

The six-vertex height function converges to the Gaussian free field

Theorem: GFF convergence

For $c \in [\sqrt{3}, 2]$, the height function h of full plane six-vertex model converges to $\sigma \cdot GFF$ with

$$\sigma^2 = \frac{2}{\arccos \Delta} = \frac{1}{\arcsin(c/2)}.$$

Based on work w. H. Duminil-Copin, K. K. Kozłowski and P. Lammers

Rescaled multi-point correlations

$\mathbb{P}$ = the full-plane six-vertex

$$a = b = 1, \quad c \in [\sqrt{3}, 2], \quad \Delta = 1 - \frac{c^2}{2} \in [-1, -\tfrac{1}{2}].$$

Gradient height : $h \longrightarrow$ rescaled $h^{(\delta)}(u) = h(u/\delta)$.

Correlations of increments

$$\Phi_k^{(\delta)}(\mathbf{u}) = \mathbb{E} \left[\prod_{i=1}^k (h^{(\delta)}(u'_i) - h^{(\delta)}(u_i)) \right], \quad \mathbf{u} = (u_1, u'_1, \dots, u_k, u'_k).$$

$$\mathcal{D}_k = \left\{ \mathbf{u} \text{ s.t. the sets } \{u_i, u'_i\} \text{ are pairwise disjoint} \right\}$$

The full-plane GFF and its correlations

GFF: Γ = the centred Gaussian field with

$$\text{Cov}(\langle \Gamma, \varphi \rangle, \langle \Gamma, \psi \rangle) = \iint G(u, v) \varphi(u) \psi(v) \, du \, dv,$$

$$G(u, v) = G(|u - v|) = -\frac{1}{2\pi} \log |u - v|,$$

ϕ, ψ smooth, compactly supported test functions of integral zero.

GFF multi-point function

$$\Psi_k^{\text{GFF}}(\mathbf{u}) = \mathbb{E}^{\text{GFF}} \left[\prod_{i=1}^k (\Gamma(u'_i) - \Gamma(u_i)) \right],$$

harmonic in each coordinate on $\mathcal{D}_k$.

Main result

Theorem: GFF convergence

For $a = b = 1$, $c \in [\sqrt{3}, 2]$, set

$$\sigma^2 = \frac{2}{\arccos \Delta} = \frac{1}{\arcsin(c/2)}.$$

For every $k \geq 1$,

$$\Phi_k^{(\delta)} \longrightarrow \sigma^k \Psi_k^{\text{GFF}} \quad \text{uniformly on compacts of } \mathcal{D}_k.$$

Also $\langle h^{(\delta)}, \varphi \rangle \rightarrow \sigma \langle \Gamma, \varphi \rangle$, for test functions φ .

Properties of $\Phi_k^{(\delta)}$

Symmetry for pairs, antisymmetry $u_i \leftrightarrow u'_i$, additivity.

Regularity, mixing, compactness (via percolation)

$\Phi_k^{(\delta)}$ **uniformly bounded** and **equicontinuous** on compacts.

$$\Rightarrow \exists \delta_n \searrow 0 \text{ s.t. } \quad \Phi_k^{(\delta_n)} \longrightarrow \Psi_k \quad \text{for all } k.$$

For a limiting family Ψ_k , as $u_1 \rightarrow u_2$,

$$\Psi_k(\mathbf{u}) = \Psi_2(u_1, u'_1, u_2, u'_2) \Psi_{k-2}(u_3, u'_3, \dots, u_k, u'_k) + O(1).$$

Rotation invariance (via isoradial FK-percolation and Yang-Baxter)

If J isometry of $\mathbb{R}^2$

$$\Phi_k^{(\delta)}(J\mathbf{u}) - \Phi_k^{(\delta)}(\mathbf{u}) \longrightarrow 0, \quad \text{hence } \Psi_k(J\mathbf{u}) = \Psi_k(\mathbf{u}).$$

Theorem 1: two-point dichotomy (via transfer matrix and rotation)

Exactly one holds:

- $\Phi_2^{(\delta)} \rightarrow \sigma^2 \Psi_2^{\text{GFF}}$ for one $\sigma \geq 0$.
- Two subsequences converge to $\sigma^2 \Psi_2^{\text{GFF}}$ and $(\sigma')^2 \Psi_2^{\text{GFF}}$, with $\sigma, \sigma' \geq 0$ distinct.

Theorem 2: extension to multipoint (via transfer matrix)

$$\Phi_2^{(\delta_n)} \rightarrow \sigma^2 \Psi_2^{\text{GFF}} \quad \implies \quad \Phi_k^{(\delta_n)} \rightarrow \sigma^k \Psi_k^{\text{GFF}} \quad \text{for every } k.$$

Theorem 3: identification of σ (via percolation and Bethe ansatz)

If $\Phi_k^{(\delta_n)} \rightarrow \sigma^k \Psi_k^{\text{GFF}}$ for every k , then

$$\boxed{\sigma^2 = -\frac{1}{f''(0)} = \frac{2}{\arccos \Delta} .}$$

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The cylinder and the joint spectrum

Normalised transfer matrix on $\text{Cyl}_L = \mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z})$, balanced.

$$T_L = \frac{V^{(0)}}{\Lambda_0^{(0)}}, \quad S_L = \text{cyclic shift}, \quad T_L S_L = S_L T_L.$$

T_L is real symmetric, S_L is unitary. In a joint orthonormal basis,

$$T_L \psi_j = (1 - a_j) \psi_j, \quad S_L \psi_j = e^{-ib_j} \psi_j,$$

$$0 \leq a_j < 2, \quad b_j \in \frac{2\pi}{L} \mathbb{Z} \pmod{2\pi}, \quad a_0 = b_0 = 0.$$

Perron–Frobenius eigenvector: ψ_0 .

Two-point correlations

U_L = horizontal height increment observable; $v = u + (x, y)$.

$$\begin{aligned}\mathbb{E}_{\text{Cyl}_L}[(h(u + e_1) - h(u))(h(v + e_1) - h(v))] \\ = - \sum_{j>0} \langle \psi_0, U_L \psi_j \rangle (1 - a_j)^{x-1} e^{-ib_j y} \langle \psi_j, U_L \psi_0 \rangle\end{aligned}$$

Two increments correlation, cylinder

If $(x_1, y_1) = u'_1 - u_1$, $(x'_1, y'_1) = u_2 - u'_1$, $(x_2, y_2) = u'_2 - u_2$.

$$\Phi_2(\mathbf{u}) = \int ((1-a)^{x_2} e^{-ib y_2} - 1)(1-a)^{x'_1} e^{-ib y'_1} (1 - (1-a)^{x_1} e^{-ib y_1}) d\mu_L(a, b)$$

with $\mu_L = \sum_{j>0} \frac{|\langle \psi_0, U_L \psi_j \rangle|^2}{a_j^2} \delta_{(a_j, b_j)}$ positive measure on $\mathbb{R}_+ \times \mathbb{R}$

Two-point correlations: limit

If $(x_1, y_1) = u'_1 - u_1$, $(x'_1, y'_1) = u_2 - u'_1$, $(x_2, y_2) = u'_2 - u_2$.

Two increments correlation, cylinder

$$\Phi_2(\mathbf{u}) = \int ((1-a)^{x_2} e^{-ib y_2} - 1) (1-a)^{x'_1} e^{-ib y'_1} (1 - (1-a)^{x_1} e^{-ib y_1}) d\mu_L(a, b).$$

with $\mu_L = \sum_{j>0} \frac{|\langle \psi_0, U_L \psi_j \rangle|^2}{a_j^2} \delta_{(a_j, b_j)}$ positive measure on $\mathbb{R}_+ \times \mathbb{R}$

Two increments correlation, scaling limit

$$\Psi_2(\mathbf{u}) = \int (e^{-ax_2 - ib y_2} - 1) e^{-ax'_1 - ib y'_1} (1 - e^{-ax_1 - ib y_1}) d\mu(a, b).$$

with μ positive measure on $\mathbb{R}_+ \times \mathbb{R}$, symmetric in b , with bounds.

Two point function representation

$$\begin{aligned}\Psi_2(\mathbf{u}) &= \int (e^{-ax_2 - iby_2} - 1) e^{-ax'_1 - iby'_1} (1 - e^{-ax_1 - iby_1}) \, \mathrm{d}\mu(a, b) \\ &= \mathcal{I}(u_2 - u_1) + \mathcal{I}(u'_2 - u'_1) - \mathcal{I}(u_2 - u'_1) - \mathcal{I}(u'_2 - u_1).\end{aligned}$$

$$\text{where } \mathcal{I}(x, y) = \int (e^{-ax - iby} - e^{-a}) \, \mathrm{d}\mu(a, b).$$

$$\Psi_2^{\text{GFF}}(\mathbf{u}) = G(|u_2 - u_1|) + G(|u'_2 - u'_1|) - G(|u_2 - u'_1|) - G(|u'_2 - u_1|).$$

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$$\text{Rotation invariance} \quad \boxed{\mathcal{I}(x, y) = \mathcal{I}\left(\sqrt{x^2 + y^2}, 0\right) =: \mathcal{I}\left(\sqrt{x^2 + y^2}\right)}$$

Regularity:

$$\begin{aligned}\mu\{\alpha \leq a \leq 2\alpha\} &\leq C, \\ \mu\{0 < a \leq \alpha, \beta \leq |b| \leq 2\beta\} &\leq C(\alpha/\beta)^\eta, \quad C, \eta > 0.\end{aligned}$$

Applying Laplacian to $\mathcal{I}(x, y) = \int (e^{-ax-iby} - e^{-a}) \, \mathrm{d}\mu(a, b)$.

$$\Delta \mathcal{I}(x, y) = \int (a^2 - b^2) e^{-ax-iby} \, \mathrm{d}\mu(a, b).$$

$\mathcal{I}$ harmonic $\iff \operatorname{supp} \mu \subset \{a^2 = b^2\}$.

The GFF measure

If μ supported on $\{a^2 = b^2\}$, with first marginal $\frac{\sigma^2}{4\pi a} \, \mathrm{d}a$, then

$$\mathcal{I}(x, y) = -\frac{\sigma^2}{2\pi} \log \sqrt{x^2 + y^2} = \sigma^2 G(\sqrt{x^2 + y^2})$$

and $\Psi_2 = \sigma^2 \Psi_2^{\text{GFF}}$.

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One-sided spectral concentration

Positivity, the regularity bounds and rotational invariance give

$$\mu\{a^2 < b^2\} = 0.$$

Second concentration

Set $\Xi(r) = -r\mathcal{I}'(r) = r \int a e^{-ar} \, \mathrm{d}\mu(a, b)$. Then

$$\Xi'(r) = -r \int (a^2 - b^2) e^{-ar} \, \mathrm{d}\mu(a, b) \leq 0.$$

Moreover

- Ξ is positive and bounded.
- Ξ constant $\Rightarrow \operatorname{supp} \mu \subset \{a^2 = b^2\} \Rightarrow \Psi_2 = 2\pi\Xi \Psi_2^{\text{GFF}}$.
- If Ψ_2 not GFF $\Rightarrow \Xi$ is strictly decreasing.

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$$\Psi_2(r\mathbf{u}) \longrightarrow \sigma_*^2 \Psi_2^{\text{GFF}}(\mathbf{u}), \quad \text{as } r \rightarrow 0 \text{ or } \infty$$

$$\text{with } \sigma_0^2 = 2\pi \lim_{r \rightarrow 0} \Xi(r) \text{ and } \sigma_\infty^2 = 2\pi \lim_{r \rightarrow \infty} \Xi(r)$$

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$$\Phi_2^{(\delta_n)} \rightarrow \sigma^2 \Psi_2^{\text{GFF}} \quad \Longrightarrow \quad \Phi_k^{(\delta_n)} \rightarrow \sigma^k \Psi_k^{\text{GFF}} \quad \text{for every } k.$$

Main idea

Fix $\Psi_k = \lim_n \Phi_k^{(\delta_n)}$ for all k .

Fact:

If $\Psi_2 = \sigma^2 \Psi_2^{\text{GFF}}$ and each Ψ_k is harmonic, then

$$\Psi_k = \sigma^k \Psi_k^{\text{GFF}} \quad \text{for every } k.$$

Sketch: Induction on k :

If Ψ_k harmonic and has these blowups:

$$\Psi_k(\mathbf{u}) = \Psi_2(u_1, u'_1, u_2, u'_2) \Psi_{k-2}(u_3, u'_3, \dots, u_k, u'_k) + O(1).$$

$$\text{then } \Psi_k = \sigma^k \Psi_k^{\text{GFF}}.$$

□

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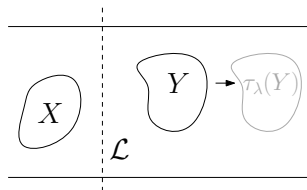
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$$\Psi_k(\mathbf{u}) = \sigma^k \Psi_2^{\text{GFF}}(u_1, u'_1, u_2, u'_2) \Psi_{k-2}^{\text{GFF}}(u_3, u'_3, \dots, u_k, u'_k) + O(1).$$

$$\text{then } \Psi_k = \sigma^k \Psi_k^{\text{GFF}}.$$

□

Towards harmonicity: spectral representation for general observables



Local observables X, Y on opposite sides of a vertical line $\mathcal{L}$

$$\mathbb{E}_{\text{Cyl}_L}[X \tau_n Y] = \int (1-a)^n d\mu_{X,Y,L}(a, b).$$

$\mu_{X,Y,L}$ complex measure on $\mathbb{R}_+ \times \mathbb{R}$.

Reflection positivity and Cauchy–Schwarz

The measure $\mu_{X,X^\dagger,L}$ is positive, and

$$\|\mu_{X,Y,L}\|^2 \leq \|\mu_{X,X^\dagger,L}\| \cdot \|\mu_{Y^\dagger,Y,L}\| \leq \mathbb{E}[XX^\dagger] \mathbb{E}[Y^\dagger Y].$$

$\dagger$ reflection w.r.t. $\mathcal{L}$ acting on height gradient.

Harmonicity for left-most point u_1

Laplacian:
$$\Delta_r \varphi(x) := \int_{C_r} (\varphi(x) - \varphi(s)) \, \mathrm{d}s$$

Fix $\mathbf{u}$ with u_1 to the left of $\mathcal{L}$, all others to the right. Set

$$X_n = \Delta_r h^{(\delta_n)}(u_1) \qquad \text{and} \qquad Y_n = \prod_{i=2}^k (h^{(\delta_n)}(u'_i) - h^{(\delta_n)}(u_i)).$$

Reflection positivity gives

$$|\mathbb{E}[X_n Y_n]|^2 \leq \underbrace{\mathbb{E}[X_n X_n^\dagger]}_{\rightarrow 0} \underbrace{\mathbb{E}[Y_n^\dagger Y_n]}_{\text{bounded}}.$$

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Conclusion

$\Delta_r \Psi_k(\mathbf{u}) = 0$ for all $r > 0$ small enough (Laplacian in u_1)

Full harmonicity

Fix $\mathbf{u}$ with $u_1, \dots, u'_j$ left of $\mathcal{L}$ and $u_{j+1}, \dots, u'_k$ right of $\mathcal{L}$.

$\tau_\lambda(\mathbf{u})$ is $\mathbf{u}$ with $u_{j+1}, \dots, u'_k$ translated perpendicularly to $\mathcal{L}$ by λ .

Proposition

For r small enough, $\lambda \mapsto \Delta_r \Psi_k(\tau_\lambda(\mathbf{u}))$ analytic in $\lambda > 0$.

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Idea: $X_n := \Delta_r \Phi_j^{(\delta_n)}(u_1, \dots, u'_j), \quad Y_n := \Phi_j^{(\delta_n)}(u_{j+1}, \dots, u'_k)$

$$\Delta_r \Phi^{(\delta_n)}(\tau_\lambda(\mathbf{u})) = \mathbb{E}[X_n \tau_{\lambda/\delta} Y_n] = \int (1-a)^{\lambda/\delta} d\mu_{X_n, Y_n}.$$

$$\|\mu_{X_n, Y_n}\|^2 \leq \underbrace{\mathbb{E}[X_n X_n^\dagger]}_{\text{bounded}} \underbrace{\mathbb{E}[Y_n^\dagger Y_n]}_{\text{bounded}}.$$

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Compactness $\Delta_r \Psi_k(\tau_\lambda(\mathbf{u})) = \int e^{-a\lambda} d\mu \quad \text{with } \|\mu\| < \infty. \square$

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Corollary

If $\Delta_r \Psi_k(\tau_\lambda(\mathbf{u})) = 0$ for an interval of $\lambda > 0$, then $\Delta_r \Psi_k(\mathbf{u}) = 0$

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Proof of full harmonicity:

$$\mathbf{u} \xrightarrow{\text{rotations} + \text{analytic push}} \tilde{\mathbf{u}} \dots \text{ with } u_1 \text{ left of others.}$$

Then $\Delta_r \Psi_k(\tilde{\mathbf{u}}) = 0$ by previous slide

Then $\Delta_r \Psi_k(\mathbf{u}) = 0$ by analyticity and rotational invariance

□

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Heuristic: matching the quadratic costs

A slope α in a box of side N has free-energy cost

$$\exp(N^2[f(\alpha) - f(0)]) \simeq \exp\left(\frac{1}{2}N^2 f''(0)\alpha^2\right).$$

For $\sigma\Gamma$, the Dirichlet-energy cost of the same linear profile is

$$\exp\left(-\frac{1}{2\sigma^2} \int |\nabla h|^2\right) = \exp\left(-\frac{1}{2\sigma^2} N^2 \alpha^2\right).$$

Matching the coefficients suggests

$$\boxed{\sigma^2 = -\frac{1}{f''(0)}}.$$

Proof outline for the variance relation.

Problems

- Torus partition function v. full plane probabilities.
- Large deviation events, no convergence to GFF (order of limits $M \rightarrow \infty, L \rightarrow \infty, \alpha \rightarrow 0$).
- Point-wise events v. mollified convergence.

Solutions:

- Compare partition-function ratio to global ridge probability.
- Decompose global ridge probability into product of many local ridge probabilities (of fixed height k):
 $O(LM\alpha^2)$ events at scale $1/\alpha$.
- Compare local ridge probabilities to mollified local ridge probabilities; then take $k \rightarrow \infty$ and $\alpha \rightarrow 0$.