The Wulff crystal of self-dual FK-percolation becomes round when approaching criticality

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Abstract

The study of the phase transition in planar FK-percolation on the square lattice has seen significant recent breakthroughs. The model undergoes a change in the nature of its phase transition at $q = 4$, transitioning from a continuous to a discontinuous regime. The aim of this article is to investigate the behaviour of the model in the discontinuous regime as $q > 4$ approaches the continuous transition point 4 from above, while maintaining the critical parameter $p = p_c(q)$. We prove that in this limit, the correlation length becomes isotropic. The core of the proof builds upon the recently established rotational invariance of the large-scale features of the model at $q = 4$ [DCKK$^+$20].

Contents

1 Introduction

1.1 Definition of the model and main result

The *random-cluster model*, introduced by Fortuin and Kasteleyn and hence also referred to as *Fortuin–Kasteleyn (FK) percolation*, is a central model of statistical mechanics. We consider it here on the two-dimensional square lattice $\mathbb{Z}^2$, where it exhibits a first-order or higher-order phase transition, depending on the cluster-weight q. We recall its definition and a few basic properties below. For more background, we refer the reader to the monograph [Gri06] and, for an exposition of more recent results, to the lecture notes [Man25].

We consider the square lattice $(\mathbb{Z}^2, \mathbb{E})$, that is, the graph with vertex set $\mathbb{Z}^2$ and edges between nearest neighbours. We will slightly abuse notation and denote the graph itself by $\mathbb{Z}^2$. Let $G = (V, E)$ be a subgraph of the square lattice. A percolation configuration ω on G is an element of $\{0,1\}^E$. An edge $e \in E$ is said to be *open* in ω if $\omega_e = 1$ and *closed* otherwise. A configuration ω can be seen as a subgraph of G with vertex set V and edge set $\{e \in E : \omega_e = 1\}$. When speaking of *connections* in ω, we view ω as a graph. A *cluster* is a maximal connected component of ω (it may be an isolated vertex). When G is finite, let $o(\omega)$ and $c(\omega)$ denote the number of open and closed edges in ω, respectively. Furthermore, let $k(\omega)$ denote the number of clusters in ω. Then the random-cluster measure on G with parameters $p \in [0,1]$ and $q > 0$, and free boundary conditions is a measure on percolation configurations given by

$$\phi^0_{G,p,q}[\omega] = \frac{1}{Z^0_{G,p,q}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)},$$

where $Z^0_{G,p,q}$ is a normalising constant called the *partition function*.

For $q \geq 1$, the model can be extended to infinite volume where it exhibits a phase transition. More precisely, the family of measures $\phi^0_{G,p,q}$ converges weakly as G tends to $\mathbb{Z}^2$. The limiting measure, denoted by $\phi^0_{\mathbb{Z}^2,p,q}$, undergoes a phase transition in the sense that there exists a critical threshold $p_c := p_c(q) \in (0,1)$ such that the probability that there exists an infinite cluster under $\phi^0_{\mathbb{Z}^2,p,q}$ is 0 for all $p < p_c$ and 1 if $p > p_c$. It has been shown [BDC12] that the critical threshold is $p_c = \sqrt{q}/(1+\sqrt{q})$.

The behaviour at the critical parameter p_c is of great interest. One important question is whether the phase transition is *continuous* or *discontinuous*, meaning here whether the probability that 0 is contained in an infinite cluster is continuous or discontinuous as a function of p. The phase transition was shown to be continuous for $1 \leq q \leq 4$ [DCST17] and discontinuous when $q > 4$ [DCGH+21]. Alternative proofs of these two regimes were obtained in [GL25] and [RS20], respectively.

The behaviour at criticality differs drastically between the two regimes. In the regime $1 \leq q \leq 4$, the measure at the point of phase transition is unique and exhibits properties indicative of scale invariance, such as the Russo–Seymour–Welsh property (RSW); it is expected to have a non-trivial, conformally invariant scaling limit. In contrast, for $q > 4$, there are multiple measures at the point of phase transition, with either sub- or super-critical behaviour, but no critical measure exists [GM23]. In particular, in the first case the correlation length of the model diverges when approaching the critical point, while in the second case it remains bounded.

It was recently proved in [DCKK+20] that for $1 \leq q \leq 4$, the model exhibits asymptotic rotational invariance at criticality, in line with its conjectured conformal invariance. More precisely, the model and its rotation by an arbitrary angle θ may be coupled so that, at large scales, they produce similar configurations with high probability. In particular, any subsequential scaling limit of the critical model is invariant under all rotations. This result will be instrumental in the proof of our main theorem.

In this paper, we will be particularly interested in the discontinuous phase transition, $q > 4$. For the exposition of our results, we need to introduce two central quantities. Fix $\theta \in [0, 2\pi)$ and define e_θ as the unit vector with angle θ to the horizontal axis. For $q \geq 1$ and $p \leq p_c$, the *correlation length* of the model in direction θ is defined as the limit

$$\xi_{p,q}(\theta) := \Big(\lim_{n\to\infty} -\tfrac{1}{n} \log \phi^0_{\mathbb{Z}^2,p,q}\big[0 \leftrightarrow \lfloor n e_\theta \rfloor\big]\Big)^{-1},$$

where $\lfloor ne_\theta \rfloor$ is the vertex of $\mathbb{Z}^2$ closest to $ne_\theta \in \mathbb{R}^2$. Similarly, we define the *point-to-hyperplane decay rate* by

$$\zeta_{p,q}(\theta) := \Big(\lim_{n\to\infty} -\tfrac{1}{n} \log \phi^0_{\mathbb{Z}^2,p,q}[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}] \Big)^{-1},$$

where $\mathcal{H}^\theta_{\geq n}$ is the half-space defined by

$$\mathcal{H}^\theta_{\geq n} := \{x \in \mathbb{R}^2 : \langle x, e_\theta\rangle \geq n\} \quad \text{for } n \geq 0.$$

The existence of both limits follows from standard sub-additivity arguments. The two quantities are related via a so-called *convex duality* (see e.g. [Ott22]). More precisely, it holds that

$$\big(\xi_{p,q}(\theta)\big)^{-1} = \sup_{\theta'\in[0,2\pi)} \big(\zeta_{p,q}(\theta')\big)^{-1} \langle e_\theta, e_{\theta'}\rangle.$$

A hallmark of the regime $q > 4$ is the finiteness of $\xi_{p_c,q}(\theta)$ and $\zeta_{p_c,q}(\theta)$ for all $\theta \in [0, 2\pi)$. It should be mentioned that $\xi_{p,q}(\theta)$ and $\zeta_{p,q}(\theta)$ are both finite for all $p < p_c$ and $q \geq 1$ [BDC12, DCRT19]. However, when $p = p_c$ and $q \in [1, 4]$, both quantities are infinite [DCST17].

The asymptotic rotational invariance of the model in the regime $1 \leq q \leq 4$ is a strong indication that $\xi_{p_c,q}$ and $\zeta_{p_c,q}$ become isotropic as $q \searrow 4$. Our main result confirms this. Note that while both $\xi_{p_c,q}(\theta)$ and $\zeta_{p_c,q}(\theta)$ diverge for all θ as $q \searrow 4$, our result states that these quantities become asymptotically isotropic when renormalised.

Theorem 1.1. *For all $\varepsilon > 0$, there exists $q_0 > 4$ such that for all $q \in (4, q_0]$ and any $\theta_1, \theta_2 \in [0, 2\pi)$, we have*

$$\left| \frac{\xi_{p_c,q}(\theta_1)}{\xi_{p_c,q}(\theta_2)} - 1 \right| < \varepsilon \quad \textit{and} \quad \left| \frac{\zeta_{p_c,q}(\theta_1)}{\zeta_{p_c,q}(\theta_2)} - 1 \right| < \varepsilon.$$

Theorem 1.1 may also be expressed in terms of the *Wulff shape*, the polar set of the inverse correlation length:

$$\mathcal{W}_q := \bigcap_{\theta\in[0,2\pi)} \{x \in \mathbb{R}^2 : \langle x, e_\theta\rangle \leq \xi_{p_c,q}(\theta)^{-1}\}.$$

Then Theorem 1.1 implies the following result.

Corollary 1.2. *When $q \searrow 4$, $\mathcal{W}_q/\sqrt{\mathrm{Vol}(\mathcal{W}_q)}$ tends to the unit disk $\mathbb{U} = \{x \in \mathbb{R}^2 : |x| \leq 1\}$.*

The Wulff shape describes the asymptotic shape of a cluster when conditioned to have a large volume. Indeed, write C for the cluster of the origin and denote its cardinality by $|\mathsf{C}|$. Then, for any $\varepsilon > 0$,

$$\phi_{\mathbb{Z}^2,p_c,q}\Big[\mathrm{d}_{\mathrm{Hausdorff}}\Big(\frac{1}{\sqrt{n}} \cdot \mathsf{C}, \frac{1}{\sqrt{\mathrm{Vol}(\mathcal{W}_q)}} \cdot \mathcal{W}_q \Big) > \varepsilon \,\Big|\, |\mathsf{C}| \geq n \Big] \to 0$$

as $n \to \infty$, where $\mathrm{d}_{\mathrm{Hausdorff}}$ denotes the Hausdorff distance [Cer06]. Thus, Corollary 1.2 states that, as $q \searrow 4$, the typical shape of a cluster conditioned to be large becomes round.

1.2 Strategy of the proof

Similar to the rotational invariance proved in [DCKK+20], our result is accompanied by a universality result relating random-cluster models on different isoradial graphs. To that end, we briefly describe an inhomogeneous FK-percolation model on some distorted embedding of the square lattice $\mathbb{Z}^2$. A more detailed discussion of this topic will be postponed until Section 2.

Fix $q > 4$ and $\alpha \in (0, \pi)$. Let $\mathbb{L}(\alpha)$ be the embedding of $\mathbb{Z}^2$ in which horizontal edges have length $2\cos(\alpha/2)$, vertical edges have length $2\sin(\alpha/2)$, and which is rotated by an angle of $\alpha/2$ — see Figure 1. Define the inhomogeneous random-cluster model on finite subgraphs of $\mathbb{L}(\alpha)$ with cluster weight q and edge-parameters p_{h} and p_{v} for the horizontal and vertical edges, respectively, given by

$$\frac{p_{\text{h}}}{1-p_{\text{h}}} = \frac{1-p_{\text{v}}}{p_{\text{v}}} = \sqrt{q}\frac{\sinh(r\alpha)}{\sinh(r(\pi-\alpha))} \quad \text{with} \quad r := \tfrac{1}{\pi}\cosh^{-1}\big(\tfrac{\sqrt{q}}{2}\big),$$

in the same way as the model on $\mathbb{Z}^2$; we also refer to (2.1) for a more general definition.

Write $\phi^0_{\mathbb{L}(\alpha),q}$ for the infinite-volume measure on $\mathbb{L}(\alpha)$ with the edge weights p_{h} and p_{v} as above and free boundary conditions. We call $\mathbb{L}(\alpha)$ an *isoradial rectangular lattice* and $\phi^0_{\mathbb{L}(\alpha),q}$ its associated random-cluster model.

Below, we will use estimates on lattices $\mathbb{L}(\alpha)$ for different values of $\alpha \in (0, \pi)$. We will use the phrase "uniform in α on compacts of $(0, \pi)$" to mean that estimates are uniform on any interval of the type $(\varepsilon, \pi - \varepsilon)$ with $\varepsilon > 0$, but potentially not on $(0, \pi)$. All constants hereafter will be uniform in α on compacts of $(0, \pi)$. The main results may be shown to be uniform over the whole interval $(0, \pi)$, but this requires more care and is not essential for our purposes; for further details, see [DCLM18, Sec. 5].

It was shown in [DCLM18] that, for any $q \geq 1$ fixed, the random-cluster models exhibit the same type of phase transition across all isoradial rectangular lattices $\mathbb{L}(\alpha)$. In particular, as for the critical random-cluster model on $\mathbb{Z}^2$, the phase transition is discontinuous for $q > 4$. Thus, for any direction $\theta \in [0, 2\pi)$ and any angle $\alpha \in (0, \pi)$, the correlation length and the point-to-hyperplane decay rate may be defined and are bounded, uniformly in $\theta \in [0, 2\pi)$ and α on compacts of $(0, \pi)$:

$$\begin{aligned} \xi_{\alpha,q}(\theta) &= \Big(\lim_{n\to\infty} -\tfrac{1}{n}\log\phi^0_{\mathbb{L}(\alpha),q}\big[0 \leftrightarrow \lfloor ne_\theta\rfloor\big]\Big)^{-1} < \infty \quad \text{and} \\ \zeta_{\alpha,q}(\theta) &= \Big(\lim_{n\to\infty} -\tfrac{1}{n}\log\phi^0_{\mathbb{L}(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big]\Big)^{-1} < \infty. \end{aligned} \tag{1.1}$$

Moreover, while the convex duality in [Ott22] is stated for the square lattice $\mathbb{Z}^2$, the same proof still applies to isoradial rectangular lattices $\mathbb{L}(\alpha)$, i.e., the relation

$$\big(\xi_{\alpha,q}(\theta)\big)^{-1} = \sup_{\theta'\in[0,2\pi)} \big(\zeta_{\alpha,q}(\theta')\big)^{-1}\langle e_\theta, e_{\theta'}\rangle. \tag{1.2}$$

still holds. Lastly, notice that for $\alpha = \frac{\pi}{2}$, one obtains a slight modification[1] of $\xi_{p_c,q}$ and $\zeta_{p_c,q}$ as previously defined.

The models $\phi^0_{\mathbb{L}(\alpha),q}$ for different values of α may be related by modifying the lattice step by step via the so-called *star-triangle transformation*. Features that are stable under these transformations can then be transferred from one measure to the other. This strategy was used successfully for FK-percolation in [GM14, DCLM18, DCKK+20] to transfer RSW estimates and prove universality of the models for $q \in [1, 4]$ and will be used here to obtain asymptotic universality for our quantities of interest.

[1] Indeed, $\mathbb{L}(\pi/2)$ is the rotation by $\pi/4$ of $\sqrt{2}\mathbb{Z}^2$ and the measure $\phi^0_{\mathbb{L}(\alpha),q}$ is the free measure with $p = p_c(q)$ on this graph.

Theorem 1.3. *For all* $\varepsilon > 0$*, there exists* $q_0 > 4$ *such that for* $q \in (4, q_0]$*, all* $\alpha \in (\varepsilon, \pi - \varepsilon)$ *and any* $\theta \in [0, 2\pi)$*, it holds that*

$$\left|\frac{\xi_{\alpha,q}(\theta)}{\xi_{\pi/2,q}(\theta)} - 1\right| < \varepsilon \quad \text{and} \quad \left|\frac{\zeta_{\alpha,q}(\theta)}{\zeta_{\pi/2,q}(\theta)} - 1\right| < \varepsilon.$$

We will focus on proving the statement for the point-to-hyperplane decay rate ζ. Indeed, this quantity is simpler to control under the local modifications we apply to the model. The analogous statement for the correlation length ξ will then be obtained via the convex duality (1.2).

We close this section by observing that Theorem 1.3 directly implies Theorem 1.1.

Proof of Theorem 1.1. We prove the statement for the correlation length ξ — the analogous result for ζ is obtained in the same way.

Fix $\varepsilon > 0$ and $0 < \varepsilon' < \frac{\varepsilon}{2+\varepsilon}$. Choose q_0 as in Theorem 1.3 according to ε' and let $q \in (4, q_0]$. Fix $\theta_1 \in (0, \frac{\pi}{4}]$ and $\theta_2 \in [\frac{\pi}{4}, \frac{\pi}{2}]$, then set $\alpha = \theta_1 + \theta_2 \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. Observe that $e^{i\alpha/2}\mathbb{R}$ is an axis of symmetry of $\mathbb{L}(\alpha)$ and therefore, $\phi^0_{\mathbb{L}(\alpha),q}$ is invariant under orthogonal reflections with respect to said axis. In particular, we have $\xi_{\alpha,q}(\theta_1) = \xi_{\alpha,q}(\theta_2)$. Theorem 1.3 then implies

$$\frac{\xi_{\pi/2,q}(\theta_1)}{\xi_{\pi/2,q}(\theta_2)} = \frac{\xi_{\pi/2,q}(\theta_1)/\xi_{\alpha,q}(\theta_1)}{\xi_{\pi/2,q}(\theta_2)/\xi_{\alpha,q}(\theta_2)} \in \left(\frac{1-\varepsilon'}{1+\varepsilon'}, \frac{1+\varepsilon'}{1-\varepsilon'}\right).$$

In particular,

$$\left|\frac{\xi_{\pi/2,q}(\theta_1)}{\xi_{\pi/2,q}(\theta_2)} - 1\right| < \frac{2\varepsilon'}{1-\varepsilon'} < \varepsilon.$$

The above extends to all angles $\theta_1, \theta_2 \in [0, 2\pi)$ using the invariance of $\phi^0_{\mathbb{L}(\pi/2),q}$ under reflections with respect to the horizontal, vertical, and diagonal axes. Finally, note that $\xi_{p_c,q}(\theta) = \frac{1}{\sqrt{2}}\xi_{\pi/2,q}(\theta + \frac{\pi}{4})$. □

1.3 Near-critical FK-percolation with $q \leq 4$

One may view the topic of the present paper in the context of near-critical FK-percolation on $\mathbb{Z}^2$. Most commonly, the near-critical regime refers to FK-percolation on $\mathbb{Z}^2$ with fixed $q \in [1, 4]$ and $p \neq p_c$, at scales where the model transitions from a critical to an off-critical behaviour. We direct the reader to [DCM22] for details and only mention here that the near-critical FK percolation with $1 \leq q \leq 4$ exhibits similar features as the critical one, most importantly (RSW).

It is expected that, as for the critical phase, the near-critical regime exhibits a form of asymptotic invariance under rotations. In particular, we expect that, for any $1 \leq q \leq 4$ and any two angles θ_1, θ_2,

$$\frac{\xi_{p,q}(\theta_1)}{\xi_{p,q}(\theta_2)} \to 1 \quad \text{and} \quad \frac{\zeta_{p,q}(\theta_1)}{\zeta_{p,q}(\theta_2)} \to 1 \qquad \text{as } p \nearrow p_c(q). \tag{1.3}$$

Theorem 1.1 may be viewed as a manifestation of (1.3) when the critical regime is approached along the line $q \searrow 4$ and $p = p_c(q)$. The authors have not managed to adapt the strategy below to also prove (1.3) and we believe a key ingredient is missing. Indeed, the *exact* validity of the star-triangle transformation (or its manifestation as track-exchanges; see Proposition 2.2 below) is essential to the present argument. These transformations are no longer exact when $p \neq p_c$, which we believe is a fundamental obstruction to adapting the argument.

It should be mentioned that [DC13] proved (1.3) for a variant of Bernoulli percolation (corresponding to $q = 1$) through different means. Indeed, this work builds on the existence and rotational invariance of the near-critical scaling limit [GPS18], which in turn relies on the description of the scaling limit of the critical phase [Smi01, CN06]. This approach has not yet been adapted to $q > 1$ and will likely require significant new ideas.

Organisation of the paper

Section 2 contains a review of some basic results for FK-percolation on isoradial graphs. In Section 3, we define half-plane measures and track-exchanges for such measures, as this framework is technically more convenient for our purposes. Finally, in Section 4, we prove the main result, namely Theorem 1.3.

2 Preliminaries

In this section, we introduce isoradial graphs and the random-cluster models associated to them.

2.1 Isoradial graphs

A rhombic tiling $\mathbb{G}^\diamond$ is a tiling of the plane by rhombi of edge-length 1. Any such graph is bipartite and we can divide its vertices in two sets of non-adjacent vertices $\mathbb{V}_\bullet$ and $\mathbb{V}_\circ$. The *isoradial graph* $\mathbb{G}$ associated to $\mathbb{G}^\diamond$ is the graph with vertex set $\mathbb{V}_\bullet$ and edge set given by the diagonals of the faces of $\mathbb{G}^\diamond$ between vertices of $\mathbb{V}_\bullet$. If the roles of $\mathbb{V}_\bullet$ and $\mathbb{V}_\circ$ are exchanged, we obtain the dual $\mathbb{G}^*$ of $\mathbb{G}$, which is also isoradial. The term isoradial refers to the fact that each face of $\mathbb{G}$ can be inscribed in a circle of radius 1. The rhombic tiling $\mathbb{G}^\diamond$ is called the *diamond graph* of $\mathbb{G}$.

A *track* of $\mathbb{G}^\diamond$ is a bi-infinite sequence of adjacent faces $(r_i)_{i\in\mathbb{Z}}$ of $\mathbb{G}^\diamond$, such that each pair r_i and r_{i+1} shares an edge, and all such edges are parallel. The angle formed by any such edge with the horizontal axis is called the *transverse angle* of the track.

The graphs considered in this paper are of a specific type: we assume that all faces of $\mathbb{G}^\diamond$ have horizontal top and bottom edges — we call these *isoradial rectangular lattices*. As a result, the diamond graphs consist of two families of tracks: *horizontal tracks* t_i with transverse angles $\alpha_i \in (0,\pi)$, and *vertical tracks* s_j, each with transverse angle 0. Each track of one family intersects all tracks of the other family, but no two tracks from the same family intersect.

For a sequence of track angles $\boldsymbol{\alpha} = (\alpha_i)_{i\in\mathbb{Z}} \in (0,\pi)^{\mathbb{Z}}$, denote by $\mathbb{L}(\boldsymbol{\alpha})$ the graph as above whose horizontal tracks have transverse angles α_i in increasing vertical order. When $\alpha_i = \alpha$ for every i, we simply write $\mathbb{L}(\alpha) = \mathbb{L}(\boldsymbol{\alpha})$. Note that $\mathbb{L}(\alpha)$ corresponds to a distorted embedding of the square lattice $\mathbb{Z}^2$, with $e^{i\alpha/2}\mathbb{R}$ and $e^{i(\alpha+\pi)/2}\mathbb{R}$ as axes of symmetry; see Figure 1 for an illustration.

We will use the same notation for finite or half-infinite sequences $\boldsymbol{\alpha}$ to indicate lattices $\mathbb{L}(\boldsymbol{\alpha})$ covering a horizontal strip or half-plane. For technical reasons (specifically for the results of [DCLM18] to adapt readily), we will always work with sequences $\boldsymbol{\alpha}$ containing at most two values. This restriction is not essential and may be removed by revisiting [DCLM18].

When considering isoradial graphs $\mathbb{G} = (\mathbb{V},\mathbb{E})$, we use the notation $\Lambda_n = [-n,n]^2$ and identify it with the subgraph spanned by the vertices of $\mathbb{V}$ contained in Λ_n. We write $\partial\Lambda_n$ for the set of vertices $v \in \mathbb{V}\cap\Lambda_n$ that have a neighbour in $\mathbb{V}\cap\Lambda_n^c$. Furthermore, we write $\Lambda_n(z)$ for the translation of Λ_n by $z \in \mathbb{R}^2$.

2.2 Definition and elementary properties of the isoradial random-cluster model

The isoradial embedding of a graph $\mathbb{G}$ produces different edge-weights — called *isoradial weights* — for the edges of $\mathbb{G}$ as a function of their length. Indeed if e is an edge of $\mathbb{G}$ and θ_e is the angle of the rhombus of $\mathbb{G}^\diamond$ containing e and not bisected by e, we set

$$p_e := \begin{cases} \frac{\sqrt{q}\sin(r(\pi-\theta_e))}{\sin(r\theta_e)+\sqrt{q}\sin(r(\pi-\theta_e))}, & q<4,\\ \frac{2\pi-2\theta_e}{2\pi-\theta_e}, & q=4,\\ \frac{\sqrt{q}\sinh(r(\pi-\theta_e))}{\sinh(r\theta_e)+\sqrt{q}\sinh(r(\pi-\theta_e))}, & q>4,\end{cases} \quad \text{where} \quad r := \begin{cases} \frac{1}{\pi}\cos^{-1}\left(\frac{\sqrt{q}}{2}\right), & q\le 4,\\ \frac{1}{\pi}\cosh^{-1}\left(\frac{\sqrt{q}}{2}\right), & q>4.\end{cases}$$

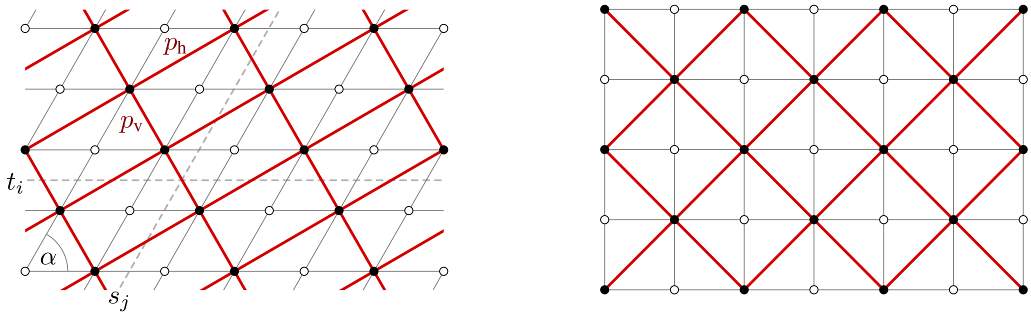

Figure 1: *Left:* An isoradial rectangular lattice $\mathbb{L}(\alpha)$. The underlying diamond graph is drawn in thin black lines, while the actual lattice is drawn in red. The white points denote vertices of the dual lattice. Two tracks — one horizontal and one vertical — are shown as dashed grey lines. Note the different parameters $p_{\rm v}$ and $p_{\rm h}$ for the two edge orientations. *Right:* For $\alpha = \pi/2$, the diamond graph is $\mathbb{Z}^2$ and the lattice $\mathbb{L}(\pi/2)$ is the rotation of $\sqrt{2}\mathbb{Z}^2$ by $\frac{\pi}{4}$. In this case, the model is homogenous, as $p_{\rm v} = p_{\rm h} = \sqrt{q}/(1+\sqrt{q})$.

For a finite subgraph $G = (V, E)$ of an isoradial graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, we define its vertex boundary ∂G as the set of vertices in V incident to a vertex in $\mathbb{V} \setminus V$. A *boundary condition* ξ on G is then given by a partition of the set ∂G. We say that two vertices of G are *wired* if they belong to the same element of the partition ξ. When no boundary vertices are wired together, we obtain *free* boundary conditions, which are denoted by 0.

Definition 2.1. For a finite subgraph $G = (V, E)$ of an isoradial graph $\mathbb{G}$, $q \geq 1$, and a boundary condition ξ, the *random-cluster measure* on G with cluster weight q and boundary conditions ξ is the measure $\phi^{\xi}_{G,q}$ on $\{0,1\}^E$ given by

$$\phi^{\xi}_{G,q}[\omega] = \frac{1}{Z^{\xi}_{G,q}} q^{k(\omega^{\xi})} \prod_{e \in E} p_e^{\omega_e}(1-p_e)^{1-\omega_e}, \tag{2.1}$$

where the p_e are the isoradial weights defined above, $k(\omega^{\xi})$ is the number of connected components in the graph ω^{ξ} which is obtained from ω by identifying vertices which are wired in ξ, and $Z^{\xi}_{G,q}$ is a normalising constant called the partition function.

We will mostly consider the random-cluster model on infinite isoradial graphs $\mathbb{G}$ with free boundary conditions obtained by taking the limit of the measures with free boundary conditions on larger and larger finite graphs G tending to $\mathbb{G}$. We write $\phi^0_{\mathbb{G},q}$ for the measure in infinite volume that is obtained this way. Notice that the measures $\phi^0_{\mathbb{L}(\alpha),q}$ defined in the introduction are indeed those obtained by the isoradial weights above.

Let us finally mention that the basic properties of the random-cluster model extend readily to this setting and we refer to [Gri06] for proofs in the homogenous setting. The following standard properties will be used repeatedly without mention.

Monotonic properties. Fix a finite subgraph $G = (V, E)$ of an isoradial graph $\mathbb{G}$, $q \geq 1$ and a boundary condition ξ. We say that an event A is *increasing* if for any $\omega \leq \omega'$[2], $\omega \in A$ implies $\omega' \in A$. The *FKG inequality* states that for any increasing events A and B,

$$\phi^{\xi}_{G,q}[A \cap B] \geq \phi^{\xi}_{G,q}[A]\phi^{\xi}_{G,q}[B]. \tag{FKG}$$

[2]Here, $\omega \leq \omega'$ refers the partial ordering on $\{0,1\}^E$ given by $\omega \leq \omega'$ if $\omega_e \leq \omega'_e$ for every edge $e \in E$.

For two boundary conditions ξ and ξ' such that $\xi \leq \xi'$, meaning that any pair of vertices wired in ξ is also wired in ξ', the random-cluster model with boundary conditions ξ is dominated by the one with boundary conditions ξ', i.e., for any increasing event A,

$$\phi^{\xi'}_{G,q}[A] \geq \phi^{\xi}_{G,q}[A]. \tag{CBC}$$

The latter inequality will be referred to as *comparison between boundary conditions*.

Spatial Markov property. For G as above, a subgraph H of G, $q \geq 1$ and a boundary condition ξ on G, we have

$$\phi^{\xi}_{G,q}[\omega \text{ on } H \,|\, \omega \text{ on } G \setminus H] = \phi^{\zeta}_{H,q}[\omega \text{ on } H], \tag{SMP}$$

where ζ is the boundary condition induced by ω^ξ on $G \setminus H$ — that is, vertices of ∂H are wired together in ζ if they are connected in ω^ξ in $G \setminus H$.

A direct consequence of (SMP) is the *finite energy property*, meaning that the probability of an edge being open is bounded away from 0 and 1, uniformly in the state of all other edges. That is, for any edge e, there exists a constant $c_{\text{FE}} > 0$ (only depending on the angle θ_e) such that

$$\phi^{\xi}_{G,q}[\omega_e = 1 \,|\, \omega \text{ on } G \setminus \{e\}] \in [c_{\text{FE}}, 1 - c_{\text{FE}}]. \tag{FE}$$

Moreover, when considering lattices $\mathbb{L}(\boldsymbol{\alpha})$, c_{FE} is uniformly positive on compacts of $(0, \pi)$.

The track-exchange operator. Fix $q \geq 1$ and consider a finite sequence of angles $\alpha_1, \dots, \alpha_n$ and let $\mathbb{S}$ denote the strip of an isoradial rectangular lattice with horizontal tracks $t_1, \dots, t_n$ with transverse angles $\alpha_1, \dots, \alpha_n$ and vertical tracks of transverse angles 0. Fix $1 < i \leq n$ and let $\mathbb{S}'$ denote the strip obtained from the same sequence of angles, but with α_i and α_{i-1} exchanged.

The boundary vertices of $\mathbb{S}$ and $\mathbb{S}'$ are those below t_1 and above t_n; they are the same in the two strips. Let ξ be a boundary condition for these two strips. The measures $\phi^{\xi}_{\mathbb{S},q}$ and $\phi^{\xi}_{\mathbb{S}',q}$ may be defined as the unique limits of measures on $\mathbb{S} \cap \Lambda_N$ as $N \to \infty$. Their uniqueness is an immediate consequence of the finite energy property (FE).

Proposition 2.2. *There exists a random map $\mathbf{T}_i$ from percolation configurations on $\mathbb{S}$ to percolation configurations on $\mathbb{S}'$ such that*

- *if $\omega \sim \phi^{\xi}_{\mathbb{S},q}$, then $\mathbf{T}_i(\omega) \sim \phi^{\xi}_{\mathbb{S}',q}$;*
- *ω and $\mathbf{T}_i(\omega)$ agree on all edges outside of t_i and t_{i-1};*
- *on edges of t_i and t_{i-1}, $\mathbf{T}_i(\omega)$ only depends on the values of ω on edges of t_i and t_{i-1} (and potentially on an additional source of randomness, independent of the rest of the configuration ω);*
- *vertices outside of $t_i \cap t_{i-1}$ are connected in the same way in ω and $\mathbf{T}_i(\omega)$;*
- *the modification is local: there exists $c > 0$ such that, for $\omega \sim \phi^{\xi}_{\mathbb{S},q}$, the state of $\mathbf{T}_i(\omega)$ on any edge e is determined[3] by ω restricted to $\Lambda_r(e)$ with probability at least $1 - e^{-cr}$.*

The map $\mathbf{T}_i$ will be called the *track-exchange operator*. The precise construction of the coupling between ω and $\mathbf{T}_i(\omega)$ is not essential, nor unique. One such coupling may be obtained by compositions of the star-triangle transformation; we refer the reader to [DCKK+20] for details. We would merely like to remark that Proposition 2.2 is an indication of the exact integrability of the model defined by the isoradial weights. We will also use $\mathbf{T}_i$ as a transformation of lattices, and write $\mathbb{S}' = \mathbf{T}_i(\mathbb{S})$.

[3]The transformation may use extra randomness, which may be encoded by i.i.d. uniform variables U_e on $[0, 1]$ for $e \in t_{i-1} \cup t_i$. The precise statement is that $(\mathbf{T}_i(\omega))_e$ may be determined with probability exponentially close to 1 from the knowledge of ω and the variables $U_\cdot$ on $\Lambda_r(e)$.

2.3 Isoradial random-cluster model with $q = 4$

Recall from (1.1) that the random-cluster model on isoradial rectangular lattices with $q > 4$ shares the features of the model on $\mathbb{Z}^2$, in particular its free infinite-volume measure exhibits exponential decay of connection probabilities. Similarly, for $q = 4$, the random-cluster measure on isoradial rectangular lattices behaves like the critical measure on $\mathbb{Z}^2$. We review its main characteristics here.

While these properties hold across the entire regime $q \in [1, 4]$, our primary focus will be on the case $q = 4$ and they will be stated as such. This specific choice is motivated by our ultimate goal to transfer some estimates from $q = 4$ to the regime $q > 4$ with q sufficiently close to 4.

In the following, let $\mathbb{L}$ be a lattice of the type $\mathbb{L}(\boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \in (0, \pi)^{\mathbb{Z}}$ is a sequence containing at most two values. All constants below will be uniform in the values of $\boldsymbol{\alpha}$ on compacts of $(0, \pi)$. It was proved in [DCLM18] that, for $q = 4$, there exists a unique infinite-volume measure on $\mathbb{L}$, denoted by $\phi_{\mathbb{L},4}$.

Duality. Recall that if $\mathbb{G}$ is an isoradial graph, then so is its dual $\mathbb{G}^*$. To each configuration ω on $\mathbb{G}$, we associate a dual configuration ω^* on $\mathbb{G}^*$ defined by taking $\omega^*_{e^*} = 1 - \omega_e$ for all edges e of $\mathbb{G}$, where e^* is the unique edge of $\mathbb{G}^*$ intersecting e. The uniqueness of the infinite-volume measure on $\mathbb{L}$ implies that the dual of the isoradial measure on $\mathbb{L}$ is the isoradial measure on the dual lattice $\mathbb{L}^*$, i.e., if $\omega \sim \phi_{\mathbb{L},4}$, then $\omega^* \sim \phi_{\mathbb{L}^*,4}$.

RSW property. Arguably the most useful ingredient in the study of critical planar percolation models is the *Russo–Seymour–Welsh (RSW)* estimate. It states that the probabilities of crossing rectangles of a given aspect ratio but arbitrary scale are uniformly bounded away from 0 and 1. Moreover, these bounds are uniform, even when boundary conditions are imposed at a macroscopic distance from the rectangle. More precisely, for $\rho, \varepsilon > 0$, there exists $c = c(\rho, \varepsilon) > 0$ such that for any event A depending on the edges at distance at least εn from the rectangle $[0, \rho n] \times [0, n]$,

$$c \leq \phi_{\mathbb{L},4}\big[\mathscr{C}([0, \rho n] \times [0, n]) \,|\, A\big] \leq 1 - c, \tag{RSW}$$

where $\mathscr{C}([0, \rho n] \times [0, n])$ denotes the event that there exists a path of open edges in $[0, \rho n] \times [0, n]$ from $\{0\} \times [0, n]$ to $\{\rho n\} \times [0, n]$. We refer to [DCLM18, Thm. 1.1] for a full proof.

We now discuss some consequences of (RSW).

Mixing. One consequence we will use repeatedly is the so-called mixing property. For every $\varepsilon > 0$, there exist $c_{\text{mix}}, C_{\text{mix}} \in (0, \infty)$ such that for every $r \leq R/2$, every event A depending on edges in Λ_r, and every event B depending on edges outside Λ_R, we have that

$$\big|\phi_{\mathbb{L},4}[A \cap B] - \phi_{\mathbb{L},4}[A]\phi_{\mathbb{L},4}[B]\big| \leq C_{\text{mix}}\big(\tfrac{r}{R}\big)^{c_{\text{mix}}}\phi_{\mathbb{L},4}[A]\phi_{\mathbb{L},4}[B]. \tag{Mix}$$

This property can be derived from (RSW) using the same arguments as in the homogeneous model, see e.g. [DCM22, Cor. 2.10].

Arm events and flower domains. Fix some angle $\theta \in [0, 2\pi)$ and define the following arm events. For $r \leq R$ and $z \in \mathbb{L}$, let $A_3^{\text{hp}(\theta)}(z; r, R)$ be the event that there exist three non-intersecting paths $\gamma_1, \gamma_2, \gamma_3$ in the annulus $\Lambda_R(z) \setminus \Lambda_r(z)$, between $\partial\Lambda_r(z)$ and $\partial\Lambda_R(z)$, all contained in the half-space $\langle \cdot, e_\theta \rangle \leq \langle z, e_\theta \rangle$, arranged in clockwise order and such that $\gamma_1, \gamma_3 \in \omega^*$ and $\gamma_2 \in \omega$. We call this the three-arm event in the half-plane orthogonal to e_θ and call the paths $\gamma_1, \gamma_2, \gamma_3$ arms.

We will now briefly introduce the notion of *flower domains*. These domains are particularly suitable for studying arm events. We refer the reader to [DCM22] for a more comprehensive treatment.

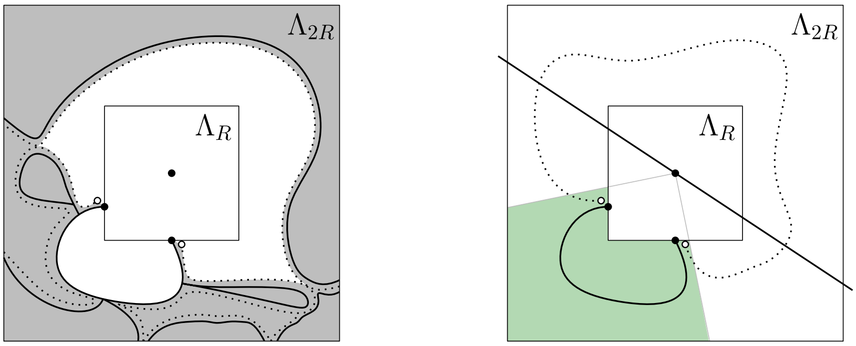

Figure 2: *Left:* the flower domain from Λ_{2R} to Λ_R is obtained by exploring all primal–dual interfaces starting on $\partial\Lambda_{2R}$ until they exit Λ_{2R} or reach Λ_R. The explored region is grey and the flower domain is the white domain; its boundary is formed of one primal and one dual petal. *Right:* a good flower domain with the unique primal petal contained in the green region.

Given $R \geq 1$ and a configuration ω, let $\mathcal{E}$ be the union of Λ_{2R}^c and all the primal/dual interfaces starting on $\partial\Lambda_{2R}$ explored inwards until they either exit Λ_{2R} or enter Λ_R. Let $\mathcal{F}$ be the connected component of the origin in $\mathcal{E}^c$. The boundary of $\mathcal{F}$ is formed either of a primal or dual circuit, or of an even number of alternating primal and dual arcs called *petals*. We call $\mathcal{F}$ the flower domain revealed from Λ_{2R} to Λ_R. See Figure 2 for an example.

For $\eta > 0$, the flower domain $\mathcal{F}$ is said to be *η-well-separated* if the endpoints of its petals are at a distance of at least ηR from each other. In particular, a flower domain with a single petal is η-well-separated by default.

It was proved in [DCM22, Lemma 3.2] that the flower domain from Λ_{2R} to Λ_R is well-separated with positive probability under $\phi^{\xi}_{\mathbb{L}\cap\Lambda_{2R}}$ for any boundary condition ξ. This can be extended to accommodate measures conditioned on arm events [GMM26].

Let us give a more precise statement in the case of the three-arm event in the half plane. We say that a flower domain is *good* if it has exactly two petals (one primal and one dual), is $\frac{1}{4}$-well-separated, and its primal petal is contained in the cone with apex 0, bisector $-e_\theta$, and aperture $\pi/2$ (see Figure 2).

Lemma 2.3 ([GMM26]). *There exists $c > 0$ such that for every $4r \leq R$, any $z \in \mathbb{L}$, and any configuration ω_0 on $\Lambda_{4r}(z)^c$ that allows for the occurrence of $A_3^{\mathrm{hp}(\theta)}(z;1,R)$, we have*

$$\phi_{\mathbb{L},4}\big[\mathcal{F} \text{ is good} \,|\, \omega = \omega_0 \text{ on } \Lambda_{4r}(z)^c \text{ and } A_3^{\mathrm{hp}(\theta)}(z;1,R)\big] \geq c,$$

where $\mathcal{F}$ is the flower domain revealed from $\Lambda_{2r}(z)$ to $\Lambda_r(z)$.

Arm exponents. Another classical consequence of (RSW) is that the probabilities of arm events may be bounded by polynomials with strictly positive exponents.

The following arm exponent bounds will be useful in our arguments. We call $\mathbb{L} = \mathbb{L}(\boldsymbol{\alpha})$ *periodic* if $\boldsymbol{\alpha}$ is a periodic sequence.

Proposition 2.4. *There exist constants $c, C > 0$ independent of θ and $\mathbb{L}$ such that, for all $1 \leq r < R$ and $z \in \mathbb{L}$,*

$$\tfrac{1}{C}\big(\tfrac{r}{R}\big)^2 \leq \phi_{\mathbb{L},4}\big[A_3^{\mathrm{hp}(\theta)}(z;r,R)\big] \leq C\big(\tfrac{r}{R}\big)^2, \qquad \textit{if } \mathbb{L} \textit{ is periodic, and} \tag{2.2}$$

$$\phi_{\mathbb{L},4}\big[A_3^{\mathrm{hp}(\theta)}(z;r,R)\big] \leq C\big(\tfrac{r}{R}\big)^{1+c} \qquad \textit{for all } \mathbb{L} \textit{ as above.} \tag{2.3}$$

In the first case, C may depend on $\mathbb{L}$, but is uniform for sequences $\boldsymbol{\alpha}$ of period 2 with values in compacts of $(0,\pi)$.

Moreover, the arm event probabilities satisfy a quasi-multiplicativity *property, i.e., for any $r \leq \rho \leq R$, we have*

$$\tfrac{1}{C} \leq \frac{\phi_{\mathbb{L},4}\big[A_3^{\mathrm{hp}(\theta)}(z;r,R)\big]}{\phi_{\mathbb{L},4}\big[A_3^{\mathrm{hp}(\theta)}(z;r,\rho)\big]\phi_{\mathbb{L},4}\big[A_3^{\mathrm{hp}(\theta)}(z;\rho,R)\big]} \leq C. \tag{2.4}$$

We refer to [DCKK+20, Prop. 3.4] for a proof of (2.2), (2.3) and merely note that they are consequences of (RSW) and certain symmetries of the lattice. Note that (2.2) requires the invariance of $\mathbb{L}$ under two independent translations, without which, only the weaker bound (2.3) may be obtained[4]. The quasi-multiplicativity (2.4) is a standard consequence of (RSW) and the arm-separation property; see [CDCH16, GMM26] for details.

2.4 Incipient infinite cluster in the half-plane

In this section, we introduce the *incipient infinite cluster* (henceforth IIC) measures with three arms in the half-plane. Fix two angles $\alpha, \beta \in (0,\pi)$ and write $\mathbb{L}_{\text{mix}}$ for the lattice of type $\mathbb{L}(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} \in (0,\pi)^{\mathbb{Z}}$ alternating between α and β, i.e., $\boldsymbol{\alpha} = (\alpha_i)_{i\in\mathbb{Z}}$ with $\alpha_i = \beta$ for i even and $\alpha_i = \alpha$ for i odd. Note that the (RSW) property applies to $\mathbb{L}_{\text{mix}}$.

Recall that the horizontal tracks of $\mathbb{L}_{\text{mix}}$ are denoted by $(t_k)_{k\in\mathbb{Z}}$ and the vertical ones by $(s_k)_{k\in\mathbb{Z}}$. We assume the vertex between t_0 and t_1 and between s_0 and s_1 to be a primal one and consider it to be the origin of $\mathbb{R}^2$. For integers i,j, define the *cell* (i,j) as the set of primal and dual vertices lying between the vertical tracks s_{2i-1} and s_{2i+1} and the horizontal tracks t_{2j-1} and t_{2j+1}. Note that the cells are centred around rhombi of angle β. To each cell, we associate its lower-left lattice point, which, by our convention, is a primal vertex — see Figure 3.

Fix $\theta \in [0,2\pi)$. For a finite cluster C of a configuration ω on $\mathbb{L}_{\text{mix}}$, let $\mathrm{Ext}_\theta(\mathsf{C})$ be the lattice point that maximises the scalar product with e_θ and whose associated cell intersects C. If multiple such maximisers exist, choose the one with the largest vertical coordinate. We call $\mathrm{Ext}_\theta(\mathsf{C})$ the *extremum in direction e_θ* of C — note that it is possible for $\mathrm{Ext}_\theta(\mathsf{C})$ to not be part of C. Finally, let

$$\mathrm{E}_\theta(\mathsf{C}) = \langle \mathrm{Ext}_\theta(\mathsf{C}), e_\theta\rangle$$

be the corresponding extremal coordinate of C in direction e_θ. See Figure 3.

Write x_n for the primal vertex of $\mathbb{L}_{\text{mix}}$ closest to $-ne_\theta$ and let C_{x_n} denote the cluster containing x_n. We define the IIC with three arms in the half-plane as the limiting measure

$$\phi^{\mathrm{IIC},\theta}_{\mathbb{L}_{\text{mix}},4}[\,\cdot\,] := \lim_{n\to\infty} \phi_{\mathbb{L}_{\text{mix}},4}\left[\,\cdot\,|\, \mathrm{Ext}_\theta(\mathsf{C}_{x_n}) = 0\right],$$

where the limit uses the weak convergence with respect to the product topology. In other words, the probability of any local event converges. Under $\phi^{\mathrm{IIC},\theta}_{\mathbb{L}_{\text{mix}},4}$, there exists an infinite cluster which is the limit of the clusters C_{x_n}. We call this the incipient infinite cluster; its extremal coordinate in the direction e_θ is 0.

The measure $\phi^{\mathrm{IIC},\theta}_{\mathbb{L}_{\text{mix}},4}$ describes the local environment around extrema of large clusters. Indeed, it may be shown that the neighbourhood of an extremum of a typical large cluster is distributed according to $\phi^{\mathrm{IIC},\theta}_{\mathbb{L}_{\text{mix}},4}$, even when the cluster is conditioned on (reasonable) large scale features. The existence of the limit is a manifestation of the following mixing property, which in turn is a consequence of (RSW).

Proposition 2.5. *There exist constants $C, c_{\mathrm{IIC}} > 0$ such that the following holds for all $r \leq R/2$. For any configurations ω_0 on Λ_R^c, any $x \in \Lambda_R^c$ and any event A depending only on edges in Λ_r,*

$$\Big|\phi^{\mathrm{IIC},\theta}_{\mathbb{L}_{\text{mix}},4}[A] - \phi_{\mathbb{L}_{\text{mix}},4}\left[A\,|\,\omega=\omega_0 \text{ on } \Lambda_R^c, \mathrm{Ext}_\theta(\mathsf{C}_x)=0\right]\Big| \leq C\big(\tfrac{r}{R}\big)^{c_{\mathrm{IIC}}},$$

[4]Using the universality result of [DCKK+20], (2.2) may be extended to all $\mathbb{L}$ as above, but this requires additional work and is not necessary.

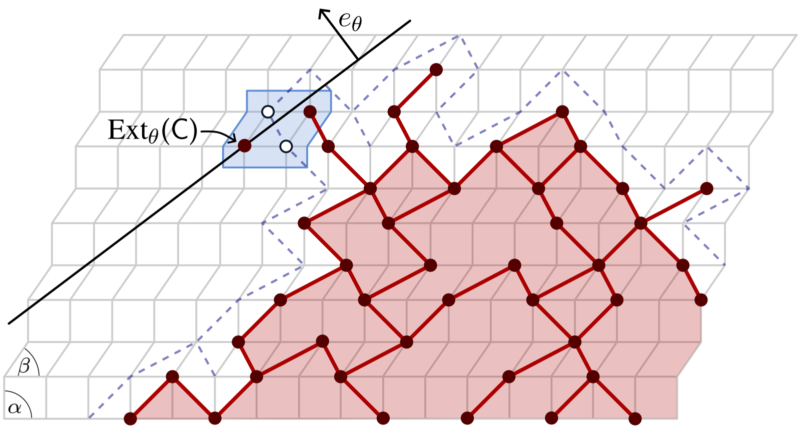

Figure 3: A cluster C with its extremum $\mathrm{Ext}_\theta(\mathsf{C})$ in direction e_θ. In this example, $\mathrm{Ext}_\theta(\mathsf{C})$ does not belong to C itself. The cell associated to the extremum is shaded in light blue and the line $\langle \cdot, e_\theta \rangle = \mathrm{E}_\theta(\mathsf{C})$ is drawn in black.

as long as the conditioning is non-degenerate[5].

A model-specific construction of the IIC measure can be found in [DCKK+20, Oul22]; for its associated polynomial mixing rate, we refer to [GPS13, Prop. 3.1].

Track-exchanges and drift. A crucial step in the proof of the asymptotic rotational invariance for $1 \leq q \leq 4$ of [DCKK+20] is to analyse how the extremum of an IIC cluster is affected by track-exchanges.

Let ω^{IIC} be sampled from $\phi^{\mathrm{IIC},\theta}_{\mathbb{L}_{\mathrm{mix}},4}$ and write $\mathsf{C}^{\mathrm{IIC}}$ for the incipient infinite cluster of ω^{IIC}. Define $\mathbf{S}_{\mathrm{even}}$ as the transformation obtained by simultaneously applying the track-exchanges $\mathbf{T}_{2k}$ for all $k \in \mathbb{Z}$, and $\mathbf{S}_{\mathrm{odd}}$ as the transformation obtained by applying the track exchanges $\mathbf{T}_{2k-1}$ for $k \in \mathbb{Z}$. Since the track-exchanges appearing in the transformations act on disjoint tracks and since the infinite-volume measures on $\mathbb{L}_{\mathrm{mix}}$ and its transforms are unique, one may perform these exchanges simultaneously on ω^{IIC}.

When $\mathbf{S}_{\mathrm{even}}$ is applied to $\mathbb{L}_{\mathrm{mix}}$, each track of angle α is exchanged with the track of angle β directly above it. In particular, the vertices at the bottom of the α-tracks remain fixed. Hence, any cluster containing one of these vertices admits a well-defined image under $\mathbf{S}_{\mathrm{even}}$. Analogously, when $\mathbf{S}_{\mathrm{odd}}$ is applied to $\mathbf{S}_{\mathrm{even}}(\mathbb{L}_{\mathrm{mix}})$, the tracks of angle α are again exchanged with the tracks of angle β directly above them, and clusters containing at least two edges thus have a natural image after the transformation. Note that $(\mathbf{S}_{\mathrm{odd}} \circ \mathbf{S}_{\mathrm{even}})(\mathbb{L}_{\mathrm{mix}})$ is a translate of $\mathbb{L}_{\mathrm{mix}}$ with the same cell structure.

Due to large clusters being preserved, $\mathsf{C}^{\mathrm{IIC}}$ has a corresponding cluster in $(\mathbf{S}_{\mathrm{odd}} \circ \mathbf{S}_{\mathrm{even}})(\omega^{\mathrm{IIC}})$, which we denote by $\tilde{\mathsf{C}}^{\mathrm{IIC}}$. We then define the *IIC increment* by

$$\Delta^{\mathrm{IIC}}\mathrm{E}_\theta = \mathrm{E}_\theta(\tilde{\mathsf{C}}^{\mathrm{IIC}}) - \mathrm{E}_\theta(\mathsf{C}^{\mathrm{IIC}}).$$

Note that $\mathbf{S}_{\mathrm{even}}(\mathbb{L}_{\mathrm{mix}})$ is also a translate of $\mathbb{L}_{\mathrm{mix}}$. As such, one might expect the application of $\mathbf{S}_{\mathrm{even}}$ and the subsequent application of $\mathbf{S}_{\mathrm{odd}}$ to have identical effects. This is not the case: $\mathbf{S}_{\mathrm{even}}(\mathbb{L}_{\mathrm{mix}})$ and $\mathbb{L}_{\mathrm{mix}}$ are translates of each other, but their partition into cells differ, which affects the definition of Ext_θ. This is the reason for applying both transformations before considering the increment.

[5]The conditioning on $\{\omega = \omega_0 \text{ on } \Lambda_R^c\}$ is always degenerate, but should be understood as imposing certain boundary conditions on the restriction of the measure to Λ_R. Here, by non-degenerate, we mean that there exists at least one configuration in Λ_R such that $\{\mathrm{Ext}_\theta(\mathsf{C}_x) = 0\}$ is realised.

The expected increment $\mathbb{E}[\Delta^{\mathrm{IIC}}\mathrm{E}_\theta]$ — where $\mathbb{E}$ refers to the expectation taken with respect to the coupling between ω^{IIC} and $(\mathbf{S}_{\mathrm{odd}} \circ \mathbf{S}_{\mathrm{even}})(\omega^{\mathrm{IIC}})$ — captures the *drift* of the extremum of a large cluster under the transformation $(\mathbf{S}_{\mathrm{odd}} \circ \mathbf{S}_{\mathrm{even}})$. It was proved in [DCKK+20] that this drift vanishes, i.e., for all $\theta \in [0, 2\pi)$,

$$\mathbb{E}[\Delta^{\mathrm{IIC}}\mathrm{E}_\theta] = 0. \tag{2.5}$$

This fact plays a central role in establishing universality among isoradial rectangular lattices for $1 \leq q \leq 4$.

3 Half-plane measures

The strategy of proving universality by successively transforming one lattice into another via a sequence of track-exchanges is faced with additional difficulties for $q > 4$. In this regime, the boundary conditions influence the model at infinite distance. Indeed, suppose a lattice $\mathbb{L}$ is transformed into $\mathbb{L}'$ by a sequence of track-exchanges, which we denote by $\mathbf{S}$. The transformation affects the boundary conditions in an uncontrollable manner, which, in this case, are crucial. In particular, it is unclear whether the push-forward of $\phi^0_{\mathbb{L},q}$ by $\mathbf{S}$ is $\phi^0_{\mathbb{L}',q}$. This problem does not appear in the regime $1 \leq q \leq 4$ due to the uniqueness of the infinite-volume measure. To circumvent this issue, we will work with half-plane measures with free boundary conditions.

From here onwards, we consider lattices $\mathbb{L}(\boldsymbol{\alpha})$ where $\boldsymbol{\alpha} = (\alpha_i)_{i\geq 1}$ is a half-infinite sequence of angles; as in the previous section, these sequences will contain at most two distinct values. These lattices cover the upper half-plane $\mathbb{R} \times \mathbb{R}_{\geq 0}$. For such a lattice, write $\partial\mathbb{L}(\boldsymbol{\alpha})$ for the set of its vertices lying on the horizontal axis $\mathbb{R} \times \{0\}$.

3.1 Uniqueness of the half-plane free measure

We call a half-infinite sequence of angles $\boldsymbol{\alpha} = (\alpha_i)_{i\geq 1}$ *periodic* if there exists $k \in \mathbb{N}$ such that $\alpha_{k+i} = \alpha_i$ for all $i \geq 1$.

Proposition 3.1. *Fix $q \geq 1$ and a half-infinite periodic sequence of angles $\boldsymbol{\alpha}$. For any sequence ξ_n of boundary conditions on $\mathbb{L}(\boldsymbol{\alpha}) \cap \Lambda_n$ with the property that ξ_n is free on $\partial\mathbb{L}(\boldsymbol{\alpha})$, the measures $\phi^{\xi_n}_{\mathbb{L}(\boldsymbol{\alpha})\cap\Lambda_n,q}$ converge as $n \to \infty$ to a measure $\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q}$, which we call the* free half-plane measure.

Proof. Write $\phi^{1/0}_{\mathbb{L}(\boldsymbol{\alpha})\cap\Lambda_n,q}$ for the isoradial random-cluster measure on $\mathbb{L}(\boldsymbol{\alpha})\cap\Lambda_n$ with free boundary conditions on $[-n, n] \times \{0\}$ and wired boundary conditions for the rest of the boundary. Let $\phi^{1/0}_{\mathbb{L}(\boldsymbol{\alpha}),q}$ be the half-plane random-cluster measure which is the weak (decreasing) limit of $\phi^{1/0}_{\mathbb{L}(\boldsymbol{\alpha})\cap\Lambda_n,q}$ for $n \to \infty$.

Similarly, write $\phi^0_{\mathbb{L}(\boldsymbol{\alpha})\cap\Lambda_n,q}$ for the random-cluster measure on $\mathbb{L}(\boldsymbol{\alpha}) \cap \Lambda_n$ with free boundary conditions everywhere and denote its weak (increasing) limit by $\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q}$.

By (CBC), it suffices to show that

$$\phi^{1/0}_{\mathbb{L}(\boldsymbol{\alpha}),q} = \phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q},$$

which in turn follows from the fact that, under $\phi^{1/0}_{\mathbb{L}(\boldsymbol{\alpha}),q}$, there exists a.s. no infinite cluster. The absence of an infinite cluster in $\phi^{1/0}_{\mathbb{L}(\boldsymbol{\alpha}),q}$ follows from the arguments of [GH00, GM23], with the periodicity of the sequence of angles playing a particular role. We give a brief sketch for completeness.

Consider the increasing limit ϕ of the downward translates of $\phi^{1/0}_{\mathbb{L}(\boldsymbol{\alpha}),q}$. Then ϕ is a percolation measure on a periodic isoradial lattice $\mathbb{L}$. Furthermore, ϕ itself is invariant under the translations which map $\mathbb{L}$ to itself. The dual of ϕ produces configurations on the horizontal translate of $\mathbb{L}$ by 1. Furthermore, due to the order in which the different parts of the boundary were taken to infinity, ϕ is stochastically dominated by its dual (translated by $(-1, 0)$).

Due to the planarity and periodicity of $\mathbb{L}$, the non-coexistence theorem of [She05, DCRT19] applies, and we conclude that, under ϕ, either the primal or the dual configuration contains no infinite cluster. By the domination above, the primal configuration contains a.s. no infinite cluster, which extends to $\phi^{1/0}_{\mathbb{L}(\boldsymbol{\alpha}),q}$ by stochastic domination. □

Corollary 3.2 (Track-exchange for half-plane measures)**.** *Fix $q \geq 1$. Let $\boldsymbol{\alpha}$ be a half-infinite periodic sequence of angles and $A \subset \mathbb{N}$ be a periodic set containing no consecutive integers. Write $\boldsymbol{\alpha}'$ for the half-infinite periodic sequence obtained from $\boldsymbol{\alpha}$ by exchanging α_{i-1} and α_i for each $i \in A$. If ω is sampled according to $\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q}$ and ω' is the configuration obtained by applying each $(\mathbf{T}_i)_{i\in A}$ to ω, then ω' is distributed according to $\phi^0_{\mathbb{L}(\boldsymbol{\alpha}'),q}$.*

Proof. Let $\mathbb{S}_K$ be the strip of $\mathbb{L}(\boldsymbol{\alpha})$ formed of the horizontal tracks $t_1, \dots, t_K$ and similarly let $\mathbb{S}'_K$ be the strip of $\mathbb{L}(\boldsymbol{\alpha}')$ containing the tracks $t_1, \dots, t_K$. Write $\phi^0_{\mathbb{S}_K,q}$ and $\phi^0_{\mathbb{S}'_K,q}$ for the measures on $\mathbb{S}_K$ and $\mathbb{S}'_K$, respectively, with free boundary conditions.

Write $\mathbf{S}$ for the composition of all transformations $(\mathbf{T}_i)_{i\in A}$. Assume now that K is such that $K + 1 \notin A$. Then, we may apply all track exchanges $(\mathbf{T}_i)_{i\in A; i\leq K}$ to $\mathbb{S}_K$ to obtain $\mathbb{S}'_K$. For simplicity, we call the composition of these transformations also $\mathbf{S}$, since the transformations $(\mathbf{T}_i)_{i\in A; i>K}$ do not apply to $\mathbb{S}_K$ and hence are considered trivial in this setting.

If ω is sampled according to $\phi^0_{\mathbb{S}_K,q}$, then, by Proposition 2.2, the law of $\mathbf{S}(\omega)$ is given by $\phi^0_{\mathbb{S}'_K,q}$. Proposition 3.1 then implies that

$$\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q} = \lim_{K\to\infty} \phi^0_{\mathbb{S}_K,q} \quad \text{and} \quad \phi^0_{\mathbb{L}(\boldsymbol{\alpha}'),q} = \lim_{K\to\infty} \phi^0_{\mathbb{S}'_K,q}.$$

The conclusion follows. □

Corollary 3.3. *Fix $q \geq 1$ and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two half-infinite periodic sequences of angles with $\alpha_i = \beta_i$ for all $i \leq n$. Then, for any event H depending only on the edges on the tracks $t_1, \dots, t_n$,*

$$\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q}[H] = \phi^0_{\mathbb{L}(\boldsymbol{\beta}),q}[H].$$

Proof. Let $\mathbb{S}(\theta_1, \dots, \theta_K)$ denote the strip with K horizontal tracks of transverse angles $\theta_1, \dots, \theta_K$ and vertical tracks of transverse angle 0. Let $\phi^\xi_{\mathbb{S}(\theta_1,\dots,\theta_K),q}$ denote the measure on this strip with boundary conditions ξ.

Fix H as in the statement and $\varepsilon > 0$. Then, by Proposition 3.1 applied to both $\mathbb{L}(\boldsymbol{\alpha})$ and $\mathbb{L}(\boldsymbol{\beta})$, there exists $K \geq n$ such that

$$\big|\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q}[H] - \phi^\xi_{\mathbb{S}(\alpha_1,\dots,\alpha_K),q}[H]\big| \leq \varepsilon \quad \text{and} \quad \big|\phi^0_{\mathbb{L}(\boldsymbol{\beta}),q}[H] - \phi^\xi_{\mathbb{S}(\beta_1,\dots,\beta_K),q}[H]\big| \leq \varepsilon \tag{3.1}$$

for any boundary conditions ξ that are free on the bottom of the strip.

Consider now the strips

$$\mathbb{S} := \mathbb{S}(\alpha_1, \dots, \alpha_K, \beta_{n+1}, \dots, \beta_K) \quad \text{and} \quad \mathbb{S}' := \mathbb{S}(\beta_1, \dots, \beta_K, \alpha_{n+1}, \dots, \alpha_K).$$

Due to (3.1) and (SMP),

$$\big|\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q}[H] - \phi^0_{\mathbb{S},q}[H]\big| \leq \varepsilon \quad \text{and} \quad \big|\phi^0_{\mathbb{L}(\boldsymbol{\beta}),q}[H] - \phi^0_{\mathbb{S}',q}[H]\big| \leq \varepsilon. \tag{3.2}$$

Since $\alpha_i = \beta_i$ for all $i \leq n$, there exists a sequence of track exchanges that act only above the track t_n and turn $\mathbb{S}$ into $\mathbb{S}'$. The configuration on $t_1, \dots, t_n$ does not change when applying this sequence of transformations, so we find

$$\phi^0_{\mathbb{S},q}[H] = \phi^0_{\mathbb{S}',q}[H].$$

Combining this with (3.2), we conclude that $|\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q}[H] - \phi^0_{\mathbb{L}(\boldsymbol{\beta}),q}[H]| \leq 2\varepsilon$. Finally, since $\varepsilon > 0$ is arbitrary, we find $\phi^0_{\mathbb{L}(\boldsymbol{\alpha}),q}[H] = \phi^0_{\mathbb{L}(\boldsymbol{\beta}),q}[H]$. □

3.2 Rate of decay of connection probabilities in the half-plane

For $\alpha \in (0, \pi)$, write $\mathbb{L}_+(\alpha)$ for the half-plane isoradial lattice with constant transverse angle α for the tracks $(t_n)_{n\geq 1}$. Since we will exclusively be working with half-plane measures, we now introduce the half-plane analogue of ζ. For $\theta \in [0, 2\pi)$, set

$$\zeta^{\text{hp}}_{\alpha,q}(\theta) = \Big(\lim_{n\to\infty} -\tfrac{1}{n} \log \phi^0_{\mathbb{L}_+(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big]\Big)^{-1}.$$

The existence of the limit, similarly to that of $\zeta_{\alpha,q}(\theta)$, is proved using the super-additivity of the sequence

$$\max_{x \in \mathcal{H}^\theta_{\geq n}} \log \phi^0_{\mathbb{L}_+(\alpha),q}[0 \leftrightarrow x].$$

Finally, observe that, due to (1.1) and the finite energy property (FE),

$$0 < \zeta^{\text{hp}}_{\alpha,q}(\theta) < \infty$$

for any $q > 4$, $\alpha \in (0, \pi)$, and $\theta \neq 3\pi/2$.

We call a direction θ *upper-half-plane aiming* for $\mathbb{L}(\alpha)$ if there exists an infinite number of integers $n \geq 1$ such that the point $x \in \mathcal{H}^\theta_{\geq n}$ maximising $\phi^0_{\mathbb{L}(\alpha),q}[0 \leftrightarrow x\,]$ lies in the upper half-plane. The upper-half-plane aiming directions may be shown to be those dual — in the sense of (1.2) — to $\theta \in [0, \pi]$. For $\mathbb{L}(\pi/2)$, all $\theta \in [0, \pi]$ are upper-half-plane aiming due to the invariance of the model with respect to vertical reflections. This is not necessarily the case for $\mathbb{L}(\alpha)$, as this particular symmetry may be lost.

We note two fundamental facts that are due to the invariance of the models $\phi^0_{\mathbb{L}(\alpha),q}$ under rotation by π. For any $\theta \in [0, \pi)$ and $\alpha \in (0, \pi)$,

$$\zeta_{\alpha,q}(\theta) = \zeta_{\alpha,q}(\pi + \theta) \quad \text{and} \quad \text{at least one of } \theta \text{ and } \pi + \theta \text{ is upper-half-plane aiming.}$$

Finally, we state the essential link between $\zeta^{\text{hp}}_{\alpha,q}(\theta)$ and $\zeta_{\alpha,q}(\theta)$.

Proposition 3.4. *Fix $\alpha \in (0, \pi)$, $q > 4$, and $\theta \in [0, 2\pi)$. Then*

$$\zeta_{\alpha,q}(\theta) \geq \zeta^{\text{hp}}_{\alpha,q}(\theta), \tag{3.3}$$

with equality if θ is upper-half-plane aiming for $\mathbb{L}(\alpha)$.

Proof. By (CBC) and inclusion of events,

$$\phi^0_{\mathbb{L}_+(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big] \leq \phi^0_{\mathbb{L}(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big].$$

Taking the logarithm, dividing by $-n$ and taking the limit $n \to \infty$, we obtain (3.3).

Assume now that θ is upper-half-plane aiming for $\mathbb{L}(\alpha)$. Fix $\varepsilon > 0$. Then, for r larger than some threshold,

$$\phi^0_{\mathbb{L}(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq r}\big] \geq \exp\big(-\tfrac{r}{\zeta_{\alpha,q}(\theta)-\varepsilon}\big).$$

Consider the point $x \in \mathcal{H}^\theta_{\geq r}$ that maximises $\phi^0_{\mathbb{L}(\alpha),q}[0 \leftrightarrow x]$. In light of (1.1), there exists a constant $C > 0$ independent of r such that

$$\phi^0_{\mathbb{L}(\alpha),q}[0 \leftrightarrow x] \geq \frac{1}{Cr} \exp\big(- \tfrac{r}{\zeta_{\alpha,q}(\theta)-\varepsilon}\big).$$

By further increasing r and using the fact that θ is upper-half-plane aiming, we may consider x to be in the upper-half-plane and such that

$$\phi^0_{\mathbb{L}(\alpha),q}[0 \leftrightarrow x] \geq \exp\big(- \tfrac{r}{\zeta_{\alpha,q}(\theta)-2\varepsilon}\big).$$

Finally, we may find $R \geq 1$ large enough such that

$$\phi^0_{\Lambda_R\cap\mathbb{L}(\alpha),q}[0 \leftrightarrow x] \geq \exp\big(- \tfrac{r}{\zeta_{\alpha,q}(\theta)-3\varepsilon}\big). \tag{3.4}$$

Now fix $z \in \mathbb{L}_+(\alpha)$ at a distance at least R from the horizontal line. Then, for any n sufficiently large, by (FKG) and (CBC)

$$\begin{aligned}\phi^0_{\mathbb{L}_+(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big] &\geq \phi^0_{\mathbb{L}_+(\alpha),q}[0 \leftrightarrow z]\phi^0_{\mathbb{L}_+(\alpha),q}[z \leftrightarrow z+\ell x]\\ &\geq \phi^0_{\mathbb{L}_+(\alpha),q}[0 \leftrightarrow z]\phi^0_{\Lambda_R\cap\mathbb{L}(\alpha),q}[0 \leftrightarrow x]^\ell\end{aligned}$$

when ℓ is a sufficiently large integer such that $z + \ell x \in \mathcal{H}^\theta_{\geq n}$. Due to the choice of x, one may choose $\ell \leq \frac{n+C_0}{r}$ for some constant C_0 that depends on θ, r, x, and z, but not on n. Using (3.4), we conclude the existence of a constant $c_1 > 0$ independent of n such that

$$\phi^0_{\mathbb{L}_+(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big] \geq c_1 \exp\big(- \tfrac{n}{\zeta_{\alpha,q}(\theta)-3\varepsilon}\big).$$

Taking the logarithm, dividing by $-n$ and taking n to infinity, we conclude that

$$\zeta^{\mathrm{hp}}_{\alpha,q}(\theta) \geq \zeta_{\alpha,q}(\theta) - 3\varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily and in light of (3.3), we conclude that the two quantities are equal. □

4 Universality: proof of Theorem 1.3

We briefly outline the idea behind proving Theorem 1.3. Our aim will be to compare $\zeta^{\mathrm{hp}}_{\alpha,q}(\theta)$ for different values of α and $\theta \neq 3\pi/2$ fixed. More precisely, we will show the following.

Proposition 4.1. *For all $\varepsilon > 0$, there exists $q_0 > 4$ such that for $q \in (4, q_0]$, all $\alpha, \beta \in (\varepsilon, \pi - \varepsilon)$ and any $\theta \in [0, 2\pi)$ with $\theta \neq 3\pi/2$,*

$$\left| \frac{\zeta^{\mathrm{hp}}_{\alpha,q}(\theta)}{\zeta^{\mathrm{hp}}_{\beta,q}(\theta)} - 1 \right| < \varepsilon.$$

We will see in Section 4.3 how to pass from the half-plane connection rates to the full-plane connection rates, and hence prove Theorem 1.3. The rest of the section is dedicated to proving Proposition 4.1.

The aim is to show that connection probabilities in $\omega \sim \phi^0_{\mathbb{L}_+(\alpha),q}$ and $\omega' \sim \phi^0_{\mathbb{L}_+(\beta),q}$ are close to each other when $q > 4$ is taken sufficiently close to 4. To achieve this, we couple these two configurations and construct a sequence of intermediate configurations, each obtained from the previous one by a sequence of track-exchanges.

In this coupling, we keep track of the extremal coordinate $\mathrm{E}_\theta(\mathsf{C})$ of the cluster C of the origin in direction e_θ. By taking $q > 4$ close enough to 4, we argue that the effect of track-exchanges on $\mathrm{E}_\theta(\mathsf{C})$ are almost identical in law to $\Delta^{\mathrm{IIC}}\mathrm{E}_\theta$, and therefore have vanishingly small expectation. This will allow us to compare the probabilities in the initial and final configurations ω and ω' to have $\mathrm{E}_\theta(\mathsf{C}) \geq n$ for large values of n. Ultimately, our goal is to show that the exponential rate of decay of these two quantities is almost equal.

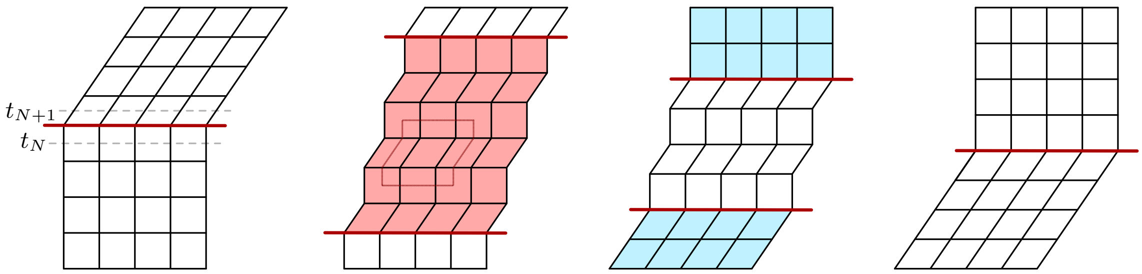

Figure 4: The track-exchanges performed on a single period. From left to right: The initial lattice $\mathbb{L}_0$ with tracks of angle $\alpha = \pi/2$ at the bottom and $\beta < \pi/2$ at the top. The first two tracks to be exchanged are marked with dashed grey lines. Applying $\mathbf{S}_t \circ \cdots \circ \mathbf{S}_0$ transforms $\mathbb{L}_0$ into $\mathbb{L}_{t+1}$ and a mixed block (colored in red) starts appearing in the middle. The cells in the mixed block at even timesteps are centred around rhombi with angle β. After more transformations, a β-block starts forming at the bottom and an α-block at the top (both marked in blue). By time $2N$, the β-block and the α-block have been exchanged completely. The interfaces of each lattice are marked with thick red lines.

4.1 The coupling

We will now fix some notation necessary for the proof of Proposition 4.1 and describe the coupling in detail. Fix two angles $\alpha, \beta \in (0, \pi)$ and $N \geq 1$ even. All constants below will be uniform in α and β in compacts of $(0, \pi)$ and in N.

Write $\mathbb{L}_0$ for the lattice $\mathbb{L}_+(\boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is the sequence given by

$$\alpha_i = \begin{cases} \alpha & \text{for } 2kN < i \leq (2k+1)N,\ k \geq 0, \\ \beta & \text{otherwise.} \end{cases}$$

We can partition the lattice into blocks of N tracks of constant angle. By means of the track-exchange operator, we can define a sequence of lattices $(\mathbb{L}_t)_{t \geq 0}$ where the blocks of angle α are eventually exchanged with those of angle β.

Recall that we write $\mathbf{T}_i$ for the track-exchange operator exchanging the tracks t_i and t_{i-1}. If t_i and t_{i-1} have the same transverse angle, we set $\mathbf{T}_i = \text{id}$. For $t \geq 1$, apply the following sequence of track exchanges to $\mathbb{L}_t$ in order to obtain $\mathbb{L}_{t+1}$:

$$\mathbf{S}_t = \begin{cases} \mathbf{T}_3 \circ \mathbf{T}_5 \circ \ldots \mathbf{T}_{2N-1} \circ \mathbf{T}_{2N+3} \circ \mathbf{T}_{2N+5} \circ \ldots & \text{if } t \text{ is odd,} \\ \mathbf{T}_2 \circ \mathbf{T}_4 \circ \ldots & \text{if } t \text{ is even.} \end{cases}$$

That is, for $t \geq 1$, set $\mathbb{L}_t = \mathbf{S}_{t-1} \circ \cdots \circ \mathbf{S}_0(\mathbb{L}_0)$. See Figure 4 for an illustration; note that since N is even, $\mathbf{S}_0$ is trivial and $\mathbb{L}_1 = \mathbb{L}_0$.

The transformations $\mathbf{S}_t$ for odd times t contains all track-exchanges $\mathbf{T}_i$ with i odd, *except* for $i - 1 \in 2N\mathbb{Z}$. This ensures that there is never any exchange of tracks between blocks of $2N$ successive tracks, and ultimately ensures that all lattices $\mathbb{L}_t$ are $2N$-periodic.

Each period is formed of a *mixed block* of alternating tracks of angles α and β sandwiched between an α-block and a β-block; the sizes of the blocks depend on t and may be null. Lines separating different blocks of the lattice (including the horizontal axis) will be referred to as *interfaces*.

After $2N$ transformations, the mixed block disappears and the β-block and α-block have been exchanged completely. We refer again to Figure 4.

Fix $q > 4$. We will associate a sequence of configurations $(\omega_t)_{t \geq 0}$ to the lattices $(\mathbb{L}_t)_{t \geq 0}$ with the property that $\omega_t \sim \phi^0_{\mathbb{L}_t, q}$ for all $t \geq 0$. This produces a *coupling* of the measures $(\phi^0_{\mathbb{L}_t, q})$. The coupling is constructed as follows. First, sample ω_0 according to $\phi^0_{\mathbb{L}_0,q}$. Then, for $t \geq 0$, assuming that $\omega_0, \dots, \omega_t$ are already defined, set

$$\omega_{t+1} = \mathbf{S}_t(\omega_t).$$

Corollary 3.2 ensures that ω_t indeed has law $\phi^0_{\mathbb{L}_t,q}$ for all $t \geq 1$. We write $\mathbb{P}$ for the probability measure governing the random sequence $(\omega_t)_{t\geq 0}$.

Fix an angle $\theta \in [0, 2\pi) \setminus \{3\pi/2\}$. For each configuration ω_t, write C_t for the cluster of the origin in ω_t and recall that $\mathrm{Ext}_\theta(\mathsf{C}_t)$ (resp. $\mathrm{E}_\theta(\mathsf{C}_t)$) denotes the extremum (resp. extremal coordinate) of C_t in direction e_θ. Since θ remains fixed throughout, we will henceforth omit it from the notation.

We will be interested in the evolution of $(\mathrm{E}(\mathsf{C}_t))_{t\geq 0}$. In order to keep track of the dynamics of the process, we define, for $t \geq 0$ even, the increments

$$\Delta_t \mathrm{E} = \mathrm{E}(\mathsf{C}_{t+2}) - \mathrm{E}(\mathsf{C}_t).$$

The following estimate, bounding the expected increment, conditionally on the extremal coordinate being large, is a key step in the proof of Proposition 4.1.

Proposition 4.2. *For each $t \geq 0$ even, $\Delta_t\mathrm{E}$ is a.s. bounded by 4. Moreover, for any $\delta, \varepsilon > 0$, there exists $q_0 > 4$ such that for any $q \in (4, q_0]$, n, N sufficiently large and any $1 \leq t \leq N(1-\delta)$ or $(1+\delta)N \leq t \leq 2N$, it holds that*

$$\mathbb{E}\big[\Delta_t\mathrm{E} \,|\, \mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big] \geq -\delta. \tag{4.1}$$

Moreover, q_0 may be chosen independently of θ, n and N (both sufficiently large) and uniformly in α, β in compacts of $(0, \pi)$.

Note that we do not follow the precise evolution of $\mathrm{E}(\mathsf{C}_t)$, but rather a "rounding" of $\mathrm{E}(\mathsf{C}_t)$ at an arbitrary precision. Indeed, the exact value of $\mathrm{E}(\mathsf{C}_t)$ may dictate the exact position of $\mathrm{Ext}(\mathsf{C}_t)$, which may lead to a degenerate conditioning. See the proof of Lemma 4.5 for the use of this rounding.

We excluded t between $(1-\delta)N$ and $(1+\delta)N$ from (4.1) for convenience. Indeed, in these time steps, the mixed block comes close to the horizontal axis, which entails some additional complications. While we believe (4.1) to also apply in this case, we omit it as it is not strictly necessary for the rest of the proof.

As a consequence of Proposition 4.2, we establish that the coupling is indeed such that the corresponding point-to-hyperplane connection probabilities in $\mathbb{L}_0$ and $\mathbb{L}_{2N}$ differ only negligibly if q is chosen sufficiently close to 4 and the distance n to the hyperplane is large enough.

Corollary 4.3. *For any $\eta > 0$ and $C \geq 1$, there exists $q_0 > 4$ such that for $q \in (4, q_0]$ and all n sufficiently large, if we set $N = Cn$, we have*

$$\mathbb{P}\big[\mathrm{E}(\mathsf{C}_{2N}) \geq (1-\eta)n\big] \geq \eta\mathbb{P}\big[\mathrm{E}(\mathsf{C}_0) \geq n\big]. \tag{4.2}$$

Moreover, q_0 may be chosen independently of θ and uniformly in α and β in compacts of $(0, \pi)$.

Proof. Fix $\eta > 0$ and $C \geq 1$. With no loss of generality, we may assume $\eta < 1/2C$. Choose constants $\varepsilon, \delta > 0$ so that

$$5\delta + \varepsilon < \tfrac{\eta}{2C} \quad \text{and} \quad \eta \leq \tfrac{5\delta+\varepsilon}{2(1+5\delta+\varepsilon)}.$$

Then choose $q_0 > 4$, sufficiently close to 4 so that Proposition 4.2 holds for the chosen values of δ, ε. Fix now n sufficiently large that (4.1) applies for all values above $\varepsilon m \geq n/2$ and set $N = Cn$.

Define the ε-rounding $e_t = \lfloor \mathsf{E}(\mathsf{C}_t)/\varepsilon \rfloor \cdot \varepsilon$ of $\mathsf{E}(\mathsf{C}_t)$ and write $\Delta_t e = e_{t+2} - e_t$ for all $t \geq 0$ even. Then, Proposition 4.2 states that, for any $0 \leq t \leq 2N$ even, except if $(1-\delta)N \leq t \leq (1+\delta)N$,

$$\mathbb{E}\big[\Delta_t e \,\big|\, e_t = \varepsilon m\big] \geq -\delta - \varepsilon \tag{4.3}$$

for all $q \in (4, q_0]$ and $\varepsilon m \geq n/2$.

Note that (4.3) does not provide a lower bound on the expected increment $\Delta_t e$ given the full past of the process. As our ultimate goal is to study the process under the conditioning $e_0 \geq n$, this may produce difficulties. To circumvent them, we will modify the process $(e_{2t})_{t\geq 0}$ to render it Markov, while maintaining the marginal laws. This is a standard procedure: define a process $(\tilde{e}_{2t})_{t\geq 0}$ on a potentially extended probability space as follows.

- Let $\tilde{e}_0 = e_0$.
- For any $t \geq 0$, let $\omega_{t+1/2}$ be a configuration with the same law as ω_t, sampled independently of the past, but such that $\lfloor \mathsf{E}(\mathsf{C}_{t+1/2})/\varepsilon \rfloor \cdot \varepsilon = \tilde{e}_t$.
- Define $\omega_{t+2} = \mathbf{S}_{t+1} \circ \mathbf{S}_t(\omega_{t+1/2})$ and use it to compute $\tilde{e}_{t+2}$.

This resampling procedure renders the process $(\tilde{e}_{2t})_{t\geq 0}$ Markov. Additionally, it is easy to check that, for any fixed, even t, e_t and $\tilde{e}_t$ have the same law. Finally, (4.3) also applies to the process $(\tilde{e}_t)_{t\geq 0}$.

Set $\tau = \inf\{t \geq 0 : \tilde{e}_{2t} < n/2\}$. By (4.3), we conclude that $\big(\tilde{e}_{2(t\wedge\tau)} + (\delta+\varepsilon)t\big)_{0\leq 2t\leq (1-\delta)N}$ and $\big(\tilde{e}_{2(t\wedge\tau)} + (\delta+\varepsilon)t\big)_{(1+\delta)N\leq 2t\leq 2N}$ are submartingales with bounded increments. Furthermore, $\tilde{e}_{(1+\delta)N\wedge 2\tau} - \tilde{e}_{(1-\delta)N\wedge 2\tau} \leq 4\delta N$, due to the deterministic bound on $\Delta_t \tilde{e}$.

Thus, by the optional stopping theorem,

$$\mathbb{E}\big[\tilde{e}_{2(N\wedge\tau)} - \tilde{e}_0 \,\big|\, \tilde{e}_0\big] \geq -(5\delta+\varepsilon)N.$$

Again, by the deterministic bound on $\Delta_t \tilde{e}$, we obtain $\tilde{e}_{2(N\wedge\tau)} - \tilde{e}_0 \leq 4N$. Applying the Markov inequality, we conclude that

$$\begin{aligned}\mathbb{P}\big[\tilde{e}_{2(N\wedge\tau)} - \tilde{e}_0 > -\eta n \,\big|\, \tilde{e}_0 \geq n\big] &\geq \mathbb{P}\big[\tilde{e}_{2(N\wedge\tau)} - \tilde{e}_0 > -2(5\delta+\varepsilon)N \,\big|\, \tilde{e}_0 \geq n\big]\\ &\geq \tfrac{5\delta+\varepsilon}{2(1+5\delta+\varepsilon)} \geq \eta.\end{aligned}$$

Since $\eta n < n/2$, when the event in the first line occurs, $\tilde{e}_{2(N\wedge\tau)} > n/2$, and thus $\tau \geq N$. We conclude from the above that

$$\mathbb{P}\big[\tilde{e}_{2N} > (1-\eta)n \,\big|\, \tilde{e}_0 \geq n\big] \geq \eta. \tag{4.4}$$

Finally, since $\tilde{e}$ and e have the same marginals, we find that

$$\begin{aligned}\mathbb{P}\big[\mathsf{E}(\mathsf{C}_{2N}) \geq (1-\eta)n\big] &\geq \mathbb{P}\big[\tilde{e}_{2N} \geq (1-\eta)n\big]\\ &\geq \eta\mathbb{P}\big[\tilde{e}_0 \geq n\big] = \eta\mathbb{P}\big[\mathsf{E}(\mathsf{C}_0) \geq n\big].\end{aligned}$$

In the last line, we assumed for simplicity that n/ε is integer-valued. This concludes the proof.

We close by remarking that, by using the independence of the increments, it may be proved that $\mathbb{P}[\tilde{e}_{2N} > (1-\eta)n \,|\, \tilde{e}_0 \geq n] \to 1$ as $n \to \infty$. For our goal, the weaker bound (4.4) suffices. □

4.2 Expected increment: proof of Proposition 4.2

Fix α, β, N, n, θ and $t \geq 0$ even. All constant below are independent of these choices, with α and β being taken in a compact of $(0, \pi)$ and n, N being large enough. Also fix $\delta, \varepsilon > 0$.

First, we argue that $\Delta_t\mathrm{E}$ is deterministically bounded by 4. Split the vertices of $\mathbb{L}_t$ and $\mathbb{L}_{t+1}$ into those placed at the top of tracks t_i with i even and those placed at the bottom of such tracks — this is a bi-partition of $\mathbb{L}_t$. The former type of vertex is not affected by the transformation $\mathbf{S}_t$, and thus the vertices of this type in C_t are the same as those in C_{t+1}. Finally, for both ω_t and ω_{t+1}, any vertex of the second category that is connected to 0 is connected to a vertex of the first category by an open edge whose length is at most 2. Repeating the same argument when applying $\mathbf{S}_{t+1}$ to ω_{t+1} (with the roles of the two types of vertices reversed), we conclude that C_{t+2} must contain a vertex in a cell neighbouring that of $\mathrm{Ext}(\mathsf{C}_t)$, and vice versa. Given that the diameter of a cell is bounded by 4, this provides the deterministic bound $\Delta_t\mathrm{E} \leq 4$.

We now turn to (4.1). Assume now that $1 \leq t \leq N(1-\delta)$ or $(1+\delta)N \leq t \leq 2N$. The expected increment can be decomposed as follows:

$$\begin{aligned}\mathbb{E}&\big[\Delta_t\mathrm{E} \,|\, \mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big]\\ &= \sum_z \mathbb{E}\big[\Delta_t\mathrm{E} \,|\, \mathrm{Ext}(\mathsf{C}_t) = z\big]\mathbb{P}\big[\mathrm{Ext}(\mathsf{C}_t) = z \,|\, \mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big],\end{aligned} \tag{4.5}$$

where the sum is taken over all points $z \in \mathbb{L}_t$ with $\langle z, e_\theta\rangle \in [n\varepsilon, (n+1)\varepsilon)$. The summands will be controlled in different ways, depending on the value of z. In the following, z is always assumed to satisfy $\langle z, e_\theta\rangle \in [n\varepsilon, (n+1)\varepsilon)$.

Choose $1 \leq r \leq R$ depending on ε and δ; the choice of r and R will be explained below. The constant $q_0 > 4$ below will depend on ε, δ and on r and R, but not on the angles θ, α, β, nor on n and N above some threshold. We henceforth assume $\varepsilon n > R$, with further lower bounds imposed below.

We distinguish three scenarios depending on the location of $z = \mathrm{Ext}(\mathsf{C}_t)$.

(1) z is in a pure block, at a distance at least 4 from any interface;

(2) z is within distance R from an interface, but not as in the first case;

(3) or z is in the mixed block, at a distance at least R from any interface.

Before describing how to bound the summands in (4.5), we state a separation lemma that will be useful in the following proofs.

Recall from Section 2.3 the definition of a good flower domain.

Lemma 4.4. *There exists $c > 0$ such that, for any $\rho \geq 1$ the following holds. There exists $q_0 > 4$ such that, for any $q \in [4, q_0)$ and any z at distance at least 4ρ from the horizontal axis,*

$$\mathbb{P}\big[\textit{the flower domain in } \omega_t \textit{ from } \Lambda_{2\rho}(z) \textit{ to } \Lambda_\rho(z) \textit{ is good } \big|\, \mathrm{Ext}(\mathsf{C}_t) = z\big] \geq c.$$

Proof. For configurations sampled according to the measure with $q = 4$ the statement holds according to Lemma 2.3. Adapting this to our setting, we find that there exists a constant $c > 0$ such that, for any $\rho \geq 1$ and z at a distance at least 4ρ from the horizontal axis, any $t \geq 0$, and any configuration $\tilde\omega$ on $\Lambda_{4\rho}(z)^c$,

$$\phi^0_{\mathbb{L}_t,4}\big[\mathcal{F} \text{ is good} \,\big|\, \omega = \tilde\omega \text{ on } \Lambda_{4\rho}(z)^c \text{ and } \mathrm{Ext}(\mathsf{C}_t) = z\big] \geq 2c, \tag{4.6}$$

where we write $\mathcal{F}$ for the flower domain revealed from $\Lambda_{2\rho}(z)$ to $\Lambda_\rho(z)$ and where the inequality holds as long as the conditioning is non-degenerate.

Now fix $\rho \geq 1$. The event in (4.6) depends only on the configuration in $\Lambda_{4\rho}(z)$ with the measure in this region being of the form $\phi^{\xi}_{\Lambda_{4\rho}(z),4}$ with boundary conditions ξ induced by $\tilde{\omega}$ and a conditioning on an event that depends on $\tilde{\omega}$.

The measures $\phi^{\xi}_{\Lambda_{4\rho}(z),q}$ are all continuous in q. As such, there exists $q_0 > 4$ such that, for any $q \in (4, q_0]$, any boundary conditions ξ and any non-degenerate event H,

$$d_{\mathrm{TV}}\big(\phi^{\xi}_{\Lambda_{4\rho}(z),q}[\,\cdot\,|\,H], \phi^{\xi}_{\Lambda_{4\rho}(z),4}[\,\cdot\,|\,H]\big) \leq c$$

The above refers to the distance in total variation between the conditioned measures. Combining the above with (4.6), we find

$$\phi^{0}_{\mathbb{L}_t,q}\big[\mathcal{F} \text{ is good} \,\big|\, \omega = \tilde{\omega} \text{ on } \Lambda_{4\rho}(z)^c \text{ and } \mathrm{Ext}(\mathsf{C}_t) = z\big] \geq c, \tag{4.7}$$

for any configuration $\tilde{\omega}$ on $\Lambda_{4\rho}(z)^c$ such that the conditioning is non-degenerate. Finally, as ω_t follows the law $\phi^{0}_{\mathbb{L}_t,q}$, (4.7) directly implies the desired bound. □

We now return to the proof of Proposition 4.2 and deal with the previously defined scenarios individually

Case (1): For z in a pure block, at a distance at least 4 from an interface, we have

$$\mathbb{P}\big[\Delta_t\mathrm{E} \geq 0 \,|\, \mathrm{Ext}(\mathsf{C}_t) = z\big] = 1. \tag{4.8}$$

Indeed, the cell of z is not affected by the transformation $(\mathbf{S}_{t+1} \circ \mathbf{S}_t)$ and therefore intersects C_{t+2}. The inequality stems from the fact that some other point may overtake the extremum and thus increase the extremal coordinate.

Case (2): We argue that the probability for $\mathrm{Ext}(\mathsf{C}_t)$ to be in case 2 under the conditioning $\mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)$ is small. Combining this with the deterministic lower bound $\Delta_t\mathrm{E} \geq -4$ shows that the contribution of points z in the second case to (4.5) is not substantially negative.

Lemma 4.5. *There exist constants $C > 1$ and $q_0 > 4$ such that, for any $q \in (4, q_0]$, if n, N are sufficiently large,*

$$\mathbb{P}\big[\mathrm{Ext}(\mathsf{C}_t) \textit{ is at a distance at most } R \textit{ from an interface, but at least } CR \textit{ from the horizontal axis} \,\big|\, \mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big] < \delta. \tag{4.9}$$

Proof of Lemma 4.5. Fix $C \geq 1$; we will see below how to choose it. We henceforth assume $N > 2CR$. Additionally, we will assume

Consider a potential realisation z of $\mathrm{Ext}(\mathsf{C}_t)$ with $\langle z, e_\theta\rangle \in [n\varepsilon, (n+1)\varepsilon)$ and z at a distance at least $4CR$ from the horizontal axis and with the minimal distance between z and an interface being smaller than R. Call the latter distance d.

Consider the following exploration procedure of the unconditioned configuration ω_t. Explore the flower domain $\mathcal{F}$ from $\Lambda_{2CR}(z)$ to $\Lambda_{CR}(z)$ and reveal the configuration on $\mathcal{F}^c$; we call this the explored region. Conditionally on the exploration, we distinguish three scenarios:

(i) $\mathcal{F}$ is not good;

(ii) the configuration in the explored region is such that, for any configuration in $\mathcal{F}$, we have $\mathrm{Ext}(\mathsf{C}_t) \neq z$;

(iii) the configuration in the explored region is such that the cluster of the origin is connected to the primal petal of $\mathcal{F}$ and is contained in $\mathcal{H}^{\theta}_{<n\varepsilon}$. In this case, depending on the configuration in $\mathcal{F}$, we may have $\mathrm{Ext}(\mathsf{C}_t) = z$.

Let $c_{\text{good}} > 0$ be the constant given by Lemma 4.4. Then there exists $\tilde{q}_0 > 4$ (depending on c_{good} and CR, but not on z or any other fixed quantities) such that

$$\mathbb{P}\big[\text{Case (i)} \,|\, \text{Ext}(\mathsf{C}_t) = z\big] \leq 1 - c_{\text{good}}$$

for any $q \in (4, \tilde{q}_0]$.

In case (ii), we have

$$\mathbb{P}\big[\text{Ext}(\mathsf{C}_t) = z \,|\, \mathcal{F}\big] = 0,$$

where the conditioning is that $\mathcal{F}$ is the result of the exploration procedure described above.

We conclude that

$$\mathbb{P}\big[\text{Case (iii) and Ext}(\mathsf{C}_t) = z\big] \geq c_{\text{good}} \cdot \mathbb{P}\big[\text{Ext}(\mathsf{C}_t) = z\big]. \tag{4.10}$$

Now fix a realisation of $\mathcal{F}$ as in case (iii). The measure for ω_t inside $\mathcal{F}$ is then given by $\phi^{\xi}_{\mathcal{F},q}$, where ξ are the boundary conditions induced by the exploration. For configurations in $\mathcal{F}$, write $\text{Ext}(\xi)$ for the extremum of the cluster of the primal petal in the direction e_θ. If ω_t is such that $\langle \text{Ext}(\xi), e_\theta \rangle \in [n\varepsilon, (n+1)\varepsilon)$, then $\text{Ext}(\mathsf{C}_t) = \text{Ext}(\xi)$.

This applies in particular to $\text{Ext}(\xi) = z$, but also to other points z' in the strip $\langle z', e_\theta \rangle \in [n\varepsilon, (n+1)\varepsilon)$. Consider now $4 < q_0 \leq \tilde{q}_0$ (depending on CR) such that, for any $4 \leq q \leq q_0$ and any $\mathcal{F}$ and z' as above,

$$\tfrac{1}{2} \leq \frac{\phi^{\xi}_{\mathcal{F},q}[\text{Ext}(\xi) = z']}{\phi^{\xi}_{\mathcal{F},4}[\text{Ext}(\xi) = z']} \leq 2.$$

The existence of such a value q_0 is guaranteed by the continuity in q of the measures $\phi^{\xi}_{\mathcal{F},q}$.

Standard applications of (RSW), (Mix) and (2.4) show that, for any $z' \in \Lambda_{CR/2}(z)$ with $\langle z', e_\theta \rangle \in [n\varepsilon, (n+1)\varepsilon)$, if we write d' for the distance between z' and the interface[6]

$$c_0 \leq \frac{\phi^{\xi}_{\mathcal{F},4}[\text{Ext}(\xi) = z']}{\phi_{\mathbb{L}_{\text{mix}},4}\big[A_3^{\text{hp}(\theta)}(z'; 1, d')\big]\,\phi^0_{\mathbb{L}_t,4}\big[A_3^{\text{hp}(\theta)}(z'; d', CR)\big]} \leq \tfrac{1}{c_0},$$

for some universal constant $c_0 > 0$.

Applying the above to $z' = z$ and using Proposition 2.4, we find

$$\phi^{\xi}_{\mathcal{F},q}[\text{Ext}(\xi) = z] \leq C_0\, d^{-2} \big(\tfrac{d}{CR}\big)^{1+c} = C_0\, d^{-1+c} (CR)^{-1-c}. \tag{4.11}$$

for some universal constant $C_0 > 0$ and with the constant $c > 0$ given by (2.3).

Conversely, for any fixed z' as above with $d' \geq CR/4$, we have

$$\phi^{\xi}_{\mathcal{F},q}\big[\text{Ext}(\xi) = z'\big] \geq c_1 (CR)^{-2}.$$

for some universal constant $c_1 > 0$. There exists a constant[7] $c_2 = c_2(\varepsilon) > 0$, independent of any other quantity except ε, such that the number of vertices $z' \in \Lambda_{CR/2}(z)$ with $\langle z', e_\theta \rangle \in [n\varepsilon, (n+1)\varepsilon)$ and $d' \geq CR/4$ is at least $c_2 CR$. Thus, summing over all such z', we conclude that

$$\phi^{\xi}_{\mathcal{F},q}\big[\langle \text{Ext}(\xi), e_\theta \rangle \in [n\varepsilon, (n+1)\varepsilon)\big] \geq c_2 c_1 (CR)^{-1}. \tag{4.12}$$

[6]We assume here that there is a unique interface intersecting $\mathcal{F}$. Due to the assumption that $N \geq 2CR$, there exist at most two such interfaces. The case of two interfaces may be treated in a similar way, but is omitted here for simplicity.

[7]It is here that it is essential that we condition on a rounding of e_t and not its exact value.

Combining (4.11) and (4.12), we conclude that

$$\begin{aligned}\mathbb{P}\big[\mathrm{Ext}(\mathsf{C}_t) = z \,|\, \mathcal{F}\big] &= \phi^{\xi}_{\mathcal{F},q}\big[\mathrm{Ext}(\xi) = z\big] \\ &\le c_3\, d^{-1+c}(CR)^{-c}\phi^{\xi}_{\mathcal{F},q}\big[\langle \mathrm{Ext}(\xi), e_\theta\rangle \in [n\varepsilon, (n+1)\varepsilon)\big] \\ &= c_3\, d^{-1+c}(CR)^{-c}\mathbb{P}\big[\mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon) \,|\, \mathcal{F}\big],\end{aligned}$$

where $c_3 = \frac{C_0}{c_1 c_2} > 0$ is a universal constant. The conditioning is again that $\mathcal{F}$ is the result of the exploration procedure described above.

As the above is valid for any explored flower domain $\mathcal{F}$ in case (iii), we conclude that

$$\mathbb{P}\big[\text{Case (iii) and } \mathrm{Ext}(\mathsf{C}_t) = z\big] \le c_3\, d^{-1+c}(CR)^{-c}\mathbb{P}\big[\mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big].$$

Finally, combining this with (4.10), we have

$$\mathbb{P}\big[\mathrm{Ext}(\mathsf{C}_t) = z \,\big|\, \mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big] \le \tfrac{1}{c_{\text{good}}}\, c_3\, d^{-1+c}(CR)^{-c}.$$

Summing over z with $d \le R$, $\langle z, e_\theta\rangle \in [n\varepsilon, (n+1)\varepsilon)$, and which are at a distance at least $4CR$ from the horizontal axis, we find

$$\begin{aligned}\mathbb{P}\big[&\mathrm{Ext}(\mathsf{C}_t) \text{ is at a distance at most } R \text{ from an interface,} \\ &\text{but at least } 4CR \text{ from the horizontal axis} \,\big|\, \mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big] \le c_4 C^{-c},\end{aligned}$$

where c_4 is a universal constant and $c > 0$ is given by (2.3). By choosing C sufficiently large, we may render the right-hand side of the above smaller than δ, thus proving (4.9) with $4C$ instead of C. □

We return now to the analysis of case (2). Let $C \ge 1$ and $q_0 > 4$ be the constants given by Lemma 4.5. Assume henceforth that $4 < q \le q_0$, and that N is such that $\delta N \sin\alpha > CR + 4$ and $\delta N \sin\beta > CR + 4$. For $t \le (1-\delta)N$ the tracks $t_0, \dots, t_{\delta N}$ are part of the frozen block of angle α; for $t \ge (1+\delta)N$, they are part of the frozen block of angle β. In both cases, due to our assumption on N, these blocks have a height at least CR. Thus, case (2) implies that the extremum is within distance R of an interface, but also at a distance at least CR from the horizontal axis. Combining the deterministic bound $\Delta_t\mathrm{E} \ge -4$ with (4.9), we find

$$\sum_{z \text{ in case (2)}} \mathbb{E}\big[\Delta_t\mathrm{E} \,|\, \mathrm{Ext}(\mathsf{C}_t) = z\big]\mathbb{P}\big[\mathrm{Ext}(\mathsf{C}_t) = z \,|\, \mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big] \ge -4\delta, \quad (4.13)$$

where the sum is taken over all possible realisations z of $\mathrm{Ext}(\mathsf{C}_t)$ included in case (2).

Case (3): The remaining points z are in the mixed block, at a distance at least R from any interface. In this scenario, we want to relate the increment $\Delta_t\mathrm{E}$ to the increment $\Delta^{\mathrm{IIC}}\mathrm{E}$ of an IIC (see Section 2.4).

To that end, we first prove that the local environment around an extremum is indistinguishable from that of an IIC extremum (up to an arbitrarily small error).

Lemma 4.6. *For any $r \ge 1$, there exists a choice of $R \ge r$ and $q_0 > 4$ such that, for any $q \in [4, q_0)$, any $t \ge 0$ even, and any $z \in \mathbb{L}_t$ in the mixed block, at a distance at least R from an interface,*

$$d_{\mathrm{TV}}\big(\mathbb{P}\big[\,\cdot\,\,|\, \mathrm{Ext}(\mathsf{C}_t) = z\big], \phi^{\mathrm{IIC}}_{\mathbb{L}_{\mathrm{mix}},4}\big) \le \delta,$$

where the two measures refer to the configuration in $\Lambda_r(z)$ and the latter is translated by z.

Note here that R and q_0 depend on r and δ, but not on z, t or any other quantity previously fixed.

Proof. Fix $r \geq 1$ and let $R \geq r$ be a constant to be fixed below. Fix a point z as in the statement.

First, observe that, in $\mathbb{L}_t$, the lattice in $\Lambda_R(z)$ is identical to $\mathbb{L}_{\text{mix}}$ — including its partition into cells. Thus, Proposition 2.5 ensures that, by choosing $R = R(r, \delta) \geq r$ sufficiently large,

$$d_{\text{TV}}\big(\phi^0_{\mathbb{L}_t,4}\left[\,\cdot \,|\, \omega = \omega_0 \text{ on } \Lambda_R(z)^c, \text{Ext}(\mathsf{C}_t) = z\right], \phi^{\text{IIC}}_{\mathbb{L}_{\text{mix}},4}\big) \leq \tfrac{\delta}{2}$$

for any configuration ω_0 on $\Lambda_R(z)^c$ for which the conditioning is not degenerate, where the two measures refer to the configuration in $\Lambda_r(z)$ and the latter is translated by z.

Now, by continuity of the measures $\phi^{\xi}_{\Lambda_R(z),q}$, there exists $q_0 = q_0(R, \delta) > 4$ such that, for any $4 < q \leq q_0$,

$$d_{\text{TV}}\big(\phi^0_{\mathbb{L}_t,4}[\,\cdot \,|\, \omega = \omega_0 \text{ on } \Lambda_R(z)^c, \text{Ext}(\mathsf{C}_t) = z], \phi^0_{\mathbb{L}_t,q}[\,\cdot \,|\, \omega = \omega_0 \text{ on } \Lambda_R(z)^c, \text{Ext}(\mathsf{C}_t) = z]\big) \leq \tfrac{\delta}{2}$$

for any ω_0 as above, with both measures referring to the configuration in the full box $\Lambda_R(z)$.

Combining the two displays above, and keeping in mind that the law of ω_t is $\phi^0_{\mathbb{L}_t,q}$, we obtain the desired conclusion. □

We now aim to control the terms $\mathbb{E}[\Delta_t \text{E} \,|\, \text{Ext}(\mathsf{C}_t) = z]$ for z deep within the mixed block. Sample ω^{IIC} according to $\phi^{\text{IIC}}_{\mathbb{L}_{\text{mix}},4}$ and translate it by z. Include this sample under the measure $\mathbb{P}$ so as to maximise the probability under $\mathbb{P}[\,\cdot \,|\, \text{Ext}(\mathsf{C}_t) = z]$ that ω_t and ω^{IIC} are identical in $\Lambda_r(z)$.

Write C^{IIC} for the cluster of z in ω^{IIC} and $\tilde{\mathsf{C}}^{\text{IIC}}$ for the corresponding cluster in $(\mathbf{S}_{t+1} \circ \mathbf{S}_t)(\omega^{\text{IIC}})$. Notice that the effect of $(\mathbf{S}_{t+1} \circ \mathbf{S}_t)$ on the local environment of $\text{Ext}(\mathsf{C}^{\text{IIC}})$ is identical to that of $(\mathbf{S}_{\text{odd}} \circ \mathbf{S}_{\text{even}})$. In particular, we have that

$$\text{E}(\tilde{\mathsf{C}}^{\text{IIC}}) - \text{E}(\mathsf{C}^{\text{IIC}}) = \Delta^{\text{IIC}}\text{E}.$$

There are multiple reasons why the increment $\Delta_t\text{E}$ might differ from $\Delta^{\text{IIC}}\text{E}$. We say a *coupling error* occurs if the configurations ω_t and ω^{IIC} are not identical in $\Lambda_r(z)$. An *increment error* occurs if the configurations $(\mathbf{S}_{t+1} \circ \mathbf{S}_t)(\omega^{\text{IIC}})$ and $\omega_{t+2} = (\mathbf{S}_{t+1} \circ \mathbf{S}_t)(\omega_t)$ are not identical on $\Lambda_{r/2}(z)$. Finally, an *IIC error* occurs if the extremum of $\tilde{\mathsf{C}}^{\text{IIC}}$ is not contained in $\Lambda_{r/2}(z)$.

If none of these errors occur, then ω_{t+2} is identical to $(\mathbf{S}_{t+1} \circ \mathbf{S}_t)(\omega^{\text{IIC}})$ on $\Lambda_{r/2}(z)$ and said box contains $\text{Ext}(\tilde{\mathsf{C}}^{\text{IIC}})$. In particular, $\text{Ext}(\tilde{\mathsf{C}}^{\text{IIC}}) \in \mathsf{C}_{t+2}$ and therefore

$$\text{E}(\mathsf{C}_{t+2}) \geq \text{E}(\tilde{\mathsf{C}}^{\text{IIC}}).$$

The inequality comes from the case where C_{t+2} has an extremum outside of $\Lambda_{r/2}(z)$. While we expect this to be unlikely under $\mathbb{P}[\,\cdot \,|\, \text{Ext}(\mathsf{C}_t) = z]$, we will not endeavour to bound its probability.

By collecting the error terms, we obtain

$$\begin{aligned} \mathbb{E}\big[\Delta_t\text{E} \,|\, \text{Ext}(\mathsf{C}_t) = z\big] &\geq \mathbb{E}\big[\Delta^{\text{IIC}}\text{E} - 4\mathbf{1}_{\text{error}} \,|\, \text{Ext}(\mathsf{C}_t) = z\big] \\ &\geq -4\mathbb{P}\big[\text{error} \,|\, \text{Ext}(\mathsf{C}_t) = z\big], \end{aligned} \tag{4.14}$$

where the first inequality is due to the deterministic bound on increments and the second one to the fact that that $\mathbb{E}[\Delta^{\text{IIC}}\text{E}] = 0$ — see (2.5).

We will now bound the probabilities of each type of error occurring. We start with the IIC and increment errors, as these are controlled by taking r large enough. Indeed, since the track exchanges are *local* transformations, the probability under $\mathbb{P}[\,\cdot \,|\, \text{Ext}(\mathsf{C}_t) = z]$ that an increment error occurs is bounded by Ce^{-cr} for universal constants C and c — see Proposition 2.2. For an IIC error to occur, $\mathsf{C}^{\rm IIC}$ should contain a point $z' \notin \Lambda_{r/2}(z)$ with

$$\langle z', e_\theta\rangle \geq \langle z, e_\theta\rangle - 2.$$

By a union bound on the possible values of z' and using Proposition 2.4, the probability of an IIC error occurring may be bounded by Cr^{-1} for some universal constant C. This computation is identical to the corresponding one in [DCKK+20]. Thus, by taking $r \geq 1$ sufficiently large we may ensure that

$$\mathbb{P}\big[\text{increment or IIC error} \,|\, \mathrm{Ext}(\mathsf{C}_t) = z\big] \leq \delta.$$

Finally, with r fixed, Lemma 4.6 states that one may choose $R \geq r$ large enough and $q_0 > 4$ such that,

$$\mathbb{P}\big[\text{coupling error} \,|\, \mathrm{Ext}(\mathsf{C}_t) = z\big] \leq \delta.$$

Inserting the last two bounds in (4.14), we find that, for $R \geq r \geq 1$ chosen as above, and for $q \in (4, q_0]$, with q_0 as above,

$$\mathbb{E}\big[\Delta_t \mathrm{E} \,|\, \mathrm{Ext}(\mathsf{C}_t) = z\big] \geq -8\delta. \tag{4.15}$$

Conclusion of proof of (4.1): Take $R \geq r \geq 1$ as dictated by case (3), so that (4.15) holds. Consider now $q_0 > 4$ sufficiently close to 4 so that (4.8), (4.13), and (4.15) hold for all $4 < q < q_0$, with the values of r, R chosen above. Then, summing these bounds we find

$$\mathbb{E}\big[\Delta_t \mathrm{E} \,|\, \mathrm{E}(\mathsf{C}_t) \in [n\varepsilon, (n+1)\varepsilon)\big] \geq -12\delta,$$

as required. □

4.3 Deducing universality: proof of Theorem 1.3

With Corollary 4.3 at hand, we are almost ready to prove our main result. We will prove Proposition 4.1 and then see how it implies Theorem 1.3.

Proof of Proposition 4.1. Fix $\varepsilon > 0$ and $\alpha, \beta \in (\varepsilon, \pi - \varepsilon)$ and $\theta \neq \frac{3\pi}{2}$. All constants below will be independent of α, β and θ, but may depend on ε.

Due to (1.1), we may fix C such that, for any $q > 4$ and any $n \geq 1$ large enough[8],

$$\phi^0_{\mathbb{L}_+(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big] \leq 2\phi^0_{\mathbb{L}_+(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n} \text{ below } t_{Cn}\big]$$

and the same for $\mathbb{L}_+(\beta)$.

Set $N = Cn$ and define the lattices $\mathbb{L}_t$ accordingly. Then, by the above and Corollary 3.3,

$$\begin{aligned}\mathbb{P}\big[\mathrm{E}(\mathsf{C}_0) \geq n\big] &\geq \tfrac{1}{2}\phi^0_{\mathbb{L}_+(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big] \quad \text{and}\\ \mathbb{P}\big[\mathrm{E}(\mathsf{C}_{2N}) \geq n(1-\varepsilon)\big] &\leq 2\phi^0_{\mathbb{L}_+(\beta),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n(1-\varepsilon)}\big].\end{aligned} \tag{4.16}$$

Let $q_0 > 4$ be given by Corollary 4.3 for $\eta = \varepsilon$ and C fixed as above. Fix $q \in (4, q_0]$. By (4.2) and (4.16), we have

$$\phi^0_{\mathbb{L}_+(\beta),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n(1-\varepsilon)}\big] \geq \tfrac{\varepsilon}{4}\phi^0_{\mathbb{L}_+(\alpha),q}\big[0 \leftrightarrow \mathcal{H}^\theta_{\geq n}\big] \tag{4.17}$$

[8]The lower bound on n may depend on q, as we will take n to infinity first.

for all n sufficiently large. By taking the logarithm, dividing by $-n(1-\varepsilon)$, and taking the limit $n \to \infty$ in (4.17), we obtain

$$(1-\varepsilon) \cdot \zeta^{\mathrm{hp}}_{\beta,q}(\theta)^{-1} \leq \zeta^{\mathrm{hp}}_{\alpha,q}(\theta)^{-1}.$$

By inverting the roles of α and β in the coupling and repeating the same argument, we obtain the opposite bound. □

Proof of Theorem 1.3. We start with proving the statement about ζ. Fix $\varepsilon > 0$ and let $q_0 > 4$ be given by Proposition 4.1. Henceforth consider $q \in (4, q_0]$. Let $\theta \in [0, 2\pi)$ and choose $\tilde{\theta} \in \{\theta, \theta + \pi\}$ (mod 2π) to be upper-half-plane aiming for $\mathbb{L}(\alpha)$. Then, by Proposition 3.4 and Proposition 4.1 applied to $\tilde{\theta}$,

$$\zeta_{\alpha,q}(\theta) = \zeta_{\alpha,q}(\tilde{\theta}) = \zeta^{\mathrm{hp}}_{\alpha,q}(\tilde{\theta}) \leq (1+\varepsilon)\, \zeta^{\mathrm{hp}}_{\frac{\pi}{2},q}(\tilde{\theta}) \leq (1+\varepsilon)\, \zeta_{\frac{\pi}{2},q}(\tilde{\theta}) = (1+\varepsilon)\, \zeta_{\frac{\pi}{2},q}(\theta).$$

Applying the same reasoning with the roles of α and $\pi/2$ exchanged we obtain the opposite bound. Note that the value of $\tilde{\theta}$ may a priori change for this second case.

Finally, the statement about ξ is readily deduced from that about ζ using (1.2). □

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