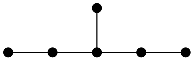
Problem: This strategy can not distinguish between seeds with same degree sequences.



S_1



S_2

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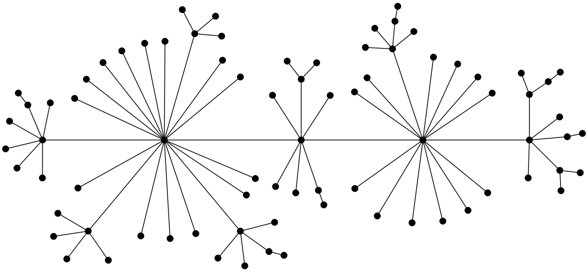
Conclusion: Need some geometric observable to distinguish $T_n^{(S_1)}$ form $T_n^{(S_2)}$

How to define a limit of T_n? Convergence in Gromov–Hausdorff topology?



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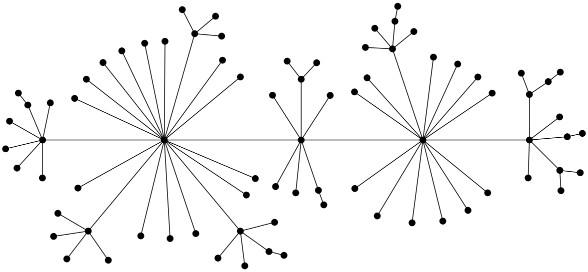

Diameter of T_n: $\log n$.   Maximal degree of T_n: $\sqrt{n}$.

No non-trivial compact scaling limit!

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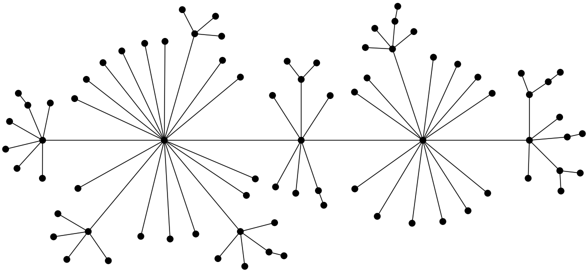

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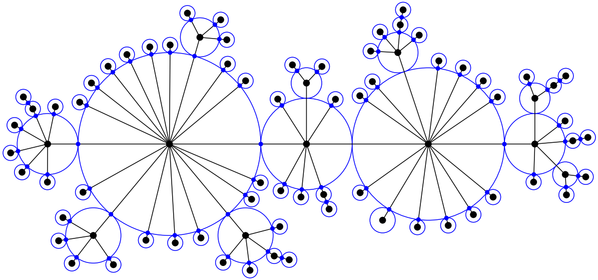

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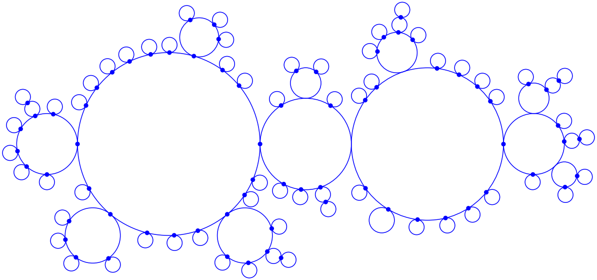

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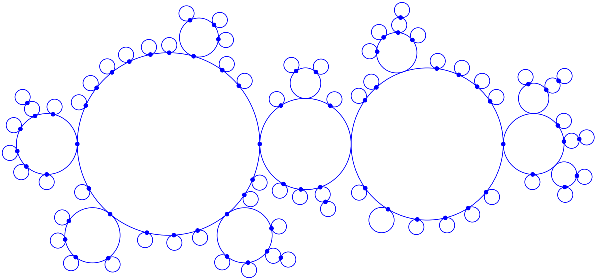

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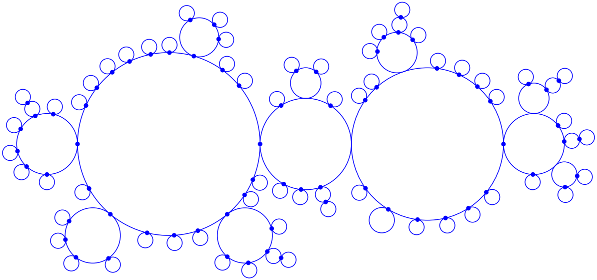

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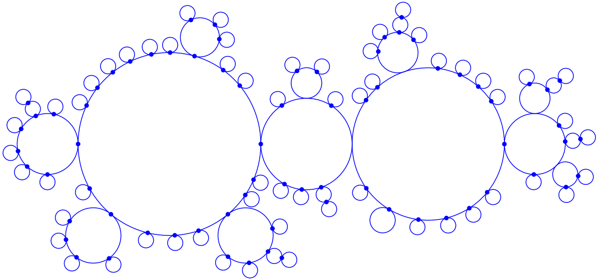

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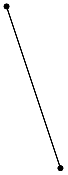
The plane LPAM and Rémy's algorithm



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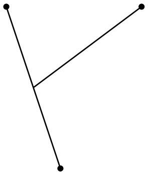

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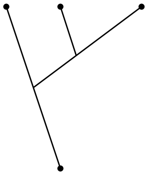

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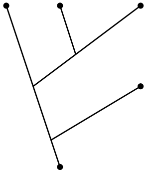

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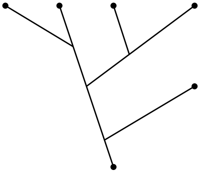

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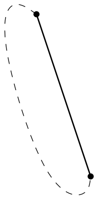

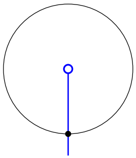

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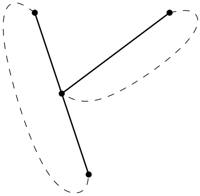

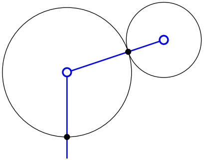

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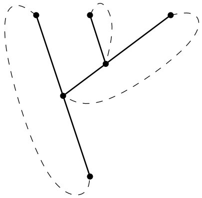

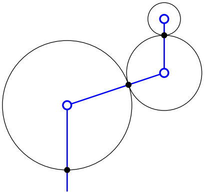

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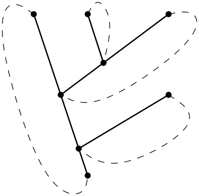

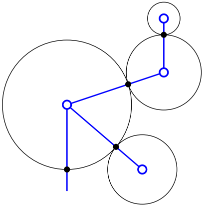

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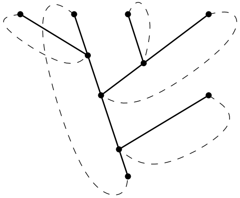

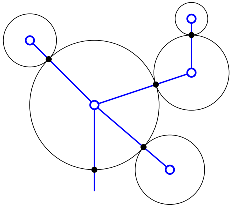

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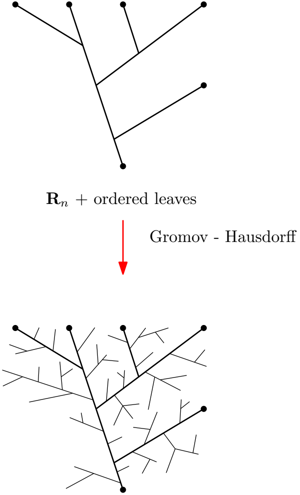

CRT

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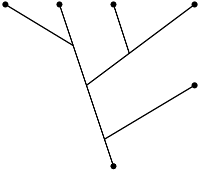
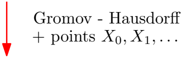
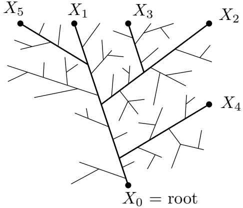


$\mathbf{R}_n$ + ordered leaves





CRT + uniform points

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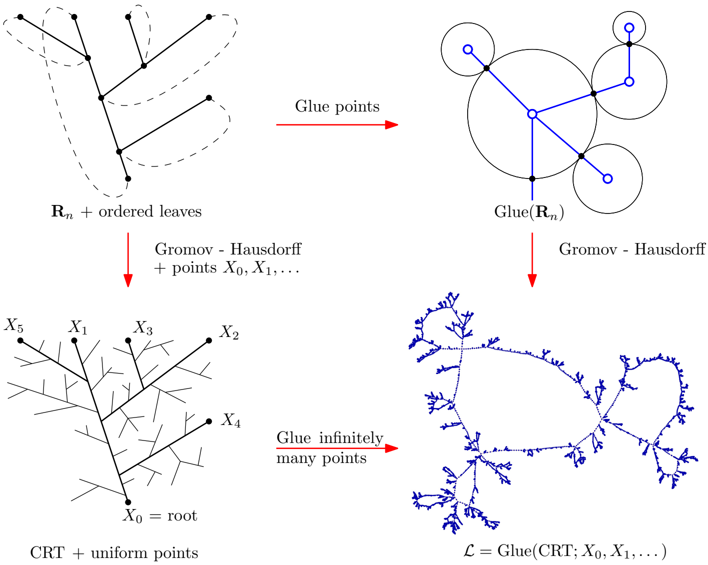

$\mathbf{R}_n = n^{th}$ step in Rémy's algorithm. $X_0^n, \dots, X_n^n =$ leaves in order of appearance.

Theorem (Rémy '85; Curien & Haas '13)

Then $\mathbf{R}_n$ is a uniform tree with n edges and $X_0^n, \dots, X_n^n$ is a uniform ordering of its leaves.
Moreover, for any k fixed,

$$n^{-1/2} \cdot (\mathbf{R}_n; X_0^n, \dots, X_k^n) \xrightarrow[n\to\infty]{a.s. for\ k-pointed\ G.H.} 2\sqrt{2} \cdot (CRT; X_0, \dots, X_k),$$

where $X_0, X_1, \dots$ are i.i.d. points in the CRT, chosen according to its mass measure.

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Consequence:
$n^{-1/2} \cdot \text{Glue}(\mathbf{R}_n; X_0^n, \dots, X_k^n) \xrightarrow[n\to\infty]{\text{a.s. for G.H.}} 2\sqrt{2} \cdot \text{Glue}(CRT; X_0, \dots, X_k).$

$\mathbf{R}_n = n^{th}$ step in Rémy's algorithm. $X_0^n, \dots, X_n^n$ = leaves in order of appearance.

Theorem (Rémy '85; Curien & Haas '13)

Then $\mathbf{R}_n$ is a uniform tree with n edges and $X_0^n, \dots, X_n^n$ is a uniform ordering of its leaves.
Moreover, for any k fixed,

$$n^{-1/2} \cdot (\mathbf{R}_n; X_0^n, \dots, X_k^n) \xrightarrow[n\to\infty]{a.s. for\ k-pointed\ G.H.} 2\sqrt{2} \cdot (CRT; X_0, \dots, X_k),$$

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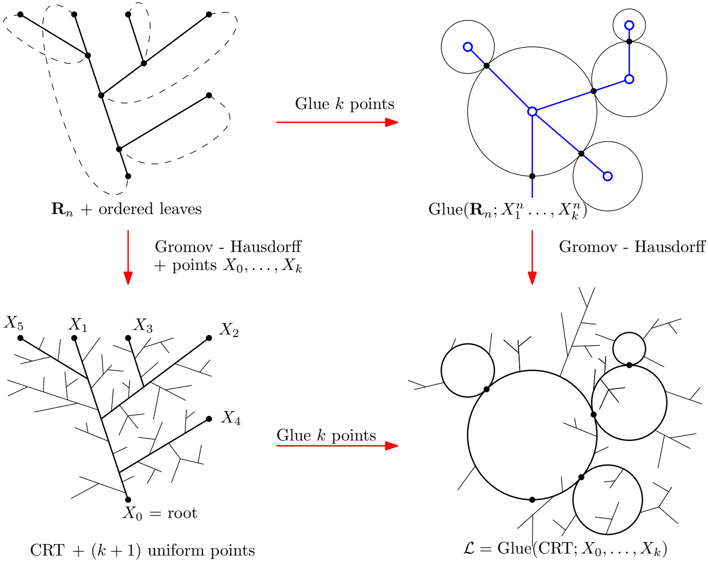
Theorem

$$n^{-1/2} \cdot \mathrm{Loop}(T_n^{-\circ}) \xrightarrow[n\to\infty]{a.s.\ for\ G.H.} 2\sqrt{2} \cdot \mathcal{L},$$

where $\mathcal{L}$ is the limit of Glue$(CRT; X_0, \dots, X_k)$ as $k \to \infty$.

The plane LPAM and Rémy's algorithm



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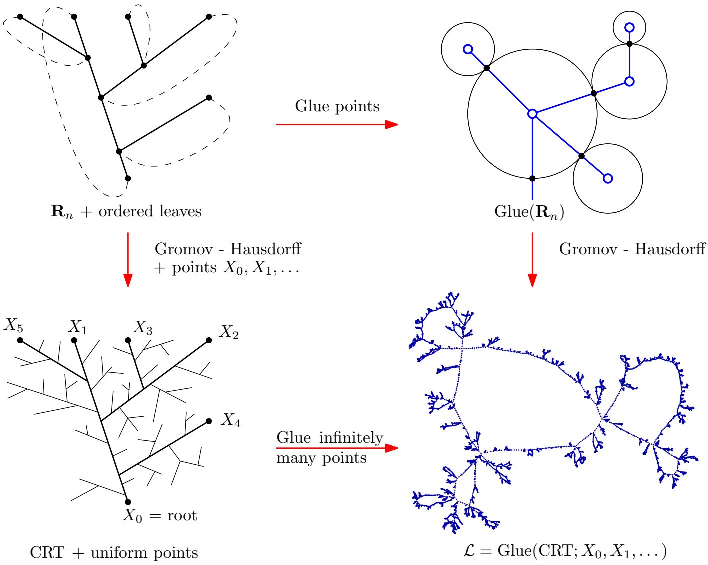
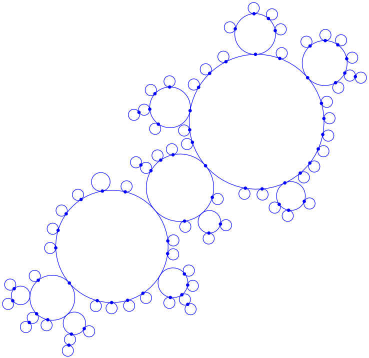
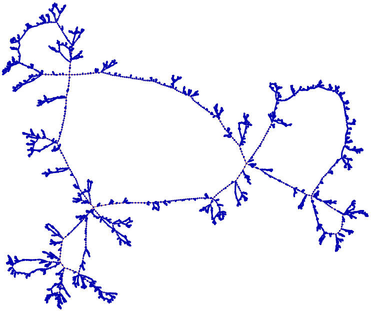


Properties of the loop tree

Theorem

$\mathcal{L}$ has a.s. Hausdorff dimension 2.





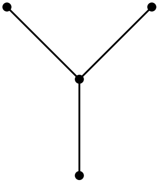
Big faces touch each other!
(In T_n the vertices of large degree are at finite distance.)

In the previous, we looked at $T_n^{\multimap}$ with seed $\multimap$.

For general seeds S, with N corners:

$T_n^{(S)}$ is obtained by:

- sample N variables $\alpha_1^n, \dots, \alpha_N^n$ with Pólya urn distribution,
- sample N independent $L_1^n, \dots, L_N^n$ of LPAMs started from $\multimap$ with resp. $\alpha_1^n, \dots, \alpha_N^n$ vertices;
- attach each L_i^n in a corner of S.



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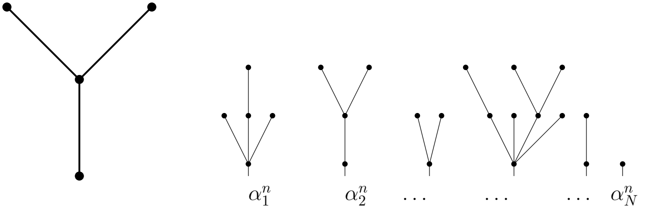

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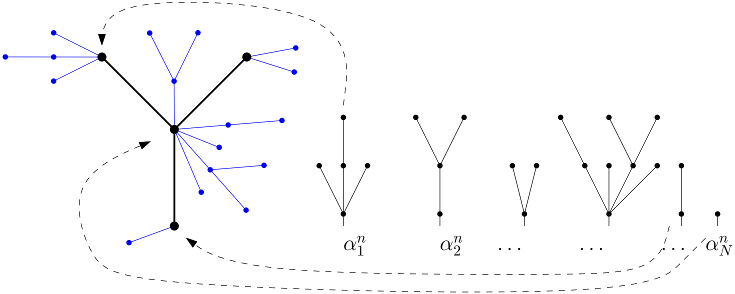

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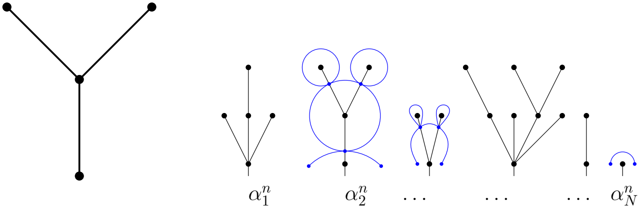

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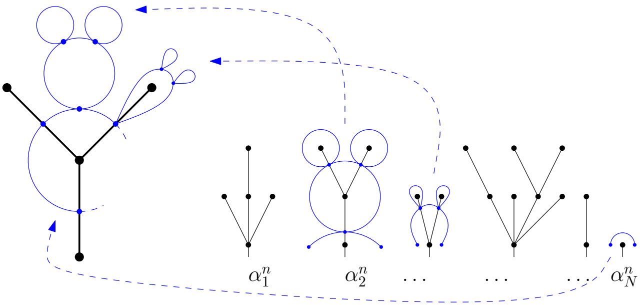

Back to distinguishing the seeds

Which observable to use?

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Number of embeddings of a given (small) tree:
A tree τ may be embedded in $D_\tau(T_n^{(S)})$ ways in $T_n^{(S)}$.

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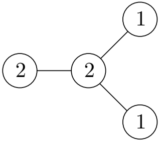
Number of embeddings of a given (small) tree:
A tree τ may be embedded in $D_\tau(T_n^{(S)})$ ways in $T_n^{(S)}$.

Do $D_\tau(T_n^{(S_1)})$ and $D_\tau(T_n^{(S_2)})$ have different asymptotics?

$$n^{-?} \cdot D_\tau(T_n^{(S)}) \to d(S)?$$

For technical conditions, we will work with decorated trees.



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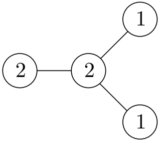

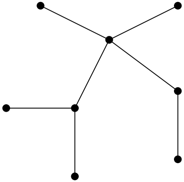

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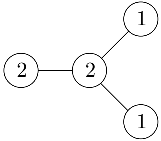

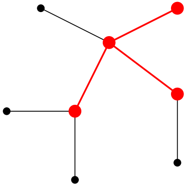

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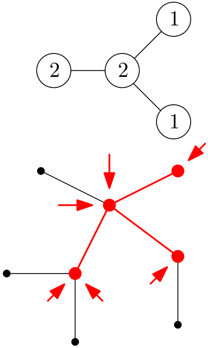

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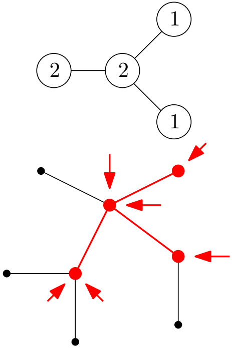
$D_\tau(T) =$ number of embeddings.



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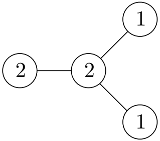

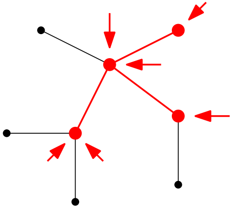

We may expect:

$$n^{-|\tau|/2} \cdot D_\tau(T_n^{(S)}) \to d(S),$$

with $d(S)$ a random variable that depends on S.
Because of the small stubs, there may be logarithmic corrections.

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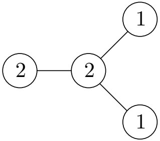
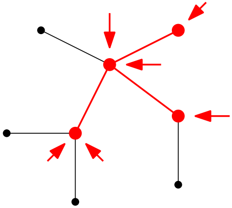


Proposition

For τ a decorated tree there exist constants $\{c_n(\tau,\tau') : \tau' \preccurlyeq \tau, n \geq 2\}$ such that

$$M_\tau(T_n^{(S)}) = \sum_{\tau' \preccurlyeq \tau} c_n(\tau,\tau') \cdot D_{\tau'}(T_n^{(S)})$$

is a martingale for any seed S, and is bounded in L^2.



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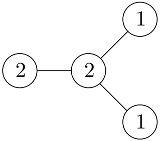

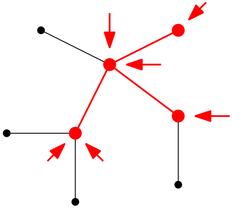

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For $S_1 \neq S_2$ with $n_0 = |S_1| = |S_2| \geq 3$, there exists a decorated tree τ such that

$$M_\tau(T_{n_0}^{(S_1)}) \neq M_\tau(T_{n_0}^{(S_2)}) \ .$$

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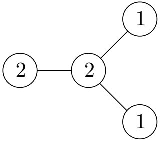

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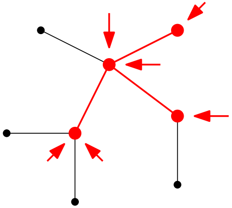

For $S_1 \neq S_2$ with $n_0 = |S_1| = |S_2| \geq 3$, there exists a decorated tree τ such that

$$\mathbb{E}[M_\tau(T_n^{(S_1)})] \neq \mathbb{E}[M_\tau(T_n^{(S_2)})].$$

Theorem

For $S_1 \neq S_2$ with $|S_1|, |S_2| \geq 3$, $\mathrm{d_{TV}}(T_n^{(S_1)}; T_n^{(S_2)})$ stays bounded away from 0. In other words $d(S_1,S_2) = \lim \mathrm{d_{TV}}(T_n^{(S_1)}; T_n^{(S_2)})$ is a distance on seeds with at least 3 vertices.

Idea of proof: recurrence for $D_\tau(T_n)$

For any τ, there exist constants $\{c(\tau,\tau') : \tau' \prec \tau\}$ such that

$$\mathbb{E}\big[D_\tau\big(T^{(S)}_{n+1}\big)\big|\,T^{(S)}_n\big] = \left(1 + \frac{|\tau|}{2n-2}\right) D_\tau\big(T^{(S)}_n\big) + \frac{1}{2n-2}\sum_{\tau' \prec \tau} c(\tau,\tau') D_{\tau'}\big(T^{(S)}_n\big).$$

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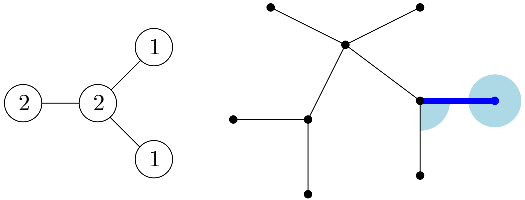

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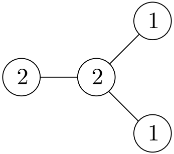

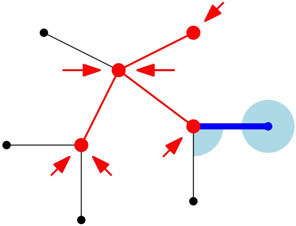

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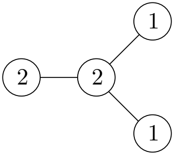

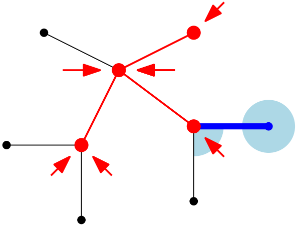

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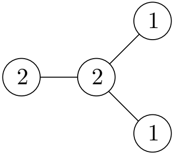

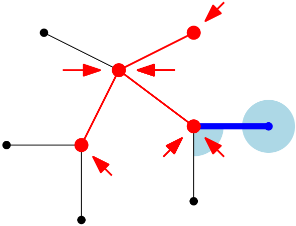

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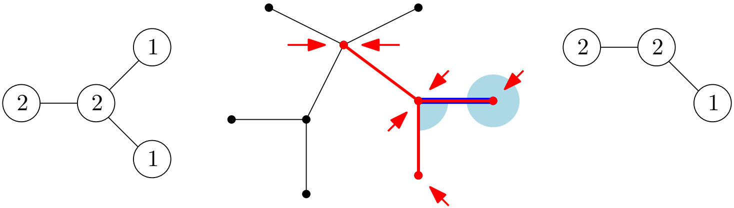

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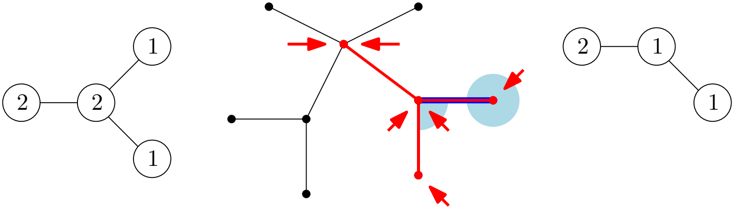

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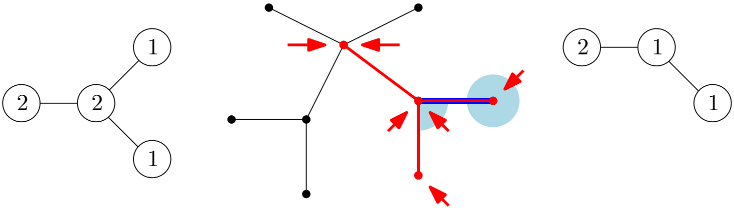

Using this recurrence formula, we show the existence of the martingales $M_\tau(T_n^{(S)})$

Related model

Affine reinforcement: For $\delta > -1$,
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Choose red edges with prob. $1+\alpha$ and the other with prob $1-\alpha$; $\alpha = 1/(2+\delta)$

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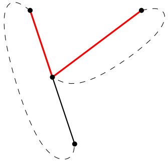

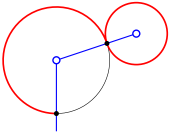

Choose red edges with prob. $1+\alpha$ and the other with prob $1-\alpha$; $\alpha = 1/(2+\delta)$

Related model

Affine reinforcement: For $\delta > -1$,
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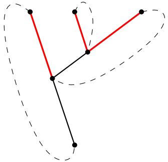

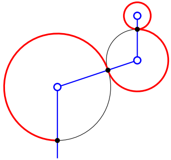

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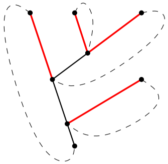

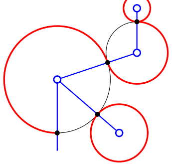

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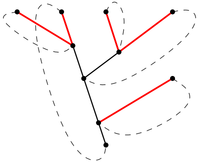

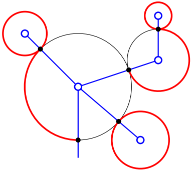

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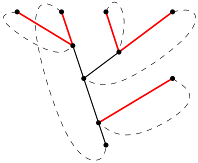

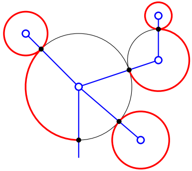

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$\mathcal{L}^{\multimap,\delta}$ should come from a fragmentation tree of Hausdorff dimension $2+\delta$.

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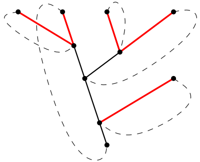

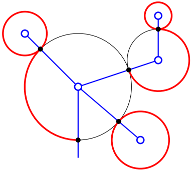

• $\mathrm{d_{TV}}(T_n^{(S_1),\delta}; T_n^{(S_2),\delta}) \not\to 0$ for $S_1 \neq S_2$.

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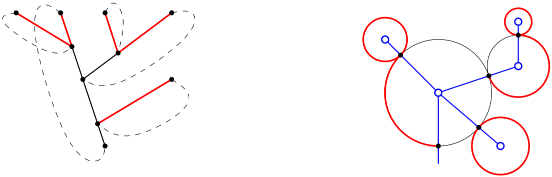

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In our proof the exchangeability of the corners played an essential role! We expect a similar result.

For $\delta = \infty$ (vertex chosen uniformly) - result obtained by Bubeck, Eldan, Mossel, Rácz.

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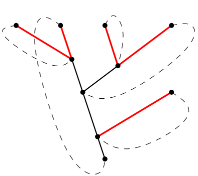

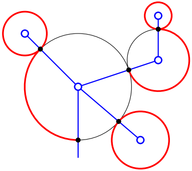

• $\mathrm{d_{TV}}(T_n^{(S_1),\delta}; T_n^{(S_2),\delta}) \not\to 0$ for $S_1 \neq S_2$.

• Is all the asymptotic information on $T_n^{(S)}$ contained in $\mathcal{L}^{(S)}$?

Thank you!