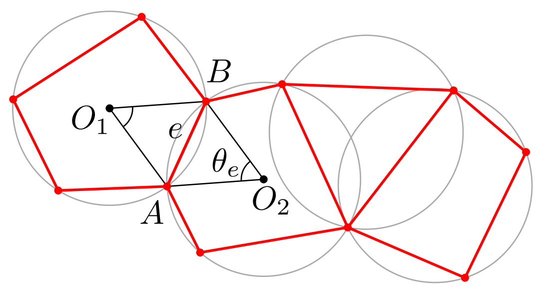
Universality for planar percolation



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Abstract

The main goal of this thesis is the study of percolation on isoradial graphs, and, more precisely, to show criticality and universality of arm exponents for these models.

An *isoradial graph* G is a planar graph embedded in the plane in such a way that every face is inscribed in a circle of radius 1. To each edge e we attach a parameter $p(e) \in [0, 1]$, which is an explicit function of the length of e. We associate to G a canonical percolation model, under which each edge e is taken open with probability $p(e)$ and closed with probability $1 - p(e)$, independently of other edges. Thus, isoradial graphs provide a large class of planar percolation models expected to be critical and to belong to the same universality class. These models include the critical homogeneous bond percolation on the square, triangular and hexagonal lattices. More generally, isoradial graphs have proved to be a particularly convenient setting for the study of various statistical mechanics models.

We will focus on two features of critical percolation models. The *box crossing property* (or RSW property) states that the probability of crossing rectangular domains of given aspect ratio is bounded away from 0 and 1, uniformly in the size of the domain. The *arm exponents* are constants that describe the asymptotic behaviour of certain unlikely events, such as that the cluster of a given vertex has large radius.

Using the *star–triangle* transformation, and its particular affinity with percolation on isoradial graphs, we manage to convert one isoradial graph into another, while preserving certain features of the percolation model. These features are related to existence of open connections; in particular we prove the universality of the box-crossing property and of the arm exponents across a large class of graphs. The box-crossing property is known to hold for certain isoradial graphs, such as the homogeneous square lattice, hence it extends to the studied models. Arm exponents however are not known to exist for any planar bond percolation model, and we make no progress on this point.

We also give a detailed account of how the box-crossing property implies criticality, as well as a certain form of isotropy of the critical phase. This is then used to prove scaling relations that relate the arm exponents to other critical exponents.

Acknowledgements

I am extremely grateful to my supervisor, Geoffrey Grimmett, who guided and inspired me throughout my thesis. Due to him I discovered a beautiful field of mathematics and learned to approach problems as a researcher, not as a student. Our collaboration was a delightful and enriching experience, which I hope will continue. I would also like to thank Wendelin Werner, who encouraged me to pursue a PhD under the supervision of Geoffrey Grimmett.

During the last years, I have enjoyed fruitful discussions with many mathematicians, of whom I would like to mention Richard Kenyon, Hugo Duminil-Copin and Christophe Garban. I apologize to those unnamed.

My deepest thanks to Nayia Constantinou; her company and support have made this PhD a very happy period. I can not overemphasize her importance during these years. My colleagues and friends, especially the ones in the CCA, have made Cambridge a stimulating and fun place to work and live.

Finally, my family and friends outside Cambridge have constantly encouraged me and helped me through the tougher periods. I am endowed to them, who, sometimes without their knowledge, have kept me going.

I thank Julia Blackwell, John Shimmon and Sheena Bridgman for their kind assistance in administrative matters. I acknowledge essential financial support from the Engineering and Physical Sciences Research Council, the Cambridge European trust, the Department of Pure Mathematics and Mathematics Statistics and the Peter Whittle Fund.

Statement of Originality

I hereby declare that my dissertation entitled "Universality for planar percolation" is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further state that no part of my dissertation has already been or is concurrently submitted for any such degree of diploma or other qualification.

This dissertation is the result of my own work, done partly in collaboration with my supervisor, Prof. G. Grimmett. Below is a more detailed account.

Chapter 1 contains the notation used throughout the dissertation as well as the main results and discussions around them.

Chapter 2 highlights the importance of the box-crossing property and of the arm exponents. It is mainly review work, and has been done mostly personally.

Chapter 3 contains a detailed presentation of isoradial graphs and of the star–triangle transformation. It is mainly a review of known material adapted to our setting. It is the result of collaboration with Prof. G. Grimmett and is partially contained in [GMa, GMb, GM11].

Chapter 4 is an initial approach we had to the problem of universality. It is original work, also published in [GMa, GMb]. It is the result of collaboration.

Chapter 5 contains the proof of the main results of the dissertation. It is original work, also published in [GM11]. It is the result of collaboration.

Contents

Chapter 1

Introduction

1.1 Overview

The idea that statistical physics models should, at large scale, be characterized by only few parameters appeared in the physics literature in the 1960's under the name *universality*.

Consider a large system of interacting particles, each taking a random state, with the states of different particles being correlated following a certain correlation structure. The intensity of the correlation is given by a parameter, usually the temperature. In very vague terms, the *renormalization group* rescales the above model, and yields an equivalent model with modified parameters. In the new model, each particle represents a group of particles of the initial model. When performing repeatedly this renormalization, the parameters degenerate, unless at certain specific points called *critical points*. In the latter case, most observables of the system become irrelevant after repeated rescaling, and only few are relevant for the large scale behaviour. In particular, systems that are different at microscopic scale may, if their differences become irrelevant, have the same large scale behaviour.

In the following decades the concept of universality became more and more widespread, also penetrating through to mathematics. Although indications of universality appear in various fields, we rarely have a good understanding of the phenomenon. From a mathematician's point of view, physics provides predictions, and arguments in favour of these predictions, but not rigorous proofs. Despite the important mathematical efforts of the last years in understanding scaling limits, only few models have been fully solved, and many await.

The first instance of universality that comes to mind to a probabilist is surely the central limit theorem. If $(X_i)_{i\in\mathbb{N}}$ are i.i.d random variables of mean 0 and variance 1, then $S_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$ converges to a normal variable, regardless of the law of X_i. Let us take a further step, and consider the convergence of random walk to Brownian motion. Regardless of the law of the step (provided it's centered and has finite variance), the trajectory of the random walk converges to a Brownian path. As the limit of a renormalization process, the Brownian path is *scale invariant*. In this case the only parameter relevant for the limit is the dimension.

In two dimensions, in addition to universality and scale invariance, statistical physics models should, at large scale, exhibit *conformal invariance*. Oded Schramm has observed that, if a scaling limit abides to this prediction, then its interfaces have to converge to one of the random curves called SLE (Schramm-Loewner evolution). For $\kappa > 0$, SLE_κ is a family of random curves indexed by a simply connected domain and two points on its boundary; it is conformally invariant and has the domain Markov property. Since these curves should describe the limits of all critical planar statistical physics models, all such models may be indexed using only the parameter κ.

From a probabilist's point of view, the simplest interacting particle system should be percolation, precisely because it lacks interaction. In its most common form, it is a one-parameter system which exhibits a phase transition similar to that of most systems in statistical physics. The fact that different regions of space have independent behaviour is particularly convenient when studying percolation. But the partition function, which usually allows a simple understanding of the system, is, in this case, trivially equal to 1, thus rendering its study futile. For planar percolation mathematicians have developed geometrical arguments that provide remarkable results without reference to the partition functions.

Two dimensional percolation is fully understood only in the case of site percolation on the triangular lattice, where Smirnov proved the convergence of the exploration process to SLE_6 [Smi01]. Understanding critical percolation on other lattices, and confirming the universality prediction, is probably the greatest challenge in two-dimensional percolation today.

In the present dissertation, we discuss the problem of universality for the canonical percolation on so-called isoradial graphs. These graphs provide a large class of planar bond percolation models, which include standard percolation on the three most studied lattices (square, triangular and hexagonal).

Isoradial graphs have been noticed to constitute a particularly convenient setting for the study of statistical physics models, as illustrated by the recent analysis of the critical Ising model by Chelkak and Smirnov [CS10, CS12]. On the one hand, such isoradial graphs are especially harmonious in a theory of discrete holomorphic functions (introduced by Duffin, see [CS11, Duf68, Mer01]), and on the other they are well adapted to transformations of star–triangle type (explained by Kenyon [Ken04]). These two properties resonate with the intertwined concepts of conformality and universality.

Isoradial graphs appear, therefore, as the "right" embedding, that allows percolation to converge to its scaling limit. Nevertheless this remains a conjecture.

Our much more modest goals are proving criticality for isoradial percolation, and a weaker form of universality, that of critical exponents. We achieve this by means of the star–triangle transformation, which we use to transform one isoradial graph into another, while preserving certain properties related to connectedness. The spirit of our approach is very close to the idea of universality, since it shows that different models are essentially the same. In addition to the concrete results it provides, it constitutes a link between models, which could be used to also transfer other properties.

In a recent lecture in Cambridge, while talking about universality for random matrices, Terence Tao mentioned a way of proving the central limit theorem, which I find illustrative of the methods in this thesis. The idea is to take two independent sets of i.i.d variables, (X_i) and (Y_i) , each of mean 0 and variance 1, and assume the sums S_n for the (X_i) converge indeed to the normal distribution. Then we may switch one by one the variables Y_i instead of X_i, and show that S_n changes by an amount that disappears in the limit. This would then prove that the sums for (Y_i) converge to the same limit as those for (X_i). We may take (X_i) to be normal variables, so that the initial convergence is immediate.

In the same spirit, we consider an isoradial graph, which we transform locally but repeatedly by the star–triangle transformation, until we obtain a completely different graph. We show that certain large scale features are not altered by this procedure. Sadly, only some of these features are known to hold in at least one of the models involved. For such features we obtain unconditional universality, while for others we have to limit ourselves to conditional results.

1.2 Basic model and notation

1.2.1 General notation

Let $G = (V, E)$ be a countable connected graph. There are two types of percolation, site and bond, and we will focus on the second. A (bond) percolation measure $\mathbb{P}$ on G is a product measure on the sample space $\Omega = \{0, 1\}^E$. A configuration is an element $\omega \in \Omega$. An edge e is called *open* (or *ω-open*) if $\omega(e) = 1$, and *closed* otherwise. A path of G is a chain of adjacent edges of E (see Section 3.2.2 for a more precise definition). It is called *open* if all its edges are open. For $u, v \in V$, we say *u is connected to v* (in ω), written $u \leftrightarrow v$ (or $u \xleftrightarrow{G,\omega} v$), if G contains an open path from u to v; if they are not connected, we write $u \overset{G,\omega}{\not\leftrightarrow} v$. An *open cluster* of ω is a maximal set of pairwise-connected vertices. Let $C_v = \{u \in V : u \leftrightarrow v\}$ denote the open cluster containing the vertex v, and write $v \leftrightarrow \infty$ if $|C_v| = \infty$.

The intensities of the measure $\mathbb{P}$ are the probabilities $\mathbf{p} = (p_e)_{e \in E}$ given by $p_e = \mathbb{P}(e \text{ is open})$. Conversely, any family of weights $\mathbf{p} \in [0, 1]^E$ gives rise to a bond percolation measure denoted $\mathbb{P}_{\mathbf{p}}$. If the intensities are all equal to some $p \in [0, 1]$, we say $\mathbb{P}$ is *homogeneous* with intensity p. Otherwise we say it is *inhomogeneous*.

Site percolation is very similar, the only difference being that sites are declared open

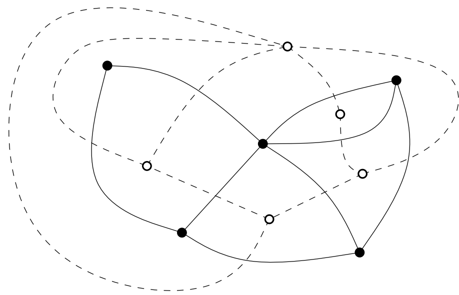

Figure 1.2.1: The graph G in solid lines, and its dual graph G^* in dashed.

or closed instead of edges. Thus site percolation measures live on $\{0,1\}^V$. The notation introduced above applies to both models.

1.2.2 Planar graphs, duality

In this work we focus on percolation on planar graphs. A graph G is called planar if it may be embedded in the plane in such a way that edges intersect only at their endpoints. Such an embedding is called a *proper* embedding. Throughout the document, when talking about a planar graph, we consider the graph, along with a proper embedding in the plane. The embedding is important for our arguments, due to their geometric nature. Thus we generally differentiate between two embeddings of the same graph.

Let G be a planar graph embedded properly in the plane $\mathbb{R}^2$. A *face* of G is a connected component of $\mathbb{R}^2 \setminus G$, where G is identified with the union of its edges and vertices. Two faces are adjacent if they share an edge.

The graph G has a *dual graph*, $G^* = (V^*, E^*)$, obtained as follows. The vertices of G^* are the faces of G. Two such vertices are connected if they correspond to adjacent faces of G. More precisely, they are connected in G^* by a number of edges of E^* equal to the number of edges of E shared by the corresponding faces of G. Thus, to each edge $e \in E$, there corresponds a unique edge $e^* \in E$. See also Figure 1.2.1.

The graph G^* is also planar, and is embedded by placing each vertex of V^* inside the corresponding face of G. An edge e^* of G^* only intersects its corresponding edge of G. Thus G^* also admits a dual, and G is one. See, for example, [Gri99, Sect. 11.2] for an account of graphical duality.

The great advantage of bond percolation on planar graphs is that we can associate to it a bond percolation on the dual graph as follows. For $\omega \in \Omega$ and $e \in E$ let $\omega^*(e^*) = 1-\omega(e)$, so that e^* is open in the dual configuration ω^* (written *open**) if and only if e is closed in the primal configuration. The notation defined for the primal is inherited by the dual. In particular, we write $u \xleftrightarrow{G,\omega^*} v$ for the event that the vertices $u, v \in V^*$ are connected in ω^*. If ω is taken according to a percolation measure $\mathbb{P}_{\mathbf{p}}$, then the configuration ω^* thus obtained also follows a percolation measure, with intensities $p_{e^*} = 1 - p_e$. We denote this dual measure $\mathbb{P}^*_{\mathbf{p}}$.

If C is a finite open cluster in a configuration ω on G, then it is surrounded by an open* circuit. This makes it possible to study planar bond percolation through geometric arguments, such as those of Section 1.5.

A similar construction exists for site percolation on planar graphs. The role of the dual graph is played by the *matching* graph, defined as follows. The vertices of the matching graph are the vertices of the original graph, and two vertices are united by an edge in the matching graph if they belong to the same face in the original graph. A vertex is considered open in the matching graph if it is closed in the original one. Thus it is common to interpret site percolation configurations as bichromatic colorings of the vertices. One colour, say red, is associated to sites open in the original graph, and the other, say blue, to those open in the matching graph.

The disadvantage of this construction is that generally the matching graph of a planar graph is not itself planar. Nevertheless, if all the faces of the original graph are triangles (we call such a graph a *triangulation*), then the matching graph is identical to the original one. This is one of the reasons why site percolation on the triangular lattice is so well understood (see Section 1.7).

1.2.3 Stochastic ordering and the FKG and BK inequalities

The following standard material is essential to the study of percolation. For proofs see for instance [Gri10, Sect. 4] and the references therein.

We start with a brief overview of stochastic ordering. Let E be a finite set, and $\Omega = \{0,1\}^E$. The set Ω has a natural partial order given by

$$\omega_1 \leq \omega_2 \quad \text{if } \omega_1(e) \leq \omega_2(e) \text{ for all } e \in E.$$

A set $A \subset \Omega$ is called *increasing* if

$$\omega_1 \leq \omega_2 \text{ and } \omega_1 \in A \Rightarrow \omega_2 \in A.$$

It is called *decreasing* if

$$\omega_1 \leq \omega_2 \text{ and } \omega_2 \in A \Rightarrow \omega_1 \in A.$$

For two probability measures η_1 and η_2 on Ω, we have the following stochastic ordering.

$$\eta_1 \leq_{\text{st}} \eta_2 \quad \text{if } \eta_1(A) \leq \eta_2(A) \text{ for all increasing sets } A \subseteq E.$$

The following result, known as Strassen's theorem, is very useful when dealing with stochastic ordering. A much more general statement than that presented next may be found in [Lin02a].

Theorem 1.2.1 ([Str65]). *Let η_1 and η_2 be probability measures on Ω. The two following statements are equivalent.*

(i) $\eta_1 \leq_{\mathrm{st}} \eta_2$,

(ii) *there exists a probability measure ν on Ω^2, with marginals η_1 and η_2, such that*

$$\nu(\{(\omega_1, \omega_2) : \omega_1 \leq \omega_2\}) = 1.$$

A probability measure η on Ω is said to be *positively associated* if

$$\eta(A \cap B) \geq \eta(A)\eta(B) \qquad \text{for all increasing events } A, B \subseteq E.$$

For two configurations $\omega_1, \omega_2 \in \Omega$ we denote $\omega_1 \vee \omega_2$ (respectively $\omega_1 \wedge \omega_2$) the pointwise maximum (respectively minimum) of ω_1 and ω_2.

Theorem 1.2.2 (FKG inequality, [FKG71]). *Let η be a strictly positive probability measure on Ω such that*

$$\eta(\omega_1 \vee \omega_2)\eta(\omega_1 \wedge \omega_2) \geq \eta(\omega_1)\eta(\omega_2), \quad \omega_1, \omega_2 \in \Omega. \tag{1.2.1}$$

Then η is positively associated.

Usually (1.2.1) is called the FKG lattice condition. See [Gri06, Sect. 2.2] and the references therein for a proof and a discussion on the FKG inequality. As a consequence, product measures are positively associated. We sometimes refer to this fact as the FKG, or Harris–FKG, inequality instead of positive association.

A second useful inequality in the study of percolation is the BK inequality, named after its authors, van den Berg and Kesten. Before stating the inequality, we need to introduce the notion of disjoint occurrence. For $\omega \in \Omega$ and $F \subseteq E$ let ω_F be the element of Ω defined by

$$\omega_F(e) = \begin{cases} \omega(e) & \text{for } e \in F, \\ 0 & \text{for } e \notin F. \end{cases} \tag{1.2.2}$$

For $A, B \subseteq \Omega$ increasing, define the set

$$A \circ B = \{\omega \in \Omega : \text{ there exists } F \subseteq E \text{ such that } \omega_F \in A \text{ and } \omega_{E \setminus F} \in B\}.$$

With this notation we have the following result.

Theorem 1.2.3 (BK inequality, [BK85]). *For η a product measure on Ω and $A, B \subseteq \Omega$*

increasing,

$$\eta(A \circ B) \le \eta(A)\eta(B). \tag{1.2.3}$$

Stochastic ordering and positive association may be extended to countably infinite sets E as discussed in [Gri06, Sect. 4.1]. In this case, the FKG and BK inequalities may be used for events that depend only on the states of finitely many coordinates of ω. This extension is particularly simple in the case of product measures; no further details are given here.

1.3 Concrete models

1.3.1 General conditions

Even though the ultimate goal of the present work is to study isoradial graphs, some results will be stated in greater generality. Nevertheless we require some minimal conditions on the graphs we work with.

We say a planar graph covers the plane if all its faces have finite diameter. If not otherwise stated, we will always consider that our planar graphs cover the plane.

Let G be a planar graph. In all our illustrations we will consider both G and its dual, G^*, to be embedded with edges as straight line segments. This is not an essential requirement in what follows. Here are two conditions that we will assume to hold for all graphs in this work.

- **Bounded edge lengths**. There exists a constant $L_e > 0$, such that all edges of G and G^* have length at most L_e.
- **Bounded vertex density**. There exist constants L_d, K_d such that, for any $(x, y) \in \mathbb{R}^2$, the number of both primal and dual vertices inside the square $[x, x+L_d] \times [y, y+L_d]$ is at least 1 and at most K_d.

Let G be a planar graph such that both G and G^* satisfy the conditions above. It follows that G is *locally finite*, in that, for any bounded domain in the plane, there are only finitely many elements (i.e. vertices and edges) of G intersecting it.

Sometimes we will work with graphs exhibiting various forms of symmetry. We give a list of terms which will be used throughout the paper.

We say G *periodic* (or translation invariant) if there exist independent non-zero vectors $\tau_1, \tau_2 \in \mathbb{R}^2$, such that G is invariant under shifts by either τ_i. A percolation measure $\mathbb{P}$ on G is said to be periodic if G is periodic and if the measure is also invariant under the shifts described above. We say G is *vertex transitive* if for any two vertices u and v there exists an automorphism of G sending u onto v.

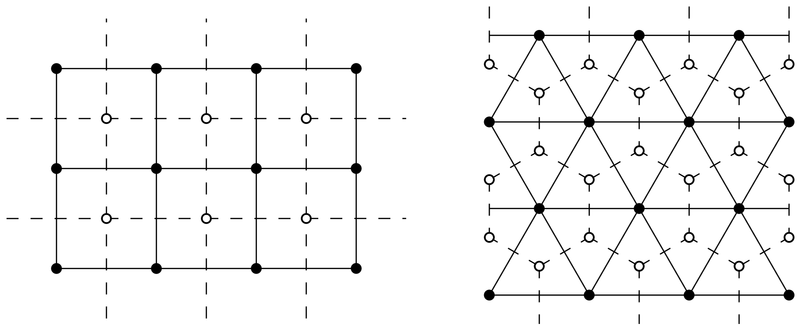

Figure 1.3.1: The square lattice and its dual square lattice. The triangular lattice and its dual hexagonal lattice.

A model $(G, \mathbb{P})$ is called *rotation invariant*, if it is invariant under rotation by some angle $\alpha \in (0, \pi)$ around some point u.

It is called *reflection invariant* if it is invariant under reflection with respect to some line d. We say it is invariant under reflection with respect to the axes, if it is invariant under reflection with respect to two perpendicular lines. We will usually assume these lines to be the axes of $\mathbb{R}^2$.

1.3.2 Lattices

In Chapter 4 we present a first approach to the problem of universality. There we do not use isoradial graphs, but rather a wide class of percolation models on three lattices which we define next. We do not attempt to give a general definition of lattices here, instead we will present the three lattices we will work with.

The square, triangular, and hexagonal (or honeycomb) lattices of Figure 1.3.1, are denoted respectively $\mathbb{Z}^2$, $\mathbb{T}$, and $\mathbb{H}$. Homogeneous percolation on these lattices is a one parameter model, and we denote $\mathbb{P}_p^{\square}$, $\mathbb{P}_p^{\triangle}$ and, respectively, $\mathbb{P}_p^{\hexagon}$ the measures with intensity $p \in [0, 1]$.

The dual of $(\mathbb{Z}^2, \mathbb{P}_p^{\square})$ is $(\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2}), \mathbb{P}_{1-p}^{\square})$, where $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ is the shift of $\mathbb{Z}^2$ by the vector $(\frac{1}{2}, \frac{1}{2})$. The dual of $(\mathbb{T}, \mathbb{P}_p^{\triangle})$ is $(\mathbb{H}, \mathbb{P}_{1-p}^{\hexagon})$.

We now turn to *inhomogeneous* percolation on the above three lattices. The edges of the square lattice are partitioned into two classes (horizontal and vertical) of parallel edges, while those of the triangular and hexagonal lattices may be split into three such classes. We allow the product measure on Ω to have different intensities on different edges, while requiring that any two parallel edges have the same intensity. Thus, inhomogeneous percolation on the square lattice has two parameters, p_0 for horizontal edges and p_1 for vertical edges, and we denote the corresponding measure $\mathbb{P}_{\mathbf{p}}^{\square}$ where $\mathbf{p} = (p_0, p_1)$. On the triangular and hexagonal lattices, the measure is defined by a triplet of parameters $\mathbf{p} = (p_0, p_1, p_2)$, and we denote these measures $\mathbb{P}_{\mathbf{p}}^{\triangle}$ and $\mathbb{P}_{\mathbf{p}}^{\hexagon}$, respectively.

The inhomogeneous models possess translation-invariance but not rotation-invariance.

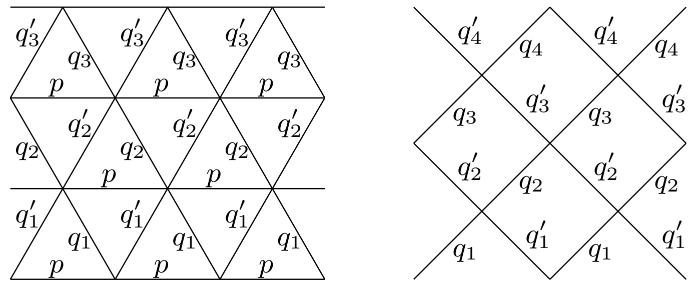

Figure 1.3.2: *Left:* The triangular lattice with the highly inhomogeneous product measure $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$. The probability for each edge to be open is described in the picture: all horizontal edges have probability p of being open, while the other edges have probability q_n (right edges of upwards pointing triangles) or q_n' (left edges of upwards pointing triangles) of being open, with n being their height. *Right:* The square lattice with a highly inhomogeneous product measure $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$, rotated by $\pi/4$. Edges inclined at angle $\pi/4$ have probability q_n of being open, while edges inclined at angle $3\pi/4$ have probability q_n' of being open, with n being their height.

Full translation-invariance is in fact inessential to the arguments of Chapter 4. To illustrate this we introduce the so-called 'highly inhomogeneous models'. They also serve as a connection between the approach of Chapter 4 and the isoradial graphs of Chapter 5.

Let $p \in (0,1)$, and let $\mathbf{q} = (q_n : n \in \mathbb{Z}) \in [0,1]^{\mathbb{Z}}$ and $\mathbf{q}' = (q_n' : n \in \mathbb{Z}) \in [0,1]^{\mathbb{Z}}$. These are the parameters of our highly inhomogeneous models on the square, triangular and hexagonal lattices.

Consider first the triangular lattice, and write $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$ for the product measure on Ω under which: any horizontal edge is open with probability p; any right (respectively, left) edge of an upwards pointing triangle is open with probability q_n (respectively, q_n'). Here, $n \in \mathbb{Z}$ denotes the height of the edge as drawn in the Figure 1.3.2. Let $\mathbb{P}^{\hexagon}_{1-p,1-\mathbf{q},1-\mathbf{q}'}$ be the measure on the hexagonal lattice that is dual to $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$.

Consider next the square lattice. The measure $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$ is defined similarly to the above, as in Figure 1.3.2. We refer to the three probability measures thus defined as *highly inhomogeneous*.

Note that the square, triangular and hexagonal lattices, embedded as in Figure 1.3.1, do indeed satisfy the conditions of Section 1.3.1.

1.3.3 Isoradial graphs

Let G be a planar graph embedded in the plane $\mathbb{R}^2$, with edges embedded as straight-line segments. It is called *isoradial* if there exists $r > 0$ such that, for every bounded face F of G, the vertices of F lie on a circle of (circum)radius r with centre in the interior of F. Note that isoradiality is a property of the planar embedding of G rather than of the graph itself. By rescaling the embedding of G, we may assume $r = 1$. In the absence of a contrary assumption, we shall assume that isoradial graphs are infinite with all faces

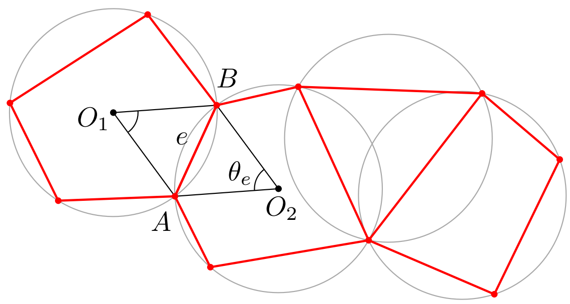

Figure 1.3.3: Part of an isoradial graph. Each face is inscribed in a circle of radius 1. With the edge e, we associate the angle θ_e.

bounded.

It was noted by Duffin [Duf68] that isoradial graphs are in two–one correspondence with rhombic tilings of the plane (i.e. there exists an explicit pairing of isoradial graphs indexed by rhombic tilings). The name 'isoradial' was coined later by Kenyon. While details of this correspondence are deferred to Section 3.1, we highlight one fact here. Let $G = (V, E)$ be isoradial. An edge $e \in E$ lies in two faces, and therefore two circumcircles. As illustrated in Figure 1.3.3, e subtends the same angle $\theta_e \in (0, \pi)$ at the centres of these circumcircles, and we define $p_e \in (0, 1)$ by

$$\frac{p_e}{1 - p_e} = \frac{\sin(\frac{1}{3}[\pi - \theta_e])}{\sin(\frac{1}{3}\theta_e)}. \tag{1.3.1}$$

We consider bond percolation on G with edge-probabilities $\mathbf{p} = (p_e : e \in E)$. This percolation measure is the canonical percolation on G, and is written $\mathbb{P}_G$.

Definition 1.3.1. *Let $\epsilon > 0$. The isoradial graph G is said to have the* bounded-angles property BAP(ϵ) *if*

$$\theta_e \in [\epsilon, \pi - \epsilon], \qquad e \in E. \tag{1.3.2}$$

It is said to have, simply, the bounded-angles property *if it satisfies* BAP(ϵ) *for some* $\epsilon > 0$.

All isoradial graphs of this paper will be assumed to have the bounded-angles property. Under this assumption, it is easy to see that the conditions of Section 1.3.1 hold.

In Section 3.1 we will introduce a second condition on isoradial graphs, called the square-grid property. Loosely speaking, the square-grid property states that there exists a square lattice structure embedded in some suitable sense in the graph. Details and examples will be given in due course. We denote $\mathcal{G}$ the family of isoradial graphs satisfying the bounded-angles property and the square-grid property.

1.4 Phase transition

1.4.1 Homogeneous square lattice: an example

The object of percolation is the study of the geometry of connected components. A first question is whether there exist infinite components.

Let us consider bond percolation on the square lattice $\mathbb{Z}^2 = (V, E)$. We present a standard argument that allows us to couple the measures $\mathbb{P}_p^{\square}$ for $p \in [0,1]$. Let $(U_e)_{e \in E}$ be a family of independent uniform variables in $[0,1]$. For $p \in [0,1]$ and $e \in E$, let $\omega^p(e) = \mathbf{1}_{\{U_e < p\}}$, where $\mathbf{1}_A$ is the indicator function of the event A. With this definition ω^p has law $\mathbb{P}_p^{\square}$, and $\omega^p \leq \omega^q$ for $p \leq q$. Hence the family of measures $(\mathbb{P}_p^{\square})_{p \in [0,1]}$ is increasing in p.

Let O denote a particular vertex of the square lattice called the origin, and define

$$\theta(p) = \mathbb{P}_p^{\square}(O \leftrightarrow \infty).$$

By the above θ is an increasing function, and we set

$$p_c(\mathbb{Z}^2) = \sup\{p : \theta(p) = 0\}.$$

By Kolmogorov's zero-one law, if $p < p_c(\mathbb{Z}^2)$, there exists $\mathbb{P}_p^{\square}$-a.s. no infinite open cluster and, if $p > p_c(\mathbb{Z}^2)$, there exists $\mathbb{P}_p^{\square}$-a.s. at least one infinite open cluster.

The parameter $p_c(\mathbb{Z}^2)$ is called the critical point of (bond percolation on) the square lattice, and $\mathbb{P}_{p_c(\mathbb{Z}^2)}^{\square}$ is called a critical percolation measure on $\mathbb{Z}^2$. Similarly we define $p_c(\mathbb{T})$ and $p_c(\mathbb{H})$.

We say the model undergoes a *phase transition* at the critical value of p. As we will later see, it is particularly interesting to study this phase transition; more precisely to study the geometry of the model for p equal or close to the critical value.

1.4.2 General graphs

While defining criticality is straightforward for homogeneous percolation, it is not obvious how to do this for inhomogeneous models. We will attempt to replicate the definition of the previous section.

Let $G = (V, E)$ be an infinite, connected graph, and let $\mathbb{P}$ be a product measure on $\{0,1\}^E$ with intensities $(p_e : e \in E)$. For $\delta \in \mathbb{R}$, we write $\mathbb{P}^\delta$ for the percolation measure with intensities $p_e^\delta := (0 \vee (p_e + \delta)) \wedge 1$. [As usual, $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.]

We say that $\mathbb{P}$ is *critical* if, for any $\delta > 0$, there exists $\mathbb{P}^{-\delta}$-a.s. no infinite open cluster, and there exists $\mathbb{P}^\delta$-a.s. at least one infinite open cluster. In the same vein, we call $\mathbb{P}$ (strictly) *supercritical* if there exists $\delta > 0$ such that there exists $\mathbb{P}^{-\delta}$-a.s. at least one infinite open cluster. Conversely, $\mathbb{P}$ is (strictly) *subcritical* if there exists $\delta > 0$ such that there exists $\mathbb{P}^\delta$-a.s. no infinite open cluster. These definitions are not standard, and we do not claim that they are the "right" ones. They merely provide the concerned reader with a clear understanding of terms that will be used frequently in what follows.

One may define subcriticality and supercriticality alternatively, purely in terms of the non-existence and, respectively, existence of an infinite component. The former definitions are stronger than the latter, hence the qualification "strictly".

An alternative definition of supercriticality, which will be used later, is to call $\mathbb{P}$ *uniformly supercritical* if there exists $\theta > 0$ such that $\mathbb{P}(v \leftrightarrow \infty) \geq \theta$ for every vertex v.

For two vectors $\mathbf{p} = (p_e)_{e\in E}$ and $\mathbf{p}' = (p'_e)_{e\in E}$, we say $\mathbf{p} \leq \mathbf{p}'$ if $p_e \leq p'_e$ for all $e \in E$. We say $\mathbf{p} < \mathbf{p}'$ if $\mathbf{p} \leq \mathbf{p}'$ and $\mathbf{p} \neq \mathbf{p}'$. The disadvantage of the above definition of criticality is that we may have two critical measures, $\mathbb{P}_\mathbf{p}$ and $\mathbb{P}_{\mathbf{p}'}$, with $\mathbf{p} < \mathbf{p}'$. Nevertheless, for most periodic models, the above can not occur.

Take G a periodic graph. Assume that each edge of G is part of a doubly infinite, non-intersecting chain of edges. Let $\mathbb{P}_\mathbf{p}$ and $\mathbb{P}_{\mathbf{p}'}$ be two periodic percolation measures on G, with $\mathbf{p}, \mathbf{p}' \in (0,1)^E$. Assume $\mathbb{P}_\mathbf{p}$ is critical, then

(a) if $\mathbf{p} < \mathbf{p}'$, then $\mathbb{P}_{\mathbf{p}'}$ is supercritical,

(b) if $\mathbf{p} > \mathbf{p}'$, then $\mathbb{P}_{\mathbf{p}'}$ is subcritical.

We will not give a proof of the above, we only note that it uses the technique of enhancement; see [Gri99, Section 3.3].

In most models, it is expected that the three phases (critical, sub- and supercritical) have very different behaviour (see Theorem 5.1.2). While the large-scale behaviour of the sub- and supercritical phases is somewhat trivial, the critical phase is expected to exhibit interesting features, such as scale invariance, and, when G is planar, conformal invariance. This statement is of course vague and may be interpreted in several ways. In the following three sections we will present some of the features expected from critical models.

1.4.3 Inhomogeneous, highly inhomogeneous and isoradial models

One of the main objectives of this work is to prove criticality for some of the models of Section 1.3.2, as well as for the isoradial graphs of Section 1.3.3. For the former, it will be convenient to use the following notation.

$$\kappa_\square(\mathbf{p}) = p_\mathrm{h} + p_\mathrm{v} - 1, \qquad \text{for } \mathbf{p} = (p_\mathrm{h}, p_\mathrm{v}), \tag{1.4.1}$$

$$\kappa_\triangle(\mathbf{p}) = p_0 + p_1 + p_2 - p_0p_1p_2 - 1, \qquad \text{for } \mathbf{p} = (p_0, p_1, p_2), \tag{1.4.2}$$

$$\kappa_{\hexagon}(\mathbf{p}) = -\kappa_\triangle(1-p_0, 1-p_1, 1-p_2), \qquad \text{for } \mathbf{p} = (p_0, p_1, p_2). \tag{1.4.3}$$

With this notation we can state the following criticality criteria.

Theorem 1.4.1. *The critical surfaces of inhomogeneous percolation models on the square, triangular, and hexagonal lattice, as presented in Section 1.3.2, are given as follows.*

(a) Square lattice: $\kappa_{\square}(\mathbf{p}) = 0$.

(b) Triangular lattice: $\kappa_{\triangle}(\mathbf{p}) = 0$.

(c) Hexagonal lattice: $\kappa_{\hexagon}(\mathbf{p}) = 0$.

The above theorem was predicted in [SE64], and discussed in [Kes82, Sect. 3.4], where part (a) was proved and examples presented in support of parts (b) and (c). The complete proof of the theorem may be found in [Gri99, Sect. 11.9]. This proof is notably different from the proof we give in Chapter 4, and we will not refer to it.

We call a triplet $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ *self-dual* if it satisfies $\kappa_{\triangle}(\mathbf{p}) = 0$. Let $\mathcal{M}$ denote the set of critical inhomogeneous bond percolation models on the square, triangular, and hexagonal lattices, as given in the theorem.

We now move on to the highly inhomogeneous models on the square, triangular, and hexagonal lattice, also presented in Section 1.3.2.

Theorem 1.4.2. *Let $p \in (0,1)$ and $\mathbf{q}, \mathbf{q}' \in [0,1)^{\mathbb{Z}}$.*

(a) *If there exists $\epsilon > 0$ such that for all $n \in \mathbb{Z}$,*

$$\kappa_{\square}(q_n, q'_n) = 0 \quad \text{and} \quad q_n, q'_n \in (\epsilon, 1-\epsilon), \tag{1.4.4}$$

then $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$ is critical.

(b) *If, for all $n \in \mathbb{Z}$, $\kappa_{\triangle}(p, q_n, q'_n) = 0$, then $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$ is critical.*

(c) *If, for all $n \in \mathbb{Z}$, $\kappa_{\hexagon}(p, q_n, q'_n) = 0$, then $\mathbb{P}^{\hexagon}_{p,\mathbf{q},\mathbf{q}'}$ is critical.*

Let $\mathcal{M}_I$ denote the set of critical highly inhomogeneous models as given in the theorem above. Also, we write $\mathcal{M}_I(\epsilon)$ for the models of $\mathcal{M}_I$ satisfying

(i) for the square lattice, $q_n, q'_n \in (\epsilon, 1-\epsilon)$ for all $n \in \mathbb{Z}$,

(ii) for the triangular and hexagonal lattices, $p \in (\epsilon, 1-\epsilon)$.

We have $\mathcal{M}_I = \cup_{\epsilon>0}\mathcal{M}_I(\epsilon)$ and $\mathcal{M} \subset \mathcal{M}_I$.

Finally, for isoradial graphs, we will prove the following.

Theorem 1.4.3. *For $G \in \mathcal{G}$, $\mathbb{P}_G$ is critical.*

The three theorems above are largely overlapping. The first theorem is a particular case of the second. Most of the models in $\mathcal{M}_I$ may be interpreted as isoradial graphs that fall under the incidence of Theorem 1.4.3. More details on this point will be provided in Section 3.1.4. More precise statements of the above theorems are given in Sections 4.1 and 5.1. The different statements of Theorems 1.4.1 and 1.4.2 reflect the structure of the proof.

The proofs of all three theorem go through geometrical constructions based on the box-crossing property, which we introduce next.

1.5 The box-crossing property

Let $G = (V, E)$ be a planar graph, and $\Omega := \{0, 1\}^E$. As usual we consider G embedded in a fixed, proper way in the plane $\mathbb{R}^2$. The 'box-crossing property' is concerned with the probabilities of open crossings of domains in $\mathbb{R}^2$. This has proved to be a very useful property indeed for the study of infinite open clusters in G; see, for example, [Gri10, Kes82].

A *(planar) domain* $\mathcal{D}$ is an open, simply connected subset of $\mathbb{R}^2$ which, for simplicity, we assume to be bounded by a Jordan curve $\partial\mathcal{D}$. Most domains of this paper are the interiors of polygons. Let $\mathcal{D}$ be a domain, and let A, B, C, D be distinct points on its boundary in anticlockwise order. Let $\omega \in \Omega$. We say that $\mathcal{D}$ has an open crossing from AD to BC if there exists an open path on G containing an arc $(\gamma_t : t \in [0, 1])$ such that: (i) $\gamma_{(0,1)} \subseteq \mathcal{D}$ (ii) γ_0 and γ_1 are on $\partial\mathcal{D}$, between A and D and between B and C respectively. Note that γ_0 and γ_1 need not be vertices of G. We will sometimes abuse notation by considering closed domains of the form $\mathcal{D} \cup \partial\mathcal{D}$. The definition of crossing is still valid in this case, and $\gamma_{(0,1)}$ is allowed to contain points of $\partial\mathcal{D}$.

A *rectangular domain* is a set $\mathcal{B} = f((0, x) \times (0, y)) \subseteq \mathbb{R}^2$, where $x, y > 0$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ comprises a rotation and a translation. The *aspect-ratio* of this rectangle is $\max\{x/y, y/x\}$. We say $\mathcal{B}$ *has open crossings* in a configuration $\omega \in \Omega$ if it has open crossings both from $f(\{0\}\times[0, y])$ to $f(\{x\}\times[0, y])$ and from $f([0, x]\times\{0\})$ to $f([0, x]\times\{y\})$. Also define the rectangular domains $\mathcal{B}(m, n) = [0, m] \times [0, n]$. A horizontal (respectively, vertical) crossing of $\mathcal{B}(m, n)$ is a crossing of $\mathcal{B}(m, n)$, from $\{0\} \times [0, n]$ to $\{m\} \times [0, n]$ (respectively, $[0, m] \times \{0\}$ to $[0, m] \times \{n\}$). Denote $\mathcal{C}_{\mathrm{h}}(\mathcal{B})$ and $\mathcal{C}_{\mathrm{v}}(\mathcal{B})$ the events that $\mathcal{B}(m, n)$ has an open horizontal (respectively, vertical) crossing.

Definition 1.5.1. *A measure $\mathbb{P}$ on Ω is said to have the* box-crossing property *if, for any $\rho > 0$, there exist $l_0 = l_0(\rho) > 0$ and $\delta = \delta(\rho) > 0$ such that, for all $l > l_0$ and all rectangular domains $\mathcal{B}$ with side-lengths l and ρl,*

$$\mathbb{P}(\mathcal{B} \textit{ has open crossings}) \geq \delta. \tag{1.5.1}$$

When working with the box-crossing property, a particularly convenient assumption is that the measure under study is positively associated, such as, for instance, the random cluster (or FK percolation) measures with $q \geq 1$. FK percolation is a family of models, similar to percolation, indexed by a cluster-weight $q > 0$. The regular percolation studied in this document is obtained for $q = 1$. For details see [Gri06]. In a standard application of the FKG inequality for positively associated measures, it suffices for the box-crossing property to consider boxes with aspect-ratio 2, and moreover only such boxes with horizontal/vertical orientation (see also Proposition 4.3.2). If (1.5.1) holds for this restricted class of boxes with $\rho = 2$ and $\delta = \delta(2)$, we say that G satisfies BXP(l_0, δ).

It was proved by Russo [Rus78] and Seymour–Welsh [SW78] that the homogeneous percolation $\mathbb{P}_p^\square$ on the square lattice with parameter $p \geq \frac{1}{2}$, satisfies the box-crossing property. It follows that, for $p = \frac{1}{2}$, both the primal and dual percolation on the square lattice have the box-crossing property. This is an essential ingredient in Kesten's proof of the fact that the critical point of bond percolation on the square lattice is $\frac{1}{2}$.

With the present tools, it is standard that the box-crossing property for a percolation measure and its dual implies criticality; a proof may be found in Section 2.1. The converse is not generally true, but it is expected to hold for most 'reasonable' models.

The result of Russo and Seymour–Welsh is commonly referred to as the RSW lemma. Strictly speaking, the RSW lemma does not solely imply the box-crossing property; it requires an input, which usually is some form of self-duality. Percolation on the square lattice with parameter $p = \frac{1}{2}$ is self-dual, and the box-crossing property follows. Other models are in the range of the RSW lemma, but do not exhibit self-duality, nor the box-crossing property. A more detailed discussion about the relationship between criticality, the box-crossing property, and the RSW lemma may be found in Section 2.2.

1.6 Critical exponents

The percolation singularity is expected to be of power-law type, and to be described by a number of so-called 'critical exponents'. These may be divided into two groups of exponents: *at criticality*, and *near criticality*. We present next the asymptotic relations defining these exponents, then discuss their existence.

First some notation. We write $f(t) \asymp g(t)$ as $t \to t_0 \in [0, \infty]$ if there exist strictly positive constants A, B such that

$$Ag(t) \leq f(t) \leq Bg(t) \tag{1.6.1}$$

in some neighborhood of t_0 (or for all large t in the case $t_0 = \infty$). For functions $f^u(t)$, $g^u(t)$ indexed by $u \in U$, we say that $f^u \asymp g^u$ *uniformly in* u (sometimes written $f^u \asymp_u g^u$) if (1.6.1) holds with constants A, B not depending on u. We write $f(t) \approx g(t)$ if $\log f(t)/\log g(t) \to 1$, and $f^u \approx g^u$ *uniformly in* u if the convergence is uniform in u.

Let $G = (V, E)$ be a graph embedded in the plane and let $\mathbb{P}_{\mathbf{p}}$ be a (critical) measure on G with intensities $\mathbf{p} \in [0, 1]^E$.

The exponents at criticality are those denoted conventionally as ρ, η, δ, and the arm exponents ρ_σ. We begin by defining the so-called *arm-events*. Let Λ_n denote the box $[-n, n]^2$ of $\mathbb{R}^2$, with boundary $\partial\Lambda_n$. For $N < n$, let $\mathcal{A}(N, n)$ be the *annulus* $[-n, n]^2 \setminus (-N, N)^2$ with inner radius N and outer radius n. The *inner* (respectively, *outer*) *boundary* of the annulus is $\partial\Lambda_N$ (respectively, $\partial\Lambda_n$). For $u \in \mathbb{R}^2$, write $\mathcal{A}^u(N, n)$ for the translate $\mathcal{A}(N, n) + u$. A *primal* (respectively, *dual*) *crossing* of $\mathcal{A}(N, n)$ is an open (respectively, open*) path whose intersection with $\mathcal{A}(N, n)$ is an arc with an endpoint in each boundary of the annulus. Primal crossings are said to have colour 1, and dual crossings colour 0.

Let $k \in \mathbb{N}$. A sequence $\sigma \in \{0,1\}^k$ is called a *colour sequence* of length k. For such σ, the arm-event $A_\sigma(N,n)$ is the event that there exist k vertex-disjoint crossings $\gamma_1, \dots, \gamma_i, \dots, \gamma_k$ of $\mathcal{A}(N,n)$ with colours σ_i taken in anticlockwise order. The corresponding event on the translated annulus $\mathcal{A}^u(N,n)$ is denoted $A^u_\sigma(N,n)$ and is said to be 'centred at u'. The value of N is largely immaterial to what follows, but $N = N(\sigma)$ is taken sufficiently large that the events $A_\sigma(N,n)$ are non-empty for $n \geq N$.

A colour sequence σ is called *monochromatic* if either $\sigma = (1,1,\dots,1)$ or $\sigma = (0,0,\dots,0)$, and *bichromatic* otherwise. It is called *alternating* if it has even length and either $\sigma = (1,0,1,0,\dots)$ or $\sigma = (0,1,0,1,\dots)$. When $\sigma = (1)$, $A_\sigma(N,n)$ is called the *one-arm-event* and denoted $A_1(N,n)$. When σ is alternating with length $k = 2j$, the corresponding event is denoted $A_{2j}(N,n)$.

The following asymptotic relations, with limits that are uniform in the choice of $v \in V$, define the exponents at criticality.

(a) volume exponent: $\mathbb{P}_{\mathbf{p}}(|C_v| = n) \approx n^{-1-1/\delta}$ as $n \to \infty$,

(b) connectivity exponent: $\mathbb{P}_{\mathbf{p}}(v \leftrightarrow w) \approx |w-v|^{-\eta}$ as $|w-v| \to \infty$,

(c) one-arm exponent: $\mathbb{P}_{\mathbf{p}}[A^v_1(N,n)] \approx n^{-\rho_1}$ as $n \to \infty$,

(d) more generally, for a colour sequence σ, the σ-arm exponent: $\mathbb{P}_{\mathbf{p}}[A^v_\sigma(N,n)] \approx n^{-\rho_\sigma}$ as $n \to \infty$, for $N \geq N_0(\sigma)$ (with $N_0(\sigma)$ not depending on v).

It is believed, but generally not proved, that the above uniformly asymptotic relations hold for suitable exponent-values, and indeed with $\approx$ replaced by the stronger relation $\asymp$.

The conventional one-arm exponent ρ is given by $\rho = 1/\rho_1$, as in [Gri99, Sect. 9.1]. When σ is alternating with length $2j$, ρ_σ is denoted ρ_{2j}, and is called the $2j$-alternating-arms exponent.

We turn now to the near-critical exponents. By subcritical exponential-decay (see Proposition 2.1.1), for $\epsilon > 0$, there exists $\xi = \xi_v(\mathbf{p} - \epsilon) \in [0,\infty)$ such that

$$-\frac{1}{n} \log \mathbb{P}_{\mathbf{p}-\epsilon}(v \leftrightarrow \partial\Lambda_n) \to 1/\xi \quad \text{as } n \to \infty,$$

where v is an arbitrary vertex. The function ξ is termed the *correlation length.*

Here are the exponents near criticality, where asymptotic relations are uniform in the choice of $v \in V$:

(a) percolation probability: $\theta(\mathbf{p}+\epsilon) := \mathbb{P}_{\mathbf{p}}(v \leftrightarrow \infty) \approx \epsilon^\beta$ as $\epsilon \downarrow 0$,

(b) correlation length: $\xi(\mathbf{p}-\epsilon) \approx \epsilon^{-\nu}$ as $\epsilon \downarrow 0$,

(c) mean cluster-size: $\mathbb{E}_{\mathbf{p}+\epsilon}(|C_v|; |C_v| < \infty) \approx |\epsilon|^{-\gamma}$ as $\epsilon \to 0$,

(d) gap exponent: for $k \geq 1$, as $\epsilon \to 0$,

$$\frac{\mathbb{E}_{\mathbf{p}+\epsilon}(|C_v|^{k+1}; |C_v| < \infty)}{\mathbb{E}_{\mathbf{p}+\epsilon}(|C_v|^{k}; |C_v| < \infty)} \approx |\epsilon|^{-\Delta}.$$

We have written $\mathbb{E}(X)$ for the mean of X under the probability measure $\mathbb{P}$, and $\mathbb{E}(X; A) = \mathbb{E}(X \mathbf{1}_A)$. In writing $\mathbf{p} \pm \epsilon$, we have assumed that $\mathbf{p} \in (\epsilon_0, 1 - \epsilon_0)$, for some $\epsilon_0 > 0$. The definition of near critical exponents may be adapted to include more general intensities, but for the present work this is irrelevant.

A critical exponent π is said to *exist* for the model $(G, \mathbb{P}_{\mathbf{p}})$ if the appropriate asymptotic relation holds uniformly in the vertex v. For a family of models $\mathcal{F}$, π is called $\mathcal{F}$*-invariant* if it exists for all $(G, \mathbb{P}) \in \mathcal{F}$, and its value is independent of the choice of $(G, \mathbb{P})$.

Critical exponents may be defined similarly for percolation models on non-planar graphs; consider for illustration d-dimensional lattices. They are believed to exist for a large class of critical percolation models, with values depending only on the dimension. Moreover, they are expected to satisfy certain relations called *scaling relations*.

We give here a more concrete conjecture concerning the existence of the critical exponents and their scaling relations.

Conjecture 1.6.1. *The critical exponents are invariant across the family of isoradial graphs endowed with the canonical percolation measure. Moreover,*

$$\eta\rho = 2, \quad 2\rho = \delta + 1, \tag{1.6.2}$$

$$\nu = \frac{1}{2 - \rho_4}, \quad \beta = \frac{2\nu}{\delta + 1}, \quad \gamma = 2\nu\frac{\delta - 1}{\delta + 1}, \quad \Delta = 2\nu\frac{\delta}{\delta + 1}. \tag{1.6.3}$$

One of the main goals of this work is to prove parts of the above conjecture. In Section 5.4 (and 4.5) we prove universality results for some exponents. More precisely, we prove that if certain arm exponents exist in one model, then they exist and are invariant across the family $\mathcal{G}$ of isoradial graphs (see Theorem 1.6.2). In a series of papers in the late 80's [Kes86, Kes87a, Kes87b] Kesten proved the scaling relations (1.6.2) and (1.6.3) for homogeneous percolation on lattices exhibiting sufficient symmetry. In Section 2 we present his proofs in greater generality, so as to apply them to our models. All our results are conditional upon the existence of the exponents.

Essentially the only two-dimensional percolation process for which critical exponents are proved to exist (and, furthermore, many of their values known explicitly) is site percolation on the triangular lattice (see [BN11, Smi01, SW01]). In accordance with the principle of universality, the values of the exponents for isoradial graphs are expected to be equal to those for site percolation on the triangular lattice. Here are the values of the exponents in this special case (which unfortunately does not belong to the class of models considered in this document).

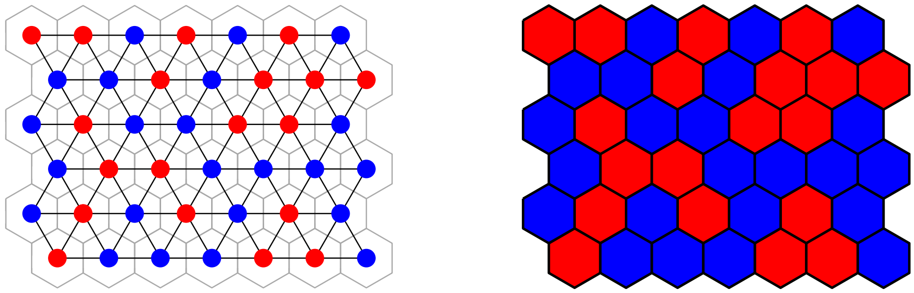

Figure 1.6.1: The site percolation on the triangular lattice in the left diagram is represented on the right as a face percolation configuration on the hexagonal lattice.

- Exponents at criticality:

$$\delta = \frac{91}{5}, \quad \eta = \frac{5}{24}, \quad \rho = \frac{48}{5}.$$

- Exponents near criticality:

$$\beta = \frac{5}{36}, \quad \nu = \frac{4}{3}, \quad \gamma = \frac{43}{18}, \quad \Delta = \frac{91}{36}.$$

- Arm exponents for σ bichromatic with length $|\sigma| > 1$:

$$\rho_\sigma = \frac{|\sigma|^2 - 1}{12}.$$

The matching graph of the triangular lattice $\mathbb{T}$, is the same triangular lattice. Thus, site percolation on the triangular lattice may be seen as a colouring with two colours (say red and blue) of the sites of the triangular lattice, or equivalently of the faces (cells) of the hexagonal lattice. See Figure 1.6.1. When $p = \frac{1}{2}$, each site has equal probability of being red or blue. Due to this special property, we may apply a technique known as colour switching to prove that the arm exponents ρ_σ are constant for all bichromatic colour sequences of given length (see [ADA99]). The monochromatic arm exponents have been studied in [BN11]. They have been proved to exist and that the k-monochromatic arm exponent is strictly between the k- and $k+1$-bichromatic arm exponents. The exact value of the monochromatic arm exponents is not known, even in the special context of site percolation on the triangular lattice.

Our main universality result for critical exponents is the following.

Theorem 1.6.2. *Let $\pi \in \{\rho\} \cup \{\rho_{2j} : j \geq 1\}$. If π exists for one model in $\mathcal{M}_I \cup \mathcal{G}$, then it is $\mathcal{M}_I \cup \mathcal{G}$-invariant.*

A more detailed version, along with several consequences, is given in Theorem 5.1.3.

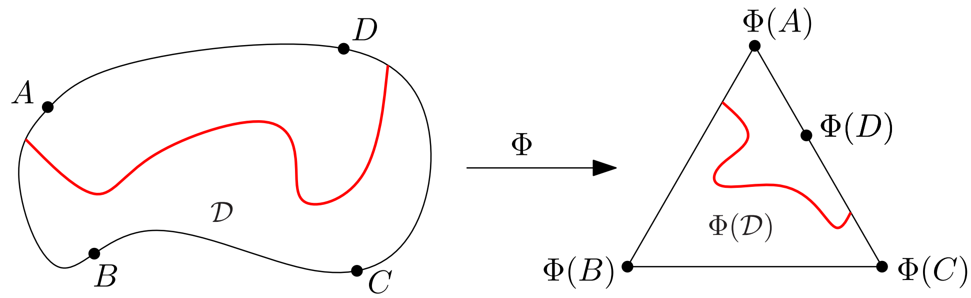

Figure 1.7.1: The Cardy-Smirnov formula. The limit of the probability that an open path in $\mathcal{D}$ joins (AB) and (CD), is the same as in the equilateral triangle $\Phi(\mathcal{D})$ with arcs $(\Phi(A)\Phi(B))$ and $(\Phi(C)\Phi(D))$, where Φ is the only conformal transformation sending A, B and C to the vertices of $\Phi(\mathcal{D})$. The formula for the limit is given by: $\frac{\Phi(D)\Phi(C)}{\Phi(A)\Phi(C)}$

1.7 Cardy's formula, conformal invariance

Let G be a planar graph with a percolation measure $\mathbb{P}$ on it. For $\delta > 0$ let G_δ be the graph G rescaled by δ and let $\mathbb{P}_\delta$ be the percolation measure $\mathbb{P}$ on G_δ.

Consider a domain $\mathcal{D}$ in the plane $\mathbb{C}$, and four points A, B, C, D distributed anti-clockwise on its boundary. We are interested in the asymptotics, as $\delta \to 0$, of the $\mathbb{P}_\delta$-probability that $\mathcal{D}$ contains an open crossing from (AB) to (CD). In the perspective of scale-invariance, we expect this probability to converge, as δ goes to 0, to a non-trivial limit. Let us, for now, consider homogeneous percolation on a periodic graph.

Cardy, in [Car92], conjectured the existence of the limit, and even gave a formula for it in terms of a hypergeometric function. His conjecture was proved in 2001 by Smirnov for critical site percolation on the triangular lattice (see [Smi01]).

Following a remark by Lennart Carleson, the formula, now known as the Cardy-Smirnov formula, is usually stated for an equilateral triangular domain $\mathcal{D}$, with vertices A, B, C, and with D an arbitrary point on AC. See Figure 1.7.1. In this case the limit of the probability that there exists a crossing from (AB) to (CD) is $\frac{DC}{AC}$.

The formula for general domains $\mathcal{D}$, is obtained by a conformal transformation of the triangular case. If A, B, C, D are distinct points on $\partial\mathcal{D}$, by the Riemann mapping theorem, there exists a unique conformal map Φ that transforms $\mathcal{D}$ in an equilateral triangle with vertices $\Phi(A) = e^{i\frac{\pi}{3}}$, $\Phi(B) = 0$ and $\Phi(C) = 1$. The limit of the crossing probability is then given by $|\Phi(D) - \Phi(C)|$. This conformal invariance feature, expected to appear in most scaling limits of critical models, is a key ingredient in the proof of convergence of the percolation interface to SLE_6. See [Wer07, Section 3] for details on the proof of this convergence.

The percolation model $(G, \mathbb{P})$ is said to satisfy Cardy's formula if, for all domains $\mathcal{D}$

with $A, B, C, D \in \partial\mathcal{D}$,

$$\mathbb{P}_\delta\left(AB \xleftrightarrow{\text{in } \mathcal{D}} CD\right) \to |\Phi(D) - \Phi(C)|, \quad \text{as } \delta \to 0, \tag{1.7.1}$$

where Φ is given as above and the convergence is uniform in the placement and orientation of $\mathcal{D}$.

Note that, unlike arm exponents, Cardy's formula is highly sensitive to the embedding of G. Is is expected that the isoradial embedding is harmonious with the canonical percolation measure it generates. We give next a conjecture that materializes this belief.

Conjecture 1.7.1. *Let G be an isoradial graph (satisfying the bounded-angles property), with canonical percolation measure $\mathbb{P}_G$. Then $(G, \mathbb{P}_G)$ satisfies Cardy's formula.*

If G is taken to be the square lattice, embedded as in Figure 1.3.1, we obtain the famous problem of proving Cardy's formula for critical homogeneous bond percolation on the square lattice. This is one of the main challenges in present percolation theory.

A weaker conjecture, in the spirit of Theorem 1.6.2, is the following.

Conjecture 1.7.2. *If Cardy's formula holds for some $G \in \mathcal{G}$, then it holds for all $G \in \mathcal{G}$.*

The above is a stronger version of universality than Theorem 1.6.2. The essential difference is that critical exponents depend very little on the embedding of the graph, while Cardy's formula is very sensitive to it. For instance, it would not be reasonable to expect Cardy's formula to hold for all models in $\mathcal{M}$, while Theorem 1.6.2 does apply to them.

The method used in proving Theorem 1.6.2 offers a perspective for Conjecture 1.7.2. Nevertheless, in the proof of Theorem 1.6.2 we have expressed arm exponents, and the box-crossing property, in terms of graph-theoretical quantities. In order to prove universality of Cardy's formula, we need to use the isoradial embedding, and our present tools are not fine enough to achieve this.

Let us get back to the box-crossing property, and see how it relates to crossings of a domain $\mathcal{D}$. Suppose both $\mathbb{P}$ and its dual, $\mathbb{P}^*$, satisfy the box-crossing property. Then, by combining box-crossings as in Figure 1.7.2, we find that the probability that there exists an open crossing in $\mathcal{D}$, from (AB) to (CD), is contained in some interval $[\epsilon, 1-\epsilon]$, with $\epsilon > 0$ only depending on $\mathcal{D}$ and on A, B, C, and D, not on scaling factor δ or on the positioning of $\mathcal{D}$. Thus, subsequential limits (as $\delta \to 0$) of the crossing-probabilities of (1.7.1) exist and are non-trivial, i.e. not 0 or 1. The problem of identifying these limits is, nevertheless, very difficult and, in most cases, still unsolved.

In light of the above observation, it is not surprising that the box-crossing property plays an important role in the proof of the Cardy-Smirnov formula. Indeed, in the proof of the formula for site percolation on the square lattice, one proves the uniform convergence of a triplet of discretely harmonic functions to a limiting triplet of harmonic functions.

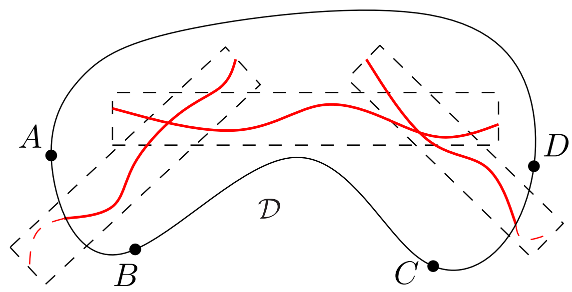

Figure 1.7.2: Combining crossings of rectangles to obtain crossings of general domains.

This is done in two steps; first one proves compactness for the family of functions, then the limit is identified via holomorphicity and boundary conditions. Using the box-crossing property, one shows that the discrete harmonic functions are Hölder continuous, with parameters that do not depend upon δ. This allows us to apply the Arzela-Ascoli criterion for compactness in L^∞ to obtain the first step of the proof.

The procedure of finding a discreetly preholomorphic (or even holomorphic) observable, showing precompactness for this observable, and proving uniqueness of the holomorphic limit using boundary conditions is the standard route for proving existence of scaling limits of critical statistical physics models. A full proof of the Cardy-Smirnov formula may be found in [Wer07, Section 2] or in [Gri10, Section 5.7].

Chapter 2

Applications of the box-crossing property

The purpose of this chapter is to present different consequences of the box-crossing property, such as criticality (Section 2.1), the separation theorem (Section 2.3) and scaling relations (Sections 2.4 and 2.5). In Section 2.2 we discuss the relation between the RSW lemma and the box-crossing property.

Throughout the chapter G will denote a planar graph, with dual G^*. We will assume G satisfies the conditions of Section 1.3.1, and all constants will depend implicitly on L_e, L_d and K_d. For simplicity suppose G is rescaled such that $L_e \leq \frac{1}{4}$, so that each face has diameter at most 1 and that $L_d \leq 1$. Also, in order to avoid trivialities, we will suppose our percolation measures to have intensities in $(0, 1)$. In certain sections we will ask the intensities to be bounded away from 0 and 1 uniformly. This will be explicitly stated.

We want to emphasize the importance of geometric arguments which do not depend on the local details of the graph. We will construct structures based on crossings of domains (usually rectangles), and will assume that the existence of such crossings is independent in disjoint domains. This is not entirely true, since the existence of crossings depends on the states of the edges entirely inside the domain, as well as of some of the edges intersecting the boundary.

Nevertheless, since all edges, primal and dual, are of bounded length, we may eliminate this dependency by imposing the existence of "buffer zones" between domains. Another way of handling this problem is to define more precisely the events we consider. Sometimes we will ask for the existence of an open crossing of a domain, when we actually mean the existence of a path crossing the domain, open on all edges contained entirely in the domain. Keeping track of these construction would overburden the proofs, so from now on we will suppose that the existence of crossings of disjoint domains are independent events.

Finally let us note that, although these constructions may seem complicated, upon careful readings of the proofs, it will be obvious how they come into play. Also note that we mostly consider the existence of open/open* circuits in annuli. These events only depend on the edges entirely contained in the annuli, hence are truly independent in disjoint annuli.

2.1 Criticality via the box-crossing property

In this section we summarise the steps needed to prove criticality for percolation measures $\mathbb{P}$, with $\mathbb{P}$ and $\mathbb{P}^*$ having the box-crossing property.

Fix a graph $G = (V, E)$, and consider a percolation measure $\mathbb{P}$ on it, with parameters $\mathbf{p} = (p_e) \in (0, 1)^E$. We remind the notation $\mathbb{P}^\nu$ for the measure with shifted parameters.

For simplicity we will assume that there exists $\epsilon_0 > 0$ such that $\mathbf{p} \in (\epsilon_0, 1 - \epsilon_0)^E$. The results presented next remain valid (with a slight modification) even when removing this condition. The condition is particularly convenient when using Russo's formula (Theorem 2.1.3). It will be obvious from the proofs that the condition may be weakened by only asking for positive density of edges with intensity bounded away from 0, and likewise for intensity bounded away from 1. This second condition is ensured by the box-crossing property.

Due to the above, if $(G, \mathbb{P})$ has the box-crossing property $\mathrm{BXP}(l_0, \delta)$ for some l_0 and $\delta > 0$, then it also satisfies $\mathrm{BXP}(1, \delta')$ for an adjusted $\delta' > 0$. Henceforth, we write $\mathrm{BXP}(\delta')$ instead of $\mathrm{BXP}(1, \delta')$.

For $v \in V$, we recall the notation C_v for the open cluster containing v, and define the radius of the cluster as

$$\mathrm{rad}(C_v) = \inf\{\mathbf{r} \geq 0 : C_v \subseteq \Lambda_r + v\}.$$

The following two propositions are the main results of this section.

Proposition 2.1.1. *Suppose $\mathbb{P}^*$ has the box-crossing property* $\mathrm{BXP}(\delta)$.

(a) *There exist $a, b > 0$ such that, for every $v \in V$,*

$$\mathbb{P}(\mathrm{rad}(C_v) \geq k) \leq ak^{-b}, \quad k \geq 0. \tag{2.1.1}$$

(b) *There exists, $\mathbb{P}$-a.s., no infinite open cluster.*

(c) *For $\nu < 0$, there exist $c, d > 0$ such that, for every $v \in V$,*

$$\mathbb{P}^\nu(|C_v| \geq k) \leq ce^{-dk}, \quad k \geq 0. \tag{2.1.2}$$

Proposition 2.1.2. *Suppose $\mathbb{P}$ has the box-crossing property* $\mathrm{BXP}(\delta)$.

(a) *There exist $a, b > 0$ such that for every $v \in V$,*

$$\mathbb{P}(\mathrm{rad}(C_v) \geq k) \geq ak^{-b}, \quad k \geq 0.$$

(b) *For $\nu > 0$ there exist $\alpha > 0$ such that for every $v \in V$,*

$$\mathbb{P}^{\nu}(v \leftrightarrow \infty) > \alpha.$$

(c) *There exists, $\mathbb{P}^{\nu}$-a.s., a unique infinite open cluster.*

Moreover, the constants in the above statements depend only on δ, not otherwise on G or $\mathbb{P}$.

These two results are well known in the case of homogeneous percolation. Our proofs are adaptations of known techniques; here we follow the proof of [Gri10, Section 5.8]. We use two important tools, Russo's formula and an influence theorem. Both of them are frequently used in percolation theory, as well as in related models. Nevertheless they are usually stated only for homogeneous measures. We next give versions adapted to our inhomogeneous models.

2.1.1 Preliminaries

The following result is the inhomogeneous version of the well-known Russo formula. For an account on Russo's formula see [Gri99, Section 2.4]; the version for inhomogeneous product measures is obtained through exactly the same computations as the one for homogeneous measures.

Let A be an increasing event in Ω. For an edge $e \in E$ and a configuration $\omega \in \Omega$ we say e is *pivotal* for A if $\omega^e \in A$ and $\omega_e \notin A$. Here ω^e and ω_e are the configurations equal to ω for all edges different of e and with $\omega^e = 1$, $\omega_e = 0$ respectively. The quantity $\mathbb{P}(e \text{ is pivotal for } A)$ is called the influence of the edge e on A, and is written $I_A(e)$. When working with $\mathbb{P}^{\nu}$ instead of $\mathbb{P}$, we write $I_A^{\nu}(e)$ for the influence of e.

Theorem 2.1.3 (Russo's formula)**.** *Let A be an increasing event defined in terms of the states of only finitely many edges of G. Then, for $e \in E$,*

$$\frac{\partial \mathbb{P}(A)}{\partial p_e} = \mathbb{P}(e \textit{ is pivotal for } A). \tag{2.1.3}$$

By summing (2.1.3) over the edges of G, we obtain, for $|\nu| < \epsilon_0$,

$$\frac{\partial \mathbb{P}^{\nu}(A)}{\partial \nu} = \sum_{e \in E} \mathbb{P}^{\nu}(e \text{ is pivotal for } A) = \sum_{e \in E} I_A^{\nu}(e). \tag{2.1.4}$$

It will therefore be useful to have an estimate of the total influence, $\sum_{e \in E} I_A^{\nu}(e)$.

This takes us to our second important tool in the poof of criticality, influence theorems and their usage in proving sharp–threshold properties. The first important influence theorem appeared in the seminal paper known as KKL, [KKL88]; many generalisations of this result followed, among them are the paper known as BKKKL, [BKK+92], and the revision of the first two by Friedgut, [Fri04]. The initial paper was limited to the study of product measures on discrete spaces $\{0,1\}^N$, the subsequent papers generalised the result to product measures on more general spaces. Versions for non-product measure later appeared in [GG06].

For our study we need an influence theorem for inhomogeneous product measures. To our knowledge such a result has not yet been stated in the literature, but one may easily be derived from known theorems. Let us first state the desired result, then discuss its proof.

Proposition 2.1.4. *There exists a constant $c \in (0,\infty)$ such that the following holds. Let A be an increasing subset of the space $\{0,1\}^N$ endowed with an inhomogeneous product measure P, such that $P(A) \in (0,1)$. Then:*

$$\sum_{i\in\{1\dots N\}} I_A(i) \geq cP(A)(1-P(A))\log\left(\frac{1}{2m}\right),$$

where $m = \max_i I_A(i)$, and the influences are computed under the measure P.

In order to prove this result we will use continuous influence theorems. Such a theorem works with the cube $[0,1]^N$ instead of the space $\{0,1\}^N$, and the reference measure is, in this case, the Lebesgue measure λ. This kind of theorem was first formulated in [BKK+92], though, as observed in [Fri04], that version contained a mistake. Friedgut gave another, slightly modified version of the same result [Fri04, Theorem 1.5]; yet another version may be found in [Gri10, Theorems 4.33 and 4.38].

We first need to explain what we mean by influence in the continuous case. For an increasing event $A \in [0,1]^N$, define the influence of the i^{th} coordinate on A as

$$I_A(i) = \lambda(\mathbf{1}_A(\omega^i) - \mathbf{1}_A(\omega_i)).$$

Here ω^i and ω_i are the elements of $[0,1]^N$ identical to ω on all coordinates except on the i^{th}, where they are equal to 1 and 0, respectively. We are now ready to state the continuous influence theorem that we will use to prove Proposition 2.1.4. This version is taken from [Gri10, Theorem 4.33].

Theorem 2.1.5. *There exists an absolute constant $c \in (0,\infty)$ such that the following* *holds. Let A be an increasing subset of the cube $[0,1]^N$ with $\lambda(A) \in (0,1)$. Then*

$$\sum_{i=1}^{N} I_A(i) \geq c\lambda(A)(1-\lambda(A)) \log\left(\frac{1}{2m}\right),$$

where $m = \max_i I_A(i)$ and the influences are computed under the measure λ. Moreover there exists $i \in \{1 \dots N\}$ such that

$$I_A(i) \geq c\lambda(A)(1-\lambda(A))\frac{\log N}{N}.$$

Proof of Proposition 2.1.4 from Theorem 2.1.5. Throughout this proof ω stands for an element of the cube $[0,1]^N$, λ denotes the Lebesgue measure on $[0,1]^N$, and P is an inhomogeneous product measure on $\{0,1\}^N$, with intensities $(p_i)_{i\in\{1\dots N\}}$. For $\omega \in [0,1]^N$, define $\tilde{\omega}$ as the element of $\{0,1\}^N$ with:

$$\tilde{\omega}(i) = \mathbf{1}_{\omega(i)\geq 1-p_i}, \qquad i = 1,\dots,N.$$

With this definition, if ω is chosen according to λ, then $\tilde{\omega}$ follows the law P. Thus, for an increasing event $\tilde{A} \subset \{0,1\}^N$, we may define $A = \{\omega \in [0,1]^N | \tilde{\omega} \in \tilde{A}\}$, and observe that $\lambda(A) = P(\tilde{A})$. Moreover A is also increasing, and the influences under λ on A are equal to the ones on $\tilde{A}$ under P:

$$\begin{aligned} I_A(i) &= \lambda(\mathbf{1}_A(\omega^i) - \mathbf{1}_A(\omega_i)) \\ &= \lambda(\mathbf{1}_{\tilde{A}}(\tilde{\omega}^i) - \mathbf{1}_{\tilde{A}}(\tilde{\omega}_i)) \\ &= P\left(i \text{ is pivotal for } \tilde{A}\right). \end{aligned}$$

Hence

$$\sum_{i\in\{1\dots N\}} I_{\tilde{A}}(i) = \sum_{i\in\{1\dots N\}} I_A(i) \geq c\lambda(A)(1-\lambda(A)) \log\left[1/(2m)\right],$$

where $m = \max_i I_A(i) = \max_i I_{\tilde{A}}(i)$. □

2.1.2 Proof of Propositions 2.1.1 and 2.1.2

Proof of Proposition 2.1.1, (a) and (b). Obviously (2.1.1) implies the non-existence of infinite components, let us therefore prove (2.1.1). Fix $\mathbb{P}$ as in Proposition 2.1.1 and choose a vertex $v \in V$. For simplicity we suppose v is placed at the origin of $\mathbb{R}^2$. For $n \geq 1$ define $\mathcal{A}_n = \mathcal{A}(2^n, 2^{n+1})$ as the square annulus centered at v, with inner radius 2^n and outer radius 2^{n+1}. Let H_n be the event that there exists a dual open circuit in $\mathcal{A}_n$, surrounding e.

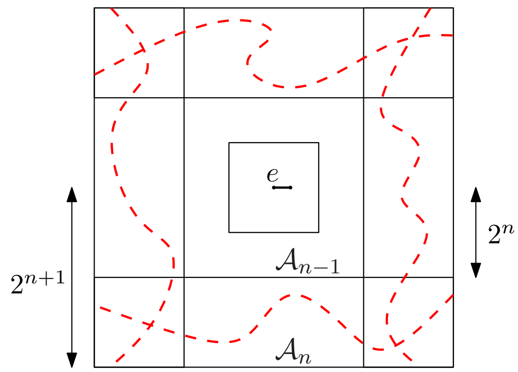

Figure 2.1.1: The annuli around v. If $\mathcal{A}_n$ contains an open* circuit, then the open cluster of v has radius at most 2^{n+1}. To construct such a circuit we may use the box-crossing property for the dual in the four rectangles that form $\mathcal{A}_n$.

The events $(H_n)_{n\geq 0}$ are independent since the annuli $\mathcal{A}_n$ are disjoint and H_n only depends on the edges entirely contained in $\mathcal{A}_n$. Moreover, using the box-crossing property for $\mathbb{P}^*$ and the FKG inequality, we deduce that there exists a constant $c_0 = c_0(\delta) > 0$ such that $\mathbb{P}(H_n) \geq c_0$ for $n \geq 0$ (see Figure 2.1.1).

If H_n occurs, then C_v is contained in $\Lambda_{2^{n+1}}$, since it can not cross the open circuit in $\mathcal{A}_n$. Thus

$$\mathbb{P}\left[\mathrm{rad}(C_v) \geq 2^n\right] \leq \mathbb{P}\left[\bigcap_{k<n} H_k^c\right] \leq (1-c_0)^n,$$

and (2.1.1) follows. □

Before proving Proposition 2.1.1 (c), we prove Proposition 2.1.2.

Proof of Proposition 2.1.2. Take $\mathbb{P}$ as in Proposition 2.1.2. Point (a) is obtained by a standard construction involving crossings of $2^k \times 2^{k+1}$ rectangles, with $k = 1, \dots, \log N$. For more details see the proof of (2.5.19).

We turn to point (b). First we use sharp-threshold to show that, for $\nu > 0$, the $\mathbb{P}^\nu$-probabilities of crossings of boxes of fixed aspect ratio tend to 1 as the size of the box tends to infinity.

Fix an aspect ratio $\alpha \geq 1$, and $\eta \in (0, \epsilon_0)$, and consider horizontal crossings of the box $\mathcal{B}(\alpha N, N)$ for $N \geq 2$. Denote H_N the event that such a crossing exists and, let $I_N^\eta(e)$ be the influence of the edge e on the event H_N, under the measure $\mathbb{P}^\eta$.

Since $\mathbb{P}$ satisfies BXP(δ), by Proposition 2.1.1 (a), there exist constants $a, b > 0$ such that, for any dual vertex v,

$$\mathbb{P}[\mathrm{rad}(C_v^*) > n] \leq an^{-b}.$$

This also holds for $\mathbb{P}^\eta$ by monotonicity.

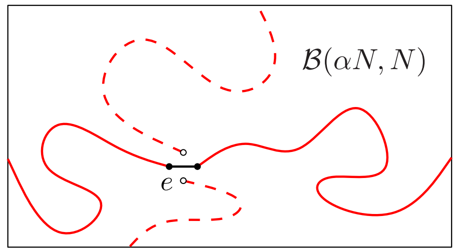

Figure 2.1.2: For the edge e to be pivotal for H_n, it needs to be connected by open paths to the lateral sides of $\mathcal{B}(\alpha N, N)$ and by open* paths to the top and bottom of the box.

For an edge e to be pivotal, open paths must join it to the lateral sides of $\mathcal{B}(\alpha N, N)$, and open* paths must join it to the top and to the bottom of the box, as in Figure 2.1.2. Let $(u, v) = e^*$, then

$$I_N^\eta(e) \le \mathbb{P}^\eta\left[\mathrm{rad}(C_u^*) \ge \tfrac{N}{2} - 1\right] + \mathbb{P}^\eta\left[\mathrm{rad}(C_v^*) \ge \tfrac{N}{2} - 1\right] \le a' N^{-b'},$$

where $a', b' > 0$ are constants obtained from a and b, and which do not depend on e. Using Proposition 2.1.4, we obtain

$$\frac{d\mathbb{P}^\eta(H_N)}{d\eta} \ge c_0 \mathbb{P}^\eta(H_N)(1 - \mathbb{P}^\eta(H_N)) \log N, \tag{2.1.5}$$

for some $c_0 > 0$. Since $\mathbb{P}$ satisfies BXP(δ), there exists $c_1 > 0$ (independent of N) such that $\mathbb{P}(H_N) \ge c_1$. For $\nu \in (0, \epsilon_0]$, by integrating (2.1.5) between 0 and ν, we obtain

$$\mathbb{P}^\nu(H_N) \ge 1 - N^{-c_0 c_1 \nu} \xrightarrow[N \to \infty]{} 1. \tag{2.1.6}$$

The above computation did not depend on the positioning and orientation of the box, hence the bound (2.1.6) holds for all rectangular boxes of aspect ratio α.

In addition to the convergence of crossing probabilities to 1, (2.1.6) offers a bound on the speed of convergence. We may then conclude by an argument similar to that of (2.5.19). For illustration we choose an alternative route, via a block argument that only uses the convergence.

Fix $\nu \in (0, \epsilon_0)$, and consider some $N > 0$. A *block* is one of the $4N \times N$ rectangles of the right diagram of Figure 2.1.3. The blocks form a network similar to the square lattice. Call a block *good* if it contains an open crossing in the long direction, along with two open crossings in the short direction contained in the squares at its ends (see the left diagram of Figure 2.1.3). The states of different blocks are not generally independent since blocks may overlap. The system of blocks thus created corresponds to a finite-range dependent bond percolation on the square lattice.

Standard arguments (for instance a counting argument) show that the critical point

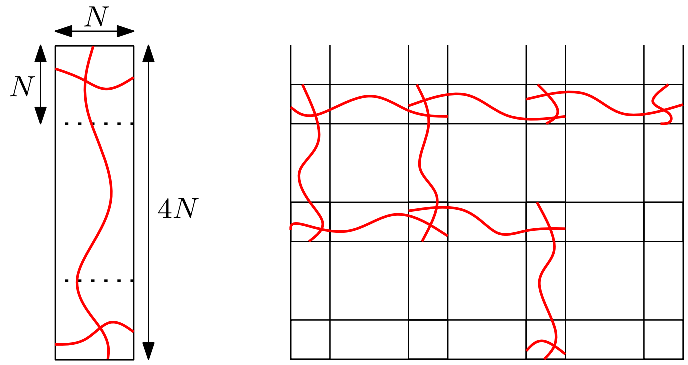

Figure 2.1.3: *Left:* for a block to be good, it needs to have an open crossing in the long direction, and two open crossings in the short direction contained in the squares at its ends. *Right:* a configuration of good blocks with the underlying open paths.

of this block percolation model is strictly less than 1. In other words, there exists some $p_c(\text{block}) < 1$ such that, if the probability for any block to be good is higher than $p_c(\text{block})$, then there exists almost surely an infinite connected component of good blocks. Moreover, the probability for a given block to be contained in such an infinite component is bounded away from 0, uniformly in the choice of the block.

By (2.1.6), when N tends to infinity, the probability for the blocks to be good tends uniformly to 1. Thus, for N is large enough, there exists a.s. an infinite connected component of good blocks. By the definition of good block, this implies the existence of an infinite path of open edges in the graph G. Moreover, for $v \in V$, there exists a uniform lower bound (uniform in the choice of v) for the probability that there exists an infinite open path within distance $4N$ of v. Since every edge has probability at least $\epsilon_0 + \nu$ of being open, v is connected to this infinite path with uniformly positive probability. This concludes the proof of the existence of an infinite component under $\mathbb{P}^\nu$.

The uniqueness of the infinite component follows by the fact that, under $\mathbb{P}^\nu$, there are a.s. infinitely many annuli $\mathcal{A}(2^n, 2^{n+1})$ containing open circuits. Note that we do not require the machinery of the classical uniqueness result of [BK89]. $\square$

Finally we prove Proposition 2.1.1 (c). The arguments we use are a combination of the sharp–threshold technique of the previous proof and the following lemma taken from [Kes81, Thm 1].

As in the previous proof we will only use the convergence in (2.1.6), with $\mathbb{P}^*$ instead of $\mathbb{P}$. If we allowed ourselves to use the speed of convergence, then the result would immediately follow. We choose this longer proof for future reference.

Lemma 2.1.6. *Let G be a planar graph endowed with a percolation measure P, with intensities bounded away from 0 and 1 by $\epsilon_1 > 0$. There exists an absolute constant c_0*

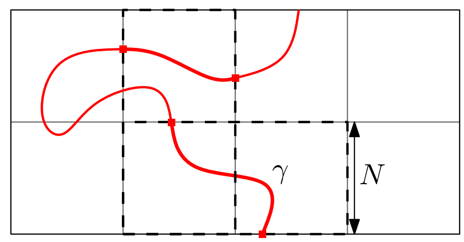

Figure 2.1.4: If the rectangle $\mathcal{B}(4N, 2N)$ is crossed vertically by γ, then γ contains two disjoint crossing of $2N \times N$ rectangles in the short direction.

such that

$$P\left[\mathcal{C}(\mathcal{B}(4N, 2N))\right] \leq \frac{100}{\epsilon_1} \sup_f P\left[\mathcal{C}(f(\mathcal{B}(2N, N))\right]^2, \tag{2.1.7}$$

where $\mathcal{C}(\mathcal{B})$ is the event that $\mathcal{B}$ contains an open crossing in the "short" direction, and the supremum is taken over all function f composed of a translation and $\frac{\pi}{2}$-rotation.

The proof of this lemma is deferred until the end of the section. The graph and the measure in the lemma are not necessarily those of Proposition 2.1.1.

Proof of Proposition 2.1.1 (c). Take $\mathbb{P}$ as in Proposition 2.1.1, and fix $\nu \in (-\frac{\epsilon_0}{2}, 0)$. By the box-crossing property for $\mathbb{P}^*$ and the theory of influence (same as in the proof of Proposition 2.1.2), for N large enough,

$$\mathbb{P}^\nu\left[f(\mathcal{B}(2N, N)) \text{ has an open}^* \text{ crossing in the long direction}\right] \geq 1 - \frac{\epsilon_0}{400},$$

for any function f composed of a translation and a rotation. But if such a crossing exists, then there exists no open crossing in the short direction. Using Lemma 2.1.6 repeatedly, we obtain

$$\mathbb{P}^\nu\left[\mathcal{C}\left(\mathcal{B}\left(2^{k+1}N, 2^k N\right)\right)\right] \leq 2^{-k}\frac{\epsilon_0}{400}. \tag{2.1.8}$$

This also holds for any rotation and translation of $\mathcal{B}(2^{k+1}N, 2^k N)$. The conclusion, (2.1.2), follows easily. □

Proof of Lemma 2.1.6. Consider the rectangle $\mathcal{B}(4N, 2N)$. Split $\mathcal{B}(4N, 2N)$ into eight $N \times N$ squares as in Figure 2.1.4, and call a tiling rectangle the union of any two adjacent squares. There are ten tiling rectangles altogether, four vertical ones and six horizontal ones.

Suppose there exists an open crossing γ of $\mathcal{B}(4N, 2N)$, from $[0, 4N] \times \{0\}$ to $[0, 4N] \times \{2N\}$. Orient γ from the bottom to the top of $\mathcal{B}(4N, 2N)$. Then γ contains two disjoint crossings (in the short direction) of tiling rectangles, one before its first intersection with

$[0, 4N] \times \{N\}$, and one after its last. The tiling rectangles containing the two crossings need not be different.

Consider two tiling rectangles B_1, B_2. By the BK inequality, the probability that there exists an open path containing disjoint crossings of B_1 and B_2 is bounded above by $\epsilon_1 P[\mathcal{C}(B_1)]P[\mathcal{C}(B_2)]$, where the factor ϵ_1 comes from the possible edge common to the two crossings (see Figure 2.1.4). By considering all combinations of two tiling rectangles, we obtain (2.1.7). $\square$

2.2 The the RSW lemma and the box-crossing property

2.2.1 Discussion

Let G be a planar graph embedded in the plane, and $\mathbb{P}$ be a percolation measure on G. Heuristically, the RSW lemma states that the probability of crossing a $2N \times N$ rectangle in the long direction may be bounded below by a positive function, ϕ, of the probability of crossing a $N \times N$ square. Moreover, it is sometimes useful to have $\phi(p) \to 1$ as $p \to 1$.

Later in this section we give precise RSW statements for models that are periodic and invariant under rotation and reflection with respect to the axes. Before doing so, we would like to discuss the relation between the RSW lemma, self-duality, criticality and the box-crossing property.

Consider homogeneous bond percolation on the square lattice with intensity p. Russo, and Seymour and Welsh proved in [Rus78, SW78] a RSW lemma for this model (see Lemma 2.2.1). When $p = \frac{1}{2}$, the model is self-dual, hence the probability of crossing a $N \times N$ square is (roughly) $\frac{1}{2}$. Using the RSW lemma, we deduce the box-crossing property for $\mathbb{P}^{\square}_{\frac{1}{2},\frac{1}{2}}$. Criticality follows as in Section 2.1.

More generally, if a model satisfies some form of the RSW lemma, and is self dual, then the box-crossing property and criticality follow as above. The RSW property by itself is not sufficient to imply criticality, it requires an input, which usually comes in the form of self-duality.

While the RSW lemma presented later does not use self-duality other than as an input, there are variations on the RSW result which are based on self-duality. Some require considerably less symmetry than the one presented here (see [BR10]). Note that our models are generally not self-dual, hence the methods of [BR10] do not apply to them.

Let us now address the different question of when does criticality imply the box-crossing property. We claim that for a model $(G, \mathbb{P})$, which is periodic and invariant under rotation and reflection with respect to the axes, criticality implies the box-crossing property for both the primal and the dual measures. This may be shown as follows.

For simplicity suppose $(G, \mathbb{P})$ is invariant under rotation by $\pi/2$ around 0, and under translation by $(1, 0)$ and $(0, 1)$. The same reasoning works in the general setting, with adaptations as in Lemma 2.2.2.

First we show that, for n large enough, there exists $c_0 > 0$ such that

$$\mathbb{P}\left[\mathcal{C}_{\mathrm{v}}(\mathcal{B}(2n, n))\right] \geq c_0. \tag{2.2.1}$$

Suppose the converse. For $\nu > 0$, using an argument similar to Lemma 2.1.6, we find that, if $\mathbb{P}^\nu\left[\mathcal{C}_{\mathrm{v}}(\mathcal{B}(2n, n))\right]$ is less than some universal constant $c_1 > 0$, then the cluster size has exponential decay, as in Proposition 2.1.1 (c). By our assumption, we may find n such that $\mathbb{P}\left[\mathcal{C}_{\mathrm{v}}(\mathcal{B}(2n, n))\right] < \frac{c_1}{2}$. Then, for $\nu > 0$ small enough, $\mathbb{P}^\nu\left[\mathcal{C}_{\mathrm{v}}(\mathcal{B}(2n, n))\right] < c_1$. Hence there exists $\mathbb{P}^\nu$-a.s. no infinite cluster. This contradicts the criticality of $\mathbb{P}$, and (2.2.1) is proved.

To conclude, we use the RSW lemma (see Lemmas 2.2.1 and 2.2.2) for the dual and primal model to obtain the box-crossing property for $\mathbb{P}$. The same argument may be used to obtain the box-crossing property for $\mathbb{P}^*$.

2.2.2 Statements of the RSW lemmas

We now give two RSW lemmas for models exhibiting sufficient symmetry. Although identical in spirit, the two differ due to the characteristics of the model.

Let G be a planar graph and $\mathbb{P}$ be a percolation measure on it. Suppose $(G, \mathbb{P})$ is periodic, invariant under rotation and under reflection with respect to two perpendicular lines. We remind the reader that G is locally finite, and we will use this implicitly in the geometrical considerations that follow.

Take $\theta \in (0, \pi)$ to be the minimal angle such that G is invariant under rotation by angle θ. Then $\theta = \frac{2\pi}{k}$ for some $k \geq 3$. First we claim that, due to periodicity,

$$\theta \in \left\{\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}\right\}. \tag{2.2.2}$$

This is obtained as follows. Let $x \in \mathbb{R}^2$ be a point such that $(G, \mathbb{P})$ is invariant under rotation by angle θ around x, and let τ_θ denote this rotation. For $u \in \mathbb{R}^2$, let σ_u be the translation by u. Take u such that G is invariant under σ_u and

$$|u| = \inf\{|v| : G \text{ is invariant under } \sigma_v\}. \tag{2.2.3}$$

By rotation invariance, G is also invariant under translation by $\tau_\theta^j(u)$ for $j \in \mathbb{Z}$, and by $\tau_\theta^j(u) \pm u$. By choice of u we have $|\tau_\theta^j(u) \pm u| \geq |u|$. This implies $k \in \{3, 4, 6\}$, whence (2.2.2).

We distinguish two cases.

(i) $\theta = \frac{\pi}{2}$,

(ii) $\theta \in \{\frac{\pi}{3}, \frac{2\pi}{3}\}$.

The square lattice is representative of the first, whereas the triangular and hexagonal lattices are examples of the second. These are not the only graphs exhibiting such symmetries, non vertex-transitive examples may be constructed.

In the first case we may suppose that $(G, \mathbb{P})$ is invariant under reflection with respect to the line $\mathbb{R} \times \{0\}$. It is also invariant under rotation by $\frac{\pi}{2}$ around a point x. It is not always the case that x is on the line $\mathbb{R} \times \{0\}$, but by periodicity we may always choose $x \notin \mathbb{R} \times \{0\}$. By translation and rescaling we may take $x = (\frac{1}{2}, \frac{1}{2})$. It is then easy to check that $(G, \mathbb{P})$ is invariant under translation by $(2, 0)$ and $(0, 2)$, and that it is also invariant under reflection with respect to $\{k\} \times \mathbb{R}$ and $\mathbb{R} \times \{k\}$ for $k \in \mathbb{Z}$.

The RSW lemma in this case may be written as follows.

Lemma 2.2.1 (RSW)**.** *For $\rho > 1$ there exists a function $\phi_\rho : (0, 1] \to (0, 1]$ such that, for $n \geq 4$,*

$$\mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}(2\rho n, 2n))\right] \geq \phi_\rho\left(\mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}(2n, 2n))\right]\right). \tag{2.2.4}$$

Moreover $\phi(p) \to 1$ as $p \to 1$.

We now move on to the second case. Suppose $(G, \mathbb{P})$ is invariant under rotation by $\frac{2\pi}{3}$ and reflection with respect to two perpendicular lines, l_1, l_2, parallel to the axes $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$, respectively. By rotation invariance, it is also invariant under reflection with respect to a line l_3, that makes an angle $\frac{\pi}{3}$ with l_1. Translate the plane such that $0 = l_2 \cap l_3$. The lines l_2 and l_3 intersect in 0 at an angle $\frac{\pi}{6}$. Moreover they are both axes of symmetry for $(G, \mathbb{P})$. It follows, by repeated reflections, that $(G, \mathbb{P})$ is invariant under rotation by $\frac{\pi}{3}$ around 0 and under reflection with respect to all lines forming an angle $k\frac{\pi}{6}$ with $\mathbb{R} \times \{0\}$ $(k \in \mathbb{Z})$. Finally, $(G, \mathbb{P})$ is also invariant under translation by a vector u, which may be taken on $\mathbb{R} \times \{0\}$. By rescaling we take $u = (1, 0)$.

The RSW lemma for this case is very similar to the one for $\theta = \frac{\pi}{2}$, the only difference is that we have to work with parallelograms instead of rectangles. Let u_k be the rotation of u by $k\frac{\pi}{3}$ around 0. Define $\mathcal{B}^{\triangle}(m, n)$ to be the parallelogram with sides mu_0, nu_1. The events $\mathcal{C}_{\mathrm{h}}(\mathcal{B}^{\triangle})$, $\mathcal{C}_{\mathrm{v}}(\mathcal{B}^{\triangle})$ are defined as for $\mathcal{B}$.

Lemma 2.2.2 (RSW)**.** *For $\rho > 1$ there exists a function $\phi_\rho : (0, 1] \to (0, 1]$ such that, for $n \geq 1$,*

$$\mathbb{P}\left[\mathcal{C}_{\mathrm{h}}\left(\mathcal{B}^{\triangle}(4\rho n, 4n)\right)\right] \geq \phi_\rho\left(\mathbb{P}\left[\mathcal{C}_{\mathrm{h}}\left(\mathcal{B}^{\triangle}(4n, 4n)\right)\right]\right) \tag{2.2.5}$$

Moreover $\phi(p) \to 1$ as $p \to 1$.

2.2.3 Proofs of Lemmas 2.2.1 and 2.2.2

The two lemmas, as well as their proofs, differ only slightly due to the different symmetries. We give a complete proof of Lemma 2.2.1 and only sketch that of Lemma 2.2.2.

Proof of Lemma 2.2.1. We follow the standard proof of [Gri99, Thm. 11.70], which in turn follows that of [Rus81]. The minor differences with [Gri99] come from the more general setting.

Fix $n \geq 2$ and consider the following rectangles:

$$\begin{aligned} B_1 &= [-n+2, n+1] \times [0, 2n], \\ B_2 &= [1, 2n+1] \times [0, 2n], \\ B_3 &= [1, 2n] \times [0, 2n]. \end{aligned}$$

We remind the reader that $(G, \mathbb{P})$ is invariant under

(i) rotation by $\pi/2$ around $\left(\frac{1}{2}, \frac{1}{2}\right)$,

(ii) translation by $(2,0)$ and $(0,2)$,

(iii) reflection with respect to the lines $\mathbb{R} \times \{k\}$ and $\{k\} \times \mathbb{R}$ for $k \in \mathbb{Z}$.

Thus we have

$$\mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B_1)\right], \mathbb{P}\left[\mathcal{C}_{\mathrm{v}}(B_2)\right], \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B_3)\right] \geq \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}(2n, 2n))\right]. \tag{2.2.6}$$

Let H_1 (respectively H_1') be the event that there exists a horizontal crossing of B_1, which, when oriented from left to right, has its last intersection with the line $\{1\} \times \mathbb{R}$ below or at (respectively above or at) height n. By reflection invariance $\mathbb{P}(H_1) = \mathbb{P}(H_1')$. Also H_1 and H_1' are increasing events. By the FKG inequality

$$1 - \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B_1)\right] = \mathbb{P}\left[H_1^c \cap (H_1')^c\right] \geq (1 - \mathbb{P}(H_1))^2.$$

Hence

$$\mathbb{P}(H_1) \geq 1 - \sqrt{1 - \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B_1)\right]}. \tag{2.2.7}$$

The argument used to obtain (2.2.7) is sometimes called the square root trick.

Let Γ be a path on G (not assumed open) crossing B_1 horizontally, and let x_1 be the last intersection point of Γ with the line $\{1\} \times \mathbb{R}$. Suppose x_1 is below or at height n. Let Γ^- be the set of edges which intersect B_1, and are below or part of Γ. Suppose H_1 occurs and let γ_1 be the lowest open path crossing B_1 horizontally. By choice of γ_1, the measure $\mathbb{P}(.|\gamma_1 = \Gamma)$ is identical to $\mathbb{P}$ outside Γ^-.

Let Γ_l be the sub-path of Γ between x_1 and its endpoint on $\{n+1\} \times \mathbb{R}$, and define Γ_r as the reflection of Γ_l with respect to the line $\{n+1\} \times \mathbb{R}$. Let H_2 be the event that there exists an open path γ_2 in B_2, above $\Gamma_l \cup \Gamma_r$, with one endpoint on $[1, 2n+1] \times \{2n\}$ and one on Γ_l, By an argument similar to the square root trick used above, and involving

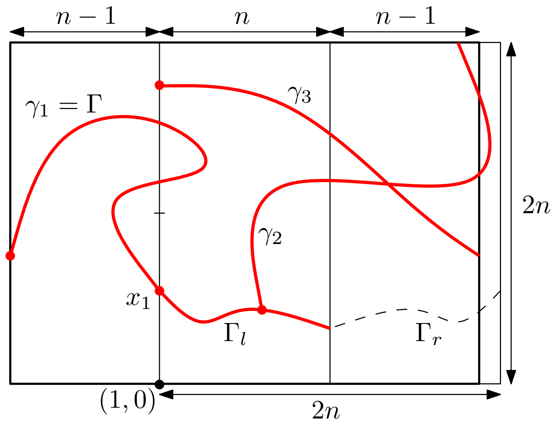

Figure 2.2.1: The events $\{\gamma_1 = \Gamma\}$, H_2 and H_3. Either γ_2 intersects γ_3, or γ_3 intersects the line $\{2n\} \times [0, 2n]$. In both cases there exists an open horizontal crossing of $[-n + 2, 2n] \times [0, 2n]$.

the reflection invariance with respect to $\{n+1\} \times \mathbb{R}$,

$$\mathbb{P}(H_2 | \gamma_1 = \Gamma) \geq 1 - \sqrt{1 - \mathbb{P}\left[\mathcal{C}_{\mathrm{v}}(B_2)\right]}. \tag{2.2.8}$$

Let H_{12} be the event that there exists a open path γ as in the definition of H_1, with the additional requirement that $\gamma_L \overset{B_2}{\longleftrightarrow} [1, 2n+1] \times \{2n\}$, where γ_L is defined as Γ_L. If $\gamma_1 = \Gamma$ and H_2 occurs, then H_{12} also occurs, with, for instance, $\gamma = \gamma_1$ and the connection to the top of B_2 provided by γ_2. By summing (2.2.8), for Γ ranging over the possible values of γ_1, we obtain

$$\mathbb{P}(H_{12}) \geq \left(1 - \sqrt{1 - \mathbb{P}\left[\mathcal{C}_{\mathrm{v}}(B_2)\right]}\right)\left(1 - \sqrt{1 - \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B_1)\right]}\right). \tag{2.2.9}$$

Finally let H_3 be the event that there exists a open horizontal crossing, γ_3, of B_3, with its left endpoint on $\{n+1\} \times [n, 2n]$. We have

$$\mathbb{P}(H_3) \geq 1 - \sqrt{1 - \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B_3)\right]}. \tag{2.2.10}$$

If both H_{12} and H_3 occur, then there exists an open horizontal crossing of $[-n+2, 2n] \times [0, 2n]$. See also Figure 2.2.1. Moreover both events are increasing, hence, by the FKG inequality and (2.2.6), (2.2.9) and (2.2.10),

$$\mathbb{P}\big[\mathcal{C}_{\mathrm{h}}([-n+2, 2n] \times [0, 2n])\big] \geq \left(1 - \sqrt{1 - \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B(2n, 2n))\right]}\right)^3. \tag{2.2.11}$$

The right hand side of the above is strictly positive if $\mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B(2n, 2n))\right] > 0$. It also tends to 1 as $\mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(B(2n, 2n))\right] \to 1$.

Note that $[-n+2, 2n] \times [0, 2n]$ is a rectangle with height $2n$ and length $3n-2 \geq \frac{5}{4}(2n)$.

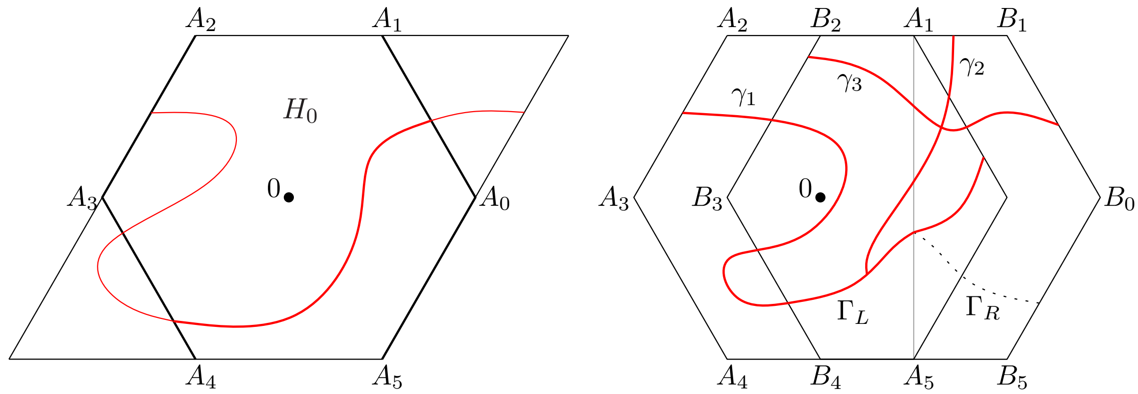

Figure 2.2.2: *Left:* A horizontal crossing of $\mathcal{C}_{\mathrm{h}}(\mathcal{B}^{\triangle}(4n, 4n))$ contains a crossing of H_0 from (A_5, A_1) to (A_2, A_4). *Right:* γ_1 is a crossing as in $\mathcal{C}(H_0)$, which last intersect (B_2, B_4) below B_3. The open path γ_2 links (B_1, B_2) to Γ_L, inside H_1. Its existence is obtained by a square root trick using the reflection invariance with respect to A_1A_5. Finally γ_3 is a crossing of H_1, between (B_5, B_1) and (B_2, B_3). Together, γ_1, γ_2 and γ_3 induce a horizontal crossing of $H_0 \cup H_1$.

Using (2.2.11) and the periodicity of G, we further combine horizontal crossing of translates of $[-n+2, 2n]$ with vertical crossings of translates of B_2 to obtain Lemma 2.2.1. □

Proof of Lemma 2.2.2. The proof is very similar to the previous one. We sketch it very briefly. Fix n and let A_i be the point $2nu_i$, for $i = 0, \dots, 5$. Let H_0 be the hexagon with vertices $A_0, \dots, A_5$ and $\mathcal{C}(H_0)$ be the event that there exists an open crossing in H_0, from (A_5, A_1) to (A_2, A_4). Note that, due to translation invariance and to the considerations of Figure 2.2.2,

$$\mathbb{P}\left[\mathcal{C}(H_0)\right] \geq \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}^{\triangle}(4n, 4n))\right]. \tag{2.2.12}$$

Let H_1 be the translate of H_0 by $(n, 0)$, with vertices $B_0, \dots, B_5$. Using the square root trick, the FKG inequality and rotation and translation invariance, we may also show that

$$\mathbb{P}\left[(B_1, B_2) \xleftrightarrow{H_0} (B_4, B_5)\right] \geq \left(1 - \sqrt{1 - \mathbb{P}\left[\mathcal{C}(H_0)\right]}\right)^2. \tag{2.2.13}$$

By the same argument as in the previous proof, we show that

$$\begin{aligned}
&\mathbb{P}\left[(A_2, A_4) \xleftrightarrow{H_0 \cup H_1} (B_5, B_1)\right] \\
&\quad \geq \left(1 - \sqrt{1 - \mathbb{P}\left[\mathcal{C}(H_0)\right]}\right)^2 \left(1 - \sqrt{1 - \mathbb{P}\left[(B_1, B_2) \xleftrightarrow{H_0} (B_4, B_5)\right]}\right).
\end{aligned}$$

See Figure 2.2.2 for the geometric construction we use. We mention that for this proof we require reflection invariance with respect to both the horizontal and the vertical axes. We give no further details of the proof. □

2.3 Separation theorem

In this section we present and discuss a general result concerning arm events, usually called the separation theorem. It basically says that, conditionally on $A_\sigma(N,n)$, the endpoints of the arms are far away from each other, in such a way that they can be extended via box crossings.

The result first appeared in [Kes87b], then was rewritten several times. We will adapt Nolin's version from his review [Nol08] of Kesten's work. In both papers the result is presented in the context of homogeneous site percolation, nevertheless it is actually valid in a much more general context, in particular in the context of bond percolation on graphs satisfying the conditions of Section 1.3.1. The theorem relies heavily on the box-crossing property, thus illustrating its importance.

2.3.1 Notation

In order to state the theorem we need to first introduce some notation. In the whole section we will work with an arm event of the type $A_\sigma(N,n)$ for some fixed colour sequence σ of length k (not necessarily alternating). All constants in the following statements depend implicitly on k and σ.

Consider a box $B_N := [0,N] \times [0,4N]$ and a constant $\eta \in (0,1)$. The notions defined here refer to crossings of B_N and more particularly to their properties near their endpoints. We will focus on horizontal crossings and their endpoints on the right side of B_N, i.e. on $\{N\} \times [0,4N]$.

A primal (respectively dual) *η-fence* is a set Γ of connected open (respectively, open*) paths comprising the union of:

(i) a horizontal crossing of B_N, with endpoint $z = (N,y)$ on the right side of B_N,

(ii) a vertical crossing of the box $[N,(1+\sqrt{\eta})N] \times [y-\eta N, y+\eta N]$,

(iii) a connection between the above two crossings, contained in $\Lambda_{\sqrt{\eta}N} + z$.

A *η-well-separated* sequence of fences is a sequence $(\Gamma_i)_{i\in 1,\dots,K}$ such that:

(i) each Γ_i is a η-fence (primal or dual),

(ii) the Γ_i are pairwise disjoint,

(iii) if we call z_i the right extremity of the crossing of B_N associated to Γ_i, the points $(z_i)_{i\in 1,\dots,K}$ are at distance at least $\sqrt{\eta}N$ from each-another and from the corners of B_N.

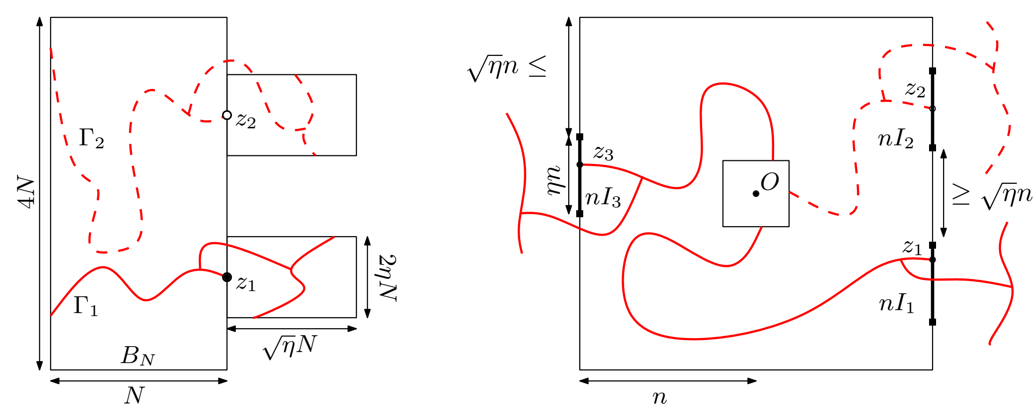

Figure 2.3.1: *Left:* two η-fences in B_N. *Right:* the event $A_\sigma^{\varnothing,I}(N,n)$ with $\sigma = (1,1,0)$. Each arm Γ_i is a fence with landing point z_i in nI_i.

A sequence of η-well-separated fences may contain fences of both colours (i.e. primal and dual fences). For illustrations of both definitions see Figure 2.3.1.

The definitions of fence and of well-separateness may be adapted in the obvious way to crossings of annuli, on both their interior and exterior boundary (the factor N will then refer to the interior, respectively exterior, radius of the annulus). See Figure 2.3.1. Note that we may ask Γ to be simultaneously a η-interior-fence and a η'-exterior-fence of $\mathcal{A}(N,n)$. In this case, we ask that the crossing of $\mathcal{A}(N,n)$ contained in Γ, have additional paths near both its interior and exterior endpoints, with factors η, N and η', n respectively.

We say that a set of disjoint crossings $(\Gamma_i)_i$ of B_N can be made into η-well-separated fences if there exists a set of η-well-separated fences $(\tilde{\Gamma}_i)_i$, such that each $\tilde{\Gamma}_i$ has the same left-most extremity and the same colour as Γ_i. We say that B_N is η-*separable* if any sequence of disjoint crossings of B_N can be made into η-well-separated fences.

An η-*landing-sequence* is a sequence of closed sub-intervals $I = (I_i : i = 1, 2, \dots, k)$ of $\partial\Lambda_1$, taken in anticlockwise order, such that each I_i has length η, and the minimal distance between any two intervals, and between any interval and a corner of Λ_1, is greater than $\sqrt{\eta}$. We shall assume that

$$0 < k(\eta + 2\sqrt{\eta}) < 8, \tag{2.3.1}$$

so that η-landing-sequences exist.

Let η, η' satisfy (2.3.1), and let I (respectively, J) be an η-landing-sequence (respectively, η'-landing-sequence). Write $A_\sigma^{I,J}(N,n)$ for the event that there exists a sequence of η-interior-, η'-exterior-fences $(\Gamma_i : i = 1, 2, \dots, k)$ in the annulus $\mathcal{A}(N,n)$, with colours prescribed by σ, such that, for all i, the interior (respectively, exterior) endpoint of Γ_i lies in NI_i (respectively, nJ_i). Let $A_\sigma^{I,\varnothing}(N,n)$ (respectively, $A_\sigma^{\varnothing,J}(N,n)$) be given similarly in terms of η-interior-fences (respectively, η'-exterior-fences). Note that

$$A_\sigma^{I,J}(N,n) \subseteq A_\sigma^{\varnothing,J}(N,n), A_\sigma^{I,\varnothing}(N,n) \subseteq A_\sigma(N,n). \tag{2.3.2}$$

These definitions are illustrated in Figure 2.3.1.

2.3.2 Statement of theorem

Now that the notation is in place, we are ready to state the main result of this section.

Theorem 2.3.1 (Separation theorem)**.** *Let $k \in \mathbb{N}$, and $\sigma \in \{0,1\}^k$. For $\delta, l_0 > 0$, and $\eta_0 > 0$, there exist constants $c > 0$ and $n_1 \geq 0$ such that: for all $(G, \mathbb{P})$, with $\mathbb{P}$ and $\mathbb{P}^*$ satisfying the box-crossing property* BXP(l_0,δ)*, all $\eta, \eta' > \eta_0$ satisfying* (2.3.1)*, all η-landing-sequences I and η'-landing-sequences J, and all $N \geq n_1$ and $n \geq 2N$, we have*

$$\mathbb{P}\left[A_\sigma^{I,J}(N,n)\right] \geq c\mathbb{P}\left[A_\sigma(N,n)\right].$$

Amongst the consequences of Theorem 2.3.1 is the following.

Corollary 2.3.2. *Let G be a planar graph and $\mathbb{P}$ be a percolation measure. Suppose $\mathbb{P}$ and $\mathbb{P}^*$ satisfy the box-crossing property* BXP(l_0,δ)*. For $k \in \mathbb{N}$ and $\sigma \in \{0,1\}^k$, there exists $c = c(\delta,\sigma) > 0$ and $n_0 = n_0(l_0) \geq 0$ such that, for all $N \geq n_0$ and $n \geq 2N$,*

$$\mathbb{P}\left[A_\sigma\left(N,2n\right)\right] \geq c\mathbb{P}\left[A_\sigma\left(N,n\right)\right].$$
$$\mathbb{P}\left[A_\sigma\left(\tfrac{N}{2},n\right)\right] \geq c\mathbb{P}\left[A_\sigma\left(N,n\right)\right].$$

Proof. We prove the first inequality, the second is similar.

Let $\eta = \eta(k)$ be such that there exists an η-landing sequence of length k, entirely situated on the right side of $\partial\Lambda_1$. Take $(I_i, i = 1,\dots,k)$ such a landing sequence. Let n_1 be given by the separation theorem applied to $(G,\mathbb{P})$ for this value of η. For $N \geq n_1 \vee l_0$ and $n \geq 2N$, let H_n be the event that, for each $i \in \{1,\dots,k\}$, the rectangle $[n,2n] \times I_i$ contains a horizontal crossing of colour σ_i.

By the box-crossing property BXP(l_0,δ), there exists $c_0 = c_0(\delta) > 0$ such that $\mathbb{P}(H_n) \geq c_0$. By the upcoming Lemma 2.3.3

$$\mathbb{P}[A_\sigma(N,2n)] \geq \mathbb{P}[A_\sigma^{\varnothing,I}(N,n) \cap H_n] \geq c_0c_1\mathbb{P}[A_\sigma(N,n)],$$

where $c_1 = c_1(\epsilon,k) > 0$ is given by the separation theorem. □

2.3.3 Proof of the separation theorem

In the proof of Theorem 2.3.1 the typical events consist of the existence of certain open and open* paths. The usual FKG inequality is not enough to control the probabilities of intersections of such events. Before the actual proof we state an enhanced version of the FKG inequality, adapted to our setting. The following lemma is taken from [Nol08, Lemma 12], and we direct the reader to the original work for the proof.

Lemma 2.3.3. *Consider A^+, $\tilde{A}^+$ two increasing events and A^-, $\tilde{A}^-$ two decreasing events on $\Omega = \{0,1\}^E$. Assume that there exist three disjoint finite sets of edges $\mathcal{A}$, $\mathcal{A}^+$ and $\mathcal{A}^-$, such that A^+,A^-, $\tilde{A}^+$ and $\tilde{A}^-$ depend only on the edges in, respectively $\mathcal{A}\cup\mathcal{A}^+$, $\mathcal{A}\cup\mathcal{A}^-$, $\mathcal{A}^+$ and $\mathcal{A}^-$. Then we have*

$$P[\tilde{A}^+ \cap \tilde{A}^- | A^+ \cap A^-] \geq P[\tilde{A}^+]P[\tilde{A}^-],$$

for any product measure P on Ω.

The proof of Theorem 2.3.1 is long and intricate and we would like to focus on the structure. Hence we have split it into a sequence of lemmas.

Fix a planar graph G with a percolation measure $\mathbb{P}$, and assume $\mathbb{P}$ satisfies $\mathrm{BXP}(l_0, \delta)$ for some $l_0, \delta > 0$. All constants in the following statements depend implicitly on σ, δ and l_0, but not otherwise on $(G, \mathbb{P})$.

For the sake of clarity we will limit ourselves to the case of the exterior boundary; the same may be adapted to the interior boundary. For $\eta > 0$, denote $A^\eta_\sigma(N, n)$ the event that there exists a sequence of η-well-separated fences $(\Gamma_i)_{i\in\{1,\dots,k\}}$ in $\mathcal{A}(N, n)$, with colours given by σ.

We skip the explanation of why we may restrain ourselves to the case where n and N are integer powers of 2. We remind the reader that B_N denotes the rectangle $[0, N] \times [0, 4N]$.

Lemma 2.3.4. *For $\nu > 0$ there exist $\eta' = \eta'(\nu) > 0$ and $N_0 = N_0(\nu) \in \mathbb{N}$ such that for all $N \geq N_0$*

$$\mathbb{P}[B_N \textit{ is } \eta'\textit{-separable}] > 1 - \frac{\nu}{4}$$

It will be obvious from the proof that η' can be chosen to be increasing in ν. This lemma is the engine room of the proof of Theorem 2.3.1; we will admit it for now and prove it in the next subsection. Here is a consequence.

Lemma 2.3.5. *Take $\nu > 0$ and $\eta' = \eta'(\nu)$ given by Lemma 2.3.4. Then, for $n > N \geq N_0(\nu)$,*

$$\mathbb{P}[A_\sigma(2^N, 2^n)] \leq \mathbb{P}[A^{\eta'}_\sigma(2^N, 2^n)] + \nu\mathbb{P}[A_\sigma(2^N, 2^{n-1})],$$

and

$$\mathbb{P}[A_\sigma(2^N, 2^n)] \leq \sum_{0\leq j<n-N} \nu^j \mathbb{P}[A^{\eta'}_\sigma(2^N, 2^{n-j})].$$

In the preceding lemma, as well as in the following ones, we consider $\mathbb{P}[A_\sigma^{\eta'}(n,n)] = 1$; this proves to be a coherent convention.

Proof. For the first equation note that, under the event $A_\sigma(2^N, 2^n) \setminus A_\sigma^{\eta'}(2^N, 2^n)$, one of the four $2^{n-1} \times 2^{n+1}$-rectangles forming $\mathcal{A}(2^{n-1}, 2^n)$ is not η'-separable (on its outward facing side). The latter is an event that, by Lemma 2.3.4, is of probability at most ν. Moreover, the fact that one of these boxes is not η'-separable is independent of the states of the edges in $\mathcal{A}(2^N, 2^{n-1})$. Thus

$$\begin{aligned}
&\mathbb{P}\left[A_\sigma(2^N, 2^n) \setminus A_\sigma^{\eta'}(2^N, 2^n)\right] \\
&\qquad \leq \mathbb{P}\left[\{\text{one of the rectangles is not } \eta'\text{-separable}\} \cap A_\sigma(2^N, 2^{n-1})\right] \\
&\qquad \leq \nu \mathbb{P}\left[A_\sigma(2^N, 2^{n-1})\right].
\end{aligned}$$

This proves the first inequality.

The second inequality is obtained by repeatedly applying the first, until we reach the event $A_\sigma^{\eta'}(2^N, 2^N)$, which has probability 1. $\square$

Lemma 2.3.6. *For $\eta' > 0$ satisfying (2.3.1), there exists $C_0 = C_0(\eta') > 0$ such that for $j \geq N \geq 0$ there exists a η'-landing sequence I' with*

$$\mathbb{P}[A_\sigma^{\eta'}(2^N, 2^j)] \leq C_0 \mathbb{P}[A_\sigma^{\varnothing, I'}(2^N, 2^j)].$$

Proof. First suppose $j > N$. For given η' we may find a finite family of η'-landing sequences such that any set of k η'-well separated fences of $\mathcal{A}(2^N, 2^n)$ lands in at least one of the landing sequences of the family. Then C_0 is given by the inverse of the number of sequences in the family.

If $j = N$ both probabilities are, by convention, 1. $\square$

Lemma 2.3.7. *For $\eta' > 0$ there exist constants $C_1 = C_1(\eta') > 0$ and $N_1 = N_1(\eta') \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ and $j \geq N_1(\eta')$, for any η'-landing sequence I', for any $\eta \geq \eta'$ and any η-landing sequence I,*

$$\mathbb{P}[A_\sigma^{\varnothing, I'}(2^N, 2^j)] \leq C_1 \mathbb{P}[A_\sigma^{\varnothing, I}(2^N, 2^{j+1})].$$

Proof. This is done through an explicit construction using crossings of boxes as illustrated in Figure 2.3.2. By the box-crossing property and Lemma 2.3.3 we obtain

$$\mathbb{P}\left[A_\sigma^{\varnothing, I}(2^n, 2^{j+1}) \middle| A_\sigma^{\varnothing, I'}(2^n, 2^j)\right] \geq C_1(\eta').$$

$\square$

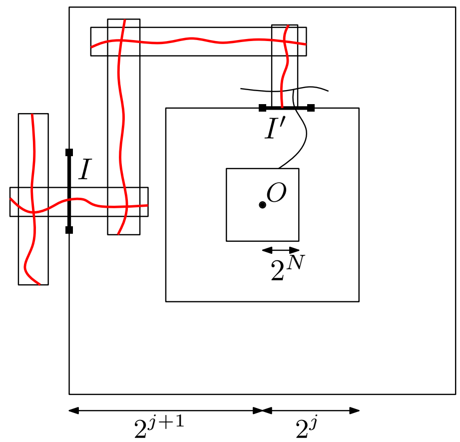

Figure 2.3.2: The extension of a fence from $\mathcal{A}(2^N, 2^j)$ to $\mathcal{A}(2^N, 2^{j+1})$. All rectangles have aspect ratio controlled by η'. Since η' may be small and we may need to fit k disjoint such construction in $\mathcal{A}(2^j, 2^{j+1})$, we need a lower bound on j. Thus we impose $j \geq N_1$ in lemmas 2.3.7 and 2.3.8.

The following lemma is a particular case of Lemma 2.3.7; we state it separately only to emphasize the steps of the proof.

Lemma 2.3.8. *For $\eta_0 > 0$ there exist constants $C_2 = C_2(\eta) > 0$ and $N_2 = N_2(\eta) \in \mathbb{N}$ such that for all $\eta \geq \eta_0$, $N \in \mathbb{N}$, $j \geq N_2(\eta)$, and any η-landing sequence I,*

$$\mathbb{P}\left[A_\sigma^{\varnothing,I}(2^N, 2^j)\right] \leq C_2 \mathbb{P}\left[A_\sigma^{\varnothing,I}(2^N, 2^{j+1})\right].$$

Let us now see how to use the above lemmas to conclude.

Proof of Theorem 2.3.1. Fix $\eta_0 > 0$. Consider the quantities $C_2(\eta_0) > 0$ and N_2 given by Lemma 2.3.8 applied to η_0.
Let $\nu = \frac{C_2}{2}$; Lemma 2.3.4, applied with this value of ν, yields quantities $\eta' > 0$ and N_0. Since η' is increasing in ν, we may choose $\eta' < \eta_0$.
Lemma 2.3.6 applied to η' yields a constant $C_0 > 0$.
Lemma 2.3.7 applied to η' yields a constant $C_1 > 0$ and a rank N_1.

We have written this so as to stress the fact that all constants in the computation depend only on η_0. Consider now some $\eta \geq \eta_0$, $n \geq N \geq \max\{N_0, N_1, N_2\}$ and a η-

landing sequence I. By the above lemmas we have

$$
\begin{aligned}
&\mathbb{P}\left[A_\sigma(2^N,2^{n+1})\right] \\
&\quad\le \mathbb{P}\left[A_\sigma(2^N,2^{n})\right] \\
&\quad\le \sum_{0\le j\le n-N} \nu^j \mathbb{P}\left[A_\sigma^{\eta'}(2^N,2^{n-j})\right] && \text{by Lemma 2.3.5} \\
&\quad\le \sum_{0\le j\le n-N} \nu^j C_0 \mathbb{P}\left[A_\sigma^{\varnothing,I'}(2^N,2^{n-j})\right] && \text{by Lemma 2.3.6 } (I' \text{ depends on } j) \\
&\quad\le \sum_{0\le j\le n-N} \nu^j C_0 C_1 \mathbb{P}\left[A_\sigma^{\varnothing,I}(2^N,2^{n-j+1})\right] && \text{by Lemma 2.3.7} \\
&\quad\le \sum_{0\le j\le n-N} \nu^j C_0 C_1 C_2^j \mathbb{P}\left[A_\sigma^{\varnothing,I}(2^N,2^{n+1})\right] && \text{by Lemma 2.3.8 for } \eta>\eta_0 \\
&\quad\le 2C_0C_1 \mathbb{P}\left[A_\sigma^{\varnothing,I}(2^N,2^{n+1})\right] && \text{since } \nu C_2 \le \frac{1}{2}.
\end{aligned}
$$

The above string of inequalities yields the desired result. □

2.3.4 Proof of Lemma 2.3.4

There are two parts in the proof of this lemma. First we show that, with high probability, the crossings can be made to land far from the corners of B_N, then we transform the crossings into fences. Both parts are based on constructions using circuits in concentric annuli. We will use constants $C_i > 0$ which arise from box crossing constructions and depend solely on δ. Fix $\nu > 0$, and work in the box $B_N = [0,N]\times[0,4N]$, where N is large, we will see later how large.

Crossings land far from corners. Denote Z^+ (respectively Z^-) the upper right (respectively lower right) corner of the box B_N. Consider some small $\eta > 0$ (we will see later how small), and say Z^+ is *protected* (or η-protected) if there exist two paths, one open and one open*, both at distance at least $\sqrt{\eta}N$ from Z^+, that separate Z^+ from the left side of B_N (in B_N). See Figure 2.3.3, right diagram. By the box-crossing property, there exists $C_0 = C_0(\delta) > 0$ such that

$$\mathbb{P}\left[\mathcal{A}_{Z^+}\left(\sqrt{\eta}N2^k,\sqrt{\eta}N2^{k+1}\right) \text{ contains an open/open}^* \text{ circuit}\right] \ge C_0,$$

for any k as long as $\sqrt{\eta}N2^k \ge l_0$.

Suppose $\sqrt{\eta}N \ge l_0$ and consider $K\in\mathbb{N}$ such that $\sqrt{\eta}N2^{K+1} < N$. If one of the annuli $\mathcal{A}_{Z^+}(\sqrt{\eta}N2^k,\sqrt{\eta}N2^{k+1})$, with $1\le k\le K$, contains an open circuit, and another an open* circuit, then the corner Z^+ is protected. Hence

$$\mathbb{P}[Z^+ \text{ is protected}] \ge 1-2(1-C_0)^K.$$

By taking K as large as possible in the above expression, we obtain

$$\mathbb{P}[Z^+ \text{ is not protected}] \leq 2(1-C_0)^{-\frac{2+\ln\eta}{4\ln 2}}.$$

The right hand side is smaller than ν if

$$\eta \leq \exp\left(-\frac{2\ln\frac{\nu}{2}\ln 2}{\ln(1-C_0)} - 2\right) =: \eta_1(\nu).$$

In the above computation we have used that $N \geq \frac{l_0}{\sqrt{\eta}} =: N_1(\eta)$.

To conclude, for any $\eta < \eta_1$ and $N \geq N_1$,

$$\mathbb{P}[Z^+ \text{ is not } \eta\text{-protected}] \leq \nu.$$

The same holds for Z^-, with the same values of η_1 and N_1.

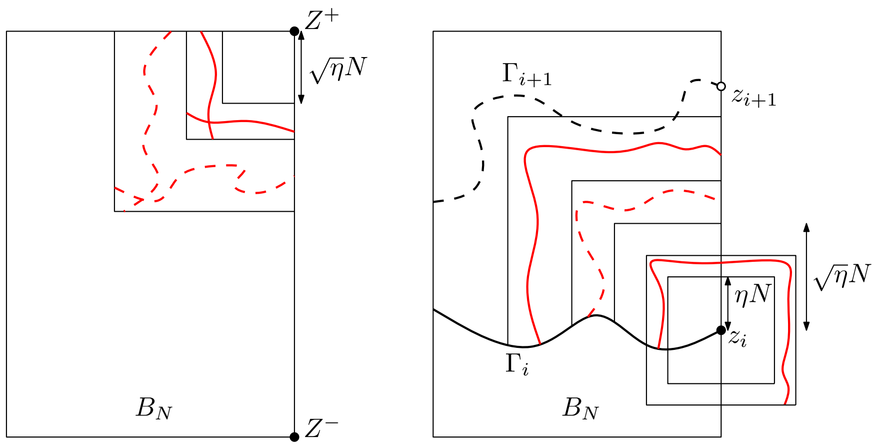

Figure 2.3.3: *Left:* The corner Z^+ is protected. *Right:* The point z_i is protected. The innermost path guarantees the fact that Γ_i is a fence; the two outer paths guarantee that $|z_{i+1} - z_i| \geq \sqrt{\eta}N$.

Crossings may be made into fences. Let I be the total number of disjoint crossings of B_N, both open and open*. First we bound I. For $T \geq 1$ and $N \geq l_0$, by the box-crossing property and the BK inequality,

$$\mathbb{P}[I \geq T] \leq \mathbb{P}[I \geq 1]^T \leq (1-C_1)^T,$$

with $C_1 = C_1(\delta) > 0$ coming from the box-crossing property for $\mathbb{P}$ and $\mathbb{P}^*$. Choose $T \geq \frac{\nu}{\ln(1-C_1)}$, such that the above probability is smaller than ν.

Let $\nu' = \frac{\nu}{T}$. We will now show that, provided η is small enough, the probability that each crossing of B_N may be made into a η-fence is greater than $1-\nu'$.

Let $(\Gamma_i)_{1\leq i\leq I}$ denote the disjoint crossings of B_N, both open and open*, in increasing order (choose Γ_1 to be the lowest crossing of B_N, Γ_2 the lowest crossing of B_N which lies strictly above Γ_1, etc.). Let $(z_i)_i$ denote their endpoints on the right side of B_N. For some $K \in \mathbb{N}$ (we will se later how to choose it), we say z_i is η-protected if:

(i) one of the annuli $\{\mathcal{A}_{z_i}(\eta N 2^k, \eta N 2^{k+1}) : 0 \leq k < K\}$ contains path of the same colour as Γ_i, above Γ_i, and connecting if to a vertical crossing along the right side of the annulus. See the innermost annulus around z_i in the right diagram of Figure 2.3.3;

(ii) there are two annuli in $\{\mathcal{A}_{z_i}(\sqrt{\eta} N 2^k, \sqrt{\eta} N 2^{k+1}) : 0 \leq k < K\}$ containing an open, respectively open*, path, connecting Γ to the line $\{N\} \times \mathbb{R}$ (as in the right diagram Figure 2.3.3).

Assume η and K are such that $\sqrt{\eta} < 2^{-K}$. Then, if z_i is η-protected, Γ_i may be made into a η-fence and $|z_{i+1} - z_i| \geq \sqrt{\eta} N$. Moreover, the two events defining a protected point depend on disjoint regions of the plane, hence are independent. For any path γ crossing B_N (in G or G^*) and any $i \in \mathbb{N}$, the event $\Gamma_i = \gamma$ only depends on the states of the edges in B_N below γ. Thus, above γ and outside B_N, the measure conditioned on $\Gamma_i = \gamma$ is equal to the regular percolation measure $\mathbb{P}$; in particular the box-crossing property holds in this region. Using this, and constructions of partial circuits in annuli as in Figure 2.3.3, we deduce that

$$\mathbb{P}[z_i \text{ is not } \eta\text{-protected}] \leq 3(1 - C_0)^K,$$

where $C_0 > 0$ does not depend on η, K or N, and N is large enough for the box-crossing property to hold in all rectangles involved. More precisely $N \geq \frac{l_0}{\eta} =: N_2(\eta)$.

Finally choose

$$K = K(\nu') = \left\lceil \frac{\ln \nu'}{\ln(1 - C_0)} \right\rceil,$$

and $\eta_2 = \eta_2(\nu') > 0$, such that $\sqrt{\eta_2} \leq 2^{-K}$. Then, for $\eta \leq \eta_2$ and $N \geq N_2(\eta)$,

$$\mathbb{P}[z_i \text{ is not } \eta\text{-protected}] \leq \nu'.$$

Conclusion. Using the above facts we deduce that, for $\eta \leq \min\{\eta_1, \eta_2\}$ and $N \geq \max\{N_1(\eta), N_2(\eta), l_0\}$,

$$\begin{aligned}
\mathbb{P}\left[B_N \text{ is not } \eta\text{-separable}\right] \leq & \mathbb{P}\left[Z^+ \text{ is not } \eta\text{-protected}\right] + \\
& \mathbb{P}\left[Z^- \text{ is not } \eta\text{-protected}\right] + \\
& \mathbb{P}\left[I \geq T\right] + \\
& \sum_{1 \leq i < T} \mathbb{P}\left[i \leq I \text{ and } z_i \text{ is not } \eta\text{-protected}\right] \\
& \leq 3\nu + T\nu' = 4\nu.
\end{aligned}$$

This is the required result, with 4ν instead of $\frac{\nu}{4}$.

2.4 Scaling relations at criticality

In this section we prove the scaling relations (1.6.2), with minimal assumptions on the model.

Let G be a planar graph embedded in the plane, and $\mathbb{P}$ be a bond percolation measure on it. We assume that G satisfies the conditions of Section 1.3.1, but no symmetry is required.

Suppose G is such that $0 \in \mathbb{R}^2$ is a vertex of G. For $n \geq 0$, denote the probabilities of the one-arm event centered at 0 by

$$\pi_1(n) = \mathbb{P}(0 \leftrightarrow \partial\Lambda_n),$$

with the convention $\pi_1(0) = 1$. For $v \in V$, write $\pi_1^v(n)$ for the probabilities of the similar one-arm events centered at v. We will assume in this section that there exists a constant $c_\pi > 0$ such that, for $n \geq 0$ and $v \in V$,

$$c_\pi^{-1}\pi_1^v(n) \leq \pi_1(n) \leq c_\pi \pi_1^v(n). \tag{2.4.1}$$

The above is immediate for periodic models, but is a significant assumption in other situations.

Theorem 2.4.1. *Suppose both $\mathbb{P}$ and $\mathbb{P}^*$ satisfy the box-crossing property. If ρ or η exist for $(G, \mathbb{P})$, then η, ρ and δ exist for $(G, \mathbb{P})$, and*

$$\eta\rho = 2 \quad \text{and} \quad 2\rho = \delta + 1. \tag{2.4.2}$$

The theorem also holds for site percolation with only minor changes in the proof. The proof which is presented next follows Kesten's arguments from [Kes86, Kes87a], with small changes due to the more general context.

We assume $\mathbb{P}$ has the box-crossing property BXP$(1, \delta_0)$ for some $\delta_0 > 0$. All constants in the rest of the section implicitly depend on c_π, δ_0 and on the constant K_d of Section 1.3.1, but, unless explicitly stated, not otherwise on $(G, \mathbb{P})$. The constants c_i in different statements are generally unrelated. We will use the phrase n large enough to mean $n \geq n_0$ with n_0 only depending on c_π, δ_0 and K_d. Before the actual proof we give a helpful bound for π_1.

Lemma 2.4.2. *There exists a constant $c > 0$ such that, for $n \geq 1$ and $v \in V$,*

$$\pi_1^v(n) \geq \frac{c}{\sqrt{n}}.$$

As a consequence, if ρ exists, then $\rho \geq 2$.

It will also be useful to note that, due to the box-crossing property, there exists a constant $c > 0$ such that, for $n \geq 1$,

$$\pi_1(2n) \geq c\pi_1(n). \tag{2.4.3}$$

Proof of Lemma 2.4.2. Fix $n \geq 1$ and consider the rectangular domain $\mathcal{B}(2n+1, 2n) = [0, 2n+1] \times [0, n]$. By the box-crossing property there exists a constant $c_1 > 0$, not depending on n, such that

$$\mathbb{P}\left[\mathcal{C}_{\mathrm{v}}(\mathcal{B}(2n+1, 2n))\right] \geq c_1.$$

Let $\mathcal{S}$ denote the strip $[n, n+1] \times [0, 2n]$. If $\mathcal{C}_{\mathrm{v}}\left[\mathcal{B}(2n+1, 2n)\right]$ occurs, then there exists at least one vertex v in $\mathcal{S}$, with two disjoint open paths linking it to the left (respectively, right) side of the box $\mathcal{B}(2n+1, 2n)$. Call such a vertex a *linked* vertex. Then

$$\begin{aligned}\sum_{v \in \mathcal{S}} \mathbb{P}(v \text{ is linked}) &\geq \mathbb{P}(\text{there exists } v \in \mathcal{S} \text{ linked}) \\ &\geq \mathbb{P}\left[\mathcal{C}_{\mathrm{v}}(\mathcal{B}(2n+1, 2n))\right] \geq c_1.\end{aligned}$$

By the conditions in Section 1.3.1, the strip $\mathcal{S}$ contains at most $K_d n$ vertices. Also, by the BK inequality, for any $v \in \mathcal{S}$,

$$\mathbb{P}(v \text{ is linked}) \leq \left(\pi_1^v(n)\right)^2 \leq c_\pi^2 \left(\pi_1(n)\right)^2.$$

In conclusion

$$K_d n \left(\pi_1(n)\right)^2 \geq \frac{c_1}{c_\pi^2},$$

which concludes the proof of the lemma. $\square$

The proof of Theorem 2.4.1 is based on the following propositions taken from [Kes87a]. Henceforth v will denote a vertex of G, and $|v|$ will be the euclidian distance between 0 and v.

Proposition 2.4.3. *If one of the following two limits exists*

$$-\frac{1}{\rho} = \lim_{n \to \infty} \frac{\log \pi_1(n)}{\log n}, \quad -\eta = \lim_{|v| \to \infty} \frac{\log \mathbb{P}(0 \leftrightarrow v)}{\log |v|}, \tag{2.4.4}$$

then they both exist and $\eta = \frac{2}{\rho}$.

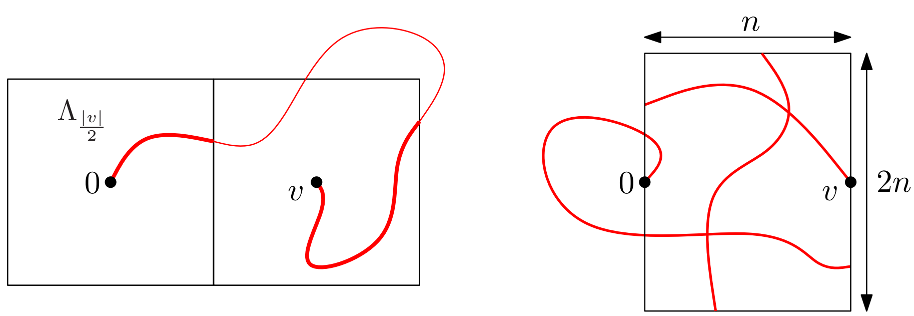

Figure 2.4.1: *Left:* The existence of a path from 0 to v implies the existence of disjoint arm events centered at 0 and v. *Right:* The horizontal paths form R and L. Together with the vertical crossing of H they form a path from 0 to v.

Proposition 2.4.4. (a) *For any $\epsilon > 0$, there exists $\lambda > 0$ such that, for $v \in V$,*

$$\mathbb{P}\left[\frac{|C_v|}{n^2\pi_1(n)} \leq \lambda \middle| \mathrm{rad}(C_v) \geq n\right] < \epsilon, \qquad \textit{for } n \geq 1. \tag{2.4.5}$$

(b) *For $\lambda > 1$ and $t \geq 1$, there exists $c(t)$ depending only on t such that, for $v \in V$,*

$$\mathbb{P}\left[\frac{|C_v|}{n^2\pi_1(n)} \geq \lambda \middle| n \leq \mathrm{rad}(C_v) \leq 2n\right] < c(t)\lambda^{-t}, \qquad \textit{for } n \geq 1. \tag{2.4.6}$$

Corollary 2.4.5. *If the limits of Proposition 2.4.3 exist, then*

$$-\frac{1}{\delta} = \lim_{n\to\infty} \frac{\log \mathbb{P}(|C_v| > n)}{\log n}$$

exists uniformly in v, and $\delta = 2\rho - 1$.

Theorem 2.4.1 follows directly from Proposition 2.4.3 and Corollary 2.4.5.

Proof of Proposition 2.4.3. Fix $v \in V$. By rotating G we may suppose $v \in \mathbb{R} \times \{0\}$. This rotation may affect $\pi_1(n)$, but only by a bounded multiplicative factor (see (2.4.3)).

First suppose $0 \leftrightarrow v$. Then there exist arms from 0 and v, respectively, to distance $|v|/2$ away. Moreover these are contained in disjoint parts of the plane. See the left diagram of Figure 2.4.1. By (2.4.1) and (2.4.3),

$$\mathbb{P}(0 \leftrightarrow v) \leq \mathbb{P}\left[\mathrm{rad}(C_0) \geq \frac{|v|}{2}\right] \mathbb{P}\left[\mathrm{rad}(C_v) \geq \frac{|v|}{2}\right] \leq c_1\pi_1\left(|v|\right)^2, \tag{2.4.7}$$

with c_1 not depending on v.

Conversely, let $n = |v|$ and define the events

$$L = \left\{0 \xleftrightarrow{\Lambda_n} \{n\} \times [-n, n]\right\} \quad \text{and} \quad R = \left\{v \xleftrightarrow{\Lambda_n + v} \{0\} \times [-n, n])\right\}.$$

Then, by the box-crossing property and (2.4.1), there exists $c_2 > 0$ such that

$$\mathbb{P}(L) \geq c_2\pi_1(n), \qquad \mathbb{P}(R) \geq c_2\pi_1(n).$$

Let H be the event that the rectangle $[0,n]\times[-n,n]$ contains a vertical open crossing. By the box-crossing property, $\mathbb{P}(H) \geq c_3$ for some $c_3 > 0$, independent of n. Finally, by the FKG inequality and the geometrical consideration of Figure 2.4.1,

$$\mathbb{P}(0 \leftrightarrow v) \geq \mathbb{P}(L \cap R \cap H) \geq c_3 c_2^2 \pi_1(n)^2. \tag{2.4.8}$$

Inequalities (2.4.7) and (2.4.8) imply the proposition. $\square$

Let us assume Proposition 2.4.4 for now, and prove Corollary 2.4.5. The proof of Proposition 2.4.4 is presented in the next section.

Proof of Corollary 2.4.5. Fix $\epsilon \in (0,1)$ and λ as in Proposition 2.4.4 (a). Then, for $n \geq 1$ and $v \in V$,

$$\begin{aligned}\mathbb{P}\left[|C_v| \geq \lambda n^2\pi_1(n)\right] &\geq \mathbb{P}\left[|C_v| \geq \lambda n^2\pi_1(n) \middle| \mathrm{rad}(C_v) \geq n\right]\mathbb{P}\left[\mathrm{rad}(C_v) \geq n\right] \\ &\geq (1-\epsilon)c_\pi^{-1}\pi_1(n).\end{aligned}$$

Using the above and (2.4.4), we obtain

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log \mathbb{P}\left[|C_v| \geq n\right]}{\log n} &\geq \liminf_{n\to\infty} \frac{\log \mathbb{P}\left[|C_v| \geq \lambda n^2\pi_1(n)\right]}{\log \lambda n^2 \pi_1(n)} \\ &= \frac{1}{1-2\rho}.\end{aligned} \tag{2.4.9}$$

We turn to the converse inequality. Fix $\epsilon > 0$ and, for $n \geq 1$, set

$$k_0 = \left\lceil (1-\epsilon)\frac{\log n}{(\log 2)(2-\frac{1}{\rho})} \right\rceil. \tag{2.4.10}$$

By our assumption, $\pi_1(n) = n^{-\frac{1}{\rho}+o(1)}$, hence

$$2^{2k_0}\pi_1(2^{k_0}) = n^{1-\epsilon+o(1)}. \tag{2.4.11}$$

For n large enough, we use Proposition 2.4.4 (b), with $t = 2$ and $\lambda = \frac{n}{2^{2k}\pi_1(2^k)} > 1$, for the following computation.

$$\begin{aligned}
&\mathbb{P}\left[|C_v| \geq n\right] \\
&\quad \leq \pi_1^v(2^{k_0}) + \sum_{k<k_0} \pi_1^v(2^k)\mathbb{P}\left[|C_v| \geq n \middle| 2^k \leq \mathrm{rad}(C_v) < 2^{k+1}\right] \\
&\quad \leq c_\pi^{-1}\pi_1(2^{k_0})\left[1 + c_2 \sum_{k<k_0} \frac{\pi_1(2^k)}{\pi_1(2^{k_0})}\left(\frac{2^{2k}\pi_1(2^k)}{n}\right)^2\right] \qquad \text{by (2.4.6)} \\
&\quad = c_\pi^{-1}\pi_1(2^{k_0})\left[1 + c_2\left(\frac{2^{2k_0}\pi_1(2^{k_0})}{n}\right)^2 \sum_{k<k_0} 2^{k-k_0}\left(\frac{2^{k-k_0}\pi_1(2^k)}{\pi_1(2^{k_0})}\right)^3\right]. \qquad (2.4.12)
\end{aligned}$$

Since $\pi_1(n) = n^{-\frac{1}{\rho}+o(1)}$ and $\rho \geq 2$ (see Lemma 2.4.2), the sum in (2.4.12) is bounded above by a constant c_3, uniformly in k_0. Thus

$$\mathbb{P}\left[|C_v| \geq n\right] \leq c_\pi^{-1}\pi_1(2^{k_0})\left[1 + c_2 c_3\left(\frac{2^{2k_0}\pi_1(2^{k_0})}{n}\right)^2\right].$$

Using (2.4.10), the above implies

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left[|C_v| \geq n\right]}{\log n} \leq \frac{1-\epsilon}{1-2\rho}. \qquad (2.4.13)$$

Finally, since $\epsilon > 0$ is arbitrary, (2.4.9) and (2.4.13) imply the corollary. $\square$

2.4.1 Proof of Proposition 2.4.4

This section is an adaptation of the arguments of [Kes86]. The proof of Proposition 2.4.4 is based on certain moments estimates for $|C_0|$, such as those given in Lemma 2.4.6. This lemma is interesting not only for its results, but also for its proof, which illustrates arguments that will be used to obtain various similar estimates.

Lemma 2.4.6. *For $t \geq 1$, there exist constants $C(t), C'(t) > 0$, such that, for all $n \in \mathbb{N}$,*

$$\mathbb{E}\left[|C_v|^t | n \leq \mathrm{rad}(C_v) \leq 2n\right] \geq C(t)[n^2\pi_1(n)]^t, \qquad (2.4.14)$$

$$\mathbb{E}\left[|C_v|^t | n \leq \mathrm{rad}(C_v) \leq 2n\right] \leq C'(t)[n^2\pi_1(n)]^t. \qquad (2.4.15)$$

In the proof of Lemma 2.4.6, we will use the following inequality.

Lemma 2.4.7. *There exists a constant $c > 0$ such that, for all $n \in \mathbb{N}$,*

$$n\pi_1(n) \leq \sum_{k=0}^{n} \pi_1(k) \leq cn\pi_1(n). \qquad (2.4.16)$$

Proof of Lemma 2.4.7. The first inequality is trivial since π is decreasing. We turn to the second.

Fix $n \in \mathbb{N}$, and let $\mathcal{S}$ denote the strip $[0,1] \times [0,2n]$. For $v \in V$ define $R(v)$ as the event that v is linked by an open path to the half space $[n,\infty) \times \mathbb{R}$. For $v \in \mathcal{S}$, we have $\mathbb{P}[R(v)] \leq c_1\pi_1(n)$ for some $c_1 > 0$, not depending on v or n. Hence, if we denote S_n the number of vertices $v \in \mathcal{S}$ such that $R(v)$ occurs, then

$$\mathbb{E}(S_n) \leq c_1 K_d \pi_1(n). \tag{2.4.17}$$

We recall the notation $\mathcal{C}_{\mathrm{h}}(\mathcal{B}(n,n))$ for the event that there exists an open horizontal crossing of $\mathcal{B}(n,n) = [0,n] \times [0,n]$. If $\mathcal{C}_{\mathrm{h}}(\mathcal{B}(n,n))$ occurs, let γ denote the lowest open horizontal crossing of $\mathcal{B}(n,n)$. Let Γ be a path crossing $\mathcal{B}(n,n)$ horizontally and $z = (z_1, z_2)$ be the highest point of Γ in $\mathcal{S}$. Denote Γ^- the set of edges of G which intersect $\mathcal{B}(n,n)$ and which are below or contained in Γ. By choice of γ, the measure $\mathbb{P}(.|\gamma = \Gamma)$ is equal to $\mathbb{P}$ outside Γ^-.

For $v = (x,y) \in V$ and $k \geq 1$, let $H_k(v)$ be the event that there exists an open circuit in $\mathcal{A}^v(k+1, 2k+1)$, which is connected to v by an open path contained in $[x-k, x+k] \times [y-k, y+2k+1]$. By the box-crossing property there exists $c_2 > 0$, such that, for $v \in V$ and $k \geq 1$, $\mathbb{P}[H_k(v)] \geq c_2\pi_1(k)$. Let

$$H_k^\Gamma(v) = \{\omega \in \Omega : \exists \sigma \in H_k(v) \text{ with } \omega_e = \sigma_e \text{ for } e \in E \setminus \Gamma^-\}.$$

With the above definition, we have

$$\mathbb{P}[H_k^\Gamma(v) | \gamma = \Gamma] \geq \mathbb{P}[H_k(v)] \geq c_2\pi_1(k). \tag{2.4.18}$$

For $k = 1, \ldots, n$, let v_k be a vertex in the square $[0,1] \times [z_2 + k, z_2 + k + 1] \subseteq \mathcal{S}$. Such a vertex exists by the conditions of Section 1.3.1. If $\gamma = \Gamma$ and $H_k^\Gamma(v_k)$ occurs, then v_k is linked to γ, hence to $[n,\infty) \times \mathbb{R}$, by an open path. Thus $R(v_k)$ also occurs. See Figure 2.4.2. By (2.4.18),

$$\begin{aligned}
\mathbb{P}[R(v_k)] &\geq \sum_\Gamma \mathbb{P}[H_k^\Gamma(v_k) | \gamma = \Gamma]\mathbb{P}[\gamma = \Gamma] \\
&\geq c_2\pi_1(k)\mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}(n,n))\right] \\
&\geq c_2c_3\pi_1(k),
\end{aligned}$$

where $c_3 > 0$ is given by the box-crossing property, and does not depend on n or k. Hence,

$$\mathbb{E}(S_n) \geq c_2c_3 \sum_{k=1}^n \pi_1(k).$$

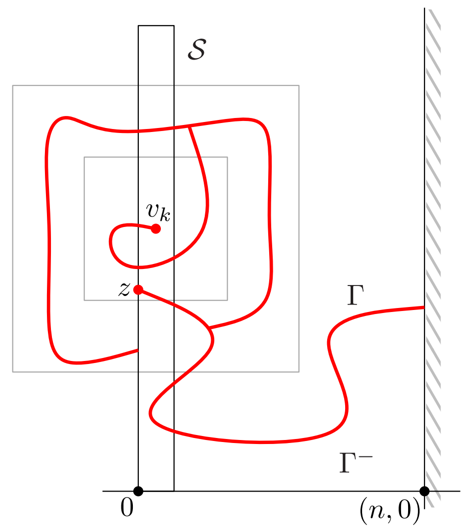

Figure 2.4.2: The intersection of the events $\gamma = \Gamma$ and $H_k^\Gamma(v_k)$ ensures that v_k is connected to $[n,\infty) \times \mathbb{R}$.

Together with (2.4.17), the above implies (2.4.16), with the sum starting at $k = 1$. The term $\pi_1(0)$ may be incorporated by increasing the constant (see also Lemma 2.4.2). $\square$

Proof of Lemma 2.4.6. First we prove (2.4.14), and, for simplicity, we take $v = 0$. The constants in the following proof do not depend on this choice.

Fix $n \geq 2$ and let H_n be the event that $\mathcal{A}(n, \frac{3}{2}n)$ contains an open circuit and that $\mathcal{A}(\frac{3}{2}n, 2n)$ contains an open* circuit. By the box-crossing property for $\mathbb{P}$ and $\mathbb{P}^*$, there exists $c_1 > 0$, not depending on n, such that $\mathbb{P}(H_n) \geq c_1$.

For $v \in \Lambda_n$ let $R(v)$ be the event that there exist an open path linking v to $\partial\Lambda_{\frac{3}{2}n}$. By the box-crossing property for $\mathbb{P}$, there exists $c_2 > 0$, not depending on n or v, such that

$$\mathbb{P}[R(v)] \geq c_2 \pi_1(n).$$

Note that H_n is increasing in the edges of $\Lambda_{\frac{3}{2}n}$, and that $R(v)$ only depends on the states of these edges. Hence, for $v \in \Lambda_n$,

$$\mathbb{P}[R(v) \cap R(0) \cap H_n] \geq c_1 c_2^2 (\pi_1(n))^2.$$

But if $R(v) \cap R(0) \cap H_n$ occurs, then $v \in C_0$ and $n \leq \mathrm{rad}(C_0) \leq 2n$. In conclusion

$$\mathbb{E}\left[|C_0|; n \leq \mathrm{rad}(C_0) \leq 2n\right] \geq \sum_{v \in \Lambda_n} \mathbb{P}[R(v) \cap R(0) \cap H_n] \geq c_1 c_2^2 n^2 (\pi_1(n))^2. \qquad (2.4.19)$$

Note that $\mathbb{P}[n \leq \text{rad}(C_0) \leq 2n] \leq \pi_1(n)$. By dividing (2.4.19) by $\pi_1(n)$, we obtain

$$\mathbb{E}\left[|C_0| \big| n \leq \text{rad}(C_0) \leq 2n\right] \geq c_1 c_2^2 n^2 \pi_1(n). \tag{2.4.20}$$

This is (2.4.14) with $t = 1$ and $n \geq 2$. The case $n = 1$ is obtained by adjusting the constants. We may extend the result to $t \geq 1$ using Jensen's inequality for positive random variables Z:

$$\mathbb{E}(Z^t) \geq [\mathbb{E}(Z)]^t.$$

We now turn to (2.4.15). As before we take $v = 0$. The constants in the following do not depend on this choice. The vertex 0 will sometimes also be denoted v_0.

By Jensen's inequality, it suffices to prove (2.4.15) for $t \in \mathbb{N}$. Fix such a t. In the following, $c_i, i \in \mathbb{N}$ will denote constants that may depend on t but not on n. We have

$$\begin{aligned}\mathbb{E}\left[|C_0|^t \big| n \leq \text{rad}(C_0) \leq 2n\right] &\leq \frac{1}{\mathbb{P}\left[n \leq \text{rad}(C_0) \leq 2n\right]} \mathbb{E}[|C_0|^t; n \leq \text{rad}(C_0) \leq 2n] \\ &\leq \frac{c_1}{\pi_1(n)} \sum_{v_1, \dots, v_t \in \Lambda_{2n}} \mathbb{P}\left[v_1, \dots, v_t \in C_0; \text{rad}(C_0) \geq n\right].\end{aligned}$$

The sum above is over all t-uplets of vertices $(v_1, \dots, v_t) \in (\Lambda_{2n})^t$. To these we add the vertex $v_0 = 0$. For such a set of vertices $(v_0, \dots, v_t)$, let $r_i = \lfloor \min\{\frac{1}{4}\|v_i - v_j\|_\infty : j \neq i\} \rfloor$, where $\|.\|_\infty$ denotes the L^∞ norm in the $\mathbb{R}^2$, and $\lfloor x \rfloor$ is the greatest integer below x. We claim that there exist c_2 such that, for all choices of $v_1, \dots, v_t$,

$$\mathbb{P}\left[v_1, \dots, v_t \in C_0; \text{rad}(C_0) \geq n\right] \leq c_2 \pi_1(n) \prod_{i=1}^{t} \pi_1(r_i). \tag{2.4.21}$$

Let us prove this claim. Fix the vertices $v_1, \dots, v_t$, and let H be the event that, for each $i \in \{1, \dots, t\}$, the annulus $\mathcal{A}^{v_i}(r_i, 2r_i)$ contains an open circuit (if $r_i = 0$, we do not require the existence of any path). By the box-crossing property, $\mathbb{P}(H) > c_3$, with $c_3 > 0$ only depending on t, not on $v_1, \dots, v_t$ or n. If $v_1, \dots, v_t \in C_0$, $\text{rad}(C_0) \geq n$ and H occurs, then there exist disjoint open paths γ_i such that γ_0 connects v_0 to $\partial\Lambda_n$ and, for $i \geq 1$, γ_i connects v_i to $\partial\Lambda_{r_i} + v_i$. See also Figure 2.4.3. By the BK and FKG inequalities,

$$\begin{aligned}\pi_1(n) \prod_{i=1}^{t} \pi_1^{v_i}(r_i) &\geq \mathbb{P}\left[\{v_1, \dots, v_t \in C_0\} \cap \{\text{rad}(C_0) \geq n\} \cap H\right] \\ &\geq c_3 \mathbb{P}\left[\{v_1, \dots, v_t \in C_0\} \cap \{\text{rad}(C_0) \geq n\}\right].\end{aligned}$$

In conjunction with (2.4.1), the above implies (2.4.21).

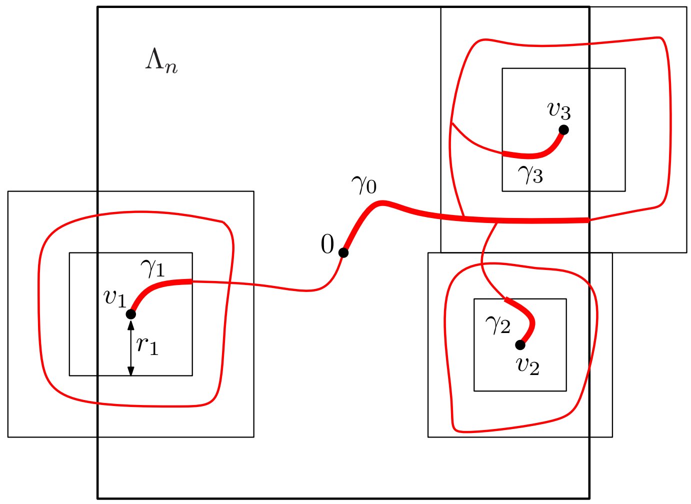

Figure 2.4.3: If $v_1, \dots, v_t \in C_0$, $\mathrm{rad}(C_0) \geq n$ and H occurs, then there exist open paths γ_i connecting v_i to $\partial\Lambda_{r_i}$ and γ_0 connecting 0 and $\partial\Lambda_n$. Moreover these paths are disjoint.

Finally, this leads to

$$\mathbb{E}\left[|C_0|^t \middle| n \leq \mathrm{rad}(C_0) \leq 2n\right] \leq c_4 \sum_{v_1,\dots,v_t \in \Lambda_{2n}} \prod_{i=1}^{t} \pi_1(r_i).$$

In order to prove (2.4.15), it suffices show the existence of a constant $c = c(t)$, such that

$$\sum_{v_1,\dots,v_t \in \Lambda_{2n}} \prod_{i=1}^{t} \pi_1(r_i) \leq cn^{2t}\pi_1(n)^t. \tag{2.4.22}$$

For that purpose, we group the terms of the sum by the $t+1$-uplet $(r_0, \dots, r_t)$.

Let us first consider the case $t = 1$. For $v_1 \in \Lambda_{2n}$, we have $r_0 = r_1 \leq n$, and $v_1 \in \mathcal{A}(4r_0, 4r_0{+}4)$. Hence, for an imposed value of r_0, there are at most $32K_d(r_0{+}1) \leq c_5 n$ choices for v_1. By (2.4.16),

$$\sum_{v_1 \in \Lambda_{2n}} \pi_1(r_1) \leq c_5 \sum_{r_1=0}^{n} n\pi_1(r_1) \leq c_6 n^2 \pi_1(n).$$

Let us also sketch the proof for $t = 2$. For any two vertices, v_1, v_2, two of the three quantities r_0, r_1, r_2 are equal, and smaller than the third. Let (r_0, r_1, r_2) be such a triplet. By analysing separately the cases $r_0 = r_1 \leq r_2$ and $r_0 \geq r_1 = r_2$, we find that there are at most $c_7 n^2$ vertices $v_1, v_2 \in \Lambda_{2n}$ which yield this particular triplet, where c_7 is a constant that only depends on K_d. In conclusion

$$\begin{aligned}\sum_{v_1,v_2\in\Lambda_{2n}}\pi_1(r_1)\pi_1(r_2) &\le c_7n^2\left(2\sum_{0\le r_1\le r_2\le n}\pi_1(r_1)\pi_1(r_2)+\sum_{0\le r_1\le r_0\le n}\pi_1(r_1)^2\right)\\ &\le 2c_7n^2\left(\sum_{r=0}^{n}\pi_1(r)\right)^2+c_7n^3\sum_{r=0}^{n}\pi_1(r)\\ &\le c_8n^4\pi_1(n)^2.\end{aligned}$$

This concludes the proof in the case $t=2$. Inequality (2.4.15) is only used in the proof of Proposition 2.4.4 with $t=1,2$. We do not prove (2.4.22) for $t\ge 3$ here, we only mention that the combinatorial argument used to estimate the sum in (2.4.22) is similar, but more complex, as it needs to take into account more situations. □

We are finally ready to prove Proposition 2.4.4.

Proof of Proposition 2.4.4. Part (b) is a simple application of Markov's inequality. For $\lambda>0$, by (2.4.15), we have

$$\begin{aligned}\mathbb{P}\left[|C_v]\ge\lambda n^2\pi_1(n)\Big|n\le \mathrm{rad}(C_v)\le 2n\right] &\le \frac{\mathbb{E}\left[|C_v]^t\big|n\le\mathrm{rad}(C_v)\le 2n\right]}{(\lambda n^2\pi_1(n))^t}\\ &\le C(t)\lambda^{-t}.\end{aligned}$$

Part (a) requires more work. For simplicity we shall prove (2.4.5) for $v=0$. It will be apparent that the constant used in the proof do not depend on this choice.

We wish to prove that

$$\mathbb{P}\left[|C_0|\ge\lambda n^2\pi_1(n)\big|\mathrm{rad}(C_0)\ge n\right]\xrightarrow[\lambda\to 0]{}1,$$

uniformly in n. Let $K=\lfloor\log_2 n\rfloor$ and split the ball Λ_n in disjoint concentric annuli $\mathcal{A}(2^k,2^{k+1})$, with $0\le k<K$.

For $k\in\{0,\dots,K-1\}$, let Y_n be the number of vertices in $\mathcal{A}(2^k,2^{k+1})$, connected by an open path inside $\mathcal{A}(2^k,2^{k+1})$ to an open circuit of $\mathcal{A}(2^k,2^{k+1})$. We claim that there exists constants $c_1,c_2>0$, independent of k, such that, for $0\le k<K$,

$$\mathbb{E}(Y_k)\ge c_1 2^k\pi_1(2^k)\quad\text{and}\quad \mathbb{E}(Y_k^2)\le c_2[2^k\pi_1(2^k)]^2. \tag{2.4.23}$$

The second inequality is proved by a combinatorial argument similar to the one used for (2.4.15). For the first inequality, let H_k be the event that there exists an open circuit in $\mathcal{A}(2^{k+\frac{2}{3}},2^{k+1})$. By the box-crossing property, $\mathbb{P}(H_k)$ is bounded away from 0, uniformly in $k\ge 3$. As in the proof of (2.4.14), if H_k occurs, then each vertex in $\mathcal{A}(2^{k+\frac{1}{3}},2^{k+\frac{2}{3}})$

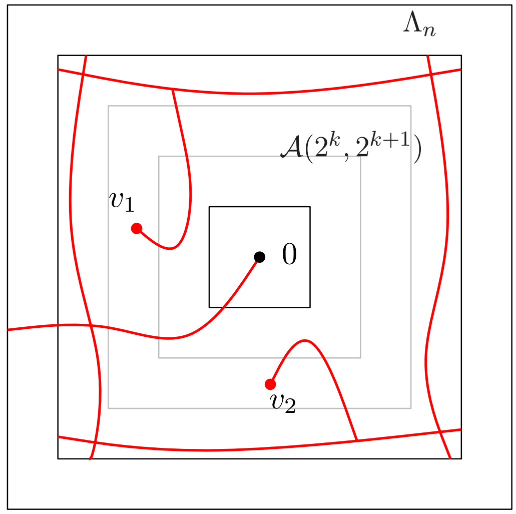

Figure 2.4.4: The ball Λ_n is split into concentric annuli. The red circuit forms the event H_k and the vertices v_1, v_2 contribute to Y_k.

has probability at least $c_3\pi_1(2^k)$ to be connected to an open circuit in $\mathcal{A}(2^{k+\frac{2}{3}}, 2^{k+1})$. See Figure 2.4.4.

We now use the (2.4.23) in a one-sided Chebyshev inequality as follows. For $s \geq \frac{1}{2}$ we have

$$\begin{aligned}\mathbb{P}\left[Y_k \leq \tfrac{c_1}{2}2^k\pi_1(2^k)\right] &\leq \mathbb{P}\left[Y_k \leq \tfrac{1}{2}\mathbb{E}(Y_k)\right]\\ &\leq \mathbb{P}\left[(Y_k - s\mathbb{E}(Y_k))^2 \geq \left(s-\tfrac{1}{2}\right)^2\mathbb{E}(Y_k)^2\right]\\ &\leq \frac{\mathrm{Var}(Y_k) + (s-1)^2\mathbb{E}(Y_k)^2}{\left(s-\frac{1}{2}\right)^2\mathbb{E}(Y_k)^2}.\end{aligned}$$

In order to minimize the right-hand side above, we take $s = 2\frac{\mathrm{Var}(Y_k)}{\mathbb{E}(Y_k)^2} + 1$, and obtain

$$\mathbb{P}\left[Y_k \leq \tfrac{c_1}{2}2^k\pi_1(2^k)\right] \leq \frac{4\mathrm{Var}(Y_k)}{4\mathrm{Var}(Y_k) + \mathbb{E}(Y_k)^2}.$$

By (2.4.23) and the above, there exists $c_3 > 0$ such that, for $0 \leq k < K$,

$$\mathbb{P}\left[Y_k \geq \tfrac{c_1}{2}2^k\pi_1(2^k)\right] \geq c_3. \tag{2.4.24}$$

Note that Y_k only depends on the configuration inside $\mathcal{A}(2^k, 2^{k+1})$, hence the variables $(Y_k : k = 0, \ldots, K-1)$ are independent. By (2.4.24), we have

$$\mathbb{P}\left[\sum_{k=0}^{K-1} Y_k \leq \lambda n^2\pi_1(n)\right] \xrightarrow[\lambda\to 0]{} 0,$$

uniformly in n.

Finally, note that both $\sum_{k=0}^{K-1} Y_k$ and $\mathrm{rad}(C_0)$ are increasing functions of the configuration. Hence

$$\mathbb{P}\left[\sum_{k=0}^{K-1} Y_k \leq \lambda n^2 \pi_1(n) \middle| \mathrm{rad}(C_0) \geq n\right] \xrightarrow[\lambda \to 0]{} 0,$$

uniformly in n. But, if $\mathrm{rad}(C_0) \geq n$, then each vertex contributing to Y_k is connected to 0, and $|C_0| \geq \sum_{k=0}^{K-1} Y_k$. This concludes the proof of Proposition 2.4.4 (a). □

2.5 Scaling relations near criticality

In this section we sketch the proof of the scaling relations (1.6.3) for models with sufficient symmetry. We follow Kesten's method from [Kes87b], also reviewed in [Nol08]. The purpose of this section is to present the main ideas in the proof and to highlight the points where symmetry is required.

Let G be a planar graph embedded in the plane, and $\mathbb{P}$ be a percolation measure on it. Suppose $(G, \mathbb{P})$ is periodic, rotation invariant by an angle $\theta \in (0, \pi)$, and invariant under reflection with respect to two perpendicular axes. Let $\mathbf{p}$ be the intensities of $\mathbb{P}$. By periodicity, there exists $\epsilon_0 > 0$ such that $\mathbf{p} \in (\epsilon_0, 1 - \epsilon_0)^E$.

Theorem 2.5.1. *Suppose $\mathbb{P}$ and $\mathbb{P}^*$ satisfy the box-crossing property. If ρ and ρ_4 exist for $(G, \mathbb{P})$, then ν, β, γ and Δ exist for $(G, \mathbb{P})$, and*

$$\nu = \frac{1}{2 - \rho_4}, \quad \beta = \frac{\rho_1}{2 - \rho_4}, \quad \gamma = \frac{2(1 - \rho_1)}{2 - \rho_4}, \quad \Delta = \frac{2 - \rho_1}{2 - \rho_4}. \tag{2.5.1}$$

The above, along with the scaling relations at criticality of Theorem 2.4.1, imply (1.6.3).

In the sketch of the proof we will sometimes make implicit assumptions about the local structure of the graph, namely about the behaviour of arm events at low scale. These assumptions are necessary only to avoid overly complicated statements. The following arguments concern essentially the behaviour at large scale; keeping track of the local details of the graph would overburden the proof.

As in Section 2.2, we distinguish two cases, $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{3}$. For simplicity assume we are in the former, and that $(G, \mathbb{P})$ is invariant under translation by $(1, 0)$ and $(0, 1)$, and under reflection with respect to the axes of $\mathbb{R}^2$. The case $\theta = \frac{\pi}{3}$ is similar. Also assume that $\mathbb{P}$ and $\mathbb{P}^*$ satisfy the box-crossing property $\mathrm{BXP}(1, \delta)$ for some $\delta > 0$.

The following notation will be useful. For a vertex u and $n \geq 0$, let $A_1^u(n) = \{\mathrm{rad}(C_u) \geq n\}$. For an edge $e = (u, v)$, with dual edge $e^* = (u^*, v^*)$, let $A_4^e(n)$ be the event that there exits 4 arms of alternating colours, originating from u, u^*, v and v^*, respectively, and landing on $\partial\Lambda_n + e \cap e^*$. We write simply $A_1(n)$ for the event $A_1^u(n)$ with u being the vertex closest to 0, and $A_4(n)$ for the event $A_4^{e_0}(n)$, where e_0 is some arbitrary fixed edge. One may imagine for simplicity that 0 is a vertex, although this is not necessarily the case. For $k \in \{1, 4\}$, we write

$$\pi_k(n) = \mathbb{P}\left[A_k(n)\right].$$

These are the probabilities of arm events centered at 0 and e_0 respectively. By periodicity, they are comparable to the probabilities of the corresponding events centered at any other point, as in (2.4.1). Since we assume the existence of ρ_1 and ρ_4,

$$\pi_k(n) = n^{-\rho_k + o(1)}, \qquad k \in \{1, 4\}. \tag{2.5.2}$$

The proof of Theorem 2.5.1 occupies the rest of the section and is split into several steps. Throughout the proof ϵ will be taken in $(-\frac{\epsilon_0}{2}, \frac{\epsilon_0}{2})$, and $\mathbb{P}^\epsilon$ will denote the measure with intensities $(p_e + \epsilon)_{e \in E}$.

Correlation length L. For ϵ small, $\mathbb{P}^\epsilon$ may be viewed as a perturbation of $\mathbb{P}$. Thus, at small scale, $\mathbb{P}^\epsilon$ is similar to a critical measure. At large scale it behaves sub- or supercritically, depending on whether $\epsilon < 0$ or $\epsilon > 0$. In loose terms, the scale at which the measure stops behaving critically is called the *correlation length* associated to ϵ. The definition of the correlation length given in Section 1.6 is one of several possible definitions. A more convenient one for our proof is in terms of crossing probabilities. Here is a precise definition.

Fix a constant $\varsigma \in (0, \frac{\delta}{2})$, which should be considered small, we will see later how small. For $|\epsilon| < \frac{\epsilon_0}{2}$, let

$$L_\varsigma(\epsilon) = \begin{cases} \inf\{n \in \mathbb{N} : \mathbb{P}^\epsilon\left[\mathcal{C}_\mathrm{h}(\mathcal{B}(n,n))\right] \le \varsigma\}, & \text{for } \epsilon < 0, \\ \inf\{n \in \mathbb{N} : \mathbb{P}^\epsilon\left[\mathcal{C}_\mathrm{h}(\mathcal{B}(n,n))\right] \ge 1 - \varsigma\}, & \text{for } \epsilon \ge 0. \end{cases} \tag{2.5.3}$$

Thus $L_\varsigma(\epsilon)$ is the smallest scale at which the probabilities of crossings of squares degenerate, and ς is a threshold for this degeneracy. By the box-crossing property and the sharp-threshold theory,

$$L_\varsigma(\epsilon) < \infty \text{ for } \epsilon \ne 0 \quad \text{and} \quad L_\varsigma(\epsilon) \xrightarrow[\epsilon \to 0]{} \infty.$$

Next we will study $\mathbb{P}^\epsilon$ at scales smaller (respectively, larger) than $L_\varsigma(\epsilon)$, and show that it behaves indeed critically (respectively, sub- or supercritically). We will also link $L_\varsigma(\epsilon)$ to the correlation length ξ introduced in Section 1.6. We will generally assume that ϵ is small, so that L is large enough to allow us to use box crossing arguments.

Henceforth $c_i, i \in \mathbb{N}$ denote strictly positive constants that may depend on δ and ς, but, unless otherwise stated, not on $(G, \mathbb{P})$ in any other way. For functions $f, g : \mathbb{R} \to (0, \infty)$, we recall the notation $f \asymp_\varsigma g$ for the fact that there exist constants $c_1, c_2 > 0$ depending only on ς, such that

$$\frac{f(x)}{g(x)} \in (c_1, c_2), \qquad \text{for all } x \in \mathbb{R}.$$

When no ambiguity is possible, we write L for $L_\varsigma(\epsilon)$.

Box crossings below the correlation length. At scales smaller than L, $\mathbb{P}^\epsilon$ behaves like a critical percolation measure, in particular it satisfies the box-crossing property. A more precise statement follows.

Fix ϵ. There exists $c_1 = c_1(\varsigma) > 0$, not depending on ϵ, such that, for $1 \le n < L_\varsigma(\epsilon)$, and all boxes B of size $2n \times n$, aligned with the axes,

$$\begin{aligned} &\mathbb{P}^\epsilon(B \text{ has an open crossing in the long direction}) \ge c_1, \\ &\mathbb{P}^\epsilon(B \text{ has an open}^* \text{ crossing in the long direction}) \ge c_1. \end{aligned} \tag{2.5.4}$$

As a consequence, the results of Sections 2.3 and 2.4, in particular the separation theorem, are also valid for $\mathbb{P}^\epsilon$, at scales smaller than L.

Let us sketch the proof of (2.5.4). Consider the case $\epsilon < 0$, the case $\epsilon > 0$ is identical by passing to the dual. The probabilities of open* crossings in $\mathbb{P}^\epsilon$ are greater than in $\mathbb{P}$, whence the second inequality of (2.5.4). We move on to the first. By definition of L, for $1 \le n < L$,

$$\mathbb{P}^\epsilon\left[\mathcal{C}_{\mathrm{h}}(B(n,n))\right] \ge \varsigma.$$

Also, $(G, \mathbb{P})$ has sufficient symmetry for the RSW lemma to hold; see Lemma 2.2.1. Hence

$$\mathbb{P}^\epsilon\left[\mathcal{C}_{\mathrm{h}}(B(2n,n))\right] \ge c_1, \tag{2.5.5}$$

with $c_1 > 0$ depending only on ς. Inequality (2.5.5) may be extended to crossing of general boxes by the rotation and translation invariance of $(G, \mathbb{P})$.

Arm events below the correlation length. The arm events at scales lower than L also behave similarly in $\mathbb{P}$ and $\mathbb{P}^\epsilon$. More precisely, for $n < L_\varsigma(\epsilon)$ and $k \in \{1, 4\}$,

$$\mathbb{P}^\epsilon\left[A_k(n)\right] \asymp_\varsigma \pi_k(n). \tag{2.5.6}$$

We focus on the case $k = 1$ and $\epsilon > 0$. The case $k = 1$ and $\epsilon < 0$ is identical by considering the dual. The case $k = 4$ is slightly more complex since $A_4(n)$ is not increasing, and we require an improved version of Russo's formula to compute the derivative of probabilities

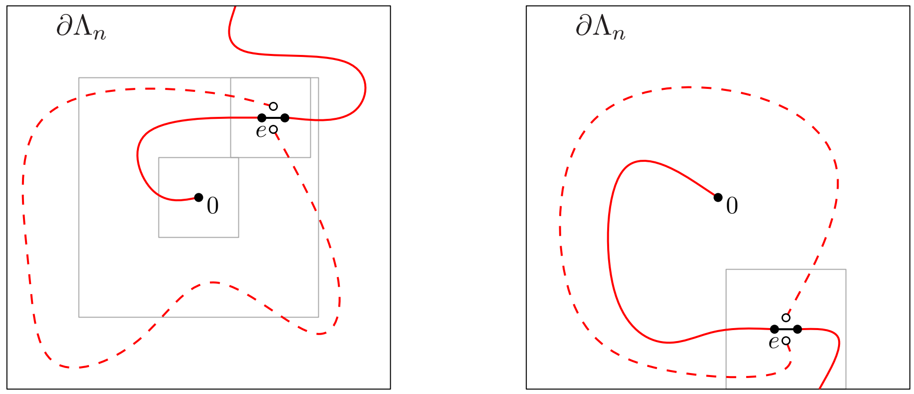

Figure 2.5.1: *Left:* The edge $e \in \Lambda_{\frac{n}{2}}$ is pivotal for $A_1(n)$. This implies the existence of a four arm event in the gray ball around e, and of one arm events in the gray ball around 0 and in the outer annulus. *Right:* If e is close to $\partial\Lambda_n$ and pivotal for $A_1(n)$, then $A_4^e(\text{dist}(e, \partial\Lambda_n))$ occurs.

of such events. Nevertheless, the additional difficulties in this case are purely technical. No further details are given here, see [Kes87b, Nol08] for the full proof.

The main idea is to use Russo's formula and the separation theorem to relate the logarithmic derivative of $\mathbb{P}^\epsilon[A_1(n)]$ to the derivative of the probability of crossing a square box of size n. For $n < L$, the latter probability does not increase too much when going from $\mathbb{P}$ to $\mathbb{P}^\eta$. This allows us to bound the logarithmic derivative of $\mathbb{P}^\epsilon[A_1(n)]$, and (2.5.6) follows.

The actual proof is quite intricate; it requires several technical tricks, but also a remarkable estimate on the five-arm exponent (see Lemma 2.5.2). We will try to give a heuristic explanation, and only sketch the actual proof.

For an edge e and $A \subseteq \mathbb{R}^2$, let $\text{dist}(e, A)$ denote the L^∞-distance from e to A. Let $|e|$ denote the distance from e to 0.

Fix $\eta \in [0, \epsilon]$ and $n < L_\varsigma(\epsilon)$. We will use repeatedly the box-crossing property at scales smaller than $L_\varsigma(\epsilon)$, i.e. (2.5.4), and its consequences, the separation theorem and Corollary 2.3.2.

We start off by computing the derivative of the $\mathbb{P}^\eta$-probability of the one-arm event. For $e \in \Lambda_{\frac{n}{2}}$, by the considerations of Figure 2.5.1 and the BK inequality, we have

$$\begin{aligned}\mathbb{P}^\eta[e \text{ is pivotal for } A_1(n)] &\leq c_1 \mathbb{P}^\eta\left[A_4^e\left(\tfrac{1}{2}|e|\right)\right] \mathbb{P}^\eta\left[A_1\left(\tfrac{1}{2}|e|\right)\right] \mathbb{P}^\eta\left[A_1\left(\tfrac{3}{2}|e|, n\right)\right] \\ &\leq c_2 \mathbb{P}^\eta[A_4(|e|)]\, \mathbb{P}^\eta[A_1(n)],\end{aligned}$$

where the second inequality is obtained using the separation theorem to connect the arm inside $\Lambda_{\frac{|e|}{2}}$ to the arm in $\mathcal{A}(\frac{3}{2}|e|, n)$. Similarly, for $e \in \mathcal{A}(\frac{n}{2}, n)$, we have

$$\mathbb{P}^\eta \left[e \text{ is pivotal for } A_1(n) \right] \leq c_3 \mathbb{P}^\eta \big[A_4 \left(\mathrm{dist}(e, \partial\Lambda_n) \right) \big] \mathbb{P}^\eta \left[A_1(n) \right].$$

By Russo's formula,

$$\frac{\partial \log \mathbb{P}^\eta \left[A_1(n) \right]}{\partial \eta} \leq c_4 \sum_{e \in \Lambda_n} \mathbb{P}^\eta \big[A_4 \left(|e| \wedge \mathrm{dist}(e, \partial\Lambda_n) \right) \big]. \tag{2.5.7}$$

We turn to the derivative of crossing probabilities. Let $\mathcal{C}_\mathrm{h}(\Lambda_n)$ be the event that Λ_n contains a horizontal open crossing. If $A_4^e(\frac{n}{4})$ occurs for some edge $e \in \Lambda_{\frac{n}{2}}$, then, through the separation theorem and the box-crossing property, we may, with positive probability, extend the arms to make e pivotal for $\mathcal{C}_\mathrm{h}(\Lambda_n)$. Hence

$$n^2 \mathbb{P}^\eta \left[A_4(n) \right] \leq c_5 \frac{\partial \mathbb{P}^\eta \left[\mathcal{C}_\mathrm{h}(\Lambda_n) \right]}{\partial \eta}. \tag{2.5.8}$$

So, in order to bound the increase of $\log \mathbb{P}^\eta \left[A_1(n) \right]$, we would like to bound the right-hand side of (2.5.7) by $n^2 \mathbb{P}^\eta \left[A_4(n) \right]$. For e far from 0 and $\partial\Lambda_n$, $\mathbb{P}^\eta \left[A_4 \left(|e| \wedge \mathrm{dist}(e, \partial\Lambda_n) \right) \right]$ is comparable to $\mathbb{P}^\eta \left[A_4(n) \right]$. But for e close to the center or to the boundary of Λ_n, the former is significantly higher than the latter. Dealing with this problem is the main difficulty in the proof of (2.5.6).

The contribution to $\frac{\partial \log \mathbb{P}^\eta [A_1(n)]}{\partial \eta}$ of the edges close to $\partial\Lambda_n$ is overestimated in (2.5.7). A quite simple trick will allow us to correct this. On the other hand, the contribution of the edges close to the center is indeed greater than $\mathbb{P}^\eta \left[A_4(n) \right]$, and we will need a fine analysis to deal with them. First we eliminate the terms with e close to $\partial\Lambda_n$.

For $\eta \in [0, \epsilon]$ let $\overline{\mathbb{P}}^\eta$ be the measure with intensities $p_e + \eta$ inside $\Lambda_{\frac{n}{2}}$, and p_e outside. Then $\mathbb{P} \leq_\mathrm{st} \overline{\mathbb{P}}^\eta \leq_\mathrm{st} \mathbb{P}^\eta$, hence $\overline{\mathbb{P}}^\eta$ also satisfies (2.5.4). In particular,

$$\mathbb{P}^\epsilon \left[A_1(n) \right] \asymp_\varsigma \mathbb{P}^\epsilon \left[A_1 \left(\frac{n}{2} \right) \right] = \overline{\mathbb{P}}^\epsilon \left[A_1 \left(\frac{n}{2} \right) \right] \asymp_\varsigma \overline{\mathbb{P}}^\epsilon \left[A_1(n) \right].$$

So in order to prove (2.5.6), it suffices to show

$$\log \frac{\overline{\mathbb{P}}^\epsilon \left[A_1(n) \right]}{\mathbb{P} \left[A_1(n) \right]} = \int_0^\epsilon \frac{\partial \log \overline{\mathbb{P}}^\eta \left[A_1(n) \right]}{\partial \eta} d\eta \leq c, \tag{2.5.9}$$

for some constant c that only depends on ς, not on n or ϵ.

As for (2.5.7), we have

$$\frac{\partial \log \overline{\mathbb{P}}^\eta \left[A_1(n) \right]}{\partial \eta} \leq c_6 \sum_{e \in \Lambda_{\frac{n}{2}}} \overline{\mathbb{P}}^\eta \left[A_4^e(|e|) \right]. \tag{2.5.10}$$

Using an inequality similar to (2.5.7), we find that

$$\int_0^\epsilon \sum_{e\in\Lambda_{\frac{n}{2}}} \overline{\mathbb{P}}^\eta \left[A_4^e(n)\right] d\eta \le c_7. \tag{2.5.11}$$

In order to go back to the periodic measure $\mathbb{P}^\eta$, we split (2.5.10) as follows:

$$\int_0^\epsilon \sum_{e\in\Lambda_{\frac{n}{2}}} \overline{\mathbb{P}}^\eta \left[A_4^e(|e|)\right] d\eta \le \int_0^\epsilon \sum_{e\in\Lambda_{\frac{n}{4}}} \overline{\mathbb{P}}^\eta \left[A_4^e(|e|)\right] d\eta + \int_0^\epsilon \sum_{e\in\mathcal{A}(\frac{n}{4},\frac{n}{2})} \overline{\mathbb{P}}^\eta \left[A_4^e(|e|)\right] d\eta.$$

By (2.5.11) the second term is bounded by a constant. In the first term the events only depend on the configuration inside $\Lambda_{\frac{n}{2}}$, where $\overline{\mathbb{P}}^\eta$ is identical to $\mathbb{P}^\eta$. It therefore remains bound

$$\int_0^\epsilon \sum_{e\in\Lambda_{\frac{n}{4}}} \mathbb{P}^\eta \left[A_4^e(|e|)\right] d\eta \asymp_\varsigma \int_0^\epsilon \sum_{k=1}^{\frac{n}{4}} k\mathbb{P}^\eta \left[A_4(k)\right] d\eta \tag{2.5.12}$$

by a constant. In light of (2.5.8), this comes down to showing that the terms with large k contribute significantly to the sum above. Suppose we could approximate $\mathbb{P}^\eta\left[A_4^e(k)\right]$ by $k^{-\alpha}$, for some $\alpha > 0$. Then we would be able to bound $\sum_{k=1}^{\frac{n}{4}} k\mathbb{P}^\eta\left[A_4^e(k)\right]$ by $n^2\mathbb{P}^\eta\left[A_4^e(n)\right]$, provided $\alpha < 2$. So, loosely speaking, we need to show that the four-arm exponent is strictly smaller than 2. We do this by considering the following five-arm event. Let $\sigma = (1,1,0,1,0)$, and write A_5 for A_σ.

Lemma 2.5.2. *Let H be a planar graph and P be a percolation measure on it. Suppose (H,P) is periodic and that P and P^* satisfy the box-crossing property* BXP(δ')*. Then there exist constants $c, c' > 0$, depending only on δ', such that, for $1 \le N \le n$,*

$$c\left(\frac{n}{N}\right)^{-2} \le P\left[A_5(N,n)\right] \le c'\left(\frac{n}{N}\right)^{-2}.$$

In other words, using only the box-crossing property, we deduce that the five-arm exponent is 2. Of course this does not imply that $\mathbb{P}^\eta\left[A_4^e(k)\right] \asymp k^{-\alpha}$ with $\alpha < 2$. Nevertheless we manage to use Lemma 2.5.2, and bound (2.5.12), through some technical manipulations, which are briefly presented next. The proof of Lemma 2.5.2 may be found at the end of this section.

For $k \le \frac{n}{4}$ and $e \in \Lambda_{\frac{n}{4}}$, we may use the separation theorem to find

$$\mathbb{P}^\eta\left[e \text{ pivotal for } \mathcal{C}_{\mathrm{h}}(\Lambda_n)\right] \ge c_8\mathbb{P}^\eta\left[A_4^e(k)\right]\mathbb{P}^\eta\left[A_4(k,n)\right]. \tag{2.5.13}$$

By Reimer's inequality (an enhanced version of the BK inequality, see [Rei00]),

$$\mathbb{P}^{\eta}\left[A_5(k,n)\right] \leq \mathbb{P}^{\eta}\left[A_4(k,n)\right]\mathbb{P}^{\eta}\left[A_1(k,n)\right].$$

Also, by the box-crossing property, there exists $\alpha = \alpha(\varsigma) > 0$ such that,

$$\mathbb{P}^{\eta}\left[A_1(k,n)\right] \leq c_9\left(\frac{n}{k}\right)^{-\alpha}.$$

Hence we find that

$$\mathbb{P}^{\eta}\left[A_4(k,n)\right] \geq c_{10}\left(\frac{n}{k}\right)^{-2+\alpha}.$$

This plays the role of the bound on the four-arm exponent. When putting this together with (2.5.13), we obtain

$$\mathbb{P}^{\eta}\left[e \text{ pivotal for } \mathcal{C}_{\mathrm{h}}(\Lambda_n)\right] \geq c_{11}\mathbb{P}^{\eta}\left[A_4^e(k)\right]\left(\frac{n}{k}\right)^{-2+\alpha}.$$

Finally, by integrating the above and using Russo's formula, we find

$$\int_0^{\epsilon}\mathbb{P}^{\eta}\left[A_4(k)\right]d\eta \leq c_{12}n^{-\alpha}k^{-2+\alpha} \tag{2.5.14}$$

We now input (2.5.14) in (2.5.12), and deduce that

$$\int_0^{\epsilon}\sum_{k=1}^{\frac{n}{4}}k\mathbb{P}^{\eta}\left[A_4(k)\right]d\eta \leq c_{12}\sum_{k=1}^{\frac{n}{4}}n^{-\alpha}k^{\alpha-1} \leq c_{13}.$$

This concludes the proof of (2.5.6).

Asymptotics for the correlation length. Using (2.5.6) and Russo's formula, we are able to obtain an asymptotic equivalent for $L_{\varsigma}(\epsilon)$ as $\epsilon \to 0$. As before, we may restrict ourselves to the case $\epsilon > 0$.

Suppose we could prove that, for $n \leq L_{\varsigma}(\epsilon)$ and $\eta \in (0,\epsilon)$,

$$\frac{\partial\mathbb{P}^{\eta}\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}(n,n))\right]}{\partial\eta} = \sum_{e\in\mathcal{B}(n,n)}\mathbb{P}^{\eta}\left[e \text{ is pivotal for } \mathcal{C}_{\mathrm{h}}(\mathcal{B}(n,n))\right] \asymp_{\varsigma} n^2\pi_4(n). \tag{2.5.15}$$

Then, by integrating the above, we would obtain

$$1 \asymp_{\varsigma} \mathbb{P}^{\epsilon}\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}(L,L))\right] - \mathbb{P}\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}(L,L))\right] \asymp_{\varsigma} \epsilon L^2\pi_4(L)$$

Finally, using (2.5.2), this implies

$$L_\varsigma(\epsilon) \approx \epsilon^{-\frac{1}{2-\rho_4}}, \qquad \text{as } \epsilon \to 0. \tag{2.5.16}$$

But (2.5.15) is not entirely true. We have seen before that the contribution to the derivative of $\mathbb{P}^\eta\left[\mathcal{C}_{\rm h}(\mathcal{B}(n,n))\right]$ of the edges in the bulk (i.e. far from $\partial\mathcal{B}(n,n)$) is indeed of order $\pi_4(n)$, but the edges close to the boundary could, in principle, have significantly higher impact. We deal with this problem as in the proof of (2.5.6). The following lemma will be useful.

Lemma 2.5.3. *Let P be a measure on G, and assume P and P^* satisfy the box-crossing property* BXP(δ')*. Then, for any $c>0$, there exists $\alpha>0$ depending only on c and δ', such that, for n,m large enough,*

$$\begin{aligned}
&\left|P\left[\mathcal{C}_{\rm h}\left(\mathcal{B}(n,m)\right)\right] - P\left[\mathcal{C}_{\rm h}\left(\mathcal{B}((1-\alpha)n,m)\right)\right]\right| \le c,\\
&\left|P\left[\mathcal{C}_{\rm h}\left(\mathcal{B}(n,m)\right)\right] - P\left[\mathcal{C}_{\rm h}\left(\mathcal{B}(n,(1-\alpha)m)\right)\right]\right| \le c.
\end{aligned}$$

The second inequality is a consequence of the first when applied to P^*. Lemma 2.5.3 may be viewed as a simplified version of Lemma 2.3.4. Indeed, if the horizontal crossings of $\mathcal{B}((1-\alpha)n,m)$ may be made into a fences and $\mathcal{B}((1-\alpha)n,m)$ has a horizontal open crossing, then the slightly longer box $\mathcal{B}(n,m)$ is also crossed horizontally by an open path.

Let $\alpha=\alpha(\varsigma)>0$ be such that the lemma holds for any measure between $\mathbb{P}$ and $\mathbb{P}^\epsilon$ with $c=\frac{1}{9}\varsigma$ and $m,n\le L$. Denote $\overline{\mathbb{P}}^\eta$ the measure with intensities $p_e+\eta$ for edges $e\in\mathcal{B}(n,n)$, with $\operatorname{dist}(e,\partial\mathcal{B}(n,n))\ge\alpha n$, and p_e for all other edges. Using Lemma 2.5.3 for $\overline{\mathbb{P}}^\epsilon$ and $\mathbb{P}^\epsilon$, we have

$$\begin{aligned}
1 &\asymp_\varsigma \mathbb{P}^\epsilon\left[\mathcal{C}_{\rm h}(\mathcal{B}(L,L))\right] - \mathbb{P}\left[\mathcal{C}_{\rm h}(\mathcal{B}(L,L))\right]\\
&\asymp_\varsigma \overline{\mathbb{P}}^\epsilon\left[\mathcal{C}_{\rm h}(\mathcal{B}(L,L))\right] - \mathbb{P}\left[\mathcal{C}_{\rm h}(\mathcal{B}(L,L))\right]\\
&\asymp_\varsigma \int_0^\eta \sum_{e\in\mathcal{B}(L,L)} \overline{\mathbb{P}}^\eta\left[e \text{ is pivotal for } \mathcal{C}_{\rm h}(\mathcal{B}(L,L))\right] d\eta\\
&\asymp_\varsigma \eta L^2\pi_4(L).
\end{aligned}$$

This allows us to deduce (2.5.16) as described above.

Above the correlation length. For scales larger than L, $\mathbb{P}^\epsilon$ behaves subcritically for $\epsilon<0$ and supercritically for $\epsilon>0$.

First we analyse the case $\epsilon<0$. By the RSW lemma (Lemma 2.2.1) for the dual model

$$\mathbb{P}^\epsilon\left[\mathcal{C}_{\rm v}(B(2L,L))\right] \le \phi(\varsigma),$$

where $\phi(x) \xrightarrow[x\to 0]{} 0$. In particular, for ς small enough, we have $\phi(\varsigma) \leq \frac{\epsilon_0}{400}$; henceforth we will assume this is the case. We may then use Lemma 2.1.6 to argue that, for $k \geq 0$,

$$\mathbb{P}^\epsilon\left[\mathcal{C}_{\mathrm{v}}(\mathcal{B}(2^{k+1}L, 2^k L))\right] \leq 2^{-k}. \tag{2.5.17}$$

This step requires rotation and translation invariance. Using (2.5.17) we show through standard geometrical arguments that there exists $c_1 > 0$ such that, for $n \geq L$,

$$\mathbb{P}^\epsilon\left[\mathrm{rad}(C_0) \geq n\right] \leq e^{-c_1 \frac{n}{L}} \mathbb{P}^\epsilon\left[\mathrm{rad}(C_0) \geq L\right]. \tag{2.5.18}$$

We turn to the case $\epsilon > 0$. Using the same arguments as above, but applied to the dual model, we have

$$\mathbb{P}^\epsilon\left[\mathcal{C}_{\mathrm{h}}(\mathcal{B}(2^{k+1}L, 2^k L))\right] \geq 1 - 2^{-k}.$$

For $k \geq 0$ define the events H_k as follows:

$$\begin{aligned} H_k &= \mathcal{C}_{\mathrm{h}}(\mathcal{B}(2^{k+1}L, 2^k L)), &\qquad \text{for } k \text{ even},\\ H_k &= \mathcal{C}_{\mathrm{v}}(\mathcal{B}(2^k L, 2^{k+1}L)), &\qquad \text{for } k \text{ odd}. \end{aligned}$$

Since $P(H_k) \geq 1 - 2^{-k}$,

$$\mathbb{P}^\epsilon\left(\bigcap_{k\geq 0} H_k\right) > c_2,$$

where $c_2 > 0$ is a universal constant. If all $H_k, k \geq 0$ occur simultaneously, then there exists an infinite cluster intersecting Λ_L. By the above and the box-crossing property in $\mathbb{P}^\epsilon$, we deduce that

$$\mathbb{P}^\epsilon(0 \leftrightarrow \infty) \geq c_3 \mathbb{P}^\epsilon\left[\mathrm{rad}(C_0) \geq L\right]. \tag{2.5.19}$$

Also, by the same type of argument, there exists c_4 such that, for $k \geq 0$,

$$\mathbb{P}^\epsilon\left[0 \not\leftrightarrow \infty \middle| \mathrm{rad}(C_0) \geq 2^k L\right] \leq c_4 2^{-k}. \tag{2.5.20}$$

Near-critical exponents. Now that we have understood the behaviour of $L_\varsigma(\epsilon)$ and that of $\mathbb{P}^\epsilon$ with respect to $L_\varsigma(\epsilon)$, we are ready to study the near-critical exponents of (2.5.1). For simplicity we do this for $v = 0$.

Correlation length ξ. For $\epsilon < 0$ and $n \geq L$, by (2.5.18),

$$\log \mathbb{P}^{\epsilon}[\mathrm{rad}(C_0) \geq n] \leq -c_1 \frac{n}{L} + \log \mathbb{P}^{\epsilon}[\mathrm{rad}(C_0) \geq n].$$

Conversely, by (2.5.4),

$$\mathbb{P}^{\epsilon}\left[\mathcal{C}_{\mathrm{h}}\left(\mathcal{B}\left(L, \tfrac{1}{2}L\right)\right)\right] \geq c_2.$$

From the above, by combining box crossings, we obtain

$$\log \mathbb{P}^{\epsilon}[\mathrm{rad}(C_0) \geq n] \geq -c_3 \frac{n}{L} + \log \mathbb{P}^{\epsilon}\left[\mathrm{rad}(C_0) \geq L\right].$$

Hence

$$\frac{1}{\xi(\epsilon)} = \lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}^{\epsilon}[\mathrm{rad}(C_0) \geq n] \asymp_{\varsigma} \frac{1}{L_{\varsigma}(\epsilon)}.$$

In conclusion

$$\nu = \frac{1}{2-\rho_4}. \tag{2.5.21}$$

Percolation probability. For $\epsilon > 0$, by (2.5.6), (2.5.16) and (2.5.19),

$$\mathbb{P}^{\epsilon}(0 \leftrightarrow \infty) \asymp_{\varsigma} \pi_1\left(L_{\varsigma}(\epsilon)\right) \approx \epsilon^{\frac{\rho_1}{2-\rho_4}},$$

as $\epsilon \to 0$. Hence

$$\beta = \frac{\rho_1}{2-\rho_4}.$$

Moments for the cluster size. For $t \in \mathbb{N}$, we claim

$$\mathbb{E}^{\epsilon}\left[|C_0|^t; |C_0| < \infty\right] \asymp_{\varsigma} L_{\varsigma}(\epsilon)^{2t} \pi_1(L_{\varsigma}(\epsilon))^{t+1}. \tag{2.5.22}$$

Once (2.5.22) is proved, using (2.5.6) and (2.5.16), we deduce

$$\gamma = \frac{2(1-\rho_1)}{2-\rho_4}, \qquad \Delta = \frac{2-\rho_1}{2-\rho_4}.$$

Fix ϵ. The idea behind (2.5.22) is to split the space into squares of size comparable to the correlation length, $L_{\varsigma}(\epsilon)$, and to use the estimates of Section 2.4.1 in each such square.

For $m, n \in \mathbb{Z}$, let

$$\mathcal{S}_{m,n} = [(2m-1)L, (2m+1)L) \times [(2n-1)L, (2n+1)L),$$

and $Y_{m,n} = |C_0 \cap \mathcal{S}_{m,n}|$. Then $|C_0| = \sum_{m,n} Y_{m,n}$. Next we prove upper and lower bounds on the moments of $Y_{m,n}$.

The squares $\mathcal{S}_{m,n}$ have size comparable to the correlation length, and the box-crossing property holds inside each of them. Hence we may use arguments similar to those in the proof of Lemma 2.4.6, to show that, for $t \geq 1$,

$$\mathbb{E}^\epsilon\left[Y_{0,0}^t; |C_0| < \infty\right] \asymp_\varsigma L^{2t}\pi_1(L)^{t+1}.$$

For $(m,n) \neq (0,0)$, we have

$$\mathbb{E}^\epsilon\left[Y_{m,n}^t; |C_0| < \infty\right] \leq \mathbb{E}^\epsilon\left[|\{v \in \mathcal{S}_{m,n} : v \leftrightarrow \partial\mathcal{S}_{m,n}\}|^t\right]\mathbb{P}^\epsilon\left[0 \leftrightarrow \partial\mathcal{S}_{m,n}; |C_0| < \infty\right].$$

As in the proof of (2.4.15), may show that

$$\mathbb{E}^\epsilon\left[|\{v \in \mathcal{S}_{m,n} : v \leftrightarrow \partial\mathcal{S}_{m,n}\}|^t\right] \leq c_4 L^{2t}\pi_1(L)^t.$$

Finally, using (2.5.18) for $\epsilon < 0$, and (2.5.20) for $\epsilon > 0$, we find

$$\mathbb{P}^\epsilon\left[0 \leftrightarrow \partial\mathcal{S}_{m,n}; |C_0| < \infty\right] \leq \pi_1(L)e^{-c_5(m\vee n)}.$$

In the above, c_4 and c_5 are strictly positive constants that only depend on ς and t, not on ϵ or (m,n). We are now ready to conclude. First we have

$$\mathbb{E}^\epsilon\left[|C_0|^t; |C_0| < \infty\right] \geq \mathbb{E}^\epsilon\left[Y_{0,0}^t; |C_0| < \infty\right] \asymp_\varsigma L^{2t}\pi_1(L)^{t+1}.$$

For the converse we use the following convexity inequality

$$\begin{aligned}
\mathbb{E}^\epsilon\left[|C_0|^t; |C_0| < \infty\right] &= \mathbb{E}^\epsilon\left[\left(\sum_{m,n\in\mathbb{Z}} Y_{m,n}\right)^t; |C_0| < \infty\right] \\
&\leq \left(\sum_{m,n\in\mathbb{Z}} \mathbb{E}^\epsilon\left[Y_{m,n}^t; |C_0| < \infty\right]^{\frac{1}{t}}\right)^t \\
&\leq c_6\left(\sum_{m,n\in\mathbb{Z}} L^2\pi_1(L)^{\frac{t+1}{t}}e^{-c_5\frac{m\vee n}{t}}\right)^t \\
&\leq c_7 L^{2t}\pi_1(L)^{t+1}.
\end{aligned}$$

This concludes the proof of (2.5.22), and that of Theorem 2.5.1.

Conclusions We have mentioned at the beginning of the section that we consider the case where $(G,\mathbb{P})$ is invariant under rotation by $\theta = \frac{\pi}{2}$. For the case $\theta = \frac{\pi}{3}$ it is more convenient to define the correlation length L in terms of crossing probabilities of the parallelograms $\mathcal{B}^{\triangle}$, as in Lemma 2.2.2. The rest of the proof is identical.

We would like to emphasize the importance of the symmetries of $(G, \mathbb{P})$. We have seen in Section 2.2 how these symmetries come into play in the proof of the RSW lemma. The RSW lemma was used in our proofs to link the crossing probabilities of general domains to those of squares, below and above the correlation length. We have also used translation and rotation invariance in proving exponential decay above the correlation length for $\epsilon < 0$ (see (2.5.17)). Finally, periodicity also comes into play to show that the probabilities of the one- and four arm events of radius n centered at different points are comparable. This will also be used for the five-arm event in the upcoming proof of Lemma 2.5.2.

The box-crossing property tells us that the critical measure is somewhat isotropic. When we move away from criticality this isotopy may be lost. Hence, the probabilities of crossing domains at certain scales may degenerate differently depending on the positioning and shape of the domain. If the model has sufficient symmetry, this problem disappears. This allows us to define a correlation length that truly separates the critical scale from the sub/supercritical scale. To our knowledge, there is no way to span this gap in the absence of symmetry.

Proof of Lemma 2.5.2 This proof is independent from the rest of the arguments presented in this section. The idea of the proof is that, inside Λ_n, there is at most one point with 5 arms originating from it. Conversely, such a point exists with positive probability. Details are given next.

Let H and P be as in Lemma 2.5.2. For $n \geq 0$, and $e = (u, v) \in \Lambda_n$, with dual edge $e^* = (u^*, v^*)$, let $A_5(e, \partial\Lambda_n)$ be the event that e is open, and that there exists vertex-disjoint paths $\gamma_1, \dots, \gamma_5$, taken in anticlockwise order, with colours $1, 1, 0, 1, 0$, originating from u, u, u^*, v and v^*, respectively, and landing on $\partial\Lambda_n$. We assume that H is such that $A_5^e(n)$ is non-empty.

Let $I_1, \dots, I_5$ be a landing sequence, and define the event $A_5^I(e, \partial\Lambda_n)$ as $A_5(e, \partial\Lambda_n)$, with the additional requirement that each γ_i lands in nI_i. The separation theorem may be adapted to A_5, and, since (H, P) is periodic, we deduce that, for n large enough and $e \in \Lambda_{\frac{n}{2}}$,

$$P\left[A_5^I(e, \partial\Lambda_n)\right] \asymp_{\delta'} P\left[A_5(e, \partial\Lambda_n)\right] \asymp_{\delta'} P\left[A_5(1, n)\right]. \tag{2.5.23}$$

Also, for $N \leq n$ large enough,

$$P\left[A_5(N, n)\right] \asymp_{\delta'} \frac{P\left[A_5(1, n)\right]}{P\left[A_5(1, N)\right]}.$$

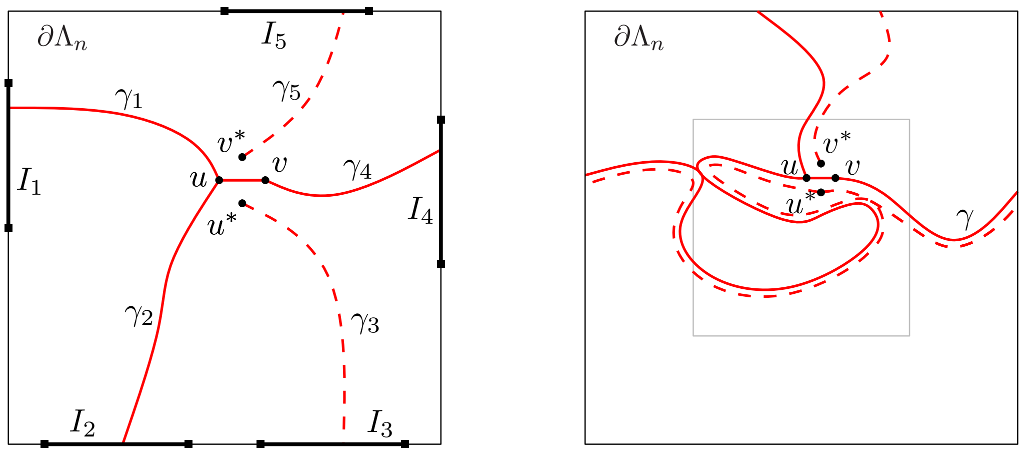

Figure 2.5.2: *Left:* The event $A_5^I(e, \partial\Lambda_n)$. For any other edge $e' \in \Lambda_n$, $A_5^I(e', \partial\Lambda_n)$ can not occur. If e' is between γ_i and γ_{i+1}, then there exists no path of colour σ_{i+3} joining e' to I_{i+3} (we use mod 5 convention for the indices). *Right:* If H_1 and H_2 both occur, then there exists an edge e on γ for which $A_5(e, \partial\Lambda_n)$ occurs.

Hence it suffices to prove

$$P\left[A_5(1,n)\right] \asymp_{\delta'} n^{-2}.$$

First we prove an upper bound. Let e be an edge in $\Lambda_{\frac{n}{2}}$ such that $A_5^I(e, \partial\Lambda_n)$ occurs. Then, by a careful inspection of the different possibilities, we conclude that there exists no other edge $e' \in \Lambda_{\frac{n}{2}}$, such that $A_5^I(e', \partial\Lambda_n)$ occurs. See the left diagram of Figure 2.5.2.

We turn to the lower bound. We show that, with positive probability, there exists $e \in \Lambda_{\frac{n}{2}}$, such that $A_5(e, \partial\Lambda_n)$ occurs. Let H_1 be the event that there exists an open* horizontal crossing of $[-n, n] \times [-\frac{n}{2}, 0]$ and an open horizontal crossing of $[-n, n] \times [0, \frac{n}{2}]$. Let γ_1^* be the lowest crossing of the first type, and γ be the lowest open horizontal crossing above γ_1^*.

By the box-crossing property for P and P^*, $P(H_1) \asymp_{\delta'} 1$. We now condition on H_1 and on the path γ. As in previous arguments, we use the fact that, above γ, P is not affected by this conditioning. Let H_2 be the event that there exists an open and an open* path, inside $[-\frac{n}{2}, 0] \times [-N, N]$, and $[0, \frac{n}{2}] \times [-N, N]$ respectively, that connect γ to the top of Λ_n. Again, by the box-crossing property, $P(H_2 | H_1, \gamma) \asymp_{\delta'} 1$.

Assume H_1 and H_2 both occur, and let γ_2^* be a open* crossing as in the definition of H_2. Orient γ from left to right, and let u be the last vertex on γ before γ_2^* that is connected by an open path to the top of Λ_n. Let v be the next vertex on γ after u, and $e = (u, v)$. Then $e \in \Lambda_{\frac{n}{2}}$, and $A_5(e, \partial\Lambda_n)$ occurs. See also the right diagram of Figure 2.5.2.

In conclusion

$$\sum_{e\in\Lambda_{\frac{n}{2}}} P\left[A_5(e,\partial\Lambda_n)\right] \geq c_1,$$

with $c_1 > 0$ only depending on δ', not on n. Together with (2.5.23), this provides the necessary lower bound. This concludes the proof of Lemma 2.5.2.

Chapter 3

Isoradial graphs and the star–triangle transformation

In this chapter we give a detailed description of isoradial graphs and their relation to the star–triangle transformation. The star–triangle transformation, presented in Section 3.2, is the key tool in the proofs of Chapters 4 and 5.

3.1 Isoradial graphs and rhombic tilings

3.1.1 Isoradial graphs

We begin by restating in more detail the definitions of Section 1.3.3.

Let $G = (V, E)$ be a planar graph embedded in the plane $\mathbb{R}^2$, with edges embedded as straight-line segments with intersections only at vertices. It is called *isoradial* if, for every bounded face F of G, the vertices of F lie on a circle of (circum)radius 1 with centre in the interior of F. In the absence of a contrary statement, we shall assume that isoradial graphs are infinite with all faces bounded. The term isoradial graph may be misleading, as it does not only refer to a graph, it refers to a graph with a fixed embedding.

Let $G = (V, E)$ be isoradial. Each edge $e = \langle A, B\rangle$ of G lies in two faces, with circumcentres O_1 and O_2. Since the two circles have equal radii, the quadrilateral AO_1BO_2 is a rhombus. Therefore, the angles AO_1B and BO_2A are equal, and we write $\theta_e \in (0, \pi)$ for their common value. See Figure 1.3.3.

Definition 3.1.1. *Let $\epsilon > 0$. The isoradial graph G is said to have the* bounded-angles property BAP(ϵ) *if*

$$\theta_e \in [\epsilon, \pi - \epsilon], \qquad e \in E. \tag{3.1.1}$$

It is said to have, simply, the bounded-angles property *if it satisfies* BAP(ϵ) *for some $\epsilon > 0$.*

All isoradial graphs of this paper will be assumed to have the bounded-angles property. The area of the rhombus AO_1BO_2 equals $\sin\theta_e$ and, under BAP(ϵ),

$$\sin\epsilon \le |AO_1BO_2| \le 1. \tag{3.1.2}$$

When G is isoradial, there is a canonical product measure, denoted $\mathbb{P}_G$, associated with its embedding, namely that with $p_e = p_{\theta_e}$, and

$$\frac{p_\theta}{1-p_\theta} = \frac{\sin(\frac{1}{3}[\pi-\theta])}{\sin(\frac{1}{3}\theta)}. \tag{3.1.3}$$

Note that $p_\theta + p_{\pi-\theta} = 1$, and that G has the bounded-angles property BAP(ϵ) if and only if

$$p_{\pi-\epsilon} \le p_e \le p_\epsilon, \qquad e \in E. \tag{3.1.4}$$

3.1.2 Rhombic tilings

A *rhombic tiling* is a planar graph embedded in $\mathbb{R}^2$ such that every face is a rhombus of side-length 1. Rhombic tilings have featured prominently in the theory of planar tilings, both periodic and aperiodic. A famous example is the aperiodic rhombic tiling of Penrose [Pen78], and the generalizations of de Bruijn [Bru81a, Bru81b] and others. The reader is referred to [GS87, Sen95] for general accounts of the theory of tiling.

There is a two–one correspondence between isoradial graphs and rhombic tilings of the plane, which we review next. Let $G = (V, E)$ be an isoradial graph. The *diamond graph* $G^\diamond$ is defined as follows. The vertex-set of $G^\diamond$ is $V^\diamond := V \cup C$, where C is the set of circumcentres of faces of G; elements of V shall be called *primal* vertices, and elements of C *dual* vertices. Edges are placed between pairs $v \in V$, $c \in C$ if and only if c is the centre of a circumcircle of a face containing v. Thus $G^\diamond$ is bipartite. Since G is isoradial, the diamond graph $G^\diamond$ is a rhombic tiling, and is illustrated in Figure 3.1.1.

From the diamond graph $G^\diamond$ we may find both G and its planar dual G^*. Write V_1 and V_2 for the two sets of vertices in the bipartite $G^\diamond$. For $i = 1, 2$, let G_i be the graph with vertex-set V_i, two points of which are joined by an edge if and only if they lie in the same face of $G^\diamond$. One of the graphs G_1, G_2 is G and the other is its dual G^*. It follows in particular that G^* is isoradial. Let $e \in E$ and let e^* denote its dual edge. The pair e, e^* are diagonals of the same rhombus of $G^\diamond$ and are thus perpendicular.

Let $e^* \in E^*$ be the dual edge (in the embedding described above) crossing the primal edge $e \in E$. Then $\theta_{e^*} = \pi - \theta_e$, so that $p_e + p_{e^*} = 1$ by (3.1.3). In conclusion, the canonical measure $\mathbb{P}_{G^*}$ is dual to the primal measure $\mathbb{P}_G$. By (3.1.4),

$$G^* \text{ satisfies BAP}(\epsilon) \text{ if and only if } G \text{ satisfies BAP}(\epsilon). \tag{3.1.5}$$

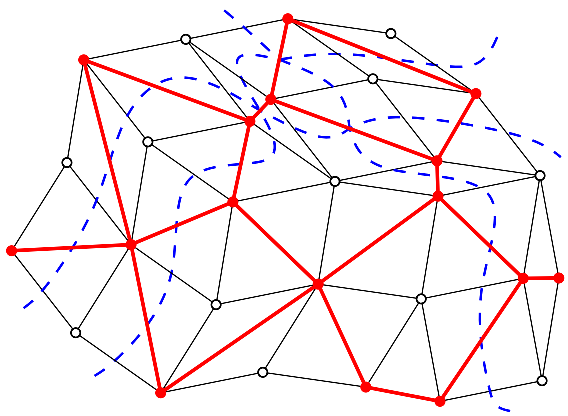

Figure 3.1.1: The isoradial graph G is drawn in red, and the associated diamond graph $G^\diamond$ in black. The primal vertices of $G^\diamond$ are those of G; the dual vertices are centres of faces of G. A track is a doubly infinite sequence of adjacent rhombi sharing a common vector, and may be represented by a path, drawn in blue. Two tracks meet in an edge of G lying in some face of $G^\diamond$.

The above construction may be applied to any rhombic tiling T to obtain a primal/dual pair of isoradial graphs.

3.1.3 Track systems

Rhombic tilings have attracted much interest, especially since the discovery by Penrose [Pen74, Pen78] of his celebrated aperiodic tiling. Penrose's rhombic tiling was elaborated by de Bruijn [Bru81a, Bru81b], who developed the following representation in terms of 'ribbons' or '(train) tracks'. Let $G = (V, E)$ be isoradial. An edge e_0 of $G^\diamond$ belongs to two rhombi r_0, r_1 of $G^\diamond$. Write e_{-1} (respectively, e_1) for the edge of r_0 (respectively, r_1) opposite e_0, so that e_{-1}, e_0, e_1 are parallel unit-line-segments. The edge e_{-1} (respectively, e_1) belongs to a further rhombus r_{-1} (respectively, r_2) that is distinct from r_0 (respectively, r_1). By iteration of this procedure, we obtain a doubly-infinite sequence of rhombi $(r_i : i \in \mathbb{Z})$ such that the intersections $(r_i \cap r_{i+1} : i \in \mathbb{Z})$ are distinct, parallel unit-line-segments. We call such a sequence a *(train) track*. We write $\mathcal{T}(G)$ for the set of tracks of G, and note that $\mathcal{T}(G) = \mathcal{T}(G^*)$. The track construction is illustrated in Figure 3.1.1.

A track $(r_i : i \in \mathbb{Z})$ is sometimes illustrated as an arc joining the midpoints of the line-segments $r_i \cap r_{i+1}$ in sequence. The set $\mathcal{T}$ may therefore be represented as a family of doubly-infinite arcs which, taken together with the intersections of arcs, defines a graph. We shall denote this graph by $\mathcal{T}$ also. A vertex v of $G^\diamond$ is said to be *adjacent* to a track $(r_i : i \in \mathbb{Z})$ if it is a vertex of one of the rhombi r_i.

It was pointed out by de Bruijn, and is easily checked, that the rhombi in a track are distinct. Furthermore, two distinct tracks may have no more than one rhombus in common. Since each rhombus belongs to exactly two tracks, it is the unique intersection of these two tracks.

Kenyon and Schlenker [KS05] have showed a converse theorem. Let Q be an infinite planar graph embedded in the plane with the property that every face has four sides. One may define the tracks of Q by an adaptation of the above definition: a track exits a face across the edge opposite to its entry. Then Q may be deformed continuously into a rhombic tiling if and only if (i) no track intersects itself, and (ii) no two tracks intersect more than once.

A track t is said to be *oriented* if it is endowed with a direction. As an oriented track t is followed in its given direction, it crosses sides of rhombi which are parallel. Viewed as vectors from right to left, these sides constitute a unit vector $\tau(t)$ of $\mathbb{R}^2$ called the *transverse vector* of t. The transverse vector makes an angle with the x-axis called the *transverse angle* of t, with value in the interval $[0, 2\pi)$.

Definition 3.1.2. *Let $I \in \mathbb{N}$. We say that an isoradial graph G has the* square-grid property SGP(I) *if its track-set $\mathcal{T}$ may be partitioned into three sets $\mathcal{T} = S \cup T_1 \cup T_2$ satisfying the following.*

(a) *For $k = 1, 2$, T_k is a set $(t_k^i : i \in \mathbb{Z})$ of distinct non-intersecting tracks indexed by $\mathbb{Z}$.*

(b) *For $k = 1, 2$ and $s \in \mathcal{T} \setminus T_k$, the tracks of T_k intersect s in their lexicographic order.*

(c) *For $k = 1, 2$, $i \in \mathbb{Z}$, and $s \in T_{3-k}$, the number of track-intersections on s between its intersections with t_k^i and t_k^{i+1} is strictly less than I.*

An isoradial graph G is said to have the square-grid property (SGP) if it satisfies SGP(I) for some $I \in \mathbb{N}$. As before, $\mathcal{G}$ denotes the set of all isoradial graphs with the bounded-angles property and the square-grid property. More specifically, we write $\mathcal{G}(\epsilon, I)$ for the set of G satisfying BAP(ϵ) and SGP(I).

Two tracks belonging to the same T_k are said to be *parallel*. Thus, G has the square-grid property if one may partition its tracks into three families: two doubly infinite families of parallel tracks, and a third family of "additional" tracks, S. Tracks from different families must intersect. Condition (c) requires that two tracks belonging to a family T_k remain, in some sense, close to each other. See also Figure 3.1.2.

We refer to $T_1 \cup T_2$ as a *square grid* of G, assumed implicitly to satisfy (c) above. A square grid is a subset of tracks with the topology of the square lattice (and satisfying (c)).

Since the square-grid property pertains to the diamond graph $G^\diamond$ rather than to G itself,

$$G \text{ satisfies SGP}(I) \text{ if and only if } G^* \text{ satisfies SGP}(I). \tag{3.1.6}$$

Let $G \in \mathcal{G}$ have square grid $T_1 \cup T_2$. It may be seen by the bounded-angles property that, for $k = 1, 2$, every $x \in \mathbb{R}^2$ lies either in some track of T_k or in the region of $\mathbb{R}^2$ 'between' two consecutive elements of T_k.

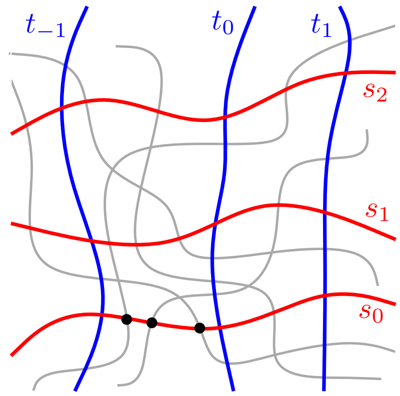

Figure 3.1.2: A track system with the square-grid property. The blue and red tracks form T_1 and T_2, respectively. The tracks in S are gray. The number of intersections on s_0 between t_{-1} and t_0 is bounded by I.

3.1.4 Examples

Here are four families of isoradial graphs with the square-grid property, and one without.

Highly inhomogeneous models as isoradial graphs

The models of $\mathcal{M}_I$ may be viewed as isoradial graphs. The square, triangular, and hexagonal lattices, embedded as in Figure 1.3.1, are indeed isoradial graphs, and the measure associated by (1.3.1) is the critical homogeneous measure. More generally we may embed the three lattices isoradially in such a way as to obtain the measures of $\mathcal{M}_I$. Thus, each model in $\mathcal{M}_I$ corresponds to an isoradial embedding of one of the three lattices. Nevertheless, a model in $\mathcal{M}_I$ differs from its isoradial version, but only by its embedding.

Take for instance the triangular lattice $\mathbb{T}$ and a measure $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$ of $\mathcal{M}_I$ on it. There exists an isoradial embedding G of the triangular lattice, with associated percolation measure $\mathbb{P}_G$, such that, for any edge e,

$$\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}(e \text{ is open}) = \mathbb{P}_G(e \text{ is open}).$$

Examples of isoradial embeddings corresponding to models in $\mathcal{M}$ and $\mathcal{M}_I$ are given in Figure 3.1.3.

It may be shown that the box-crossing property and the universality of arm exponents are equivalent in the two embeddings. See Propositions 3.1.3 and 3.1.4 for more precise statements. In the context of the previous example, the following holds:

(i) $(\mathbb{T}, \mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'})$ has the box-crossing property if and only if $(G, \mathbb{P}_G)$ does,

(ii) the euclidian metric is equivalent to the graph distance on both $\mathbb{T}$ and G.

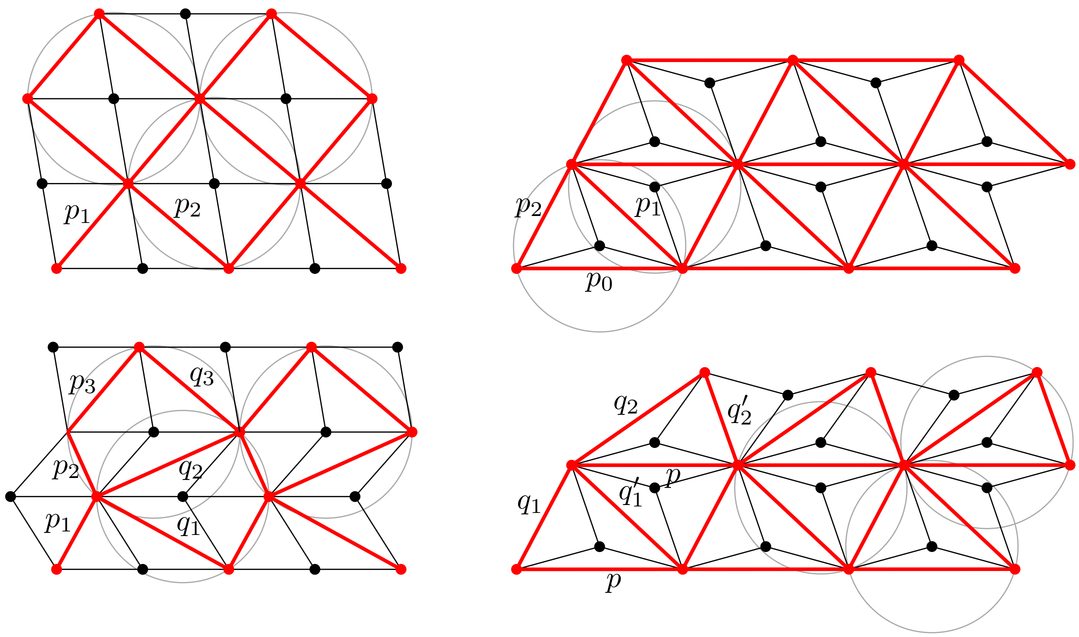

Figure 3.1.3: Isoradial embeddings corresponding to inhomogeneous (top) and highly inhomogeneous (bottom) measures on the square and triangular lattices. The lattices are drawn in red and the diamond graphs in black. These model differ from the ones presented in Section 1.3.2 only by their embedding.

It is easy to check that track systems of the square, triangular and hexagonal lattices satisfy the square-grid property SGP(2). For $\epsilon > 0$, the models in $\mathcal{M}(\epsilon)$ correspond to isoradial graphs in $\mathcal{G}(\epsilon', 2)$, with $\epsilon' > 0$ depending only on ϵ. So do the models on the square lattice in $\mathcal{M}_I(\epsilon)$. For $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'} \in \mathcal{M}_I(\epsilon)$ it may be that $\inf\{q_n : q_n > 0\} = 0$. Such a measure does not correspond to a graph of $\mathcal{G}$.

In conclusion, the results of Chapter 5 (Theorems 5.1.1 and 5.1.3) imply those of Chapter 4 (Theorems 4.1.1 and 4.1.4), except for some of the highly inhomogeneous models on the triangular and hexagonal lattices.

Isoradial square lattices

An isoradial embedding of the square lattice is called an *isoradial square lattice* The track-system of such a graph is simply a square grid, and *vice versa*.

Periodic graphs

Let G be an isoradial embedding of a periodic connected graph H (the embedding itself need not be periodic). The track system $\mathcal{T}$ of G (viewed as a set of arcs) is determined by the structure of H. Since H is periodic, so is $\mathcal{T}$ (viewed as a graph). Therefore, $\mathcal{T}$ may be embedded homeomorphically into $\mathbb{R}^2$ in a periodic manner. After re-scaling, we may assume that $\mathcal{T}$ is invariant under any unit shift of $\mathbb{R}^2$ in the direction of a coordinate vector. In fact, $\mathcal{T}$ may be thought of as the lifting to the universal cover of a track-system on a torus.

As observed in [KS05, Sect. 5.2], any oriented track t has an asymptotic angle $\alpha(t) \in S^1$, and in addition the reversed track has direction $\pi + \alpha(t)$. Let $t \in \mathcal{T}$, viewed as a subset of $\mathbb{R}^2$. There exists $(a, b) \in \mathbb{Z}^2$, $(a, b) \neq (0, 0)$, such that t is invariant under the shift $\tau_{a,b} : z \mapsto z + (a, b)$. We have that $\tan \alpha(t) = b/a$. By periodicity, the set of all angles (modulo π) of $\mathcal{T}$ is finite, and we write it as $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $m \geq 1$.

Let T_k be the set of tracks with asymptotic angle (modulo π) α_k. By periodicity, each T_k is a set of tracks indexed by $\mathbb{Z}$, and may be ordered according to their crossings of the line with polar coordinates $\theta = \theta_0$ with $\theta_0 \neq \alpha_k$ for all k. Since tracks $t_k \in T_k$, $t_l \in T_l$ (with $k \neq l$) have different asymptotic angles, they must intersect.

It remains to show that any $t, t' \in T_k$ do not intersect (whence, in particular, $m \geq 2$). Suppose the converse, that there exist $k \in \{1, 2, \ldots, m\}$ and $t, t' \in T_k$ such that t and t' intersect at some point $J \in \mathbb{R}^2$. Since t and t' have the same angle α_k, there exists $(a, b) \in \mathbb{Z}^2$ such that t and t' are invariant under $\tau_{a,b}$. Therefore, they intersect at $J + n(a, b)$ for all $n \in \mathbb{Z}$, in contradiction of the fact that they may have at most one intersection.

For any distinct pair T_k, T_l, part (c) of the square-grid property holds by periodicity.

We have proved not only that G has the square-grid property, but the stronger fact that its track-set may be partitioned into m classes of parallel tracks.

Rhombic tilings via multigrids

The following 'multigrid' construction was introduced and studied by de Bruijn [Bru81a, Bru81b, Bru86]. A *grid* is a set of parallel lines in $\mathbb{R}^2$ with some common perpendicular unit-vector v. A *multigrid* is a family of grids with pairwise non-parallel perpendiculars. Suppose there are $m \geq 2$ grids, with perpendiculars $v_1, v_2, \ldots, v_m$. The kth grid is given in terms of a set $C_k = \{c_k^i : i \in \mathbb{Z}\}$ of reals, specifically as the set of all $z \in \mathbb{R}^2$ with $z.v_k = c_k^i$ as i ranges over $\mathbb{Z}$. It is assumed that the c_k^i are strictly increasing in i, with $c_k^i/i \to 1$ as $i \to \pm\infty$.

With the lines of the kth grid, duly oriented, we associate a unit vector w_k. It is explained in [Bru86] how, under certain conditions on the C_k, v_k, w_k, one may 'dualize' the multigrid to obtain a rhombic tiling of $\mathbb{R}^2$. The track-set of the ensuing tiling is a homeomorphism of the multigrid with transverse vectors w_k. Under the additional assumption that the differences $|c_k^{i+1} - c_k^i|$ are uniformly bounded away from 0 and ∞, all such tilings have both the bounded-angles property and the square-grid property. The results of this paper apply to the associated isoradial graphs.

Penrose's rhombic tiling may be obtained thus with $m = 5$, the v_k being vectors forming a regular pentagon, with $w_k = v_k$, and $C_k = \{i + \gamma_k : i \in \mathbb{Z}\}$ with an appropriate vector (γ_k). Other choices of the parameters yield a broader class of aperiodic rhombic tilings of

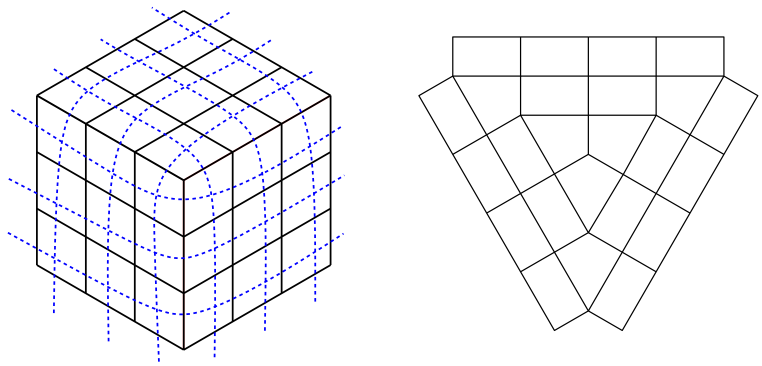

Figure 3.1.4: Part of a rhombic tiling without the square-grid property, and one of the two corresponding isoradial graphs. The square-grid property fails since no two of the three families of non-intersecting tracks are doubly infinite.

the plane. See [Bru81a, Bru81b]. Percolation on Penrose tilings has been considered in [Hof98].

A track-system with no square grid

Figure 3.1.4 is an illustration of a track-system without the square-grid property.

3.1.5 Equivalence of metrics

Let G be an isoradial graph. It will be convenient to use both the Euclidean metric $|\cdot|$ and the graph-metric $d^\diamond$ on $G^\diamond$. For $n \in N$ and $u \in G^\diamond$ we write $\Lambda_u^\diamond(n)$ for the ball of $d^\diamond$-radius n centred at u:

$$\Lambda_u^\diamond(n) = \{v \in G^\diamond : d^\diamond(u,v) \le n\}.$$

Proposition 3.1.3. *Let $\epsilon > 0$. There exists $c_d = c_d(\epsilon) > 0$ such that, for any isoradial graph $G = (V,E)$ satisfying* BAP(ϵ),

$$c_d^{-1}|v - v'| \le d^\diamond(v,v') \le c_d|v - v'|, \qquad v, v' \in G^\diamond. \tag{3.1.7}$$

Proof. Let u, v be distinct vertices of $G^\diamond$. Since each edge of $G^\diamond$ has length 1, $d^\diamond(u,v) \ge |u - v|$. Conversely, let S_{uv} be the set of all faces of $G^\diamond$ (viewed as closed sets of $\mathbb{R}^2$) that intersect the straight-line segment uv of $\mathbb{R}^2$ joining u to v. Since the diameter of any such face is less than 2, every point of the union of S_{uv} is within Euclidean distance 2 of uv. By BAP(ϵ) and (3.1.2), every face has area at least $\sin\epsilon$, and therefore $|S_{uv}| \le 4(|u-v|+4)/\sin\epsilon$. Similarly, there exists $\delta = \delta(\epsilon) > 0$ such that $|u-v| \ge \delta$. The edge-set of elements of S_{uv} contains a path of edges of $G^\diamond$ from u to v, whence

$$d^\diamond(u,v) \le \frac{8}{\sin\epsilon}(|u-v|+4) \le \frac{8(\delta+4)}{\delta\sin\epsilon}|u-v|,$$

as required. □

3.1.6 The box-crossing property for graphs in $\mathcal{G}$

This section begins with a definition of the rectangular domains of an isoradial graph $G \in \mathcal{G}$, using the topology of its square grid.

Let (t,t') be an ordered pair of non-intersecting tracks of G. A point $x \in \mathbb{R}^2$ is said to be 'strictly between' t and t' if, with these tracks viewed as arcs of $\mathbb{R}^2$, there exists an unbounded path of $\mathbb{R}^2$ from x that intersects t but not t', and *vice versa*. A face F of $G^\diamond$ is said to be *between* t and t' if: either F is a rhombus of t, or every point of F is strictly between t and t'. Note that this usage of 'between' is not reflexive: there are faces between t and t' that are not between t' and t. A vertex or edge of $G^\diamond$ is said to be 'between' t and t' if it belongs to some face between t and t'. The *domain* between t and t' is the union of the (closed) faces between t and t'. It is useful to think of a domain as either a subgraph of $G^\diamond$, or as a closed region of $\mathbb{R}^2$.

Suppose $G \in \mathcal{G}$ has a square grid $S \cup T$, with $S = (s_j : j \in \mathbb{Z})$ and $T = (t_i : i \in \mathbb{Z})$. We call tracks in S (respectively, T) *horizontal* (respectively, *vertical*). For $i_1, i_2, j_1, j_2 \in \mathbb{Z}$ we define $\mathcal{D} = \mathcal{D}(t_{i_1}, t_{i_2}; s_{j_1}, s_{j_2})$ to be the intersection of the domains between t_{i_1} and t_{i_2} and between s_{j_1} and s_{j_2}.

We say that $\mathcal{D}$ is crossed horizontally if G contains an open path π such that: (i) every edge of π lies in $\mathcal{D}$, and (ii) the first edge crosses t_{i_1} and the last vertex is adjacent to t_{i_2}. Write $\mathcal{C}_{\mathrm{h}}(\mathcal{D}) = \mathcal{C}_{\mathrm{h}}(t_{i_1}, t_{i_2}; s_{j_1}, s_{j_2})$ for the event that $\mathcal{D}$ is crossed horizontally, with a similar definition of the vertical-crossing event $\mathcal{C}_{\mathrm{v}}(\mathcal{D})$. See Figure 3.1.5 for an illustration of the above notions.

The purpose of the following proposition is to restate the box-crossing property in terms of the geometry of the square grid. Considering the structure of isoradial graphs in $\mathcal{G}$, if $(G, \mathbb{P}_G)$ has the box-crossing property, then it satisfies BXP$(3,\delta)$ for some $\delta > 0$. Henceforth, for isoradial graphs, we will write BXP(δ) for BXP$(3,\delta)$.

Proposition 3.1.4. *Let $\epsilon > 0$, $I \in \mathbb{N}$, and let $G \in \mathcal{G}(\epsilon, I)$. The graph G has the box-crossing property if and only if there exists $\delta > 0$ such that, for $N \in \mathbb{N}$ and $i, j \in \mathbb{Z}$,*

$$\mathbb{P}_G\big[\mathcal{C}_{\mathrm{h}}(t_i, t_{i+2N}; s_j, s_{j+N})\big], \mathbb{P}_G\big[\mathcal{C}_{\mathrm{v}}(t_i, t_{i+N}; s_j, s_{j+2N})\big] \ge \delta. \tag{3.1.8}$$

Moreover, if (3.1.8) *holds, then G satisfies* BXP(δ') *with δ' depending on δ, ϵ, I and not further on G.*

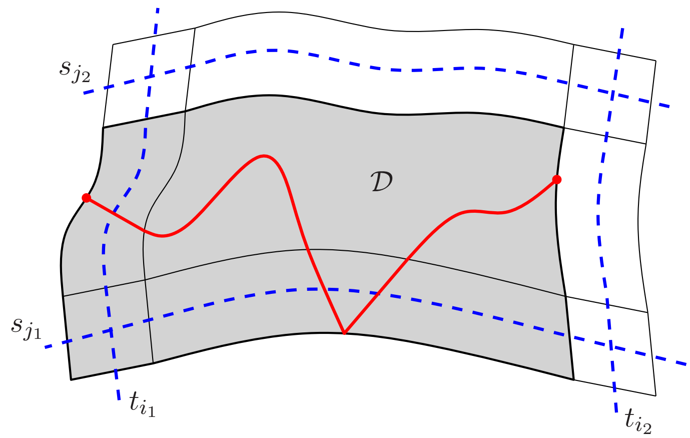

Figure 3.1.5: The shaded domain $\mathcal{D} = \mathcal{D}(t_{i_1}, t_{i_2}; s_{j_1}, s_{j_2})$ is crossed horizontally.

Proof. We prove only the final sentence of the proposition. The converse (that the box-crossing property implies (3.1.8) for some $\delta > 0$) holds by similar arguments, and will not be used this document. Let $G \in \mathcal{G}(\epsilon, I)$, and assume (3.1.8) with $\delta > 0$.

Let $N \in \mathbb{N}$. For $i, j \in \mathbb{Z}$, the *cell* $C_{i,j}$ is the domain $\mathcal{D}(t_{iN}, t_{(i+1)N}; s_{jN}, s_{(j+1)N})$. The cells have disjoint interiors and cover the plane. Two distinct cells $C = C_{i,j}$, $C' = C_{k,l}$ are said to be *adjacent* if (i, j) and (k, l) are adjacent vertices of the square lattice, in which case we write $C \sim C'$. More specifically, we write $C \sim_{\mathrm{h}} C'$ (respectively, $C \sim_{\mathrm{v}} C'$) if $|i - k| = 1$ (respectively, $|j - l| = 1$). With the adjacency relation $\sim$, the graph having the set of cells as vertex-set is isomorphic to the square lattice.

Each cell has perimeter at most $4IN$, and therefore diameter not exceeding $2IN$. A cell contains at least N^2 faces of $G^\diamond$, and thus (by (3.1.2)) has total area at least $N^2 \sin\epsilon$.

For $\mu \in \mathbb{N}$ with $\mu \geq 2I$, let $u = (-\mu N, 0)$ and $v = (\mu N, 0)$ viewed as points in the plane. Let U_{uv}^N be the union of the set S_{uv}^N of cells that intersect the straight-line segment uv with endpoints u, v. Let R be the tube $uv + [-2IN, 2IN]^2$. Thus R has area $8IN(\mu N + 2IN)$, and $U_{uv}^N \subseteq R$. Since each cell has area at least $N^2 \sin\epsilon$, the cardinality of S_{uv}^N satisfies

$$|S_{uv}^N| \leq \frac{8IN(\mu N + 2IN)}{N^2 \sin\epsilon} = \frac{8I(\mu + 2I)}{\sin\epsilon}. \tag{3.1.9}$$

There exists a chain of cells $C_1, \dots, C_K \in S_{uv}^N$ such that $u \in C_1$, $v \in C_K$ and $C_k \sim C_{k+1}$ for $k = 1, 2, \dots, K-1$; see Figure 3.1.6. Let $k \in \{1, 2, \dots, K-1\}$, and assume $C_k \sim_{\mathrm{h}} C_{k+1}$. Let H_k be the event that C_k and C_{k+1} are crossed vertically, and $C_k \cup C_{k+1}$ is crossed horizontally. A similar definition holds when $C_k \sim_{\mathrm{v}} C_{k+1}$, with vertical and horizontal interchanged. By (3.1.8) and the Harris–FKG inequality, $\mathbb{P}_G(H_k) \geq \delta^3$.

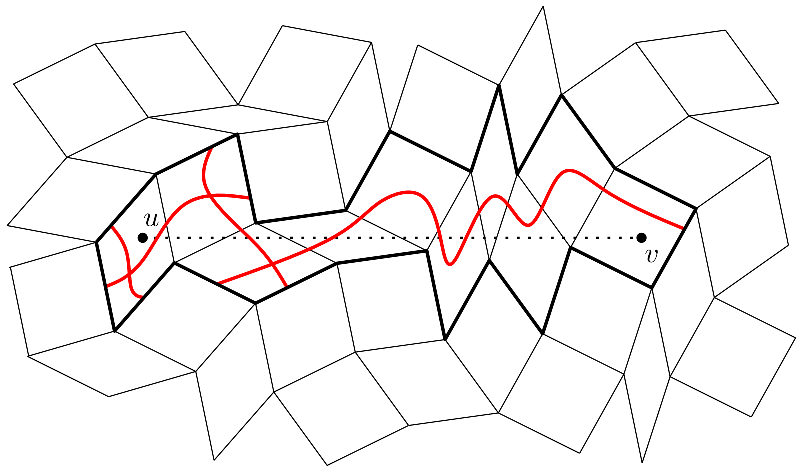

Figure 3.1.6: The region U_{uv}^N is outlined in bold, and contains a chain of cells joining u to v. The events H_k are drawn explicitly for the first two contiguous pairs of cells.

By the Harris–FKG inequality, the fact that $K \le |S_{u,v}^N|$, and (3.1.9),

$$\mathbb{P}_G\left(\bigcap_{k=1}^{K-1} H_k\right) \ge \delta^{3K} \ge \delta^{24I(\mu+2I)/\sin\epsilon}. \tag{3.1.10}$$

If the event on the left side occurs, the rectangle

$$S_{\mu,N} := \big[-(\mu-2I)N, (\mu-2I)N\big] \times [-2IN, 2IN]$$

of $\mathbb{R}^2$ is crossed horizontally.

Let $R_k = [-k,k] \times [-\frac{1}{2}k, \frac{1}{2}k]$ where $k \ge 8I$. Pick N such that $4IN \le k \le 8IN$, so that R_k is 'higher' and 'shorter' than $S_{10I,N}$. By (3.1.10) with $\mu = 10I$,

$$\mathbb{P}_G\left(R_k \text{ is crossed horizontally}\right) \ge \delta'', \tag{3.1.11}$$

where $\delta'' = \delta^{288I^2/\sin\epsilon}$. Smaller values of k are handled by adjusting δ'' accordingly.

The same argument is valid for translates and rotations of the line-segment uv, and the proof is complete. $\square$

3.1.7 Isoradial square lattices

An *isoradial square lattice* is an isoradial embedding of the square lattice $\mathbb{Z}^2$. Isoradial square lattices, and only these graphs, have a square grid as track-system.

Let G be an isoradial square lattice. The diamond graph $G^\diamond$ possesses two families of parallel tracks, namely the horizontal tracks $(s_j : j \in \mathbb{Z})$ and the vertical tracks $(t_i : i \in \mathbb{Z})$. The graph $G^\diamond$, and hence the pair (G, G^*) also, may be characterized in terms of two

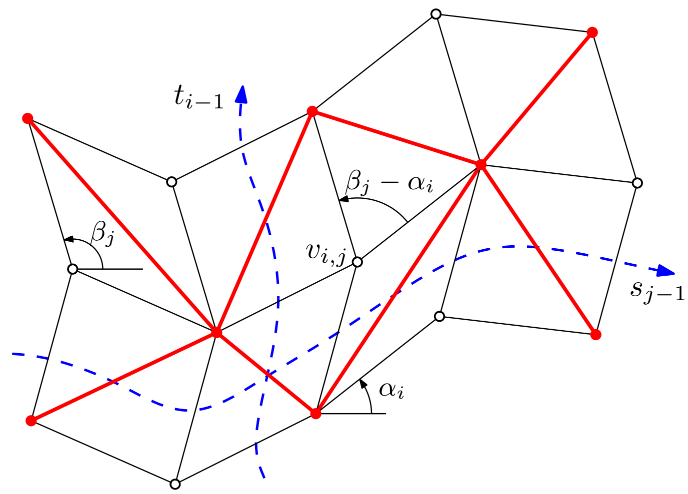

Figure 3.1.7: An isoradial square lattice (in red) with the associated diamond graph. The diamond graph is isomorphic to $\mathbb{Z}^2$, and its embedding is characterized by two sequences $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ of angles.

vectors of angles linked to the transverse vectors. First, we orient s_0 in an arbitrary way (interpreted as 'rightwards'). As we proceed in the given direction along s_0, the crossing tracks t_i are numbered in increasing sequence, and are oriented from right to left (interpreted as 'upwards'). Similarly, as we proceed along t_0, the crossing tracks s_j are numbered in increasing sequence and oriented from left to right. Let $\tau(s_j)$ (respectively, $\tau(t_i)$) be the transverse vector of s_j (respectively, t_i) with transverse angle β_j (respectively, θ_i). Rather than working with the θ_i, we work instead with $\alpha_i := \theta_i - \pi$ as illustrated in Figure 3.1.7. Write $\boldsymbol{\alpha} = (\alpha_i : i \in \mathbb{Z})$ and $\boldsymbol{\beta} = (\beta_j : j \in \mathbb{Z})$, and note that $\alpha_i \in [-\pi, \pi)$, $\beta_j \in [0, 2\pi)$. We will generally assume that G is rotated in such a way that $\alpha_0 = 0$, so that $\beta_j \in [0, \pi]$ and $\beta_j - \pi \le \alpha_i \le \beta_j$ for $i, j \in \mathbb{Z}$.

The vertex of $G^\diamond$ adjacent to the four tracks t_{i-1}, t_i, s_{j-1}, s_j is denoted $v_{i,j}$. If not otherwise stated, we shall assume that the tracks are labelled in such a way that the vertex $v_{0,0}$ is a primal vertex of $G^\diamond$.

Tracks t_i, s_j intersect in a rhombus of $G^\diamond$ with sides $\tau(t_i)$, $\tau(s_j)$, $-\tau(t_i)$, $-\tau(s_j)$ in clockwise order, and thus its internal angles are $\beta_j - \alpha_i$ and $\pi - (\beta_j - \alpha_i)$. Thus, G satisfies the bounded-angles property BAP(ϵ) if and only if

$$\beta_j - \alpha_i \in [\epsilon, \pi - \epsilon], \qquad i, j \in \mathbb{Z}. \tag{3.1.12}$$

Conversely, for two vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ satisfying (3.1.12), we may construct the diamond graph denoted $G^\diamond_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as in Figure 3.1.7. This gives rise to an isoradial square lattice denoted $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ (and its dual) satisfying BAP(ϵ). We write $\mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for the canonical measure of $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

We introduce now some notation to be used later. For a set W of vertices of $G^\diamond$, we define the *height* of W by

$$h(W) = \sup\big\{j : \exists i \text{ with } v_{i,j} \in W\big\}.$$

This definition extends in an obvious way to sets of edges.

In Section 5.2.2 is described an operation of so-called 'track-exchange' on isoradial square lattices. This introduces a potential for confusion between the *label* and the *level* of a track. In the $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ above, we say that s_j is (initially) at *level* j. The level of s_j may change under track-exchange, but $v_{i,j}$ shall always refer to the vertex between levels $j-1$ and j in the new graph.

Due to this potential confusion, it will be convenient to use a different notation for domains in square lattices than for general graphs. For $M_1, M_2, N_1, N_2 \in \mathbb{Z}$ with $M_1 \le M_2$, $N_1 \le N_2$, let $B(M_1, M_2; N_1, N_2)$ be the subgraph of G induced by the subset of vertices $\{v_{i,j} : M_1 \le i \le M_2,\ N_1 \le j \le N_2\}$. For $M, N \in \mathbb{N}$, we use the abbreviated notation $B(M,N) = B(-M, M; 0, N)$. A *horizontal crossing* of $B = B(M_1, M_2; N_1, N_2)$ is an open path of B linking some vertex v_{M_1,n_1} to some vertex v_{M_2,n_2}; a *vertical crossing* links some v_{m_1,N_1} to some v_{m_2,N_2}. We write $\mathcal{C}_{\mathrm{h}}[B]$ (respectively, $\mathcal{C}_{\mathrm{v}}[B]$) for the event that a box B contains a horizontal (respectively, vertical) crossing. For a vertex $v_{i,j}$ of G, we write $B + v_{i,j}$ for the translate $\{v_{r,s} : v_{r-i,s-j} \in B\}$.

When applied to G, we have that

$$B(M_1, M_2; N_1, N_2) = \mathcal{D}(t_{M_1}, t_{M_2}; s_{N_1}, s_{N_2}),$$

since s_{N_1} and s_{N_2} are the tracks at levels N_1 and N_2 respectively. As mentioned before, the latter will not always be the case. Use of the notation B emphasizes that domains are defined in terms of tracks at specific levels, rather than of tracks with specific labels.

The following lemma will be used in Section 5.2.

Lemma 3.1.5. *Let $G = (V, E)$ be an isoradial square lattice satisfying the bounded-angles property* $\mathrm{BAP}(\epsilon)$ *and the following.*

(a) *For $\rho \ge 1$, there exists $\eta(\rho) > 0$ such that*

$$\mathbb{P}_G\big(\mathcal{C}_{\mathrm{h}}[B(\lfloor \rho N \rfloor, N) + v]\big) \ge \eta(\rho), \qquad N \in \mathbb{N},\ v \in V.$$

(b) *There exist $\rho_0, \eta_0 > 0$ such that*

$$\mathbb{P}_G\big(\mathcal{C}_{\mathrm{v}}[B(N, \lfloor \rho_0 N \rfloor) + v]\big) \ge \eta_0, \qquad N \ge \rho_0^{-1},\ v \in V.$$

Then there exists $\delta = \delta(\rho_0, \eta_0, \eta(1), \eta(2\rho_0^{-1}), \epsilon) > 0$ such that G has the box-crossing property $\mathrm{BXP}(\delta)$.

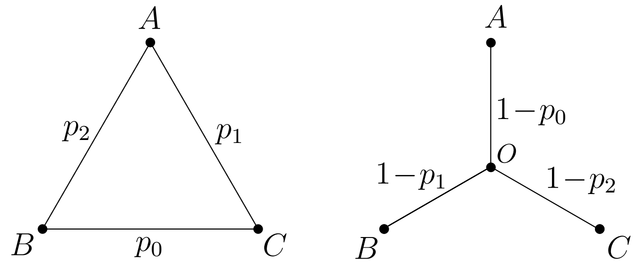

Figure 3.2.1: The star–triangle transformation

Outline proof. Assume (a) and (b) hold. Just as in the proof of Proposition 4.3.2, the crossing probabilities of boxes of G with aspect-ratio 2 and horizontal/vertical orientations are bounded away from 0 by a constant that depends only on the aspect-ratios of the boxes illustrated in Figure 4.3.1. (Here, the boxes in question are those of G viewed as an isoradial square lattice, that is, boxes of the form $B(\cdot;\cdot)$ defined before the lemma.) Therefore, the hypothesis of Proposition 3.1.4 holds with suitable constants, and the claim follows from its conclusion. □

3.2 The star–triangle transformation

In Section 3.2.1 we review the basic action of the star–triangle transformation, then, in Section 3.2.3 we show its harmony with isoradial embeddings. The star–triangle transformation is a central tool in the proofs of Chapters 4 and 5. To physicists, the star–triangle transformation is better known as the Yang-Baxter equation.

3.2.1 Star–triangle transformation

The star–triangle transformation was discovered first in the context of electrical networks, and adapted by Onsager and Kramers–Wannier to the Ising model. In its base form, it is a graph-theoretic transformation between the hexagonal lattice and the triangular lattice. Its importance stems from the fact that a variety of probabilistic models are conserved under this transformation, including the critical percolation, Potts, and random-cluster models. The methods of this paper extend to all such systems, but we concentrate here on percolation, for which we summarize its manner of operation as in [Gri99, Sect. 11.9].

Consider the triangle $\Delta = (V, E)$ and the star $\Delta' = (V', E')$ of Figure 3.2.1. Let $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$ be a triplet of parameters. Write $\Omega = \{0, 1\}^E$ with associated product probability measure $\mathbb{P}^{\triangle}_{\mathbf{p}}$ with intensities p_i (as in the left diagram of Figure 3.2.1), and $\Omega' = \{0, 1\}^{E'}$ with associated measure $\mathbb{P}^{\bigcirc}_{1-\mathbf{p}}$, with intensities $1 - p_i$ (as in the right diagram of Figure 3.2.1). Let $\omega \in \Omega$ and $\omega' \in \Omega'$. For each graph we may consider open connections between its vertices, and we abuse notation by writing, for example, $x \xleftrightarrow{\Delta,\omega} y$ for the *indicator function* of the event that x and y are connected in Δ by an open path of ω. Thus connections in Δ are described by the family $(x \xleftrightarrow{\Delta,\omega} y : x, y \in V)$ of random variables, and similarly for Δ'.

Proposition 3.2.1 (Star–triangle transformation)**.** *Let* $\mathbf{p} \in [0,1)^3$ *be such that*

$$\kappa_{\triangle}(p_0, p_1, p_2) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1. \tag{3.2.1}$$

The families

$$\left(x \xleftrightarrow{\Delta,\omega} y : x, y = A, B, C\right), \quad \left(x \xleftrightarrow{\Delta',\omega'} y : x, y = A, B, C\right),$$

have the same law.

The proof is an elementary computation, and may be found in [Gri99, Sect. 11.9]. Next we explore couplings of the two measures. Let $\mathbf{p} \in [0,1)^3$ satisfy (3.2.1), and let Ω (respectively, Ω') have associated measure $\mathbb{P}^{\triangle}_{\mathbf{p}}$ (respectively, $\mathbb{P}^{\pentagon}_{1-\mathbf{p}}$) as above. There exist random mappings $T : \Omega \to \Omega'$ and $S : \Omega' \to \Omega$ such that $T(\omega)$ has law $\mathbb{P}^{\pentagon}_{1-\mathbf{p}}$, and $S(\omega')$ has law $\mathbb{P}^{\triangle}_{\mathbf{p}}$. Such mappings are given in Figure 3.2.2, and we shall not specify them more formally here. Note from the figure that $T(\omega)$ is deterministic for seven of the eight elements of Ω; only in the eighth case does $T(\omega)$ involve further randomness. Similarly, $S(\omega')$ is deterministic except for one special ω'. Each probability in the figure is well defined since $P := (1-p_0)(1-p_1)(1-p_2) > 0$.

Proposition 3.2.2 (Star–triangle coupling)**.** *Let* $\mathbf{p}$ *be self-dual and let* S *and* T *be given as in Figure 3.2.2. With* ω *and* ω' *sampled as above,*

(a) $T(\omega)$ *has the same law as* ω'*,*

(b) $S(\omega')$ *has the same law as* ω*,*

(c) *for* $x, y \in \{A, B, C\}$*,* $x \xleftrightarrow{G,\omega} y$ *if and only if* $x \xleftrightarrow{G',T(\omega)} y$*,*

(d) *for* $x, y \in \{A, B, C\}$*,* $x \xleftrightarrow{G',\omega'} y$ *if and only if* $x \xleftrightarrow{G,S(\omega')} y$*.*

The maps S and T act on configurations on stars and triangles. They act simultaneously on the duals of these graph elements, illustrated in Figure 3.2.3. Let $\omega \in \Omega$, and define $\omega^*(e^*) = 1 - \omega(e)$ for each primal/dual pair e/e^* of the left side of the figure. The action of T on Ω induces an action on the dual space Ω^*, and it is easily checked that this action preserves ω^*-connections. The map S behaves similarly. This property of the star–triangle transformation has been generalized and studied in [BR10] and the references therein.

3.2.2 The star–triangle transformation and open paths

Since the star–triangle transformations S and T preserve connections, they also preserve open paths, as described below. Let us first give a precise definition of paths.

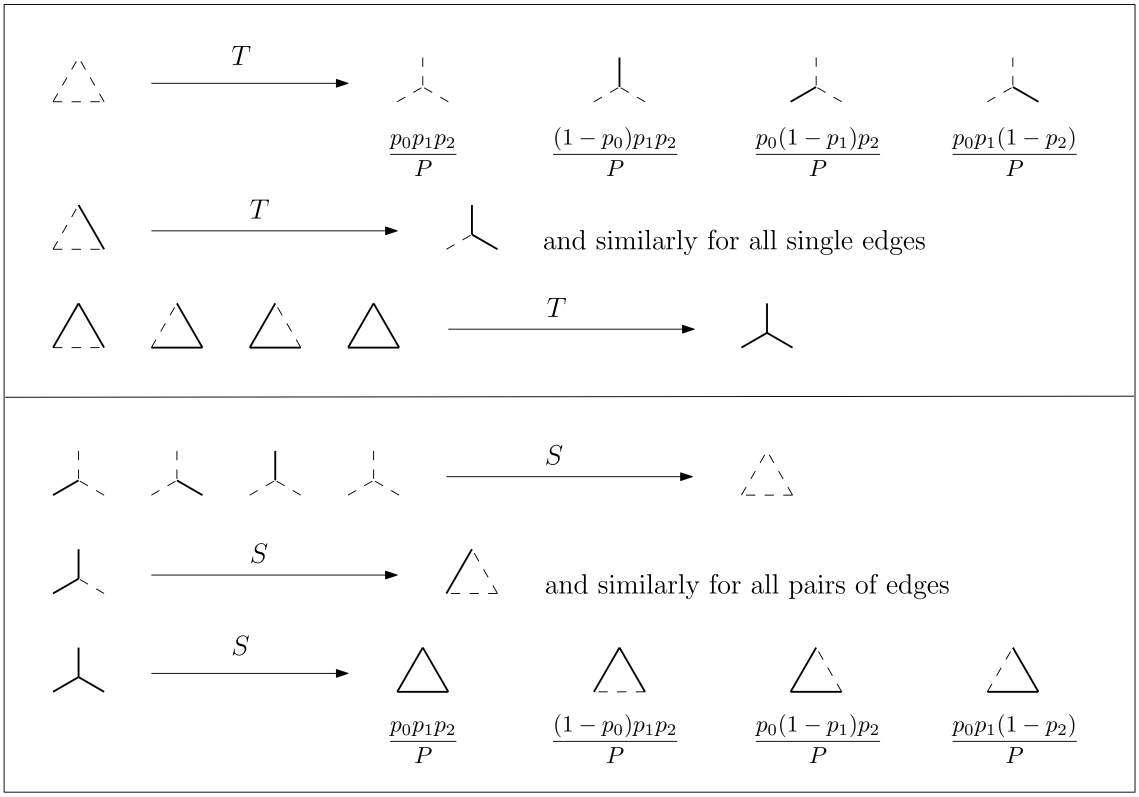

Figure 3.2.2: The random maps T and S and their transition probabilities, with $P := (1-p_0)(1-p_1)(1-p_2)$. Since $\kappa_\triangle(\mathbf{p}) = 0$, the probabilities in the first and last rows sum to 1.

A *path* $\Gamma = (\Gamma_t)$ in $\mathbb{R}^2$ is a continuous function $\Gamma : [a,b] \to \mathbb{R}^2$ for some real interval $[a,b]$. Note that a path Γ may in general have self-intersections, and there may be subintervals of $[a,b]$ on which Γ is constant. Let $\phi : [c,d] \to [a,b]$ be continuous and strictly increasing with $\phi(c) = a$ and $\phi(d) = b$. We term the path $\Gamma_\phi = (\Gamma_{\phi(t)})$ a *reparametrization* of Γ over $[c,d]$.

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^2$. The space of paths may be metrized by

$$d_{path}(\Gamma, \Pi) = \inf \left\{ \sup_{t\in[0,1]} \left|\Gamma'_t - \Pi'_t\right| \right\},$$

where the infimum is over all reparametrizations Γ' (respectively, Π') of Γ (respectively, Π) over $[0,1]$. Note that d_{path} is not a metric since $d_{path}(\Gamma, \Gamma') = 0$ if Γ' is a reparametrization of Γ, and thus the corresponding metric acts on a space of equivalence classes of paths (see [AB99, eqn (2.1)]). We shall use the fact that, if two paths (parametrized over $[0,1]$)

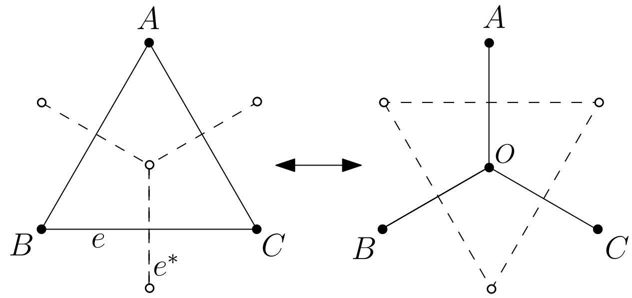

Figure 3.2.3: The star–triangle transformation acts simultaneously on primal and dual graph elements.

satisfy $d_{path}(\Gamma, \Pi) < \delta$, then

$$\Gamma \subseteq \Pi^{\delta}, \quad |\Gamma_0 - \Pi_0| \leq \delta, \quad |\Gamma_1 - \Pi_1| \leq \delta,$$

where

$$A^{\delta} := \{x + y : x \in A, \, |y| \leq \delta\}.$$

Let $G = (E, V)$ be a planar graph. A path $\gamma : [a, b] \to \mathbb{R}^2$ is called a path on G if its image in $\mathbb{R}^2$ only uses edges and vertices of G. Moreover, γ is such that $[a, b]$ may be split into a finite family of intervals $\{[a_k, a_{k+1}] : 0 \leq k \leq K\}$, with every interval being mapped by γ onto either an edge or a vertex of G. In other words γ may be represented as a chain of edges, with possible stationary points at vertices. Henceforth all paths will be paths on graphs (we allow loops and repeated edges). Such a path is called *open* (in a given configuration) if it traverses only open edges.

Let $\mathbb{P}$ be a percolation measure on G. Suppose $G = (V, E)$ contains a triangle $\Delta = ABC$, and that $\mathbb{P}$ has intensities $\mathbf{p} = (p_0, p_1, p_2)$ on the edges of Δ, as in Figure 3.2.1. Let $T(G)$ be the graph obtained from G by replacing Δ with the star Δ' with center O. For a configuration $\omega \in \Omega = \{0, 1\}^E$, $T(\omega)$ is a random configuration on $T(G)$, identical to ω outside Δ' and given by the coupling described in Figure 3.2.2 on Δ'. By Proposition 3.2.2 the operation described above preserves open connections.

Let $\omega \in \Omega$ be a configuration of open edges of G and γ be an ω-open path. We will describe how we associate to γ a $T(\omega)$-open path on $T(G)$, which we call $T(\gamma)$. Suppose for simplicity that γ has no stationary points, and parametrize it such that it passes through the sequence $\gamma_0, \gamma_1, \ldots, \gamma_K$ of vertices of G, in order. Hence each $(\gamma_t : k \leq t \leq k+1)$ is an open edge of G. The path $T(\gamma)$ is also parametrized by $[0, K]$ and is obtained as follows. If k is such that $(\gamma_t : k \leq t \leq k+1)$ is not an edge of Δ, then $T(\gamma)$ is identical to γ on $[k, k+1]$. If it is an edge of Δ, say BC, we set $T(\gamma)_k = B$, $T(\gamma)_{k+\frac{1}{2}} = O$, $T(\gamma)_{k+1} = C$, and interpolate linearly between these points. By the coupling of ω and $T(\omega)$, $T(\gamma)$ is

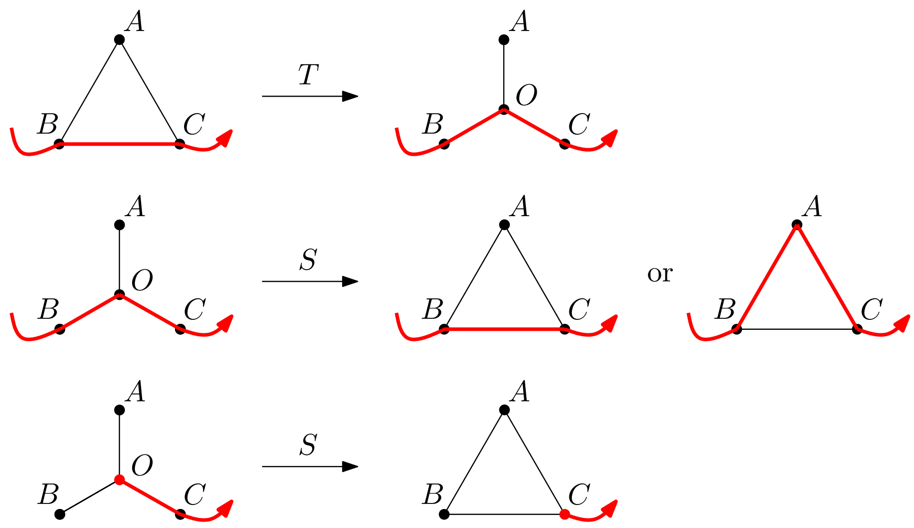

Figure 3.2.4: The action of T and S on the red open path.

indeed a $T(\omega)$ open path, with same endpoints as γ.

Suppose now that G contains a star Δ', and that ω, γ are as above. As for T, we define $S(\gamma)$ to be equal to γ outside Δ'. For the sections of γ that intersect Δ' we proceed as follows. Let k be such that $\gamma_k = O$. If $k = 0$, then $\gamma_1 \in \{A, B, C\}$, and let $S(\gamma)$ be stationary on $[0, 1]$, equal to γ_1. Similarly, when $k = K$, $S(\gamma)_t = \gamma_{K-1}$ for $t \in [K-1, K]$. Finally, if $0 < k < K$, then we have two cases, either $\gamma_{k-1} = \gamma_{k+1}$ or $\gamma_{k-1} \neq \gamma_{k+1}$. In the first case, $S(\gamma)$ is equal to γ_{k-1} on $[k-1, k+1]$. In the second case, suppose $\gamma_{k-1} = B$ and $\gamma_{k+1} = C$. If the edge BC is $S(\omega)$-open, then set $S(\gamma)_{k-1} = B$, $S(\gamma)_{k+1} = C$ and interpolate linearly. If BC is $S(\omega)$-closed, by the coupling of Figure 3.2.2, both edges AB and AC are $S(\omega)$-open. We then set $S(\gamma)_{[k-1,k]} = BA$ and $S(\gamma)_{[k,k+1]} = AC$. This defines $S(\gamma)$ as a $S(\omega)$-open path on $S(G)$.

The action of S and T on open paths is described in Figure 3.2.4.

3.2.3 The star–triangle transformation for isoradial graphs

Let $G = (V, E)$ be an isoradial graph, and let Δ be a triangle of G with vertices A, B, C. Seen as a transformation between graphs, the star–triangle transformation changes Δ into a star Δ' with a new central vertex $O \in \mathbb{R}^2$. It turns out that O may be chosen in such a way that the new graph, denoted G', also is isoradial. The right way of seeing this is via the diamond graph $G^\diamond$, as illustrated in Figure 3.2.5. This construction has its roots in the Z-invariant Ising model of Baxter [Bax82, Bax86], studied in the context of isoradial graphs by Mercat [Mer01], Kenyon [Ken04], and Costa-Santos [CS06] (see also [BdT10]).

The triangle Δ comprises the diagonals of three rhombi of $G^\diamond$. These rhombi form the interior of a *hexagon* with primary vertices A, B, C and three further dual vertices.

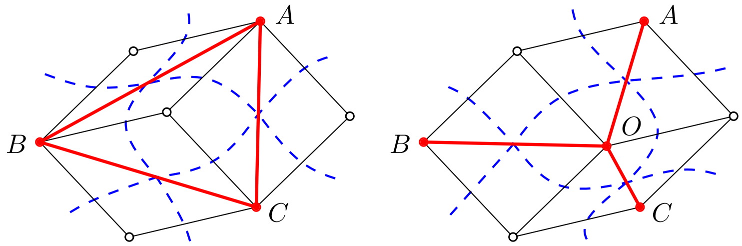

Figure 3.2.5: The triangle on the left is replaced by the star on the right. The new vertex O is the circumcentre of the three dual vertices of the surrounding hexagon of $G^\diamond$.

Let O be the circumcentre of these dual vertices. Three new rhombi are formed from the hexagon augmented by O (as shown). The star Δ' has edges AO, BO, CO, and the ensuing graph is isoradial (since it stems from a rhombic tiling).

By an examination of the angles in the figure, the intensities associated to Δ by (1.3.1) satisfy (3.2.1), and the star–triangle transformation may be applied to Δ. Moreover, this transformation yields the canonical measure on Δ'. That is, the star–triangle transformation maps $\mathbb{P}_G$ to $\mathbb{P}_{G'}$. Furthermore, for $\epsilon > 0$,

$$G \text{ satisfies BAP}(\epsilon) \text{ if and only if } G' \text{ satisfies BAP}(\epsilon). \tag{3.2.2}$$

The same holds when applying the star–triangle transformation to a star contained in an isoradial graph.

We shall sometimes view the star–triangle transformation as acting on the rhombic tiling $G^\diamond$ rather than on G, and thereby it acts simultaneously on G and its dual G^*.

The star–triangle transformation of Figure 3.2.5 is said to act on the *track-triangle* formed by the tracks on the left side, and to *slide* one of the tracks illustrated there *over the intersection* of the other two, thus forming the track-triangle on the right side.

A natural question when dealing with percolation on isoradial graphs is why do we associate parameters to edges via (1.3.1), and not another formula. In light of the above, we may give a explanation.

Suppose we wish to associate to every isoradial graph G a canonical critical percolation measure $\mathbb{P}_G$ with parameters $p_e = \phi(\theta_e)$, where θ_e is given as in Figure 1.3.3. Equivalently we could ask p_e to be a function of the length of e; the expression in terms of θ is more harmonious with the computations.

Since we want $\mathbb{P}_G$ to be critical, it is reasonable to expect that the canonical measure associated to the dual graph G^* is the dual measure of $\mathbb{P}_G$. Hence we want ϕ to satisfy

$$\phi(\pi - \theta) = 1 - \phi(\theta), \qquad \text{for } \theta \in [0, \pi]. \tag{3.2.3}$$

It is also reasonable to ask that the star–triangle transformation may be applied to triangles in $(G, \mathbb{P}_G)$. Thus we expect

$$\phi(\theta_1) + \phi(\theta_2) + \phi(\theta_3) - \phi(\theta_1)\phi(\theta_2)\phi(\theta_3) = 1, \qquad \text{for } \theta_1 + \theta_2 + \theta_3 = 2\pi. \tag{3.2.4}$$

If, in addition, we assume that ϕ is continuous on $[0, \pi]$, then ϕ is uniquely determined by (3.2.3) and (3.2.4), and is such that

$$\frac{\phi(\theta)}{1 - \phi(\theta)} = \frac{\sin(\frac{1}{3}[\pi - \theta])}{\sin(\frac{1}{3}\theta)}.$$

Chapter 4

Universality for inhomogeneous lattices: a first approach

4.1 Results

This chapter summarizes the proofs of Theorems 1.4.1, 1.4.2, and 1.6.2 for the inhomogeneous and highly inhomogeneous models on the square, triangular and hexagonal lattices.

The approach described here is that of [GMa, GMb]. Although the different methods of Chapter 5 yield more general results, we include the following material as an illustration of another possible approach. Both methods rely on the star–triangle transformation, but use it in different ways.

Recall the notation $\mathcal{M}$ and $\mathcal{M}_I$ for the inhomogeneous and highly inhomogeneous models which satisfy the hypothesis of Theorems 1.4.1 and 1.4.2, respectively. Since $\mathcal{M} \subset \mathcal{M}_I$, we will state the following theorems for $\mathcal{M}_I$.

Theorem 4.1.1. *For $\epsilon > 0$ there exists $\delta, l_0 > 0$ such that all models in $\mathcal{M}_I(\epsilon)$ satisfy the box-crossing property* $\mathrm{BXP}(l_0, \delta)$.

For models in $\mathcal{M}_I$, due to the geometry of the lattices, l_0 may always be taken to be twice the length of the edges. In the rest of the chapter we write $\mathrm{BXP}(\delta)$ instead of $\mathrm{BXP}(l_0, \delta)$.

Theorem 4.1.1 implies criticality for the models in $\mathcal{M}_I$ (Theorems 1.4.1 and 1.4.2) via Propositions 2.1.1 and 2.1.2. In Section 4.6 prove the following slightly more general results.

Theorem 4.1.2. *Let $p \in (0,1)$ and $\mathbf{q}, \mathbf{q}' \in [0,1)^{\mathbb{Z}}$.*

(a) *If*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\triangle}(p, q_n, q'_n) \leq 0, \tag{4.1.1}$$

then there exists, $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$*-a.s., no infinite open cluster.*

(b) *If there exists* $\nu > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\triangle}(p, q_n, q'_n) \leq -\nu, \tag{4.1.2}$$

then there exist $c, d > 0$ *such that, for every vertex* v*,*

$$\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}(|C_v| \geq k) \leq ce^{-dk}, \quad k \geq 0.$$

(c) *If there exists* $\nu > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\triangle}(p, q_n, q'_n) \geq \nu, \tag{4.1.3}$$

then $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$ *is uniformly supercritical.*

The same holds for $\mathbb{P}^{\hexagon}_{p,\mathbf{q},\mathbf{q}'}$ *with* $\kappa_{\hexagon}$ *in place of* $\kappa_{\triangle}$*.*

Theorem 4.1.3. *Let* $\mathbf{q}, \mathbf{q}' \in (0,1)^{\mathbb{Z}}$*.*

(a) *If there exists* $\epsilon > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\square}(q_n, q'_n) \leq 0 \quad \textit{and} \quad q_n, q'_n \leq 1 - \epsilon, \tag{4.1.4}$$

then there exists, $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$*-a.s., no infinite open cluster.*

(b) *If there exists* $\nu > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\square}(q_n, q'_n) \leq -\nu,$$

then there exist $c, d > 0$ *such that, for every vertex* v*,*

$$\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}(|C_v| \geq k) \leq ce^{-dk}, \quad k \geq 0.$$

(c) *If there exists* $\nu > 0$ *such that, for all* n*,*

$$\kappa_{\square}(q_n, q'_n) \geq \nu,$$

then $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$ *is uniformly supercritical.*

Finally, we have a universality result for arm exponents across $\mathcal{M}_I$.

Theorem 4.1.4. *For every* $\pi \in \{\rho\} \cup \{\rho_{2j} : j \geq 1\}$*, if* π *exists for some model* $M \in \mathcal{M}_I$*, then it is* $\mathcal{M}_I$*-invariant.*

The above may be used in conjunction with Theorems 2.4.1, 2.5.1, and 4.1.1 to obtain universality results for other critical exponents as in Theorem 5.1.3.

The rest of the chapter is structured as follows. Section 4.3 contains the proof of Theorem 4.1.1 for the inhomogeneous models $\mathcal{M}$; the extension to the highly inhomogeneous models of $\mathcal{M}_I$ is sketched in Section 4.4. Sections 4.5 and 4.6 contain the proofs of Theorem 4.1.4 and Theorems 4.1.2, 4.1.3, respectively. The proofs in Sections 4.3 - 4.5 are based on the lattice transformations presented in Section 4.2.

4.2 Lattice transformations via the star–triangle transformation

We show next how to use the star–triangle transformation to convert the triangular lattice into the square lattice and vice-versa. The transformation will transport self-dual measures on the first lattice to measures on the second lattice. This permits the transportation of the box-crossing property from one lattice to the other. This general approach was introduced by Baxter and Enting [BE78] in a study of the Ising model, and has since been developed under the name Yang–Baxter equation, [McC10, PAY06].

Henceforth it is convenient to work with so-called *mixed lattices* that combine the square lattice with either the triangular or hexagonal lattice. We shall be precise about the manner in which a mixed lattice is embedded in $\mathbb{R}^2$. Let $i \in \mathbb{R}$, and let $I = \mathbb{R} \times \{i\}$ be the horizontal line of $\mathbb{R}^2$ with *height* i, called the *interface*; above I consider the triangular lattice and below I the square lattice. Our triangular lattice comprises equilateral triangles with side length $\sqrt{3}$, and our square lattice comprises rectangles whose horizontal (respectively, vertical) edges have length $\sqrt{3}$ (respectively, 1), as illustrated in the leftmost diagram of Figure 4.2.1. The embedding is specified up to horizontal translation and, in order to precise, we assume that the point $(0, i)$ is a vertex of the lattice. We call the ensuing graph the *mixed triangular lattice* $\mathbb{L}$ with interface $I = I_{\mathbb{L}}$.

The *mixed hexagonal lattice* $\mathbb{L}$ with interface $I = I_{\mathbb{L}}$ is similarly composed of a regular hexagonal lattice (of side length 1) above I and a square lattice below I (with edge-lengths as above), as drawn in the central diagram of Figure 4.2.1.

We define the *height* $h(A)$ of a subset $A \subseteq \mathbb{R}^2$ as the supremum of the y-coordinates of elements of A. A mixed lattice $\mathbb{L}$ may be identified with the subset of $\mathbb{R}^2$ belonging to its edge-set. Thus, for a mixed lattice $\mathbb{L}$, $h(I_{\mathbb{L}})$ is the height of its interface.

We next define two transformations, $T^{\triangle}$ and $T^{\triangledown}$ acting on a mixed triangular lattice $\mathbb{L}$.

(a) $T^{\triangle}$ transforms all upwards pointing triangles of $\mathbb{L}$ into stars, with centres at the circumcentres of the equilateral triangles.

(b) $T^{\triangledown}$ transforms all downwards pointing triangles into stars.

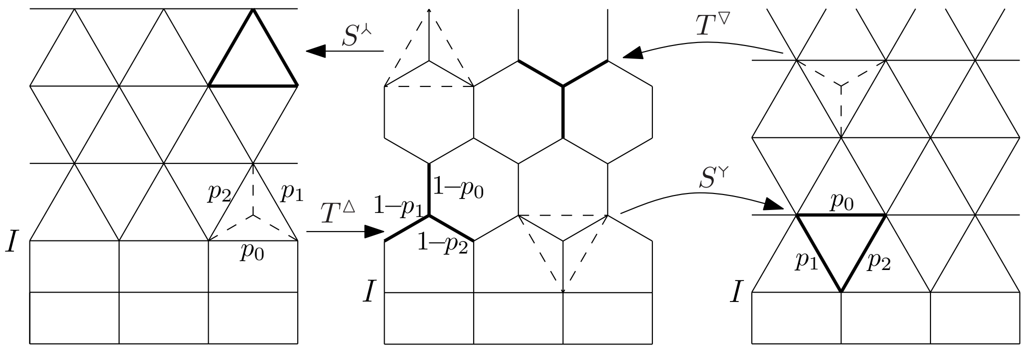

Figure 4.2.1: Transformations $S^{\curlywedge}$, $S^{\curlyvee}$, $T^{\vartriangle}$, and $T^{\triangledown}$ of mixed lattices. The transformations map the zones with dashes to the bold triangles/stars. The interface-height decreases by 1 from the leftmost to the rightmost graph.

It is easily checked (and illustrated in Figure 4.2.1) that each transformation maps a mixed triangular lattice to a mixed hexagonal lattice.

We define similarly the transformations $S^{\curlywedge}$ and $S^{\curlyvee}$ on a mixed hexagonal lattice; these transform all upwards (respectively, downwards) pointing stars into triangles. They transform a mixed hexagonal lattice to a mixed triangular lattice.

The concatenated operators $S^{\curlywedge} \circ T^{\triangledown}$ and $S^{\curlyvee} \circ T^{\vartriangle}$ map the mixed triangular lattice $\mathbb{L}$ to another mixed triangular lattice, but with a different interface height:

$$\begin{aligned} h(I_{S^{\curlywedge} \circ T^{\triangledown}\mathbb{L}}) &= h(I_{\mathbb{L}}) + 1, \\ h(I_{S^{\curlyvee} \circ T^{\vartriangle}\mathbb{L}}) &= h(I_{\mathbb{L}}) - 1. \end{aligned}$$

Loosely speaking, repeated application of $S^{\curlywedge} \circ T^{\triangledown}$ transforms $\mathbb{L}$ into the square lattice, while repeated application of $S^{\curlyvee} \circ T^{\vartriangle}$ transforms it into the triangular lattice.

We now extend the domains of the above maps to include configurations. Let $\mathbb{L} = (V, E)$ be a mixed triangular lattice with $\Omega_E = \{0,1\}^E$, and let $\omega \in \Omega_E$. The image of $\mathbb{L}$ under $T^{\vartriangle}$ is written $T^{\vartriangle}\mathbb{L} = (T^{\vartriangle}V, T^{\vartriangle}E)$ and we write $\Omega_{T^{\vartriangle}E} = \{0,1\}^{T^{\vartriangle}E}$. Let $\mathbf{p} \in [0,1)^3$ be self-dual. Let $T^{\vartriangle}(\omega)$ be chosen (randomly) from $\Omega_{T^{\vartriangle}E}$ by independent applications of the kernel T within every upwards pointing triangle of $\mathbb{L}$. Note that the random map T depends on the choice of $\mathbf{p}$.

By Proposition 3.2.2, for any two vertices A, B on $\mathbb{L}$, we have:

$$\left(A \xleftrightarrow{\mathbb{L},\omega} B\right) \Leftrightarrow \left(A \xleftrightarrow{T^{\vartriangle}\mathbb{L},T^{\vartriangle}(\omega)} B\right). \tag{4.2.1}$$

The corresponding statements for $T^{\triangledown}$, $S^{\curlywedge}$, and $S^{\curlyvee}$ are valid also, with one point of note. In applying the transformations $S^{\curlywedge}$, $S^{\curlyvee}$ to a mixed hexagonal lattice, the points A and B in the corresponding versions of (4.2.1) must not be centres of transformed stars, since these points disappear during the transformations.

Let $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ be self-dual, and let $S^{\curlywedge}$, $S^{\curlyvee}$, $T^{\triangle}$, $T^{\triangledown}$ be given accordingly. We identify next the probability measures on the mixed lattices that are preserved by the operation of these transformations.

Let $\mathbb{L} = (V, E)$ be a mixed (triangular or hexagonal) lattice. The probability measure denoted $\mathbb{P}_{\mathbf{p}}$ on Ω_E is product measure whose intensity $p(e)$ at edge e is given as follows.

(a) $p(e) = p_0$ if e is horizontal,

(b) $p(e) = 1 - p_0$ if e is vertical,

(c) $p(e) = p_1$ if e is the right edge of an upwards pointing triangle,

(d) $p(e) = p_2$ if e is the left edge of an upwards pointing triangle,

(e) $p(e) = 1 - p_2$ if e is the right edge of an upwards pointing star,

(f) $p(e) = 1 - p_1$ if e is the left edge of an upwards pointing star.

When it becomes necessary to emphasize the lattice $\mathbb{L}$ in question, we shall write $\mathbb{P}_{\mathbf{p}}^{\mathbb{L}}$.

Proposition 4.2.1. *If $\mathbf{p} \in [0,1)^3$ is self-dual in that $\kappa_{\triangle}(\mathbf{p}) = 0$, then $\mathbb{P}_{\mathbf{p}}$ is preserved by the transformations $S^{\curlywedge}$, $S^{\curlyvee}$, $T^{\triangle}$, and $T^{\triangledown}$. That is, if U is any of these four transformations acting on the mixed lattice $\mathbb{L} = (V, E)$, then*

$$\omega \in \Omega_E \text{ has law } \mathbb{P}_{\mathbf{p}}^{\mathbb{L}} \quad \Leftrightarrow \quad U(\omega) \text{ has law } \mathbb{P}_{\mathbf{p}}^{U\mathbb{L}}.$$

As in Section 3.2.3, the transformations $T^{\triangle}$, $T^{\triangledown}$, $S^{\curlywedge}$ and $S^{\curlyvee}$ may be extended to open paths. We view these transformations as dynamical modifications of open paths, hence we say a path *drifts* under the transformations.

Let ω be an edge-configuration on a mixed triangular lattice $\mathbb{L}$. Let γ be an ω-open path on $\mathbb{L}$, and consider the action of the map $T^{\triangle}$ (illustrated in Figure 4.2.2). The image lattice $T^{\triangle}\mathbb{L}$ is endowed with the edge-configuration $T^{\triangle}(\omega)$. The star–triangle transformations contributing to $T^{\triangle}$ act on disjoint parts of $\mathbb{L}$, hence we may define γ as the path obtained by the procedure described in Section 3.2.3 applied separately in each triangle affected by $T^{\triangle}$. We obtain thus a $T^{\triangle}(\omega)$-open path, which we denote $T^{\triangle}(\gamma)$. Note that $T^{\triangle}(\gamma)$ is equal to γ in the square part of $\mathbb{L}$ (excluding the interface) and has the same endpoints as γ. The same holds for $T^{\triangledown}$.

We turn now to a mixed hexagonal lattice $\mathbb{H}$ under the transformation $S^{\curlyvee}$ (the same argument holds for $S^{\curlywedge}$). Let ω be an edge-configuration on $\mathbb{H}$, and γ an open path. As before, through the construction of Section 3.2.3, we define a $S^{\curlyvee}(\omega)$-open path $S^{\curlyvee}(\gamma)$. The part of γ lying below the interface is not affected by $S^{\curlyvee}$, but if its endpoints are in the hexagonal part of $\mathbb{H}$, then they may drift under the action of $S^{\curlyvee}$.

An illustration of the transformations is given in Figure 4.2.2.

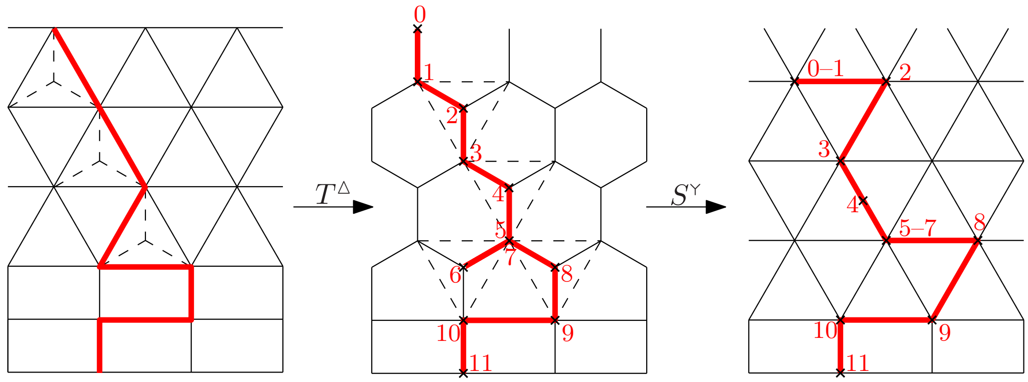

Figure 4.2.2: Transformations of lattice-paths. The transformation $T^{\triangle}$ acts deterministically on open paths, each edge of a triangle being transformed into two segments of an upwards pointing star. When applying $S^{\curlyvee}$, the segment labelled from 0 to 1 contracts to one point, as does that labelled from 5 to 7.

The following proposition acts as a (basic) control on the drift of open paths under the transformations $T^{\triangle}$, $T^{\triangledown}$, $S^{\curlywedge}$ and $S^{\curlyvee}$.

Proposition 4.2.2. *Let γ be an open path on a mixed lattice. We have that*

(a) $d_{path}(\gamma, T^{\triangle}(\gamma)) \leq \frac{1}{2}$ *and* $d_{path}(\gamma, T^{\triangledown}(\gamma)) \leq \frac{1}{2}$,

(b) $d_{path}(\gamma, S^{\curlywedge}(\gamma)) \leq 1$ *and* $d_{path}(\gamma, S^{\curlyvee}(\gamma)) \leq 1$,

(c) $d_{path}(\gamma, (S^{\curlywedge} \circ T^{\triangledown})(\gamma)) \leq 1$ *and* $d_{path}(\gamma, (S^{\curlyvee} \circ T^{\triangle})(\gamma)) \leq 1$,

whenever the transformations are matched to the mixed lattice.

Proof. This follows by examination of the different cases in the transformations, and is illustrated in Figures 3.2.4 and 4.2.2. □

4.3 Proof of Theorem 4.1.1 for $\mathcal{M}$

4.3.1 Outline of the proof

Theorem 4.1.1 for the inhomogeneous models in $\mathcal{M}$ is an immediate consequence of the following theorem. Recall that a triplet $\mathbf{p} \in [0,1)^3$ is *self-dual* if $\kappa_{\triangle}(\mathbf{p}) = 0$, with $\kappa_{\triangle}$ given in (1.4.2).

Theorem 4.3.1. *Let $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ be self-dual.*

(a) *If $\mathbb{P}^{\square}_{(p_0, 1-p_0)}$ has the box-crossing property, then so does $\mathbb{P}^{\triangle}_{\mathbf{p}}$.*

(b) *If $p_0 > 0$, and if $\mathbb{P}^{\triangle}_{\mathbf{p}}$ has the box-crossing property, then so does $\mathbb{P}^{\square}_{(p_0, 1-p_0)}$.*

(c) *$\mathbb{P}^{\triangle}_{\mathbf{p}}$ has the box-crossing property if and only if $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$ has it.*

Since $\mathbb{P}^{\square}_{(\frac{1}{2},\frac{1}{2})}$ has the box-crossing property, we have by Theorem 4.3.1(a) that $\mathbb{P}^{\triangle}_{(\frac{1}{2},p_1,p_2)}$ has the box-crossing property for all self-dual triplets $(\frac{1}{2},p_1,p_2)$. As $(\frac{1}{2},p_1,p_2)$ ranges within the set of self-dual triplets, p_1 ranges over the interval $[0,\frac{1}{2}]$. By Theorem 4.3.1(b), for all $p_1 \in (0,\frac{1}{2})$, $\mathbb{P}^{\square}_{(p_1,1-p_1)}$ has the box-crossing property. We then use Theorem 4.3.1(a) again to deduce that $\mathbb{P}^{\triangle}_{\mathbf{p}}$ has the box-crossing property for all self-dual triplets $\mathbf{p}$. Finally, the conclusion may be extended to the hexagonal lattice by Theorem 4.3.1(c).

Theorem 4.3.1(a, b) is proved in the remainder of this section. Part (c) is an immediate consequence of a single application of the star–triangle transformation, and no more will be said about this. We assume henceforth that all lattices are embedded in $\mathbb{R}^2$ in the style of Figure 4.2.1.

4.3.2 Specific notation

Before the proof of Theorem 4.3.1 it will be useful to introduce some notation specific to this chapter.

Let $G = (E, V)$ be a planar graph and let $\omega \in \Omega_E = \{0,1\}^E$. Let $\mathcal{C}_{\mathrm{h}}(m,n)$ (respectively, $\mathcal{C}_{\mathrm{v}}(m,n)$) be the event that there is an open horizontal (respectively, vertical) crossing of the box $B_{m,n} := [-m,m] \times [0,n]$ of $\mathbb{R}^2$. Suppose now that G is invariant under translation by the non-zero real vectors $(a,0)$ and $(0,b)$ for some least positive a and b. A probability measure P on Ω_E is called *translation-invariant* if it is invariant under the actions of these translations.

Lemma 4.3.2. *A translation–invariant, positively associated probability measure P on Ω_E has the box-crossing property if and only if the following hold for some N_0:*

(a) *For $\rho \geq 1$, there exists $\eta(\rho) > 0$ such that, for all $N \geq N_0$,*

$$P\left[\mathcal{C}_{\mathrm{h}}(\rho N, N)\right] \geq \eta(\rho). \tag{4.3.1}$$

(b) *There exist $\rho_0, \eta_0 > 0$ such that , for all $N \geq N_0$,*

$$P\left[\mathcal{C}_{\mathrm{v}}(N, \rho_0 N)\right] > \eta_0. \tag{4.3.2}$$

Moreover there exists $\delta = \delta(\rho_0, \eta_0, \eta(1), \eta(2\rho_0^{-1})) > 0$ and $N_1 \geq 0$ such that P has the box-crossing property $\mathrm{BXP}(N_1, \delta)$.

Remark 4.3.3. If the measure P of Proposition 4.3.2 is not translation–invariant, the proposition remains valid with (4.3.1)–(4.3.2) replaced by the same inequalities *uniformly* for all translates of the relevant rectangles.

Proof. This is sketched. It is trivial that the box-crossing property implies (4.3.1) and (4.3.2). Conversely, suppose (4.3.1) and (4.3.2) hold. The positive association permits

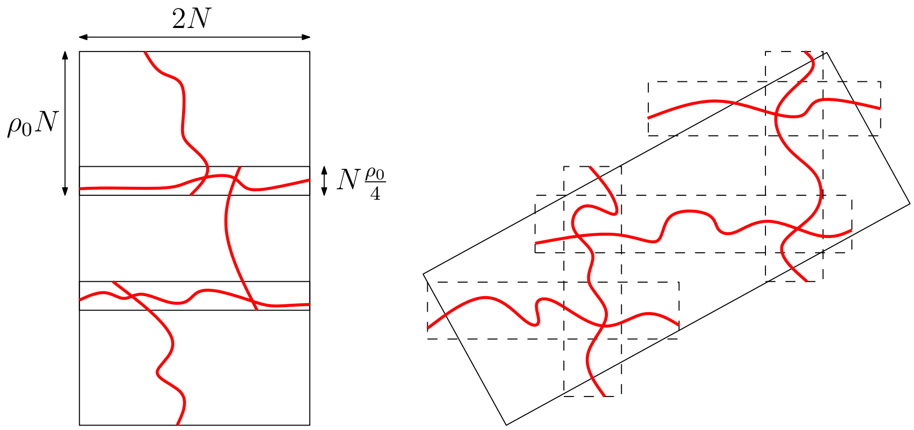

Figure 4.3.1: *Left:* Vertical crossings of copies of $B_{N,\rho_0 N}$ and horizontal crossings of copies of $B_{N,N\frac{\rho_0}{4}}$ may be combined to obtain vertical crossings of boxes with arbitrary aspect ratio. *Right:* Crossings of the type $\mathcal{C}_{\mathrm{h}}(\alpha n, n)$ and $\mathcal{C}_{\mathrm{v}}(n, \alpha n)$ may be combined to obtain crossings of boxes with general inclination.

the combination of box-crossings to obtain crossings of larger boxes. The claim is now obtained as illustrated in Figure 4.3.1. □

4.3.3 Proof of Theorem 4.3.1(a)

It suffices to assume $p_0 > 0$, since the hypothesis does not hold when $p_0 = 0$. By Proposition 4.3.2, it suffices to prove the following two propositions.

Proposition 4.3.4. *Let $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ be self-dual with $p_0 > 0$. For $\alpha > 1$ and $N \in \mathbb{N}$,*

$$\mathbb{P}^{\triangle}_{\mathbf{p}}[\mathcal{C}_{\mathrm{h}}((\alpha-1)N, 2N)] \geq \mathbb{P}^{\square}_{(p_0,1-p_0)}[\mathcal{C}_{\mathrm{h}}(\alpha N, N)].$$

Proposition 4.3.5. *Let $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ be self-dual with $p_0 > 0$. There exist $\beta = \beta(p_0) > 0$, and $\rho_N = \rho_N(\beta) > 0$ satisfying $\rho_N \to 1$ as $N \to \infty$, such that*

$$\mathbb{P}^{\triangle}_{\mathbf{p}}[\mathcal{C}_{\mathrm{v}}(2N, \beta N)] \geq \rho_N \mathbb{P}^{\square}_{(p_0,1-p_0)}[\mathcal{C}_{\mathrm{v}}(N, N)], \quad N \in \mathbb{N}.$$

The constant β is given by

$$\beta := \frac{1 - \sqrt{1 - p_0(1-p_0)}}{1 - p_0}, \tag{4.3.3}$$

and $\rho_N = \rho_N(\beta)$ may be calculated explicitly by the final argument of this subsection.

Proof of Proposition 4.3.4. Let $\mathbf{p} \in [0,1)^3$ be self-dual with $p_0 > 0$, and let $\alpha > 1$ and

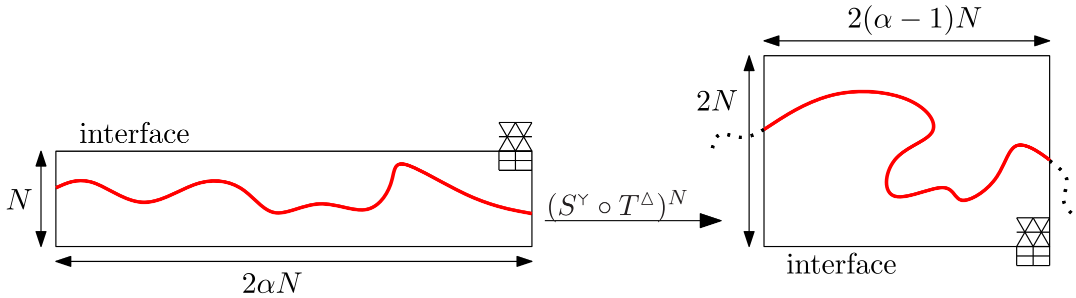

Figure 4.3.2: Transformation of a horizontal crossing of $B_{\alpha N,N}$ by $(S^{\curlyvee} \circ T^{\triangle})^N$. The interface moves down N steps. The path drifts by at most distance N and cannot go below the interface of the image lattice.

$N \in \mathbb{N}$. Let $\mathbb{L} = (V, E)$ be a mixed triangular lattice with interface-height $h(I_{\mathbb{L}}) = N$, and write $\mathbb{P}_{\mathbf{p}}$ for the associated product measure on $\mathbb{L}$. Since $B_{\alpha N,N} = [-\alpha N, \alpha N] \times [0, N]$ is beneath the interface,

$$\mathbb{P}^{\square}_{(p_0,1-p_0)}\left[\mathcal{C}_{\mathrm{h}}(\alpha N, N)\right] = \mathbb{P}^{\mathbb{L}}_{\mathbf{p}}\left[\mathcal{C}_{\mathrm{h}}(\alpha N, N)\right].$$

Let $\omega \in \mathcal{C}_{\mathrm{h}}(\alpha N, N)$. We claim that there exists a horizontal open crossing of $B_{(\alpha-1)N,2N}$ in $(S^{\curlyvee} \circ T^{\triangle})^N(\omega)$, as illustrated in Figure 4.3.2.

Let γ be an open path of $\mathbb{L}$, parametrized by $[0,1]$, that crosses $B_{\alpha N,N}$ horizontally. By Proposition 4.2.2, $d_{path}(\gamma, \gamma(N)) \leq N$ where $\gamma(N) := (S^{\curlyvee} \circ T^{\triangle})^N(\gamma)$, whence,

$$|\gamma_0 - \gamma(N)_0| \leq N, \tag{4.3.4}$$

$$|\gamma_1 - \gamma(N)_1| \leq N, \tag{4.3.5}$$

$$\gamma(N) \subseteq \gamma^N \subseteq B^N_{\alpha N,N}. \tag{4.3.6}$$

Since γ contains no vertex with strictly negative y-coordinate, and the transformations do not act in this region, neither does $\gamma(N)$. Hence,

$$\gamma(N) \subseteq \gamma^N \cap \mathbb{R} \times [0, \infty) \subset \mathbb{R} \times [0, 2N].$$

Taken with (4.3.4)–(4.3.5), we deduce that $\gamma(N)$ contains an open path γ' that crosses $B_{(\alpha-1)N,2N}$ in the horizontal direction.

Since $B_{(\alpha-1)N,2N}$ lies entirely in the triangular part of $(S^{\curlyvee} \circ T^{\triangle})^N\mathbb{L}$, we have by Proposition 4.2.1 that

$$\begin{aligned}\mathbb{P}^{\mathbb{L}}_{\mathbf{p}}\left[\mathcal{C}_{\mathrm{h}}(\alpha N, N)\right] &\leq \mathbb{P}^{(S^{\curlyvee} \circ T^{\triangle})^N\mathbb{L}}_{\mathbf{p}}\left[\mathcal{C}_{\mathrm{h}}((\alpha-1)N, 2N)\right] \\ &= \mathbb{P}^{\triangle}_{\mathbf{p}}\left[\mathcal{C}_{\mathrm{h}}((\alpha-1)N, 2N)\right],\end{aligned}$$

and the proposition is proved. □

Proof of Proposition 4.3.5. Consider the box $B_{N,N}$ in the mixed triangular lattice $\mathbb{L}$ with interface-height $h(I_{\mathbb{L}}) = N$. We follow the strategy of the previous proof by considering the action of $S^{\curlyvee} \circ T^{\triangle}$ on a vertical open crossing γ of the box. In N applications of $S^{\curlyvee} \circ T^{\triangle}$, the lattice within the box is transformed from square to triangular. By Proposition 4.2.2(c), the image of γ may drift by distance 1 or less at each step. Drift of γ in the horizontal direction can be accommodated within a box that is wider in that direction. Vertical drift is however more troublesome. Whereas the lower endpoint of γ is unchanged by N applications of $S^{\curlyvee} \circ T^{\triangle}$, its upper endpoint may be reduced in height by 1 at each such application. If this were to occur at every application, both endpoints of the final path would be on the x-axis. This possibility will be controlled by proving that the downward velocity of the upper endpoint is strictly less than 1.

Let $\mathbf{p} \in [0,1)^3$ be self-dual with $p_0 > 0$, and write $\mathbb{L}^k = (S^{\curlyvee} \circ T^{\triangle})^k \mathbb{L}$ for $0 \le k \le N$. The lattice $\mathbb{L}^k$ has edge-set E^k and configuration space $\Omega^k = \{0,1\}^{E^k}$. Let $\mathbb{P}^k_{\mathbf{p}}$ denote the probability measure on Ω^k given before Proposition 4.2.1. Recall from that proposition that $S^{\curlyvee} \circ T^{\triangle}$ acts as a *random* mapping from Ω^k to Ω^{k+1}, via the 'kernel' given in Figure 3.2.2. We shall assume that sequential applications of this kernel are independent of one another and of the choice of initial configuration. More specifically, let $(\omega^k : k \ge 0)$ satisfy:

(a) ω^k is a random configuration from Ω^k,

(b) the sequence $(\omega^k : k \ge 0)$ has the Markov property,

(c) given $\{\omega^0, \omega^1, \dots, \omega^k\}$, ω^{k+1} may be expressed as $\omega^{k+1} = S^{\curlyvee} \circ T^{\triangle}(\omega^k)$,

(d) the law of ω^0 is $\mathbb{P}^0_{\mathbf{p}}$.

Let $\mathbb{P}$ denote the joint law of the sequence $(\omega^0, \omega^1, \dots)$. By Proposition 4.2.1, the law of ω^k is $\mathbb{P}^k_{\mathbf{p}}$.

Let $D^k = B_{N+k,\infty} = [-N-k, N+k] \times [0,\infty)$ viewed as a subgraph of $\mathbb{L}^k$, and call the line $\mathbb{R} \times \{0\}$ the *base* of $\mathbb{R}^2$. We shall work with the sequence $(h^k : 1 \le k \le N)$ of random variables given by

$$h^k := \sup\big\{h : \exists x_1, x_2 \in \mathbb{R} \text{ with } (x_1, 0) \xleftrightarrow{D^k, \omega^k} (x_2, h)\big\}.$$

Note that h^k acts on Ω^k.

Since $\mathbb{L}^N$ is entirely triangular in the upper half-plane, it suffices to show the existence of $\rho_N = \rho_N(\beta) > 0$ such that $\rho_N \to 1$ and

$$\mathbb{P}(h^N \ge \beta N) \ge \rho_N \mathbb{P}(h^0 \ge N), \tag{4.3.7}$$

with β as in (4.3.3). The remainder of this subsection is devoted to proving this.

Lemma 4.3.6. *For $0 \le k < N$, the following two statements hold:*

$$h^{k+1} \ge h^k - 1, \tag{4.3.8}$$

$$\mathbb{P}(h^{k+1} \ge h + \tfrac{1}{2} \mid h^k = h) \ge \beta, \quad h \ge 0. \tag{4.3.9}$$

Proof. We may assume that $h^k < \infty$ for $0 \le k \le N$, since the converse has zero probability. Let $k < N$, and let $\gamma^k = \gamma^k(\omega^k)$ be the leftmost path in D^k that reaches some point at height h^k. By Proposition 4.2.2(c), $\mathbb{L}^{k+1}$ possesses an open vertical crossing of B_{N+k+1,h^k-1}, so that $h^{k+1} \ge h^k - 1$. Inequality (4.3.8) is proved, and we turn to (4.3.9).

Let $0 \le k < N$, and let $\mathcal{G}$ be the set of all paths Γ of $\mathbb{L}^k$ such that there exists $h > 0$ with:

(a) all vertices of Γ lie in $B_{N+k,h}$,

(b) Γ has one endpoint (denoted Γ_0) in $\mathbb{R} \times \{0\}$,

(c) its other endpoint (denoted Γ_1) lies in $\mathbb{R} \times \{h\}$.

For $\Gamma \in \mathcal{G}$, there is a unique such h, denoted $h(\Gamma)$.

Let $\Gamma \in \mathcal{G}$, and let $L(\Gamma)$ be the closed sub-region of $[-N-k, N+k] \times [0, h(\Gamma)] \subseteq \mathbb{R}^2$ lying 'to the left' of Γ. Let $\mathcal{G}(\Gamma)$ be the subset of $\mathcal{G}$ containing all paths Γ' with $h(\Gamma') = h(\Gamma)$ and $\Gamma' \subseteq L(\Gamma)$. We write $\Gamma' < \Gamma$ if $\Gamma' \subseteq L(\Gamma)$ and $\Gamma' \ne \Gamma$.

Suppose that $p_1 \le p_2$. The endpoint Γ_1 is the lower *left* corner of some upwards pointing triangle denoted $ABC = ABC(\Gamma)$, where $A = \Gamma_1$ and O is its centre. If $p_2 > p_1$, we work instead with the similar triangle of which Γ_1 is the lower *right* corner, and the ensuing argument is exactly similar. See Figure 4.3.3.

We claim that

$$\mathbb{P}(BC \text{ is } \omega^k\text{-closed} \mid \gamma^k = \Gamma) \ge 1 - p_1, \quad \Gamma \in \mathcal{G}. \tag{4.3.10}$$

Since the marginal of $\mathbb{P}$ on Ω^k is $\mathbb{P}^k_{\mathbf{p}}$, it suffices to show that

$$\mathbb{P}^k_{\mathbf{p}}(BC \text{ closed} \mid \gamma^k = \Gamma) \ge 1 - p_1, \quad \Gamma \in \mathcal{G}. \tag{4.3.11}$$

This is proved as follows. Let $\Gamma \in \mathcal{G}$. Then $\{\gamma^k = \Gamma\} = F \cap G \cap \{\Gamma \text{ open}\}$ where F is the event that there exists no $\Gamma' < \Gamma$ such that every edge of $\Gamma' \setminus \Gamma$ is open, and G is the event that there exists no $\Gamma'' \in \mathcal{G}$ with $h(\Gamma'') > h(\Gamma)$ and every edge of $\Gamma'' \setminus \Gamma$ is open. Since $F \cap G$ is a decreasing event that is independent of the states of edges in Γ, we have by the positive association of $\mathbb{P}^k_{\mathbf{p}}$ that

$$\begin{aligned}\mathbb{P}^k_{\mathbf{p}}(\gamma^k = \Gamma \mid BC \text{ closed}) &= \mathbb{P}^k_{\mathbf{p}}(\Gamma \text{ open})\mathbb{P}^k_{\mathbf{p}}(F \cap G \mid BC \text{ closed})\\ &\ge \mathbb{P}^k_{\mathbf{p}}(\Gamma \text{ open})\mathbb{P}^k_{\mathbf{p}}(F \cap G) = \mathbb{P}^k_{\mathbf{p}}(\gamma^k = \Gamma).\end{aligned}$$

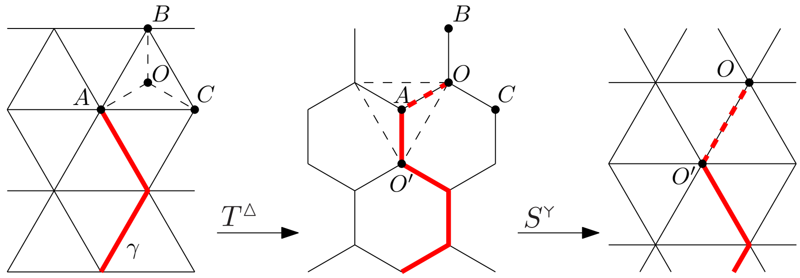

Figure 4.3.3: An illustration of the action of $S^{\curlyvee} \circ T^{\triangle}$ when $\gamma^k = \Gamma$. The top endpoint A of Γ is preserved under $T^{\triangle}$. If $\omega^k(BC) = 0$, there is a strictly positive probability that AO is open in $T^{\triangle}(\omega^k)$, in which case $h^{k+1} \geq h^k + \frac{1}{2}$.

Therefore,

$$\begin{aligned}\mathbb{P}^k_{\mathbf{p}}(BC \text{ closed} \mid \gamma^k = \Gamma) &= \mathbb{P}^k_{\mathbf{p}}(\gamma^k = \Gamma \mid BC \text{ closed}) \frac{\mathbb{P}^k_{\mathbf{p}}(BC \text{ closed})}{\mathbb{P}^k_{\mathbf{p}}(\gamma^k = \Gamma)} \\ &\geq \mathbb{P}^k_{\mathbf{p}}(BC \text{ closed}) = 1 - p_1,\end{aligned}$$

and (4.3.10) is proved.

Consider the state of the edge AO in the configuration $T^{\triangle}(\omega^k)$. By Figure 3.2.2, for any $\omega \in \Omega^k$ with $\omega(BC) = 0$,

$$\mathbb{P}^k_{\mathbf{p}}\big(AO \text{ open in } T^{\triangle}(\omega) \,\big|\, \omega^k = \omega\big) \geq \frac{p_0 p_2}{(1-p_0)(1-p_2)}.$$

It follows that

$$\mathbb{P}\big(h^{k+1} \geq h^k + \tfrac{1}{2} \,\big|\, \omega^k = \omega\big) \geq \frac{p_0 p_2}{(1-p_0)(1-p_2)} 1_{\{\omega(BC)=0\}}, \quad \omega \in \Omega^k.$$

Recall that $BC = BC(\gamma^k(\omega))$. Therefore, for $\Gamma \in \mathcal{G}$,

$$\begin{aligned}\mathbb{P}\big(h^{k+1} \geq h^k + \tfrac{1}{2} \,\big|\, \gamma^k = \Gamma\big) &\geq \frac{p_0 p_2}{(1-p_0)(1-p_2)} \mathbb{P}(\omega^k(BC) = 0 \mid \gamma^k = \Gamma) \\ &\geq \frac{(1-p_1) p_0 p_2}{(1-p_0)(1-p_2)},\end{aligned}$$

by (4.3.10).

Now p_0 is fixed, $p_1 \leq p_2$, and $\kappa_{\triangle}(\mathbf{p}) = 0$. Hence, the last ratio is a minimum when $p_1 = p_2$, whence

$$\frac{(1-p_1) p_0 p_2}{(1-p_0)(1-p_2)} \geq \frac{1 - \sqrt{1 - p_0(1-p_0)}}{1 - p_0} = \beta,$$

and the claim of the lemma follows. □

There are at least two ways to complete the proof of Proposition 4.3.5, of which one involves controlling the mean of $h^{k+1} - h^k$. We take a second route here, via a small standard lemma. For a real-valued discrete random variable X, we write $\mathcal{L}(X)$ for its law, and $\mathcal{S}(X) := \{x \in \mathbb{R} : P(X = x) > 0\}$ for its *support*. The inequality $\leq_{\text{st}}$ denotes stochastic domination.

Lemma 4.3.7. *Let (X_0, X_1) and (Y_0, Y_1) be pairs of real-valued discrete random variables such that:*

(a) $X_0 \leq_{\text{st}} Y_0$,

(b) *for $x \in \mathcal{S}(X_0)$, $y \in \mathcal{S}(Y_0)$ with $x \leq y$, the conditional laws of X_1 and Y_1 satisfy $\mathcal{L}(X_1 \mid X_0 = x) \leq_{\text{st}} \mathcal{L}(Y_1 \mid Y_0 = y)$.*

Then $X_1 \leq_{\text{st}} Y_1$.

Proof. We include a proof for completeness. By Strassen's Theorem (see [Lin02b, Sect. IV.1]), there exists a probability space and two random variables X_0', Y_0', distributed respectively as X_0 and Y_0, such that $P(X_0' \leq Y_0') = 1$. Now,

$$
\begin{aligned}
P(X_1 > u) &= \sum_{x \leq y} P(X_1 > u \mid X_0 = x) P(X_0' = x, Y_0' = y) \\
&\leq \sum_{x \leq y} P(Y_1 > u \mid Y_0 = y) P(X_0' = x, Y_0' = y) \\
&= P(Y_1 > u),
\end{aligned}
$$

where the summations are restricted to $x \in \mathcal{S}(X_0)$ and $y \in \mathcal{S}(Y_0)$. □

Let $(H^k : k \geq 0)$ be a Markov process with $H^0 = h^0$ and transition probabilities

$$
P(H^{k+1} = j \mid H^k = i) = \begin{cases} \beta & \text{if } j = i + \frac{1}{2}, \\ 1 - \beta & \text{if } j = i - 1, \end{cases} \tag{4.3.12}
$$

with β as above. By Lemma 4.3.6 and an iterative application of Lemma 4.3.7,

$$
\mathbb{P}(h^N \geq \beta N) \geq P(H^N \geq \beta N).
$$

Since h^0 and H^0 have the same distribution,

$$
\begin{aligned}
\frac{\mathbb{P}(h^N \geq \beta N)}{\mathbb{P}(h^0 \geq N)} &\geq \frac{P(H^N \geq \beta N)}{P(H^0 \geq N)} \\
&\geq P(H^N \geq \beta N \mid H^0 \geq N) =: \rho_N(\beta).
\end{aligned}
$$

Now, (H_k) is a random walk with mean step-size $-1 + 3\beta/2$. By the law of large numbers, $\rho_N \to 1$ as $N \to \infty$. In addition, $\rho_N > 0$, and (4.3.7) follows. □

4.3.4 Proof of Theorem 4.3.1(b)

By Proposition 4.3.2, it suffices to prove the following two propositions.

Proposition 4.3.8. *Let $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ be self-dual with $p_0 > 0$. There exists $\beta = \beta(p_0) \in \mathbb{N}$ and $N_0 = N_0(p_0) \in \mathbb{N}$ such that, for $\alpha \in \sqrt{3}\mathbb{N}$ with $\alpha > \beta$, and $N \geq N_0$,*

$$\mathbb{P}^{\square}_{(p_0,1-p_0)}\big[\mathcal{C}_{\mathrm{h}}((\alpha-\beta)N, \beta N)\big] \geq (1-\alpha e^{-N})\mathbb{P}^{\triangle}_{\mathbf{p}}[\mathcal{C}_{\mathrm{h}}(\alpha N, N)]. \tag{4.3.13}$$

Proposition 4.3.9. *Let $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ be self-dual. For $\alpha > 0$ and $N \in 2\mathbb{N}$,*

$$\mathbb{P}^{\square}_{(p_0,1-p_0)}\big[\mathcal{C}_{\mathrm{v}}((\alpha+\tfrac{1}{2})N, \tfrac{1}{2}N)\big] \geq \mathbb{P}^{\triangle}_{\mathbf{p}}[\mathcal{C}_{\mathrm{v}}(\alpha N, N)]. \tag{4.3.14}$$

Proof of Proposition 4.3.8. Let $\mathbf{p}$ satisfy the hypothesis of the proposition. The idea is to consider repeated applications of the transformation $S^{\curlywedge} \circ T^{\triangledown}$ to an open horizontal crossing of a box in the triangular part of a mixed lattice. The interface moves upwards, and the crossing may 'drift' upwards at each step. A new technique is required to control the rate of this drift. This will be achieved by bounding the vertical displacement of the path by a certain growth process.

We partition the plane into vertical columns

$$\mathcal{C}_n = \big(n\sqrt{3}, (n+1)\sqrt{3}\big) \times \mathbb{R}, \quad n \in \mathbb{Z},$$

of width $\sqrt{3}$. Let $\mathbb{L} = (V, E)$ be a mixed lattice, and $\omega \in \Omega_E$. The $\mathcal{C}_n$ correspond to the columns of the square sublattice of $\mathbb{L}$, as illustrated in Figure 4.3.4.

For any (parametrized) open path $\Gamma = (\Gamma_t : a \leq t \leq b)$ on $\mathbb{L}$, let

$$H_n(\Gamma) = \sup\big\{h(\Gamma_t) : t \text{ such that } \Gamma_t \in \mathcal{C}_n\big\}$$

be its *height* in $\mathcal{C}_n$. (The supremum of the empty set is taken to be $-\infty$.) Note that $h(\Gamma) = \sup_n H_n(\Gamma)$. The growth of the $H_n(\Gamma)$ may be bounded as follows under the action of the random map $S^{\curlywedge} \circ T^{\triangledown}$.

For future use, we define $\eta : (0,1) \to (0,1)$ by

$$\eta(x) = \Big(1 + x - \sqrt{1 - x + x^2}\Big)^2, \tag{4.3.15}$$

and note that η is increasing.

Lemma 4.3.10. *Let $\mathbb{L}$ be a mixed triangular lattice, and let ω, Γ be as above. There exists a family of independent Bernoulli random variables $(Y_n : n \in \mathbb{Z})$ with parameter $1 - \eta(p_0)$, such that, for all $n \in \mathbb{Z}$,*

$$H_n\big((S^{\curlywedge} \circ T^{\triangledown})(\Gamma)\big) \leq \max\big\{H_{n-1}(\Gamma), H_n(\Gamma) + Y_n, H_{n+1}(\Gamma)\big\}.$$

We delay the proof of this lemma until later in this subsection.

Let $\mathbb{L}^0 = (V, E)$ be the mixed triangular lattice with interface-height $h(I_{\mathbb{L}^0}) = 0$, and let $\omega^0 \in \Omega_E$. Let $\alpha \in \sqrt{3}\mathbb{N}$, and let γ^0 be an open path of $\mathbb{L}^0$ in the box $B_{\alpha N,N}$. We shall use the notation introduced at the start of the proof of Proposition 4.3.5, with the difference that the transformation $S^{\curlyvee} \circ T^{\vartriangle}$ there is replaced here by $S^{\curlywedge} \circ T^{\triangledown}$. Thus, $\mathbb{L}^k = (S^{\curlywedge} \circ T^{\triangledown})^k \mathbb{L}$, and ω^k is the edge-configuration on $\mathbb{L}^k$ given by $\omega^k = S^{\curlywedge} \circ T^{\triangledown}(\omega^{k-1})$ for $k \geq 1$. Recall that ω^k is a random function of ω^{k-1} generated via the kernel of Figure 3.2.2, and we assume as before that sequential applications of this kernel are independent. We shall study the heights of the image paths $\gamma^k = (S^{\curlywedge} \circ T^{\triangledown})^k(\gamma^0)$.

As before, if ω^0 is chosen according to $\mathbb{P}^0_{\mathbf{p}}$, then the law of ω^k is $\mathbb{P}^k_{\mathbf{p}}$. The law of the sequence $(\omega^k : k \geq 0)$ is written $\mathbb{P}$, although for the moment we take ω^0 to be fixed and write $\mathbb{P}(\cdot \mid \omega^0)$ for the corresponding conditional measure.

We shall show that the speed of growth of the maximal height of γ^k is strictly less than 1. This will be proved by constructing a certain growth process that dominates (stochastically) the family $(H_n(\gamma^k) : n \in \mathbb{Z},\, k \geq 0)$.

Let $\zeta \in (0, 1)$. Let $(Y^k_n : n \in \mathbb{Z},\, k \geq 0)$ be a family of independent Bernoulli random variables with parameter $1 - \zeta$. The Markov process $\mathbf{X}^k := (X^k_n : n \in \mathbb{Z})$ is given as follows.

(a) The initial value $\mathbf{X}^0$ is given by

$$X^0_n = \begin{cases} N & \text{for } n \in [-\alpha N/\sqrt{3}, \alpha N/\sqrt{3}], \\ -\infty & \text{for } n \notin [-\alpha N/\sqrt{3}, \alpha N/\sqrt{3}]. \end{cases}$$

(b) For $k \geq 0$, conditional on $\mathbf{X}^k$, the vector $\mathbf{X}^{k+1}$ is given by

$$X^{k+1}_n = \max\{X^k_{n-1}, X^k_n + Y^k_n, X^k_{n+1}\}, \quad n \in \mathbb{Z}.$$

Lemma 4.3.11. *Let $\zeta \in (0, 1)$. There exist $\beta, N_0 \in \mathbb{N}$ depending on ζ only (independent of α, N) such that, for $\alpha \in \sqrt{3}\mathbb{N}$ and $N \geq N_0$,*

$$P\left(\max_n X^{\beta N}_n \leq \beta N\right) \geq 1 - \alpha e^{-N}.$$

We postpone the proof of this lemma, first completing that of Proposition 4.3.8. Let $\zeta = \eta(p_0)$, and let β and N_0 be given as in Lemma 4.3.11. Since $H_n(\gamma^0) \leq X^0_n$ for all n, we have by Lemma 4.3.10 that, given ω^0, $h(\gamma^k)$ is dominated stochastically by $\max_n X^k_n$. By Lemma 4.3.11,

$$\mathbb{P}\big(h(\gamma^{\beta N}) \leq \beta N \,\big|\, \omega^0\big) \geq 1 - \alpha e^{-N}, \quad N \geq N_0. \tag{4.3.16}$$

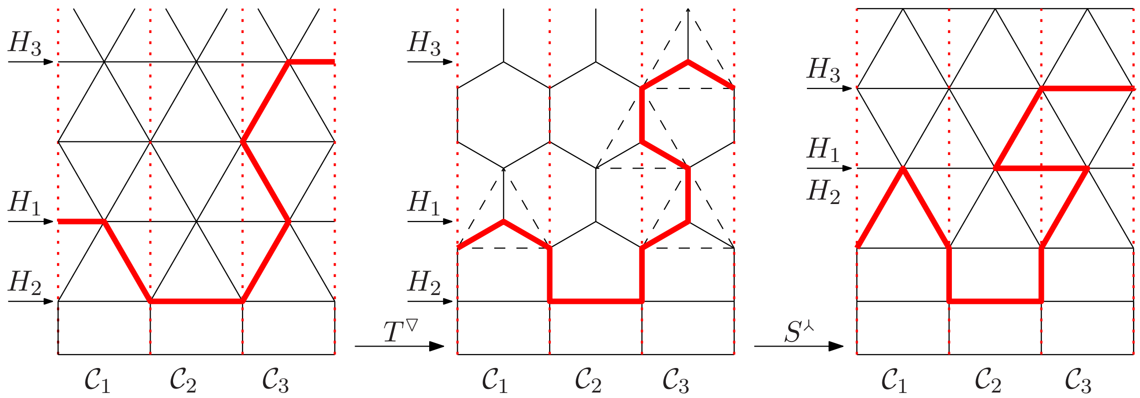

Figure 4.3.4: The evolution of the heights of a crossing within columns, when applying $T^\triangledown$ and $S^\curlywedge$. The heights in each column are the same in the first and second lattice. In the third: H_1 increases by 1; H_2 increases by 2; H_3 does not change.

Since $h(I_{\mathbb{L}^0}) = 0$ and $h(I_{\mathbb{L}^N}) = N$,

$$\mathbb{P}^{\triangle}_{\mathbf{p}}[\mathcal{C}_{\mathrm{h}}(\alpha N, N)] = \mathbb{P}\big(\omega^0 \in \mathcal{C}_{\mathrm{h}}(\alpha N, N)\big),$$
$$\mathbb{P}^{\square}_{(p_0,1-p_0)}[\mathcal{C}_{\mathrm{h}}((\alpha-\beta)N, \beta N)] = \mathbb{P}\big(\omega^{\beta N} \in \mathcal{C}_{\mathrm{h}}((\alpha-\beta)N, \beta N)\big).$$

Hence,

$$\frac{\mathbb{P}^{\square}_{(p_0,1-p_0)}[\mathcal{C}_{\mathrm{h}}((\alpha-\beta)N, \beta N)]}{\mathbb{P}^{\triangle}_{p}[\mathcal{C}_{\mathrm{h}}(\alpha N, N)]} \geq \mathbb{P}\big[\omega^{\beta N} \in \mathcal{C}_{\mathrm{h}}((\alpha-\beta)N, \beta N) \,\big|\, \omega^0 \in \mathcal{C}_{\mathrm{h}}(\alpha N, N)\big]. \quad (4.3.17)$$

Let $\omega^0 \in \mathcal{C}_{\mathrm{h}}(\alpha N, N)$ and let γ^0 be an ω^0-open crossing of $B_{\alpha N,N}$. By Proposition 4.2.2, the leftmost point of $\gamma^{\beta N}$ lies to the left of $B_{(\alpha-\beta)N,\beta N}$, and the rightmost point to the right of that box. Moreover $\gamma^{\beta N}$ is contained in the upper half-plane, since the lower half-plane is in the square-lattice part of every $\mathbb{L}^k$. If, in addition, $h(\gamma^{\beta N}) \leq \beta N$, then $\gamma^{\beta N}$ contains a $\omega^{\beta N}$-open horizontal crossing of $B_{(\alpha-\beta)N,\beta N}$. In conclusion,

$$\begin{aligned}\mathbb{P}\Big(\omega^{\beta N} \in \mathcal{C}_{\mathrm{h}}((\alpha-\beta)N, \beta N) \,\Big|\, \omega^0 \in \mathcal{C}_{\mathrm{h}}(\alpha N, N)\Big)& \\ \geq \mathbb{P}\left(h(\gamma^{\beta N}) \leq \beta N \,\middle|\, \omega^0 \in \mathcal{C}_{\mathrm{h}}(\alpha N, N)\right)& \\ \geq 1 - \alpha e^{-N}, \qquad N \geq N_0,&\end{aligned}$$

by (4.3.16). The claim follows by (4.3.17). $\square$

Proof of Lemma 4.3.10. We recall two properties of the transformations $S^\curlywedge$ and $T^\triangledown$ when applied to an ω-open path Γ. In constructing $T^\triangledown(\Gamma)$, we apply $T^\triangledown$ to downwards pointing triangles of $\mathbb{L}$ containing either one or two edges of Γ. As illustrated in Figure 3.2.2, $T^\triangledown$

acts deterministically on such triangles, and hence $T^\triangledown(\Gamma)$ is specified by knowledge of Γ. By inspection of Figure 4.3.4 or otherwise,

$$H_n(T^\triangledown(\Gamma)) = H_n(\Gamma), \quad n \in \mathbb{Z}. \tag{4.3.18}$$

The situation is less simple when applying $S^\curlywedge$ to $T^\triangledown(\Gamma)$. Let $\mathcal{S}$ be the set of upwards pointing stars of $T^\triangledown\mathbb{L}$, and let $(Z_\mathrm{l}^s, Z_\mathrm{r}^s : s \in \mathcal{S})$ be independent Bernoulli random variables with parameter

$$\nu := \sqrt{1-\nu_0} \quad \text{where} \quad \nu_0 := 1 - \frac{p_1p_2}{(1-p_1)(1-p_2)}.$$

For $s \in \mathcal{S}$, let $\underline{Z}^s = \min\{Z_\mathrm{l}^s, Z_\mathrm{r}^s\}$, noting that

$$P(\underline{Z}^s = 1) = \nu^2 = 1 - \nu_0. \tag{4.3.19}$$

We call $s \in \mathcal{S}$ a *horizontal star* (for Γ) if $T^\triangledown(\Gamma)$ includes the two non-vertical edges of s.

By (4.3.18), any changes in the H_n occur only when applying $S^\curlywedge$. The height $H_n(\Gamma)$ may grow under the application of $S^\curlywedge \circ T^\triangledown$ for either of two reasons: (i) the highest part of Γ within $\mathcal{C}_n$ may move upwards, or (ii) part of Γ in a neighbouring column may drift into $\mathcal{C}_n$ (in which case, we say it 'invades' $\mathcal{C}_n$). These two possibilities will be considered separately.

Let $n \in \mathbb{Z}$. Assume first that

$$H_n(\Gamma) \le \max\bigl\{H_{n-1}(\Gamma), H_{n+1}(\Gamma)\bigr\} - 1. \tag{4.3.20}$$

By Proposition 4.2.2, the part of Γ within $\mathcal{C}_n$ cannot drift upwards by more than 1. By considering the ways in which parts of Γ may invade $\mathcal{C}_n$, we find that such invasions may occur only horizontally, and not diagonally upwards (see Figure 4.3.4). Combining these two observations, we deduce under (4.3.20) that

$$H_n(S^\curlywedge \circ T^\triangledown(\Gamma)) \le \max\bigl\{H_{n-1}(\Gamma), H_{n+1}(\Gamma)\bigr\}. \tag{4.3.21}$$

Suppose next that

$$H_n(\Gamma) \ge \max\bigl\{H_{n-1}(\Gamma), H_{n+1}(\Gamma)\bigr\}. \tag{4.3.22}$$

By Proposition 4.2.2, (4.3.18), and the above remark concerning invasion,

$$H_n(S^\curlywedge \circ T^\triangledown(\Gamma)) \le H_n(\Gamma') + 1 = H_n(\Gamma) + 1,$$

where $\Gamma' = T^\triangledown(\Gamma)$. Assume that $H_n(S^\curlywedge(\Gamma')) = H_n(\Gamma') + 1$. Then there must exist a star $s \in \mathcal{S}$ such that:

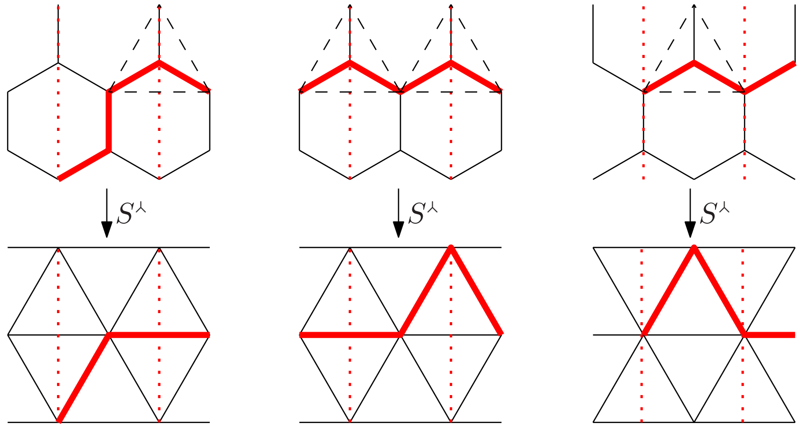

Figure 4.3.5: Three examples of growth of path-height within a column under the action of $S^{\curlywedge}$, under the assumption $H_n(\Gamma') \geq \max\{H_{n-1}(\Gamma'), H_{n+1}(\Gamma')\}$. *Left:* The base of the marked triangle is present in the image, and the height does not increase. *Middle*: The base of the rightmost marked triangle is absent. The heights in the central and right columns increase. There is a strictly positive probability that both marked bases are present, and that the height in the central column does not increase. *Right:* The base of the marked triangle is absent, and the height increases by 1.

(a) s is a horizontal star for Γ,

(b) s intersects $\mathcal{C}_n$,

(c) $H_n(T^{\triangledown}(\Gamma)) = h(O)$ where O is the centre of s,

(d) the base of $S^{\curlywedge}(s)$ is closed in $S^{\curlywedge} \circ T^{\triangledown}(\omega)$.

(See the middle and rightmost cases of Figure 4.3.5 for illustrations.)

Let s satisfy (a), (b), and (c), and write A for the highest vertex of s, so that $T^{\triangledown}(\Gamma)$ includes the edges BO and CO. The edge BC is open in $S^{\curlywedge} \circ T^{\triangledown}(\omega)$ with (conditional) probability

$$\begin{cases} 1 & \text{if } AO \text{ is closed in } T^{\triangledown}(\omega), \\ \nu_0 & \text{if } AO \text{ is open in } T^{\triangledown}(\omega). \end{cases}$$

See also Figure 3.2.2. This conditional probability is achieved by declaring BC to be open if and only if: either AO is closed in $T^{\triangledown}(\omega)$, or AO is open in $T^{\triangledown}(\omega)$ and $\underline{Z}^s = 0$. With this coupling,

$$\text{if (d) above holds, then } \underline{Z}^s = 1, \text{ and hence } Z^s_{\mathrm{l}} = Z^s_{\mathrm{r}} = 1.$$

We return to (4.3.22). If the highest part of Γ in $\mathcal{C}_n$ comprises a single horizontal star s, as on the right of Figure 4.3.5,

$$H_n(S^{\curlywedge} \circ T^{\triangledown}(\Gamma)) - H_n(\Gamma) \leq \max\{Z^s_{\mathrm{l}}, Z^s_{\mathrm{r}}\} =: Y_n. \tag{4.3.23}$$

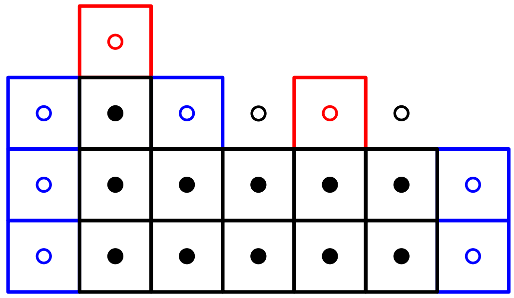

Figure 4.3.6: The black squares represent the bricks at step k in the growth process. The blue and red squares are the additions at time $k+1$. The lateral extensions (blue) occur with probability 1, and the vertical extensions (red) with probability $1-\zeta$.

If, on the other hand, the highest part of Γ in $\mathcal{C}_n$ corresponds to two stars, s_1 and s_2, that also intersect $\mathcal{C}_{n-1}$ and $\mathcal{C}_{n+1}$ respectively (as in the first and second diagrams of the figure),

$$H_n(S^{\curlywedge} \circ T^{\triangledown}(\Gamma)) - H_n(\Gamma) \leq \max\{Z_{\mathrm{r}}^{s_1}, Z_{\mathrm{l}}^{s_2}\} =: Y_n. \tag{4.3.24}$$

Recalling the properties of the Z_{l}^{s}, Z_{r}^{s}, we have that the Y_n are independent Bernoulli variables with parameter $1-\eta'$ where

$$\eta' := \left(1-\sqrt{1-\nu_0}\right)^2 = \left(1-\sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}}\right)^2. \tag{4.3.25}$$

The proof is completed by the elementary exercise of showing that $\eta' \geq \eta(p_0)$. □

Proof of Lemma 4.3.11. The process $\mathbf{X} = (\mathbf{X}^k : k \geq 0)$ may be represented physically as follows. Above each integer is a pile of bricks, illustrated in Figure 4.3.6. At each epoch of time, each column gains a random number of bricks. If a column is as least as high as its two nearest neighbouring columns, a brick is added with probability $1-\zeta$. Otherwise, bricks are added to the column to match the height of its higher neighbour.

We study the process via the times at which bricks are placed at vertices. For each pair A, B of neighbours in the upper half-plane $\mathbb{Z}\times\mathbb{Z}_0$ of the square lattice with the usual embedding, we place a directed edge denoted AB from A to B, and similarly a directed edge BA from B to A. Let $\mathcal{E}$ be the set of all such directed edges. The random variables $(\tau_{AB} : AB \in \mathcal{E})$ are assumed independent with distributions as follows.

$$\tau_{AB} = \begin{cases} 1 & \text{if } AB \text{ is horizontal,} \\ 0 & \text{if } AB \text{ is directed downwards,} \end{cases}$$

and τ_{AB} has the geometric distribution with parameter $1-\zeta$ if AB is directed upwards, that is,

$$P(\tau_{AB} = r) = \zeta^{r-1}(1-\zeta), \quad r \geq 1.$$

Thinking about τ_{AB} as the time for the process to pass along the edge AB, we define the *passage-time* from C to D by

$$\tau(C,D) = \inf\left\{\tau(\vec{\Gamma}) := \sum_{e\in\vec{\Gamma}} \tau_e : \vec{\Gamma} \in \mathcal{P}_{C,D}\right\},$$

where $\mathcal{P}_{C,D}$ is the set of all directed paths from C to D.

Let $\alpha \in \sqrt{3}\mathbb{N}$ and $L_i := [-\alpha N/\sqrt{3}, \alpha N/\sqrt{3}] \times \{i\}$. The initial state G_0 of this growth process is the set $\bigcup_{i=0}^{N} L_i$. It is easily seen that the state G_k at time k comprises exactly the set of all vertices D such that there exists $C \in L_N$ with $\tau(C,D) \leq k$.

Let $\beta > 3$ be an integer, to be chosen later. By the above,

$$P(h(G_{\beta N}) \geq \beta N) \leq \sum_{\substack{C,D:\\ C\in L_N,\, h(D)=\beta N}} P(\tau(C,D) \leq \beta N). \tag{4.3.26}$$

Now, $\tau(C,D) \leq \beta N$ if and only if there exists a directed path $\vec{\Gamma} \in \mathcal{P}_{C,D}$ with passage-time not exceeding βN, so that

$$P(h(G_{\beta N}) \geq \beta N) \leq \sum_{\vec{\Gamma}\in\mathcal{P}_N} P(\tau(\vec{\Gamma}) \leq \beta N), \tag{4.3.27}$$

where $\mathcal{P}_N$ is the set of directed paths whose endpoints C, D are as in (4.3.26). Consider such a path $\vec{\Gamma}$, and let u, d, h be the numbers of its upward, downward, and horizontal edges, respectively. Since upward and horizontal edges have passage-times at least 1, we must have $u + h \leq \beta N$. By considering the heights of the first and last vertices, $u - d = (\beta - 1)N$. Therefore, $\vec{\Gamma}$ has no more than $(\beta + 1)N$ edges in total, of which at least $(\beta - 1)N$ are upward.

There are $|L_N| \leq 2\alpha N$ possible choices for C, so that

$$|\mathcal{P}_N| \leq 2\alpha N 4^{2N}\binom{(\beta+1)N}{2N}. \tag{4.3.28}$$

For $\vec{\Gamma} \in \mathcal{P}_N$, $\tau(\vec{\Gamma})$ is no smaller than the sum of the passage-times of its upward edges. Therefore,

$$P(\tau(\vec{\Gamma}) \leq \beta N) \leq P(S \leq \beta N), \tag{4.3.29}$$

where S is the sum of $(\beta - 1)N$ independent random variables with the Geom$(1 - \zeta)$

distribution. It is elementary that

$$P(S \leq \beta N) = P\big(T \geq (\beta - 1)N\big),$$

where T has the binomial distribution $\mathrm{bin}(\beta N, 1-\zeta)$. By Markov's inequality (as in the proof of Cramér's Theorem),

$$\limsup_{N\to\infty} P\big(T \geq (\beta-1)N\big)^{1/N} \leq \beta \left(\frac{\beta(1-\zeta)}{\beta-1}\right)^{\beta}, \tag{4.3.30}$$

when $\beta(1-\zeta) < \beta - 1$, that is, $\beta > 1/\zeta$.

By (4.3.27)–(4.3.30), there exists $N_0 = N_0(\beta,\zeta)$ such that, for $N \geq N_0$,

$$P(h(G_{\beta N}) \geq \beta N) \leq 2\alpha N 4^{2N} \binom{(\beta+1)N}{2N} \left\{ 2\beta \left(\frac{\beta(1-\zeta)}{\beta-1}\right)^{\beta} \right\}^{N}.$$

By Stirling's formula, there exists $c = c(\zeta)$ and $N_1 = N_1(\beta,\zeta)$ such that, for $N \geq N_1$,

$$P(h(G_{\beta N}) \geq \beta N) \leq \alpha \left\{ c\beta^3 \left(\frac{\beta(1-\zeta)}{\beta-1}\right)^{\beta} \right\}^{N}. \tag{4.3.31}$$

Choose $\beta = \beta(\zeta)$ sufficiently large that the last term is smaller than αe^{-N}, and the proof is complete. $\square$

This concludes the proof of Lemma 4.3.11 and thus of Proposition 4.3.8.

Proof of Proposition 4.3.9. Let $N \in 2\mathbb{N}$. Let $\mathbb{L} = (V,E)$ be the mixed triangular lattice with interface-height 0, so that

$$\mathbb{P}^{\triangle}_{\mathbf{p}}\big[\mathcal{C}_{\mathrm{v}}(\alpha N, N)\big] = \mathbb{P}^{\mathbb{L}}_{\mathbf{p}}\big[\mathcal{C}_{\mathrm{v}}(\alpha N, N)\big].$$

Let $\omega \in \Omega_E$, and let γ be an ω-open vertical crossing of $B_{\alpha N,N}$. In $\frac{1}{2}N$ applications of $S^{\curlywedge} \circ T^{\triangledown}$, the images of the lower endpoint of γ remain in the square part of the lattice, and thus are immobile. By Proposition 4.2.2, $(S^{\curlywedge} \circ T^{\triangledown})^{N/2}(\gamma)$ contains a vertical crossing of $B_{(\alpha+\frac{1}{2})N,N/2}$ that is open in $(S^{\curlywedge} \circ T^{\triangledown})^{N/2}(\omega)$. Since $B_{(\alpha+\frac{1}{2})N,N/2}$ lies entirely within the square part of $(S^{\curlywedge} \circ T^{\triangledown})^{N/2}\mathbb{L}$, we deduce that

$$\begin{aligned}\mathbb{P}^{\square}_{(p_0,1-p_0)}\left[\mathcal{C}_{\mathrm{v}}((\alpha+\tfrac{1}{2})N, \tfrac{1}{2}N)\right] &= \mathbb{P}^{(S^{\curlywedge}\circ T^{\triangledown})^{N/2}\mathbb{L}}_{\mathbf{p}}\left[\mathcal{C}_{\mathrm{v}}((\alpha+\tfrac{1}{2})N, N)\right] \\ &\geq \mathbb{P}^{\triangle}_{\mathbf{p}}[\mathcal{C}_{\mathrm{v}}(\alpha N, N)],\end{aligned}$$

and the claim is proved. $\square$

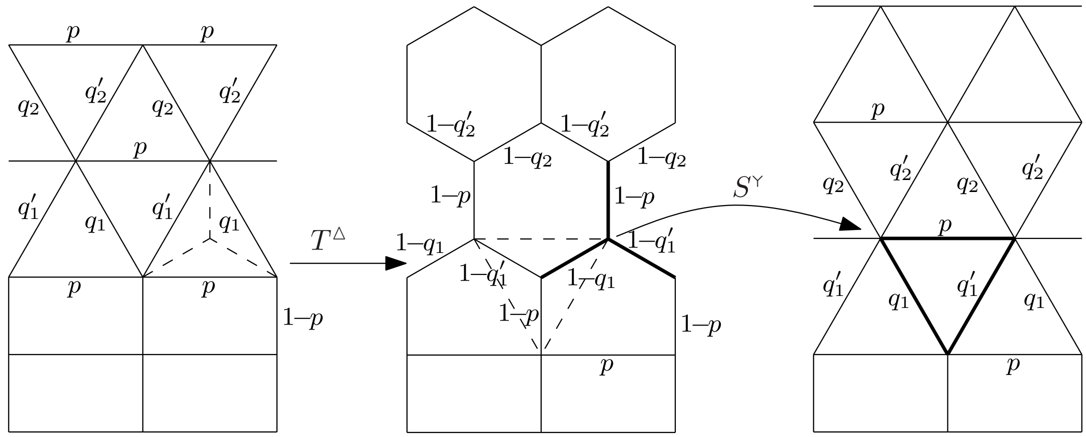

Figure 4.4.1: A mixed triangular lattice (left) with the highly inhomogeneous measure above the interface. The transformation $S^{\curlyvee} \circ T^{\triangle}$ moves the interface down by one unit. Every triangle is parametrized by a self-dual triplet.

4.4 Proof of Theorem 4.1.1 for $\mathcal{M}_I$

We will only sketch how to adapt the proofs of Section 4.3 to incorporate the highly inhomogeneous models.

The proof of Theorem 4.1.1 for the highly inhomogeneous models on $\mathbb{T}$ and $\mathbb{H}$ follows exactly that of Section 4.3.3 on noting that: each triangle of the mixed triangular lattice of Figure 4.4.1 has three edges with parameters forming a self-dual triplet, and the constants of Propositions 4.3.4 to 4.3.9 depend only (in the current setting) on the value of p and not otherwise on $\mathbf{q}$ and $\mathbf{q}'$. The hexagonal-lattice case follows by a single application of the star–triangle transformation.

We now focus on highly inhomogeneous models on the square lattice. Let $\mathbf{q} = 1 - \mathbf{q}'$ satisfy (1.4.4) with $\epsilon > 0$, and let $p = 1 - p' = \frac{1}{2}\epsilon$. We may pick $r_n \in (0, 1)$ such that $\kappa_\triangle(p, q_n, r_n) = 0$ for all n, and we write $r'_n = 1 - r_n$. By the above the measure $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{r}}$ has the box-crossing property, and we propose to transport this property to the square-lattice measure $\mathbb{P}_{\mathbf{q},\mathbf{q}'}$ via the star-triangle transformation.

Let $\mathbb{L} = (V, E)$ be the mixed triangular lattice on the left of Figure 4.4.2, and denote by $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$ the product measure given there. Under $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$, all triangles in $\mathbb{L}$ have self-dual triplets. Thus, $T^{\triangledown}$ acts on Ω_E endowed with $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$ in the manner of Section 4.2 (with parameters varying between triangles), and the ensuing measure is given in the middle figure. Then $S^{\curlywedge}$ acts on edge-configurations of $T^{\triangledown}\mathbb{L}$ (with parameters varying between stars). The ensuing lattice $(S^{\curlywedge} \circ T^{\triangledown})\mathbb{L}$ is illustrated on the right, and it may be noted that the corresponding measure is precisely that of $\mathbb{L}$ shifted upwards and rightwards.

In the triangular part of $\mathbb{L}$, $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$ corresponds to the measure $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{r}}$, while in the square part it corresponds to $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$. By Theorem 4.1.1 for the highly inhomogeneous models on $\mathbb{T}$,

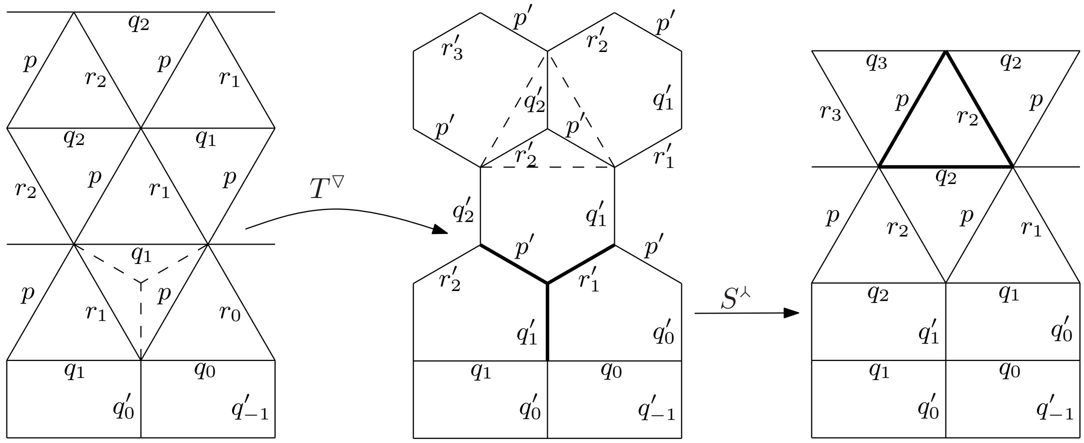

Figure 4.4.2: *Left*: The measure $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$ on $\mathbb{L}$. In the triangular part the measure is $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{r}}$ on a rotated lattice, and in the square part it is $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$. *Middle, right*: Application of $S^{\curlywedge} \circ T^{\triangledown}$ transforms $\mathbb{L}$ to a copy of itself shifted upwards and sideways.

$\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{r}}$ has the box-crossing property, and thus it remains to adapt the proofs of Propositions 4.3.8 and 4.3.9.

Proposition 4.3.9 holds because of its non-probabilistic bound for the drift of a path under $S^{\curlywedge} \circ T^{\triangledown}$. Its proof is easily adapted to give, as there, that, for $\alpha > 0$ and $N \in 2\mathbb{N}$,

$$\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}\big[\mathcal{C}_{\mathrm{v}}((\alpha + \tfrac{1}{2})N, \tfrac{1}{2}N)\big] \geq \mathbb{P}^{\triangle}_{\mathbf{q},\mathbf{r},p}[\mathcal{C}_{\mathrm{v}}(\alpha N, N)].$$

The proof of Proposition 4.3.8 requires the probabilistic estimate of Lemma 4.3.10. This hinges on the application of $S^{\curlywedge}$ to configurations on upwards pointing stars. The key fact is that $\eta(p_0) > 0$, with η as in (4.3.15) and p_0 the parameter associated with a horizontal edge in the triangular lattice. In the present situation, such edges have parameters q_n. Since $q_n \geq \epsilon$, we have that $\eta(q_n) \geq \eta(\epsilon) > 0$. This results in an altered version of Lemma 4.3.10 with $\eta(p_0)$ replaced by $\eta(\epsilon)$. The proof continues as before, and a version of (4.3.13) results. Theorem 4.1.1 for highly inhomogeneous models on the square lattice is proved.

4.5 Universality of arm exponents

4.5.1 Outline of proof

The main goal of this section is to prove the following proposition.

Proposition 4.5.1. *Let $k \in \{1, 2, 4, \dots\}$ and $\epsilon > 0$. There exist constants $c_i = c_i(k, \epsilon) > 0$*

and $N_0 = N_0(k, \epsilon)$ *such that for any model* $(\mathbb{L}, \mathbb{P}) \in \mathcal{M}_I(\epsilon)$ *and any* $n \geq 2N \geq 2N_0$,

$$c_1 \mathbb{P}^{\square}_{\frac{1}{2},\frac{1}{2}}[A_k(N,n)] \leq \mathbb{P}[A_k(N,n)] \leq c_2 \mathbb{P}^{\square}_{\frac{1}{2},\frac{1}{2}}[A_k(N,n)]. \tag{4.5.1}$$

Theorem 4.1.4 follows directly from the above, and the rest of the Section is dedicated to Proposition 4.5.1. Its proof is structured as follows. We use transformations similar to those in the proof of Theorem 4.1.1 to transport arm events from one model to another. To do that we introduce in Section 4.5.2 a modified version of the mixed lattices used in Section 4.3, and the corresponding transformations. In Section 4.5.3 we give an alternative definition of arm events, adapted to our context, and relate it to the regular definition. In Section 4.5.4 we use the modified arm events to prove Proposition 4.5.1.

For the remainder of this section $\epsilon > 0$ is fixed and, unless otherwise stated, all constants $c_i > 0$, $N_0 \in \mathbb{N}$ depend only on ϵ and on the number k of arms in the event under study. We use the expression 'for $n > N$ large enough' to mean: for $n \geq c_0 N$ and $N > N_0$.

4.5.2 Mixed lattices: a second version

Whereas the mixed lattices of Section 4.2 were suited for proving the box-crossing property, a slightly altered hybrid is useful for studying arm exponents.

Let $m \geq 0$, and consider the mixed lattice $\mathbb{L}^m = (V^m, E^m)$ drawn on the left of Figure 4.5.1, formed of a horizontal strip of the square lattice centred on the x axis of height $2m$, with the triangular lattice above and beneath it. The embedding of each lattice is otherwise as in Section 4.2: the triangular lattice comprises equilateral triangles of side length $\sqrt{3}$, and the square lattice comprises rectangles with horizontal (respectively, vertical) dimension $\sqrt{3}$ (respectively, 1). We require also that the origin of $\mathbb{R}^2$ be a vertex of the mixed lattice.

Let $\mathbf{p} \in [0,1)^3$, and let $\mathbb{P}^m_{\mathbf{p}}$ be the product measure on $\Omega^m = \{0,1\}^{E^m}$ for which edge e is open with probability $p(e)$ given by:

(a) $p(e) = p_0$ if e is horizontal,

(b) $p(e) = 1 - p_0$ if e is vertical,

(c) $p(e) = p_1$ if e is the right edge of an upwards pointing triangle,

(d) $p(e) = p_2$ if e is the left edge of an upwards pointing triangle.

Suppose further that $\mathbf{p}$ is self-dual, in that $\kappa_\triangle(\mathbf{p}) = 0$, and let $\omega^m \in \Omega^m$. We denote by $T^\triangle$ (respectively, $T^\triangledown$) the transformation T of Figure 3.2.2 applied to an upwards (respectively, downwards) pointing triangle. Write T^+ for the transformation of ω^m obtained by applying $T^\triangle$ to every upwards pointing triangle in the upper half plane, and $T^\triangledown$ similarly in the lower half plane; sequential applications of star–triangle transformations are required to be independent of one another.

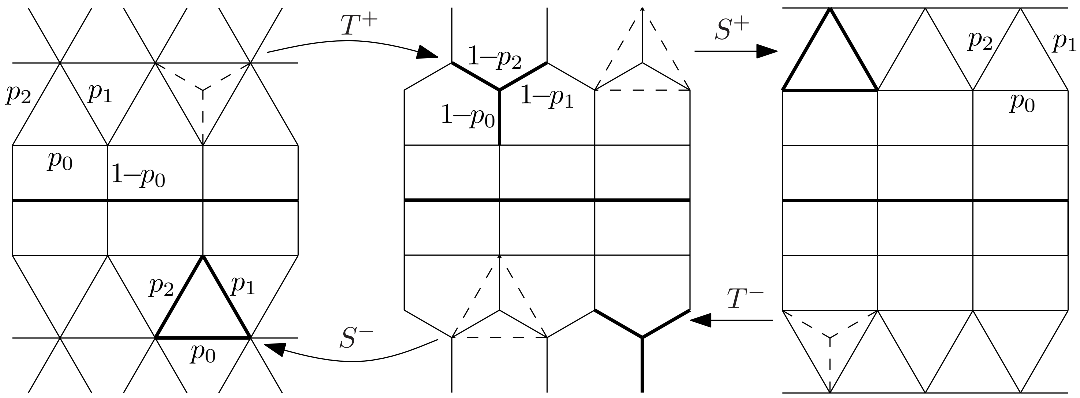

Figure 4.5.1: The transformation $S^+ \circ T^+$ (respectively, $S^- \circ T^-$) transforms $\mathbb{L}^1$ into $\mathbb{L}^2$ (respectively, $\mathbb{L}^2$ into $\mathbb{L}^1$). They map the dashed graphs to the bold graphs.

Similarly, we denote by $S^{\curlywedge}$ (respectively, $S^{\curlyvee}$) the transformation S of Figure 3.2.2 applied to an upwards (respectively, downwards) pointing star. Write S^+ for the transformation of $(T^+\mathbb{L}^m, T^+(\omega^m))$ obtained by applying $S^{\curlywedge}$ to all upwards pointing stars in the upper half-plane and similarly $S^{\curlyvee}$ in the lower half-plane. It may be checked that $\omega^{m+1} = S^+ \circ T^+(\omega^m)$ lies in Ω^{m+1} and has law $\mathbb{P}_{\mathbf{p}}^{m+1}$. That is, viewed as a transformation acting on measures, we have $(S^+ \circ T^+)\mathbb{P}_{\mathbf{p}}^m = \mathbb{P}_{\mathbf{p}}^{m+1}$.

The transformations T^- and S^- are defined similarly, and illustrated in Figure 4.5.1. As in that figure, for $m \geq 0$,

$$\begin{aligned} (S^+ \circ T^+)\mathbb{L}^m &= \mathbb{L}^{m+1}, \quad (S^+ \circ T^+)\mathbb{P}_{\mathbf{p}}^m = \mathbb{P}_{\mathbf{p}}^{m+1}, \\ (S^- \circ T^-)\mathbb{L}^{m+1} &= \mathbb{L}^m, \quad (S^- \circ T^-)\mathbb{P}_{\mathbf{p}}^{m+1} = \mathbb{P}_{\mathbf{p}}^m. \end{aligned}$$

We turn to the operation of these two transformations on open paths, and will concentrate on $S^+ \circ T^+$; similar statements are valid for $S^- \circ T^-$. Let $\omega^m \in \Omega^m$, and let π be an ω^m-open path of $\mathbb{L}^m$. As in Section 4.2, the image of π under $S^+ \circ T^+$ contains some ω^{m+1}-open path π'. Furthermore, π' lies within the 1-neighborhood of π viewed as a subset of $\mathbb{R}^2$, and has endpoints within unit Euclidean distance of those of π. Any vertex of π in the square part of $\mathbb{L}^m$ is unchanged by the transformation. The corresponding statements hold also for open* paths of the dual of $\mathbb{L}^m$. These facts will be useful in observing the effect of $S^+ \circ T^+$ on the arm events.

Let $\mathbb{L} = (V, E)$ be a mixed lattice duly embedded in $\mathbb{R}^2$, and write V_0 for the subset of V lying on the x-axis. Let $\omega \in \Omega = \{0,1\}^E$. For $R \subseteq \mathbb{R}^2$ and $A, B \subseteq R \cap V_0$, we write $A \xleftrightarrow{R,\omega} B$ (with negation written $A \overset{R,\omega}{\not\longleftrightarrow} B$) if there exists an ω-open path joining some $a \in A$ and some $b \in B$ using only edges that intersect R. We remind the notation $R^1 = \{r + d : r \in R,\ |d| \leq 1\}$.

Proposition 4.5.2. *Let $m \geq 0$, $\omega \in \Omega^m$, $R \subseteq \mathbb{R}^2$, and $u, v \in R \cap V_0$. For $\tau \in \{S^+ \circ$*

$T^+, S^- \circ T^-\}$,

(a) *if* $u \xleftrightarrow{R,\omega} v$, *then* $u \xleftrightarrow{R^1,\tau(\omega)} v$,

(b) *if* $u \overset{R^1,\omega}{\not\longleftrightarrow} v$, *then* $u \overset{R,\tau(\omega)}{\not\longleftrightarrow} v$.

Proof. (a) Let $\tau = S^+ \circ T^+$; the case $\tau = S^- \circ T^-$ is similar (we assume $m \geq 1$ where necessary). If $u \xleftrightarrow{R,\omega} v$, there exists an ω-open path π of $\mathbb{L}$ from u to v using edges that intersect R. Since u, v are not moved by τ, the image $\tau(\pi)$ contains a $\tau(\omega)$-open path of $\tau\mathbb{L}$ from u to v. It is elementary that τ transports paths through a distance not exceeding 1 (see Proposition 4.2.2). Therefore, every edge of $\tau(\pi)$ intersects R^1.
(b) Suppose $u \xleftrightarrow{R,\tau(\omega)} v$. By considering the star–triangle transformations that constitute the mapping τ (as in part (a)), we have that $u \xleftrightarrow{R^1,\omega} v$. □

As in Section 4.4, we may also define highly inhomogeneous measures on the mixed lattices $\mathbb{L}^n$. The transformations T^+, T^-, S^+ and S^- are defined similarly, with star–triangle transformations depending on the local parameters of the lattices. We extend $\mathcal{M}_I(\epsilon)$ to accommodate the highly inhomogeneous models on the mixed lattices.

4.5.3 Modified arm-events

Let $\mathbb{L}$ be one of the square, triangular, and hexagonal lattices, or a hybrid thereof as in Section 4.5.2. Let $x_i = (i\sqrt{3}, 0)$, $i \geq 0$, denote the vertices common to these lattices to the right of the origin, and $y_i = ((i+\frac{1}{2})\sqrt{3}, \frac{1}{2})$, $i \geq 0$, the vertices of the dual lattice $\mathbb{L}^*$ corresponding to the faces of $\mathbb{L}$ lying immediately above the edge $x_i x_{i+1}$. We recall the notation $\Lambda_n = [-n,n]^2 \subseteq \mathbb{R}^2$, with boundary $\partial\Lambda_n$, and that C_x (respectively, C^*_y) denotes the open cluster of $\mathbb{L}$ containing x (respectively, the open* cluster of $\mathbb{L}^*$ containing y). For $n \geq 1$ and any connected subgraph C of either $\mathbb{L}$ or $\mathbb{L}^*$, we write $C \cap \partial\Lambda_r \neq \varnothing$ if C contains vertices in both Λ_r and $\mathbb{R}^2 \setminus (-r,r)^2$. Note that we may have $C \cap \partial\Lambda_r \neq \varnothing$ even when there are no vertices of C belonging to $\partial\Lambda_r$.

For $j, n \in \mathbb{N}$ with $j \geq 2$, let

$$\begin{aligned}
\bar{A}_1(n) &= \{C_{x_0} \cap \partial\Lambda_n \neq \varnothing\},\\
\bar{A}_2(n) &= \{C_{x_0} \cap \partial\Lambda_n \neq \varnothing,\ C^*_{y_0} \cap \partial\Lambda_n \neq \varnothing\},\\
\bar{A}_{2j}(n) &= \bigcap_{0 \leq i < j} \left\{C_{x_i} \cap \partial\Lambda_n \neq \varnothing,\ \text{and } x_i \overset{\Lambda_n,\omega}{\not\longleftrightarrow} \{x_0, x_1, \dots, x_{i-1}\}\right\}.
\end{aligned}$$

We write $\bar{A}^{\mathbb{L}}_k(n)$ when the role of $\mathbb{L}$ is to be stressed. Note the condition of disconnection in the definition of $\bar{A}_{2j}(n)$: it is required that the x_i are not connected by open paths of edges all of which intersect Λ_n.

A proof of the following elementary lemma is sketched at the end of this subsection.

Proposition 4.5.3. *Let* $(\mathbb{L}, \mathbb{P}) \in \mathcal{M}_I(\epsilon)$ *and* $k \in \{1, 2, 4, 6, \dots\}$. *There exists* $N_0 = N_0(k) \in \mathbb{N}$ *and* $c_i = c_i(\epsilon, N, k) > 0$ *such that*

$$\mathbb{P}[\bar{A}_k(n)] \leq \mathbb{P}[A_k(N, n)] \leq c_0 \mathbb{P}[\bar{A}_k(n)], \tag{4.5.2}$$

$$\mathbb{P}[\bar{A}_k(n)] \leq c_1 \mathbb{P}[\bar{A}_k(2n)]. \tag{4.5.3}$$

for $n \geq N \geq N_0$.

Proof. First, a note concerning the event $\bar{A}_{2j}(n)$ with $j \geq 2$. If $\omega \in \bar{A}_{2j}(n)$, the vertices x_i, $0 \leq i < j$, are connected to $\partial\Lambda_n$ by open paths. We claim that j such open paths may be found that are vertex-disjoint and interspersed by j open* paths joining the y_i to $\partial\Lambda_n$. This will imply the existence of $2j$ arms of alternating types joining $\{x_0, y_0, x_1, y_1, \dots, x_{j-1}\}$ to $\partial\Lambda_n$, such that the open primal paths are vertex-disjoint, and the open* dual paths are vertex-disjoint except at the y_i. The claim may be seen as follows (see also the left diagram of Figure 4.5.2). The dual edge e with endpoints $\pm y_0$ is necessarily open*. By exploring the boundary of C_{x_0} at e, one may find two open* paths denoted π_0, π_0', joining y_0 to $\partial\Lambda_n$, and vertex-disjoint except at y_0. Let $0 \leq r \leq j-2$. Since $x_r, x_{r+1} \xleftrightarrow{\Lambda_n, \omega} \partial\Lambda_n$ and $x_r \overset{\Lambda_n,\omega}{\not\longleftrightarrow} x_{r+1}$, we may similarly explore the boundary of C_{x_r} to find an open* path π_r of Λ_n that joins y_r to $\partial\Lambda_n$, and is vertex-disjoint from either π_0 or π_0', and in addition from π_s, $s \neq r$. The dual paths $\pi_0', \pi_0, \pi_1, \dots, \pi_{j-2}$ are the required open* arms. The first inequality in (4.5.2) follows immediately.

For the second inequality in (4.5.2), as well as for (4.5.3), we will need to use the box-crossing property and the separation theorem.

First we note that both $\mathbb{P}$ and $\mathbb{P}^*$ have the box-crossing property BXP(δ), with a constant $\delta = \delta(\epsilon) > 0$ that depends only on ϵ, not otherwise on $\mathbb{L}$ and $\mathbb{P}$. If $\mathbb{L}$ is one of the square, triangular, or hexagonal lattices, then the above is proved in Theorem 4.1.1. For a mixed lattice $\mathbb{L}^m$, the box-crossing property holds in both the square and triangular sections of the lattice, in order to deduce it in the whole of the plane we need a short argument which we detail in the next two paragraphs.

Suppose for simplicity that we work with an inhomogeneous measure $\mathbb{P}^m_{\mathbf{p}}$ with $p_0 \in (\epsilon, 1-\epsilon)$. Recall the notation $B_{M,N} = [-M, M] \times [0, N]$, and denote by $\mathcal{C}_{\mathrm{h}}(B_{M,N})$ (respectively, $\mathcal{C}_{\mathrm{v}}(B_{M,N})$) the event that there exists a horizontal (respectively, vertical) open crossing of $B_{M,N}$ (with a similar notation $\mathcal{C}^*_{\mathrm{h}}$, $\mathcal{C}^*_{\mathrm{v}}$ for dual crossings). For every translation f, $f(B_{M,3N})$ contains a rectangle with dimensions $2M \times N$ lying in either the square or triangular part of $\mathbb{L}^m$. Thus

$$\mathbb{P}^m_{\mathbf{p}}\big[\mathcal{C}_{\mathrm{h}}(f(B_{M,3N}))\big] \geq \min\big\{\mathbb{P}^{\triangle}_{\mathbf{p}}\big[\mathcal{C}_{\mathrm{h}}(M,N)\big], \mathbb{P}^{\square}_{(p_0,1-p_0)}\big[\mathcal{C}_{\mathrm{h}}(M,N)\big]\big\} \geq \delta', \tag{4.5.4}$$

with an adjusted value of $\delta' = \delta'(\epsilon) > 0$, given by the box-crossing property. The dual model lives on a mixed square/hexagonal lattice and the same inequality holds with $\mathcal{C}_{\mathrm{h}}$

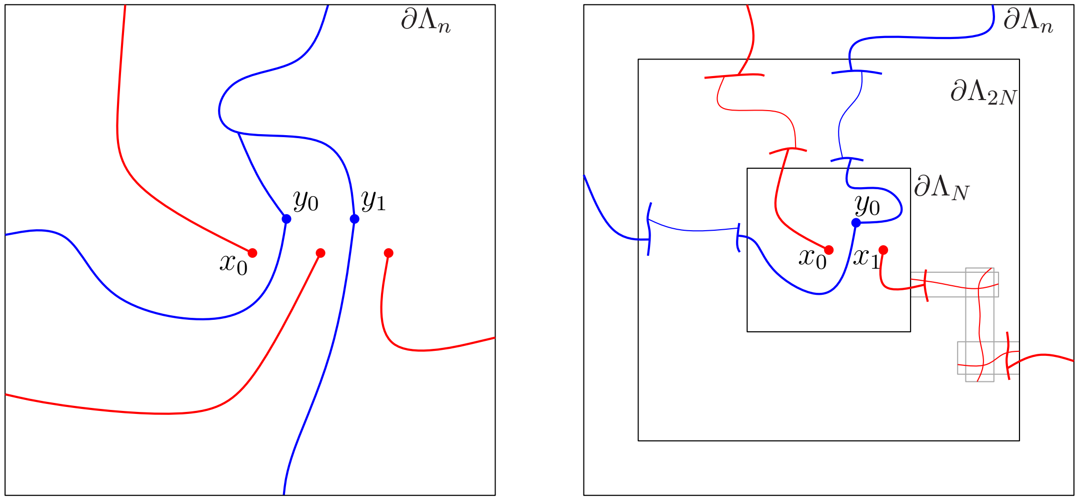

Figure 4.5.2: *Left:* The event $\bar{A}_{2j}(n)$ implies the existence of j primal arms (red) and j dual arms (blue) extending to $\partial\Lambda_n$. *Right:* Combining $\bar{A}^I(N)$ and $A^{I,\varnothing}(2N,n)$ to form $\bar{A}(n)$. The primal fences of $\bar{A}^I(N)$ and $A^{I,\varnothing}(2N,n)$ are the thick red paths. The dual ones are the thick blue paths. They are connected inside $\mathcal{A}(N,2N)$ by the thin paths forming H_N. These may be constructed via crossings of boxes of determined aspect ratio, as shown for the primal arm originating at x_1.

replaced by $\mathcal{C}_{\mathrm{h}}^*$.

For vertical crossings we may adapt the proof of Proposition 4.3.9 to obtain

$$\mathbb{P}_{\mathbf{p}}^m[\mathcal{C}_{\mathrm{v}}(f(B_{3N,N}))] \geq \mathbb{P}_{\mathbf{p}}^{\triangle}[\mathcal{C}_{\mathrm{v}}(B_{N,2N})] \geq \delta'', \tag{4.5.5}$$

where f is any translation and $\delta'' = \delta''(\epsilon) > 0$ is given by the box-crossing property for $\mathbb{P}_{\mathbf{p}}^{\triangle}$. The same inequality holds with $\mathcal{C}_{\mathrm{v}}$ replaced by $\mathcal{C}_{\mathrm{v}}^*$. Inequalities (4.5.4), (4.5.4), along with Proposition 4.3.2, imply the box-crossing property for $\mathbb{P}$. The same is valid for the dual measure $\mathbb{P}^*$.

We now come back to the proof of (4.5.2). Since $\mathbb{P}$ and $\mathbb{P}^*$ satisfy the box-crossing property, we may use the separation theorem. Fix $\eta = \eta(k) > 0$, so that (2.3.1) holds, and let I be a η-landing sequence of length k. It will be convenient to introduce the notation $\bar{A}^I(n)$ for the event $\bar{A}(n)$ with the additional requirement that the k arms are fences with landing points in I. The definition is similar to that of $A_k^{\varnothing,I}(N,n)$, and by a straightforward adaption of the separation theorem, there exist $c_1 > 0$ such that, for N large enough,

$$\mathbb{P}[\bar{A}_k(N)] \leq c_1 \mathbb{P}[\bar{A}_k^I(N)]. \tag{4.5.6}$$

Moreover, by the separation theorem, for $n \geq N$ large enough,

$$\mathbb{P}[A_k^{I,\varnothing}(2N,n)] \leq c_2 \mathbb{P}[A_k(2N,n)]. \tag{4.5.7}$$

Fix $N = N(\epsilon, k)$ such that both (4.5.6) and (4.5.7) hold. Let H_N be the event described in the right diagram of Figure 4.5.2 by the thin paths. It only depends on the configuration inside $\mathcal{A}(N, 2N)$. For n large enough, we have

$$\bar{A}_k^I(N) \cap A_k^I(2N,n) \cap H_N \subset \bar{A}_k(n)$$

Finally, by the box-crossing property for $\mathbb{P}$ and $\mathbb{P}^*$, we bound the probability of H_N by a constant $c_3(\epsilon, k) > 0$, and by Lemma 2.3.3

$$\mathbb{P}\left[\bar{A}_k(n)\right] \geq \mathbb{P}\left[\bar{A}_k^I(N)\right] \mathbb{P}\left[A_k^I(2N,n)\right] \mathbb{P}\left[H_N\right] \geq c_1 c_2 c_3 \mathbb{P}\left[\bar{A}_k(N)\right] \mathbb{P}\left[A_k(2N,n)\right].$$

A careful inspection of the local properties of the lattice shows that there exists $c_4 = c_4(\epsilon, k)$ such that

$$\mathbb{P}\left[\bar{A}_k(N)\right] \geq c_4.$$

This concludes the proof of (4.5.2).

The proof of (4.5.3) is exactly similar to that of Corollary 2.3.2 and uses inequality (4.5.6) along with a construction using box-crossings. We do not give further details here. $\square$

4.5.4 Proof of Proposition 4.5.4

The proof of the universality of the box-crossing property was based on a technique that transforms one of these lattices into the other while preserving primal and dual connections. The same technique will be used here to prove the following results.

Proposition 4.5.4. *Fix $k \in \{1, 2, 4, 6, \dots\}$ and $\epsilon > 0$. There exist constants $c_1, c_2, n_0 > 0$ such that, for $\mathbf{p} \in [0,1)^3$, self-dual, with $p_0 \in (\epsilon, 1-\epsilon)$, and $n \geq n_0$,*

$$c_1 \mathbb{P}_{\mathbf{p}}^{\triangle}\left[\bar{A}_k(n)\right] \leq \mathbb{P}_{(p_0, 1-p_0)}^{\square}\left[\bar{A}_k(n)\right] \leq c_2 \mathbb{P}_{\mathbf{p}}^{\triangle}\left[\bar{A}_k(n)\right].$$

The above is enough to prove Theorem 4.1.4 for $\mathcal{M}$. In order to extend the theorem to $\mathcal{M}_I$ we need a similar statement for highly inhomogeneous models.

Proposition 4.5.5. *Let $p \in (\epsilon, 1-\epsilon)$ and $\mathbf{q}, \mathbf{q}' \in [0,1]^{\mathbb{Z}}$ be such that*

$$\kappa_\triangle(p, q_n, q_n') = 0, \qquad \text{for } n \in \mathbb{Z}. \tag{4.5.8}$$

For any $k \in \{1, 2, 4, 6, \dots\}$ and there exist $c_i, n_1 > 0$, depending only on ϵ and k, such *that, for all* $n \geq n_1$,

$$c_0 \mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}\left[\bar{A}_k(n)\right] \leq \mathbb{P}^{\square}_{(p,1-p)}\left[\bar{A}_k(n)\right] \leq c_1 \mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}\left[\bar{A}_k(n)\right], \tag{4.5.9}$$

$$c_0 \mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}\left[\bar{A}_k(n)\right] \leq \mathbb{P}^{\square}_{\mathbf{q},1-\mathbf{q}}\left[\bar{A}_k(n)\right] \leq c_1 \mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}\left[\bar{A}_k(n)\right]. \tag{4.5.10}$$

At the end of the Section we will give the proof of Proposition 4.5.4. The similar proof Proposition 4.5.5 is omitted. Before we do this, let us prove Proposition 4.5.1 using Propositions 4.5.4 and 4.5.5.

Proof of Proposition 4.5.1. This is done in several steps.

We say a measure $\mathbb{P}$ satisfies (4.5.11) if there exist constants $c_1, c_2, n_0 > 0$ such that, for $n \geq n_0$

$$c_1 \mathbb{P}^{\square}_{\frac{1}{2},\frac{1}{2}}\left[\bar{A}_k(n)\right] \leq \mathbb{P}\left[\bar{A}_k(n)\right] \leq c_2 \mathbb{P}^{\square}_{\frac{1}{2},\frac{1}{2}}\left[\bar{A}_k(n)\right]. \tag{4.5.11}$$

Fix $\epsilon > 0$. By Proposition 4.5.4 $\mathbb{P}^{\triangle}_{(\frac{1}{2},p_1,p_2)}$ satisfies (4.5.11) for all $p_1, p_2 \in [0,1]$ such that $\kappa_{\triangle}(\frac{1}{2}, p_1, p_2) = 0$. By Proposition 4.5.3, so does $\mathbb{P}^{\triangle}_{(p_1,p_2,\frac{1}{2})}$

Through another application of Proposition 4.5.4, $\mathbb{P}^{\square}_{(p_1,1-p_1)}$ satisfies (4.5.11) for all $p_1 \in (\epsilon, \frac{1}{2})$. By (4.5.2), we have proved (4.5.1) for the models of $\mathcal{M}(\epsilon)$ on the square lattice. A third application of Proposition 4.5.4, together with (4.5.2), extend (4.5.1) to all models in $\mathcal{M}(\epsilon)$.

We use (4.5.9), along with (4.5.2), to deduce (4.5.1) for models in $\mathcal{M}_I(\epsilon)$ on the square lattice, and, via (4.5.10) and (4.5.2), we extend (4.5.1) to all models in $\mathcal{M}_I(\epsilon)$.

Note that all constants in the comparison inequalities above come from Propositions 4.5.3, 4.5.4 and 4.5.5, and only depend on ϵ. □

The proof of Proposition 4.5.4 relies on the following lemma, in which the measure $\mathbb{P}_{\mathbf{p}}$ is utilized within the star–triangle transformations comprising the map τ. Let $k \in \{1, 2, 4, 6, \ldots\}$.

Lemma 4.5.6. *Let* $\mathbb{L} = (V, E)$ *be a mixed lattice as defined in Section 4.5.2, and let* $\mathbb{P}_{\mathbf{p}}$ *be a self-dual measure on* $\Omega = \{0,1\}^E$*. For* $n/\sqrt{3} > k + 2$ *and* $\tau \in \{S^+ \circ T^+, S^- \circ T^-\}$,

$$\tau \bar{A}^{\mathbb{L}}_k(n) \subseteq \bar{A}^{\tau\mathbb{L}}_k(n-1).$$

The proof of the lemma is deferred to the end of this section. Let $\mathbf{p}$ be self-dual, with $p_0 \in (\epsilon, 1-\epsilon)$. Let c and N_1 be as in Proposition 4.5.3. By making n applications of

$\tau = S^+ \circ T^+$ to $\mathbb{L}^0$, we deduce that $\tau^n A_k^{\mathbb{L}^0}(2n) \subseteq A_k^{\mathbb{L}^n}(n)$. Therefore, for $n \geq N_1$,

$$\begin{aligned}
\mathbb{P}^{\square}_{(p_0,1-p_0)}[A_k(n)] &= \mathbb{P}^n_{\mathbf{p}}[A_k(n)] \\
&\geq \mathbb{P}^0_{\mathbf{p}}[A_k(2n)] \quad \text{by Lemma 4.5.6} \\
&= \mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(2n)] \\
&\geq c\mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)] \quad \text{by (4.5.3)} .
\end{aligned}$$

This proves the first inequality of Proposition 4.5.4.

Fix $n \geq \max\{k\sqrt{3}, N_1\}$, and consider the event $A_k(n)$ on the lattice $\mathbb{L}^n$. If we apply n times the transformation $S^- \circ T^-$ to $\mathbb{L}^n$, we obtain via Lemma 4.5.6 applied to the event $A_k(2n)$ that:

$$\begin{aligned}
\mathbb{P}^{\square}_{(p_0,1-p_0)}[A_k(n)] &= \mathbb{P}^n_{\mathbf{p}}[A_k(n)] \\
&\leq c^{-1}\mathbb{P}^n_{\mathbf{p}}[A_k(2n)] \quad \text{by (4.5.3)} \\
&\leq c^{-1}\mathbb{P}^0_{\mathbf{p}}[A_k(n)] \quad \text{by Lemma 4.5.6} \\
&= c^{-1}\mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)].
\end{aligned}$$

Proposition 4.5.4 is proved.

Proof of Lemma 4.5.6. Let $k \in \{1, 4, 6, \ldots\}$, we shall consider the case $k = 2$ separately. Let $\tau \in \{S^+ \circ T^+, S^- \circ T^-\}$ and $\omega \in A_k^{\mathbb{L}}(n)$. Note that the points x_r, $r = 0, 1, \ldots,$ are invariant under τ.

It is explained in Section 4.2 (see also Section 3.2.2) that the image $\tau(\pi)$ of an ω-open path π contains a $\tau(\omega)$-open path of $\tau\mathbb{L}$ lying within distance 1 of π. Therefore, for $n/\sqrt{3} > 2r + 2$, if $C_{x_r}(\omega) \cap \partial\Lambda_n \neq \varnothing$, then $C_{x_r}(\tau(\omega)) \cap \partial\Lambda_{n-1} \neq \varnothing$. The proof when $k = 1$ is complete, and we assume now that $k \geq 4$. Let $j = k/2$ and $n/\sqrt{3} > k + 2$. By Proposition 4.5.2, $x_r \overset{\Lambda_{n-1},\tau(\omega)}{\not\longleftrightarrow} x_s$ for $0 \leq r < s \leq j - 1$, whence $\tau(\omega) \in A_k^{\tau\mathbb{L}}(n-1)$.

Finally, let $k = 2$. Let $\tau \in \{S^+ \circ T^+, S^- \circ T^-\}$ and $\omega \in A_2^{\mathbb{L}}(n)$. Let γ (respectively, γ^*) be an open primal (respectively open* dual) path starting at x_0 (respectively y_0), that intersects $\partial\Lambda_n$. Since x_0 and y_0 are unchanged under τ, they are contained, respectively, in $\tau(\gamma)$ and $\tau(\gamma^*)$. By the remarks in Section 3.2 concerning the operation of τ on open* dual paths, we conclude that $C_{x_0} \cap \partial\Lambda_{n-1} \neq \varnothing$ in $\tau\mathbb{L}$, and similarly $C^*_{y_0} \cap \partial\Lambda_{n-1} \neq \varnothing$ in $\tau\mathbb{L}^*$. The proof is complete. □

4.6 Proofs of Theorems 4.1.2 and 4.1.3

Since $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$ is increasing in $\mathbf{q}$ and $\mathbf{q}'$, and since the non-existence of an infinite component is a decreasing event, Theorem 4.1.2(a) follows from Proposition 2.1.1(b).

Turning to part (b) of Theorem 4.1.2, assume (4.1.2) holds with $\delta > 0$. Let $\epsilon = \frac{1}{4}\delta$

and note from (4.1.2) that $p, q_n, q'_n < 1 - \epsilon$ for $n \in \mathbb{Z}$. Therefore, $p + \epsilon, q_n + \epsilon, q'_n + \epsilon < 1$ for all n, and

$$\kappa_\triangle(p + \epsilon, q_n + \epsilon, q'_n + \epsilon) \leq 0, \quad n \in \mathbb{Z}.$$

By Theorem 4.1.1 and the monotonicity of measures, the measure of the dual process, $\mathbb{P}^{\hexagon}_{1-p-\epsilon, 1-\mathbf{q}-\epsilon, 1-\mathbf{q}'-\epsilon}$, has the box-crossing property. The claim follows by Proposition 2.1.1(c) with $\nu = \epsilon$.

Assume finally that (4.1.3) holds with $\delta > 0$. Let $\epsilon = \frac{1}{3}\min\{\delta, p\}$ and write

$$x^+ = \max\{x, 0\}, \quad \widehat{x} = x1_{\{x \geq \epsilon\}}.$$

Then

$$\kappa_\triangle\big((p - \epsilon)^+, (q_n - \epsilon)^+, (q'_n - \epsilon)^+\big) \geq 0, \quad n \in \mathbb{Z}.$$

By Theorem 4.1.1 and the monotonicity of measures, the associated product measure on the triangular lattice has the box-crossing property. By Proposition 2.1.2(b) with $\nu = \epsilon$ we have that $\mathbb{P}_{\widehat{p}, \widehat{\mathbf{q}}, \widehat{\mathbf{q}}'}$ is supercritical. By monotonicity of measures, $\mathbb{P}_{p, \mathbf{q}, \mathbf{q}'}$ is supercritical as claimed.

The same arguments are valid for the hexagonal lattice.

Finally, consider Theorem 4.1.3, and assume (4.1.4). Let $\nu_n = (1 - q_n - q'_n)/2$, and apply Theorem 4.1.1 to the self-dual measure $\mathbb{P}^{\square}_{\mathbf{q}+\boldsymbol{\nu}, \mathbf{q}'+\boldsymbol{\nu}}$. Part (a) then follows by Proposition 2.1.1(b). The proofs of (b, c) hold as for the triangular lattice.

Chapter 5

Universality for isoradial graphs

5.1 Results

We recall the notation $\mathcal{G}(\epsilon, I)$ for the class of isoradial graphs with the bounded-angles property BAP(ϵ) and the square-grid property SGP(I) (Section 3.1). The main technical result of this chapter is the following. Criticality and universality will follow.

We recall from Section 3.1.6 the fact that, for isoradial graphs, we write BXP(δ) for BXP($3, \delta$).

Theorem 5.1.1. *For $\epsilon > 0$ and $I \in \mathbb{N}$, there exists $\delta = \delta(\epsilon, I) > 0$ such that if G satisfies* BAP(ϵ) *and* SGP(I), *$\mathbb{P}_G$ satisfies* BXP(δ)

Note that, if $G \in \mathcal{G}(\epsilon, I)$, then $G^* \in \mathcal{G}(\epsilon, I)$ also. The following criticality result follows by Propositions 2.1.1 and 2.1.2.

Theorem 5.1.2 (Criticality). *Let $G = (V, E) \in \mathcal{G}(\epsilon, I)$, and let $\nu > 0$. All constants in the following depend only on ϵ, I and ν, not otherwise on G.*

(a) *There exist $a, b, c, d > 0$ such that, for $v \in V$,*

$$ak^{-b} \leq \mathbb{P}_G\big(\mathrm{rad}(C_v) \geq k\big) \leq ck^{-d}, \qquad k \geq 1.$$

(b) *There exists, $\mathbb{P}_G$-a.s., no infinite open cluster.*

(c) *There exist $f, g > 0$ such that, for $v \in V$,*

$$\mathbb{P}_G^{-\nu}(|C_v| \geq k) \leq fe^{-gk}, \qquad k \geq 0.$$

(d) *There exists $h > 0$ such that, for $v \in V$,*

$$\mathbb{P}_G^{\nu}(v \leftrightarrow \infty) > h.$$

(e) *There exists, $\mathbb{P}^{\nu}_{G}$-a.s., exactly one infinite open cluster.*

Our universality theorem is presented next.

Theorem 5.1.3 (Universality)**.**

(a) *Let $\pi \in \{\rho\} \cup \{\rho_{2j} : j \geq 1\}$. If π exists for some $G \in \mathcal{G}$, then it is $\mathcal{G}$-invariant.*

(b) *If either ρ or η exists for some $G \in \mathcal{G}$, then ρ, η, δ are $\mathcal{G}$-invariant and satisfy* (1.6.2).

(c) *If ρ and ρ_4 exist for some $G \in \mathcal{G}$, then ν, β, γ and Δ are invariant in the set of graphs of $\mathcal{G}$ which are periodic and invariant under rotation and reflection. Also the exponents satisfy* (1.6.3).

Point (a) will be proved in Section 5.4. Points (b) and (c) of the above are direct consequences of (a) and of Theorems 2.4.1 and 2.5.1.

Finally, we make some comments on the proofs. There are two principal steps in the proof of Theorem 5.1.1. Firstly, using a technique involving star–triangle transformations, the box-crossing property is transported from the homogeneous square lattice to an arbitrary isoradial embedding of the square lattice (with the bounded-angles property). Secondly, the square-grid property is used to transport the box-crossing property to general isoradial graphs. This method may be used also to show the invariance of certain arm exponents across the class of such isoradial graphs, as in Theorem 5.1.3 (a). The basic approach is similar to that of Chapter 4, but the geometrical constructions used here differ in substantial regards from the previous. The following use of the star–triangle transformation is inspired by work of Kenyon [Ken04].

5.2 Proof of Theorem 5.1.1: Isoradial square lattices

5.2.1 Outline of proof

The proof for isoradial square lattices is based on Proposition 5.2.1 below. We recall from Section 3.1.7 the notation $G_{\boldsymbol{\alpha},\xi}$ for the isoradial square lattice generated by the sequences of angles $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$. For $\xi \in [0, 2\pi)$, we write $G_{\boldsymbol{\alpha},\xi}$ for the isoradial square lattice generated by the angle-sequence $\boldsymbol{\alpha}$ and the constant sequence (ξ).

Proposition 5.2.1. *Let $\delta, \epsilon > 0$. There exists $\delta' = \delta'(\delta, \epsilon) > 0$ such that the following holds. Let $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be an isoradial square lattice satisfying* BAP(ϵ)*, and let $\xi \in [0, 2\pi)$ be such that $\boldsymbol{\alpha}$ and the constant sequence (ξ) satisfy* BAP(ϵ)*,* (3.1.12)*. If $G_{\boldsymbol{\alpha},\xi}$ satisfies* BXP(δ)*, then $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ satisfies* BXP(δ')*.*

Corollary 5.2.2. *Let $\epsilon > 0$. There exists $\delta = \delta(\epsilon) > 0$ such that every isoradial square lattice satisfying* BAP(ϵ) *has the box-crossing property* BXP(δ)*.*

Since $\mathcal{G}(\epsilon, 1)$ is the set of isoradial square lattices satisfying BAP(ϵ), the corollary is equivalent to Theorem 5.1.1 with $I = 1$.

By Lemma 3.1.5, Proposition 5.2.1 follows from the forthcoming Propositions 5.2.4 and 5.2.8, dealing respectively with horizontal and vertical crossings. Both these propositions rely on a technique called track-exchange, which we present in Section 5.2.2.

Remark 5.2.3. The material in Section 5.2.4, and specifically Proposition 5.2.8, may be circumvented by use of Theorem 4.1.1, where the box-crossing property is proved for highly inhomogeneous square lattices. We do not take this route here since it would reduce the integrity of the current proof, and would require the reader to be familiar with the method of Chapter 4.

Here is an outline of the alternative approach. An isoradial square lattice $G_{\boldsymbol{\alpha},\xi}$ satisfying BAP(ϵ) has the measure of a highly inhomogeneous square lattice of $\mathcal{M}_I(p_\epsilon)$. By Theorem 4.1.1, such a lattice has the box-crossing property. Moreover, the box-crossing property is equivalent in the isoradial and the $\mathbb{Z}^2$ embedding (with δ differing by a factor bounded uniformly in ϵ, see Propositions 3.1.4 and 4.3.2). By Proposition 5.2.4, horizontal box-crossings may be transported from $G_{\boldsymbol{\alpha},\xi}$ to the more general isoradial square lattice $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Similarly, by interchanging the roles of the horizontal and vertical tracks of $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we obtain the existence of vertical box-crossings in that lattice. Such crossing probabilities are now combined, using Proposition 3.1.4, to obtain Theorem 5.1.1 for $\mathcal{G}(\epsilon, 1)$.

Proof of Corollary 5.2.2. Let $\epsilon > 0$ and let $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ satisfy BAP(ϵ).

First, assume that one of the two sequences $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ is constant. Without loss of generality we may take $\boldsymbol{\alpha}$ to be constant, and by rotation of the graph, we shall assume $\boldsymbol{\alpha} \equiv 0$. There exists $\delta > 0$ such that the homogeneous square lattice $G_{0,\pi/2}$ satisfies BXP(δ) (see, for example, [Gri99, Sect. 1.7]). By Proposition 5.2.1 with $\xi = \frac{1}{2}\pi$, $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ satisfies BXP(δ') for some $\delta' = \delta'(\delta, \epsilon) > 0$.

Consider now the case of general $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$. By the above, $G_{\boldsymbol{\alpha},\beta_0}$ satisfies BXP(δ'). By Proposition 5.2.1 with $\xi = \beta_0$, $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ satisfies BXP(δ'') for some $\delta'' = \delta''(\delta', \epsilon) > 0$. $\square$

The following is fixed for the rest of this section. Let $\epsilon > 0$, and let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be sequences of angles satisfying BAP(ϵ), (3.1.12). Let ξ be an angle such that $\boldsymbol{\alpha}$ and (ξ) satisfy BAP(ϵ), (3.1.12). All constants in this section may depend on ϵ, but not further on $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, ξ unless otherwise stated.

5.2.2 Track-exchange in an isoradial square lattice

Let G be an isoradial square lattice. The tracks of G are to be viewed as doubly-infinite sequences of rhombi with a common vector. In this section, we describe a procedure for interchanging two consecutive parallel tracks.

Consider a vertical strip $G = G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ of the square lattice, where $\boldsymbol{\alpha} = (\alpha_i : -M \leq i \leq N)$ and $\boldsymbol{\beta} = (\beta_j : j \in \mathbb{Z})$ are vectors of angles satisfying BAP(ϵ), (3.1.12). Thus every *finite*

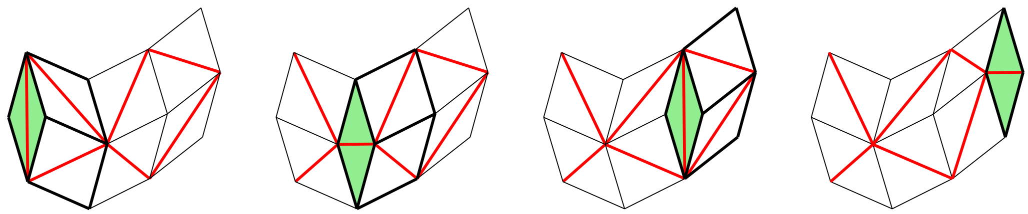

Figure 5.2.1: A new rhombus is introduced on the left (marked in green). This is then 'slid' along the pair of tracks by a sequence of star–triangle transformations, until it reaches the right side where it is removed.

face of G has circumradius 1. There are two types of tracks in G, the finite horizontal tracks (s_j), and the infinite vertical tracks (t_i). We explain next how to exchange two adjacent horizontal tracks by a sequence of star–triangle transformations, employing a process that is implicit in [Ken04]. Track s_j has transverse angle β_j, as illustrated in Figure 3.1.7, and the 'exchange' of two tracks may be interpreted as the interchange of their transverse angles.

We write Σ_j for the operation that exchanges the tracks at levels $j-1$ and j. When applied to G, Σ_j exchanges s_{j-1} and s_j, and we describe Σ_j by reference to $G^\diamond$. If $\beta_j = \beta_{j-1}$, there is nothing to do, and Σ_j interchanges the labels of the tracks without changing the transverse angles. Assume $\beta_j > \beta_{j-1}$. We insert a new rhombus on the left side of the strip formed of s_{j-1} and s_j, marked in green in Figure 5.2.1. This creates a hexagon in $G^\diamond$, containing either a triangle or a star of G. The star–triangle transformation is applied within this hexagon, thereby moving the new rhombus to the right. By repeated star–triangle transformations, we 'slide' the new rhombus along the two tracks from left to right. When it reaches the right side, it is removed. In the new graph, the original tracks s_{j-1} and s_j have been exchanged (or, more precisely, the transverse angles of the tracks at levels $j-1$ and j have been interchanged). Let Σ_j be the transformation thus described, and say that Σ_j 'goes from left to right' when $\beta_j > \beta_{j-1}$. If $\beta_j < \beta_{j-1}$, we construct Σ_j 'from right to left'.

Viewed as an operation on graphs, Σ_j replaces an isoradial graph G by another isoradial graph $\Sigma_j(G)$. It operates also on configurations, as follows. Let ω be an edge-configuration of G, and assign a random state to the new 'green' edge with the distribution appropriate to the isoradial embedding. The star–triangle transformations used in Σ_j are independent applications of the kernels T and S of Figure 3.2.2. The ensuing configuration on $\Sigma_j(G)$ is written $\Sigma_j(\omega)$. Thus Σ_j is a random operator on ω, with randomness stemming from the extra edge and the star–triangle transformations. Note that Σ_j is not a local transformation, in that the state of an edge in $\Sigma_j(G)$ depends on the states of certain distant edges.

Let σ_j denote the permutation that exchanges the $j-1$ and jth terms of a sequence.

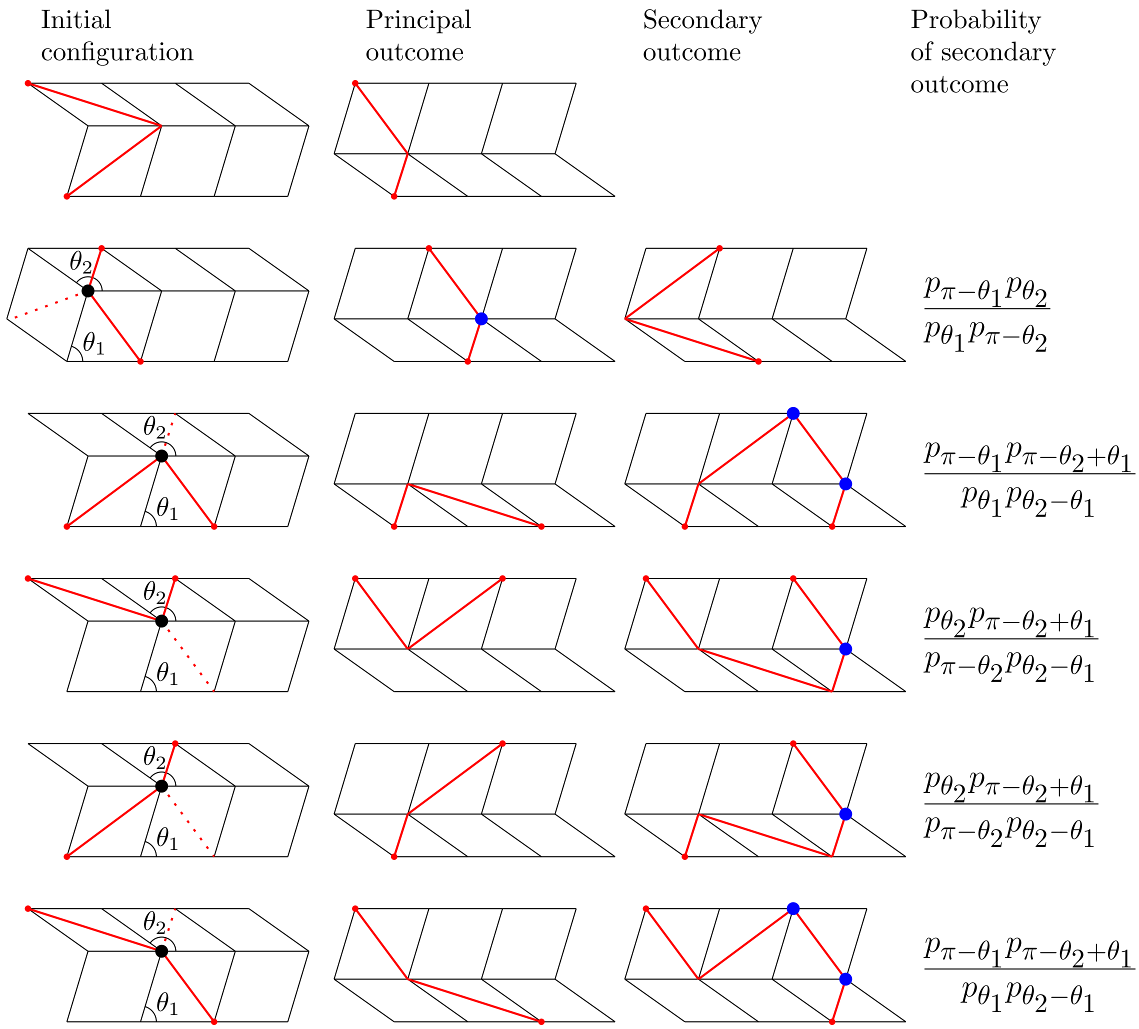

Figure 5.2.2: The six possible ways in which γ may intersect the strip in two edges between height $j-1$ and $j+1$, and the corresponding actions of Σ_j. In five cases, the resulting configuration can be non-deterministic. If the dotted edge is closed, the resulting configuration is in the second column. If it is open, the resulting configuration is that of the third column with the given probability (recall from (3.1.3) that $p_{\pi-\alpha} = 1 - p_\alpha$). The movement of black vertices can cause the height increases marked in blue. The tracks s_k are drawn as horizontal for simplicity, and $\theta_1 = \beta' - \alpha_m$, $\theta_2 = \beta - \alpha_m$, where $v_{m,j}$ denotes the black vertex, and β'/β is the transverse angle of the lower/upper track.

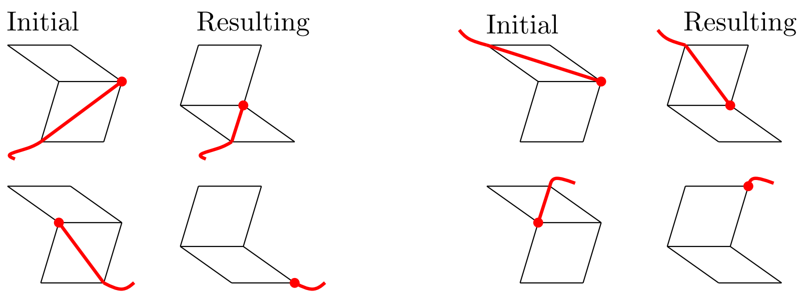

Figure 5.2.3: If an endpoint of γ lies between the two tracks, the corresponding edge is sometimes contracted to a single point.

We may write

$$\Sigma_j(G_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = (G_{\boldsymbol{\alpha},\sigma_j\boldsymbol{\beta}}, \mathbb{P}_{\boldsymbol{\alpha},\sigma_j\boldsymbol{\beta}}).$$

When applying the Σ_j in sequence, we distinguish between the *label* s_j of a track and its *level*. Thus, Σ_j interchanges the tracks currently at levels $j-1$ and j.

We consider next the transportation of open paths. Let ω be a configuration on $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, and let γ be an ω-open path. The action of a star–triangle transformation on γ is discussed in detail in Section 3.2.2. The transformation Σ_j comprises three steps: the addition of an edge to $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, a series of star–triangle transformations, and the removal of an edge. The first step does not change γ, and the effect of the second step is discussed in Section 3.2.2 and the following paragraphs. If the removed edge is in the image of the path γ at the moment of removal, we say that Σ_j *breaks* γ. Thus, $\Sigma_j(\gamma)$ is an open path of $\Sigma_j(G)$ whenever Σ_j does not break γ. In applying the Σ_j, we shall choose the strip-width $M+N$ sufficiently large that open paths of the requisite type do not reach the boundary, and therefore are not broken.

Finally, we summarise in Figures 5.2.2–5.2.3 the action of Σ_j on the path γ, with M and N chosen sufficiently large. Consider two tracks s', s at respective levels $j-1$ and j, with transverse angles β' and β. Edges of γ lying outside levels $j-1$ and j are unchanged by Σ_j. The intersection of γ with these two tracks forms a set of open sub-paths of length either 1 or 2; there are four possible types of length 1, and six of length 2. We do not describe this in detail, but refer the reader to the figures, which are drawn for the case $\beta > \beta'$. The path γ may cross the tracks in more than one of the diagrams on the left of Figure 5.2.2, and the image path contains an appropriate subset of the edges in the listed outcomes. Note that, if the intersections of γ with s and s' are at distance at least 2 from the lateral boundaries, then Σ_j does not break γ,

In the special case when $\beta = \beta'$, Σ_j interchanges the labels of s' and s but alters neither embedding nor configuration. In this degenerate case, we set $\Sigma_j(\gamma) = \gamma$, and note that Figure 5.2.2 remains accurate.

5.2.3 Horizontal crossings

We recall from Section 3.1.7 the notation $B(M, N)$ for the subgraph of an isoradial square lattice induced by the vertices $\{v_{i,j} : -M \le i \le M, 0 \le j \le N\}$.

Proposition 5.2.4. *There exist $\lambda, N_0 \in \mathbb{N}$, depending on ϵ only, such that, for $\rho \in \mathbb{N}$ and $N \ge N_0$,*

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\big(\mathcal{C}_{\mathrm{h}}[B((\rho-1)N, \lambda N)]\big) \\
&\qquad \ge (1-\rho e^{-N})\mathbb{P}_{\boldsymbol{\alpha},\xi}\big(\mathcal{C}_{\mathrm{h}}[B(\rho N, N)]\big) \\
&\qquad\qquad \times \mathbb{P}_{\boldsymbol{\alpha},\xi}\big(\mathcal{C}_{\mathrm{v}}[B(-\rho N, -(\rho-1)N; 0, N)]\big) \\
&\qquad\qquad \times \mathbb{P}_{\boldsymbol{\alpha},\xi}\big(\mathcal{C}_{\mathrm{v}}[B((\rho-1)N, \rho N; 0, N)]\big).
\end{aligned}$$

Proof. We shall make repeated track-exchanges to transform $G_{\boldsymbol{\alpha},\xi}$ into $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, while maintaining the existence of an open path of requisite type.

Fix $\rho \in \mathbb{N}$ with $\rho > 1$, and $\lambda, N_0 \in \mathbb{N}$ to be chosen later, and let $N \ge N_0$. Let

$$\widetilde{\beta}_j = \begin{cases} \xi & \text{if } j < N, \\ \beta_{j-N} & \text{if } j \ge N. \end{cases}$$

We refer to the part of $G = G_{\boldsymbol{\alpha},\widetilde{\boldsymbol{\beta}}}$ above height N as the *irregular block*, and that with height between 0 and N as the *regular* block. The regular block may be viewed as part of $G_{\boldsymbol{\alpha},\xi}$, and the irregular block as part of $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. We will only be interested in the graph above height 0.

We work on a vertical strip $\{v_{i,j} : -M \le i \le M\}$ of G with width $2M$, where

$$M = (\rho + 2\lambda + 1)N, \tag{5.2.1}$$

and we truncate $\boldsymbol{\alpha}$ to a finite sequence $(\alpha_i : -M \le i \le M-1)$.

We will work with graphs obtained from G by a sequential application of the transformations Σ_j of Section 5.2.2, and to this end we let

$$U_k = \Sigma_k \circ \Sigma_{k+1} \circ \cdots \circ \Sigma_{N+k-1}, \qquad k \ge 1. \tag{5.2.2}$$

Note that U_k moves the track at level $N+k-1$ to level $k-1$, while raising the tracks at levels $k-1, \dots, N+k-2$ by one level each (see Figure 5.2.4). We propose to apply $U_1, U_2, \dots, U_{\lambda N}$ to G in turn, thereby moving part of the irregular block beneath the regular block.

Let E_N be the event that there exists an open path of G within $B(\rho N, N)$, with endpoints $v_{x_0,0}$ and $v_{x_1,0}$ for some $x_0 \in [-\rho N, -(\rho-1)N]$ and $x_1 \in [(\rho-1)N, \rho N]$. By the definition of $\widetilde{\boldsymbol{\beta}}$, $B(\rho N, N)$ is entirely contained in the regular block of G. By the

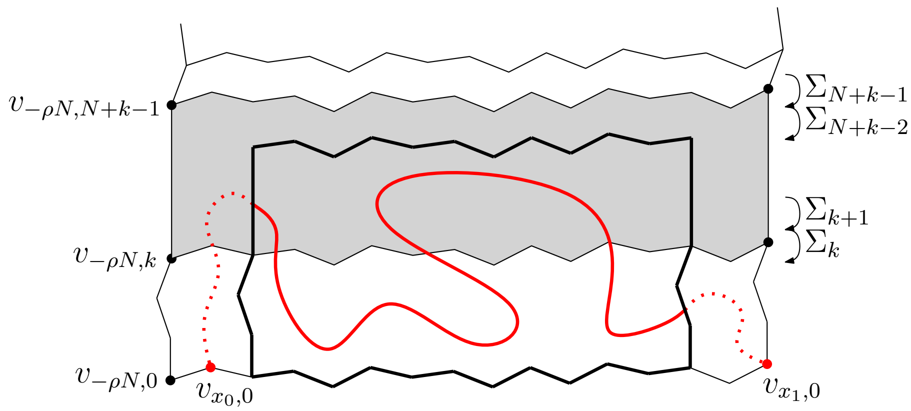

Figure 5.2.4: The transformation U_k raises the (shaded) regular block by one unit, and moves the track above by N units downwards.

Harris–FKG inequality,

$$\begin{aligned}\mathbb{P}_{\boldsymbol{\alpha},\widetilde{\boldsymbol{\beta}}}(E_N) \geq \mathbb{P}_{\boldsymbol{\alpha},\xi}\big(\mathcal{C}_{\mathrm{h}}[B(\rho N, N)]\big)& \\ \times \mathbb{P}_{\boldsymbol{\alpha},\xi}\big(\mathcal{C}_{\mathrm{v}}[B(-\rho N, -(\rho-1)N; 0, N)]\big)& \\ \times \mathbb{P}_{\boldsymbol{\alpha},\xi}\big(\mathcal{C}_{\mathrm{v}}[B((\rho-1)N, \rho N; 0, N)]\big)&.\end{aligned} \tag{5.2.3}$$

Let ω^0 be a configuration on G, chosen according to $\mathbb{P}_G$. For $k \in \mathbb{N}$, let $G^0 = G$ and

$$G^k = U_k \circ \cdots \circ U_1(G), \quad \omega^k = U_k \circ \cdots \circ U_1(\omega^0).$$

The family $(\omega^k : k \geq 0)$ is a sequence of configurations on the G^k with associated law denoted $\mathbb{P}$. Note that $\mathbb{P}$ is given in terms of the law of ω^0, and of the randomizations contributing to the U_i. The marginal law of ω^k under $\mathbb{P}$ is $\mathbb{P}_{G^k}$.

For $\omega^0 \in E_N$, and let γ^0 be a path in $B(\rho N, N)$ with endpoints $v_{x_0,0}$ and $v_{x_1,0}$ for some $x_0 \in [-\rho N, -(\rho-1)N]$ and $x_1 \in [(\rho-1)N, \rho N]$. Let $\gamma^k = U_k \circ \cdots \circ U_1(\gamma^0)$. The path evolves as we apply the U_k sequentially, and most of this proof is directed at studying the sequence $\gamma^0, \gamma^1, \ldots, \gamma^{\lambda N}$.

First we show that the path is not broken by the track-exchanges. For $0 \leq k \leq \lambda N$, set

$$D^k = \big\{v_{x,y} \in (G^k)^\diamond : |x| \leq (\rho+1)N + 2k - y,\ 0 \leq y \leq N + k\big\}.$$

The proof of the following elementary lemma is summarised at the end of this section.

Lemma 5.2.5. *For $0 \leq k \leq \lambda N$, γ^k is an open path contained in D^k.*

The set $\{v_{x,0} : x \in \mathbb{Z}\}$ of vertices of $G^\diamond$ is invariant under the U_k, whence the endpoints of the γ^k are constant for all k. It follows that the horizontal span of $\gamma^{\lambda N}$ is at least

$2(\rho-1)N$.

If $\gamma^{\lambda N}$ has maximal height not exceeding λN, then it contains a $\omega^{\lambda N}$-open horizontal crossing of $B((\rho-1)N, \lambda N)$. The graph $G^{\lambda N}$ agrees with $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ within $B((\rho-1)N, \lambda N)$, so

$$\mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\big(\mathcal{C}_{\mathrm{h}}[B((\rho-1)N, \lambda N)]\big) \geq \mathbb{P}\big(h(\gamma^{\lambda N}) \leq \lambda N \,\big|\, E_N\big)\mathbb{P}(E_N).$$

By (5.2.3), it suffices to show the existence of $\lambda, N_0 \in \mathbb{N}$ such that,

$$\mathbb{P}\big(h(\gamma^{\lambda N}) \leq \lambda N \,\big|\, \omega^0\big) \geq 1-\rho e^{-N}, \qquad N \geq N_0,\ \omega^0 \in E_N, \tag{5.2.4}$$

and the rest of the proof is devoted to this. The basic idea is similar to the corresponding step of Chapter 4 (Proposition 4.3.8), but the calculations are more elaborate.

Let $\omega^0 \in E_N$ and let γ^0 be as above. We observe the evolution of the heights of the images of γ^0 within each column. For $n \in \mathbb{Z}$ and $0 \leq k \leq \lambda N$, set

$$\mathcal{C}_n = \{v_{n,y} : y \in \mathbb{Z}\}, \quad h_n^k = \begin{cases} h(\gamma^k \cap \mathcal{C}_n) & \text{if } \gamma^k \cap \mathcal{C}_n \neq \varnothing, \\ -\infty & \text{otherwise.} \end{cases}$$

Thus, $h(\gamma^{\lambda N}) = \sup\{h_n^{\lambda N} : n \in \mathbb{Z}\}$.

The process $(h_n^k : n \in \mathbb{Z})$, $k = 0, 1, \ldots, \lambda N$, has some lateral drift depending on the directions of the track-exchanges Σ_j. We will modify it in order to relate it to the growth process of Proposition 4.3.8. The track above the regular block is transported by U_k through the regular block, and thus all Σ_j contributing to U_k are in the same direction. Let $(d_k : k \geq 0)$ be given by $d_0 = 0$ and

$$d_{k+1} = \begin{cases} d_k + 1 & \text{if } \beta_k > \xi, \\ d_k & \text{if } \beta_k = \xi, \\ d_k - 1 & \text{if } \beta_k < \xi, \end{cases}$$

and set $H_n^k = h_{n+d_k}^k$. The rest of the proof is devoted to the process $\mathbf{H}^k = (H_n^k : n \in \mathbb{Z})$, $k = 0, 1, \ldots, \lambda N$.

We introduce some notation to be used in the proof. A sequence $\mathbf{R} = (R_n : n \in \mathbb{Z}) \in (\mathbb{Z} \cup \{-\infty\})^{\mathbb{Z}}$ is termed a *range*. The *height in column* n of $\mathbf{R}$ is the value R_n, and the *height* of the range is $\sup\{R_n : n \in \mathbb{Z}\}$. For two ranges $\mathbf{R}^1$, $\mathbf{R}^2$, we write $\mathbf{R}^1 \leq \mathbf{R}^2$ if $R_n^1 \leq R_n^2$ for $n \in \mathbb{Z}$. The *maximum* of a family of ranges is the pointwise supremum sequence. The range $\mathbf{R}$ is called *regular* if

$$|R_{n+1} - R_n| \leq 1, \qquad n \in \mathbb{Z}. \tag{5.2.5}$$

The *mountain* at a point $(n, r) \in \mathbb{Z}^2$ is defined to be the range $\mathbf{M}(n, r) = (M(n, r)_l :$

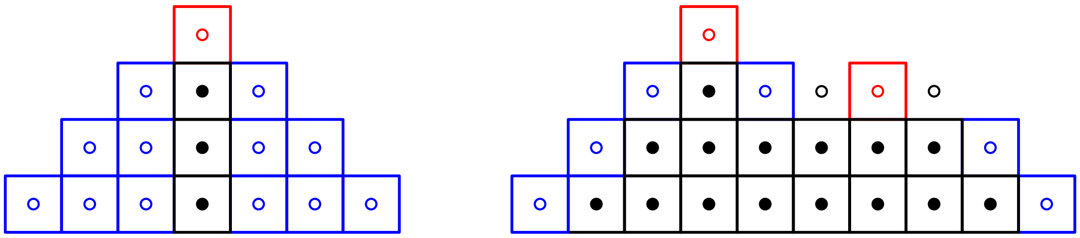

Figure 5.2.5: *Left:* One step in the evolution of $\mathbf{H}$. The initial range $\mathbf{H}^0$ has only one occupied column (black). The blue/black squares form the mountain of the black column. The red square is added at random. *Right:* One step in the evolution of the $\mathbf{X}^k$ (or $\mathbf{H}^k$ when regular). The black squares are the configuration at step k, the blue squares are the additions at time $k+1$ due to the covering, and the red squares are the random additions.

$l \in \mathbb{Z}$) given by

$$M(n,r)_l = \begin{cases} r - |n-l| + 1 & \text{for } l \neq n, \\ r & \text{for } l = n. \end{cases}$$

Note that mountains have flat tops of width 3 centred at (n,r), and sides with gradient ± 1. The *covering* of a range $\mathbf{R}$ is the range $C(\mathbf{R})$ formed as the union of the mountains of each of its elements:

$$C(\mathbf{R}) = \max\left\{\mathbf{M}(n, R_n) : n \in \mathbb{Z}\right\}.$$

We note that $\mathbf{R} \le C(\mathbf{R})$ with sometimes strict inclusion, and also that $\mathbf{R}$ and $C(\mathbf{R})$ have the same height. If $\mathbf{R}$ is regular, the heights of $\mathbf{R}$ and $C(\mathbf{R})$ in any given column differ by at most 1. See Figure 5.2.5 for an illustration of these definitions. We return to the study of $(\mathbf{H}^k)$.

Lemma 5.2.6. *There exists $\eta = \eta(\epsilon) \in (0,1)$, and a family of independent Bernoulli random variables $(Y_n^k : n \in \mathbb{Z},\ 0 \le k < \lambda N)$ with common parameter η such that*

$$H_n^{k+1} \le \max\{C(\mathbf{H}^k)_n, H_n^k + Y_n^k\}, \qquad n \in \mathbb{Z},\ 0 \le k < \lambda N. \tag{5.2.6}$$

The (Y_n^k) are random variables used in the star–triangle transformations, and the probability space may be enlarged to accommodate these variables.

The proof of the lemma is deferred until later in the section. Meanwhile, we continue the proof of (5.2.4) by following that of Proposition 4.3.8. Let (Y_n^k) be as in Lemma 5.2.6, and let $\mathbf{X}^k := (X_n^k : n \in \mathbb{Z})$, $k = 0, 1, \ldots, \lambda N$, be the Markov chain given as follows.

(a) The initial value $\mathbf{X}^0$ is the regular range given by

$$X_n^0 = \begin{cases} N & \text{for } n \in [-\rho N, \rho N]\,, \\ N + \rho N - |n| & \text{for } n \notin [-\rho N, \rho N]\,. \end{cases}$$

(b) For $k \geq 0$, conditionally on $\mathbf{X}^k$, the range $\mathbf{X}^{k+1}$ is given by

$$X_n^{k+1} = \max\left\{X_{n-1}^k, X_n^k + Y_n^k, X_{n+1}^k\right\}, \qquad n \in \mathbb{Z}. \tag{5.2.7}$$

We show first, by induction, that $\mathbf{X}^k \geq \mathbf{H}^k$ for all k. It is immediate that $\mathbf{X}^0$ is regular, and that $\mathbf{X}^0 \geq \mathbf{H}^0$. Suppose that $\mathbf{X}^k \geq \mathbf{H}^k$. By (5.2.7), each range $\mathbf{X}^k$ is regular and

$$\mathbf{X}^{k+1} \geq C(\mathbf{X}^k) \geq C(\mathbf{H}^k). \tag{5.2.8}$$

By (5.2.6), $H_n^{k+1} > C(\mathbf{H}^k)_n$ only if $Y_n^k = 1$. Since $X_n^k \geq H_n^k$, we have in this case that

$$X_n^{k+1} \geq X_n^k + 1 \geq H_n^k + 1 = H_n^{k+1}. \tag{5.2.9}$$

By (5.2.8)–(5.2.9), $\mathbf{X}^{k+1} \geq \mathbf{H}^{k+1}$, and the induction step is complete.

The $\mathbf{X}^k$ are controlled via the following lemma.

Lemma 5.2.7. *There exist $\lambda, N_0 \in \mathbb{N}$, depending on η only, such that*

$$\mathbb{P}\left(\max_n X_n^{\lambda N} \leq \lambda N\right) \geq 1 - \rho e^{-N}, \qquad \rho \in \mathbb{N},\ N \geq N_0.$$

Sketch proof. It is very similar to that of Lemma 4.3.11. A small difference arises through the minor change of the initial value $\mathbf{X}^0$, but this is covered by the inclusion of smaller-order terms in (4.3.31). □

Let λ and N_0 be given thus. For $N \geq N_0$ and $\omega^0 \in E_N$,

$$\begin{aligned} \mathbb{P}\left(h(\gamma^{\lambda N}) \leq \lambda N \,\middle|\, \omega^0\right) &\geq \mathbb{P}\left(\max_n X_n^{\lambda N} \leq \lambda N\right) \\ &\geq 1 - \rho e^{-N}. \end{aligned}$$

This concludes the proof of Proposition 5.2.4. □

Proof of Lemma 5.2.6. Let $k \geq 0$ and let ω be a configuration on G^k. Let γ be an open path on G^k that visits no vertex within distance 2 of the sides of G^k and with $h(\gamma) \leq N+k$. We abuse notation slightly by defining $\mathbf{H}^k$ and $\mathbf{H}^{k+1}$ as in the proof of Proposition 5.2.4 with γ and $U_{k+1}(\gamma)$ instead of γ^k and γ^{k+1}, respectively. That is,

$$H_n^k = h(\gamma \cap \mathcal{C}_{n+d_k}), \quad H_n^{k+1} = h(U_{k+1}(\gamma) \cap \mathcal{C}_{n+d_{k+1}}), \qquad n \in \mathbb{Z}.$$

We will prove that there exists a family of independent Bernoulli random variables ($Y_n : n \in \mathbb{Z}$), independent of ω, with some common parameter $\eta = \eta(\epsilon) > 0$ to be specified later, such that

$$H_n^{k+1} \leq \max\{C(\mathbf{H}^k)_n, H_n^k + Y_n\}, \qquad n \in \mathbb{Z}. \tag{5.2.10}$$

Once this is proved, the i.i.d. family ($Y_n^k : n \in \mathbb{Z},\ 0 \leq k < \lambda N$) may be constructed step by step, by applying the above to the pair ω^k, γ^k for $0 \leq k < \lambda N$. By Lemma 5.2.5, the assumptions on γ are indeed satisfied by each γ^k. By the independence of ($Y_n : n \in \mathbb{Z}$) and ω above, the family ($Y_n^k : n, k$) satisfies the conditions of the lemma.

It remains to prove (5.2.10) for fixed k. If $\beta_k = \xi$, no track-exchange takes place, hence $\mathbf{H}^{k+1} = \mathbf{H}^k$ and (5.2.10) holds. Suppose $\beta_k \neq \xi$. Without loss of generality we may suppose $\beta_k > \xi$, so that $d_{k+1} = d_k + 1$ and the track-exchanges in the application of $U := U_{k+1}$ are all from left to right. To simplify notation we shall assume $d_k = 0$.

Equation (5.2.10) is proved in two steps. First, we will show that

$$H_n^{k+1} \leq \max\{H_{n-i}^k - |i| + 1 : i \in \mathbb{Z}\}. \tag{5.2.11}$$

This equation is a weaker version of (5.2.10) in which each Y_n is replaced by 1.

We prove (5.2.11) by analysing the individual track-exchanges of which U is composed. For $k \leq j \leq N + k$, let $\Psi_j = \Sigma_{j+1} \circ \cdots \circ \Sigma_{N+k}$. Thus, Ψ_{N+k} is the identity, $\Psi_k = U$, and $\Psi_{j-1} = \Sigma_j \circ \Psi_j$. Recall that the diamond graph is bipartite, with the primal and dual vertices as vertex-sets. A vertex $v_{n,r}$ is said to be contained in a range $\mathbf{R}$ if $r \leq R_n$. A set of vertices is contained in $\mathbf{R}$ if every member is thus contained.

Let the sequence ($\mathbf{L}^j : j = N + k, N + k - 1, \ldots, k$) of ranges be defined recursively as follows. First, $\mathbf{L}^{N+k} = \mathbf{H}^k$. We obtain $\mathbf{L}^{j-1}$ from $\mathbf{L}^j$ by increasing its height in certain columns: for each primal vertex $v_{n,j}$ contained in $\mathbf{L}^j$, the heights in columns $n + 1$ and $n + 2$ increase to $j + 1$ and j, if not already at that height or greater.

We claim that $\Psi_j(\gamma)$ is contained in $\mathbf{L}^j$ for $N + k \geq j \geq k$, which is to say that

$$h(\Psi_j(\gamma) \cap \mathcal{C}_n) \leq L_n^j, \qquad n \in \mathbb{Z}. \tag{5.2.12}$$

The above holds for $j = N + k$ by the definition of $\mathbf{L}^{N+k}$, and we proceed by (decreasing) induction on j as follows. The path $\Psi_{j-1}(\gamma)$ is obtained by applying Σ_j to $\Psi_j(\gamma)$, as illustrated in Figure 5.2.2. Possible increases in column heights are marked in blue. Since the black vertices in Figure 5.2.2 are contained in $\mathbf{L}^j$, the blue ones are contained in $\mathbf{L}^{j-1}$. This concludes the induction.

Therefore, $U(\gamma) = \Psi_k(\gamma)$ is contained in $\mathbf{L}^k$, and hence inequality (5.2.11) follows once we have proved that

$$L_{n+1}^k \leq \max\{H_{n-i}^k - |i| + 1 : i \geq -1\}. \tag{5.2.13}$$

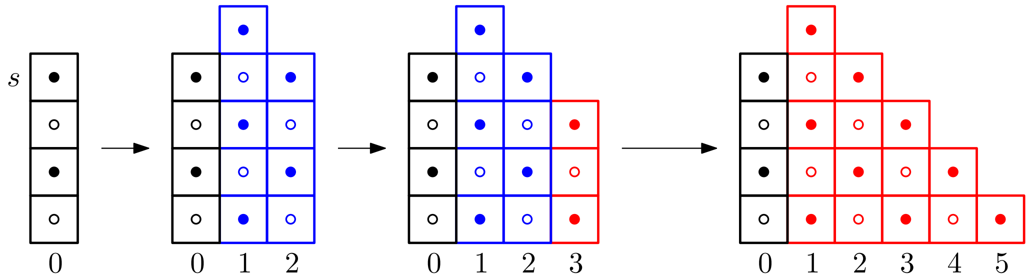

Figure 5.2.6: An illustration of the sequence $\widetilde{\mathbf{L}}(0,s)^j$ beginning with the initial column $\widetilde{\mathbf{L}}(0,s)^{N+k} = \Delta(0,s)$. This column is unchanged up to and including $j = s$, and then it evolves as illustrated.

This we shall do by observing that the sequence $(\mathbf{L}^j)$ is, in a certain sense, additive with respect to its initial state. We think of $\mathbf{L}^{N+k}$ as a union of columns, each of whose evolutions may be followed individually.

Let $r, s \in \mathbb{Z}$ be such that $s \leq N+k$ and $r+s$ is even, so that $v_{r,s}$ is a primal vertex. Let $\Delta(r,s)$ be the range comprising a single column of height s at position r. Consider the sequence $(\widetilde{\mathbf{L}}(r,s)^j)$ with the same dynamics as $(\mathbf{L}^j)$ but with initial state $\widetilde{\mathbf{L}}(r,s)^{N+k} = \Delta(r,s)$. The evolution of $\widetilde{\mathbf{L}}(r,s)^j$ is illustrated in Figure 5.2.6. We have that $\widetilde{\mathbf{L}}(r,s)^j = \Delta(r,s)$ for $N+k \geq j \geq s$ and, for $s > j \geq k$,

$$\widetilde{L}(r,s)^j_m = \begin{cases} -\infty & \text{if } m < r \text{ or } m > r+s-j+1, \\ s & \text{if } m = r, \\ s-(m-r)+2 & \text{if } r < m \leq r+s-j+1. \end{cases}$$

The range $\mathbf{L}^j$ is obtained by combining the contributions of the columns of $\mathbf{H}^k$, in that

$$\mathbf{L}^j = \max\left\{\widetilde{\mathbf{L}}(r, H^k_r)^j : r \in \mathbb{Z}\right\}, \qquad N+k \geq j \geq k. \tag{5.2.14}$$

A rearrangement of the above with $j = k$ implies (5.2.13); (5.2.11) follows by extending the maximum in (5.2.13) over $i \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$ be such that

$$H^k_n + 1 \leq \max\{H^k_{n-i} - |i| + 1 : i \in \mathbb{Z} \setminus \{0\}\}. \tag{5.2.15}$$

Then (5.2.11) implies $H^{k+1}_n \leq C(\mathbf{H}^k)_n$, whence (5.2.10) holds for this particular value of n.

It remains to prove (5.2.10) when (5.2.15) fails. Assume n does not satisfy (5.2.15), so that (5.2.11) implies $H^{k+1}_n \leq H^k_n + 1$. We shall prove that

$$H^{k+1}_n \leq H^k_n + Y_n, \tag{5.2.16}$$

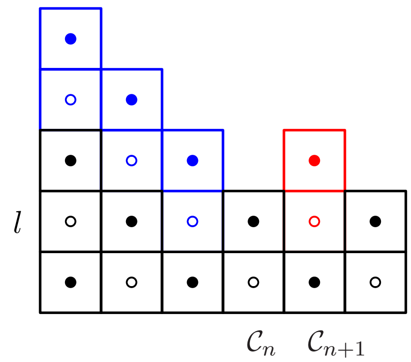

Figure 5.2.7: The environment around $v_{n,l}$. By (5.2.18), the black blocks contain $\mathbf{H}^k$. The range $\mathbf{L}^l$, and hence the path $\Psi_l(\gamma)$, is contained in the aggregate range shown. The height in $\mathcal{C}_{n+1}$ increases only if the red block appears when applying Σ_l.

where the Y_n are independent Bernoulli random variables with respective parameters

$$\eta_k(n) := \frac{p_{\pi-\xi+\alpha_n} p_{\pi-\beta_k+\xi}}{p_{\xi-\alpha_n} p_{\beta_k-\xi}}, \tag{5.2.17}$$

(with p_θ given in (3.1.3)), and which are independent of ω.

Let $l = H_n^k$. We first analyse the action of $\Psi_l = \Sigma_{l+1} \circ \cdots \circ \Sigma_{N+k}$, and then that of Σ_l.

The vertex $v_{n,l}$ is necessarily primal. Since (5.2.15) fails,

$$H_{n-i}^k \leq l + |i| - 1, \qquad i \in \mathbb{Z} \setminus \{0\}.$$

Since each $v_{n-i,l+|i|-1}$ is a dual vertex, we have the strengthened inequality

$$H_{n-i}^k \leq l + |i| - 2, \qquad i \in \mathbb{Z} \setminus \{0\}. \tag{5.2.18}$$

See Figure 5.2.7 for an illustration of the environment around $v_{n,l}$.

By (5.2.12), and (5.2.18) substituted into (5.2.14),

$$h(\Psi_l(\gamma) \cap \mathcal{C}_{n-i}) \leq L_{n-i}^l \leq l + i, \qquad i \geq -1. \tag{5.2.19}$$

Note that Σ_l is the final track-exchange with the potential to add vertices to the path at height $l+1$. Hence, $H_n^{k+1} = l+1$ only if $v_{n+1,l+1}$ is contained in $\Psi_{l-1}(\gamma)$, or, equivalently, only if the height in $\mathcal{C}_{n+1}$ increases to $l+1$ when applying Σ_l to $\Psi_l(\gamma)$.

By (5.2.19) with $i = 0, 1$, the only cases in which this may happen are those of the third and sixth lines of Figure 5.2.2 (with $v_{n,l}$ the black vertex). (See Figure 5.2.8 for a more detailed illustration of the third case.) Moreover, the height in $\mathcal{C}_{n+1}$ increases only if the secondary outcome occurs. In both cases, the secondary outcome occurs with probability $\eta_k(n)$ if the edge $e = \langle v_{n,l}, v_{n+1,l+1} \rangle$ is open, and does not occur if e is closed. We therefore

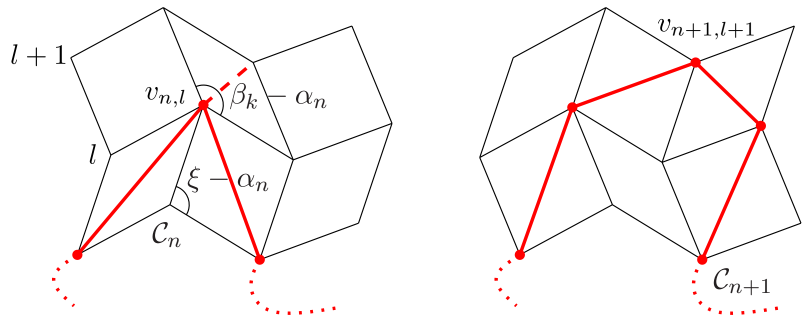

Figure 5.2.8: The third case of Figure 5.2.2. If the dashed edge in the initial configuration (left) is open then, with probability $\eta_k(n)$, the resulting configuration is that on the right side.

provide ourselves with a Bernoulli random variable Y_n, with parameter $\eta_k(n)$, for use in the former situation. We have that $H_n^{k+1} = H_n^k + 1$ only if $Y_n = 1$, and (5.2.16) follows.

Let $A = \xi - \alpha_n$ and $B = \beta_k - \xi$. By (5.2.17),

$$\begin{aligned}\eta_k(n) &= \frac{p_{\pi-A}p_{\pi-B}}{p_A p_B} = \frac{\sin(\frac{1}{3}A)\sin(\frac{1}{3}B)}{\sin(\frac{1}{3}[\pi - A])\sin(\frac{1}{3}[\pi - B])} \\ &= \frac{\cos(\frac{1}{3}[A-B]) - \cos(\frac{1}{3}[A+B])}{\cos(\frac{1}{3}[A-B]) - \cos(\frac{1}{3}[2\pi - A - B])} =: g(A,B).\end{aligned}$$

By assumption, $B > 0$, and so by (3.1.12),

$$\epsilon \le A \le A + B \le \pi - \epsilon. \tag{5.2.20}$$

There exists $c(\epsilon) > 0$ such that, subject to (5.2.20),

$$\cos(\tfrac{1}{3}[A-B]) \ge \cos(\tfrac{1}{3}[A+B]) \ge \cos(\tfrac{1}{3}[2\pi - A - B]) + c(\epsilon).$$

Therefore,

$$\eta := \sup\{g(A,B) : \epsilon \le A \le A + B \le \pi - \epsilon\}$$

satisfies $\eta < 1$, and this concludes the proof of the lemma. $\square$

Proof of Lemma 5.2.5. We sketch this. Since $B(\rho N, N) \subseteq D^0$, we have that $\gamma^0 \subseteq D^0$. It suffices to show that, for $0 \le k < \lambda N$ and γ an open path in D^k, U_k does not break γ and $U_k(\gamma) \subseteq D^{k+1}$.

By considering the individual track-exchanges of which U_k is composed, it may be seen that $\Psi_j(\gamma)$ is an open path contained in D^{k+1} for all j (with $\Psi_j = \Sigma_{j+1} \circ \Psi_{j+1}$ as in the last proof). In considering how $\Psi_j(\gamma)$ is obtained from $\Psi_{j+1}(\gamma)$, it is useful to inspect the different cases of Figure 5.2.2, and in particular those involving blue points. The path may be displaced laterally and, during the sequential application of track-exchanges, the

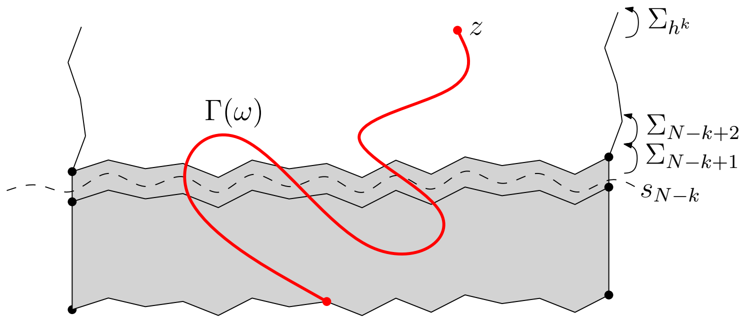

Figure 5.2.9: The transformation V_k moves s_{N-k} upwards by N units.

drift may be extended laterally as it is propagated downwards. The shapes of the D^i have been chosen in such a way that $\Psi_j(\gamma)$ is contained in D^{k+1} for all j. The argument is valid regardless of the direction of Σ_j. □

5.2.4 Vertical crossings

Proposition 5.2.8. *Let $\delta = \frac{1}{2}p^4_{\pi-\epsilon} \in (0, \frac{1}{2})$. There exists $c_N = c_N(\delta) > 0$ satisfying $c_N \to 1$ as $N \to \infty$ such that*

$$\mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\big(\mathcal{C}_{\mathrm{v}}[B(4N,\delta N)]\big) \geq c_N \mathbb{P}_{\boldsymbol{\alpha},\xi}\big(\mathcal{C}_{\mathrm{v}}[B(N,N)]\big), \qquad N \in \mathbb{N}.$$

Proof. The notation of Section 5.2.3 will be used. We work on the graph $G_{\boldsymbol{\alpha},\widetilde{\boldsymbol{\beta}}}$ of the proof of Proposition 5.2.4, and use transformations Σ_j to transport a vertical crossing from the regular block to the irregular section.

Let $N \in \mathbb{N}$, and recall that $G_{\boldsymbol{\alpha},\widetilde{\boldsymbol{\beta}}}$ is a vertical strip of the original graph G of width $2M$. For this proof we take $M = 5N$. For $k \in \{0, 1, \ldots, N-1\}$, set

$$V_k = \Sigma_{2N-k} \circ \cdots \circ \Sigma_{N-k+1}. \tag{5.2.21}$$

The map V_k exchanges the track at level $N-k$ with the N tracks immediately above it. The sequential action of $V_0, V_1, \ldots, V_{N-1}$ moves the regular block upwards track by track, see Figure 5.2.9.

Let ω^0 be a configuration on $G^0 := G_{\boldsymbol{\alpha},\widetilde{\boldsymbol{\beta}}}$ chosen according to its canonical measure $\mathbb{P}_{\boldsymbol{\alpha},\widetilde{\boldsymbol{\beta}}}$, and let

$$\begin{aligned}
G^k &= V_{k-1} \circ \cdots \circ V_0(G_{\boldsymbol{\alpha},\widetilde{\boldsymbol{\beta}}}),\\
\omega^k &= V_{k-1} \circ \cdots \circ V_0(\omega^0),\\
D^k &= \big\{v_{x,y} \in (G^k)^\diamond : |x| \leq N + 2k + y,\ 0 \leq y \leq 2N\big\},\\
h^k &= \sup\big\{h \leq N : \exists x_1, x_2 \in \mathbb{Z} \text{ with } v_{x_1,0} \xleftrightarrow{D^k,\omega^k} v_{x_2,h}\big\}.
\end{aligned}$$

That is, h^k is the greatest height of an open path of G^k starting in $\{v_{x,0} : x \in \mathbb{Z}\}$ and lying in the trapezium D^k. The law $\mathbb{P}$ of the sequence $(\omega^k : k \in \mathbb{N}_0)$ is a combination of the law of ω^0 with those of the star–triangle transformations comprising the V_k.

The box $B(N, N)$ is contained in D^0, and lies entirely in the regular block of G^0. The box $B(4N, \delta N)$ contains the part of D^N between heights 0 and δN, and lies in the irregular section of G^N ($\delta < \frac{1}{2}$ is given in the proposition). Therefore, it suffices to prove the existence of $c_N = c_N(\delta) > 0$ such that $c_N \to 1$ and

$$\mathbb{P}(h^N \geq \delta N) \geq c_N \mathbb{P}(h^0 \geq N). \tag{5.2.22}$$

The remainder of this section is devoted to the proof of (5.2.22).

Let $(\Delta_i : i \in \mathbb{N})$ be independent random variables with common distribution

$$P(\Delta = 0) = 2\delta, \quad P(\Delta = -1) = 1 - 2\delta. \tag{5.2.23}$$

The Δ_i are independent of all random variables used in the construction of the percolation processes of this section. We set

$$H^k = H^0 + \sum_{i=1}^{k} \Delta_i, \tag{5.2.24}$$

where H^0 is an independent copy of h^0, independent of the Δ_i. The inequalities $\leq_{\mathrm{st}}$, $\geq_{\mathrm{st}}$ refer to stochastic ordering.

Lemma 5.2.9. *Let* $0 \leq k < N$*. If* $h^k \geq_{\mathrm{st}} H^k$*, then* $h^{k+1} \geq_{\mathrm{st}} H^{k+1}$*.*

Inequality (5.2.22) is deduced as follows. Evidently, $h^0 \geq_{\mathrm{st}} H^0$ and, by Lemma 5.2.9, $h^N \geq_{\mathrm{st}} H^N$. In particular,

$$\mathbb{P}(h^N \geq \delta N) \geq P(H^N \geq \delta N).$$

Since h^0 and H^0 have the same distribution,

$$\begin{aligned}\frac{\mathbb{P}(h^N \geq \delta N)}{\mathbb{P}(h^0 \geq N)} &\geq \frac{P(H^N \geq \delta N)}{P(H^0 \geq N)} \\ &\geq P(H^N \geq \delta N \mid H^0 \geq N) =: c_N(\delta).\end{aligned}$$

Now, (H^k) is a random walk with mean step-size $2\delta - 1$. By the law of large numbers, $c_N \to 1$ as $N \to \infty$. In addition, $c_N > 0$, and (5.2.22) follows. □

Proof of Lemma 5.2.9. Let $0 \leq k < N$. We apply V_k to G^k, and study the effects of the track-exchanges in V_k. For $N - k \leq j \leq 2N - k$, let $\Psi_j = \Sigma_j \circ \cdots \circ \Sigma_{N-k+1}$, and let D_j^k

be the subgraph of $\Psi_j(G^k)^\diamond$ induced by vertices $v_{x,y}$ with $0 \le y \le 2N$ and

$$|x| \le \begin{cases} N + 2k + y + 2 & \text{if } y \le j, \\ N + 2k + y + 1 & \text{if } y = j + 1, \\ N + 2k + y & \text{if } y > j + 1. \end{cases} \tag{5.2.25}$$

The D^k_j increase with j, and $D^k \subseteq D^k_{N-k}$, $D^k_{2N-k} \subseteq D^{k+1}$.

Let $\omega^k_j = \Psi_j(\omega^k)$ and

$$h^k_j = \sup\big\{h \le N : \exists x_1, x_2 \in \mathbb{Z} \text{ with } v_{x_1,0} \xleftrightarrow{D^k_j, \omega^k_j} v_{x_2,h}\big\},$$

noting that

$$h^k \le h^k_{N-k}, \quad h^{k+1} \ge h^k_{2N-k}. \tag{5.2.26}$$

First, we prove that, for $N - k \le j < 2N - k$,

$$h^k_{j+1} \ge h^k_j - 1, \tag{5.2.27}$$

$$h^k_{j+1} \ge h^k_j \qquad \text{if } h^k_j \ne j + 1, \tag{5.2.28}$$

$$\mathbb{P}(h^k_{j+1} \ge h \mid h^k_j = h) \ge 2\delta \qquad \text{if } h = j + 1. \tag{5.2.29}$$

Fix j such that $N - k \le j < 2N - k$. Let γ be an ω^k_j-open path of $\Psi_j(G^k)$, lying in D^k_j, with one endpoint at height 0 and the other at height h^k_j.

By consideration of Figure 5.2.2, $\Sigma_{j+1}(\gamma)$ is a ω^k_{j+1}-open path contained in D^k_{j+1}. The lower endpoint of γ is not affected by Σ_{j+1}. The upper endpoint is affected only if it is at height $j+1$, in which case its height decreases by at most 1 (see Figure 5.2.3). This proves (5.2.27) and (5.2.28), and we turn to (5.2.29).

Let $\mathcal{P}_j$ be the set of paths γ of $\Psi_j(G^k)$, contained in D^k_j, such that there exists $h > 0$ with:

(a) γ has one endpoint in $\{v_{x,0} : x \in \mathbb{Z}\}$,

(b) its other endpoint lies in $\{v_{x,h} : x \in \mathbb{Z}\}$, and

(c) with the exception of its endpoints, all vertices of γ have heights between 1 and $h-1$.

For $\gamma \in \mathcal{P}_j$, there is a unique such h, which equals its height $h(\gamma)$.

We perform a preliminary computation. Let $\gamma, \gamma' \in \mathcal{P}_j$. We write $\gamma' < \gamma$ if $\gamma' \ne \gamma$, $h(\gamma') = h(\gamma)$, and γ' contains no edge strictly to the right of γ within $\{v_{x,y} : x \in \mathbb{Z},\ 0 \le y \le h(\gamma)\}$. Note that

$$h^k_j = \sup\big\{h(\gamma) : \gamma \in \mathcal{P}_j,\ \gamma \text{ is } \omega^k_j\text{-open}\big\},$$

and denote by $\Gamma = \Gamma(\omega_j^k)$ the ω_j^k-open path of $\mathcal{P}_j$ that is the minimal element of $\{\gamma \in \mathcal{P}_j : h(\gamma) = h_j^k,\ \gamma \text{ is } \omega_j^k\text{-open}\}$ with respect to the order $<$.

We have that

$$\{\Gamma(\omega_j^k) = \gamma\} = \{\gamma \text{ is } \omega_j^k\text{-open}\} \cap N_\gamma, \qquad \gamma \in \mathcal{P}_j, \tag{5.2.30}$$

where N_γ is the decreasing event that:

(a) there is no $\gamma' \in \mathcal{P}_j$ with $h(\gamma') > h(\gamma)$, all of whose edges not belonging to γ are ω_j^k-open,

(b) there is no $\gamma' < \gamma$ with $h(\gamma') = h(\gamma)$, all of whose edges not belonging to γ are ω_j^k-open.

Note that N_γ is independent of the event $\{\gamma \text{ is } \omega_j^k\text{-open}\}$.

Let F be a set of edges of $\Psi_j(G^k)$, disjoint from γ, and let C_F be the event that every edge in F is ω_j^k-closed. Let $\mathbb{P}_j^k$ denote the marginal law of ω_j^k, and p_e the edge-probability of the edge e of $\Psi_j(G^k)$. By (5.2.30) and the Harris–FKG inequality,

$$\begin{aligned}\mathbb{P}(C_F \mid \Gamma = \gamma) &= \frac{\mathbb{P}_j^k(\Gamma = \gamma \mid C_F)}{\mathbb{P}_j^k(\Gamma = \gamma)}\mathbb{P}_j^k(C_F) \\ &= \frac{\mathbb{P}_j^k(N_\gamma \mid C_F)}{\mathbb{P}_j^k(N_\gamma)}\mathbb{P}_j^k(C_F) \\ &\geq \mathbb{P}_j^k(C_F) = \prod_{f \in F}(1 - p_f),\end{aligned} \tag{5.2.31}$$

where we have extended the domain of $\mathbb{P}$ to include the intermediate subsequence of $\omega^k = \omega_{N-k}^k, \omega_{N-k+1}^k, \ldots, \omega_{2N-k}^k = \omega^{k+1}$.

Let $\gamma \in \mathcal{P}_j$ with $h(\gamma) = j + 1$ and suppose $\Gamma(\omega_j^k) = \gamma$. Without loss of generality, we may suppose that Σ_{j+1}, applied to $\Psi_j(G^k)$, goes from left to right; a similar argument holds otherwise.

Let $z = v_{x,j+1}$ denote the upper endpoint of γ and let z' denote the other endpoint of the unique edge of γ leading to z. Either $z' = v_{x+1,j}$ or $z' = v_{x-1,j}$. In the second case, it is automatic as in Figure 5.2.3 that $h(\Sigma_{j+1}(\gamma)) \geq j + 1$.

Assume that $z' = v_{x+1,j}$, as illustrated in Figure 5.2.10, and let $F = \{e_1, e_2, e_3, e_4\}$ where

$$\begin{aligned} e_1 &= \langle v_{x,j+1}, v_{x-1,j+2}\rangle, \quad e_2 = \langle v_{x-1,j+2}, v_{x-2,j+1}\rangle, \\ e_3 &= \langle v_{x-2,j+1}, v_{x-1,j}\rangle, \quad e_4 = \langle v_{x-1,j}, v_{x,j+1}\rangle, \end{aligned}$$

are the edges of the face of $\Psi_j(G^k)$ to the left of z. By definition of $\mathcal{P}_j$, F is disjoint from γ. By studying the three relevant star–triangle transformations contributing to Σ_{j+1} as

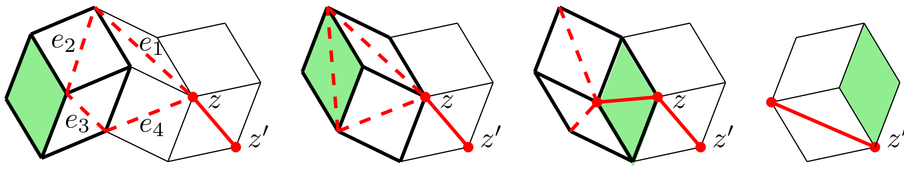

Figure 5.2.10: Three star–triangle transformations contributing to Σ_{j+1}, from left to right. The dashed edges are closed, the bold edges are open. The first and last passages occur with probability 1, and the second with probability $p_{e_1}p_{e_4}/(1-p_{e_1})(1-p_{e_4})$.

illustrated in Figure 5.2.10, we find as in Figure 3.2.2 that

$$\begin{aligned}\mathbb{P}\big(h(\Sigma_{j+1}(\gamma)) \geq j+1 \,\big|\, \Gamma = \gamma\big) &\geq \frac{p_{e_1}p_{e_4}}{(1-p_{e_1})(1-p_{e_4})}\mathbb{P}(C_F \mid \Gamma = \gamma)\\ &\geq \frac{p_{e_1}p_{e_4}}{(1-p_{e_1})(1-p_{e_4})}\mathbb{P}(C_F),\end{aligned}$$

by (5.2.31).

In summary, we have that

$$\begin{aligned}\mathbb{P}(h^k_{j+1} \geq h^k_j \mid \Gamma = \gamma) &\geq \frac{p_{e_1}p_{e_4}}{(1-p_{e_1})(1-p_{e_4})}\prod_{f\in F}(1-p_f) \qquad (5.2.32)\\ &= p_{e_1}p_{e_4}(1-p_{e_2})(1-p_{e_3})\\ &\geq p^4_{\pi-\epsilon} = 2\delta,\end{aligned}$$

by (3.1.4). The proof of (5.2.29) is complete.

It remains to show that (5.2.27)–(5.2.29) imply the lemma. Suppose $h^k \geq_{\rm st} H^k$. We shall bound (stochastically) the h^k_j by a Markov chain, as follows. Let $(X_j : j = N-k,\dots,2N-k)$ be an inhomogeneous Markov chain taking values in $\mathbb{N}_0$, with transition probabilities given by

$$\begin{aligned}X_{j+1} = X_j \quad &\text{if } X_j \neq j+1,\\ P(X_{j+1} = x \mid X_j = j+1) &= \begin{cases} 2\delta & \text{if } x = j+1,\\ 1-2\delta & \text{if } x = j.\end{cases}\end{aligned}$$

One may construct a random variable Δ'_{k+1} with law given by (5.2.23), independent of X_{N-k}, such that $X_{2N-k} - X_{N-k} \geq \Delta'_{k+1}$.

By (5.2.27)–(5.2.29), for all j,

$$\mathbb{P}(h^k_{j+1} \geq x \mid h^k_j = y) \geq P(X_{j+1} \geq x \mid X_j = z), \quad x,y,z \in \mathbb{N}_0,\ z \leq y.$$

Let $X_{N-k} = H^k$. By the induction hypothesis, $H^k \leq_{\rm st} h^k \leq h^k_{N-k}$, whence $X_{2N-k} \leq_{\rm st}$

h^k_{2N-k} by Lemma 4.3.7 iterated. Therefore,

$$h^{k+1} \geq h^k_{2N-k} \geq_{\mathrm{st}} X_{2N-k} \geq X_{N-k} + \Delta'_{k+1} =_{\mathrm{st}} H^{k+1},$$

as claimed. □

5.3 Proof of Theorem 5.1.1: The general case

Let $G \in \mathcal{G}(\epsilon, I)$. By SGP($I$), there exist two families $(s_j : j \in \mathbb{Z})$ and $(t_i : i \in \mathbb{Z})$ of tracks forming a square grid of G. A star–triangle transformation is said to act 'between s_k and s_0' if the three faces of $G^\diamond$ on which it acts are between s_k and s_0. (Recall from Section 3.1.6 that such faces may belong to s_k but not to s_0). A path is said to be between s_0 and s_k if it comprises only edges between s_0 and s_k (that is, edges belonging to faces between s_0 and s_k). A vertex of $G^\diamond$ is said to be just below s_0 if it is adjacent to s_0 and between s_{-1} and s_0.

Let $E_N = E_N(G)$ be the event that there exists an open path γ on G such that:

(a) γ is between s_0 and s_N,

(b) the endpoints of γ are just below s_0,

(c) one endpoint is between t_{-2N} and t_{-N} and the other between t_N and t_{2N}.

We claim that there exists $\delta = \delta(\epsilon, I) > 0$, independent of G and N, such that

$$\mathbb{P}_G(E_N) \geq \delta, \qquad N \geq 1. \tag{5.3.1}$$

Since such a path γ contains a horizontal crossing of the domain $\mathcal{D} = \mathcal{D}(t_{-N}, t_N; s_0, s_N)$, (5.3.1) implies

$$\mathbb{P}_G\big[\mathcal{C}_{\mathrm{h}}(t_{-N}, t_N; s_0, s_N)\big] \geq \delta.$$

Since δ depends only on ϵ and I, the corresponding inequality holds for crossings of translations of $\mathcal{D}$, and also with the roles of the (s_j) and (t_i) reversed. By Proposition 3.1.4, the claim of the theorem follows from (5.3.1), and we turn to its proof.

The method is as follows. Consider the graph G between s_0 and s_N. By making a finite sequence of star–triangle transformations between s_N and s_0, we shall move the s_j downwards in such a way that the section of the resulting graph, lying both between t_{-2N} and t_{2N} and between the images of s_0 and s_N, forms a box of an isoradial square lattice. By Corollary 5.2.2, this box is crossed horizontally with probability bounded away from 0. The above star–triangle transformations are then reversed to obtain a horizontal crossing of $\mathcal{D}$ in the original graph G.

Since a finite sequence of star–triangle transformations changes G at only finitely many places, we may retain the track-notation s_j, t_i throughout their application. We say

$s_j, s_{j+1}, \dots, s_{j+k}$ are *adjacent between* t_{N_1} *and* t_{N_2} if there exists no track-intersection in the domain $\mathcal{D}(t_{N_1}, t_{N_2}; s_j, s_{j+k})$ except those on $s_j, s_{j+1}, \dots, s_{j+k}$. The proof of the next lemma is deferred until later in this section.

Lemma 5.3.1. *There exists a finite sequence* $(T_k : 1 \le k \le K)$ *of star–triangle transformations, each acting between* s_N *and* s_0*, such that, in* $T_K \circ \cdots \circ T_1(G)$*, the tracks* $s_0, \dots, s_N$ *are adjacent between* t_{-2N} *and* t_{2N}*.*

Let $(T_k : 1 \le k \le K)$ be given thus, and write $G^0 = G$ and $G^k = T_k \circ \cdots \circ T_1(G^0)$. Let S_k be the inverse transformation of T_k, as in Section 3.2.1, so that $S_k(G^k) = G^{k-1}$. Since the track notation is retained for each G^k, the event E_N is defined on each such graph. By a careful analysis of its action, we may see that S_k preserves E_N for $k = K, K-1, \dots, 1$. The details are provided in the next paragraph.

Let $1 \le k \le K$ and let γ be an open path of G^k satisfying (a)–(c) above. Since T_k does not move s_0, $S_k(\gamma)$ has the same endpoints as γ. Furthermore, T_k acts between s_N and s_0. Thus the three faces of $(G^k)^\diamond$ on which S_k acts are either all strictly below s_N, or two of them are part of s_N and the third is above. In the first case $S_k(\gamma)$ may differ from γ but is still contained between s_0 and s_N; in the second case S_k does not influence γ. In conclusion, $S_k(\gamma)$ is an open path on G^{k-1} that satisfies (a)–(c).

Since the canonical measure is conserved under a star–triangle transformation, the remark above implies

$$\mathbb{P}_G(E_N) \ge \mathbb{P}_{G^K}(E_N). \tag{5.3.2}$$

It remains to prove a lower bound for $\mathbb{P}_{G^K}(E_N)$.

Write $(r_i : i \in \mathbb{Z})$ for the sequence of all tracks other than the s_j, indexed and oriented according to their intersections with s_0, with $r_0 = t_0$, and including the t_i in increasing order. Let β_j be the transverse angle of s_j, and $\pi + \alpha_i$ that of r_i. Since each r_i intersects each s_j, the vectors $\boldsymbol{\alpha} = (\alpha_i : i \in \mathbb{Z})$, $\boldsymbol{\beta} = (\beta_j : j \in \mathbb{Z})$ satisfy (3.1.12), and hence $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is an isoradial square lattice satisfying BAP(ϵ). By Corollary 5.2.2, there exists $\delta' = \delta'(\epsilon) > 0$ such that $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ satisfies the box-crossing property BXP(δ').

The track-system of G^K inside $\mathcal{D}(t_{-2N}, t_{2N}; s_0, s_N)$ is isomorphic to a rectangle of $\mathbb{Z}^2$, and comprises the horizontal tracks $s_0, s_1, \dots, s_N$, crossed in order by those r_i between (and including) t_{-2N} and t_{2N}. Thus, G^K agrees with $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ inside this domain.

Consider the following boxes of $(G^K)^\diamond$:

$$\begin{aligned} V_1 &= \mathcal{D}(t_{-2N}, t_{-N}; s_0, s_N), \\ V_2 &= \mathcal{D}(t_N, t_{2N}; s_0, s_N), \\ H &= \mathcal{D}(t_{-2N}, t_{2N}; s_0, s_N). \end{aligned}$$

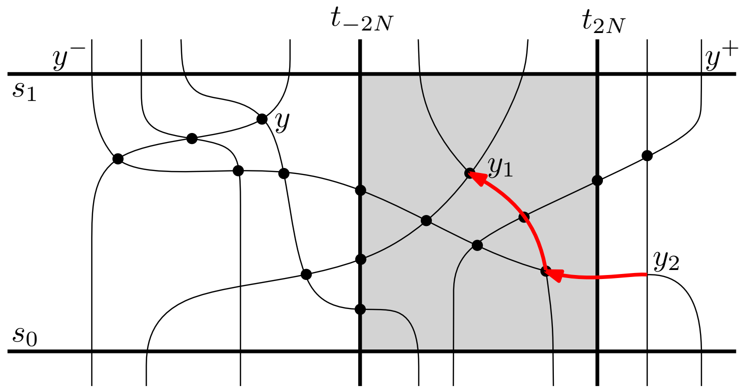

Figure 5.3.1: The black points are indicated. The path γ from y_2 to y_1 is drawn in red. The points y and y_1 are maximal, and are not comparable. The region $\mathcal{R}$ is shaded.

By the Harris–FKG inequality,

$$\begin{aligned} \mathbb{P}_{G^K}(E_N) &\geq \mathbb{P}_{G^K}\left[\mathcal{C}_{\mathrm{v}}(V_1) \cap \mathcal{C}_{\mathrm{v}}(V_2) \cap \mathcal{C}_{\mathrm{h}}(H)\right] \\ &\geq \mathbb{P}_{G^K}[\mathcal{C}_{\mathrm{v}}(V_1)]\mathbb{P}_{G^K}[\mathcal{C}_{\mathrm{v}}(V_2)]\mathbb{P}_{G^K}[\mathcal{C}_{\mathrm{h}}(H)]. \end{aligned} \tag{5.3.3}$$

The boxes V_1, V_2 in $(G^K)^\diamond$ may be regarded as boxes in $G^\diamond_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, and have height N and width at least N. Similarly, the box H has height N and width at most $4IN$. By BXP(δ') and (5.3.3), there exists $\delta = \delta(\epsilon, I) > 0$ such that $\mathbb{P}_{G^K}(E_N) \geq \delta$, and (5.3.2) is proved.

Proof of Lemma 5.3.1. We shall prove the existence of a finite sequence $(T_k : 1 \leq k \leq K)$ of star–triangle transformations, each acting between s_1 and s_0, such that, in $T_K \circ \cdots \circ T_1(G)$, the tracks s_0, s_1 are adjacent between t_{-2N} and t_{2N}. The general claim follows by iteration.

In this proof we work with the graph G only through its track-set $\mathcal{T}$. Tracks will be viewed as arcs in $\mathbb{R}^2$. A *point* of $\mathcal{T}$ is the intersection of two tracks, and we write $\mathcal{P}$ for the set of points.

Let $\mathcal{N}$ be the set of tracks that are not parallel to s_0. Any $r \in \mathcal{N}$ intersects both s_0 and s_1 exactly once, and we orient such r in the direction from its intersection with s_0 to that with s_1.

An oriented path γ on the track-set $\mathcal{T}$ is called *increasing* if it uses only tracks in $\mathcal{N}$ and it conforms to their orientations. For points $y_1, y_2 \in \mathcal{P}$, we write $y_1 \geq y_2$ if there exists an increasing path γ from y_2 to y_1. By the properties of $\mathcal{T}$ given in Section 3.1.3, the relation $\geq$ is reflexive, antisymmetric, and transitive, and is thus a partial order on $\mathcal{P}$.

Let $\mathcal{R}_k$ be the closed region of $\mathbb{R}^2$ delimited by t_{-k}, t_k, s_0, s_1, illustrated in Figure 5.3.1. A point $y \in \mathcal{P}$ is coloured *black* if it is strictly between s_0 and s_1, and in addition $y \geq y'$ for some y' in $\mathcal{R} := \mathcal{R}_{2N}$ or on its boundary. In particular, any point in the interior of $\mathcal{R}$ or of its left/right boundaries is black. We shall see that the black points are precisely

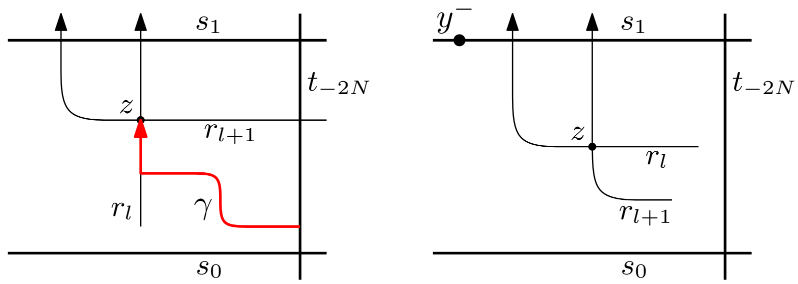

Figure 5.3.2: *Left:* The oriented track r_{l+1} crosses r_l from right to left, in contradiction of the choice of γ as highest. *Right:* The track r_{l+1} crosses r_l from left to right.

those to be 'moved' above s_1 by the star–triangle transformations T_k.

We prove first that the number B of black points is finite. By BAP(ϵ), the number of tracks intersecting $\mathcal{R}$ is finite. Let y^+ (respectively, y^-) be the rightmost (respectively, leftmost) point on s_1 that is the intersection of s_1 with a track r that intersects $\mathcal{R}$. We claim that

$$\text{if } r \in \mathcal{T} \text{ has a black point, then it intersects } s_1 \text{ between } y^- \text{ and } y^+. \tag{5.3.4}$$

Assume (5.3.4) for the moment. Since a black point is the unique intersection of two tracks, and since (5.3.4) implies that there are only finitely many tracks with black points, we have that $B < \infty$.

We prove (5.3.4) next. Let y be a black point. If $y \in \mathcal{R}$, (5.3.4) follows immediately. Thus we may suppose, without loss of generality, that y is strictly to the left of $\mathcal{R}$. There exists an increasing path γ, starting at a point on the left boundary of $\mathcal{R}$ and ending at y. Take γ to be the 'highest' such path. Let $(r_l : 1 \le l \le L)$ be the tracks used by γ in order, where $L < \infty$. We will prove by induction that, for $l \ge 1$,

$$r_l \text{ intersects } s_1 \text{ between } y^- \text{ and } y^+. \tag{5.3.5}$$

Clearly (5.3.5) holds with $l = 1$ since r_1 intersects $\mathcal{R}$.

Suppose $1 \le l < L$ and (5.3.5) holds for r_l, and let $z = r_l \cap r_{l+1}$. If r_{l+1} intersects $\mathcal{R}$ (before or after z), (5.3.5) follows trivially. Suppose r_{l+1} does not intersect $\mathcal{R}$. There are two possibilities: either r_{l+1} crosses r_l from right to left, or from left to right. The first case is easily seen to be impossible, since it contradicts the choice of γ as highest. Hence, r_{l+1} crosses r_l from left to right (see Figure 5.3.2). The part of the oriented track r_{l+1} after z is therefore above the corresponding part of r_l. Since r_{l+1} intersects s_1 after z, and does not intersect $\mathcal{R}$, the intersection of r_{l+1} and s_1 lies between y^- and y^+, and the induction step is complete.

In conclusion r_L intersects s_1 between y^+ and y^-. Let r denote the other track containing y. By the same reasoning, r intersects r_L from left to right, whence it also intersects

s_1 between y^+ and y^-. This concludes the proof of (5.3.4), and we deduce that $B < \infty$.

If $B = 0$, there is no point in the interior of either $\mathcal{R}$ or its left/right sides, whence s_0, s_1 are adjacent between t_{-2N} and t_{2N}.

Suppose $B \geq 1$. We will show that B may be reduced by one by a star–triangle transformation acting between s_0 and s_1, and the claim of the lemma will follow by iteration.

Since $B < \infty$, there exists a black point that is maximal in the partial order $\geq$, and we pick such a point $y = r_1 \cap r_2$. By the maximality of y, the tracks r_1, r_2, s_1 form a track-triangle. By applying the star–triangle transformation to this track-triangle as in Figure 3.2.5, the point y is moved above s_1, and the number of black points is decreased. This concludes the proof of Lemma 5.3.1. □

5.4 Universality for arm exponents

5.4.1 Outline of proof

We recall the isoradial embedding $G_{0,\pi/2}$ of the homogeneous square lattice, with associated measure denoted $\mathbb{P}_{0,\pi/2}$.

Proposition 5.4.1. *Let $k \in \{1, 2, 4, \dots\}$, $\epsilon > 0$, and $I \in \mathbb{N}$. There exist constants $c_i = c_i(k, \epsilon, I) > 0$ and $N_0 = N_0(k, \epsilon, I) \in \mathbb{N}$ such that, for $N \geq N_0$, $n \geq c_0 N_0$, $G \in \mathcal{G}(\epsilon, I)$, and any vertex u of $G^\diamond$,*

$$c_1 \mathbb{P}_{0,\pi/2}[A_k(N, n)] \leq \mathbb{P}_G[A_k^u(N, n)] \leq c_2 \mathbb{P}_{0,\pi/2}[A_k(N, n)].$$

Part (a) of Theorem 5.1.3 is an immediate consequence. Sections 5.4.2–5.4.5 are devoted to the proof of Proposition 5.4.1. In Section 5.4.2 is presented a modified definition of the arm-events, adapted to the context of an isoradial graph. This is followed by Proposition 5.4.2, which asserts in particular the equivalence of the two types of arm-events. The proof of Proposition 5.4.1 follows, using the techniques of the proof of Theorem 5.1.1; the proof for isoradial square lattices is in Section 5.4.3, and for general graphs in Section 5.4.4. Section 5.4.5 contains the proof of Proposition 5.4.2.

For the remainder of this section, $\epsilon > 0$ and $I \in \mathbb{N}$ shall remain fixed. Unless otherwise stated, constants $c_i > 0$, $N_0 \in \mathbb{N}$ depend only on ϵ, I, and on the number k of arms in the event under study. We use the expression 'for $n > N$ large enough' to mean: for $n \geq c_0 N$ and $N > N_0$.

5.4.2 Modified arm-events

Let $G \in \mathcal{G}(\epsilon, I)$, $k \in \{1, 2, 4, \dots\}$, and let s be a track and u be a vertex of $G^\diamond$, adjacent to s. For $n \geq N$, we define the 'modified arm-event' $\widetilde{A}_k^{u,s}(N, n)$ as follows. For simplicity of notation, we omit explicit reference to u and s when no ambiguity results, but in such a case we say that $\widetilde{A}_k(N,n)$ is 'centred at u'. Recall the notation $\Lambda_u^\diamond(n)$ from Section 3.1.5, and the constant c_d of (3.1.7). A vertex $u \in G^\diamond$ is said to satisfy (5.4.1) if it is primal and its open cluster C_u satisfies

$$C_u \subseteq \Lambda_u^\diamond(3c_d^2 n), \tag{5.4.1}$$

and to satisfy (5.4.1)* if it is dual and (5.4.1) holds with C_u replaced by C_u^*.

The modified arm-events $\widetilde{A}_k(N,n) = \widetilde{A}_k^{u,s}(N,n)$ are defined thus:

(i) For $k=1$, $\widetilde{A}_1(N,n)$ is the event that there exist vertices $x_1 \in \Lambda_u^\diamond(N)$ and $y_1 \notin \Lambda_u^\diamond(n)$, both adjacent to s, on the same side of s as u and satisfying (5.4.1), such that $x_1 \leftrightarrow y_1$.

(ii) For $k=2$, $\widetilde{A}_2(N,n)$ is the event that there exist vertices $x_1, x_1^* \in \Lambda_u^\diamond(N)$ and $y_1, y_1^* \notin \Lambda_u^\diamond(n)$, all adjacent to s and on the same side of s as u, such that:

   (a) x_1 and y_1 satisfy (5.4.1), and x_1^* and y_1^* satisfy (5.4.1)*,

   (b) $x_1 \leftrightarrow y_1$ and $x_1^* \leftrightarrow^* y_1^*$.

(iii) For $k = 2j \geq 4$, $\widetilde{A}_k(N,n)$ is the event that there exist vertices $x_1, \dots, x_j \in \Lambda_u^\diamond(N)$ and $y_1, \dots, y_j \notin \Lambda_u^\diamond(n)$, all adjacent to s and on the same side of s as u, such that:

   (a) each x_i and y_i satisfies (5.4.1),

   (b) $x_i \leftrightarrow y_i$ and $x_i \not\leftrightarrow x_{i'}$ for $i \neq i'$.

The technical assumption (5.4.1) will be useful in Section 5.4.3, when applying star–triangle transformations to isoradial square lattices.

The following proposition contains three statements, the third of which relates the modified arm-events to those of Section 1.6. All arm-events A_k and $\widetilde{A}_k$ considered here are centred at the same vertex $u \in G^\diamond$. The event $\widetilde{A}_k(N,n)$ is to be interpreted in terms of any of the tracks to which u is adjacent.

Proposition 5.4.2. *There exist constants $c_i > 0$ such that, for $n > N$ large enough,*

$$\mathbb{P}_G[A_k(N,2n)] \leq \mathbb{P}_G[A_k(N,n)] \leq c_1 \mathbb{P}_G[A_k(N,2n)], \tag{5.4.2}$$

$$\mathbb{P}_G[A_k(N,n)] \leq \mathbb{P}_G[A_k(2N,n)] \leq c_2 \mathbb{P}_G[A_k(N,n)], \tag{5.4.3}$$

$$c_3 \mathbb{P}_G[A_k(N,n)] \leq \mathbb{P}_G[\widetilde{A}_k(N,n)] \leq c_4 \mathbb{P}_G[A_k(N,n)]. \tag{5.4.4}$$

By (5.4.4), for $n > N$ large enough, there exist constants $c_5, c_6 > 0$ such that, if u is adjacent to the tracks s and t,

$$c_5 \mathbb{P}_G[\widetilde{A}_k^{u,s}(N,n)] \leq \mathbb{P}_G[\widetilde{A}_k^{u,t}(N,n)] \leq c_6 \mathbb{P}_G[\widetilde{A}_k^{u,s}(N,n)].$$

The proof of Proposition 5.4.2 is deferred to Section 5.4.5. It relies on the separation theorem of Section 2.3.

5.4.3 Proof of Proposition 5.4.1: Isoradial square lattices

Let G be an isoradial square lattice satisfying the bounded-angles property BAP(ϵ), and let u be a vertex of $G^\diamond$. As usual, the horizontal tracks are labelled $(s_j : j \in \mathbb{Z})$ and the vertical tracks $(t_i : i \in \mathbb{Z})$.

As explained in Section 3.1.7, $G = G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for angle-sequences $\boldsymbol{\alpha} = (\alpha_i : i \in \mathbb{Z})$, $\boldsymbol{\beta} = (\beta_j : j \in \mathbb{Z})$ satisfying (3.1.12). We label $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in such a way that $u = v_{0,0}$, whence u is adjacent to t_0 and s_0 (here we do not require $v_{0,0}$ to be primal). The latter track may change its label through track-exchanges. Let ξ be such that $\boldsymbol{\alpha}$ and the constant sequence (ξ) satisfy BAP(ϵ), (3.1.12). All arm-events in the following are centred at $u = v_{0,0}$.

Lemma 5.4.3. *There exist constants $c_i > 0$ such that, for $n > N$ large enough,*

$$c_1 \mathbb{P}_{\boldsymbol{\alpha},\xi}[A_k(N,n)] \leq \mathbb{P}_G[A_k(N,n)] \leq c_2 \mathbb{P}_{\boldsymbol{\alpha},\xi}[A_k(N,n)].$$

Proof. Let $N, n \in \mathbb{N}$ be picked (later) such that N and n/N are large, and write $M = \lceil 3c_d^2 n \rceil$. For $0 \leq m \leq M$, let G^m be the isoradial square lattice with angle-sequences $\widetilde{\boldsymbol{\alpha}} = (\alpha_i : -4M \leq i \leq 4M)$ and $\widetilde{\boldsymbol{\beta}}^m$, with

$$\widetilde{\beta}_j^m = \begin{cases} \xi & \text{if } -m \leq j < m, \\ \beta_{j+m} & \text{if } -(m+M) \leq j < -m, \\ \beta_{j-m} & \text{if } m \leq j < m+M, \\ \xi & \text{if } j < -(m+M) \text{ or } j \geq m+M. \end{cases} \tag{5.4.5}$$

Thus G^m is obtained from G by taking the horizontal tracks s_j, $-(m+M) \leq j < m+M$, splitting them with a band of height $2m$, and filling the rest of space with horizontal tracks having transverse angle ξ. By the choice of ξ, each G^m satisfies BAP(ϵ). Moreover, inside $\Lambda_u^\diamond(M)$, G^0 is identical to G, and G^M is identical to $G_{\boldsymbol{\alpha},\xi}$.

For $0 \leq m < M$, let

$$U_m = (\Sigma_{m+1} \circ \cdots \circ \Sigma_{m+M}) \circ (\Sigma_{-(m+1)} \circ \cdots \circ \Sigma_{-(m+M)}),$$

where the Σ_j are given in Section 5.2.2. Under U_m, the track at level $m+M$ is moved to the position directly above that at level $m-1$, and the level $-(m+M+1)$ track below the level $-m$ track. We have that

$$U_m(G^m) = G^{m+1}.$$

Let ω^0 be a configuration on G^0 such that $\widetilde{A}_k^{u,s_0}(N,n)$ occurs. Set $j=1$ when $k=1$, and $j=k/2$ when $k\geq 2$. There exist vertices $x_1,\dots,x_j$, $y_1,\dots,y_j$ and, when $k=2$, x_1^*, y_1^*, all lying in the set $\{v_{m,0}: m\in\mathbb{Z}\}$ of vertices of $G^\diamond$, such that:

(a) $x_i \xleftrightarrow{G^0,\omega^0} y_i$ and $x_i \mathrel{\not\xleftrightarrow{G^0,\omega^0}} x_{i'}$ for $i\neq i'$,

(b) $x_1^* \xleftrightarrow{G^0,\omega^0}{}^{*}\, y_1^*$, when $k=2$,

(c) $d^\diamond(v_{0,0},x_i)\leq N$, $d^\diamond(v_{0,0},y_i)>n$,

(d) $d^\diamond(v_{0,0},x_1^*)\leq N$, $d^\diamond(v_{0,0},y_1^*)>n$, when $k=2$,

(e) $C_{x_i}\subseteq \Lambda_u^\diamond(M)$ and, when $k=2$, $C^*_{x_1^*}\subseteq\Lambda_u^\diamond(M)$.

As we apply $U_{M-1}\circ\cdots\circ U_0$ to (G^0,ω^0), the images of paths from each of x_i, y_i, and x_1^*, y_1^* retain their starting points.

Each $\Lambda_u^\diamond(r)$ has a diamond shape. By an argument similar to that of Lemma 5.2.5, for $0\leq m\leq M$,

$$C_{x_i}(\omega^m)\subseteq \Lambda_u^\diamond(M+2m), \quad C^*_{x_1^*}(\omega^m)\subseteq\Lambda_u^\diamond(M+2m).$$

Moreover, since $C_{x_i}(\omega^m)$ and $C^*_{x_1^*}(\omega^m)$ do not extend to the left/right boundaries of G^m, these clusters neither break nor merge with one another. Therefore,

(a) $x_i \xleftrightarrow{G^M,\omega^M} y_i$ and $x_i \mathrel{\not\xleftrightarrow{G^M,\omega^M}} x_{i'}$ for $i\neq i'$,

(b) $x_1^* \xleftrightarrow{G^M,\omega^M}{}^{*}\, y_1^*$, when $k=2$,

so that $\omega^M\in A_k(c_dN, c_d^{-1}n)$. This step is similar to that of Section 4.5.3. In conclusion, there exists $c_3>0$ such that

$$\begin{aligned}\mathbb{P}_G[A_k(N,n)] &\leq c_3\mathbb{P}_{G^0}[\widetilde{A}_k(N,n)] \qquad\text{by (5.4.4)}\\ &\leq c_3\mathbb{P}_{G^M}[A_k(c_dN,c_d^{-1}n)].\end{aligned}$$

Since the intersection of any G^m with $\mathcal{A}(c_dN,c_d^{-1}n)$ is contained in $\Lambda_u^\diamond(M)$, we have by the discussion after (5.4.5) that there exists $c_4>0$ with

$$\begin{aligned}\mathbb{P}_G[A_k(N,n)] &\leq c_3\mathbb{P}_{\boldsymbol{\alpha},\xi}[A_k(c_dN,c_d^{-1}n)]\\ &\leq c_3c_4\mathbb{P}_{\boldsymbol{\alpha},\xi}[A_k(N,n)],\end{aligned}$$

by (5.4.2) and (5.4.3), iterated. The second inequality of Lemma 5.4.3 is proved.

Turning to the first inequality, let ω^M be a configuration on G^M such that $\widetilde{A}_k(N,n)$ occurs (the arm event is defined in terms of $v_{0,0}$ and the horizontal track at level 0). It may be seen as above that $\omega^0=U_{M-1}\circ\cdots\circ U_0(\omega^M)$ is a configuration on G^0 contained in $A_k(c_d N, c_d^{-1} n)$. Furthermore,

$$\mathbb{P}_{\boldsymbol{\alpha},\xi}[A_k(N,n)] \le c_3 \mathbb{P}_{G^M}[\widetilde{A}_k(N,n)] \le c_3 c_4 \mathbb{P}_G[A_k(N,n)].$$

The proof is complete. □

Corollary 5.4.4. *There exist constants $c_i > 0$ such that, for $n > N$ large enough and any isoradial square lattice $G_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathcal{G}(\epsilon, I)$,*

$$c_1 \mathbb{P}_{0,\pi/2}[A_k(N,n)] \le \mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[A_k(N,n)] \le c_2 \mathbb{P}_{0,\pi/2}[A_k(N,n)]. \tag{5.4.6}$$

Proof. If $\boldsymbol{\alpha}$ is a constant vector (α_0), (5.4.6) follows by Lemma 5.4.3 with $\xi = \alpha_0 + \pi/2$.

For $\boldsymbol{\alpha}$ non-constant, we apply Lemma 5.4.3 with $\xi = \beta_0$, thus bounding the arm-event probabilities for $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by those for $G_{\boldsymbol{\alpha},\beta_0}$. Now, $G_{\boldsymbol{\alpha},\beta_0}$ is of the type analysed above, and the conclusion follows. □

5.4.4 Proof of Proposition 5.4.1: The general case

Let $G \in \mathcal{G}(\epsilon, I)$, and let $(s_j : j \in \mathbb{Z})$ and $(t_i : i \in \mathbb{Z})$ be two families of tracks forming a square grid of G, duly oriented. Write $(r_i : i \in \mathbb{Z})$ for the sequence of all tracks other than the s_j, indexed and oriented according to their intersections with s_0, with $r_0 = t_0$, and including the t_i in increasing order. Let β_j be the transverse angle of s_j, and $\pi + \alpha_i$ that of r_i. Since each r_i intersects each s_j, the vectors $\boldsymbol{\alpha} = (\alpha_i : i \in \mathbb{Z})$, $\boldsymbol{\beta} = (\beta_j : j \in \mathbb{Z})$ satisfy (3.1.12), and hence $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is an isoradial square lattice satisfying BAP(ϵ). As in Lemma 5.3.1, we may retain the labelling of tracks throughout the proof. Let u be the vertex adjacent to s_0 and t_0, below and respectively left of these tracks. All arm-events in the following are centred at the vertex u and expressed in terms of the track s_0.

Lemma 5.4.5. *There exist constants $c_1, c_2 > 0$ such that, for $n > N$ large enough,*

$$c_1 \mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[A_k(N,n)] \le \mathbb{P}_G[A_k(N,n)] \le c_2 \mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[A_k(N,n)].$$

This lemma, together with Corollary 5.4.4, implies Proposition 5.4.1 for arm events centred at u. By the square-grid property, any vertex is within bounded distance of one of the tracks $(s_j : j \in \mathbb{Z})$. This allows us to extend the conclusion to arm events centred at any vertex.

Proof. Let $n \in \mathbb{N}$ and $M = \lceil c_d n \rceil$. By Lemma 5.3.1, applied in two stages above and below s_0, there exists a finite sequence R^+ of star–triangle transformations such that, in $G^M := R^+(G)$, the tracks $s_{-M}, \dots, s_M$ are adjacent between t_{-M} and t_M. Moreover, no star–triangle transformation in R^+ involves a rhombus lying in s_0. The sequence R^+ has an inverse sequence denoted R^-. Note that G^M agrees with $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ inside $\Lambda_n + u \subseteq \Lambda_u^{\diamond}(M)$.

Let ω be a configuration on G belonging to $\widetilde{A}_k(N,n)$, and let vertices x_i, y_i be given accordingly. Consider the image configuration $\omega^M = R^+(\omega^0)$ on G^M. By considering the action of the transformation R^+, we may see that

(a) $x_i \xleftrightarrow{G^M,\omega^M} y_i$ and $x_i \not\xleftrightarrow{G^M,\omega^M} x_{i'}$ for $i \neq i'$,

(b) $x_1^* \xleftrightarrow{G^M,\omega^M}{}^* y_1^*$, when $k=2$.

Taken together with (3.1.7), this implies that $\omega^M \in A_k(c_d N, c_d^{-1} n)$. Therefore, there exist $c_i > 0$ such that

$$\begin{aligned}
\mathbb{P}_G[A_k(N,n)] &\leq c_3 \mathbb{P}_G[\widetilde{A}_k(N,n)] && \text{by (5.4.4)}\\
&\leq c_3 \mathbb{P}_{G^M}[A_k(c_d N, c_d^{-1} n)]\\
&= c_3 \mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[A_k(c_d N, c_d^{-1} n)]\\
&\leq c_3 c_4 \mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[A_k(N,n)] && \text{by (5.4.2) and (5.4.3),}
\end{aligned}$$

and the second inequality of the lemma is proved.

Conversely, let ω^M be a configuration on G^M belonging to $\widetilde{A}_k(N,n)$. By applying the inverse transformation, we obtain the configuration $\omega = R^-(\omega^M)$ on G. As above,

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[A_k(N,n)] &= \mathbb{P}_{G^M}[A_k(N,n)]\\
&\leq c_3 \mathbb{P}_{G^M}[\widetilde{A}_k(N,n)] && \text{by (5.4.4)}\\
&\leq c_3 \mathbb{P}_G[A_k(c_d N, c_d^{-1} n)]\\
&\leq c_3 c_4 \mathbb{P}_G[A_k(N,n)] && \text{by (5.4.2) and (5.4.3).}
\end{aligned}$$

This concludes the proof of the first inequality of the lemma. □

5.4.5 Proof of Proposition 5.4.2

This section is devoted to the proof of Proposition 5.4.2, and is not otherwise relevant to the rest of the paper. The two main ingredients of the proof are the separation theorem (Theorem 2.3.1) and the equivalence of metrics, (3.1.7).

Proof of Proposition 5.4.2. Inequalities (5.4.2) and (5.4.3) follow from Corollary 2.3.2 and the box-crossing property for G (Theorem 5.1.1).

Consider the first inequality of (5.4.4) (the second is easier to prove). The idea is as follows. Suppose that $A_k(N,n)$ occurs (together with some additional assumptions). One may construct a bounded number of open or open* box-crossings in order to obtain $\widetilde{A}_k(N,n)$. These two arm-events are given in terms of annuli defined via different metrics — the Euclidean metric and $d^\diamond$, respectively — but the radii of these annuli are comparable by (3.1.7).

Assume $n/N \geq 2$, and let

$$M = c_d^{-1}N, \quad m = c_d n, \tag{5.4.7}$$

with c_d as in (3.1.7). Let $k \in \{1, 2, 4, \dots\}$, $\sigma = (1, 0, 1, 0, \dots)$, and consider the corresponding arm-event $A_k(M, m)$. All constants in the following proof may depend on k, ϵ, and I but, unless otherwise specified, on nothing else. All arm-events that follow are assumed centred at the vertex u adjacent to a track s. By translation, we may assume that u is the origin of $\mathbb{R}^2$. In order to gain some control over the geometry of s, we may assume, without loss of generality, that its transverse angle β satisfies $\beta \in [\frac{1}{4}\pi, \frac{3}{4}\pi]$.

Let $\eta = \eta(k) > 0$ satisfy (2.3.1), and let J be an η-landing sequence of length k, entirely contained in $\{1\} \times [0, 1]$, with J_1 being the lowest interval. Henceforth assume $M \geq N_1$, where N_1 is given in Theorem 2.3.1 with $\eta_0 = \eta$. By that theorem, there exists $c_0 > 0$ such that

$$\mathbb{P}_G[A_k^{J,J}(M, m)] \geq c_0 \mathbb{P}_G\left[A_k(M, m)\right]. \tag{5.4.8}$$

Let (M, v_i) be the lower endpoint of MJ_i, and (M, w_i) the upper. Let H_M be the event that, for $i \in \{1, 2, \dots, k\}$, the following crossings of colour σ_i exist:

(a) a horizontal crossing of $[-w_i, M] \times [v_i, w_i]$,

(b) a vertical crossing of $[-w_i, -v_i] \times [-w_i, w_i]$,

(c) for i odd, a horizontal crossing of $[-w_i, w_i] \times [-w_i, -v_i]$,

(d) for i even, a horizontal crossing of $[-w_i, w_k] \times [-w_i, -v_i]$.

If $k \geq 4$, we require also an open* vertical crossing of $[v_k, w_k] \times [-w_k, 0]$. The event H_M depends only on the configuration inside Λ_M, and is illustrated in Figure 5.4.1.

Let (m, v_i) be the lower endpoint of mJ_i, and (m, w_i) the upper. Let K_m be the event that, for $i \in \{1, 2, \dots, k\}$, the following crossings of colour σ_i exist:

(a) a horizontal crossing of $[m, (m + w_i)] \times [v_i, w_i]$,

(b) a vertical crossing of $[(m + v_i), m + w_i] \times [-(m + w_i), w_i]$,

(c) a horizontal crossing of $[-(m + w_i), m + w_i] \times [-(m + w_i), -(m + v_i)]$,

(d) if i is odd, a vertical crossing of $[-(m + w_i), -(m + v_i)] \times [-(m + w_i), m + w_i]$,

(e) if i is even, a vertical crossing of $[-(m + w_i), -(m + v_i)] \times [-(m + w_i), m + w_k]$.

We require in addition the following:

(f) when $k = 1$, an open* circuit in $\mathcal{A}(2m, 3m)$,

(g) when $k = 2$, an open circuit in $\mathcal{A}(2m, 3m)$,

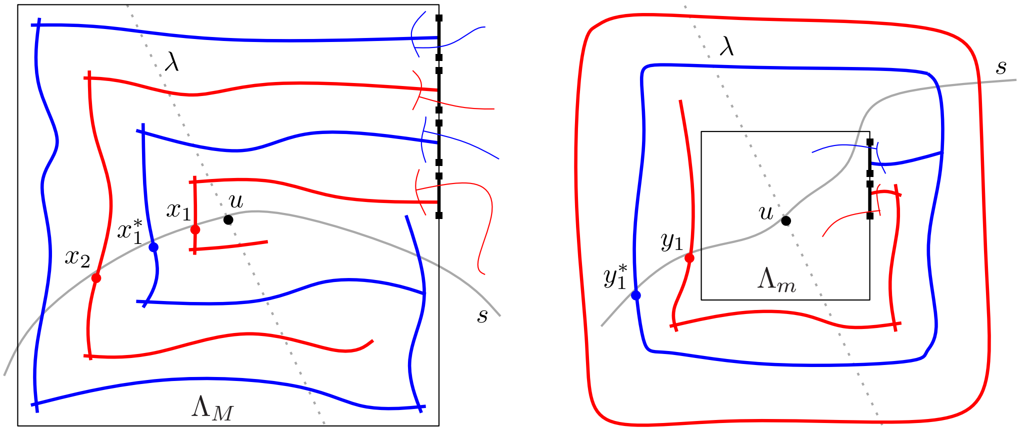

Figure 5.4.1: *Left:* The event H_M for $k = 4$. The red paths are open, the blue paths are open*. The thin coloured paths are parts of the interior fences of $A_k^{J,J}(M,m)$. *Right:* The event K_m for $k = 2$, together with parts of the exterior fences of the arm-event. The track s intersects the open/open* crossings just above the points labelled x_i and y_i.

(h) when $k \geq 2$, an open* circuit in $\mathcal{A}(m + v_k, m + w_k)$.

The event K_m depends only on the configuration inside $\mathcal{A}(m, 3m)$, and is illustrated in Figure 5.4.1.

Set $j = 1$ when $k = 1$, and $j = k/2$ when $k \geq 2$. We claim that, on $H_M \cap K_m \cap A_k^{J,J}(M,m)$, there exist vertices $x_1, \dots, x_j$, $y_1, \dots, y_j$ and, when $k = 2$, x_1^*, y_1^*, all adjacent to s and on the same side of s as u, such that:

(a) $x_i \in \Lambda_M$, $y_i \notin \Lambda_m$ and, when $k = 2$, $x_1^* \in \Lambda_M$, $y_1^* \notin \Lambda_M$,

(b) $x_i \leftrightarrow y_i$ and $x_i \not\leftrightarrow x_{i'}$ for $i \neq i'$,

(c) $x_1^* \leftrightarrow^* y_1^*$ when $k = 2$,

(d) $C_{x_i} \subseteq \Lambda_{3m}$ and, when $k = 2$, $C^*_{x_1^*} \subseteq \Lambda_{3m}$.

This claim holds as follows. The crossings in the definition of H_M (respectively, K_m) may be regarded as extensions of the arms of $A_k^{J,J}(M,m)$ inside Λ_M (respectively, outside Λ_m). Let λ be the straight line with inclination $\beta \in [\frac{1}{4}\pi, \frac{3}{4}\pi]$, passing through u. Since s corresponds to a chain of rhombi with common sides parallel to λ, it intersects λ only in the edge of $G^\diamond$ crossing s and containing u. Therefore, the part of s to the left of λ necessarily intersects all the above extensions. These intersections provide the x_i, y_i and, when $k = 2$, x_1^*, y_1^*. The remaining statements above are implied by the definitions of the relevant events.

By (3.1.7), (5.4.1), and (5.4.7),

$$H_M \cap K_m \cap A_k^{J,J}(M,m) \subseteq \widetilde{A}_k(c_d M, c_d^{-1} m).$$

By Lemma 2.3.3,

$$\mathbb{P}_G\left[H_M \cap K_m \cap A_k^{J,J}(M,m)\right] \geq \mathbb{P}_G(H_M)\mathbb{P}_G(K_m)\mathbb{P}_G\left[A_k^{J,J}(M,m)\right].$$

The events H_M and K_m are given in terms of crossings of boxes with aspect-ratios independent of M and m. Therefore, there exists $c_1 > 0$ such that, for m and M large enough, $\mathbb{P}_G(H_M) \geq c_1$ and $\mathbb{P}_G(K_m) \geq c_1$. In conclusion, by (5.4.7), there exists $c_5 > 0$ such that, for $n/N \geq 2$,

$$\begin{aligned}\mathbb{P}_G[\widetilde{A}_k(N,n)] &\geq \mathbb{P}_G(H_M)\mathbb{P}_G(K_m)\mathbb{P}_G[A_k^{J,J}(M,m)] \\ &\geq c_1^2 c_0 \mathbb{P}_G[A_k(M,m)] && \text{by (5.4.8)} \\ &\geq c_1^2 c_0 c_5 \mathbb{P}_G[A_k(c_d M, c_d^{-1} m)] \\ &= c_1^2 c_0 c_5 \mathbb{P}_G[A_k(N,n)] && \text{by (5.4.7)}\end{aligned}$$

where the third inequality holds by iteration of (5.4.2)–(5.4.3). The first inequality of (5.4.4) follows.

The second inequality is simpler. Set $M = c_d N$ and $m = c_d^{-1} n$. By the equivalence of the euclidian and graph distance (3.1.7), $\widetilde{A}_k(c_d^{-1}M, c_d m) \subseteq A_k(M,m)$. By iteration of (5.4.2)–(5.4.3), there exists $c_6 > 0$ such that, for $m > M$ large enough,

$$\begin{aligned}\mathbb{P}_G[\widetilde{A}_k(c_d^{-1}M, c_d m)] &\leq \mathbb{P}_G[A_k(M,m)] \\ &\leq c_6 \mathbb{P}_G[A_k(c_d^{-1}M, c_d m)].\end{aligned}$$

This concludes the proof of Proposition 5.4.2. $\square$

List of Notation

Against each entry is the page at which the notation was introduced.

Set notation :

$\lvert . \rvert$	98	Euclidian norm on $\mathbb{R}^2$
$\lVert . \rVert_\infty$	64	L^∞ norm on $\mathbb{R}^2$
A^δ	99	Fattening of a set: $\{a + x : \lvert x \rvert \leq \delta\}$
$A + v$	25	Translate of a set: $\{a + v : a \in A\}$
$\mathcal{A}(N, n)$, $\mathcal{A}^u(N, n)$	25	Annulus of inner $\lVert . \rVert_\infty$-radius N and outer $\lVert . \rVert_\infty$-radius n (centered at u)
$B_{m,n}$	109	$[-m, m] \times [0, n]$; planar domains used in Chapter 4
$\mathcal{B}(m, n)$	24	$[0, m] \times [0, n]$; rectangular planar domain
d_{path}	98	Distance between paths
$\mathcal{D}$	24	Planar domain
$\partial\mathcal{D}$	24	Boundary of the domain $\mathcal{D}$
$h(.)$	105	Height
Λ_r	25	Ball of radius r in $(\mathbb{R}^2, \lVert . \rVert_\infty)$

Graph notation :

$\boldsymbol{\alpha}$, $\boldsymbol{\beta}$	94	Sequences of transverse angles
$B(m, n)$	95	Domains defined in terms of tracks in isoradial square lattices
BAP(ϵ)	20	Bounded angles condition with bound ϵ
c_d	90	Constant in the equivalence between $d^\diamond$ and $\lvert . \rvert$
$d^\diamond$	90	Graph distance on $G^\diamond$
$\mathcal{D}(t_1, t_2; s_1, s_2)$	91	Domain between tracks t_1 and t_2 and between s_1 and s_2
e, e^*	14	Pair of primal, dual edges of G, G^*
$G = (V, E)$	13	Graph, usually planar
$G^* = (V^*, E^*)$	14	Dual graph of G
$G^\diamond$	84	Diamond graph of the isoradial graph G
$G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$	94	Isoradial square lattice

Γ, γ	98	Paths on graphs
$\mathbb{H}$	18	Hexagonal lattice
L_e	17	Bound on the length of edges of graphs
L_d, K_d	17	Bounds on the density of vertices of graphs
$\mathbb{L}$	105	Mixed lattice
$\Lambda_u^\diamond(r)$	90	$d^\diamond$-ball of radius r, centered at u
$\mathrm{SGP}(I)$	86	Square grid property with bound I
$\mathbb{T}$	18	Triangular lattice
$\mathcal{T}(G)$	85	Track system of the isoradial graph G
θ_e	20	Angle associated to the edge e of an isoradial graph
$\mathbb{Z}^2$	18	Square lattice

Percolation notation :

$\leftrightarrow$, $\xleftrightarrow{G,\omega}$	13	Open connection in ω
$\overset{G,\omega}{\not\longleftrightarrow}$	13	Negation of $\xleftrightarrow{G,\omega}$
$\overset{*}{\leftrightarrow}$	14	Open* connection (in the dual graph)
$A_\sigma^u(N,n)$, $A_k(N,n)$	26	Arm-events
$A_\sigma^{I,J}(N,n)$	49	Arm-event with imposed landing sequences
$\bar{A}_k^{\mathbb{L}}(n)$	128	Arm-event adapted to mixed lattices
$\widetilde{A}_k^{u,s}(N,n)$	160	Arm-event adapted to isoradial graphs
$\mathrm{BXP}(l_0,\delta)$	24	Box-crossing property with constants δ and l_0
β, ν, γ, Δ	26	Exponents near criticality
$\mathcal{C}_\mathrm{h}(\mathcal{B})$, $\mathcal{C}_\mathrm{v}(\mathcal{B})$	24	Existence of horizontal,respectively vertical, open crossings of $\mathcal{B}$
$\mathcal{C}_\mathrm{h}(m,n)$, $\mathcal{C}_\mathrm{v}(m,n)$	109	$\mathcal{C}_\mathrm{h}(B_{m,n})$ and $\mathcal{C}_\mathrm{v}(B_{m,n})$ respectively
$\mathcal{C}_\mathrm{h}(t_1,t_2;s_1,s_2)$, $\mathcal{C}_\mathrm{v}(t_1,t_2;s_1,s_2)$	91	Existence of horizontal, respectively vertical, open crossings of $\mathcal{D}(t_1,t_2;s_1,s_2)$
C_v	13	Open cluster containing v
δ, η, ρ	26	Exponents at criticality
$\mathcal{G}$, $\mathcal{G}(\epsilon,I)$	20	Family of isoradial graphs with the bounded-angles property and the square-grid property
$\kappa_\square$, $\kappa_\triangle$, $\kappa_{\hexagon}$	22	Functions defining criticality for the square, triangular and hexagonal lattices
$\mathcal{M}$	23	Family of critical inhomogeneous models
$\mathcal{M}_I$, $\mathcal{M}_I(\epsilon)$	23	Family of critical highly inhomogeneous models
$\Omega=\{0,1\}^E$	13	Set of percolation configurations on G
ω, ω^*	14	Primal (respectively dual) configuration

p	13	Percolation intensity
$\mathbf{p}$	13	Family of percolation intensities
p_θ	20	Parameter associated to an edge e of an isoradial graph, with $\theta_e = \theta$
$\mathbb{P}$	13	Percolation measure
$\mathbb{P}_{\mathbf{p}}$	13	Percolation measure with intensities $\mathbf{p}$
$\mathbb{P}^\nu$	21	Percolation measure with shifted parameters
$\mathbb{P}_p^\square$, $\mathbb{P}_p^\triangle$, $\mathbb{P}_p^{\hexagon}$	18	Homogeneous percolation on the square, triangular and hexagonal lattices, with parameter $p \in [0,1]$
$\mathbb{P}_{\mathbf{p}}^\square$, $\mathbb{P}_{\mathbf{p}}^\triangle$, $\mathbb{P}_{\mathbf{p}}^{\hexagon}$	18	Inhomogeneous percolation on the square, triangular and hexagonal lattices, with parameters $\mathbf{p} \in [0,1]^2$, and $\mathbf{p} \in [0,1]^3$ respectively
$\mathbb{P}_{\mathbf{q},\mathbf{q}'}^\square$, $\mathbb{P}_{p,\mathbf{q},\mathbf{q}'}^\triangle$, $\mathbb{P}_{p,\mathbf{q},\mathbf{q}'}^{\hexagon}$	19	Highly inhomogeneous percolation on the square, triangular and hexagonal lattices
$\mathbb{P}_G$	20	Canonical percolation measure on the isoradial graph G
$\mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$	94	Percolation measure associated to $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}$
$\mathrm{rad}(C_v)$	34	Radius of C_v in the $\\|.\\|_\infty$ norm
ρ_σ, ρ_k	26	Arm exponents
S, T	97	Star–triangle transformations
Σ_j	138	Track exchange in isoradial square lattices

Other notation :

$\mathbf{1}_A$	21	Indicator function of the event A
$\vee$, $\wedge$	16, 21	Maximum, respectively minimum
$\lfloor x \rfloor$	64	Greatest integer not greater than x
$\lceil x \rceil$	56	Least integer not less than x
$\leq_{\mathrm{st}}$	15	Stochastic ordering
$f \asymp g$	25	f/g is bounded away from 0 and ∞
$f \asymp_c g$	25	f/g is bounded away from 0 and ∞ uniformly in c
$f \approx g$	25	$\log f(t) / \log g(t) \to 1$

Assumed notation :

$\lvert A \rvert$	-	Cardinality of the set A
$\mathbb{C}$	-	The complex plane
$\mathbb{E}$	-	Expectation
$\mathbb{N}$	-	The set of strictly positive integers
$\mathbb{N}_0$	-	The set of non-negative integers
$\mathbb{R}$	-	The set of real numbers

$\mathbb{Z}$	-	The set of integers

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