

FK-percolation, the six-vertex model and the Gaussian free field

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Abstract

These lecture notes accompany a mini-course offered at the introductory school of the 2026 thematic programme “Statistical Mechanics and Combinatorics of Discrete Planar Structures” at the ESI in Vienna. They have two complementary goals. The first is to introduce a selection of robust planar-percolation techniques: the spatial Markov property, positive association, comparison of boundary conditions, duality and Russo–Seymour–Welsh theory. The second is to explain how these qualitative probabilistic tools combine with the transfer-matrix formalism and Bethe Ansatz computations to show that the six-vertex model converges to the Gaussian free field (GFF) in a range of parameters.

We first discuss the continuous/discontinuous dichotomy for planar FK-percolation. We then use the Baxter–Kelland–Wu correspondence and the six-vertex transfer matrix to identify the type of transition: continuous for $1 \leq q \leq 4$ and discontinuous for $q > 4$. Finally, we present a roadmap for the proof that, for $-1 \leq \Delta \leq -\frac{1}{2}$, the full-plane six-vertex height function converges to an explicit multiple of the GFF.

The style is deliberately concise. Representative arguments are proved, while technically heavier inputs are isolated as theorems, proof sketches or exercises. In this way the notes are intended both as an introduction to the models and as a guide to the complete proof of the GFF convergence.

Notes on literature. The following references impact directly the content of these lecture notes; more references are provided in the text.

- The continuous/discontinuous phase transition dichotomy for FK-percolation was first proved in [DCST17]; other relevant versions may be found in [DCT20] and [Lam26].
- Establishing the regimes of continuous/discontinuous phase transition in terms of q follows the technique of [DCGH⁺21] and hinges on Bethe ansatz estimates from [DCKK⁺22]; see also [DCGH⁺18].
- The rotational invariance of FK-percolation is an essential input. While not treated here, it is worth mentioning. It was obtained in [DCKK⁺20] using techniques developed in [GM14] and [DCLM18].
- The above rotational invariance is transferred to the six-vertex model via the *M-formula* in [ACDC⁺26].
- Finally, the GFF convergence result in full plane was obtained in [DCKLM26].
- Ongoing work mentioned in Section 3.7 is pursued with E. Averous, H.-B. Chen, H. Duminil-Copin, T. He, K. Kozłowski, D. Krachun, P. Lammers and J. Xia.

Finally, the first chapters of these notes were adapted from [Man25].

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Statement about the use of AI. These notes were compiled with the help of AI, based on [Man25] and [DCKLM26]. Possible inaccuracies may persist. No AI was used in the research presented here.

Overview

In these notes we focus on two planar models and their large scale behaviour. Both models are defined on the square lattice $\mathbb{Z}^2$, or finite regions thereof. We will write $\delta\mathbb{Z}^2$ for the lattice with mesh-size δ . Fix Λ a finite subgraph of $\mathbb{Z}^2$; $\Lambda = \{-n, \dots, n\}^2$ is a good illustration. We view it as a graph with edge-set $E(\Lambda)$, vertex set Λ and faces $F(\Lambda)$.

FK-percolation As a percolation measure, FK-percolation is defined on configurations of open and closed edges $\omega \in \{0, 1\}^{E(\Lambda)}$ and we are interested in the connections induced by the open edges of ω . The probability of a configuration is given by

$$\phi_{\Lambda, p, q}[\omega] = \frac{1}{Z(\Lambda, p, q)} p^{|\omega|} (1-p)^{|E(\Lambda) \setminus \omega|} q^{k(\omega)}, \quad \omega \in \{0, 1\}^{E(\Lambda)},$$

where the exponents are the number of open edges, number of closed edges and number of connected components of ω . Here $q \geq 1$ and $p \in (0, 1)$ are parameters of the model. We will focus on the critical regime $q \in [1, 4]$ and $p = p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$.

It will be of particular interest to study the geometry of the large connected components of ω , which we do via their contours ω° (also called the loop representation of ω). A major conjecture states that the loop representation of ω converges to a continuous, non-trivial limit as the mesh size of the lattice on which ω is sampled tends to 0.

Here is a slightly more rigorous version of the conjecture. Fix some simply connected domain $\mathcal{D}$ and let $\mathcal{D}^\delta$ be a discretisation of $\mathcal{D}$ by $\delta\mathbb{Z}^2$.

Conjecture 1. *The law of ω° where $\omega \sim \phi_{\mathcal{D}^\delta, p_c(q), q}^0$ converges to the nested conformal loop ensemble $CLE_\kappa(\mathcal{D})$, where*

$$\kappa = \kappa(q) = 4\pi / \arccos(-\frac{\sqrt{q}}{2}).$$

The actual definition of $CLE_\kappa(\mathcal{D})$ is not important here. It is important to understand that it is a random, locally finite family of non-intersecting loops that is invariant under conformal transformations and has a Markov property. Such families are indexed by a positive parameter κ .

More strikingly, they are all different, which is to say that the models $\phi_{p_c(q), q}$ for different values of q belong to different universality classes.

At the time of writing, Conjecture 1 was only proved for the FK-Ising model, which is to say $q = 2$, [Smi10, CS12, CDCH⁺14, KS16, KS19]. A variant of the conjecture for Bernoulli percolation ($q = 1$) was proved in [Smi01, CN06].

Six-vertex model The six-vertex model on Λ is an assignment of integer heights h to each of the faces of Λ in such a way that heights of neighbouring faces differ by exactly 1. The internal vertices of Λ may be split into:

- vertices for which the heights of the four adjacent faces take three values (call these vertices of type a and b) and

- vertices for which the heights of the four adjacent faces take only two values (call these vertices of type c).

Note that the number of possible height functions on Λ is infinite, but only finitely many remain when quotiented by the addition of a global constant. We will henceforth view them as such; when important, we refer to these as gradients of height functions.

The probability of a (gradient of a) height function h is then

$$\mathbb{P}_{\Lambda,c}[h] = \frac{c^{\#\text{type } c \text{ vertices}}}{Z_{6V}(\Lambda, c)},$$

where $c > 0$ is a parameter. The model is also sometimes expressed in terms of

$$\Delta = 1 - c^2/2 \in (-\infty, 1).$$

The range where the model is expected to have critical behaviour is $c \in (0, 2]$ (or $\Delta \in [-1, 1)$).

Conjecture 2. *For $c \in (0, 2]$, the law of h sampled according to $\mathbb{P}_{\mathcal{D}^\delta, c}$ converges to the Gaussian Free Field (GFF) in $\mathcal{D}$ multiplied by a constant $\sigma(c) = (\arcsin \frac{c}{2})^{-\frac{1}{2}}$.*

Again, the definition of the GFF is not as important here. We only mention that it is a Gaussian process in a generalised functional space with covariance given by the Green function. As such, it is also a conformally invariant object.

We have been purposefully vague about boundary conditions, both to lighten notation and because their treatment in full generality remains mysterious to the author. We mention however that we always refer to overall “flat” boundary conditions.

Variants of Conjecture 2 were proved for the free fermion point ($\Delta = 0$, $c = \sqrt{2}$) in [Ken01] and were extended to a vicinity of these parameters in [GMT17].

We close the presentation of the six-vertex model by mentioning that the gradient of a height function may also be expressed using orientations of edges. Indeed, given h , orient any edge separating two faces such that the face on the left has higher height. Then, six configurations of orientations are possible around any internal vertex, whence the name of the model.

BKW correspondence The two models (on slightly different lattices) are related via the Baxter–Kelland–Wu (BKW) correspondence. This may be understood as follows.

Sample ω according to $\phi_{p_c(q), q}$ for some $q \geq 1$. Consider the loops of ω° and orient each loop clockwise with “probability” $e^{-\lambda}$ and counter-clockwise with “probability” e^λ , where λ is some parameter satisfying

$$\sqrt{q} = e^{2\pi\lambda} + e^{-2\pi\lambda} \quad \text{and} \quad c = e^{\pi\lambda} + e^{-\pi\lambda}.$$

The resulting orientation provides a six-vertex configuration (in terms of orientation of edges).

Again, we have been purposefully vague about boundary conditions and boundary vertices. Indeed, these are particularly tricky to handle throughout this correspondence. Another significant difficulty is that when $q \in [1, 4]$ (which is the regime of interest to us), the value of λ is purely imaginary, and thus the correspondence does not have a probabilistic interpretation. It is however key to remember that the correspondence relates FK-percolation with $q \in [1, 4]$ to the six-vertex model with $\Delta \in [-1, -\frac{1}{2}]$ by

$$\Delta = -\frac{\sqrt{q}}{2} \quad \text{and} \quad c = \sqrt{2 + \sqrt{q}}.$$

When comparing Conjectures 1 and 2, we can’t help but wonder how the same universality class for the six-vertex model relates to different universality classes for FK-percolation.

Main result. The ultimate result of these notes (Theorem 3.2) concerns the six-vertex model. As we will see, a full plane version of this model may be defined. In this setting, and for parameters corresponding to FK-percolation with $q \in [1, 4]$, we obtain the expected GFF convergence.

Theorem.[DCKLM26] *Fix $c \in [\sqrt{3}, 2]$ (that is $\Delta \in [-1, -\frac{1}{2}]$) and consider the full-plane six-vertex model on $\delta\mathbb{Z}^2$ with zero slope. Then, as $\delta \searrow 0$, the six-vertex height function converges to $\sigma\Gamma$, where*

$$\sigma^2 = \frac{2}{\arccos \Delta} = \frac{1}{\arcsin(c/2)}$$

and Γ is the full-plane Gaussian Free Field.

Chapter 1

Phase transition of planar FK-percolation

The goal of this chapter is to introduce FK-percolation and its critical behaviour in two dimensions. We work on $\mathbb{Z}^2$ or subgraphs thereof. The cluster-weight $q \geq 1$ will be fixed, and we remove it from the notation. The edge-weight p will be mostly fixed to its self-dual value $p_{\text{sd}} = \frac{\sqrt{q}}{1+\sqrt{q}}$; when this is the case, we will also omit it. General results about the random-cluster model may be found in [Gri06, Dum13, DC20].

The ultimate result of the chapter is the dichotomy between two types of critical behaviour (continuous and discontinuous phase transition) at the self-dual point. In Chapter 2 we will establish which type of phase transition the model undergoes for different values of q .

We will also place little emphasis on the fact that p_{sd} is indeed the critical point of the model. This is an easy by-product of the proof below, but not one necessary for the understanding of the rest of the document.

1.1 Finite-volume measures

Let $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^2$. Its boundary ∂G is the set of vertices incident to edges of $\mathbb{Z}^2$ not contained in E . A boundary condition ξ is a partition of ∂G ; vertices in the same part are regarded as wired together. The two extremal choices are the free boundary condition 0, for which no boundary vertices are wired together, and the wired boundary condition 1, for which all boundary vertices are wired together.

A percolation configuration is any element $\omega \in \{0, 1\}^E$. Edges with $\omega(e) = 1$ are called open, while those with $\omega(e) = 0$ are called closed. We view ω also as the set of open edges and as the graph with vertex set V and edge-set formed of the open edges; in particular $|\omega|$ is the number of open edges. Connections in ω are denoted by $\leftrightarrow$. Finally, ω^ξ denotes the graph obtained from ω by identifying together any wired vertices of the boundary.

Definition 1.1 (Random-cluster measure). For $p \in (0, 1)$, $q \geq 1$ and a boundary condition ξ , set

$$\phi_{G,p,q}^\xi[\omega] = \frac{1}{Z_{G,p,q}^\xi} p^{|\omega|} (1-p)^{|E \setminus \omega|} q^{k(\omega^\xi)}, \quad \omega \in \{0, 1\}^E,$$

where $k(\omega^\xi)$ is the number of clusters after applying the boundary wiring.

The dependence between edges is organised by the following three properties. They are elementary consequences of the cluster weight and are the basic “calculus” of FK-percolation.

Proposition 1.2 (Spatial Markov property). *If $H \subset G$, then, conditionally on the configuration in $G \setminus H$, the law in H is*

$$\phi_{G,p,q}^\xi[\cdot | \omega|_{G \setminus H}] = \phi_{H,p,q}^\zeta[\cdot], \quad (\text{SMP})$$

where ζ is the wiring of ∂H induced by the configuration $\omega|_{G \setminus H}$ and by ξ .

For two measures on $\{0, 1\}^E$, write $\mu \leq_{\text{st}} \nu$ to mean that $\mu[A] \leq \nu[A]$ for every increasing event A ; equivalently, the two measures admit a coupling in which the ν -configuration contains the μ -configuration. For two boundary conditions ζ, ξ , write $\zeta \geq \xi$ if any two vertices wired in ξ are also wired in ζ .

Proposition 1.3 (FKG and monotonicity). *For $q \geq 1$ one has*

$$\phi_{G,p,q}^\xi[A \cap B] \geq \phi_{G,p,q}^\xi[A] \phi_{G,p,q}^\xi[B] \quad (A, B \text{ increasing}), \quad (\text{FKG})$$

and, if $p' \geq p$ and $\zeta \geq \xi$, then

$$\phi_{G,p,q}^\xi \leq_{\text{st}} \phi_{G,p',q}^\zeta.$$

Combining these facts is particularly useful. If $H \subset G$, the spatial Markov property expresses the restriction to H as a mixture over induced boundary conditions; monotonicity therefore gives

$$\phi_{H,p,q}^0 \leq_{\text{st}} \phi_{G,p,q}^\xi|_H \leq_{\text{st}} \phi_{H,p,q}^1. \quad (1.1)$$

The same comparison remains valid after conditioning on information outside H . We will repeatedly “push” unknown boundary conditions in this way.

For a single edge e ,

$$\phi_{\{e\},p,q}^\xi[e \text{ open}] = \begin{cases} p, & \xi = 1, \\ \frac{p}{p + (1-p)q}, & \xi = 0. \end{cases}$$

Applying (SMP) after revealing every other edge gives finite energy and the following useful comparisons.

Corollary 1.4. *For all $q \geq 1$ and $p \in (0, 1)$,*

$$\phi_{G, \frac{p}{p+(1-p)q}, 1}^0 \leq_{\text{st}} \phi_{G,p,q}^\xi \leq_{\text{st}} \phi_{G,p,1}^0.$$

Note in the first equation that $\phi_{G,p,1}^0$ is the assignment of i.i.d. states to each edge of G . The proof is left as an exercise.

1.2 The full-plane measures

Let $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$. The monotone comparison of nested boxes implies the following limits.

Proposition 1.5. *For $p \in (0, 1)$ and $q \geq 1$,*

$$\phi_{p,q}^0 = \lim_{n \rightarrow \infty} \phi_{\Lambda_n,p,q}^0, \quad \phi_{p,q}^1 = \lim_{n \rightarrow \infty} \phi_{\Lambda_n,p,q}^1$$

exist locally. They are translation invariant and ergodic, and

$$\phi_{p,q}^0 \leq_{\text{st}} \phi_{p,q}^1, \quad \phi_{p,q}^i \leq_{\text{st}} \phi_{p',q}^i \quad (p < p', i \in \{0, 1\}).$$

Every thermodynamic limit with arbitrary boundary conditions lies between the free and wired limits.

Define

$$\theta(p, q) := \phi_{p,q}^1[0 \leftrightarrow \infty], \quad p_c(q) := \inf\{p : \theta(p, q) > 0\}.$$

The equality of the free and wired measures, and the sharp behaviour on the two sides of p_c , will not be part of the lecture development; they are isolated in the exercises below.

We will nevertheless use the bulk free energy. Boundary conditions change $\log Z$ by only a boundary-order term, so the limit

$$f(p, q) := \lim_{n \rightarrow \infty} \frac{1}{|E(\Lambda_n)|} \log Z_{\Lambda_n, p, q}^{\xi_n} \quad (1.2)$$

exists and is independent of (ξ_n) . This normalisation is convenient for the BKW correspondence in Chapter 2.

1.3 Duality for FK-percolation

Planar FK-percolation has a convenient duality property. For simplicity, we will state this property in the simple case of measures on boxes Λ_n with either free or wired boundary conditions. Write Λ_n^* for the subgraph of $(\mathbb{Z}^2)^*$ formed of the e^* edges dual to the edges e of Λ_n . For ω a percolation configuration on Λ_n , write ω^* for the percolation configuration on Λ_n^* defined by

$$\omega^*(e^*) = 1 - \omega(e) \quad \text{for any pair of primal/dual edges } e, e^*.$$

Proposition 1.6 (Duality). *Fix $n \geq 1$, $q \geq 1$, $p \in (0, 1)$ and $\xi \in \{0, 1\}$. If ω is sampled according to $\phi_{\Lambda_n, p, q}^\xi$, then ω^* has the law $\phi_{\Lambda_n^*, p^*, q}^{1-\xi}$, where $p^* \in (0, 1)$ is defined by*

$$\frac{p}{1-p} \cdot \frac{p^*}{1-p^*} = q.$$

More general duality relations may be obtained, that is, for more general subgraphs and more general boundary conditions. Most may be deduced from the above, by fixing the configuration in parts of Λ_n and applying (SMP). We will not detail these generalisations, and will only use them in isolated places.

Proof. It suffices to consider the case $\xi = 1$. A simple induction shows that

$$|\omega| + |\omega^*| = |E| \quad \text{and} \quad k(\omega^1) - k(\omega^*) = (2n-1)^2 - |\omega|. \quad (1.3)$$

The first relation is obvious. The second is immediate for the empty configuration and may be extended to other configurations by induction. Indeed, when adding an edge e to some configuration ω , either the number of connected components in the primal configuration decreases by 1 while that in the dual remains the same, or the number of connected components of the primal configuration remains the same, but that in the dual increases by one.

The duality relation follows directly from (1.3). $\square$

It is immediate to check that the only point for which $p = p^*$ is the so-called *self-dual* point

$$p_{\text{sd}} = p_{\text{sd}}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

It is natural to conjecture that $p_c = p_{\text{sd}}$ for all $q \geq 1$. This was first confirmed in [BD12] and will be deduced through different means below.

Hereafter, we study the behaviour of the model at this self-dual parameter; that it corresponds to the point of phase transition and that the phase transition is sharp will be consequences of our investigation. We start off with an RSW-type estimate for symmetric quads.

A quad is a simply connected domain $\mathcal{D}$ with four marked points on its boundary a, b, c, d in counter-clockwise order, with the boundary of $\mathcal{D}$ formed of two arcs (ab) and (cd) of the primal lattice and two arcs (bc) and (da) of the dual lattice (with a diagonal segment of length $1/\sqrt{2}$ between each arc). The *alternating* boundary condition on $(\mathcal{D}, a, b, c, d)$ — denoted *alt* — is produced by conditioning that all edges of the arcs (ab) and (cd) are open, while all other edges of the lattice outside of $\mathcal{D}$ are closed.

We say that $(\mathcal{D}, a, b, c, d)$ is symmetric if there exists an isometry of $\mathbb{R}^2$ that maps the primal lattice on the dual so that $\mathcal{D}$ is mapped to itself, with the arcs (ab) and (cd) mapped to the dual arcs (bc) and (da) .

Corollary 1.7. *Fix $q \geq 1$ and a symmetric quad $(\mathcal{D}, a, b, c, d)$. Then*

$$\phi_{\mathcal{D}, p_{\text{sd}}, q}^{\text{alt}}[(ab) \xleftrightarrow{\mathcal{D}} (cd)] \geq \frac{1}{1+q} \quad (1.4)$$

Proof. The event $\omega \in \{(ab) \xleftrightarrow{\mathcal{D}} (cd)\}$ is the complement of $\omega^* \in \{(bc) \xleftrightarrow{\mathcal{D}} (da)\}$. Moreover, if $\omega \sim \phi_{\mathcal{D}, p_{\text{sd}}, q}^{\text{alt}}$ then $\sigma(\omega^*)$ has almost the same law, where σ is the isometry for which $\mathcal{D}$ is symmetric. Indeed, $\sigma(\omega^*) \sim \phi_{\mathcal{D}, p_{\text{sd}}, q}^{\text{alt}'}$ where the boundary conditions *alt'* are identical to *alt*, except that the two wired arcs are also wired together. This difference in boundary conditions affects the weight of any configuration by a factor of at most q . Thus, the Radon–Nikodym derivative of $\phi_{\mathcal{D}, p_{\text{sd}}, q}^{\text{alt}'}$ with respect to $\phi_{\mathcal{D}, p_{\text{sd}}, q}^{\text{alt}}$ is between $1/q$ and q . It follows that

$$\begin{aligned} \phi_{\mathcal{D}, p_{\text{sd}}, q}^{\text{alt}}[\omega^* \in \{(bc) \xleftrightarrow{\mathcal{D}} (da)\}] &= \phi_{\mathcal{D}, p_{\text{sd}}, q}^{\text{alt}'}[\omega \in \{(ab) \xleftrightarrow{\mathcal{D}} (cd)\}] \\ &\leq q \phi_{\mathcal{D}, p_{\text{sd}}, q}^{\text{alt}}[\omega \in \{(ab) \xleftrightarrow{\mathcal{D}} (cd)\}]. \end{aligned}$$

The above, together with the complementarity of the two events, proves (1.4). $\square$

Remark 1.8. We will mostly use the above for quads $(\mathcal{D}, a, b, c, d)$ which are not really symmetric, but are “better” than symmetric. That is, quads so that if $\omega \notin \{(ab) \xleftrightarrow{\mathcal{D}} (cd)\}$ then $\sigma(\omega^*) \in \{(ab) \xleftrightarrow{\mathcal{D}} (cd)\}$. The proof of Corollary 1.7 applies readily to this case, as well.

1.4 Statement of the dichotomy theorem

Write H_n for the event that there exists an open circuit in $\Lambda_{2n} \setminus \Lambda_n$ that surrounds Λ_n ; also denote by H_n^* the same event for the dual model.

Theorem 1.9 (Duminil-Copin, Sidoravicius, Tassion 2017). *Fix $q \geq 1$ and let $p = p_{\text{sd}} = \sqrt{q}/(1 + \sqrt{q})$. Then exactly one of the two scenarios below occurs.*

(Con) *There exists $c > 0$ such that, for all $n \geq 1$*

$$\phi_{\Lambda_{2n} \setminus \Lambda_n}^0[H_n] \geq c \quad \text{and} \quad \phi_{\Lambda_{2n} \setminus \Lambda_n}^1[H_n^*] \geq c. \quad (1.5)$$

(DisCon) *There exists $c > 0$ such that, for all $n \geq 1$*

$$\phi^0[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn} \quad \text{and} \quad \phi^1[0 \overset{*}{\leftrightarrow} \partial\Lambda_n] \leq e^{-cn}. \quad (1.6)$$

The identification of the critical point follows by combining this theorem with standard sharpness and planar-duality arguments. Since these arguments are not needed for the proof of the dichotomy itself, they are left as Exercise 11.

Corollary 1.10 (Phase transition on $\mathbb{Z}^2$). *For each $q \geq 1$, we have*

$$p_c = p_{\text{sd}}(\mathbb{Z}^2) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Moreover, for all $p \neq p_c$, $\phi^0 = \phi^1$.

As such, Theorem 1.9 describes two possible behaviours at the critical parameter, which in turn correspond to two types of phase transitions. Indeed, (DisCon) corresponds to a discontinuous (or first order) phase transition, while (Con) corresponds to a continuous phase transition (or one of order two or higher). For $p < p_c$, define the *correlation length* as

$$\xi(p)^{-1} = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \phi_{\Lambda_n, p, q}^0[0 \leftrightarrow \partial \Lambda_n];$$

see Exercise 12 for details on why the limit exists. For $p > p_c$, set $\xi(p) = \xi(p^*)$. The exercises discuss the off-critical finiteness of ξ , uniqueness of the Gibbs measure and continuity of θ . At the critical point, the following features distinguish the two types of transition.

Corollary 1.11. *If (DisCon) occurs at p_{sd} , we have*

- *non-uniqueness of infinite-volume measure at criticality: $\phi_{p_c}^0 \neq \phi_{p_c}^1$;*
- *$p \mapsto \theta(p)$ is discontinuous at p_c , as is the edge intensity $p \mapsto \phi_p^0(e \text{ open})$;*
- *$p \mapsto f(p, q)$ is not differentiable at p_c ;*
- *the correlation length $\xi(p)$ is bounded uniformly in $p \in (0, 1)$.*

Conversely, if (Con) occurs at p_{sd} , we have

- *uniqueness of infinite-volume measure at criticality: $\phi_{p_c}^0 = \phi_{p_c}^1$;*
- *$p \mapsto \theta(p)$ and $p \mapsto \phi_p^0(e \text{ open})$ are continuous at p_c . Moreover, there exist constants $c, \alpha > 0$ such that*

$$c n^{-1} \leq \phi_{p_c}[0 \leftrightarrow \partial \Lambda_n] \leq n^{-\alpha} \quad \text{for all } n \geq 1;$$

- *$p \mapsto f(p, q)$ is differentiable at p_c .*
- *the correlation length tends to infinity as $p \rightarrow p_c$.*

The off-critical deductions are collected in the exercises. We now focus on the proof of Theorem 1.9, the heart of this chapter. Section 1.5 proves an instrumental RSW estimate (sometimes called the *pushing lemma*); Section 1.6 then proves a renormalisation inequality which implies the theorem.

1.5 RSW in strips

Throughout this section, we work with $p = p_{\text{sd}}$ and omit it from the notation. Set $\text{Strip}_N = \mathbb{Z} \times [-N, N]$. Let $1/0$ be the boundary conditions that are wired on the top of Strip_N and free on the bottom. The following RSW theorem is central to the proof of Theorem 1.9.

The measure $\phi_{\text{Strip}_N}^{1/0}$ is defined as a limit of measures on rectangles $[-M, M] \times [-N, N]$ as $M \rightarrow \infty$; the boundary conditions on the lateral sides of the rectangles are irrelevant due to the finite energy property.

Theorem 1.12 (RSW in strips). *There exists $c > 0$ such that, for any $\rho \geq 1$ and $N \geq n$,*

$$\phi_{\text{Strip}_N}^{1/0} [[-\rho n, \rho n] \times [-n, n] \text{ crossed horizontally}] \geq c^\rho. \quad (1.7)$$

The same applies to crossing probabilities for the dual model.

Several proofs of RSW properties are available, most significantly [KST23]. These usually hinge on positive association and certain symmetries of the model. Here, the model is not invariant under rotations by $\pi/2$ or vertical reflections, but is invariant under horizontal translations and reflections. This requires a particular approach, which we detail below. Additionally, in FK-percolation one should be aware of the effect of boundary conditions. Here the model has a self-duality property (Corollary 1.7) which facilitates the proof.

Theorem 1.12 is a consequence of the following lemmas. Write $\text{Rect}(m, n) = [-m, m] \times [-n, n]$ and $\mathcal{C}_h(\text{Rect}(m, n))$ and $\mathcal{C}_v(\text{Rect}(m, n))$ for the events that $\text{Rect}(m, n)$ contains a horizontal and a vertical crossing, respectively.

In the duality identities below, rectangles are treated as symmetric with respect to primal-dual changes. This is not the literally the case as integer-vertex rectangles differ from their duals by half a mesh size. These differences should be included for an exact duality identity. However, the finite-energy property allow to absorb them in the uniform constants appearing in the proof.

Lemma 1.13. (*Duality*) *For any $N \geq n$ and $m \geq 1$,*

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(m, n))] + \phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_v(\text{Rect}(m, n))] = 1. \quad (1.8)$$

Lemma 1.14. (*Non-degenerate horizontal crossings*) *There exists a constant $0 < c \leq 1/2$ such that, for any $N \geq n$*

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(cn, n))] \geq c. \quad (1.9)$$

Lemma 1.15. (*Lengthening crossings*) *There exists a constant $c > 0$ such that, for any $N \geq n$ and $m \geq 1$,*

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(2m, n))] \geq c \phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(m, n))]^4 \phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_v(\text{Rect}(m, n))]^2. \quad (1.10)$$

Of the three lemmas, the last one is the most important and hardest to prove. Before proving the lemmas, let us see how they imply Theorem 1.12.

Proof of Theorem 1.12. Fix $N \geq n$. Observe that the finite energy property implies that

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(m+1, n))] \geq \left(\frac{p}{p+(1-p)q}\right)^2 \phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(m, n))], \quad (1.11)$$

since for any ω in $\mathcal{C}_h(\text{Rect}(m, n))$, opening at most two edges produces a configuration in $\mathcal{C}_h(\text{Rect}(m+1, n))$.

Let $c_0 > 0$ be the constant given by Lemma 1.14 and let m be the largest value for which

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(m-1, n))] \geq c_0.$$

Then, by Lemma 1.14, $m \geq c_0 n$. Moreover,

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(m, n))] \geq c_1 \quad \text{and} \quad \phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_v(\text{Rect}(m, n))] \geq 1/2,$$

where $c_1 = \left(\frac{p}{p+(1-p)q}\right)^2 c_0$ is a uniformly positive constant — for the first inequality, see (1.11), for the second we use Lemma 1.13 and the fact that $c_0 \leq 1/2$.

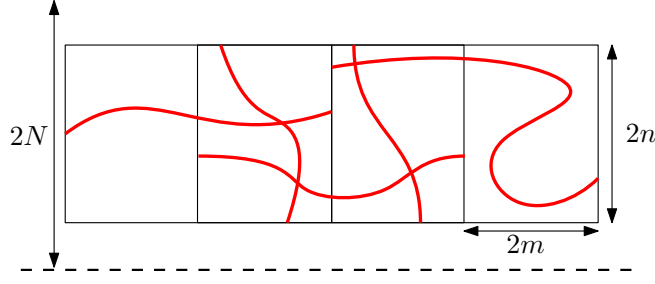


Figure 1.1: By combining vertical crossings of horizontal translations of $\text{Rect}(m, n)$ and horizontal crossings of translations of $\text{Rect}(2m, n)$, we produce horizontal crossings of $\text{Rect}(km, n)$. We work here in a horizontal strip with wired boundary conditions on the top and free on the bottom.

Lemma 1.15 now implies that

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(2m, n))] \geq c_2$$

for some uniformly positive constant c_2 . Then, by combining horizontal and vertical crossings as in Figure 1.1, we conclude that, for any $k \geq 2$,

$$\begin{aligned} \phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(km, n))] &\geq \phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(2m, n))]^{k-1} \phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_v(\text{Rect}(m, n))]^{k-2} \\ &\geq (\tfrac{1}{2}c_2)^k. \end{aligned}$$

In light of the fact that $m \geq c_0 n$, this implies (1.7).

Finally, observe that, by the duality of $\phi_{\text{Strip}_N}^{1/0}$, the probability in (1.7) is the same for primal and dual crossings. This allows us to conclude that (1.7) also holds for ω^* . $\square$

We now turn to the proof of the three lemmas.

Proof of Lemma 1.13. Note that if $\omega \notin \mathcal{C}_h(\text{Rect}(m, n))$, then $\omega^* \in \mathcal{C}_v(\text{Rect}(m, n))$, and vice-versa. Moreover, if $\omega \sim \phi_{\text{Strip}_N}^{1/0}$, then ω^* has the same law as the vertical reflection of ω . Thus

$$\phi_{\text{Strip}_N}^{1/0} [\omega \in \mathcal{C}_v(\text{Rect}(m, n))] = \phi_{\text{Strip}_N}^{1/0} [\omega^* \in \mathcal{C}_v(\text{Rect}(m, n))]$$

These two observations together yield (1.8). $\square$

Proof of Lemma 1.14. We will prove by contradiction that

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_h(\text{Rect}(n/3, n))] \geq \frac{1}{4(1+q)},$$

which suffices to deduce (1.9). Assume the opposite for some n . Then (1.8) implies that

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_v(\text{Rect}(n/3, n))] \geq \tfrac{1}{2}.$$

Consider now the translates $\text{Rect}_L = \text{Rect}(n/3, n) - (2n/3, 0)$ and $\text{Rect}_R = \text{Rect}(n/3, n) + (2n/3, 0)$. By (FKG),

$$\phi_{\text{Strip}_N}^{1/0} [\mathcal{C}_v(\text{Rect}_L) \cap \mathcal{C}_v(\text{Rect}_R)] \geq \tfrac{1}{4}.$$

Define the leftmost crossing Γ_L on Rect_L and the rightmost crossing Γ_R of Rect_R , when these two rectangles are crossed (see Figure 1.2). Let $\mathcal{D}$ be the part of Λ_n between Γ_L

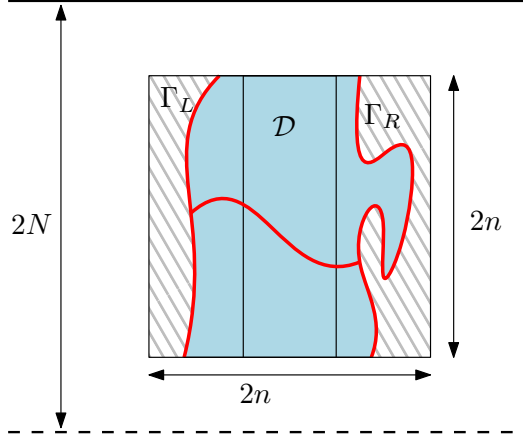


Figure 1.2: The vertical crossings Γ_L and Γ_R are the leftmost and rightmost crossings of their respective rectangles. Identifying them only has an influence on the state of the edges on them and in the hashed areas. Conditionally on Γ_L and Γ_R , the part $\mathcal{D}$ of $\text{Rect}(n, n)$ left between them (in light blue) is “narrower” than a square and has boundary conditions that are wired on its lateral sides. As such, it is crossed horizontally with probability at least $1/(1+q)$.

and Γ_R . Then, conditionally on Γ_L and Γ_R , the boundary conditions induced on $\mathcal{D}$ by the configuration outside dominate the alternating boundary conditions. This is the standard exploration principle: discovering an extremal crossing only fixes edges on the explored side, and the spatial Markov property identifies the conditional law in the unexplored region. Note that $\mathcal{D}$ is not symmetric, but if $\omega \notin \{\Gamma_L \xleftrightarrow{\mathcal{D}} \Gamma_R\}$ then $\sigma(\omega^*) \in \{\Gamma_L \xleftrightarrow{\mathcal{D}} \Gamma_R\}$, where σ is the rotation of the lattice by $\pi/2$ composed with the shift by $(1/2, 1/2)$. Corollary 1.7 (or more precisely Remark 1.8) implies that

$$\phi_{\text{Strip}_N}^{1/0}[\Gamma_L \xleftrightarrow{\mathcal{D}} \Gamma_R \mid \Gamma_L, \Gamma_R] \geq \frac{1}{1+q}.$$

Finally, notice that when the event above occurs, $\text{Rect}(n/3, n)$ is crossed horizontally. Combining the last two displays we conclude that

$$\phi_{\text{Strip}_N}^{1/0}[\mathcal{C}_h(\text{Rect}(n/3, n))] \geq \frac{1}{4(1+q)},$$

which contradicts our assumption. $\square$

Proof of Lemma 1.15. Fix m, n and N as in the statement. Since we work only with $\phi_{\text{Strip}_N}^{1/0}$, we denote this simply by ϕ .

Write a, b, c, d for the corners of $\text{Rect}(m, n)$ in counter-clockwise order, starting with the top left corner; see Figure 1.3, left diagram. Let v be the topmost point on (ab) so that

$$\phi[(av) \xleftrightarrow{\text{Rect}(m, n)} (cd)] \geq \frac{1}{2}\phi[\mathcal{C}_h(\text{Rect}(m, n))]. \quad (1.12)$$

Then the same lower bound holds for $\phi[(vb) \xleftrightarrow{\text{Rect}(m, n)} (cd)]$. Write w for the horizontal reflection of v , belonging to the side (cd) . Then, by (FKG),

$$\begin{aligned} \phi[(vb) \xleftrightarrow{\text{Rect}(m, n)} (wd)] &\geq \phi[(vb) \xleftrightarrow{\text{Rect}(m, n)} (cd)]\phi[(ab) \xleftrightarrow{\text{Rect}(m, n)} (wd)]\phi[\mathcal{C}_v(\text{Rect}(m, n))] \\ &\geq \frac{1}{4}\phi[\mathcal{C}_h(\text{Rect}(m, n))]^2\phi[\mathcal{C}_v(\text{Rect}(m, n))]. \end{aligned}$$

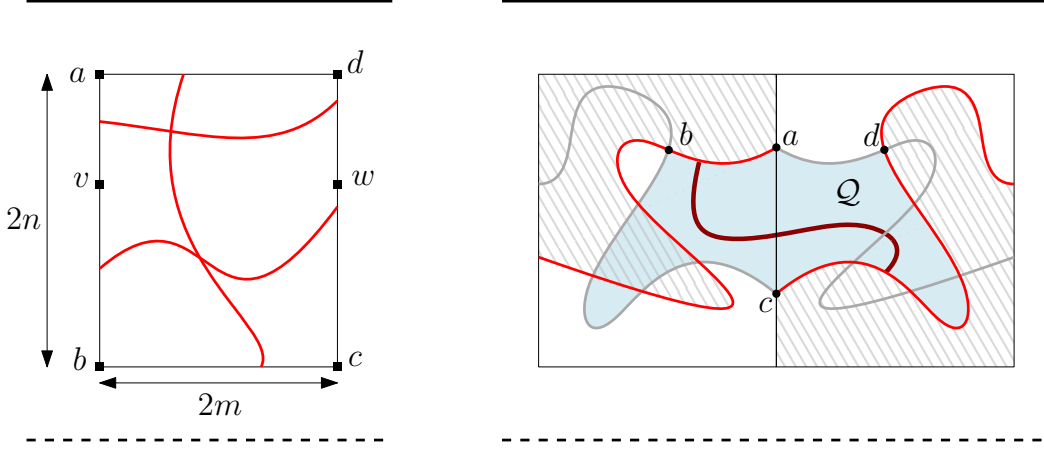


Figure 1.3: *Left:* with v chosen as in (1.12), the probability of obtaining a “diagonal” crossing, i.e., from (vb) to (wd) , is bounded away from 0 by a universal multiple of $\phi[\mathcal{C}_h(\text{Rect}(m, n))]^2 \phi[\mathcal{C}_v(\text{Rect}(m, n))]$.

Right: in two side-by-side copies of $\text{Rect}(m, n)$, if both the left and right rectangle contain diagonal crossings as in the left picture, then we may construct a symmetric domain $\mathcal{Q}$ with points a, b, c, d on its boundary, as depicted. If $\mathcal{Q}$ is crossed from (ab) to (cd) , the left side of the left rectangle is connected to the right side of the right rectangle.

We call the event on the left-hand side above $\mathcal{E}(\text{Rect}(m, n))$.

Write Rect_L for the translate of $\text{Rect}(m, n)$ by $(-m, 0)$ and Rect_R for its translate by $(m, 0)$. Then, by the above and (FKG),

$$\phi[\mathcal{E}(\text{Rect}_L) \cap \mathcal{E}(\text{Rect}_R)] \geq \frac{1}{16} \phi[\mathcal{C}_h(\text{Rect}(m, n))]^4 \phi[\mathcal{C}_v(\text{Rect}(m, n))]^2. \quad (1.13)$$

Write Γ_L for the topmost path realising $\mathcal{E}(\text{Rect}_L)$ and Γ_R for the bottommost realising $\mathcal{E}(\text{Rect}_R)$. When the events are not realised, write $\Gamma_L = \emptyset$ and $\Gamma_R = \emptyset$, respectively.

Let σ be the horizontal reflection with respect to the axis $\{0\} \times \mathbb{R}$ composed with the shift by $(\frac{1}{2}, \frac{1}{2})$. Conditionally on Γ_L and Γ_R both different from $\emptyset$, write $\mathcal{Q}$ for the quad obtained as follows. Due to the definition of $\mathcal{E}$, Γ_L necessarily intersects $\sigma(\Gamma_R)$; let b be the last such point of intersection along Γ_L , when going from left to right; set $d = \sigma(b)$. Let a and c be the endpoints of Γ_L and Γ_R , respectively, on $\{0\} \times \mathbb{R}$. Then $\mathcal{Q}$ is the quad delimited by the arcs Γ_L between a and b , $\sigma(\Gamma_R)$ between b and c , Γ_R between c and d and $\sigma(\Gamma_L)$ between d and a . See Figure 1.3, right diagram.

Then $\mathcal{Q}$ is invariant¹ under σ , with the wired arcs (ab) and (cd) being mapped by σ onto the free arcs (bc) and (da) . As such, we would like to argue that Corollary 1.7 applies, and provides a lower bound on the probability of $(ab) \xleftrightarrow{\mathcal{Q}} (cd)$ under the measure conditional on Γ_L and Γ_R . Notice, however, that parts of $\mathcal{Q}$ may have been explored when determining Γ_L and Γ_R , which complicates our analysis (see Figure 1.3, right diagram, hashed regions). Still, the measure in the unexplored part of $\mathcal{Q}$ induced by the conditioning on Γ_L and Γ_R is the measure with wired boundary conditions on (ab) and (cd) , random boundary conditions on (bc) and (da) , conditioned that all explored edges are open — indeed, the edges inside the explored region have no influence on the measure outside, only those of the boundary of the explored region do, and they are all open. As such, the measure in $\mathcal{Q}$ induced by the conditioning on Γ_L and Γ_R dominates the measure with alternating free and wired boundary conditions on the arcs of $\partial\mathcal{Q}$, for which Corollary 1.7 applies.

¹This is not formally true, but a finite energy argument allows us to apply Corollary 1.7.

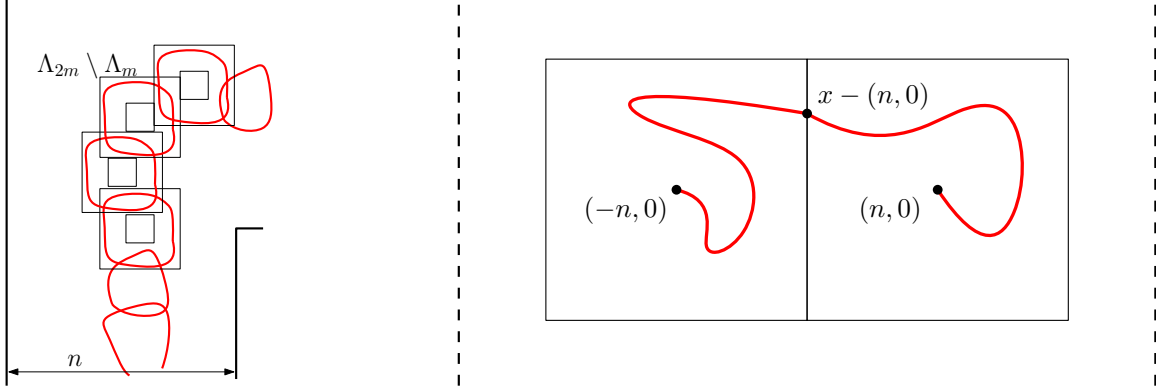


Figure 1.4: *Left:* For $m \leq n/20$, combining several instances of H_m produces H_n . Comparing boundary conditions and using (FKG), we conclude that $u_n \geq \phi_{\Lambda_{2n} \setminus \Lambda_n}^0[H_n] \geq u_m^{Cn/m}$ for some universal constant C .

Right: Consider the point x on the right side of $\partial\Lambda_n$ that is most likely to be connected to 0 in Λ_n . Then, under the measure $\phi_{\Lambda_{2N}}^0$, the point $(-n, 0)$ is connected to $x - (n, 0)$ with probability at least $\frac{1}{|\partial\Lambda_n|} \phi_{\Lambda_N}^0[0 \leftrightarrow \partial\Lambda_n]$. The same holds for connections between $(n, 0)$ and $x - (n, 0)$. When both connections occur, $(-n, 0)$ connects to $(n, 0)$.

We conclude that

$$\phi[\Gamma_L \overset{Q}{\leftrightarrow} \Gamma_R \mid \Gamma_L, \Gamma_R] \geq \frac{1}{1+q}, \quad (1.14)$$

when $\Gamma_L \neq \emptyset$ and $\Gamma_R \neq \emptyset$. Finally notice that, when the event above occurs, $\text{Rect}(2m, n)$ is crossed horizontally by an open path. Combining (1.14) with (1.13) proves (1.10). $\square$

1.6 Proof of the dichotomy: Theorem 1.9

In this section, we work with $p = p_{\text{sd}}$ and omit it from the notation. By Theorem 1.12, the comparison of boundary conditions and (FKG), we conclude that

$$\phi^1[H_n] \geq c \quad \text{and} \quad \phi^0[H_n^*] \geq c \quad (1.15)$$

for all n and some constant $c > 0$. Notice here that the boundary conditions are *favourable* to the events H_n and H_n^* , respectively.

When the boundary conditions are adverse, the events H_n and H_n^* may have much smaller probabilities. For $n \geq 1$, set

$$u_n = \phi_{\Lambda_{10n}}^0[H_n].$$

Proposition 1.16. *There exists $\delta > 0$ such that, if $u_{n_0} < \delta$ for some $n_0 \geq 1$, then*

$$u_{10kn_0} \leq 2^{-k} \quad \text{for all } k \geq 1.$$

This proposition, together with the planar constructions in the proof below, implies the dichotomy theorem.

Proof of Theorem 1.9. Proposition 1.16 implies that the sequence² $(u_n)_{n \geq 1}$ is either uniformly bounded away from 0, or converges exponentially fast to 0. When the former occurs,

²Formally, this only works directly if we limit n to powers of 10. To access all values of n , one may observe that for any $n/200 \leq m \leq n/20$, $u_n \geq u_m^C$, where C is some fixed constant; see Figure 1.4.

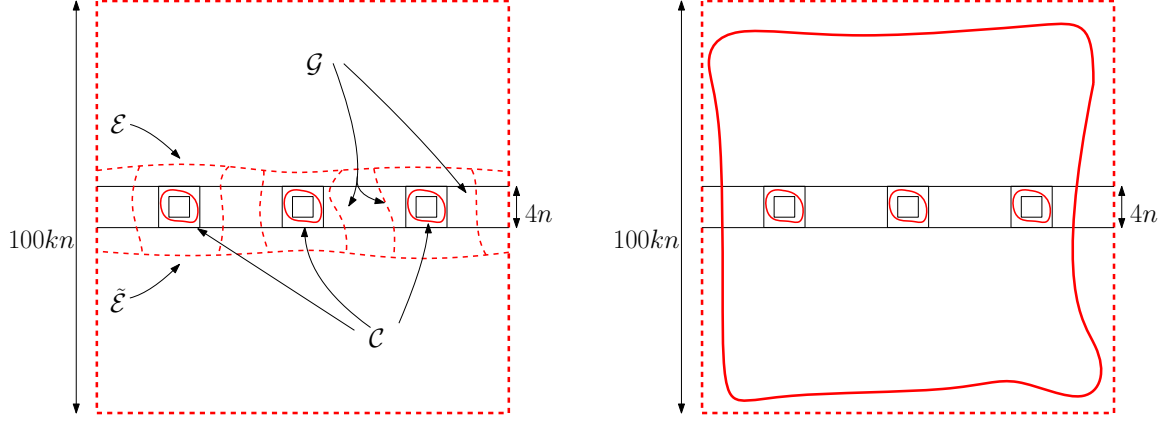


Figure 1.5: *Left:* The event $\mathcal{C} \cap \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}$ with the elements required by each event highlighted. Its probability may be bounded from above by the probability of the primal circuits of $\mathcal{C}$ occurring, conditionally on the dual paths. This produces an upper bound of u_n^k . Conversely, its probability is bounded from below in several steps: first construct $\mathcal{C}$, then, conditionally on $\mathcal{C}$, the dual paths forming $\mathcal{E} \cap \tilde{\mathcal{E}}$ and $\mathcal{G}$ appear with probability at least $(c_{\mathcal{G}} c_{\mathcal{E}})^k$. *Right:* Construct $\mathcal{C}$ by first requiring that H_{10kn} occurs. Conditionally on this event, $\mathcal{C}$ occurs with probability at least $c_{\mathcal{C}}^k$. As such, $\mathcal{C}$ has probability at least $c_{\mathcal{C}}^k u_{10kn}$.

a standard geometric construction (see Figure 1.4, left diagram) also implies that $\phi_{\Lambda_{2n} \setminus \Lambda_n}^0[H_n]$ is uniformly bounded away from 0. By self-duality, we deduce (1.5).

Assume now that the latter occurs, which is to say $u_N \leq e^{-cN}$ for all $N \geq 1$ and some $c > 0$ independent of N . We will prove (1.6).

Consider $n \leq N$, with N an integer multiple of n . Let $x \in \partial\Lambda_n$ be a maximiser of $\phi_{\Lambda_n}^0[0 \xleftrightarrow{\Lambda_n} x]$; by symmetry, we may take x on the right side of Λ_n . Using (FKG) (see also Figure 1.4, right diagram), we find that

$$\left(\frac{1}{|\partial\Lambda_n|} \phi_{\Lambda_n}^0[0 \leftrightarrow \partial\Lambda_n]\right)^2 \leq \phi_{\Lambda_n}^0[0 \xleftrightarrow{\Lambda_n} x]^2 \leq \phi_{\Lambda_{2n}}^0[(-n, 0) \leftrightarrow (n, 0)].$$

Furthermore, combining the event on the right-hand side above and its rotations CN/n times — where C is some fixed constant — we may construct H_N . Applying (1.1) to push the free boundary conditions further, we find

$$\left(\phi_{\Lambda_{2N}}^0[(-n, 0) \leftrightarrow (n, 0)]\right)^{CN/n} \leq u_N \leq e^{-cN}.$$

The last two displays allow us to conclude that there exists $c_0 > 0$ such that

$$\phi_{\Lambda_N}^0[0 \leftrightarrow \partial\Lambda_n] \leq e^{-c_0 n} \quad \text{for all } N \geq n \geq 1.$$

Taking $N \rightarrow \infty$ in the above, we conclude (1.6). $\square$

Proof of Proposition 1.16. The central element of this proof is the following *renormalisation inequality*. There exists a constant $C > 0$ such that for any $n, k \geq 1$,

$$u_{10kn} \leq (C u_n)^k. \tag{1.16}$$

Indeed, Proposition 1.16 follows from the above by taking $\delta = 1/2C$. We now focus on proving (1.16).

The constants c_* below are positive and independent of n and k . Figure 1.5 may be useful in understanding the proof. Fix $k, n \geq 1$ and assume for simplicity that k is odd.

Define the event $\mathcal{C}$ as the intersection of the translates of H_n by the $(20jn, 0)$ with $j = -(k-1)/2, \dots, (k-1)/2$. By a standard exploration argument, (FKG) and the monotonicity in boundary conditions,

$$\phi_{\Lambda_{100kn}}^0[\mathcal{C}] \geq \phi_{\Lambda_{100kn}}^0[\mathcal{C} \mid H_{10kn}] \phi_{\Lambda_{100kn}}^0[H_{10kn}] \geq (\phi^1[H_n])^k \phi_{\Lambda_{100kn}}^0[H_{10kn}] \geq c_{\mathcal{C}}^k u_{10kn},$$

where $c_{\mathcal{C}} = \inf_n \phi^1[H_n]$, which is strictly positive by (1.15).

Write $\mathcal{E}$ for the event that there exists a dual horizontal crossing in $[-100kn, 100kn] \times [2n, 4n]$ and $\tilde{\mathcal{E}}$ for the vertical reflection of this event. Then, repeated applications of (1.7) at scales $N = 100nk(3/4)^j$, producing horizontal dual crossings of translates of $[-100kn, 100kn] \times [-N/2, N/2]$, imply

$$\phi_{\Lambda_{100kn}}^0[\mathcal{E} \cap \tilde{\mathcal{E}} \mid \mathcal{C}] \geq \prod_{j=0}^{C \log k} \exp(-c \cdot 100k \cdot (\frac{4}{3})^j) \geq c_{\mathcal{E}}^k,$$

for universal constants $C, c_{\mathcal{E}} > 0$.

Finally, when $\mathcal{E} \cap \tilde{\mathcal{E}}$ occurs, write $\mathcal{G}$ for the event that there exists a dual crossings between the topmost dual path realising $\mathcal{E}$ and the bottommost path realising $\tilde{\mathcal{E}}$, in each of the squares $[20jn - 10n, 20jn - 2n] \times [-4n, 4n]$ and $[20jn + 2n, 20jn + 10n] \times [-4n, 4n]$ with $j = -(k-1)/2, \dots, (k-1)/2$. By pushing of boundary conditions (1.1) and (FKG), we conclude that

$$\phi_{\Lambda_{100kn}}^0[\mathcal{G} \mid \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{C}] \geq \phi_{\Lambda_{4n}}^{\text{alt}}[\mathcal{C}_v(\Lambda_{4n})]^{2k} \geq (\frac{1}{1+q})^{2k},$$

where the last inequality is due to (1.4).

Combine the three displays above to deduce that

$$\phi_{\Lambda_{100kn}}^0[\mathcal{C} \mid \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}] \geq \phi_{\Lambda_{100kn}}^0[\mathcal{C} \cap \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}] \geq c_{\text{tot}}^k u_{10kn},$$

where $c_{\text{tot}} > 0$ is a universal constant.

Now, when $\mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}$ occurs, each of the disjoint annuli $\Lambda_{10n} \setminus \Lambda_{2n} + (20jn, 0)$ contains a dual circuit. These may be explored from the outside, leading to independent measures in each of their interiors, with free boundary conditions. Thus

$$\phi_{\Lambda_{100kn}}^0[\mathcal{C} \mid \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}] \leq u_n^k.$$

The last two displays combine to prove the desired inequality (1.16). $\square$

1.7 Loop topology and rotational invariance

In this section, fix $q \geq 1$, $p = p_{\text{sd}}(q)$ and suppose that we are in the case (Con) of Theorem 1.9. As we will see in the next chapter, this corresponds to $q \in [1, 4]$. Write $\phi_{\mathbb{Z}^2}$ for the unique infinite volume measure with these parameters.

Loop topology. A percolation configuration ω on $\mathbb{Z}^2$ may be encoded by its loop configuration ω° . The latter is the unique partition of the edges of the medial graph of $\mathbb{Z}^2$ into loops or infinite paths in such a way that no loop or path intersects an edge of ω or ω^* . See Figure 2.3 and Section 2.2 for more details. When ω is sampled according to $\phi_{\mathbb{Z}^2}$, it has no infinite cluster, hence ω° is formed only of finite loops. These are of two types: those with primal edges on the inside and those with primal edges on the outside.

In this section, we consider ω° as an oriented loop configuration, with each loop being oriented so that the primal lies on its right side — this is not the same orientation as that

used in the BKW correspondence of Section 2.2. We will also identify ω° and ω to avoid overburdening notation.

For two loops γ and γ' of oriented loop configurations define the loop distance between them as

$$d(\gamma, \gamma') = \inf \|\gamma - \gamma'\|_\infty,$$

where the infimum is over all orientation-preserving parametrizations of γ and γ' by $\mathbb{S}^1$.

For two families of oriented loops $\mathcal{F}$ and $\mathcal{F}'$, define the Camia–Newman distance [CN06] by

$$d_{\text{CN}}(\mathcal{F}, \mathcal{F}') \leq \epsilon$$

if for each loop γ of $\mathcal{F}$ contained in $\Lambda_{1/\epsilon}$ of diameter greater than ϵ there exists a loop $\gamma' \in \mathcal{F}'$ such that $d(\gamma, \gamma') < \epsilon$, and vice versa.

Finally, if ϕ and ϕ' are probability measures producing oriented families of loops (or equivalently percolation configurations on some rescaled lattice $\delta\mathbb{Z}^2$), set

$$\mathbf{d}_{\text{CN}}(\phi, \phi') \leq \epsilon$$

if there exists a coupling P of ϕ and ϕ' producing configurations $\mathcal{F}$ and $\mathcal{F}'$ such that

$$P[d_{\text{CN}}(\mathcal{F}, \mathcal{F}') < \epsilon] > 1 - \epsilon.$$

With the distance defined above, we can formalise Conjecture 1 mentioned in the introduction.

Conjecture 3. *For $q \in [1, 4]$, the critical FK-percolation measures $\phi_{\delta\mathbb{Z}^2}$ on the rescaled lattice $\delta\mathbb{Z}^2$ converge for $\mathbf{d}_{\text{CN}}$, as $\delta \rightarrow 0$, to the nested, full plane CLE_κ measure with*

$$\kappa(q) = 4\pi / \arccos(-\frac{\sqrt{q}}{2}).$$

A similar conjecture may be expressed in finite domains. For the definitions, constructions and details of CLE_κ , see [Wer04, Wer09a, Wer09b, She09, SW12]. A fundamental property of CLE_κ , and conjecturally of scaling limits of critical statistical mechanics models, is that they exhibit *conformal invariance*.

We remark that the RSW property (1.5) feeds into the Aizenman–Burchard compactness criterion [AB99] to prove that the measures $\phi_{\delta\mathbb{Z}^2}$ are compact for $\mathbf{d}_{\text{CN}}$. See Exercises 16 and 17.

Rotational invariance. We list here an essential theorem from [DCKK⁺20]. This is the starting point of the GFF convergence for the six-vertex model. We will not prove this result, as it requires significant effort. Interested readers may consult [Man25] or the original article.

Theorem 1.17. *Fix I an isometry of $\mathbb{R}^2$ and write $\phi_{I(\delta\mathbb{Z}^2)}$ for the critical FK-percolation measure on the rescaled lattice $\delta\mathbb{Z}^2$ transformed via I . Then,*

$$\mathbf{d}_{\text{CN}}(\phi_{\delta\mathbb{Z}^2}, \phi_{I(\delta\mathbb{Z}^2)}) \xrightarrow{\delta \rightarrow 0} 0.$$

The only isometries for which the above is non-trivial are rotations by angles not multiples of $\pi/2$. Thus, we say that $\phi_{\mathbb{Z}^2}$ is *asymptotically rotationally invariant*.

The proof of Theorem 1.17 is intimately related to a universality result, which is naturally stated in the context of FK-percolation on isoradial graphs. We will describe it briefly below.

Define an inhomogeneous FK-percolation model on some distorted embedding of the square lattice $\mathbb{Z}^2$. For $\alpha \in (0, \pi)$, let $\mathbb{L}(\alpha)$ be the rectangular embedding of $\mathbb{Z}^2$ with horizontal edge length $2\cos(\alpha/2)$ and vertical edge length $2\sin(\alpha/2)$, rotated by $\alpha/2$. Consider the FK-percolation model on $\mathbb{L}(\alpha)$ with cluster-weight q and different edge-parameters p_{hor} and p_{vert} for the “horizontal” and “vertical” edges (that is, the edges of lengths $2\cos(\alpha/2)$ and $2\sin(\alpha/2)$) respectively, given by

$$\frac{p_{\text{hor}}}{1 - p_{\text{hor}}} = q \frac{1 - p_{\text{vert}}}{p_{\text{vert}}} = \sqrt{q} \frac{\sin(r\alpha)}{\sin(r(\pi - \alpha))} \quad \text{with } r := \frac{1}{\pi} \cos^{-1} \left(\frac{\sqrt{q}}{2} \right).$$

Write $\phi_{\mathbb{L}(\alpha)}$ for the unique infinite-volume measure on $\mathbb{L}(\alpha)$ with the parameters above — the uniqueness was proved in [DCLM18]. At $q = 4$, the sine ratio is interpreted by continuity as $\alpha/(\pi - \alpha)$. Observe that $\phi_{\mathbb{L}(\pi/2)}$ is a rescaled and rotated version of the homogeneous square-lattice measure; while the other models have anisotropic edge weights.

The models $\phi_{\mathbb{L}(\alpha)}$ for different values of α may be related via a series of star-triangle transformation; features that are stable under these transformations may then be transferred from $\phi_{\mathbb{L}(\pi/2)}$ to all other $\phi_{\mathbb{L}(\alpha)}$. This strategy was used in [GM14, DCLM18] for RSW estimates. A much finer control of the transformation was used in [DCKK⁺20] to prove the following.

Theorem 1.18 (Universality for isoradial rectangular lattices). *For all $\alpha \in (0, \pi)$*

$$\mathbf{d}_{\text{CN}}(\phi_{\delta\mathbb{L}(\alpha)}, \phi_{\delta\mathbb{L}(\pi/2)}) \xrightarrow{\delta \rightarrow 0} 0. \quad (1.17)$$

Observe that Theorem 1.18 readily implies Theorem 1.17. Indeed, for any $\alpha \in (0, \pi)$, $e^{i\alpha/2}\mathbb{R}$ is an axis of symmetry for $\mathbb{L}(\alpha)$, and therefore for $\phi_{\mathbb{L}(\alpha)}$. Then (1.17) implies that $\phi_{\mathbb{L}(\pi/2)}$ is asymptotically invariant under the reflection with respect to $e^{i\alpha/2}\mathbb{R}$. The composition of the reflections with respect to the horizontal axis and to $e^{i\alpha/2}\mathbb{R}$ produce the rotation by α , and we deduce the asymptotic invariance of $\phi_{\mathbb{L}(\pi/2)}$ by said rotation.

1.8 Exercises: FK-percolation on the square lattice

Basics

Exercise 1. Prove Propositions 1.2, 1.3, equation (1.1) and Corollary 1.4.

To prove (FKG) and the monotonicity in p use the following criteria for positive measures μ, ν on $\{0, 1\}^E$

$$\begin{aligned} (\mu(\omega \wedge \omega')\mu(\omega \vee \omega') \geq \mu(\omega)\mu(\omega'), \forall \omega, \omega' \in \{0, 1\}^E) &\implies (\text{FKG}), \quad \text{and} \\ (\mu(\omega \wedge \omega')\nu(\omega \vee \omega') \geq \mu(\omega)\nu(\omega'), \forall \omega, \omega' \in \{0, 1\}^E) &\implies \mu \leq_{\text{st}} \nu. \end{aligned}$$

Furthermore, it suffices to check the first criterion only for ω, ω' differing for exactly two edges.

Exercise 2. Prove Proposition 1.5 (properties of infinite volume measures).

Exercise 3. Prove the existence of the free energy of FK-percolation (1.2).

Exercise 4. Detail the proof of Proposition 1.6.

Phase transition

Exercise 5. [Convexity and exceptional parameters] Introduce the logistic parameter $t = \log(p/(1-p))$ and subtract from $|E(\Lambda_n)|^{-1} \log Z_{\Lambda_n, p, q}^\xi$ the explicit term $\log(1-p) = -\log(1+e^t)$ coming from $(1-p)^{|E|}$.

- (a) Prove that the resulting finite-volume free energies are convex in t and identify their derivatives with the edge density.
- (b) Show that the left and right derivatives of the limiting free energy are respectively the edge densities under $\phi_{p, q}^0$ and $\phi_{p, q}^1$.
- (c) Deduce that $\phi_{p, q}^0 = \phi_{p, q}^1$ except possibly at countably many values of p .
- (d) Conclude that, for $p < p'$,

$$\phi_{p, q}^0 \leq_{\text{st}} \phi_{p, q}^1 \leq_{\text{st}} \phi_{p', q}^0.$$

Exercise 6. [Russo's formula and sharp thresholds] For an increasing event A on a finite graph, differentiate its probability and prove

$$\frac{d}{dp} \phi_{G, p, q}^\xi[A] = \frac{1}{p(1-p)} \sum_{e \in E} \text{Cov}_{\phi_{G, p, q}^\xi}(\mathbf{1}_A, \omega(e)).$$

The sharp-threshold input used below is the Graham–Grimmett inequality [GG11]: for p bounded away from 0 and 1,

$$\sum_{e \in E} \text{Cov}(\mathbf{1}_A, \omega(e)) \geq c \phi[A](1 - \phi[A]) \log \frac{1}{\max_e \text{Cov}(\mathbf{1}_A, \omega(e))}. \quad (1.18)$$

Explain how small individual influences force a narrow transition window.

Exercise 7. [Subcritical uniqueness] Let A be an increasing event depending only on the edges of Λ_n . Explore the connections from $\partial\Lambda_n$ to $\partial\Lambda_N$ and show that

$$0 \leq \phi_{\Lambda_N, p, q}^1[A] - \phi_{\Lambda_N, p, q}^0[A] \leq \phi_{\Lambda_N, p, q}^1[\Lambda_n \leftrightarrow \partial\Lambda_N].$$

Conclude that $\phi_{p, q}^0 = \phi_{p, q}^1$ whenever $\phi_{p, q}^1[0 \leftrightarrow \infty] = 0$.

Using duality and the fact that $p_c = p_{\text{sd}}$, conclude that $\phi_{p, q}^0 = \phi_{p, q}^1$ for all $p \neq p_c$.

Exercise 8. [Continuity of finite-volume events] Show that $p \mapsto \phi_{G, p, q}^\xi[A]$ is continuous for every finite graph and (increasing) event A . Determine which one-sided semicontinuity statements survive after taking the infinite-volume limit.

Exercise 9. [Influence of one edge] Let $A, B \subset V(G)$ and $e \in E(G)$. Prove

$$\begin{aligned} \phi[A \leftrightarrow B \mid \omega(e) = 1] - \phi[A \leftrightarrow B \mid \omega(e) = 0] \\ \leq \phi[e \leftrightarrow A \text{ and } e \leftrightarrow B \mid \omega(e) = 1]. \end{aligned} \quad (1.19)$$

Use the spatial Markov property to extend the estimate to the influence of a finite boundary condition.

Exercise 10. [Finite-size criterion] Fix $p \in (0, 1)$ and $q \geq 1$ and omit them from notation. We work here on $\mathbb{Z}^2$. Prove that there exists $\eta > 0$ such that, if for some n

$$\phi_{\Lambda_{4n}}^1[\Lambda_n \not\leftrightarrow \partial\Lambda_{2n}] = \phi_{\Lambda_{4n}}^1[H_n^*] > 1 - \eta,$$

then there exists $c > 0$ such that, for all N large enough,

$$\phi_{\Lambda_N}^1[0 \leftrightarrow \partial\Lambda_N] < e^{-cN}.$$

In particular, conclude that $p < p_c$.

Attention: the last inequality is strict!

Exercise 11. [Sharpness from dichotomy] Assume Theorem 1.9, the sharp-threshold inequality (1.18) and Exercises 5, 9 and 10.

- (a) In case (DisCon), use duality and monotonicity to obtain exponential decay below p_{sd} and percolation above it.
- (b) In case (Con) fix $n \geq 1$ and bound the influence of any given edge e on $\Lambda_n \leftrightarrow \partial\Lambda_{2n}$ under $\phi_{\Lambda_{4n}, p, q}^0$ with $p \leq p_{\text{sd}}$. That is, bound $\text{Cov}(\mathbf{1}_{\Lambda_n \leftrightarrow \partial\Lambda_{2n}}, \omega(e))$ or equivalently

$$\phi_{\Lambda_{4n}}^0[\Lambda_n \leftrightarrow \partial\Lambda_{2n} | \omega(e) = 1] - \phi_{\Lambda_{4n}}^0[\Lambda_n \leftrightarrow \partial\Lambda_{2n} | \omega(e) = 0]$$

using (1.19)

- (c) Conclude from the above that, for any $p < p_{\text{sd}}$,

$$\phi_{p, \Lambda_{4n}}^0[\Lambda_n \leftrightarrow \partial\Lambda_{2n}] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

- (d) Still in case (Con), use (1.5) and the above to deduce that, for any $p < p_{\text{sd}}$,

$$\phi_{p, \Lambda_{4n}}^1[\Lambda_n \leftrightarrow \partial\Lambda_{2n}] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

- (e) Deduce $p_c = p_{\text{sd}}$, exponential decay away from criticality, uniqueness of the infinite-volume measure for $p \neq p_c$, and continuity of θ away from p_c .

Correlation length

Exercise 12. Fix $p \in (0, 1)$ and $q \geq 1$. We wish to prove that

$$u_n := \frac{1}{n} \log \phi_{\Lambda_n, p, q}^0[0 \leftrightarrow \partial\Lambda_n]. \tag{1.20}$$

converges. If $p < p_c$, we denote the limit by $-1/\xi(p)$.

- (a) Using (FKG) and the comparison of boundary conditions, prove that for all $m, n \geq 1$

$$\phi_{\Lambda_{n+m}, p, q}^0[0 \leftrightarrow \partial\Lambda_{n+m}] \geq \frac{1}{C \min\{n, m\}} \phi_{\Lambda_n, p, q}^0[0 \leftrightarrow \partial\Lambda_n] \phi_{\Lambda_m, p, q}^0[0 \leftrightarrow \partial\Lambda_m],$$

where C is some universal constant.

- (b) Use this fact to prove the existence of the limit in (1.20).

Hint: Apply Fekete's lemma to a slightly modified sequence, or use the standard almost-subadditivity variant in which an additive $O(\log(n + m))$ error is allowed. The logarithmic error disappears after division by n .

- (c) Show that $\phi_{p,q}^0$ exhibits exponential decay of cluster radii if and only if $\xi(p) < \infty$. Moreover prove that then

$$\phi_{p,q}^0[0 \leftrightarrow \partial\Lambda_n] = e^{-n/\xi(p)+o(n)} \quad \text{as } n \rightarrow \infty.$$

- (d) For $p > p_c$, write $\xi(p) = \xi(p^*)$ with p^* being the dual parameter to p . Observe that, $p^* < p_c$. Show that there exist universal constants $c, C > 0$ such that

$$e^{-Cn/\xi(p)} \leq \phi_{p,q}^0[0 \leftrightarrow \partial\Lambda_n \text{ but } 0 \not\leftrightarrow \infty] \leq e^{-cn/\xi(p)} \quad \text{for all } n \text{ large enough.}$$

- (e) For $p > p_c$, prove that

$$\phi_{p,q}^0[0 \leftrightarrow \partial\Lambda_n \text{ but } 0 \not\leftrightarrow \infty] = e^{-2n/\xi(p)+o(n)} \quad \text{as } n \rightarrow \infty.$$

Exercise 13. The goal of the exercise is to study the continuity of $\xi(p)$ for $p \leq p_c$.

- (a) Prove that $p \mapsto \xi(p)$ is increasing for $p < p_c$.
(b) Prove that $p \mapsto \xi(p)$ is continuous for $p < p_c$.
(c) Assume that $\xi^0(p_c) := \lim_{p \nearrow p_c} \xi(p) < \infty$ as $p \nearrow p_c$. Prove then that

$$\phi_{p_c,q}^0[0 \leftrightarrow \partial\Lambda_n] = e^{-n/\xi^0(p_c)+o(n)} \quad \text{as } n \rightarrow \infty,$$

and in particular that (DisCon) occurs at p_c .

- (d) Conclude that, if (Con) occurs at p_c , then $\xi(p) \rightarrow \infty$ as $p \nearrow p_c$.

Criticality

Exercise 14. Consider FK-percolation on $\mathbb{Z}^2$ with some $q \geq 1$, $p = p_{\text{sd}}(q)$. Assume that we are in the case (Con) of Theorem 1.9. Show that there exists $c > 0$ such that

$$cn^{-1} \leq \phi_{\Lambda_{2n}}^\xi[0 \leftrightarrow \partial\Lambda_n] \leq n^{-c} \quad \text{for all } n \geq 1 \text{ and all boundary conditions } \xi.$$

Deduce that $\phi[|\mathbf{C}_0|] = \infty$, where $\mathbf{C}_0$ is the cluster of 0, and ϕ is the unique infinite-volume measure (here used as an expectation).

Exercise 15. Fix $q \geq 1$ and $p \in (0, 1)$. Let ϕ be a Gibbs measure for FK-percolation on $\mathbb{Z}^2$ which is invariant under translations and rotations by $\pi/2$. Assume that there exists $c > 0$ such that

$$\begin{aligned} \phi[\omega \text{ contains a horizontal crossing of } [0, 2n] \times [0, n]] &\geq c & \text{and} \\ \phi[\omega^* \text{ contains a horizontal crossing of } [0, 2n] \times [0, n]] &\geq c & \text{for all } n \geq 1. \end{aligned}$$

Prove (without using Theorem 1.9) that

$$\begin{aligned} \phi[\omega \text{ contains a horizontal crossing of } [0, kn] \times [0, n]] &\geq c^{2k} & \text{and} \\ \delta < \phi[H_n] < 1 - \delta & & \text{for all } n \geq 1, \end{aligned}$$

and the same for the dual, for some $\delta > 0$. Does this imply that $\phi[0 \leftrightarrow \partial\Lambda_n] \rightarrow 0$?

Exercise 16. Fix $\phi_{\mathbb{Z}^2}$ a FK-percolation measure satisfying (1.5) and write $\phi_{\delta\mathbb{Z}^2}$ for its rescaled version. Show that exists $C > 0$ such that, for any $0 < \delta < 1$, and $\lambda \geq 1$

$$\phi_{\delta\mathbb{Z}^2}[\text{more than } \frac{\lambda}{\delta^2} \text{ clusters of diameter at least } \delta \text{ intersect } \Lambda_1] \leq Ce^{-\lambda/C}$$

Exercise 17. Fix $\phi_{\mathbb{Z}^2}$ a FK-percolation measure satisfying (1.5). Use the Aizenman–Burchard compactness criterion [AB99] to prove that the measures $\phi_{\delta\mathbb{Z}^2}$ are compact for $\mathbf{d}_{\text{CN}}$.

Exercise 18. Fix $d = 2$ and $p = p_c$ and assume we are in case (Con). Write ϕ for the unique infinite volume measure. Show the existence of constants $c, C > 0$ such that, for any $n \geq 1$ and $m \geq 1$,

$$e^{-Cm} \leq \phi[[0, nm] \times [0, n] \text{ crossed horizontally}] \leq e^{-cm}.$$

Also, show that, for any $1 \leq k \leq n$,

$$e^{-Ck^2m} \leq \phi[[0, nm] \times [0, n] \text{ crossed horizontally by } k \text{ disjoint paths}] \leq e^{-ck^2m}.$$

Hint: the second upper bound is delicate (observe that a naive argument gives $\leq e^{-ckm}$ for some universal constant $c > 0$). To obtain the required bound, start by proving the following statement:

There exists $c_0 > 0$ and $C_0 \geq 1$ such that, for all $1 \leq k \leq n$,

$$\phi[[0, C_0n/k] \times [0, n] \text{ crossed horizontally by } k \text{ disjoint paths}] \leq e^{-ck}. \quad (1.21)$$

To do this, consider $\Gamma_1, \Gamma_2, \dots$ the successive lowest horizontal crossings of $[0, t] \times [0, \infty]$ where $t = C_0n/k$ and C_0 is to be determined. Let h_ℓ be the highest point of Γ_ℓ . Using an RSW construction, prove the existence of a universal constant $c_0 > 0$ (independent of all of the above, including C_0) such that, for all ℓ

$$\phi[h_{\ell+1} - h_\ell \geq t | \omega \text{ under } \Gamma_\ell] \geq c_0.$$

Setting $X_\ell = \mathbf{1}_{\{h_{\ell+1} - h_\ell \geq t\}}$ and choosing C_0 sufficiently large, prove (1.21).

Other

Exercise 19. Consider the torus $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^2$ as a finite graph. Fix an increasing event A which is invariant under the translations by $(1, 0)$ and $(0, 1)$. Notice that, for $q \geq 1$ and $p \in (0, 1)$, we may define $\phi_{\mathbb{T}_N, p, q}$ as the FK-percolation measure on $\{0, \dots, N\}^2$ with periodic boundary conditions (that is, where we wire each $(x, 0)$ to (x, N) and each $(0, y)$ to (N, y) for $0 \leq x, y \leq N$).

Use (1.18) to prove that, for some constant $c > 0$ depending only on p and q ,

$$\frac{d}{dp} \phi_{\mathbb{T}_N, p, q}[A] \geq c \phi_{\mathbb{T}_N, p, q}[A] (1 - \phi_{\mathbb{T}_N, p, q}[A]) \log N.$$

Exercise 20. [Torus duality: a warning] Assume (DisCon). Ordinary torus FK is not exactly self-dual: the factor $q^{s(\omega)}$ in Lemma 2.3 must be retained.

- (a) Show $\phi^0[e \text{ open}] < 1/2$ using duality and the strict inequality between free and wired edge densities.
- (b) If ϕ_N is the ordinary square-torus measure, show that its symmetrisation $\hat{\phi}_N = (\phi_N + \phi_N^*)/2$ has edge density $1/2$. Explain why this does not prove the same assertion for ϕ_N .
- (c) Admit that sub-sequential limits are mixtures of ϕ^0, ϕ^1 . Deduce $\hat{\phi}_N \rightarrow (\phi^0 + \phi^1)/2$, and show this limit is not ergodic. Do not infer these mixture weights for the unsymmetrised measure.

Chapter 2

Continuous/discontinuous phase transition

Theorem 1.9 identified two types of phase transition for FK-percolation on $\mathbb{Z}^2$. The goal of this chapter is to determine which values of q correspond to which type of phase transition.

Theorem 2.1. *The phase transition of FK-percolation on $\mathbb{Z}^2$ is continuous if $1 \leq q \leq 4$, and discontinuous if $q > 4$.*

The theorem above was proved in a series of papers via different methods. Originally, the continuity was proved in [DCST17] and the discontinuity in [DCGH⁺21]. Alternative proofs of these two results were obtained in [GL23] and [RS20], respectively.

We will present here a proof of both the continuity and discontinuity regimes inspired by the method of [DCGH⁺21]. It consists in the explicit computation of the rate of decay of a certain event that allows us to distinguish the cases (Con) and (DisCon) of Theorem 1.9. The computation is done by relating critical FK-percolation to the six-vertex model, which we define precisely below. The free energy of the six-vertex model is then estimated by applying the Bethe ansatz to its transfer matrix and computing its leading eigenvalues.

This section is meant to highlight the links between FK-percolation and the six-vertex model, and the very different techniques that may be used to analyse them. A more specific take-home message is that the continuity/discontinuity of the phase transition of FK-percolation corresponds to the delocalisation/localisation of the height function of the corresponding six-vertex model, which in turn corresponds to the differentiability, or lack thereof, of the six-vertex free energy as a function of its “slope”, at zero slope.

The actual computation of the six-vertex free energies with different slope will not be detailed as it does not employ the type of tools highlighted in these notes.

2.1 Six-vertex model on the torus

Let $L, M \in 2\mathbb{N}$ and write $\mathbb{T}_{L,M} = (\mathbb{Z}/M\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z})$ for the torus of width M and height L . A six-vertex configuration on $\mathbb{T}_{L,M}$ is an assignment of directions (or arrows) to each edge of $\mathbb{T}_{L,M}$ with the restriction that any vertex has exactly two incoming and two outgoing edges; we call this restriction the *ice rule*. As a result, there are only six possible configurations at each vertex, whence the name of the model.

Each possibility carries a weight (see Figure 2.1). Consider three positive parameters $a, b, c > 0$. The weight of a configuration $\vec{\omega}$ is

$$w_{6V}(\vec{\omega}) = a^{\#\text{vertices of type } a} \cdot b^{\#\text{vertices of type } b} \cdot c^{\#\text{vertices of type } c}.$$

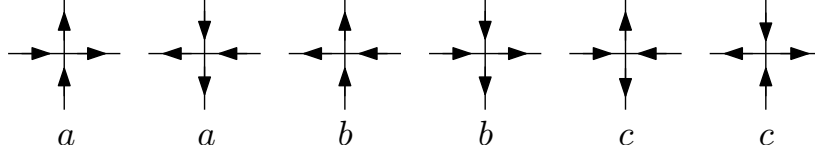


Figure 2.1: The six possible vertex configurations obeying the ice rule. The weights are chosen to be invariant under total arrow reversal.

It is standard to parametrise the model via

$$\Delta := \frac{a^2 + b^2 - c^2}{2ab},$$

as models with equal Δ are expected to behave similarly. Henceforth, we focus exclusively on the case $a = b = 1$, for which $\Delta = 1 - c^2/2 \in (-\infty, 1)$.

Preservation of arrows. Partition the horizontal edges of the torus into vertical columns. It is a direct but crucial consequence of the ice rule that, in any six-vertex configuration, the number of left-arrows is the same in each column. We call this the *preservation of left-arrows*.

Define the partition functions

$$Z_{6V}^{(k)}(\mathbb{T}_{L,M}) = \sum_{\substack{\vec{\omega} \text{ with } L/2 + k \\ \text{left-arrows per column}}} w_{6V}(\vec{\omega}) \quad \text{and} \quad Z_{6V}(\mathbb{T}_{L,M}) = \sum_{k=-L/2}^{L/2} Z_{6V}^{(k)}(\mathbb{T}_{L,M}).$$

Finally, for $\alpha \in [-1, 1]$, define the free energy of the (sloped) model as

$$f_{6V} = \lim_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{LM} \log Z_{6V}(\mathbb{T}_{L,M}) \quad \text{and} \quad f_{6V}(\alpha) = \lim_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{LM} \log Z_{6V}^{(\alpha L/2)}(\mathbb{T}_{L,M}), \quad (2.1)$$

where in the second definition we use an integer approximation of $\alpha L/2$. Simple combinatorial manipulations (see, for instance, [DCGH⁺21, proof of Cor 1.4] and Exercise 21) show that the limits exist no matter the order in which L and M are sent to infinity. Moreover, they show that

$$f_{6V} = f_{6V}(0).$$

2.2 Relation to FK-percolation: BKW correspondence

We next present a correspondence between critical FK-percolation on (a $\pi/4$ -rotated version of) the torus and the six-vertex model described above. It is sometimes called the Baxter–Kelland–Wu (or BKW) correspondence [BKW76]. Figure 2.3 sums it up.

Notice that $\mathbb{T}_{L,M}$ is a bipartite graph; consider a bipartite colouring in black and white of its vertices. Let $\mathbb{T}_{L,M}^\bullet$ be the graph containing only the black vertices of $\mathbb{T}_{L,M}$, with edges between vertices at a distance $\sqrt{2}$ of each other. Write Ω_{FK} for the set of FK-percolation configurations on $\mathbb{T}_{L,M}^\bullet$ and $\phi_{\mathbb{T}_{L,M}}$ for the associated FK-percolation measure (the parameters p and q will be fixed and omitted from the notation). As on any finite graph, the FK-percolation measure may be defined on $\mathbb{T}_{L,M}^\bullet$ with no mention of boundary conditions; alternatively $\phi_{\mathbb{T}_{L,M}}$ may be viewed as a FK-percolation measure on a rectangle with periodic boundary conditions.

Write $\mathcal{E}_k$ for the event that $\omega \in \Omega_{\text{FK}}$ contains at least k horizontally crossing clusters, where clusters are counted on the cylinder obtained by cutting $\mathbb{T}_{L,M}$ vertically above the origin — see Figure 2.2. The BKW correspondence will be used to prove the following.

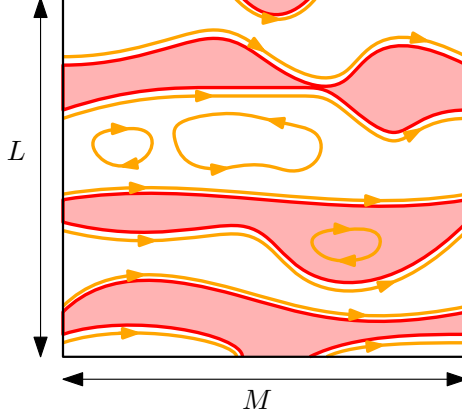


Figure 2.2: See the torus $\mathbb{T}_{L,M}$ as a horizontal cylinder. The event $\mathcal{E}_k$ (here with $k = 3$) requires the existence of k clusters of ω (in red) between the left and right sides of the cylinder. This leads to the existence of at least $2k$ paths between winding horizontally around the torus in the loop configuration ω° (in orange). If all these paths are oriented left, the resulting six-vertex configurations will have an excess of $2k$ left-arrows. Indeed, any contractible loop contributes the same number of left- and right-arrows to each column, regardless of its orientation.

Proposition 2.2 (BKW correspondence). *For $q \geq 1$, $p = p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ and $c = \sqrt{2 + \sqrt{q}}$,*

$$f_{\text{FK}} = f_{6V} + \frac{1}{4} \log q - \log(1 + \sqrt{q}) \quad \text{and} \quad (2.2)$$

$$\phi_{\mathbb{T}_{L,M}}[\mathcal{E}_{\alpha L/2}]^{1/LM} = \exp(f_{6V}(\alpha) - f_{6V}(0) + o(1)) \quad \text{for any } \alpha > 0, \quad (2.3)$$

where f_{FK} is the free energy of FK-percolation defined in (1.2) and $o(1)$ in the last line denotes a quantity converging to 0 as $M \rightarrow \infty$, then $L \rightarrow \infty$.

The rest of the section is dedicated to proving this result. Henceforth, $q \geq 1$ is fixed and we set $p = \frac{\sqrt{q}}{1+\sqrt{q}}$ and $c = \sqrt{2 + \sqrt{q}}$.

Parameters of the correspondence. First, we define a parameter linking q and c . Let λ be such that

$$e^\lambda + e^{-\lambda} = \sqrt{q}. \quad (2.4)$$

Notice that λ is real for $q \geq 4$ and purely imaginary for $1 \leq q < 4$. We choose $\lambda \geq 0$ for $q \geq 4$, and $\lambda = i \arccos(\sqrt{q}/2)$ for $1 \leq q < 4$. Changing the sign has no effect on c . Then set

$$c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}}.$$

Loop configurations. We define two more types of configurations needed in describing the BKW correspondence. An *oriented loop* on $\mathbb{T}_{L,M}$ is an oriented cycle of $\mathbb{T}_{L,M}$ which is edge-disjoint, non-self-intersecting and such that it turns by $\pm\pi/2$ at every visited vertex. We may view oriented loops as ordered collections of edges of $E(\mathbb{T}_{L,M})$, quotiented by cyclic permutations of the indices. Unoriented loops (or simply loops) are oriented loops considered up to reversal of the indices. A (oriented) loop configuration on $\mathbb{T}_{L,M}$ is a partition of $E(\mathbb{T}_{L,M})$ into (oriented) loops. Write $\Omega_{\text{Loop}}^\circ$ and $\Omega_{\text{Loop}}^\circ$ for the set of configurations of unoriented and oriented loops, respectively.

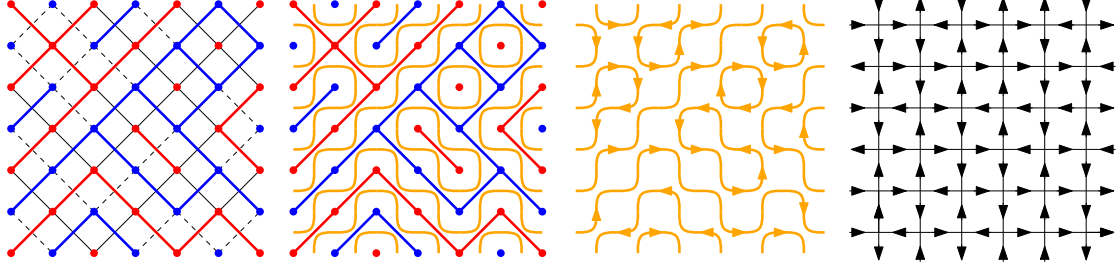


Figure 2.3: The different steps in the correspondence between the FK-percolation and six-vertex models on a torus. From left to right: A FK-percolation configuration and its dual, the corresponding loop configuration, an orientation of the loop configuration and the resulting six-vertex configuration. Note that in the first picture, there exist both a primal and dual cluster winding horizontally around the torus; this leads to two loops that wind horizontally (see second picture); if these loops are oriented in the same direction (as in the third picture), then the number of left-arrows on every column of the six-vertex configuration is $\frac{L}{2} \pm 1$.

Associate the following weights to unoriented and oriented loop configurations. For an unoriented loop configuration ω° , write $\ell(\omega^\circ)$ for the number of loops of ω° and $\ell_0(\omega^\circ)$ for the number of loops that are not contractible (on the torus) to a point. For an oriented loop configuration ω° , write $\ell_-(\omega^\circ)$ and $\ell_+(\omega^\circ)$ for the number of contractible loops of ω° which are oriented clockwise and counterclockwise, respectively. Set

$$\begin{aligned} w_o(\omega^\circ) &= \sqrt{q}^{\ell(\omega^\circ)} \cdot \left(\frac{2}{\sqrt{q}}\right)^{\ell_0(\omega^\circ)} & \text{for all } \omega^\circ \in \Omega_{\text{Loop}}^\circ; \\ w_o(\omega^\circ) &= e^{\lambda(\ell_+(\omega^\circ) - \ell_-(\omega^\circ))} & \text{for all } \omega^\circ \in \Omega_{\text{Loop}}^\circ. \end{aligned}$$

Additionally, for FK-percolation configurations ω and six-vertex configurations $\vec{\omega}$, recall the weights

$$w_{\text{FK}}(\omega) = p_{\text{sd}}^{|\omega|} (1 - p_{\text{sd}})^{|E \setminus \omega|} q^{k(\omega)} \quad \text{and} \quad w_{6V}(\vec{\omega}) = c^{\#\text{type } c \text{ vertices}}.$$

Correspondence between configurations. The correspondence between configurations is best described in Figure 2.3.

Unoriented loop configurations are in bijection with FK-percolation configurations: associate to any FK-percolation configuration ω the unique loop configuration whose loops do not intersect any edge of ω or ω^* .

An oriented loop configuration ω° is said to be coherent with the unoriented loop configuration containing the same loops. It is also said to be coherent with the six-vertex configuration whose edge-orientations are given by the orientations of the loops.

Notice that for any unoriented loop configuration ω° , there are $2^{\ell(\omega^\circ)}$ oriented loop configurations coherent with it. Similarly, for any six-vertex configuration $\vec{\omega}$, there are $2^{\#\text{vertices of type } c}$ oriented loop configurations coherent with $\vec{\omega}$.

Correspondence for weights. The following lemmas relate the weights of the different configurations. Both results are obtained via fairly direct computations which we will only sketch; full proofs are available in [DCGH⁺21].

Lemma 2.3. *For any $\omega \in \Omega_{\text{FK}}$,*

$$w_{\text{FK}}(\omega) = C \sqrt{q}^{\ell(\omega^\circ)} q^{s(\omega)} = C \left(\frac{\sqrt{q}}{2}\right)^{\ell_0(\omega^\circ)} q^{s(\omega)} \sum_{\omega^\circ \text{ coherent w. } \omega} w_o(\omega^\circ), \quad (2.5)$$

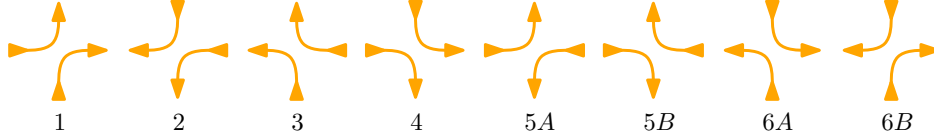


Figure 2.4: The 8 different types of vertices encountered in an oriented loop configuration.

where ω° is the loop configuration corresponding to ω and the sum is over the $2^{\ell(\omega^\circ)}$ oriented loop configurations coherent with ω° . The term $s(\omega) \in \{0, 1\}$ is the indicator that ω contains a cluster that winds around $\mathbb{T}_{L,M}$ in both the vertical and horizontal directions. Finally, $C = \frac{\sqrt{q}^{LM/2}}{(1+\sqrt{q})^{LM}}$.

The first equality is proved by induction on the number of open edges of ω . The second is obtained by observing that in the sum on the right-hand side, every contractible loop of ω° appears with its two possible orientations, thus with a total weight of $e^\lambda + e^{-\lambda} = \sqrt{q}$; non-contractible loops appear with a weight of 2, compensated by the term $(\frac{\sqrt{q}}{2})^{\ell_0(\omega)}$.

Lemma 2.4. *For any six-vertex configuration $\vec{\omega}$,*

$$w_{6V}(\vec{\omega}) = \sum_{\omega^\circ \text{ coherent w. } \vec{\omega}} w_\circ(\omega^\circ). \quad (2.6)$$

The key here is to see $w_\circ(\omega^\circ)$ as $e^{\lambda \text{wind}(\omega^\circ)/2\pi}$, where $\text{wind}(\omega^\circ)$ is the total winding of all loops of ω° . This winding may be computed locally: vertices of type *a* and *b* produce a total winding of 0, while vertices of type *c* produce a total winding of $\pm\pi$, depending on how the loops are split at the vertex. See also Figure 2.4.

Consequences for the partition functions: proof of Proposition 2.2. Summing (2.5) and (2.6) over all configurations, we easily find that

$$Z_{\text{FK}} \cdot \phi_{\mathbb{T}_{L,M}} \left[\left(\frac{2}{\sqrt{q}} \right)^{\ell_0} q^{-s} \right] = \sum_{\omega \in \Omega_{\text{FK}}} \left(\frac{2}{\sqrt{q}} \right)^{\ell_0(\omega^\circ)} q^{-s(\omega)} \cdot w_{\text{FK}}(\omega) = C \cdot Z_{6V},$$

with C given by Lemma 2.3. This is not exactly an equality between the two partition functions, due to the term $\phi_{\mathbb{T}_{L,M}} \left[\left(\frac{2}{\sqrt{q}} \right)^{\ell_0} q^{-s} \right]$. Note, however, that this term is of no significance for the free energy. Indeed, ℓ_0 is bounded by $L + M$, so

$$f_{\text{FK}} := \lim_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{LM} \log Z_{\text{FK}}(\mathbb{T}_{L,M}) = f_{6V} + \frac{1}{4} \log q - \log(1 + \sqrt{q}).$$

This proves (2.2).

We turn to (2.3), and prove a more general fact. Fix $k > 0$. For a six-vertex configuration $\vec{\omega}$, write $K(\vec{\omega})$ for the excess of left-arrows on each column, that is, the number of left-arrows on any one column minus $L/2$. Use the same notation for oriented loop configurations ω° .

When $\vec{\omega}$ and ω° are coherent, $K(\vec{\omega}) = K(\omega^\circ)$. In particular, the oriented loop configurations coherent with $\{\vec{\omega} : K(\vec{\omega}) = k\}$ are exactly those with $K(\omega^\circ) = k$.

For an unoriented loop configuration ω° , write $J_k(\omega^\circ)$ for the number of ways in which its horizontally-winding loops may be oriented to produce ω° with $K(\omega^\circ) = k$. For most ω° , we have $J_k(\omega^\circ) = 0$. Indeed, to have $J_k(\omega^\circ) > 0$, ω° needs to have at least $2k$ loops winding

horizontally around the torus (where a loop winding horizontally more than once is counted with multiplicity) — see also Figure 2.2. Finally, these loops are all counted in $\ell_0(\omega^\circ)$. Indeed, any contractible or vertically-winding loop contributes the same number of left- and right-arrows to each column; imbalances come only from loops winding horizontally around $\mathbb{T}_{L,M}$.

For an unoriented loop configuration ω° , write $\ell_h(\omega^\circ)$ for the number of non-contractible loops of ω that do not wind horizontally around the torus (in particular, they wind vertically). Observe that re-orienting the contractible loops of ω° , or those contributing to ℓ_h , does not change $K(\omega^\circ)$. Thus, applying (2.6), then (2.5), we find

$$\begin{aligned} Z_{6V}^{(k)} &= \sum_{\omega^\circ: K(\omega^\circ)=k} w_\circ(\omega^\circ) = \sum_{\omega^\circ} J_k(\omega^\circ) \frac{2^{\ell_h(\omega^\circ)}}{\sqrt{q}^{\ell_0(\omega^\circ)}} \sqrt{q}^{\ell(\omega^\circ)} \\ &= \frac{1}{C} \sum_{\omega} J_k(\omega^\circ) \frac{2^{\ell_h(\omega^\circ)}}{\sqrt{q}^{\ell_0(\omega^\circ)}} q^{-s(\omega)} w_{\text{FK}}(\omega) = \frac{Z_{\text{FK}}}{C} \phi_{\mathbb{T}_{L,M}} \left[J_k(\omega^\circ) \frac{2^{\ell_h(\omega^\circ)}}{\sqrt{q}^{\ell_0(\omega^\circ)}} q^{-s(\omega)} \right]. \end{aligned}$$

All terms in the expectation on the right-hand side, except $J_k(\omega^\circ)$, are ultimately unimportant. The important observation is that $J_k(\omega^\circ) \neq 0$ only if there are at least k primal clusters crossing $\mathbb{T}_{L,M}$ horizontally (where clusters are counted on the cylinder obtained by cutting $\mathbb{T}_{L,M}$ vertically above the origin).

Recall that we are interested in $f_{6V}(\alpha)$ which corresponds to the partition function over configurations with an excess $K(\vec{\omega}) = \alpha L/2$. Crude upper and lower bounds on the terms appearing in the probability above suffice to prove that

$$f_{6V}(\alpha) = f_{\text{FK}} + \lim_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{LM} \log \phi_{\mathbb{T}_{L,M}}[\mathcal{E}_{\alpha L/2}] - \frac{1}{4} \log q + \log(1 + \sqrt{q})$$

Together with (2.2), this proves (2.3) (also observe that $f_{6V} = f_{6V}(0)$).

2.3 Solving six-vertex via the transfer matrix

The following theorem is obtained by estimating the leading eigenvalues of blocks of the transfer matrix of the six-vertex model via the Bethe ansatz. It will be used in our case to distinguish between continuous and discontinuous phase transitions of the FK-percolation model.

Theorem 2.5 (Free energy close to zero slope). *As $\alpha \searrow 0$, we have*

$$f_{6V}(0) - f_{6V}(\alpha) = \begin{cases} \frac{1}{2} m(\Delta) \alpha + o(\alpha) & \text{if } c > 2, \\ \frac{1}{4} \arccos(\Delta) \alpha^2 + o(\alpha^2) & \text{if } 1 \leq c \leq 2, \end{cases} \quad (2.7)$$

where $\Delta = 1 - c^2/2$. In the first line, writing $\lambda = \text{arccosh}(-\Delta) > 0$, the coefficient is

$$m(\Delta) = \lambda + 2 \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \tanh(j\lambda) > 0. \quad (2.8)$$

The coefficient $m(\Delta)$ in (2.8) is the FK inverse correlation length in the convention of [DCGH⁺21] for $q > 4$.

Below, we give some details of the Bethe ansatz approach and the ingredients needed to prove Theorem 2.5. We do not provide a full proof here, as it is highly technical and does not involve tools relevant to the rest of these notes. See [DCKK⁺22] for the free-energy asymptotics; the explicit quadratic coefficient is computed in [DCKLM26, Sec. 24].

Transfer matrix formalism. We identify the possible configurations of horizontal arrows on one column by $\{\pm\}^L$. Consider the matrix $V = V_L$ defined by

$$V(\vec{x}, \vec{x}') = \sum_{\text{possible completions}} c^{\#\text{type } c \text{ vertices}} \quad \forall \vec{x}, \vec{x}' \in \{\pm\}^L,$$

where the completions refer to the possible assignments of vertical arrows between $\vec{x}$ and $\vec{x}'$ that obey the ice rule. Formally, V is an operator acting on $(\mathbb{C}^2)^{\otimes L}$, which may be viewed as a 2^L -dimensional vector space with an orthonormal basis $(\Psi_{\vec{x}})_{\vec{x} \in \{\pm\}^L}$ indexed by the possible horizontal arrow configurations on one column.

The transfer matrix is split into blocks corresponding to the number of left-arrows on each column (recall that this number is conserved). Indeed, if $\vec{x}$ and $\vec{x}'$ have a different number of left-arrows, then $V(\vec{x}, \vec{x}') = 0$. Write $V^{(k)}$ for the block with $L/2 + k$ left-arrows. Then, it is immediate to check that

$$Z_{6V}^{(k)}(\mathbb{T}_{L,M}) = \text{Tr}(V^{(k)})^M. \quad (2.9)$$

Furthermore, each such block has non-negative entries, is irreducible, and has strictly positive diagonal entries. It follows by the Perron–Frobenius theorem that each $V^{(k)}$ has a single eigenvector of maximal eigenvalue (with all other eigenvalues being of strictly smaller modulus). We call these the Perron–Frobenius eigenvector and eigenvalue, and write $\Lambda^{(k)} = \Lambda^{(k)}(L)$ for the latter.

Then, (2.9) implies

$$Z_{6V}^{(k)}(\mathbb{T}_{L,M}) = (\Lambda^{(k)}(L))^M (1 + O(e^{-\varepsilon M})),$$

for some $\varepsilon = \varepsilon(L) > 0$. Since $Z_{6V}^{(k)}(\mathbb{T}_{L,M})$ is used to determine the free energy with slope α , we find

$$f_{6V}(\alpha) = \lim_{L \rightarrow \infty} \frac{1}{L} \log \Lambda^{(\alpha L/2)}(L).$$

Statement of the Bethe ansatz for the six-vertex model. The Bethe ansatz is a general method for producing eigenvectors and eigenvalues of certain transfer matrices. It originates in the work of Bethe [Bet31]. We state it below in the context of the six-vertex model.

Recall that $\Delta = 1 - c^2/2$ and set

$$S(x, y) := e^{-ix} + e^{iy} - 2\Delta, \\ \mu := \begin{cases} \arccos(-\Delta) & \text{if } c \leq 2 \\ 0 & \text{if } c > 2 \end{cases} \quad \text{and} \quad \mathcal{D} := (-\pi + \mu, \pi - \mu).$$

Let $\Theta : \mathcal{D}^2 \rightarrow \mathbb{R}$ be the unique continuous function which satisfies $\Theta(0, 0) = 0$ and

$$\exp(-i\Theta(x, y)) = e^{i(x-y)} \cdot \frac{e^{-ix} + e^{iy} - 2\Delta}{e^{-iy} + e^{ix} - 2\Delta}.$$

For $z \neq 1$, set

$$L(z) := 1 + \frac{c^2 z}{1-z}, \quad M(z) := 1 - \frac{c^2}{1-z}.$$

Theorem 2.6 (Bethe ansatz). Fix $0 \leq k \leq L/2$ and set $n = L/2 - k$. Let $(p_1, p_2, \dots, p_n) \in \mathcal{D}^n$ be distinct and satisfy the equations

$$Lp_j = 2\pi I_j - \sum_{k=1}^n \Theta(p_j, p_k), \quad \forall j \in \{1, \dots, n\}, \quad (2.10)$$

where $I_j = j - \frac{n+1}{2}$ for $j \in \{1, \dots, n\}$. Set $\psi = \sum_{|\vec{x}|=n} \psi(\vec{x}) \Psi_{\vec{x}}$, the sum is over $\vec{x} \in \{\pm\}^L$ with n right-arrows at positions $1 \leq x_1 < \dots < x_n \leq L$ and $\psi(\vec{x})$ is given by

$$\begin{aligned} \psi(\vec{x}) &:= \sum_{\sigma \in \mathfrak{S}_n} A_\sigma \prod_{k=1}^n \exp(ip_{\sigma(k)} x_k), \\ A_\sigma &:= \varepsilon(\sigma) \prod_{1 \leq k < \ell \leq n} e^{ip_{\sigma(k)} S(p_{\sigma(k)}, p_{\sigma(\ell)})}. \end{aligned}$$

Here σ is an element of the symmetric group $\mathfrak{S}_n$. Then

$$V\psi = \Lambda\psi,$$

where

$$\Lambda = \begin{cases} \prod_{j=1}^n L(e^{ip_j}) + \prod_{j=1}^n M(e^{ip_j}) & \text{if all } p_1, \dots, p_n \text{ are nonzero,} \\ \left[2 + c^2(L-1) + c^2 \sum_{j \neq \ell} \partial_x \Theta(0, p_j) \right] \cdot \prod_{j \neq \ell} M(e^{ip_j}) & \text{if } p_\ell = 0 \text{ for some } \ell. \end{cases}$$

In words, the above explains how a solution to (2.10) may be used to produce an eigenvector and eigenvalue for V . The proof of the above is an intricate, but ultimately direct computation; a full proof may be found in [DCGH⁺18]. It is expected that there exists exactly one solution to (2.10), which produces the Perron–Frobenius eigenvector and eigenvalue of $V^{(k)}$.

Let us assume that one may find solutions $p_1, \dots, p_n$ to (2.10), and that their empirical distribution defined as $\rho_L = \frac{1}{L} \sum_{i=1}^n \delta_{p_i}$ converges to a measure $\rho(x)dx$ on $\mathcal{D}$ admitting a density — this is sometimes called *condensation* of the Bethe roots. Then (2.10) implies that

$$\begin{aligned} 2\pi\rho(x) &= 1 + \int_{-Q}^Q \partial_x \Theta(x, y) \rho(y) dy, \quad x \in (-Q, Q), \\ \int_{-Q}^Q \rho(x) dx &= \frac{1-\alpha}{2}. \end{aligned} \quad (2.11)$$

Here $[-Q, Q] \subset \overline{\mathcal{D}}$ is the occupied interval of Bethe roots; Q depends on the slope, and ρ is extended by zero outside this interval. Moreover

$$\lim_{L \rightarrow \infty} \frac{1}{L} \log \Lambda = \int_{\mathcal{D}} \log |M(e^{ix})| \rho(x) dx. \quad (2.12)$$

The condition (2.11) allows one to determine ρ , which may then be injected in (2.12) to obtain asymptotics for Λ .

We will not prove Theorem 2.6 here, nor how it applies to extract Theorem 2.5. We limit ourselves to the following observations.

- The difficulties in applying this method are: prove that (2.10) has solutions and that these solutions condense as $L \rightarrow \infty$. Show that the resulting vector is not null, and therefore that the resulting Λ is an eigenvalue. Prove that Λ is the Perron–Frobenius eigenvalue of the corresponding block of V . Finally, when this is done, one needs to compute sufficiently precise estimates on the resulting eigenvalue, using the continuous version of the Bethe equations (2.11) and (2.12).
- The Bethe ansatz changes form at $\Delta = -1$, corresponding to $c = 2$. Outside of this value, it depends smoothly on Δ .
- These difficulties were overcome in [DCGH⁺18] and [DCKK⁺22]. Proving that Λ is indeed the Perron–Frobenius eigenvalue was done by considering the model (for finite L) at the “trivial” cases $\Delta = 0$ and $\Delta = -\infty$, where the computations are explicit, then proving that $\Delta \mapsto (\psi, \Lambda)$ is an analytic function, hence preserving the positivity of ψ . The condensation is also proved using this type of evolution in Δ : it is proved that $\psi(\Delta)$ can not continuously “escape” a region of condensation.
- The estimates obtained for $\Lambda^{(k)}/\Lambda^{(0)}$ are not sufficiently fine to apply to $k = 1$ (or small values of k). For our purpose $k = \alpha L/2$ suffices, where the estimates are more robust.
- The ultimate computation of $\Lambda^{(k)}/\Lambda^{(0)}$ proves (2.7).

2.4 Deducing the type of phase transition

Combining Proposition 2.2 with Theorem 2.5 gives a direct estimate of the probability of many horizontal crossings.

Corollary 2.7. *We have*

$$\lim_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{LM} \log \phi_{\mathbb{T}_{L,M}}[\mathcal{E}_{\alpha L/2}] = \begin{cases} -\frac{1}{2}m(-\sqrt{q}/2)\alpha + o(\alpha) & \text{if } q > 4, \\ -\frac{1}{4} \arccos(-\sqrt{q}/2)\alpha^2 + o(\alpha^2) & \text{if } 1 \leq q \leq 4, \end{cases} \quad (2.13)$$

where m is given explicitly by (2.8).

This corollary implies directly Theorem 2.1.

Proof of Theorem 2.1. Fix $q \geq 1$. Define the correlation length at criticality as

$$\xi^{-1} = \xi^{-1}(p_c, q) = -\lim_n \frac{1}{n} \log \phi_{\Lambda_n}^0[0 \leftrightarrow \partial \Lambda_n] = -\lim_n \frac{1}{n} \log \phi^0[0 \leftrightarrow \partial \Lambda_n].$$

The equality of the two limits may be obtained by standard arguments, see Exercise 12. Also define

$$\pi(n) = \phi_{\Lambda_n}^0[(-n/2, 0) \leftrightarrow (n/2, 0)].$$

Then, a straightforward analysis shows that $\frac{1}{n} \log \pi(n) \rightarrow -\xi^{-1}$.

We claim that there exists a universal constant $c > 0$ such that, for all $4 \leq 2k \leq L \leq M$,

$$c^{k^2 M/L} \pi(L/4k)^{4k^2 M/L} \leq \phi_{\mathbb{T}_{L,M}}[\mathcal{E}_k] \leq L^k e^{-(k-2)M/(\xi+o(1))}. \quad (2.14)$$

with $o(1)$ denoting a quantity converging to 0 as $M \rightarrow \infty$.

For the lower bound, cut up the cylinder in horizontal rectangles $\text{Rect}_j = [0, M] \times [j \frac{L}{2k}, (j+1) \frac{L}{2k}]$ for $j = 1, \dots, 2k$. Using RSW techniques¹, we may construct k dual clusters crossing the

¹Here, (1.7) does not apply directly, but one may easily prove the same estimate in periodic rather than 1/0 boundary conditions

cylinder horizontally in the rectangles Rect_j with j even with a probability of at least $c^{k^2 M/L}$ for some constant $c > 0$. Conditionally on the existence of these clusters (or, more generally, on any configuration in the rectangles with j even), the probability that each of the rectangles Rect_j with j odd contains a horizontal crossing is at least $\pi(L/4k)^{4kM/L}$. Indeed, combining $4kM/L$ translates of $\{(-\frac{L}{8k}, 0) \leftrightarrow (\frac{L}{8k}, 0)\}$ using (FKG), one obtains a horizontal crossing of Rect_j . The lower bound in (2.14) follows.

For the upper bound, assume that $\mathcal{E}_k$ occurs. Start from an arbitrary point and explore the closest primal clusters above and below this point that cross the cylinder horizontally. When conditioning on these clusters, the measure in the rest of the cylinder has free boundary conditions. Moreover, the unexplored part contains at least $k - 2$ other crossing clusters. We explore these clusters one by one, from bottom to top. After each cluster is explored, the probability that another cluster exists is at most $L\phi^0[0 \leftrightarrow \partial\Lambda_M]$. This proves the upper bound in (2.14).

When taking $k = \alpha L/2$, then taking $M \rightarrow \infty$, $L \rightarrow \infty$ and then $\alpha \rightarrow 0$, (2.14) gives

$$\frac{1}{2}\xi^{-1} - o(1) \leq \limsup_{\substack{M \rightarrow \infty \\ L \rightarrow \infty}} -\frac{1}{\alpha LM} \log \phi_{\mathbb{T}_{L,M}}[\mathcal{E}_{\alpha L/2}] \leq \frac{1}{2}\xi^{-1} + o(1),$$

where $o(1)$ denotes a quantity tending to 0 as $\alpha \rightarrow 0$. Combine the above with (2.13) to obtain that $\xi(p_c, q) = \infty$ if and only if $1 \leq q \leq 4$. Theorem 1.9 allows us to conclude. $\square$

2.5 The M-formula

Fix $c \in [1, 2]$. Percolation tools similar to (1.5) in a graphical representation of the six-vertex model yield a unique full-plane zero-slope measure $\mathbb{P}_{\mathbb{Z}^2}$; see [She05, GL23, DCKLM26]. It is the local limit of unconditioned tori, and of increasing domains with sufficiently flat boundary conditions. Under $\mathbb{P}_{\mathbb{Z}^2}$, the height function is only defined up to global shift, and should be referred to more precisely as the gradient of the height function. Indeed, in this regime of parameters, the height function delocalises, and quantitative bounds may be obtained on its variance.

Assume next that $c \in [\sqrt{3}, 2]$ or, equivalently, $\Delta \in [-1, -\frac{1}{2}]$. By reworking the BKW correspondence and taking the size of the torus to infinity, we obtain the following formula relating observables in the six-vertex and FK-percolation models. Write ϕ for the corresponding full-plane FK-percolation measure, defined on $\sqrt{2}e^{i\frac{\pi}{4}}\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. Choose the branch of (2.4) given by $\lambda = i \arccos(\frac{\sqrt{q}}{2}) = 2i \arccos(\frac{c}{2})$. For $x \in \mathbb{R}$, set

$$\cos_\lambda(x) := \frac{\cos(x + |\lambda|)}{\cos(|\lambda|)}.$$

A test function F is any function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ that is continuous, bounded, has compact support and has zero total integral. For an FK-percolation configuration ω , recall its loop representation ω° be its loop representation. For any loop $\ell \in \omega^\circ$, write $\text{int}(\ell)$ for the region of the plane in the interior of the loop ℓ . It is a union of faces of $\mathbb{Z}^2$ — see Figure 2.3.

Theorem 2.8 (M-formula). *Let F be a test function. Then,*

$$\mathbb{E}_{\mathbb{Z}^2} \left[\exp \left(i \int_{\mathbb{R}^2} h(x) F(x) dx \right) \right] = \phi \left[\prod_{\ell \in \omega^\circ} \cos_\lambda \left(\int_{\text{int}(\ell)} F(x) dx \right) \right]. \quad (2.15)$$

In the above, h is viewed as a function from $\mathbb{R}^2 \rightarrow \mathbb{R}$ which is constant on the faces of $\mathbb{Z}^2$; the absolute additive constant is irrelevant since $\int_{\mathbb{R}^2} F(x) dx = 0$.

At the present time, this is the best tool we have to transfer information between the six-vertex and FK-percolation models. A version of this formula for (sub-sequential) scaling limits is derived in [ACDC⁺26].

2.6 Exercises: six-vertex transfer matrix

Exercise 21. Prove that the free energy of the six-vertex model is well defined, as in (2.1).

Exercise 22. Prove Lemmas 2.3 and 2.4 and conclude Proposition 2.2.

Exercise 23. Prove that each block $V^{(k)}$ of the transfer matrix does indeed have the Perron–Frobenius property.

Exercise 24. Prove the M-formula (2.15).

Exercise 25. Prouve that, under assumption (Con), there exists $C > 0$ such that

$$\phi_{\mathbb{T}_{L,M}}[\mathcal{E}_k] \geq \exp(-Ck^2M/L)$$

Conversely, under assumption (DisCon),

$$\phi_{\mathbb{T}_{L,M}}[\mathcal{E}_k] < \exp(-kM/\xi(p))$$

Chapter 3

Convergence of the six-vertex height function to the GFF

The previous chapters illustrated a first interaction between percolation and integrability. Through the BKW correspondence, the probability of observing many long FK clusters on a cylinder was related to the six-vertex free energy. The transfer matrix and the Bethe ansatz then determined the type of the FK phase transition.

We now discuss a more recent application of the same circle of ideas: the scaling limit of the six-vertex height function. The complete proof is long and contains several independent ingredients. Our goal here is to give an overview of the proof, stressing the overall structure and some novel ingredients. Some results will be stated as black boxes, while others will only be sketched. All results in this chapter are proved in [\[DCKLM26\]](#).

3.1 Height functions, correlations and the main result

The full-plane six-vertex model and its correlations. Throughout this chapter, the six-vertex weights are isotropic and will mostly be

$$a = b = 1, \quad c \in [\sqrt{3}, 2], \quad \Delta = 1 - \frac{c^2}{2} \in [-1, -\frac{1}{2}].$$

We will sometimes consider values of c beyond those above. Recall from Chapter 2 that the ice rule permits one to associate a height function with every arrow configuration. It is useful to record the convention explicitly.

We recall from Section 2.5 the notion of infinite-volume measure for the six-vertex model when $c \in [1, 2]$. There exists a measure $\mathbb{P}_{\mathbb{Z}^2}$ on six-vertex configurations which is the local limit of unconditioned torus measures as both side lengths tend to infinity in arbitrary fashion. For balanced torus measures the result we use takes $M \rightarrow \infty$ first, then $L \rightarrow \infty$. It is the unique ergodic Gibbs measure invariant under global arrow reversal.

This follows from the general theory of planar height-function Gibbs measures combined with delocalisation; see [\[She05, GL23\]](#) and the references therein. There is also a percolative route. A spin/loop representation has FKG-type monotonicity and RSW circuits. Extremal boundary conditions yield monotone limits, while circuits shield a fixed local observable from the boundary and force the extremal limits to coincide. At $c = 2$ ($\Delta = -1$), this mechanism can be seen particularly directly through the $q = 4$ BKW coupling; see Exercise 26.

We write $\mathbb{E}_{\mathbb{Z}^2}$ for expectation under this measure. The restriction $c \geq \sqrt{3}$ will enter later, through the rotation-invariance input.

Extend h to a piecewise constant function on $\mathbb{R}^2$ and, for $\delta > 0$, set

$$h^{(\delta)}(u) := h(u/\delta).$$

There is no normalising power of δ in this definition. The height fluctuations are of order one when tested against a smooth, zero-mass test function; the variance of a height difference grows logarithmically with the distance.

Definition 3.1 (Multi-point correlations). For $k \geq 1$ and $\mathbf{u} = (u_1, u'_1, \dots, u_k, u'_k) \in (\mathbb{R}^2)^{2k}$, define

$$\Phi_k(\mathbf{u}) := \mathbb{E}_{\mathbb{Z}^2} \left[\prod_{i=1}^k (h(u'_i) - h(u_i)) \right]$$

and

$$\Phi_k^{(\delta)}(\mathbf{u}) := \Phi_k(u_1/\delta, u'_1/\delta, \dots, u_k/\delta, u'_k/\delta).$$

Let $\mathcal{D}_k$ be the set of $\mathbf{u}$ for which the two-point sets $\{u_i, u'_i\}$ are pairwise disjoint.

The full-plane GFF. Write

$$G(u, v) := -\frac{1}{2\pi} \log |u - v| \quad (3.1)$$

for the full-plane Green kernel. It is only defined as a covariance modulo additive constants. One convenient definition of the full-plane GFF Γ is as the centred Gaussian process indexed by compactly supported signed measures φ of total mass zero and finite Dirichlet energy ($\iint G(u, v) d\varphi(u) d\varphi(v) < \infty$), with

$$\text{Cov}(\langle \Gamma, \varphi \rangle, \langle \Gamma, \psi \rangle) = \iint G(u, v) d\varphi(u) d\psi(v).$$

The correlation functions of GFF increments are obtained from Wick's formula. For $\mathbf{u} \in \mathcal{D}_k$, write $\Psi_k^{\text{GFF}}(\mathbf{u})$ for the sum over pairings of the k increments, each covariance being computed from (3.1). In particular,

$$\Psi_2^{\text{GFF}}(u_1, u'_1, u_2, u'_2) = G(u'_1, u'_2) - G(u'_1, u_2) - G(u_1, u'_2) + G(u_1, u_2). \quad (3.2)$$

For odd k , the function Ψ_k^{GFF} is zero. Also, it is crucial to remark that, as a linear combination of Wick factors of Green functions, Ψ_k^{GFF} is harmonic on $\mathcal{D}_k$ for all k .

Main result. We may now formally state the main result of these notes. It should be understood as $h^\delta \rightarrow \sigma\Gamma$ in law, in an appropriate space, where Γ is a full plane GFF and $\sigma = \sigma(c)$ is an explicit constant.

Theorem 3.2 (GFF convergence). *Fix $a = b = 1$ and $c \in [\sqrt{3}, 2]$. As $\delta \searrow 0$, the full-plane six-vertex height function converges to $\sigma\Gamma$, where*

$$\sigma^2 = \frac{2}{\arccos \Delta} = \frac{1}{\arcsin(c/2)}.$$

More precisely:

- for every k , $\Phi_k^{(\delta)}$ converges to $\sigma^k \Psi_k^{\text{GFF}}$, uniformly on compact subsets of $\mathcal{D}_k$;
- the joint laws of $\langle h^{(\delta)}, \varphi_i \rangle$ converge to those of $\sigma\Gamma$ for every finite family of zero-mass test measures of compact support and finite Dirichlet energy.

The full-plane restriction is important. In a bounded domain, the boundary condition may induce a global limit shape on h and one expects convergence to a zero-boundary GFF shifted by this global shape. The theorem above does not address this problem. Even when the global shape is null, the result in finite domains does not immediately follow from Theorem 3.2.

For anisotropic weights $a \neq b$, these two modes of convergence remain valid after embedding the square lattice as an appropriate rhombic lattice; see Section 3.7.

3.2 Ingredients and the logic of the proof

We prove Theorem 3.2 by proving the first point, namely the convergence

$$\Phi_k^{(\delta)} \xrightarrow[\delta \rightarrow 0]{} \sigma^k \Psi_k^{\text{GFF}},$$

uniformly on compact subsets of $\mathcal{D}_k$, for all k even. To better understand this goal, let us list some preliminary observations about the multi-point correlations $\Phi_k^{(\delta)}$

- **Basics:** $\Phi_k^{(\delta)}$ is antisymmetric under exchanging u_i and u'_i , and symmetric under permutations of the pairs (u_i, u'_i) . It has the additivity property

$$\Phi_k^{(\delta)}(u_1, u'_1, \mathbf{v}) + \Phi_k^{(\delta)}(u'_1, u''_1, \mathbf{v}) = \Phi_k^{(\delta)}(u_1, u''_1, \mathbf{v}). \quad (3.3)$$

- **Regularity and mixing:** For every even k , the family $(\Phi_k^{(\delta)})_{\delta > 0}$ is uniformly bounded and equicontinuous¹ on compacts of $\mathcal{D}_k$. As such, it is pre-compact for the uniform convergence on compacts of $\mathcal{D}_k$.

In addition, if for $\mathbf{u} = (u_1, \dots, u'_k) \in \mathcal{D}_k$ all $u'_1, u_2, \dots, u'_k$ are regarded as fixed and u_1 as variable,

$$\limsup_{u_1 \rightarrow u_2} \limsup_{\delta \rightarrow 0} \left| \Phi_k^{(\delta)}(\mathbf{u}) - \Phi_2^{(\delta)}(u_1, u'_1, u_2, u'_2) \Phi_{k-2}^{(\delta)}(u_3, \dots, u'_k) \right| < \infty. \quad (3.4)$$

As u_1 tends to infinity, the growth of $|\Phi_k^{(\delta)}(\mathbf{u})|$ may be bounded by a polylogarithm of $\|u_1\|$, uniformly in δ . Finally, these statements also hold in sufficiently large cylinders, with constants uniform in the circumference.

These properties are obtained via RSW techniques for a percolation representation of the six-vertex model. It is crucial here that $c \in [1, 2]$, as this is the only regime where FKG and RSW hold for the percolation representation. We will not detail this here, but mention that for $c = 2$, the properties above may be obtained using (1.5) for the associated FK-percolation.

The features listed above are *qualitative* properties of the model, which carry over to sub-sequential scaling limits. It is not reasonable to expect that they suffice to identify the limit.

- **Rotational invariance.** A crucial symmetry of the sub-sequential scaling limits is their invariance under rotations. More precisely, we have that, for any Euclidean isometry I , any k and any $\mathbf{u} \in \mathcal{D}_k$,

$$|\Phi_k^{(\delta)}(\mathbf{u}) - \Phi_k^{(\delta)}(I\mathbf{u})| \xrightarrow[\delta \rightarrow 0]{} 0. \quad (3.5)$$

When passing to the scaling limit, the above implies the invariance of scaling limits under all isometries.

It is generally expected that scaling limits are invariant under conformal transformations, and in particular under isometries and changes of scale. Furthermore, if conformal invariance is proved for the scaling limit, identifying the limit should not be difficult. As such, (3.5) offers part of the symmetries expected in the limit.

¹Equicontinuity is not actually true for $\Phi_k^{(\delta)}$, but for the unique function that is equal to $\Phi_k^{(\delta)}$ at centres of faces and interpolates linearly in between.

Recall from Theorem 1.17 of Section 1.7 that a similar invariance result was proved for critical FK-percolation with $q \in [1, 4]$. The six-vertex version (3.5) is obtained from Theorem 1.17 via an infinite-volume version of the M-formula (2.15); see [ACDC⁺26] for the full derivation.

The link with FK-percolation needed for (3.5) is the only input currently restricting the proof to $c \geq \sqrt{3}$ rather than $c \geq 1$.

In light of the compactness above, we may consider sub-sequential limits Ψ_k of the multi-point correlation functions $\Phi_k^{(\delta)}$. Our goal is thus to identify such limits as $\sigma^k \Psi_k^{\text{GFF}}$. A curial step is proving their harmonicity on $\mathcal{D}_k$.

The proof is split into three steps, which we illustrate by three separate theorems.

Step 1: two-point function. We first study the two-point function Φ_2 . We do not prove outright that it converges to its GFF counterpart, but prove the following dichotomy about sub-sequential scaling limits.

Theorem 3.3 (Dichotomy for two point function). *Exactly one of the following occurs.*

- (a) *There exists $\sigma \geq 0$ such that $\Phi_2^{(\delta)} \rightarrow \sigma^2 \Psi_2^{\text{GFF}}$ as $\delta \searrow 0$.*
- (b) *There exist distinct $\sigma \neq \sigma'$ and two sequences $\delta_n, \delta'_n \searrow 0$ such that*

$$\Phi_2^{(\delta_n)} \xrightarrow{n \rightarrow \infty} \sigma^2 \Psi_2^{\text{GFF}} \quad \text{and} \quad \Phi_2^{(\delta'_n)} \xrightarrow{n \rightarrow \infty} (\sigma')^2 \Psi_2^{\text{GFF}}.$$

The convergences above are uniform compacts of $\mathcal{D}_2$. Note that in case (b) it is possible to also have non-GFF scaling sub-sequential limits.

The ingredients needed for Theorem 3.3 are the transfer-matrix representation, the regularity estimates and the rotational invariance.

Step 2: multi-point functions. Next we extend the study to multi-point functions.

Theorem 3.4 (From two-point to multi-point). *Assume $\sigma \geq 0$ and $\delta_n \searrow 0$ are such that*

$$\Phi_2^{(\delta_n)} \xrightarrow{n \rightarrow \infty} \sigma^2 \Psi_2^{\text{GFF}}.$$

Then

$$\Phi_k^{(\delta_n)} \xrightarrow{n \rightarrow \infty} \sigma^k \Psi_k^{\text{GFF}} \quad \forall k \geq 2.$$

In particular, the six-vertex model converges, along the scaling sequence δ_n , to $\sigma \cdot \text{GFF}$.

Here the main idea is to extend the harmonicity of the two-point function to all multi-point functions via a technique that allows to compare the Laplacian of Ψ_k to that of Ψ_2 .

Ingredients used here are the transfer-matrix representation (specifically a form of *reflection-positivity*) and the regularity and mixing estimates. The rotational invariance is also used, but only for convenience and may be circumvented.

It should be mentioned that, in both steps above, we use the transfer-matrix representation, but do not explicitly compute any eigenvalues. As such, the method remains very soft. However, the conclusion of the two first steps is not the GFF convergence, but rather sub-sequential GFF convergences. More precisely, step 2 generalizes the dichotomy of Theorem 3.3 to the full scaling limit.

Step 3: Fixing the variance. The third step determines the variance of the limiting GFF.

Theorem 3.5 (Variance as surface tension). *Assume $\sigma \geq 0$ and $\delta_n \rightarrow 0$ are such that*

$$\Phi_k^{(\delta_n)} \xrightarrow{n \rightarrow \infty} \sigma^k \Psi_k^{\text{GFF}} \quad \forall k \geq 2.$$

Then

$$\sigma^2 = -\frac{1}{f''(0)}, \quad (3.6)$$

where f is the free energy defined in (2.1).

One may view (3.6) as a *glimpse of scale invariance*, which combines with the rotational invariance of (3.5) to provide a large set of symmetries for the scaling limit.

The ingredients coming into the proof of Theorem 3.5 are the explicit computation of the asymptotics of f computed via the Bethe ansatz (in particular its differentiability at 0) and percolation techniques that allow the comparison of σ and $f''(0)$. The latter are in spirit similar to those needed for regularity and mixing, but more sophisticated.

In the next sections we introduce the transfer matrix representation, then sketch the proof of each of the Theorems 3.3, 3.4 and 3.5. Before that, observe that these suffice to reach our goal.

Proof of Theorem 3.2. Theorem 3.4 extends the dichotomy of Theorem 3.3 to all multi-point correlations, and hence to the scaling limit. As a consequence of Theorem 3.5, point (b) of the dichotomy of is excluded. Thus, there exists a unique scaling limit, which is $\sigma \cdot \text{GFF}$, with σ given by (2.7). $\square$

3.3 The cylinder and its spectral measures

Consider sub-sequential limits $(\Psi_k)_k$ of the correlation functions $(\Phi_k^{(\delta)})_k$. We now explain the transfer matrix representation for these functions.

Fix an even integer L and consider the infinite cylinder

$$\text{Cyl}_L := \mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z}).$$

A six-vertex configuration on a torus $\mathbb{T}_{L,M}$ or on Cyl_L is called balanced if every vertical cut contains $L/2$ arrows pointing in each direction. Taking the length of a balanced torus to infinity produces a cylinder measure $\mathbb{P}_{\text{Cyl}_L}$. As $L \rightarrow \infty$, these measures converge locally to $\mathbb{P}_{\mathbb{Z}^2}$.

Let V_L be the transfer matrix in the balanced block and let Λ_L be its Perron–Frobenius eigenvalue (recall from Section 2.3 that V_L has the Perron–Frobenius property). Define the normalised transfer matrix

$$T_L := \frac{1}{\Lambda_L} V_L.$$

Let S_L denote the cyclic shift of the L horizontal arrows. The matrices T_L and S_L commute. Moreover, T_L is real symmetric and S_L is unitary (indeed $S_L^L = \text{id}$). They may therefore be simultaneously diagonalised in an orthonormal basis $(\psi_j)_{j \geq 0}$.

Due to the normalisation, the eigenvalues of T_L are contained in $(-1, 1]$ (with 1 being an eigenvalue of multiplicity one), while those of S_L are on the unit circle. Write their joint eigenvalues as

$$1 - a_j \quad \text{and} \quad e^{-ib_j},$$

with $a_j \in [0, 2)$ and $b_j \in \frac{2\pi}{L}\mathbb{Z}$ and $a_0 = b_0 = 0$

The two-point function. Let U_L be the operator² that returns the height increment between the face $f = (0, 0), (0, 1), (1, 1), (0, 0)$ and the one to its right $f + e_1$. Then, if $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two centres of faces with $v_1 > u_1$, we find

$$\mathbb{E}_{\mathbb{T}_{L,M}}[(h(u + e_1) - h(u))(h(v + e_1) - h(v)) \mid \text{balanced}] = \frac{\text{Tr}(T_L^{M-d-1} S_L^{-y} U_L S_L^y T_L^{d-1} U_L)}{\text{Tr}(T_L^M)}.$$

Passing to the infinite cylinder and writing the trace in the orthonormal basis $(\psi_j)_j$, we find

$$\begin{aligned} \mathbb{E}_{\text{Cyl}_L}[(h(u + e_1) - h(u))(h(v + e_1) - h(v))] \\ = - \sum_{j \geq 0} (1 - a_j)^{v_1 - u_1 - 1} e^{-i(v_2 - u_2)b_j} |\langle \psi_j, U_L \psi_0 \rangle|^2. \end{aligned} \quad (3.7)$$

It is more convenient to encode the above using the spectral measure on $\mathbb{R}_+ \times \mathbb{R}$ defined as

$$\mu_L = \sum_{j > 0} \frac{|\langle \psi_j, U_L \psi_0 \rangle|^2}{a_j^2} \delta_{(a_j, b_j)}.$$

Summing elementary increments, and extending to non-horizontal increments by additivity and cylinder mixing, gives

$$\begin{aligned} \mathbb{E}_{\text{Cyl}_L}[(h(u'_1) - h(u_1))(h(u'_2) - h(u_2))] \\ = \int ((1 - a)^{x_2} e^{-iby_2} - 1)(1 - a)^{x'_1} e^{-iby'_1} (1 - (1 - a)^{x_1} e^{-iby_1}) d\mu_L(a, b), \end{aligned}$$

where $x_1, x'_1, x_2 \geq 0$ and $y_1, y'_1, y_2 \in \mathbb{R}$ are the successive horizontal and vertical displacements defined by

$$(x_1, y_1) = u'_1 - u_1, \quad (x'_1, y'_1) = u_2 - u'_1, \quad (x_2, y_2) = u'_2 - u_2. \quad (3.8)$$

After sending L to infinity and then $\delta \rightarrow 0$, one obtains a positive measure μ on $\mathbb{R}_{>0} \times \mathbb{R}$, invariant under $(a, b) \mapsto (a, -b)$, such that the following holds.

Proposition 3.6 (Two-point function representation). *For horizontally ordered points $\mathbf{u} = (u_1, u'_1, u_2, u'_2)$,*

$$\begin{aligned} \Psi_2(\mathbf{u}) &= \int (e^{-ax_2 - iby_2} - 1) e^{-ax'_1 - iby'_1} (1 - e^{-ax_1 - iby_1}) d\mu(a, b). \\ &= \mathbf{I}(u_2 - u_1) - \mathbf{I}(u_2 - u'_1) - \mathbf{I}(u'_2 - u_1) + \mathbf{I}(u'_2 - u'_1), \end{aligned} \quad (3.9)$$

where $(x_1, y_1), (x'_1, y'_1), (x_2, y_2)$ are given by (3.8) and³

$$\mathbf{I}(x, y) = \int (e^{-ax - iby} - e^{-a}) d\mu(a, b). \quad (3.10)$$

Let us indicate why the passage to a limiting measure is possible. The regularity estimates imply the existence of constants $C, c > 0$ such that, uniformly in L ,

$$\begin{aligned} \mu_L\{a \in [\alpha, 2\alpha]\} &\leq C, \\ \mu_L\{0 < a \leq \alpha, \beta \leq |b| \leq 2\beta\} &\leq C(\alpha/\beta)^c, \quad \beta \geq \alpha > 0. \end{aligned} \quad (3.11)$$

Together they define a compact space of locally finite measures for an appropriate notion of weak convergence. The same bounds apply to the limiting measure μ . Also keep in mind that

$$\mu \text{ is a positive measure on } (0, \infty) \times \mathbb{R}, \text{ invariant under } (a, b) \mapsto (a, -b). \quad (3.12)$$

²This is actually a skewed transfer matrix: it has the same entries in absolute value as T_L , but with a sign dictated by the direction of a specific vertical arrow.

³We choose to define $\mathbf{I}$ using (3.10) rather than $\int e^{-ax - iby} d\mu(a, b)$ as the former is integrable while the latter is not.

More general observables. Let X and Y be local observables of the six-vertex gradient (i.e. its arrow configuration) supported on opposite sides of a vertical line of Cyl_L . Also, write τ_n for the horizontal shift by n units. A similar analysis as for the two point function, but where we insert more general operator instead of U_L , shows the following.

Proposition 3.7 (General observable representation). *For all $L \geq 2$ even, X and Y as above and $n \geq 0$,*

$$\mathbb{E}_{\text{Cyl}_L}[X \tau_n Y] = \int (1-a)^n d\mu_{X,Y,L}(a,b), \quad (3.13)$$

where $\mu_{X,Y,L}$ is a complex-valued measure on $\mathbb{R}_+ \times \mathbb{R}$ which depends on X , Y and L .

The key difference with (3.7), is that the operator U_L is replaced with operators associated to X and Y . These do not combine to yield positive coefficients as in (3.7), hence the complex-valued measure. Controlling the total variation of this measure is key, especially when taking limits. We do this below.

Let $\dagger$ be reflection in the vertical line separating the supports of X and Y , acting on height functions rather than on arrows. Write $X^\dagger$ for the observable $h \mapsto X(h^\dagger)$.

Proposition 3.8 (Reflection positivity). *For any local observables X and Y as above, the spectral measure $\mu_{X,X^\dagger,L}$ is positive. Moreover,*

$$\|\mu_{X,Y,L}\|^2 \leq \|\mu_{X,X^\dagger,L}\| \|\mu_{Y^\dagger,Y,L}\| = \mathbb{E}_{\text{Cyl}_L}[X X^\dagger] \mathbb{E}_{\text{Cyl}_L}[Y^\dagger Y]. \quad (3.14)$$

That $\mu_{X,X^\dagger,L}$ is a positive measure comes from the fact that the operator giving X is adjoint to that giving $X^\dagger$. The inequality above is an instance of Cauchy–Schwarz and the equality is provided by (3.25).

Although elementary, (3.14) is essential: one may control the total variation of $\mu_{X,Y,L}$ by controlling the expectations on the right.

3.4 The two-point dichotomy: Theorem 3.3

We start by explaining what is expected of μ . Formally applying the Laplacian in one endpoint multiplies the integrand in (3.9) by

$$a^2 + (ib)^2 = a^2 - b^2.$$

Thus a sufficient condition for Ψ_2 to be harmonic in each variable on $\mathcal{D}_2$ is

$$\text{supp } \mu \subset \{(a,b) : a^2 = b^2\}. \quad (3.15)$$

This is also essentially necessary. Moreover, if (3.15) holds and the first marginal of μ is

$$\frac{\sigma^2}{2\pi a} da, \quad (3.16)$$

then (3.9) is precisely $\sigma^2 \Psi_2^{\text{GFF}}$. Indeed, the identity

$$-\log r = \int_0^\infty \frac{e^{-ar} - e^{-a}}{a} da$$

is the Laplace representation of the logarithmic Green kernel (see Exercise 29).

We now turn to an attempt to prove (3.15), which will ultimately lead to Theorem 3.3. To begin, observe that the rotational invariance (3.5) together with (3.9) yields a strong constraint on μ in addition to (3.11) and (3.12), namely

$$\mathbf{I}(x, y) = \mathbf{I}(\sqrt{x^2 + y^2}, 0) \quad \text{for all } x > 0 \text{ and } y \in \mathbb{R}. \quad (3.17)$$

Thus (3.9) becomes

$$\Psi_2(\mathbf{u}) = \mathbf{I}(|u_2 - u_1|) + \mathbf{I}(|u'_2 - u'_1|) - \mathbf{I}(|u_2 - u'_1|) - \mathbf{I}(|u'_2 - u_1|),$$

where we abbreviate $\mathbf{I}(x) = \mathbf{I}(x, 0)$. Observe the similarity with (3.2).

The conditions above constrain μ significantly. We will not prove the result below, but mention that it comes from extending (3.17) to complex values x by analyticity.

Theorem 3.9 (One-sided spectral concentration). *Any measure μ satisfying (3.11), (3.12) and (3.17) also satisfies*

$$\mu\{(a, b) : a^2 < b^2\} = 0. \quad (3.18)$$

It is tempting to expect the rotational invariance (3.17) to also give the opposite inequality, and hence (3.15). This is false: there exist measures satisfying the regularity bounds (3.11) and (3.12) and the rotational invariance requirement (3.17) whose spectral measure has mass in $a^2 > b^2$ — see Exercise 30. The remaining argument uses scale in a subtler fashion.

Proof of Theorem 3.3. We start with a general consideration about sub-sequential limits. Consider a sub-sequential limit Ψ_2 of $\Phi_2^{(\delta)}$ as $\delta \searrow 0$. By further extracting sub-sequences, assume that μ_L converges, in the $L \rightarrow \infty$, then $\delta \searrow 0$ limit to some measure μ . Then, by Theorem 3.9, μ satisfies (3.18).

Introduce the scale-dependent stiffness

$$\Xi(x) := -x\mathbf{I}'(x). \quad (3.19)$$

Using (3.17), and (3.18) we find

$$\Xi'(x) = -x \Delta[\mathbf{I}(|u|)]_{u=(x,0)} = -x \int (a^2 - b^2) e^{-ax} d\mu(a, b) \leq 0. \quad (3.20)$$

Thus Ξ is bounded and non-increasing.

Moreover, if Ξ is constant, μ is concentrated on $a^2 = b^2$. In that case (3.17) also fixes the first marginal to be (3.16), and Ψ_2 is a GFF two-point function with $\sigma^2 = 2\pi \cdot \Xi$. See Exercise 29.

We now turn to the actual dichotomy. First, if every sub-sequential two-point limit is a GFF kernel, either all amplitudes coincide, in which case compactness gives (a), or two amplitudes differ, giving (b) immediately.

It remains to consider the case where a non-GFF sub-sequential two-point limit exists. Its representing measure μ has non-constant Ξ . The endpoint values of Ξ define two distinct amplitudes:

$$\frac{\sigma_{\text{small}}^2}{2\pi} := \lim_{x \searrow 0} \Xi(x), \quad \frac{\sigma_{\text{large}}^2}{2\pi} := \lim_{x \nearrow \infty} \Xi(x).$$

Compactness of the measures and (3.20) show that zooming in and zooming out on Ψ_2 give two GFF two-point functions with precisely these amplitudes:

$$\begin{aligned} \Psi_2^{\text{small}}(\mathbf{u}) &:= \lim_{x \searrow 0} \Psi_2(x \cdot \mathbf{u}) = \sigma_{\text{small}}^2 \Psi_2^{\text{GFF}}(\mathbf{u}) \quad \text{and} \\ \Psi_2^{\text{large}}(\mathbf{u}) &:= \lim_{x \nearrow \infty} \Psi_2(x \cdot \mathbf{u}) = \sigma_{\text{large}}^2 \Psi_2^{\text{GFF}}(\mathbf{u}). \end{aligned}$$

A diagonal choice of the mesh sizes δ_n and of the zoom factors x turns these two “limits of limits” into two genuine sub-sequential limits of the original model. Thus (b) occurs. $\square$

3.5 From two-point to multi-point correlations: Theorem 3.4

Suppose that a sequence $\delta_n \searrow 0$ has been chosen so that

$$\Phi_2^{(\delta_n)} \longrightarrow \sigma^2 \Psi_2^{\text{GFF}}. \quad (3.21)$$

Due to compactness and a diagonal argument, we may assume that all multi-point correlations $\Phi_k^{(\delta_n)}$ converge uniformly on compacts of $\mathcal{D}_k$ to limits Ψ_k .

Our first goal is to prove that all Ψ_k are harmonic in each coordinate on $\mathcal{D}_k$. We first treat a well ordered configuration.

Proposition 3.10. *Let $\mathbf{u} \in \mathcal{D}_k$ be such that u_1 is strictly to the left of a vertical line, while $u_2, u'_2, \dots, u'_k$ lie to its right. (The location of u'_1 is irrelevant by additivity.) The function Ψ_k is harmonic in u_1 at $\mathbf{u}$.*

Observe that, since Ψ_k is invariant under rotations, it follows that, at any point $\mathbf{u} \in \mathcal{D}_k$, Ψ_k is harmonic in any coordinate that is strictly outside of the convex hull of the others.

Proof. We use the mean-value defect criterion for harmonicity. Let

$$\Upsilon_r(\mathbf{u}) := \int_{C_r} \left(\Psi_k(\mathbf{u}) - \Psi_k(s, u'_1, u_2, u'_2, \dots, u_k, u'_k) \right) ds,$$

where ds denotes the uniform probability measure on a small circle $C_r(u_1)$ of radius r centred at u_1 . We assume here that r is sufficiently small for $C_r(u_1)$ to also be contained left of the line separating u_1 for the other points.

At mesh δ_n , introduce the two observables

$$\begin{aligned} X_n &:= \int_{C_r} (h(u_1/\delta_n) - h(s/\delta_n)) ds, \\ Y_n &:= \prod_{i=2}^k (h(u'_i/\delta_n) - h(u_i/\delta_n)). \end{aligned}$$

They lie on opposite sides of the vertical line. Proposition 3.8 then gives

$$|\mathbb{E}[X_n Y_n]|^2 \leq \mathbb{E}[X_n X_n^\dagger] \mathbb{E}[Y_n^\dagger Y_n]. \quad (3.22)$$

The second factor is uniformly bounded by the regularity estimates. The first factor tends to zero as $\delta_n \rightarrow 0$: it is a double mean-value defect of $\Psi_2 = \sigma^2 \Psi_2^{\text{GFF}}$. Passing to the limit in (3.22) yields $\Upsilon_r = 0$. This being true for any $r > 0$ small enough, we conclude the desired harmonicity. $\square$

We next turn to total harmonicity.

Proposition 3.11. *The function Ψ_k is harmonic in each coordinate on $\mathcal{D}_k$.*

Proof. Due to additivity (3.3), it suffices to treat points $\mathbf{u}$ for which all $|u_i - u'_i|$ are much smaller than all $|u_i - u_j|$ with $i \neq j$. Fix such a set of points $\mathbf{u}$. Let $\mathcal{L}$ be a line separating certain pairs, say $u_1, \dots, u'_j$ from all other pairs $u_{j+1}, \dots, u'_k$. Due to rotational invariance (3.5), we may assume $\mathcal{L}$ to be vertical.

For $\lambda \in \mathbb{R}$, write $\tau_\lambda(\mathbf{u})$ for the set of points obtained from $\mathbf{u}$ by translating $u_{j+1}, \dots, u'_k$ horizontally by λ . Recall from the previous proof that

$$\Upsilon_r(\mathbf{u}) := \int_{C_r} \left(\Psi_k(\mathbf{u}) - \Psi_k(s, u'_1, u_2, u'_2, \dots, u_k, u'_k) \right) ds.$$

Applying (3.25) and (3.14), we conclude that, for r sufficiently small

$$\Upsilon_r(\tau_\lambda(\mathbf{u})) = \int_{[0,\infty)} e^{-a\lambda} d\nu(a), \quad \lambda \geq 0,$$

where ν is a finite complex measure. It follows that $\lambda \mapsto \Upsilon_r(\tau_\lambda(\mathbf{u}))$ is analytic in the real, positive parameter λ . Thus

$$\Upsilon_r(\tau_\lambda(\mathbf{u})) = 0 \text{ for } \lambda > 0 \text{ in some open interval} \implies \Upsilon_r(\mathbf{u}) = 0. \quad (3.23)$$

Applying repeatedly pushings τ_λ and rotations to $\mathbf{u}$, we may modify it into a set of points $\tilde{\mathbf{u}}$ for which $\tilde{u}_1$ is outside of the convex hull of the other points. By Proposition 3.10, Υ_r is null in a neighbourhood of $\tilde{\mathbf{u}}$. Working back using (3.23) and (3.5), we conclude that $\Upsilon_r(\mathbf{u}) = 0$. $\square$

We are finally ready to conclude the proof of Theorem 3.4.

Proof of Theorem 3.4. We prove by induction that $\Psi_k = \sigma^k \Psi_k^{\text{GFF}}$ for all $k \geq 2$ even. Recall that both terms are null for k odd. The case $k = 2$ is our assumption (3.21).

Assume now that the result is valid for $k - 2 \geq 2$. Let us prove the result for k . The function Ψ_k is harmonic on $\mathcal{D}_k$ due to Proposition 3.11. Furthermore, due to the induction hypothesis, regularity and mixing (3.4), Ψ_k has the same asymptotics as $\sigma^k \Psi_k^{\text{GFF}}$ at the boundaries of $\mathcal{D}_k$. Thus Liouville's theorem then shows that $\Psi_k = \sigma^k \Psi_k^{\text{GFF}}$, which concludes the induction. $\square$

3.6 Surface tension and identification of the variance: Theorem 3.5

Assume that a scaling limit of the six-vertex height function along a sequence $\delta_n \searrow 0$ is $\sigma \cdot \text{GFF}$. Our goal here is to connect $f''(0)$ to σ . Recall that, in our range $c \in [1, 2]$, Theorem 2.5 implies that f is twice differentiable at 0, with $f''(0)$ determined by (2.7).

Here is a very rough heuristic. On a box of side length N , imposing an average slope α for the six-vertex model is expected to have a “cost”

$$\exp(N^2(f(\alpha) - f(0)) + o(N^2)) \simeq \exp\left(\frac{1}{2}N^2 f''(0)\alpha^2\right).$$

For a GFF of amplitude σ , the formal exponential cost of the linear profile of slope α is

$$\exp\left(-\frac{1}{2\sigma^2}N^2\alpha^2\right).$$

Matching the quadratic costs suggests $f''(0) = -1/\sigma^2$.

This argument cannot be used literally for several reasons.

- The free energy refers to a ratio of partition functions rather than the probability of an event. We solve this by relating $Z_{6V}^{(\alpha L/2)}(\mathbb{T}_{L,M})/Z_{6V}(\mathbb{T}_{L,M})$ to the probability under the balanced measure of having two horizontally winding paths on $\mathbb{T}_{L,M}$ of height difference at least $\alpha L/2$. We call this a “global ridge event”.
- The convergence to the GFF is available for the full-plane measure, while the events above are on tori. More importantly, the event above is expressed in terms of point-wise heights, while the convergence to the GFF is for events expressed in terms of the mollified height field.

- The order in which the limits are taken is different for the two interpretations. When relating the (rate of decrease) of the probability of the global ridge event to $f''(0)$, we first take $M \rightarrow \infty$, then $L \rightarrow \infty$ then $\alpha \rightarrow 0$. When working with convergence to the GFF, we consider events of non-vanishing probability and take the mesh size to 0.

We will now explain briefly how to handle some of these issues without going into details. This is the most technical part of [DCKLM26], though not the most unexpected.

The idea is to write the probability of the global ridge event in the first point above as a product of $O(LM\alpha^2)$ local ridge event probabilities, each of which is small but non-vanishing. Taking M and L to infinity while keeping α fixed relates the probability of local ridge events to $e^{C(f(\alpha)-f(0))/\alpha^2} \simeq \exp(-\frac{C}{2}f''(0))$, where $C > 0$ is a constant determined by the exact definition of local ridge events.

The local ridge events are at a scale $1/\alpha$, and may therefore be written as unit events on a lattice renormalised by α . Thus, taking $\alpha \rightarrow 0$ (along a specific sequence) allows to relate the probability of ridge events to similar events in the GFF.

Finally, issues about the point-wise vs. mollified fields are dealt with by taking the height of the ridges in the local ridge event sufficiently large. In such cases, we may prove that the local ridge events for the point-wise height field are close in probability to full-plane versions of these events for the mollified field.

3.7 Open questions and related results

Several natural questions are raised by this result.

Anisotropic weights. For $a \neq b$ the six vertex model is called anisotropic. It was proved in [DCKLM26] that, if embedded properly, the GFF convergence of Theorem 3.2 extends to this case, provided that $\Delta \in [-1, -\frac{1}{2}]$, with the variance depending only on Δ . This is obtained via the universality of the corresponding FK-percolation [DCKK⁺20] and the M-formula (2.15). We expect a direct proof to also be available (see also “Parameter limitations” below).

Parameter limitations. Theorem 3.2 is only valid for $\Delta \in [-1, -\frac{1}{2}]$ (or $c \in [\sqrt{3}, 2]$ for the isotropic case). For $\Delta < -1$ (i.e. $c > 2$), the height function of the six-vertex model is known to be localised [GP23], and the GFF convergence is therefore false. However, GFF convergence is expected for all $\Delta \in [-1, 1]$ (i.e. $c \in (0, 2]$). We identify two distinct limitations to our argument:

- $c \geq 1$: In the regime $c \geq 1$, the six-vertex model has a percolation representation that satisfies the FKG inequality, which is crucial to our arguments (alternatively, the FKG inequality for $|h|$ could be used, which is also valid only when $c \geq 1$). FKG, or positive association, is essential for percolation arguments, which in turn imply the qualitative features of the model, such as mixing, regularity etc.

This limitation seems essential and we do not expect to overcome it without significant novel ideas.

- $c \geq \sqrt{3}$: This corresponds to the regime for which the six-vertex model is related to FK-percolation with $q \geq 1$. We only use this once in our argument: to deduce the rotational invariance of the six-vertex model from that of FK-percolation.

We believe this limitation to be artificial. Indeed, it is reasonable to expect that the rotational invariance property may be proved directly for the six-vertex model, along the same lines as for FK-percolation.

Convergence in domains. One expects domain versions of Theorem 3.2 with flat boundary conditions. Specifically, given a simply connected domain $\mathcal{D}$ and approximations $\mathcal{D}^\delta$ of $\mathcal{D}$ by unions of faces of $\delta\mathbb{Z}^2$, we expect a convergence of the type

$$\mathbb{P}_{\mathcal{D}^\delta}^{\xi_\delta} \longrightarrow \sigma \cdot \text{GFF}_{\mathcal{D}}^\zeta \quad \text{as } \delta \rightarrow 0, \quad (3.24)$$

where ξ_δ are boundary conditions (that is assignments of heights to the faces of $\delta\mathbb{Z}^2$ outside of $\mathcal{D}^\delta$) and ζ are some boundary conditions for the GFF representing the limit of ξ_δ . We write $\text{GFF}_{\mathcal{D}}^\zeta$ for the Gaussian Free Field in $\mathcal{D}$ with boundary conditions ζ . The convergence is in law for the topology used in Theorem 3.2. Here we discuss the situation where the boundary conditions are “flat”. For instance, we may require that $\max \xi_\delta - \min \xi_\delta$ is uniformly bounded.

Such results are significantly stronger than the full-plane GFF convergence of Theorem 3.2, and are likely to be crucial for the extension of the convergence to other models (see below).

A convergence of the type (3.24) is the object of ongoing work.

Six-vertex model with slope A different regime of study is that of the six-vertex model with a global slope. This may be defined on the torus (see definition of $f_{6V}(\alpha)$ in (2.1)) or in domains with sloped boundary conditions.

It is expected that the model with a non-zero, non-degenerate slope is delocalised for all values of Δ , with a GFF-type limit. Our methods do not apply to this setting.

Extension to other models. Given the BKW correspondence, it is natural to ask how Theorem 3.2 translates to the associated FK-percolation. As the BKW correspondence is not a bijection and contains complex weights, this is a highly non-trivial question.

The special case $q = 4$ is more tractable, as the level lines of the six-vertex model are exactly the loops of FK-percolation. In this case, the critical one and two-arm exponents, as well as the mixing exponent were identified from Theorem 3.2 in [CDCH⁺26a] using the M-formula (2.15). Furthermore, if the convergence of Theorem 3.2 is extended to domains with constant boundary conditions, then the full convergence of the FK-percolation interfaces to CLE_4 should follow readily. This is the object of current investigations.

For values of $1 \leq q < 4$, obtaining corresponding results seems substantially harder. At the time of writing, the one and two arm exponents for critical FK-percolation with $1 \leq q < 4$ are in the process of being derived [CDCH⁺26b, ADCH26], while the mixing exponent ι may only be derived in the vicinity of $q = 4$ [ADCHK26]. The mixing exponent and full scaling limit remain open in the rest of the range $1 \leq q < 4$.

3.8 Exercises: spectral representations and the GFF

Exercise 26. [The full-plane measure at $\Delta = -1$] Set $c = 2$, so that the corresponding FK-percolation parameter is $q = 4$. Recall that $\mathbb{T}(L, M)$ is the torus of side-lengths L and M . Write $\phi_{\mathbb{T}(L, M)}$ for the FK percolation measure and $\mathbb{P}_{\mathbb{T}(L, M)}$ for the six vertex measure on the torus.

- (a) Prove that, for any given N and $\varepsilon > 0$, for L and M large enough, the probability under $\phi_{\mathbb{T}(L, M)}$ that there exists a loop surrounding Λ_N is at least $1 - \varepsilon$.
- (b) Conclude that $\lim_{L, M \rightarrow \infty} \mathbb{P}_{\mathbb{T}(L, M)}$ is a well defined measure on full plane six-vertex configurations.
- (c) For $M \geq L$, use FK-percolation estimates and BKW correspondence to obtain $\mathbb{P}_{\mathbb{T}(L, M)}[\text{balanced}] \geq \exp(-CM/L)$ for some universal constant C .

- (d) Conclude that the limit $\lim_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \mathbb{P}_{\mathbb{T}(L,M)}[\cdot \mid \text{balanced}]$ exists and is equal to the full plane measure defined in point (b).

Exercise 27. [Regularity at $\Delta = -1$] Set $c = 2$, so that the corresponding FK-percolation parameter is $q = 4$.

- (a) Extend the BKW correspondence to the full plane measures $\phi_{\mathbb{Z}^2}, \mathbb{P}_{\mathbb{Z}^2}$.
- (b) Using the RSW for FK-percolation (1.5), show that for every even k , the family of multi-point correlations $(\Phi_k^{(\delta)})_{\delta > 0}$ is uniformly bounded on compacts of $\mathcal{D}_k$.
- (c) Using the RSW for FK-percolation (1.5), show that for every even k , the family of multi-point correlations $(\Phi_k^{(\delta)})_{\delta > 0}$ is equicontinuous on compacts of $\mathcal{D}_k$. A suitable interpolation between centres of faces is needed here.
- (d) Conclude that the families $(\Phi_k^{(\delta)})_{\delta > 0}$ are precompact for the uniform convergence on compacts of $\mathcal{D}_k$.
- (e) Fix $k \geq 4$ even and $\mathbf{u} = (u_1, \dots, u'_k) \in \mathcal{D}_k$. Below, fix all $u'_1, u_2, \dots, u'_k$, but regard u_1 as variable. Show that

$$\limsup_{u_1 \rightarrow u_2} \limsup_{\delta \rightarrow 0} \left| \Phi_k^{(\delta)}(\mathbf{u}) - \Phi_2^{(\delta)}(u_1, u'_1, u_2, u'_2) \Phi_{k-2}^{(\delta)}(u_3, \dots, u'_k) \right| < \infty.$$

Exercise 28. [Transfer matrix and reflection positivity] Let T, S the normalised transfer matrix and the shift matrix. Recall that T is a real symmetric matrix with spectral radius one, and S is a unitary matrix which commutes with T . Let ψ be the common Peron Frobenius eigenvector with $T\psi = S\psi = \psi$.

- (a) Simultaneously diagonalise T and S and derive (3.7).
- (b) Let X, Y be observables as in Proposition 3.7. Show that

$$\mathbb{E}_{\text{Cyl}_L}[X \tau_n Y] = \int (1-a)^n d\mu_{X,Y,L}(a, b), \quad (3.25)$$

for a complex measure on $\mathbb{R}_+ \times \mathbb{R}$ to be described.

- (c) Recall the definition of $X^\dagger$. Show that the spectral coefficients for $(X, X^\dagger)$ are non-negative, which is to say that $\mu_{X, X^\dagger, L}$ is a positive measure.
- (d) Prove the total-variation estimate (3.14).

Exercise 29. [The GFF spectral measure] All measures μ considered here are positive measures on $\mathbb{R}_+ \times \mathbb{R}$ satisfying (3.11) and (3.12).

- (a) Show that if μ is the measure supported on $b = \pm a$, with total first marginal $\frac{\sigma^2}{2\pi a} da$, then (3.9) yields $\Psi_2 = \sigma^2 \Psi_2^{\text{GFF}}$.
- (b) Show that if μ is a measure supported on $\{b^2 \leq a^2\}$, with the stiffness Ξ defined in (3.19) being constant, then it is supported on $\{a^2 = b^2\}$. If in addition μ satisfies (3.17), then its first density needs to be of the form $\frac{\sigma^2}{2\pi a} da$.
- (c) For general measure μ , check that multiplication of the spectral integrand by $a^2 - b^2$ corresponds to applying the Laplacian.

Exercise 30. Fix $m > 0$. Let μ_m be the measure on $\mathbb{R}_+ \times \mathbb{R}$ given by the image of the Lebesgue measure on $\mathbb{R}$ by the map $t \mapsto (m \cosh t, m \sinh t)$. Prove that μ satisfies (3.11), (3.12) and the rotational invariance requirement (3.17). Conclude that so do mixtures of measures μ_m .

 Write μ_0 for the measure on $\{a^2 = b^2\}$ with first marginal $\frac{1}{a} da$ (this may be seen as a limit of the above as $m \rightarrow 0$). Prove that any measure μ satisfying (3.11), (3.12) and (3.17) is a mixture of $(\mu_m)_{m \geq 0}$.

Exercise 31. [Wick correlations] Let Γ be a centred Gaussian process with covariance G .

- (a) Derive (3.2) for two increments.
- (b) Write the four-increment correlation as a sum of three products of two-increment correlations.
- (c) Prove that the GFF increment correlations are harmonic in each endpoint away from the other endpoints, satisfy additivity, and have the fusion singularities (3.4).

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