Rotational invariance in critical planar lattice models

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December 25, 2020

Abstract

In this paper, we prove that the large scale properties of a number of two-dimensional lattice models are rotationally invariant. More precisely, we prove that the random-cluster model on the square lattice with cluster-weight $1 \le q \le 4$ exhibits rotational invariance at large scales. This covers the case of Bernoulli percolation on the square lattice as an important example. We deduce from this result that the correlations of the Potts models with $q \in \{2, 3, 4\}$ colors and of the six-vertex height function with $\Delta \in [-1, -1/2]$ are rotationally invariant at large scales.

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1 Introduction

1.1 Motivation

Physical systems undergoing a continuous phase transition have been the focus of much attention in the past seventy years, both on the physical and the mathematical sides. Since Onsager's revolutionary solution of the 2D Ising model, mathematicians and physicists tried to understand the delicate features of the critical phase of these systems. In the sixties, the arrival of the renormalization group (RG) formalism (see [29] for a historical exposition) led to a generic (non-rigorous) deep understanding of continuous phase transitions. The RG formalism suggests that "coarse-graining" renormalization transformations correspond to appropriately changing the scale and the parameters of the model under study. The large scale limit of the critical regime then arises as the fixed point of the renormalization transformations.

A striking consequence of the RG formalism is that the assumption that the critical fixed point is unique leads to the prediction that the scaling limit at the critical point must satisfy translation, rotation and scale invariance, which allows one to deduce some information about correlations. In [51], Polyakov outlined a set of arguments pointing towards a much stronger invariance of statistical physics models at criticality: since the scaling limit field theory is a local field, it should be invariant under any map which is locally a composition of translation, rotation and homothety, which leads to postulate full conformal invariance. In [8, 7], Belavin, Polyakov and Zamolodchikov went even further by considering massless field theories that enjoy full conformal invariance from the very start, a fact which allowed them to derive explicit expressions for their correlation functions, hence giving birth to conformal field theories. Once conformal invariance is proved, a whole world of new techniques becomes available thanks to Conformal Field Theory and the Schramm-Loewner Evolution [47], and it is therefore a problem of fundamental importance to prove conformal invariance of the scaling limits of lattice models.

Proving conformal invariance is quite difficult for most lattice models. The examples of models for which such a statement has been obtained can be counted on the fingers of one's hand: site Bernoulli percolation on the triangular lattice [58, 11, 10] (respectively for Cardy's formula, SLE(6) convergence, and CLE(6) convergence), Ising and FK-Ising models [59, 13, 39, 12, 14, 9] (respectively for the fermionic observables in FK-Ising, in Ising, the energy and the spin fields, SLE convergence, and CLE convergence), uniform spanning trees [55], dimers [42], level lines of the discrete GFF [56].

In all the mentioned cases, the proof relied under one form or another, on discrete holomorphic observables satisfying some discrete version of conformally covariant boundary value problems. Mathematicians were therefore able to prove conformal invariance directly, bypassing the road suggested by physicists consisting in first proving scaling and rotation invariance (translation invariance is obvious), and then deducing from it conformal invariance. Unfortunately, today's mathematicians' strategy is very dependent on discrete properties of the system, which explains why we are currently limited to very few instances of proofs of conformal invariance.

In this paper, we perform one step towards the strategy inspired by Field Theory and prove rotational invariance of the large-scale properties of a number of planar models at their critical point. Our strategy is quite general and applies to a number of integrable planar systems. Namely, we treat the case of the random-cluster model (also called Fortuin-Kasteleyn percolation), the Potts models, as well as the six-vertex model. We believe that the reasoning has also applications for the Askhin-Teller model and certain loop models. The proof will proceed by focusing on the random-cluster model and then extending its rotational invariance to other planar lattice models using known mapping between the models.

1.2 Definition of the random-cluster model and distance between percolation configurations

As mentioned in the previous section, the model of central interest in this paper is the random-cluster model, introduced by Fortuin and Kasteleyn around 1970 [30, 31], which we now define. For background, we direct the reader to the monograph [35] and to the lecture notes [19] for an exposition of the most recent results.

Consider the square lattice $(\mathbb{Z}^2, \mathbb{E})$, that is the graph with vertex-set $\mathbb{Z}^2 = \{(n,m) : n,m \in \mathbb{Z}\}$ and edges between nearest neighbours. In a slight abuse of notation, we write $\mathbb{Z}^2$ for the graph itself. Consider a finite subgraph G of the square lattice with vertex-set V and edge-set E. For instance, think of $G = \Lambda_n$ as being the subgraph of $\mathbb{Z}^2$ spanned by the vertex-set $\{-n,\dots,n\}^2$ (we will use the notation Λ_n throughout the paper). A percolation configuration ω on G is an element of $\{0,1\}^E$. An edge e is *open* (in ω) if $\omega_e = 1$, otherwise it is *closed*. A configuration ω can be seen as a subgraph of G with vertex-set V and edge-set $\{e \in E : \omega_e = 1\}$. When speaking of connections in ω, we view ω as a graph. A *cluster* is a connected component of ω.

Definition 1.1. The random-cluster measure on G with edge-weight $p \in [0,1]$, cluster-weight $q > 0$, and free boundary conditions is given by

$$\phi^0_{G,p,q}[\omega] := \frac{p^{|\omega|}(1-p)^{|E|-|\omega|}q^{k(\omega)}}{Z^0_{\mathrm{RCM}}(G,p,q)}, \tag{1}$$

where $|\omega| := \sum_{e\in E}\omega_e$ is the number of open edges, $k(\omega)$ is the number of connected components of the graph, and $Z^0_{\mathrm{RCM}}(G,p,q)$ is a normalising constant called the *partition function* chosen in such a way that $\phi^0_{G,p,q}$ is a probability measure.

For $q \geq 1$, the family of measures $\phi^0_{G,p,q}$ converges weakly as G tends to the whole square lattice to an infinite-volume measure $\phi^0_{p,q}$ on $\{0,1\}^{\mathbb{E}}$. The random-cluster model undergoes a phase transition [5, 26] at a critical parameter

$$p_c = p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$$

in the sense that the $\phi^0_{p,q}$-probability that there exists an infinite cluster is 0 if $p < p_c(q)$, and is 1 if $p > p_c(q)$.

It was also proved in [20, 27] that the phase transition is continuous (i.e. that the probability that 0 is connected to infinity is tending to 0 as $p \searrow p_c$) if and only if $q \leq 4$ (see also [52] for a short proof of discontinuity of the phase transition when $q > 4$). In the whole paper we restrict our attention to the range $q \in [1,4]$. For this reason,

fix $q \in [1,4]$ and $p = p_c(q)$ and drop them from notation.

We will be interested in measuring how close the large scale properties of two random percolation configurations really are. In order to do that, we introduce a rescaling of the lattice and define the random-cluster model on subgraphs of $\delta\mathbb{Z}^2$ with $\delta > 0$. To highlight on which lattice we are working, we will consistently use the subscript δ to refer to a percolation configuration on a subgraph of the lattice $\delta\mathbb{Z}^2$, and write ω_δ for such a configuration. When Ω is a simply connected domain of $\mathbb{R}^2$, write Ω_δ for the intersection of Ω with $\delta\mathbb{Z}^2$.

In [11], Camia and Newman introduced a convenient way of measuring the geometry of large clusters in a percolation configuration in the plane. Let $\mathfrak{C} = \mathfrak{C}(\Omega)$ be the collection of sets $\mathcal{F} = \mathcal{F}_0 \sqcup \mathcal{F}_1$ of two locally finite families $\mathcal{F}_0$ and $\mathcal{F}_1$ of non-self-crossing loops in some simply connected domain Ω that do not intersect each other (even between loops in $\mathcal{F}_0$ and $\mathcal{F}_1$). Define the metric on $\mathfrak{C}$,

$$d_{\mathbf{CN}}(\mathcal{F},\mathcal{F}')\le\varepsilon \iff \left(\begin{array}{c}\forall i\in\{0,1\},\forall\gamma\in\mathcal{F}_i \text{ with } \gamma\subset B(0,1/\varepsilon),\exists\gamma'\in\mathcal{F}'_i, d(\gamma,\gamma')\le\varepsilon\\ \text{and similarly when exchanging } \mathcal{F}' \text{ and } \mathcal{F}\end{array}\right),$$

where, for two loops γ_1 and γ_2, we set

$$d(\gamma_1,\gamma_2):=\inf\sup_{t\in\mathbb{S}^1}|\gamma_1(t)-\gamma_2(t)|,$$

with the infimum running over all continuous one-to-one parametrizations of the loops γ_1 and γ_2 by $\mathbb{S}^1$.

Another way of encoding the geometry of large clusters was proposed by Schramm and Smirnov in [57]. In order to define it formally, let a *quad* Q be the image of an homeomorphism from $[0,1]^2$ to $\mathbb{C}$, and let a,b,c,d be the images of the corners of $[0,1]^2$. A *crossing* of Q is a continuous path in Q going from (ab) to (cd). Let $\mathcal{Q}$ be the set of quads, endowed with the distance between quads given by

$$d_{\mathcal{Q}}(Q,Q'):=d(\partial Q,\partial Q')+|a-a'|+|b-b'|+|c-c'|+|d-d'|.$$

Call $S\subset\mathcal{Q}$ *hereditary* if whenever $Q\in S$, every quad Q' that is such that any crossing of Q contains a crossing of Q' must also belong to S. Let $\mathfrak{H}=\mathfrak{H}(\Omega)$ be the set of closed hereditary subsets of $\mathcal{Q}$. Endow $\mathfrak{H}$ with the smallest topology generated by the sets of the type $\{Q\in S\}_{Q\in\mathcal{Q}}$ and $\{S\cap U=\emptyset\}_{U \text{ open set in } \mathcal{Q}}$. The set $\mathfrak{H}$ with this topology is metrizable, and we denote the metric (whose definition is implicit) by $d_{\mathbf{SS}}(\cdot,\cdot)$.

A configuration ω can be identified with the (automatically hereditary) set $S\in\mathfrak{H}$ containing all the quads that are crossed by an open path in ω (seen as a continuous path in the plane). Similarly, ω can be seen as an element of $\mathfrak{C}$ by considering the loop representation of the model obtained as follows (see Section 3.2 for details): to each ω is associated a *dual configuration* ω^* on the dual graph, as well as a *loop configuration* $\overline{\omega}$ on the medial graph, corresponding basically to the boundaries between the primal and dual clusters. Then, we say that a loop is in $\mathcal{F}_1$ if it is the exterior boundary of a primal cluster, and in $\mathcal{F}_0$ if it is the exterior boundary of a dual cluster. Whether ω is seen as an element of $\mathfrak{H}$ or $\mathfrak{C}$ will depend on the context (it will always be clear which identification is used, if any).

1.3 Main results for the random-cluster model

Below, we state results in simply connected domains Ω with a C^1-smooth boundary, meaning that $\partial\Omega$ can be parametrized by a C^1-function whose differential does not vanish at any point[1]. By taking the limit as Ω tends to $\mathbb{R}^2$ of the results below, we also obtain the statement for the unique infinite-volume measure[2].

[1]Such a condition may be relaxed to cover any Jordan domain, yet we postpone such considerations to a later article to focus on the most interesting aspects of the problem at hand (which are already encompassed in the present framework).

[2]In fact, the proof will consist in first obtaining an infinite-volume version and then deducing from it the finite volume one.

We will identify the rotation by the angle α with the multiplication by $e^{i\alpha}$. Below, $X \sim \mu$ means a random variable X with law μ.

The main theorem of our paper is the following.

Theorem 1.2 (Rotation invariance of critical random-cluster model)**.** *Fix $q \in [1,4]$ and a simply connected domain Ω with a C^1-smooth boundary. For every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(q,\varepsilon,\Omega) > 0$ such that for every $\alpha \in (\varepsilon, \pi - \varepsilon)$ and $\delta \leq \delta_0$, there exists a coupling $\mathbb{P}$ between $\omega_\delta \sim \phi^0_{\Omega_\delta}$ and $\omega'_\delta \sim \phi^0_{(e^{-i\alpha}\Omega)_\delta}$ such that*

$$\mathbb{P}[d_{\mathbf{SS}}(\omega_\delta, e^{i\alpha}\omega'_\delta) > \varepsilon] < \varepsilon,$$
$$\mathbb{P}[d_{\mathbf{CN}}(\omega_\delta, e^{i\alpha}\omega'_\delta) > \varepsilon] < \varepsilon.$$

This theorem has a number of applications for the random-cluster model. First, the definition of the Schramm-Smirnov topology implies, in particular, that crossing probabilities are invariant under rotation in the following sense. For a quad Q, let $\{\omega \in \mathcal{C}(Q)\}$ be the event that Q is crossed in the percolation configuration ω.

Corollary 1.3 (Rotation invariance of crossing probabilities)**.** *Fix $q \in [1,4]$ and a simply connected domain Ω with a C^1-smooth boundary. For every $\varepsilon > 0$ small enough, there exists $\delta_0 = \delta_0(q,\varepsilon,\Omega) > 0$ such that for every quad Q with ε-neighborhood contained in Ω, every $\alpha \in (\varepsilon, \pi - \varepsilon)$, and every $\delta < \delta_0$,*

$$|\,\phi^0_{(e^{i\alpha}\Omega)_\delta}[\mathcal{C}(e^{i\alpha}Q)] - \phi^0_{\Omega_\delta}[\mathcal{C}(Q)]\,| \leq \varepsilon.$$

Furthermore, the condition that Ω contains the ε-neighborhood of Q can be replaced[3] *by $\Omega \supset Q$ when $1 \leq q < 4$.*

We turn to "pointwise correlations". For points $x_1, \dots, x_n$ and a partition $\mathcal{P}$ of $\{x_1, \dots, x_n\}$, let $\mathcal{E}(\mathcal{P}, x_1, \dots, x_n)$ be the event that x_i and x_j are connected if and only if they belong to the same element of $\mathcal{P}$. The following corollary will be useful when studying spin-spin correlations in the Potts model.

Corollary 1.4 (Rotation invariance of connectivity correlations)**.** *Fix $q \in [1,4]$ and a simply connected domain Ω with a C^1-smooth boundary. For every $\varepsilon > 0$ and n, there exists $\delta_0 = \delta_0(q,n,\varepsilon,\Omega) > 0$ such that for every $\alpha \in (\varepsilon, \pi - \varepsilon)$ and $\delta \leq \delta_0$, every $x_1, \dots, x_n \in \Omega_\delta$ at a distance at least ε of each other and of the boundary of Ω, and every partition $\mathcal{P}$ of $\{x_1, \dots, x_n\}$,*

$$|\phi^0_{(e^{i\alpha}\Omega)_\delta}[\mathcal{E}(\mathcal{P}, e^{i\alpha}x_1, \dots, e^{i\alpha}x_n)] - \phi^0_{\Omega_\delta}[\mathcal{E}(\mathcal{P}, x_1, \dots, x_n)]| \leq \varepsilon\, \phi^0_{\Omega_\delta}[\mathcal{E}(\mathcal{P}, x_1, \dots, x_n)],$$

where we use, in a slight abuse of notation, $e^{i\alpha}x_i$ to denote a vertex x of $(e^{i\alpha}\Omega)_\delta$ within a distance δ of the image of x_i under the rotation by the angle α.

[3] We also believe the result to be true for $q = 4$, but in this case both quantities may tend to zero (under certain conditions) as δ tends to 0.

Remark 1.5. We may also study the edge-density variables $\epsilon_e^\Omega := \omega_e - \phi_\Omega^0[\omega_e]$ and prove some rotation invariance for these variables. Obtaining this result requires some standard coupling argument that we postpone to a forthcoming paper in which we will prove additional properties of the near-critical regime of the model related to these edge-density variables.

1.4 Applications to other models

In this section, we explain some applications to other models. The list is not exhaustive, and we believe that the previous result has further implications for a wide class of 2D models at criticality.

Potts model The Potts model is one of the most classical models of ferromagnetism. When $q \in \{2, 3, 4\}$, the model undergoes a continuous phase transition, as predicted by Baxter (see e.g. the book [3]) and proved in [27]. The model is defined as follows. Let $\mathbb{T}_q$ be the simplex in $\mathbb{R}^{q-1}$ containing $(1, 0, \dots, 0)$ such that for any $a, b \in \mathbb{T}_q$,

$$a \cdot b := \begin{cases} 1 & \text{if } a = b, \\ -\frac{1}{q-1} & \text{otherwise} \end{cases}$$

(above and below $\cdot$ denotes the scalar product). Attribute a *spin* variable $\sigma_x \in \mathbb{T}_q$ to each vertex $x \in V$. A *spin configuration* $\sigma = (\sigma_x : x \in V) \in \mathbb{T}_q^V$ is given by the collection of all the spins. Introduce the Hamiltonian of σ defined by

$$H_G(\sigma) := -\sum_{xy \in E} \sigma_x \cdot \sigma_y.$$

The *Gibbs measure on G at inverse temperature $\beta \geq 0$* is defined by the formula, for every $f : \mathbb{T}_q^V \to \mathbb{R}$,

$$\mu_{G,\beta,q}[f] := \frac{1}{Z_{\text{Potts}}(G, \beta, q)} \sum_{\sigma \in \mathbb{T}_q^V} f(\sigma) \exp[-\beta H_G(\sigma)]. \tag{2}$$

Similarly to the random-cluster model, the Potts model exhibits a phase transition at inverse temperature $\beta_c(q) := \frac{q-1}{q} \log(1 + \sqrt{q})$, which separates a phase where correlations decay exponentially fast from a phase where they do not decay. We will always fix q and $\beta = \beta_c$ and therefore drop them from the subscript in the measure.

The following corollary stating the rotational invariance of the spin field is an immediate application (via the Edwards-Sokal coupling) of the corresponding one for the random-cluster model.

Corollary 1.6 (Rotation invariance of spin-spin correlations)**.** *Fix $q \in \{2, 3, 4\}$ and a simply connected domain Ω with a C^1-smooth boundary. For every $\varepsilon > 0$ and n, there exists $\delta_0 = \delta_0(q, \varepsilon, n, \Omega) > 0$ such that for every $\alpha \in (\varepsilon, \pi - \varepsilon)$ and $\delta \leq \delta_0$, every*

$\tau_1, \ldots, \tau_n \in \mathbb{T}_q$, *and every* $x_1, \ldots, x_n \in \Omega_\delta$ *at a distance at least* ε *of each other and of the boundary of* Ω,

$$|\mu_{\Omega_\delta}[\sigma_{x_i} = \tau_i, 1 \leq i \leq n] - \mu_{(e^{i\alpha}\Omega)_\delta}[\sigma_{e^{i\alpha}x_i} = \tau_i, 1 \leq i \leq n]| \leq \varepsilon\, \mu_{\Omega_\delta}[\sigma_{x_i} = \tau_i, 1 \leq i \leq n],$$

where we use, in the slight abuse of notation, $e^{i\alpha}x_i$ *to denote a vertex* x *of* $(e^{i\alpha}\Omega)_\delta$ *within a distance* δ *of the image of* x_i *under the rotation by the angle* α.

Remark 1.7. One may also deduce the rotation invariance of energy n-point correlations (i.e. the correlations of the random variables $\epsilon_e^\Omega := \sigma_x \cdot \sigma_y - \mu_\Omega[\sigma_x \cdot \sigma_y]$ for $e = xy$ an edge of G). Again, the proof of this result is postponed to a forthcoming paper.

These results are of course known for the Ising model (i.e. the $q = 2$ Potts model). In fact, in this case the existence of the scaling limit and its conformal invariance are known, see [12] for the spin field, and [39] for the energy field.

Six-vertex height function The six-vertex model on the torus is the archetypical example of an integrable model. It is defined as follows. For $N > 0$ even, let $\mathbb{T}_{N,M} := (V_{N,M}, E_{N,M})$ be the toroidal square grid graph with $N \times M$ vertices. An *arrow configuration* $\vec{\omega}$ on $\mathbb{T}_{N,M}$ is the choice of an orientation for every edge of $E_{N,M}$. We say that $\vec{\omega}$ satisfies the *ice rule*, or equivalently that it is a *six-vertex configuration*, if every vertex of $V_{N,M}$ has two incoming and two outgoing incident edges in $\vec{\omega}$. These edges can be arranged in six different ways around each vertex as depicted in Figure 1, hence the name of the model. Define the *weight* of a configuration $\vec{\omega}$ to be $W_{6\text{V}}(\vec{\omega}) := a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6}$, where n_i is the number of vertices of $V(\mathbb{T}_{N,M})$ having type i in $\vec{\omega}$. One may define an infinite-volume limit as M and then N tend to infinity, of the measures on $\mathbb{T}_{N,M}$ attributing probability proportional to $W_{6\text{V}}(\vec{\omega})$, which we call $\mathbb{P}^{6\text{V}}_{\mathbb{Z}^2}$, see [23] for details. The measure can also be seen as a measure on gradients of height functions, where we associate to $\vec{\omega}$ a height function h on the dual graph $(\mathbb{Z}^2)^*$ increasing by 1 when crossing an arrow oriented from right to left.

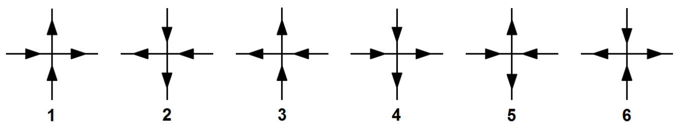

Figure 1: The 6 possibilities, or "types", of vertices in the six-vertex model.

In the next corollary, we claim a rotation invariance result for the six-vertex model in the regime $\Delta := \frac{a^2+b^2-c^2}{2ab} \in [-1, -\frac{1}{2}]$. Again, we rescale the lattice by a factor δ and call the infinite-volume measure thus obtained $\mathbb{P}^{6\text{V}}_{\delta\mathbb{Z}^2}$.

Corollary 1.8 (Rotation invariance of height function correlations)**.** *Fix* $a = b = 1$ *and* $c > 0$ *such that* $1 - c^2/2 \in [-1, -\frac{1}{2}]$. *For every* $\varepsilon > 0$ *and* n, *there exists* $\delta_0 = \delta_0(c, \varepsilon, n) >$

0 such that for every $\alpha \in (\varepsilon, \pi - \varepsilon)$, every $\delta \leq \delta_0$, and every $x_1, \ldots, x_{2n} \in (\delta\mathbb{Z}^2)^*$ at a distance between ε and $1/\varepsilon$ of each other,

$$\Big|\mathbb{E}^{6V}_{\delta\mathbb{Z}^2}\Big[\prod_{i=1}^{n}(h_{x_{2i}} - h_{x_{2i-1}})\Big] - \mathbb{E}^{6V}_{\delta\mathbb{Z}^2}\Big[\prod_{i=1}^{n}(h_{e^{i\alpha}x_{2i}} - h_{e^{i\alpha}x_{2i-1}})\Big]\Big| \leq \varepsilon,$$

where we use the slight abuse of notation $e^{i\alpha}x_i$ to denote a vertex x of $e^{i\alpha}(\delta\mathbb{Z}^2)^*$ within a distance δ of the image of x_i under the rotation by the angle α.

The previous corollary can be improved to give rotational invariance of smooth averages of the height function. We omit the details here. A more general result is mentioned in Remark 7.1.

The six-vertex model height function in the full plane is conjectured to converge to the Gaussian Free Field (GFF) whenever $\Delta \in [-1, 1)$, and one therefore expects the correlations to be not only rotationally invariant but also conformally invariant. Rotation invariance is one step in the direction of proving GFF convergence. The convergence to GFF was obtained for $\Delta = 0$ (more precisely for the directed model of dimers) in [42].

2 Proof Roadmap

In this section, we outline the proof of Theorem 1.2 and introduce several key concepts and results. This roadmap is essential for navigating the rest of the paper; the other parts of the paper may be read separately. Let us mention that all the results in the introduction are deduced from Theorem 1.2 in fairly straightforward ways in Section 7.

The main idea will be to couple the random-cluster model on the square lattice with a random-cluster model on a rotated rectangular lattice (meaning a lattice whose faces are rectangles) which has the line $e^{i\alpha/2}\mathbb{R}$ as axis of symmetry, in such a way that the Camia-Newman and Schramm-Smirnov distances between the two configurations are small. Then, one may couple the original model with the model on the rectangular lattice with this additional symmetry, use this symmetry, and then couple the obtained configuration with the original model, in such a way that the distance between the starting and final configurations is small with probability very close to one. This will therefore prove that the symmetry in question is an approximate symmetry of the original model. Together with the symmetries with respect to horizontal lines, this will imply the approximate rotational symmetry. In order to implement the scheme, we need a few additional notions.

2.1 Random-cluster model on isoradial rectangular graphs.

An *isoradial graph* $\mathbb{L}$ is a planar graph embedded in the plane in such a way that (i) every face is inscribed in a circle of radius 1 and (ii) the center of each circumcircle is contained in the corresponding face, see Fig. 2. We sometimes call the embedding *isoradial* (note that it is a property of the embedding and that the graph can have several isoradial embeddings). Isoradial graphs were introduced by Duffin in [16] in the

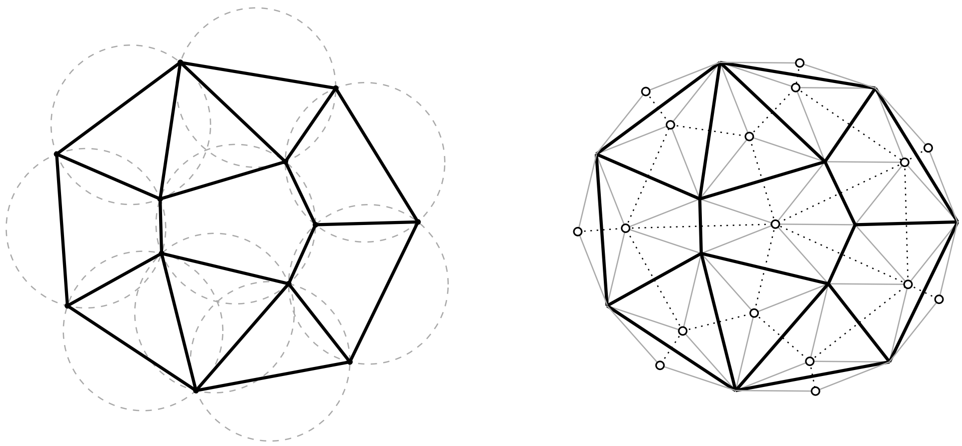

Figure 2: The black graph is (a finite part of) an isoradial graph. All its faces can be inscribed into circumcircles of radius one. The centers of the inscribing circles have been drawn in white; the dual edges are in dotted lines. The diamond graph is drawn in gray in the right picture.

context of discrete complex analysis, and later appeared in the physics literature in the work of Baxter [2], where they are called Z-invariant graphs. The term isoradial was only coined later by Kenyon, who studied discrete complex analysis on these graphs [43]. Since then, isoradial graphs have been studied extensively; we refer to [13, 45, 48] for literature on the subject.

Given an isoradial graph $\mathbb{L}$ (which we call the *primal graph*), we can construct its *dual graph* $\mathbb{L}^*$ as follows: the vertex-set is given by the circumcenters of faces of $\mathbb{L}$, and the edges connect vertices that correspond to faces of $\mathbb{L}$ that share an edge. The *diamond graph* associated to $\mathbb{L}$ has vertex-set given by the vertices of $\mathbb{L}$ and $\mathbb{L}^*$, and edge-set given by the pairs (x, u) with $x \in \mathbb{L}$ and each $u \in \mathbb{L}^*$ which is the center of a face adjacent to x. All edges of the diamond graph are of length 1, and the diamond graph is a rhombic tiling of the plane. See Figure 2 for an illustration.

A track of $\mathbb{L}$ is a bi-infinite sequence of adjacent faces $(r_i)_{i\in\mathbb{Z}}$ of the diamond graph, with the edges shared by each r_i and r_{i+1} being parallel. The angle formed by any such edge with the horizontal axis is called the *transverse angle* of the track.

Isoradial graphs considered in this paper are of a very special type, see Figure 3. They will all be isoradial embeddings of the square lattice; moreover we assume that all diamonds have bottom and top edges that are horizontal. A consequence of this assumption is that the diamond graph can be partition into *(horizontal) tracks* t_i with a constant transverse angle α_i. When the sequence of track angles is $\boldsymbol{\alpha} = (\alpha_i)_{i\in\mathbb{Z}} \in (0, \pi)^{\mathbb{Z}}$, denote the graph by $\mathbb{L}(\boldsymbol{\alpha})$. When $\alpha_i = \alpha$ for every i, simply write $\mathbb{L}(\alpha)$ and call such lattices *rectangular lattices*. Note that $\mathbb{L}(\alpha)$ is a rotated version of a rectangular lattice that has $e^{i\alpha/2}\mathbb{R}$ as axis of symmetry. In particular, $\mathbb{L}(\frac{\pi}{2})$ is simply a rescaled and rotated (by an angle of $\pi/4$) version of $\mathbb{Z}^2$.

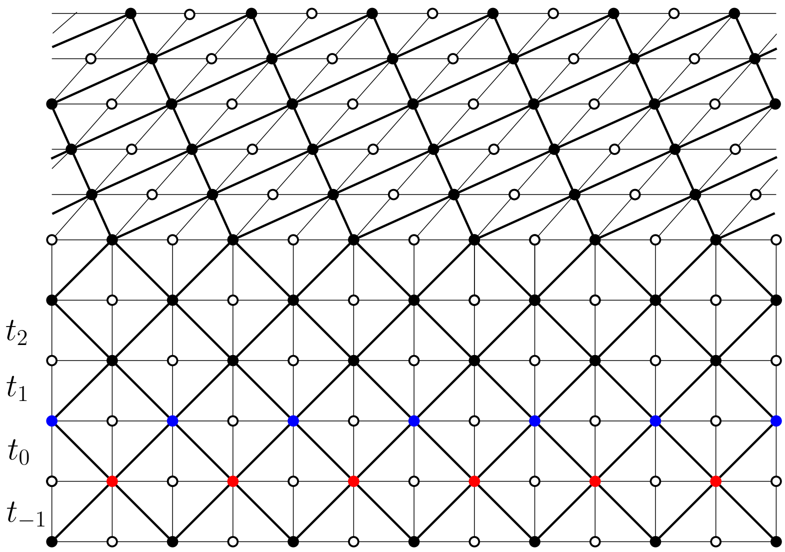

Figure 3: An example of a graph $\mathbb{L}(\boldsymbol{\alpha})$, where α_i is equal to $\frac{\pi}{2}$ for $i \leq 3$, and α above. The diamond graph is drawn in light black lines, the white points refer to the vertices of the dual lattice, and the black points and the stronger black lines refer to the primal lattice. One sees that both the lower and upper parts are portions of a rotated rectangular lattice, and that below this rectangular lattice is simply the square lattice. The vertices of t_0^- are drawn in red, and those of t_0^+ in blue.

2.2 Universality among isoradial rectangular graphs and a first version of the coupling.

As described in Section 3, isoradial graphs $\mathbb{L}(\boldsymbol{\alpha})$ are associated to a canonical set of edge-weights, therefore producing random-cluster measures $\phi_{\delta\mathbb{L}(\boldsymbol{\alpha})}$ on $\delta\mathbb{L}(\boldsymbol{\alpha})$. The next theorem states that the behaviour on different rectangular isoradial graphs is universal. In a way, this is the cornerstone of the paper.

Theorem 2.1 (Universality of critical random-cluster models on rectangular graphs)**.** *For $q \in [1, 4]$ and $\varepsilon > 0$, there exists $\delta_0 = \delta_0(q, \varepsilon) > 0$ such that for every $\delta < \delta_0$ and $\alpha \in (\varepsilon, \pi - \varepsilon)$, there exists a coupling $\mathbf{P}_{\alpha,\delta,\varepsilon}$ between $\omega \sim \phi_{\delta\mathbb{L}(\alpha)}$ and $\omega' \sim \phi_{\delta\mathbb{L}(\frac{\pi}{2})}$ such that*

$$\mathbf{P}_{\alpha,\delta,\varepsilon}[d_{\mathbf{CN}}(\omega, \omega') > \varepsilon] < \varepsilon,$$
$$\mathbf{P}_{\alpha,\delta,\varepsilon}[d_{\mathbf{SS}}(\omega, \omega') > \varepsilon] < \varepsilon.$$

This result states the universality of the scaling limit among rectangular lattices. It will be shown in Section 7 that it implies Theorem 1.2. Even though we already mentioned it before, let us recall that the proof will consist in using this theorem to relate the model on $\delta\mathbb{L}(\frac{\pi}{2})$ to the one on $\delta\mathbb{L}(\alpha)$, then use the reflection with respect to $e^{i\alpha/2}\mathbb{R}$, and finally use again the theorem to relate back the new graph to the model on a rotated version of $\delta\mathbb{L}(\frac{\pi}{2})$.

To describe the coupling of Theorem 2.1, let us first ignore the rescaling by δ and simply work with $\delta = 1$. The coupling $\mathbf{P}_{\alpha,\delta,\varepsilon}$ will then simply be the push forward by the map $x \mapsto \delta x$ of a coupling between configurations in $\mathbb{L}(\alpha)$ and $\mathbb{L}(\frac{\pi}{2})$.

A naive and simplified version of the coupling can be described fairly easily. We do it now. The construction is based on exchanging tracks by successive applications of the star-triangle transformation. Below, let $\mathbf{T}_i$ be the track-exchange operator (constructed in Section 3.5) exchanging the tracks t_i and t_{i-1}. This track exchange is seen as a random map on configurations, and a deterministic one on lattices; it maps $\mathbb{L}(\boldsymbol{\alpha})$ to $\mathbb{L}(\boldsymbol{\alpha}')$ where $\boldsymbol{\alpha}'$ is obtained from $\boldsymbol{\alpha}$ by exchanging α_i and α_{i-1}. It also maps configurations on $\mathbb{L}(\boldsymbol{\alpha})$ to possibly random configurations in $\mathbb{L}(\boldsymbol{\alpha}')$ by applying successive star-triangle operations. One of its most important features is that the push-forward of $\phi_{\mathbb{L}(\boldsymbol{\alpha})}$ by $\mathbf{T}_i$ is $\phi_{\mathbb{L}(\boldsymbol{\alpha}')}$. For readers who are not familiar with these notions, everything is detailed in Section 3.5.

Coupling: version 1

1) Let $\mathbb{L}^{(0)}$ be the lattice with angles

$$\alpha_j = \alpha_j(\alpha, N) := \begin{cases} \alpha & \text{if } j \geq N, \\ \frac{\pi}{2} & \text{if } j < N. \end{cases}$$

and sample $\omega_\delta^{(0)} \sim \phi_{\mathbb{L}^{(0)}}$.

2) Recursively for $0 \leq t < T := 2N \times \lceil 2N/\sin\alpha \rceil$, define

$$j(t) := N + (2N+1)\lfloor t/(2N) \rfloor - t$$

and

$$\begin{aligned} \mathbb{L}^{(t+1)} &:= \mathbf{T}_{j(t)}(\mathbb{L}^{(t)}), \\ \omega^{(t+1)} &:= \mathbf{T}_{j(t)}(\omega^{(t)}). \end{aligned}$$

Since the track-exchange operator $\mathbf{T}_{j(t)}$ preserves the law of the random-cluster model, we have that

$$\omega^{(t)} \sim \phi_{\mathbb{L}^{(t)}} \qquad \text{for every } t.$$

Also, note that $\omega^{(0)}$ and $\omega^{(T)}$ are not quite sampled according to $\phi_{\mathbb{L}(\pi/2)}$ and $\phi_{\mathbb{L}(\alpha)}$, but the law of the restriction to the strip $\mathbb{R} \times [-N, N]$ is the same on $\mathbb{L}^{(0)}$ and $\mathbb{L}(\frac{\pi}{2})$ (resp. $\mathbb{L}^{(T)}$ and $\mathbb{L}(\alpha)$) due to classical properties of the track-exchange operator (see Remark 3.12 for a more precise statement).

The problem with this first version of the coupling is that it lacks ergodic properties that are essential to our proof (or at least it is not straightforward to prove them). We therefore introduce below a slightly modified version of the coupling where the configuration is resampled at each step, just keeping some relevant information on $\omega^{(t)}$. Which relevant information will be dictated by the following paragraph.

2.3 The homotopy topology and the second and third versions of the coupling.

The track-exchange operator behaves well with respect to certain properties of the collection of loops in the percolation configuration, among which the inclusion between large loops/clusters and the homotopy class of large loops in punctured planes. We will therefore work with the interpretation of configurations as collection of loops $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1)$, but with a different distance than the Camia-Newman one.

For $\eta > 0$ and a loop γ, let $[\gamma]_\eta$ be its cyclic homotopy class in $\mathbb{R}^2 \setminus \mathbb{B}_\eta$, where $\mathbb{B}_\eta := \eta\mathbb{Z}^2 \cap [-1/\eta, 1/\eta]^2$. The homotopy classes will be encoded by reduced words, see Section 5.1 for a detailed definition (for an explanation of why we chose to work with homotopy classes rather than the maybe more intuitive inclusion, see Figure 4). Introduce the distance defined by

$$d_{\mathbf{H}}[\mathcal{F}, \mathcal{F}'] \le \eta \iff \left(\begin{array}{l} \forall i \in \{0,1\}, \forall \gamma \in \mathcal{F}_i \text{ surrounding at least 2 but not all points in } \mathbb{B}_\eta \\ \exists \gamma' \in \mathcal{F}'_i \text{ s.t. } [\gamma]_\eta = [\gamma']_\eta, \text{ and similarly when exchanging } \mathcal{F} \text{ and } \mathcal{F}' \end{array} \right).$$

This distance controls the Camia-Newman and Schramm-Smirnov distances, as stated in the next theorem.

Theorem 2.2 (Correspondence between different topologies)**.** *Fix $q \in [1,4]$. For every $\kappa > 0$, there exist $\eta = \eta(q,\kappa) > 0$ and $\delta_0 = \delta_0(\kappa, \eta) > 0$ such that for every $\delta < \delta_0$, and every $\alpha \in (0, \pi)$, if $\mathbf{P}$ denotes a coupling between $\omega_\delta \sim \phi^0_{\delta\mathbb{L}(\pi/2)}$ and $\omega'_\delta \sim \phi^0_{\delta\mathbb{L}(\alpha)}$,*

$$\mathbf{P}[d_{\mathbf{H}}[\omega_\delta, \omega'_\delta] \le \eta \text{ and } d_{\mathbf{SS}}[\omega_\delta, \omega'_\delta] \ge \kappa] \le \kappa,$$
$$\mathbf{P}[d_{\mathbf{H}}[\omega_\delta, \omega'_\delta] \le \eta \text{ and } d_{\mathbf{CN}}[\omega_\delta, \omega'_\delta] \ge \kappa] \le \kappa.$$

It may at first sight look strange that the shape of a large loop is well determined by its homotopy class. Indeed, one may produce arbitrarily large loops that have trivial homotopy. Yet, recall that percolation clusters are typically fractal, and that it is therefore unlikely that large parts of their contour do not contribute to the complexity of their homotopy class. For instance, one may easily see that it is very unlikely that a large loop is homotopically (almost) trivial.

With this theorem in our hands, we can reformulate Theorem 2.1 into the following theorem.

Theorem 2.3 (Universality of critical random-cluster models on rectangular graphs)**.** *For $q \in [1,4]$ and $\varepsilon > 0$, there exists $\delta_0 = \delta_0(q,\varepsilon) > 0$ such that for every $\delta < \delta_0$ and $\alpha \in (\varepsilon, \pi - \varepsilon)$, there exists a coupling $\mathbf{P}_{\alpha,\delta,\varepsilon}$ between $\omega \sim \phi_{\delta\mathbb{L}(\alpha)}$ and $\omega' \sim \phi_{\delta\mathbb{L}(\frac{\pi}{2})}$ such that*

$$\mathbf{P}_{\alpha,\delta,\varepsilon}[d_{\mathbf{H}}(\omega, \omega') > \varepsilon] < \varepsilon.$$

The trivial proof below justifies that we henceforth focus on deriving Theorem 2.3.

Proof of Theorem 2.1. Theorems 2.3 and 2.2 combine to give Theorem 2.1. ☐

The fact that we are only interested in the homotopy classes of loops suggests that we may allow ourselves to resample the configuration at every step only keeping non-trivial homotopy classes in mind. This naturally leads to the next coupling (notations are the same as in the previous section).

Below, introduce the multiset $[\cdot]_{\eta,i}(\omega)$ gathering[4] the homotopy classes in $\mathbb{R}^2 \setminus N\mathbb{B}_\eta$ (here homotopy class is meant in the sense of Remark 5.2) of the loops in $\mathcal{F}_i(\omega)$ (when ω is seen as an element of the Camia-Newman space) that surround at least two but not all points in $N\mathbb{B}_\eta$.

Coupling: second version

1) Sample $\omega^{(0)} \sim \phi_{\mathbb{L}^{(0)}}$.

2) Recursively for $0 \leq t < T$, given $\omega^{(t)}$,

- Sample $\omega^{(t+1/2)} \sim \phi_{\mathbb{L}^{(t)}}[\,\cdot\,|([\cdot]_{\eta,0}, [\cdot]_{\eta,1})(\omega^{(t+1/2)}) = ([\cdot]_{\eta,0}, [\cdot]_{\eta,1})(\omega^{(t)})]$,
- Sample $\omega^{(t+1)} := \mathbf{T}_{j(t)}(\omega^{(t+1/2)})$.

The construction still guarantees that $\omega^{(t)}$ has law $\phi_{\mathbb{L}^{(t)}}$ at each time step. Furthermore, the resampling trick keeps only the homotopy classes of loops in mind, while guaranteeing sufficient refreshment at each step. The problem with this second coupling is that we actually mislead the reader into believing that the track-exchange preserves in a reasonable fashion the homotopy classes of large loops.

What is true is that it preserves the homotopy "between loops". As a consequence, it is in fact more convenient to consider the homotopy classes of large loops in ω not with respect to points in $N\mathbb{B}_\eta$, but rather with respect to certain clusters, which we will call "marked nails" (see Section 6 for a formal definition). At this point we do not enter into precise considerations concerning these nails, but simply mention that they will be mesoscopic clusters of ω which are close to the points in $N\mathbb{B}_\eta$. It will be crucial to control how the positions of these nails evolve during the process. At this stage, and in order not to complicate the discussion too much, let us informally consider $\mathcal{H}_{\text{intro}}(\omega)$ to be the information of the *position of the marked nails*, as well as the *homotopy classes* in $\mathbb{R}^2 \setminus \{\text{marked nails}\}$ of the loops in $\mathcal{F}(\omega)$ surrounding at least 2 and not all marked nails (we will see how to make formal sense of these notions in Section 6).

Coupling: third version

1) Sample $\omega_\delta^{(0)} \sim \phi_{\mathbb{L}^{(0)}}$.

2) Recursively for $0 \leq t < T$, given $\omega^{(t)}$,

[4] Formally, it is a function from the set of homotopy classes into non-negative integers.

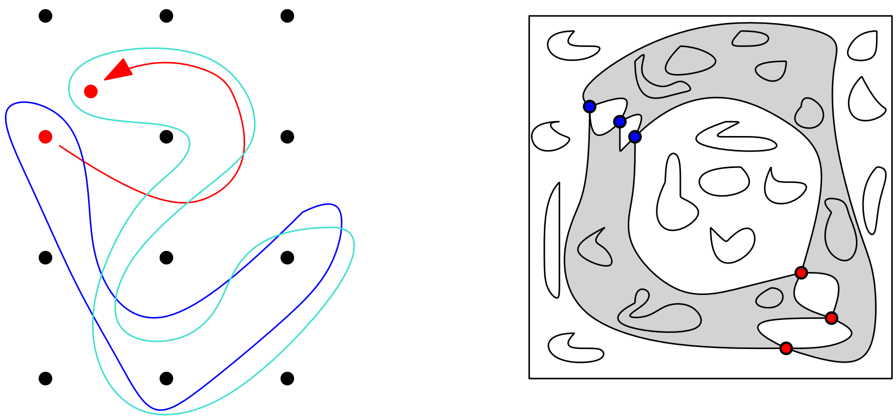

Figure 4: On the left, an example of a transformation of the red points that continuously transforms a blue loop with a certain homotopy class into another blue loop with a different homotopy class. It is therefore crucial to encode homotopy classes in a coherent way, which will be done in Section 5.1. Also, on the right, a justification of why we did not choose to keep track of a simpler property of loops, namely, the inclusion between loops. In the picture, the two loops created by opening one of the blue dots, or one of the red dots, have the same inclusion properties with respect to other loops, but have very different large scale connectivity properties (in particular they are far apart in Camia-Newman and Schramm-Smirnov distances). They do however have different homotopy classes with respect to other loops.

- Sample $\omega^{(t+1/2)} \sim \phi_{\mathbb{L}^{(t)}}[\,\cdot\,|\mathcal{H}_{\text{intro}}(\omega^{(t+1/2)}) = \mathcal{H}_{\text{intro}}(\omega^{(t)})]$,
- Sample $\omega^{(t+1)} := \mathbf{T}_{j(t)}(\omega^{(t+1/2)})$.

The true coupling will be made completely explicit in Section 6, in particular with a precise definition of the formal equivalent $\mathcal{H}(\omega)$ of $\mathcal{H}_{\text{intro}}(\omega)$. The coupling will be close (but not quite the same) to this one. There will be small technicalities related to the definition of marked nails, but all of this will be treated carefully in Section 6, and the fourth version of the coupling defined there will be the final one.

The true (and interesting) challenge with this third coupling is to manage to relate the homotopy classes in $\mathbb{R}^2 \setminus \{\text{marked nails}\}$ to those in $\mathbb{R}^2 \setminus \mathbb{B}_\eta(N)$. Indeed, the coupling will perfectly preserve the former, but these homotopy classes relate to homotopy classes of $\mathbb{R}^2 \setminus \mathbb{B}_\eta(N)$ only if the marked nails are not moving too much. The main part of the proof of Theorem 2.3 will be to show that this is indeed the case.

In order to do that, we will approximately write the global displacement of extrema for the nails as a sum of independent increments whose laws are dictated by the action of a track-exchange on Incipient Infinite Clusters with three-arms in half-planes. More precisely, fix $\alpha, \beta \in (0, \pi)$. Introduce the *(half-plane three-arm) Incipient Infinite Cluster*

(IIC) on $\mathbb{L}(\beta)$ defined informally by the formula

$$\mathbb{\Phi}[\cdot] = \phi_{\mathbb{L}(\beta)}[\,\cdot\,|\mathrm{lmax}(\infty) = 0],$$

where $\mathrm{lmax}(\infty)$ is the left-most highest vertex on the infinite cluster (the conditioning is degenerate, but can be made sense of, see the proper definition in Section 3.4).

Then, consider a series of track exchanges bringing down a track of angle α from $+\infty$ to $-\infty$. We will prove that the average height of the highest vertex on the infinite cluster after this series of track exchanges is 0. In other words, the "drift" induced by passing down a track of angle α through an environment of tracks of angle β is zero.

This result will then be combined with the fact that highest points of nails look like highest points of the infinite cluster in the half-plane three arm IIC to prove that extremal coordinates of large clusters do not move much throughout the coupling described above. To complete this, soft arguments will enable us to extend this property to extrema in the other directions.

To conclude this part, let us mention that the original idea of [36, 37, 38] was to prove that macroscopic clusters of Bernoulli percolation do not move too fast when applying the track exchanges to transform one isoradial graph into another[5]. In this paper, we refine the argument by studying the drift of large clusters through the track-exchange coupling and by extending it to general random-cluster models. In order to prove that this speed is zero, we use the integrability of the six-vertex model on the torus.

2.4 Harvesting integrability on the torus

For a vertex v, let v^+ be the vertex on the top left of v. We will see in Section 6 that proving that the drift is zero in the previous section will be related to the following result.

Fix $\alpha, \beta \in (0, \pi)$. Consider the graphs $\mathbb{L}_i = \mathbb{L}_i(\alpha, \beta)$ defined by $\alpha_j = \beta$ for $j \neq i$, and $\alpha_i = \alpha$. Let $\mathbb{\Phi}_i^2$ be the *2-rooted (half-plane three-arm) Incipient Infinite Cluster on* $\mathbb{L}_i$ defined as the random-cluster model on $\mathbb{L}_i$ conditioned on having an infinite cluster and having $\mathrm{lmax}(\infty)$ equal to 0 or 0^+ (see Section 3.4 for a formal definition).

Theorem 2.4. *For every* $q \in [1, 4]$ *and* $\alpha, \beta \in (0, \pi)$*,*

$$\mathbb{\Phi}_1^2[\mathrm{lmax}(\infty) = 0^+] = \frac{\sin\alpha}{\sin\alpha + \sin\beta} = \mathbb{\Phi}_0^2[\mathrm{lmax}(\infty) = 0]. \tag{3}$$

To prove this result, we work on the torus. For positive integers N, M with N even, let $\mathbb{T}_i(N, M)$ be the $N \times 2M$ torus with $2M$ horizontal tracks $t_{1-M}, \dots, t_M$ with angle equal to α for t_i and β for t_j with $j \neq i$. Let t_j^- (resp. t_j^+) denote the set of vertices on the bottom (resp. top) of the track t_j. Also, let $\phi_{\mathbb{T}_i(N,M)}$ be the random-cluster measure on $\mathbb{T}_i(N, M)$.

[5] In these papers, the notion of speed was not introduced nor proved to exist, but in the language of this paper, the results of [36, 37, 38] state that the absolute value of the speed is strictly smaller than 1 for Bernoulli percolation.

Fix $z_{-1}, z_0, z_1, x_1, \dots, x_k$ distinct vertices found on $t^-_{1-M} (= t^+_M)$ in that order. Let $y_1, \dots, y_k$ be the vertices of t^-_M such that $x_i = y_i^+$. Define the event (see Figure 5):

$$E(k) = E(k, z_{-1}, z_0, z_1, x_1, \dots, x_k, y_1, \dots, y_k, N, M)$$

that

(i) the only edges that are open in t_M are the edges linking x_i and y_i,

(ii) there are k disjoint clusters connecting x_i to y_i for $1 \le i \le k$ in $\mathbb{T}_i(N, M) \setminus t_M$ (note that these clusters are also disjoint in $\mathbb{T}_i(N, M)$),

(iii) z_{-1} is connected to z_1 but not to z_0 or to any of the x_i among $x_1, \dots, x_k$.

Roughly speaking, the event states that there exists k disjoint clusters "winding" vertically around the torus, along with a separate cluster forming an arch above z_0. The role of this event will be explained after Theorem 2.7.

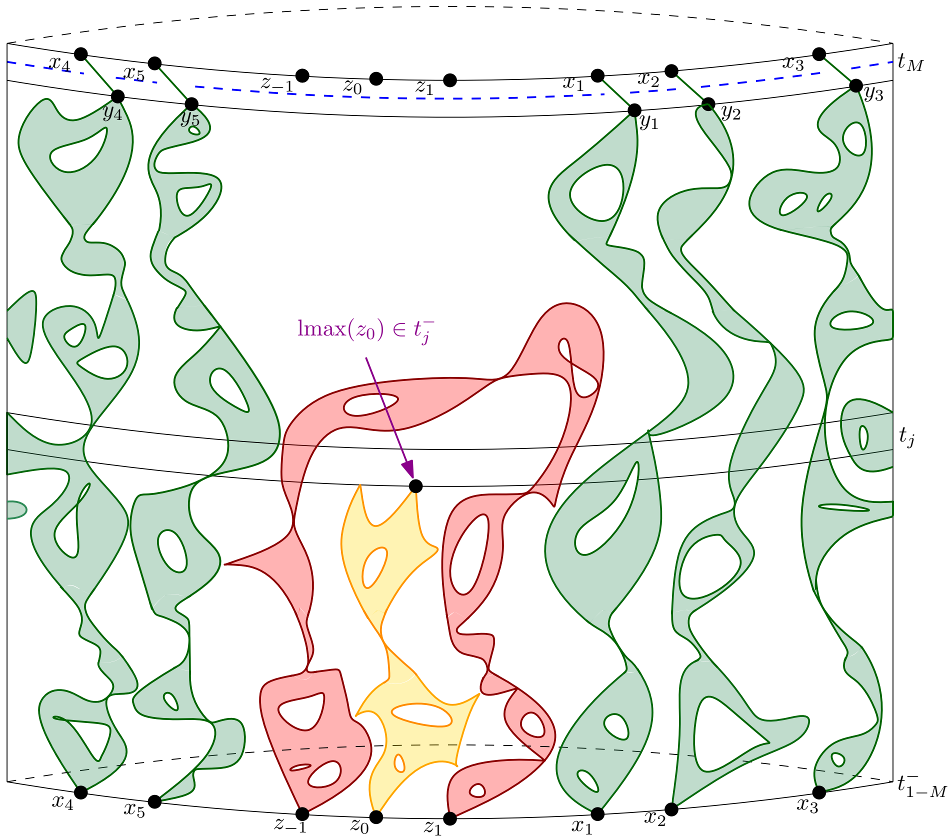

Figure 5: A picture of the event $E_j(k)$. Note that we are on a torus, hence $t^-_{1-M} = t^+_M$.

If $\mathrm{lmax}(z_0)$ denotes the left-most highest vertex of the cluster of z_0, set, for $-M < j \le M$,

$$E_j(k) := E(k) \cap \{\mathrm{lmax}(z_0) \in t_j^-\}. \tag{4}$$

The interest of these events comes from the following proposition combining two tools from exact integrability: the commutation of transfer matrices and the asymptotic behaviour of the Perron-Frobenius eigenvalues of the transfer matrix of the six-vertex model. More precisely, let $V_N(q,\theta)$ be the transfer matrix of the six-vertex on a torus of width N, with weights a, b, c given, if $\zeta \in [0, \pi/2]$ satisfies $\sqrt{q}/2 = \cos\zeta$, by the formulae

$$a \sin\tfrac{\zeta}{2} = \sin(1 - \tfrac{\theta}{\pi})\zeta \qquad b \sin\tfrac{\zeta}{2} = \sin\tfrac{\theta\zeta}{\pi} \qquad c = 2\cos\tfrac{\zeta}{2}. \tag{5}$$

Let $\lambda_N^{(k)}(\theta)$ be the Perron-Frobenius eigenvalue of the block of the transfer matrix with $N/2 + k$ up arrows (and therefore $N/2 - k$ down arrows) per row. To better grasp the signs in the next statements, note that $\lambda_N^{(k)}(\theta)$ is non-increasing in k.

Proposition 2.5. *For every $\alpha, \beta \in (0, \pi)$ and every $N \geq 2k$,*

$$\lim_{M\to\infty} \frac{\phi_{\mathbb{T}_1(N,M)}[E_1(k)]}{\phi_{\mathbb{T}_1(N,M)}[E_0(k)]} = \frac{\lambda^{(k)}(\beta)}{\lambda^{(k+3)}(\beta)} \times \frac{1 - \lambda_N^{(k+3)}(\alpha)/\lambda_N^{(k)}(\alpha)}{1 - \lambda_N^{(k+3)}(\beta)/\lambda_N^{(k)}(\beta)},$$

$$\lim_{M\to\infty} \frac{\phi_{\mathbb{T}_0(N,M)}[E_1(k)]}{\phi_{\mathbb{T}_0(N,M)}[E_0(k)]} = \frac{\lambda^{(k+3)}(\alpha)}{\lambda^{(k)}(\alpha)} \times \frac{1 - \lambda_N^{(k+3)}(\beta)/\lambda_N^{(k)}(\beta)}{1 - \lambda_N^{(k+3)}(\alpha)/\lambda_N^{(k)}(\alpha)}.$$

This proposition combines very well with the following probabilistic estimate.

Proposition 2.6. *For every $\alpha, \beta \in (0, \frac{\pi}{2})$, there exist $C, \eta > 0$ such that for $i = 0, 1$ and every $k \leq N/2$,*

$$\Big| \lim_{M\to\infty} \phi_{\mathbb{T}_i(N,M)}[E_1(k) | E_1(k) \cup E_0(k)] - \Phi_i^2[\mathrm{lmax}(\infty) = 0^+] \Big| \leq C\Big(\frac{\lambda_N^{(k)}(\beta)}{\lambda_N^{(k+3)}(\beta)} - 1 \Big)^{\eta}.$$

To interpret this proposition, think of a value of k for which $\lambda_N^{(k)}(\beta)/\lambda_N^{(k+3)}(\beta)$ is close to 1, which should be the case when N/k is large. By the definition of the events $E_0(k)$ and $E_1(k)$, there are k clusters crossing the torus from bottom to top, with an additional cluster finishing either on t_0^- or $t_1^-(= t_0^+)$. One expects the different clusters to be typically distant of roughly N/k. In particular, one may predict that none of the clusters of the x_i or $z_{\pm 1}$ comes close (meaning much closer than N/k) to the maximum of the cluster of z_0. Proving the separation property will not be straightforward, and will constitute the heart of the proof of this proposition. Now, the convergence of finite volume measures with proper conditioning to Φ_i^2 would imply that near the top of the cluster, the measure $\phi_{\mathbb{T}_i}[\,\cdot\,|E_1(k) \cup E_0(k)]$ can be coupled with Φ_i^2 with probability close to 1 when N/k is very large.

These two propositions will combine with the following statement from [17, Thm. 22] on the behaviour of the eigenvalues for the six-vertex model's transfer matrix, to prove Theorem 2.4.

Theorem 2.7. *For every $\theta \neq \pi/2$ and $\Delta \in (-1, 0)$, there exists $C = C(\Delta) < \infty$ such that, for every N, k large enough,*

$$\tfrac{1}{N} \log \lambda_N^{(k)}(\theta) = F(a, b, c) - C(\Delta) \sin\theta (1 + o(1)) (\tfrac{k}{N})^2 + O(\tfrac{1}{kN}), \tag{6}$$

where $o(1)$ is a quantity tending to zero as k/N tends to 0.

The reason for working with the events $E_j(k)$ rather than the simpler event $\{\text{lmax}(z_0) \in t_j^-\}$ is now apparent: the asymptotic in (6) is most meaningful when $N = o(k^3)$, so that the $O(\frac{1}{kN})$ becomes insignificant compared to the middle term. The arch formed by the cluster of $z_{\pm 1}$ is not strictly necessary, but will simplify the proof of Proposition 2.6.

A finer asymptotic for $\frac{1}{N} \log \lambda_N^{(1)}(\theta)$ would allow one to circumvent the introduction of $E(k)$, and would eliminate all difficulties from the proof of Proposition 2.6. Unfortunately, at the time of wiring, no such asymptotic is available.

Organization In Section 3, we recall some background on the random-cluster model on isoradial graphs and prove several technical facts that will be used later in the paper. In Section 4, we show Theorem 2.4 via Propositions 2.5 and 2.6. Section 5 proves Theorem 2.2. In Section 6, we explain how Theorem 2.3 is derived. Finally, in Section 7, we show Theorem 1.2 as well as its direct applications.

3 Preliminaries

3.1 Definition of the random-cluster model

For a graph $G = (V, E)$ included in an isoradial graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ with vertex-set V and edge-set E, *boundary conditions* ξ on G are given by a partition of the set ∂G of vertices in V incident to a vertex in $\mathbb{V} \setminus V$. We say that two vertices of G are *wired together* if they belong to the same element of the partition ξ. Recall that a *cluster* is a connected component of ω.

In the paper, we will always work with the random-cluster model on an isoradial graph with specific weights, called *isoradial weights*, associated with this graph, given by

$$p_e := \begin{cases} \dfrac{\sqrt{q}\sin(r(\pi-\theta_e))}{\sin(r\theta_e)+\sqrt{q}\sin(r(\pi-\theta_e))} & \text{if } q<4, \\ \dfrac{2\pi-2\theta_e}{2\pi-\theta_e} & \text{if } q=4, \\ \dfrac{\sqrt{q}\sinh(r(\pi-\theta_e))}{\sinh(r\theta_e)+\sqrt{q}\sinh(r(\pi-\theta_e))} & \text{if } q>4, \end{cases} \tag{7}$$

(the last case is not relevant to this paper, see below) where $r := \frac{1}{\pi}\cos^{-1}\left(\frac{\sqrt{q}}{2}\right)$ for $q \le 4$ and the same formula with cosh instead of cos for $q > 4$, and $\theta_e \in (0, \pi)$ is the angle subtended by e (see Figure 6).

Definition 3.1. The random-cluster measure with isoradial edge-weights and cluster-weight $q > 0$ on a finite graph G with boundary conditions ξ is given by

$$\phi_{G,q}^{\xi}[\omega] := \frac{q^{k(\omega^\xi)}}{Z_{\text{RCM}}^{\xi}(G,q)} \prod_{e\in E} p_e^{\omega_e}(1-p_e)^{1-\omega_e}, \tag{8}$$

where $k(\omega^\xi)$ is the number of connected components of the graph ω^ξ which is obtained from ω by identifying wired vertices together, and $Z_{\text{RCM}}^{\xi}(G,q)$ is a normalising constant called the *partition function* chosen in such a way that $\phi_{G,q}^{\xi}$ is a probability measure.

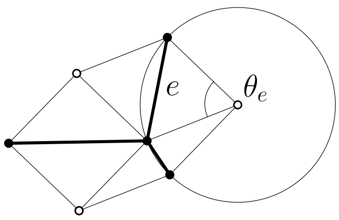

Figure 6: The edge e and its subtended angle θ_e; bold edges are those of $\mathbb{G}$, thin ones are those of the diamond graph.

Two specific families of boundary conditions will be of special interest to us. On the one hand, the *free* boundary conditions, denoted 0, correspond to no wirings between boundary vertices. On the other hand, the *wired* boundary conditions, denoted 1, correspond to all boundary vertices being wired together.

We will also consider the random-cluster model on infinite isoradial graphs $\mathbb{G}$ with free boundary conditions obtained by taking the limit of the measures with free boundary conditions on larger and larger finite graphs G tending to $\mathbb{G}$. Set $\phi_{\mathbb{G},q}$ for the measure in infinite volume, which, as shows in [18] is unique for $1 \leq q \leq 4$.

The choice of the isoradial parameters is such that the model is critical. This result was obtained in the case of the square lattice in [5] and for isoradial graphs in [18] (see also the anterior paper [6] for the case $q > 4$).

As we will always fix isoradial weights and $q \in [1,4]$*, we remove their dependency from the notation.*

3.2 Elementary properties of the random-cluster model

We will use the following standard properties of the random-cluster model. They can be found in [35], and we only recall them briefly below.

Monotonic properties. Fix G as above. An event A is called *increasing* if for any $\omega \leq \omega'$ (for the partial ordering on $\{0,1\}^E$ given by $\omega \leq \omega'$ if $\omega_e \leq \omega'_e$ for every $e \in E$), $\omega \in A$ implies that $\omega' \in A$. Fix $q \geq 1$ and some boundary conditions $\xi' \geq \xi$, where $\xi' \geq \xi$ means that any wired vertices in ξ are also wired in ξ'. Then, for every increasing events A and B,

$$\phi_G^{\xi}[A \cap B] \geq \phi_G^{\xi}[A]\phi_G^{\xi}[B], \tag{FKG}$$

$$\phi_G^{\xi'}[A] \geq \phi_G^{\xi}[A]. \tag{CBC}$$

The inequalities above will respectively be referred to as the *FKG inequality* and the *comparison between boundary conditions.*

Spatial Markov property. For any configuration $\omega' \in \{0,1\}^E$ and any $F \subset E$,

$$\phi_G^{\xi}[\cdot_{|F} \,|\, \omega_e = \omega'_e, \forall e \notin F] = \phi_H^{\xi'}[\cdot], \tag{SMP}$$

where H denotes the graph induced by the edge-set F, and ξ' are the boundary conditions on H defined as follows: x and y on ∂H are wired if they are connected in $(\omega'_{|E\setminus F})^{\xi}$.

A direct consequence of the spatial Markov property is the *finite-energy property* guaranteeing that conditioned on the states of all the other edges in a graph, the probability that an edge is open is between $p/(p+q(1-p))$ and p.

Dual model. Define (see Figure 7) the dual graph $G^* = (V^*, E^*)$ of G as follows: place dual sites at the centers of the faces of G (the external face, when considering a graph in the plane, must be counted as a face of the graph), and for every edge $e \in E$, place a dual edge between the two dual sites corresponding to faces bordering e. When the graph is isoradial, we make the following choice for the position of dual vertices in V^*: the vertex v^* corresponding to a face of G is placed at the center of the corresponding circumcircle. The dual of an isoradial graph is by construction an isoradial graph.

Given a subgraph configuration ω, construct a configuration ω^* on G^* by declaring any edge of the dual graph to be open (resp. closed) if the corresponding edge of the primal lattice is closed (resp. open) for the initial configuration. The new configuration is called the *dual configuration* of ω. The dual model on the dual graph given by the dual configurations then corresponds to a random-cluster measure with isoradial weights and dual boundary conditions. We do not want to discuss too much the details of how dual boundary conditions are defined (we refer to [35] for details and to [18] for the isoradial setting) and we simply observe that the dual of free boundary conditions are the wired ones, and vice versa.

Loop model. The loop representation of a configuration on G is supported on the *medial graph* of G defined as follows (see Figure 7). For an isoradial lattice $\mathbb{G}$, let $\mathbb{G}^\diamond$ be the graph with vertex-set given by the midpoints of edges of $\mathbb{G}$ and edges between pairs of nearest vertices. For future reference, note that the faces of $\mathbb{G}^\diamond$ contain either a vertex of $\mathbb{G}$ or one of $\mathbb{G}^*$, and that it is the dual of the diamond graph. Let $G^\diamond$ be the subgraph of $\mathbb{G}^\diamond$ spanned by the edges of $\mathbb{G}^\diamond$ adjacent to a face corresponding to a vertex of G.

Let ω be a configuration on G; recall its dual configuration ω^*. Draw self-avoiding paths on $G^\diamond$ as follows: a path arriving at a vertex of the medial lattice always takes a turn at vertices so as not to cross the open edges of ω or ω^*. The loop configuration $\overline{\omega}$ thus defined is formed of disjoint loops. Together these form a partition of the edges of $G^\diamond$.

Let us conclude this section by mentioning that we will (almost) always consider $\mathbb{G} = \mathbb{L}(\boldsymbol{\alpha})$.

3.3 Uniform bounds on crossing probabilities

As it is often the case when investigating the critical behaviour of lattice models, we will rely on uniform crossing estimates in rectangles, as well as estimates on certain universal and non-universal critical exponents. Such crossing estimates initially emerged in the study of Bernoulli percolation in the late seventies under the coined name of Russo-Seymour-Welsh theory [53, 54].

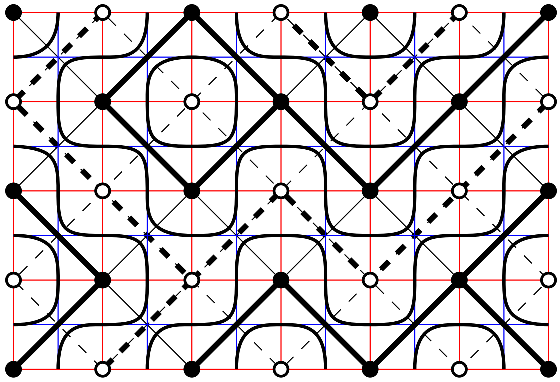

Figure 7: We depicted in black, dotted black, red and blue respectively the primal, dual, diamond and medial lattices. The primal configuration ω is in bold and the dual one ω^* in dashed bold. Finally, the loop configuration $\overline{\omega}$ is in black.

Recall the following definition: for a quad Q, let $\mathcal{C}(Q)$ be the event that Q is crossed in the percolation configuration ω (when ω is seen as an element of the Schramm-Smirnov set $\mathfrak{H}$, this corresponds to the event $\omega \in Q$).

Theorem 3.2. *For $1 \le q \le 4$ and $\rho, \varepsilon > 0$, there exists $c = c(\rho, \varepsilon) > 0$ such that for every $n \ge 1$, every $\boldsymbol{\alpha} = (\alpha_i : i \in \mathbb{Z})$ with $\varepsilon \le \alpha_i \le \pi - \varepsilon$ for every $i \in \mathbb{Z}$, every $\Omega \subset \mathbb{R}^2$ containing the εn neighborhood of $R := [0, \rho n] \times [0, n]$), and every boundary conditions ξ,*

$$c \le \phi^{\xi}_{\mathbb{L}(\boldsymbol{\alpha})\cap\Omega}[\mathcal{C}(R)] \le 1 - c. \tag{RSW}$$

Proof. This result is a direct consequence of [18, Thm. 1.1]. While that paper studies doubly-periodic isoradial graphs, the techniques in it extend to our framework with rectangular-type tracks with angles $\alpha_i \in (\varepsilon, \pi - \varepsilon)$ for every $i \in \mathbb{Z}$ (this condition guarantees a uniform bounded angle property; see comments below [18, Thm. 1.2]). □

Remark 3.3. We will repeatedly use this theorem as well as a number of its classical applications. We are aware that some proofs may be difficult to read for somebody not familiar with Russo-Seymour-Welsh type arguments. We tried to be complete but succinct, as there is a clear trade-off in the proofs below between providing a large amount of detail on classical RSW machinery, and putting emphasis on the novel arguments in this paper. We refer to the large literature on the RSW theory to see some of the classical arguments we will use in this article.

We now discuss some consequences of the above. The previous theorem has classical applications for the probability of so-called arm events. Below, $\Lambda_n \subset \mathbb{G}$ is the subgraph of $\mathbb{G}$ induced by vertices in $[-n, n]^2 \subset \mathbb{R}^2$. A self-avoiding path of type 0 or 1 connecting the inner to the outer boundary of an annulus $\Lambda_R \setminus \Lambda_{r-1}$ is called an *arm*. We say that an arm is *of type* 1 if it is composed of primal edges that are all open in ω, and *of type* 0 if it is composed of dual edges that are all open in ω^*. For $k \ge 1$ and $\sigma \in \{0,1\}^k$, define

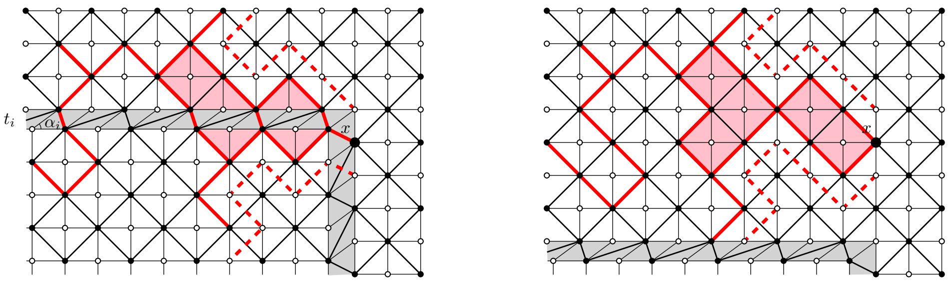

Figure 8: On the left is an isoradial lattice in the left half-plane, almost identical to $\mathbb{L}(\boldsymbol{\alpha})$. The three arm events for any point on the vertical axis have the same probability in the left and right graphs.

$A_\sigma(r,R)$ to be the event that there exist k *disjoint* arms from $\partial\Lambda_r$ to $\partial\Lambda_R$ which are of type $\sigma_1,\dots,\sigma_k$, when indexed in counterclockwise order. We also introduce $A^X_\sigma(r,R)$ to be the same event as $A_\sigma(r,R)$, except that the paths must lie in the lower half-plane $\mathbb{H}^- := \mathbb{R}\times(-\infty,0]$ if $X=\mathrm{T}$, upper-half-plane $\mathbb{H}^+ := \mathbb{R}\times[0,+\infty)$ if $X=\mathrm{B}$, and left half-plane $\mathbb{L}(\boldsymbol{\alpha})\cap((\infty,0]\times\mathbb{R})$ if $X=\mathrm{R}$.

Finally, let $A^{\mathrm{TR}}_{010}(r,R)$ be the event that there are three arms (two of type 0 and one of type 1) in the quarter plane $[-\infty,0]^2$, and $A^X_{010}(r,R)\circ A_1(r,R)$ the event that there are three arms in the corresponding half-plane, plus an additional disjoint arm of type 1 in the plane.

We will need the following two estimates.

Proposition 3.4 (Estimates on certain arm events)**.** *For every $\varepsilon>0$, there exist $c,C\in(0,\infty)$ such that for every $1\le q\le 4$, every $R\ge r\ge 1$ and every $\boldsymbol{\alpha}$ with $\alpha_i\in(\varepsilon,\pi-\varepsilon)$ for every $i\in\mathbb{Z}$,*

$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A_1(r,R)] \le C(r/R)^c, \tag{9}$$
$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A^{\mathrm{T}}_{010}(r,R)] \le C(r/R)^2, \tag{10}$$
$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A^{\mathrm{B}}_{010}(r,R)] \le C(r/R)^2, \tag{11}$$
$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A^{\mathrm{R}}_{010}(r,R)] \le C(r/R)^{1+c}, \tag{12}$$
$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A^{\mathrm{TR}}_{010}(r,R)] \le C(r/R)^{2+c}, \tag{13}$$
$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A^{\mathrm{T}}_{010}(r,R)\circ A_1(r,R)] \le C(r/R)^{2+c}. \tag{14}$$

Furthermore, if α_i is equal to $\pi/2$ except for one value of i, then we also have

$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A^{\mathrm{R}}_{010}(r,R)] \le C(r/R)^2, \tag{15}$$
$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A^{\mathrm{R}}_{010}(r,R)\circ A_1(r,R)] \le C(r/R)^{2+c}. \tag{16}$$

Proof. The first bound can be obtained from (RSW) using standard techniques from Bernoulli percolation.

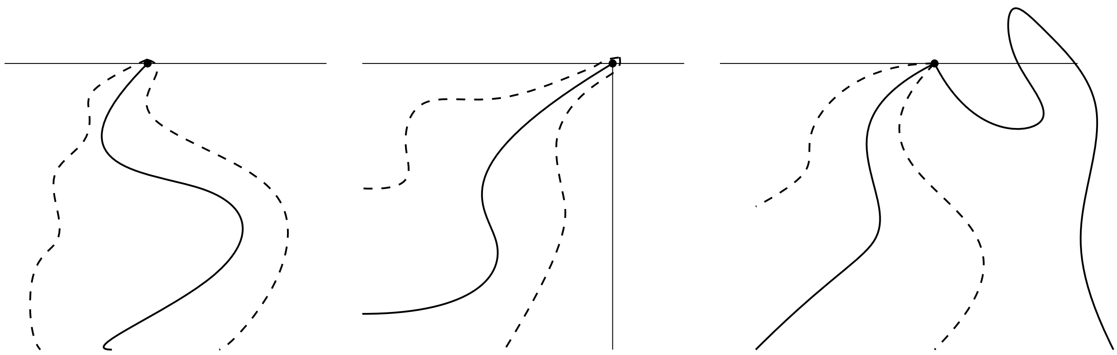

Figure 9: On the left, an instance of $A_{010}^{\mathrm{T}}(0,R)$ where the primal path is depicted by a bold path, and the dual ones by dashed paths. In the middle, an instance of $A_{010}^{\mathrm{TR}}(0,R)$. On the right, $A_{010}^{\mathrm{T}}(0,R) \circ A_1(0,R)$.

For the second and third ones, the case of the square lattice is also a direct consequence of (RSW) and standard techniques of Bernoulli percolation. Transferring the estimates to $\mathbb{L}(\boldsymbol{\alpha})$ can be done using the techniques in [18, Theorem 1.4]).

The argument involving [18, Theorem 1.4] only allows one to access exponents for half-planes delimited by straight tracks, but does not apply to arm exponents in the left half-plane of $\mathbb{L}(\boldsymbol{\alpha})$. Nevertheless, the special condition on the lattice allows one to obtain (15) via a simple trick. Indeed, assuming that $\alpha_i < \pi/2$, consider the isoradial graph on the left of Figure 8. Applying repeated star-triangle transformations (see also Figure 28 for the exact procedure), the probability of a half-plane three arm event for any vertex on the vertical axis may be shown to be the same in the left and right lattices of Figure 8, and ultimately be equal to the corresponding probability in the square lattice. Thus, (15) follows from the result for the square lattice.

For (12), note that the classical argument for Bernoulli percolation for the two-arm event in the half-plane immediately implies that

$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A_{01}^{\mathrm{R}}(r,R)] \le C(r/R). \tag{17}$$

Therefore, (12) follows by conditioning on the first two arms, and then using (RSW) and the comparison between boundary conditions to bound the probability of the third arm.

For (13), one may use (RSW) to prove that

$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A_{010}^{\mathrm{TR}}(r,R)] \le C(r/R)^c \phi_{\mathbb{L}(\boldsymbol{\alpha})}[A_{010}^{\mathrm{T}}(r,R)] \le C(r/R)^{2+c}.$$

For (14), one can condition on the first three arms, and then use (9) and the comparison between boundary conditions to bound the probability of the fourth arm. □

A second consequence of (RSW) that we will repeatedly use is the mixing property.

Proposition 3.5 (Mixing property). *For every $\varepsilon > 0$, there exist $C_{\mathrm{mix}}, c_{\mathrm{mix}} \in (0,\infty)$ such that for every $\boldsymbol{\alpha}$ with $\alpha_i \in (\varepsilon, \pi - \varepsilon)$ for every $i \in \mathbb{Z}$, every $r \le R/2$, every event*

A depending on edges in Λ_r, and every event B depending on edges outside Λ_R, we have that

$$\Big|\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A\cap B]-\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A]\phi_{\mathbb{L}(\boldsymbol{\alpha})}[B]\Big|\le C_{\rm mix}(r/R)^{c_{\rm mix}}\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A]\phi_{\mathbb{L}(\boldsymbol{\alpha})}[B].$$

Proof. The argument follows the same lines as for the square lattice, see e.g. [24, Proposition 2.9]. □

The previous properties imply the following, which we will use repeatedly.

Proposition 3.6 (Crossing in annulus with adverse boundary conditions)**.** *There exists $c>0$ such that for every $r\le R/2$, every $\Omega\subset\mathbb{L}(\boldsymbol{\alpha})$ with $\alpha_i\in(\varepsilon,\pi-\varepsilon)$ for every $i\in\mathbb{Z}$, and every boundary conditions ξ inducing free boundary conditions on $\partial\Omega\cap(\Lambda_R\setminus\Lambda_r)$,*

$$\phi^\xi_\Omega[\Lambda_r\longleftrightarrow\partial\Lambda_R]\le(r/R)^c.$$

Proof. The comparison between boundary conditions implies that

$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[\Lambda_r\longleftrightarrow\partial\Lambda_R]\le\phi^1_{\Lambda_R\setminus\Lambda_r}[\Lambda_r\longleftrightarrow\partial\Lambda_R].$$

Now, the mixing property together with (9) conclude the proof. □

Finally, we will also use the following easy claim.

Proposition 3.7 (Tight number of macroscopic clusters in a box)**.** *For $\varepsilon>0$, there exist $c,C\in(0,\infty)$ such that for every $N\ge0$ and every $\boldsymbol{\alpha}$ with $\alpha_i\in(\varepsilon,\pi-\varepsilon)$ for every $i\in\mathbb{Z}$,*

$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[\exp(c\mathbf{N}_\varepsilon)]\le C,\tag{18}$$

where $\mathbf{N}_\varepsilon$ be the number of clusters of diameter at least εN intersecting Λ_N.

Proof. The claim follows if we can show that for some constant $c_0>0$, we have that for every $k\ge0$,

$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[\mathbf{N}_\varepsilon\ge k+1|\mathbf{N}_\varepsilon\ge k]\le1-c_0.$$

Index the vertices in the box one by one and let $\mathcal{C}_i$ be the cluster of the i-th vertex (it is equal to the clusters $\mathcal{C}_j$ for every j such that the j-th vertex belongs to $\mathcal{C}_i$). Let $\mathbf{i}$ be the smallest index i such that there are k clusters of diameter at least εN among $\mathcal{C}_j$ for $1\le j\le i$. Conditioned on $\mathcal{C}_1,\dots,\mathcal{C}_{\mathbf{i}}$, the boundary conditions outside of the union of these clusters are free within the box. One easily deduces from (RSW) and the comparison between boundary conditions that the probability that there is an additional cluster of diameter at least εN is smaller than $1-c_0$, hence concluding the proof of the proposition. □

3.4 Incipient Infinite Clusters with three arms in the half-plane

In this section, we introduce the Incipient Infinite Cluster (IIC) measures with three arms in the half-plane. Let $\alpha, \beta \in (0, \pi)$ be two angles. Recall the definitions of $\mathbb{L}(\beta)$ and $\mathbb{L}_i(\alpha, \beta)$. Below, we use the shorthand notation $\mathbb{L} := \mathbb{L}(\beta)$ and $\mathbb{L}_i := \mathbb{L}_i(\alpha, \beta)$ and embed the lattices in such a way that the origin 0 is a vertex of the graph. Let $\text{lmax}(v)$ be the left-most highest vertex of the cluster of v.

Theorem 3.8. *For every $\alpha, \beta \in (0, \pi)$, there exist a measure Φ on $\mathbb{L}(\beta)$ and measures Φ_i and Φ_i^2 on $\mathbb{L}_i$ for every $i \in \mathbb{Z}$ such that for every event A depending on finitely many edges,*

$$\begin{aligned}
\Phi[A] &= \lim_{R\to\infty} \phi^0_{\mathbb{L}}[A | 0 \longleftrightarrow \partial\Lambda_R, \text{lmax}(0) = 0], \\
\Phi_i[A] &= \lim_{R\to\infty} \phi^0_{\mathbb{L}_i}[A | 0 \longleftrightarrow \partial\Lambda_R, \text{lmax}(0) = 0], \\
\Phi_i^2[A] &= \lim_{R\to\infty} \phi^0_{\mathbb{L}_i}[A | \{0 \longleftrightarrow \partial\Lambda_R, \text{lmax}(0) = 0\} \cup \{0^+ \leftrightarrow \partial\Lambda_R, \text{lmax}(0^+) = 0^+\}].
\end{aligned}$$

Proof. The proof of this theorem follows the same lines as the construction of the IIC for Bernoulli percolation once one has (RSW). We omit the details here and refer to [1, 32, 40, 46]. □

We also mention a mixing property. We state it in the way which is closest to applications.

Proposition 3.9 (Mixing property for IIC). *For every $\varepsilon > 0$, there exist $C, c > 0$ such that for every $\alpha, \beta \in (\varepsilon, \pi - \varepsilon)$, every $r \leq R/2$, every event A depending on Λ_r, every $\Omega \supset [-R, R]^2$, every $I \subset \partial\Omega$, every $x \in \partial\Omega \setminus I$, and every boundary conditions ξ, we have that*

$$\begin{aligned}
|\Phi[A] - \phi^\xi_{\Omega\cap\mathbb{L}}[A | x \not\longleftrightarrow I, \text{lmax}(x) = 0]| &\leq C(r/R)^c, \\
|\Phi_i[A] - \phi^\xi_{\Omega\cap\mathbb{L}_i}[A | x \not\longleftrightarrow I, \text{lmax}(x) = 0]| &\leq C(r/R)^c, \\
\Big|\Phi_i^2[A] - \phi^\xi_{\Omega\cap\mathbb{L}_i}\big[A \big| x \not\longleftrightarrow I, \text{lmax}(x) \in \{0, 0^+\}\big]\Big| &\leq C(r/R)^c.
\end{aligned}$$

Proof. As before, we refer to [1, 32, 40, 46] for details. □

We also introduce the measures Ψ_i where the conditioning is over the right-most bottom-most vertex of the cluster being 0. It also coincides with the symmetry with respect to the origin of the measure Φ_{-i} defined on $\mathbb{L}_{1-i}(\pi - \alpha, \pi - \beta)$. The measures Ψ_i satisfy properties corresponding to the properties above.

Finally, we introduce $\rotatebox{-90}{$\Psi$}$ to be the measure obtained as the limit as $R \to \infty$ of measures on $\mathbb{L}_0$ conditioned on the events that 0 is connected to $\partial\Lambda_R$ and is not connected to the right of the vertical line $\{(x, y) \in \mathbb{R}^2 : x = 0\}$. Again, the properties of the measures Φ_i extend to this measure.

3.5 The star-triangle and the track-exchange transformations

In this section, we present the track-exchange transformation. In order to do it, we first introduce the star-triangle transformation and then define the track-exchange transformation as the result of a sequence of star-triangle transformations.

Star-triangle transformation The *star-triangle transformation*, also known as the *Yang-Baxter relation*, was first discovered by Kennelly in 1899 in the context of electrical networks [41]. Then, it became a key relation in different models of statistical mechanics [3, 49] indicative of the integrability of the system. We do not plan to do a full review on this transformation (see for instance [18] for more details) and focus directly on the context of the random-cluster model on isoradial graphs with isoradial edge-weights.

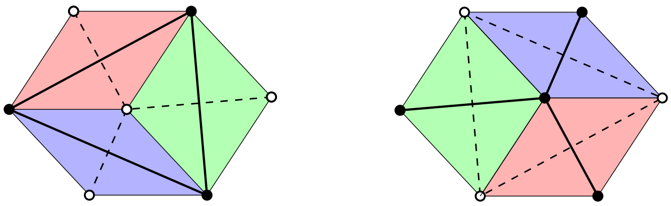

Figure 10: The three diamonds together with the drawing, on the left, of the triangle (in which case the dual graph in dashed has a star) and, on the right, of the star (in which case the dual graph has a triangle).

First of all, note that for any triangle ABC contained in an isoradial graph, there exists a unique choice of point O (namely the orthocenter) such that, if the triangle ABC is replaced by the star $ABCO$, the resulting graph is also isoradial. Conversely, for every star $ABCO$ in an isoradial graph, the graph obtained by removing this star and putting the triangle ABC is isoradial. This process of changing the graph is called the *star-triangle transformation*. Note that triangles and stars of isoradial graphs correspond to hexagons formed of three rhombi in the diamond graph. Thus, when three such rhombi are encountered in a diamond graph, they may be permuted as in Figure 10 using a star-triangle transformation.

The star-triangle transformation was first used to prove that the laws of connections between vertices of a graph G with a triangle ABC and the graph G' obtained from G with the star $ABCO$ instead of ABC are the same, except for the additional vertex O in G'. The fact that the star-triangle transformation can be used to construct a coupling between the random-cluster models on G and G' was proved in several places, see for instance [18]. The first observation that this could be done goes back to the work of [36, 37, 38], even though the identification that the star-triangle transformation was preserving the partition functions of models on isoradial graphs with isoradial weights goes long back.

Definition 3.10 (Star-triangle coupling)**.** Consider a graph G containing a triangle ABC and let G' be the graph with the star $ABCO$ instead. Introduce the *star-triangle coupling* between $\omega \sim \phi^\xi_G$ and $\omega' \sim \phi^\xi_{G'}$ defined as follows (see Figure 11):

- For every edge e which does not belong to $ABCO$, $\omega'_e = \omega_e$,
- If two or three of the edges of ABC are open in ω, then all the edges in $ABCO$ are open in ω',
- If exactly one of the edges of ABC is open in ω, say BC, then the edges BO and OC are open in ω', and the third edge of the star is closed in ω',
- If no edge of ABC is open in ω, then ω'_{OABC} has
  - no open edge with probability equal to $\frac{1-p_{OA}}{p_{OA}}\frac{1-p_{OB}}{p_{OB}}\frac{1-p_{OC}}{p_{OC}}$,
  - the edge OA is open and the other two closed with probability $q\frac{1-p_{OB}}{p_{OB}}\frac{1-p_{OC}}{p_{OC}}$,
  - similarly with cyclic permutations for B and C.

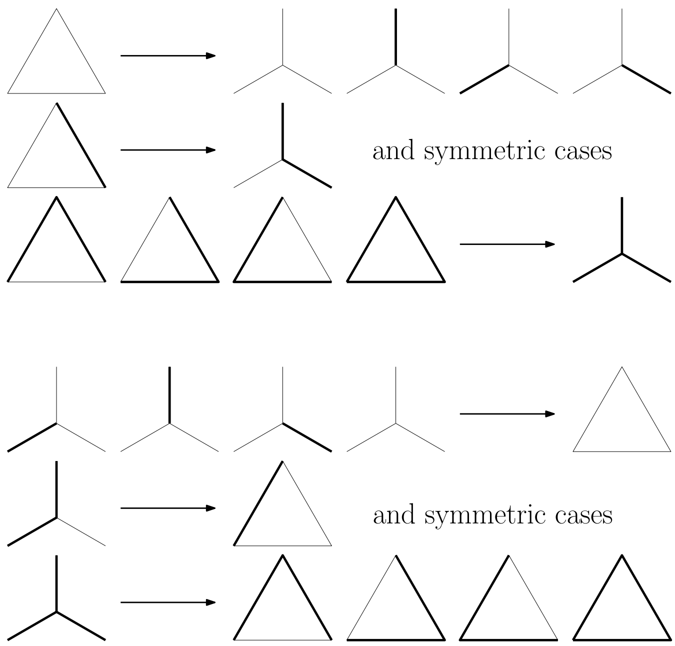

Figure 11: A picture of the possible transformations in the star-triangle coupling (the probabilities in the case of multiple outcomes are described in the definition). We also pictured the reverse map.

Let us make a few observations concerning the coupling. First, note that the transformation uses extra randomness in one case and that it is not a deterministic matching of the different configurations. Second, the coupling preserves the connectivity between the vertices, except at the vertex O. Third, in the coupling, given ω', the edges of ABC in ω are sampled as follows:

- If there is one or zero edge of $ABCO$ that is open in ω', then none of the edges in ABC is open in ω,
- If exactly two of the edges in $ABCO$ are open in ω', say AO and BO, then the edge AB is the only edge of ABC that is open in ω,

- If all the edges of $ABCO$ are open in ω', then
  - all the edges of ABC are open in ω with probability $\frac{1}{q}\frac{p_{AB}}{1-p_{AB}}\frac{p_{BC}}{1-p_{BC}}\frac{p_{CA}}{1-p_{CA}}$,
  - AB and BC are open and CA is closed with probability equal to $\frac{1}{q}\frac{p_{AB}}{1-p_{AB}}\frac{p_{BC}}{1-p_{BC}}$,
  - similarly with cyclic permutations.

Track-exchange operator The previous star-triangle operator gives rise to a track-exchange operator defined as follows. For $\mathbb{L} = \mathbb{L}(\boldsymbol{\alpha})$ and $i \in \mathbb{Z}$, let $\mathbb{L}' = \mathbb{L}(\boldsymbol{\alpha}')$ be the lattice obtained by exchanging the tracks t_i and t_{i-1} that is exchanging α_i and α_{i-1} in the sequence $\boldsymbol{\alpha}$. Index the vertices of t_{i-1}^- from left to right by $(x_k : k \in \mathbb{Z})$ and assume that $\alpha_{i-1} > \alpha_i$. Also, let $\mathbb{L}_k$ be the isoradial graph, see Figure 13, obtained by

- taking the same diamonds as $\mathbb{L}$ (or equivalently $\mathbb{L}'$) on t_j with $j \notin \{i-1, i\}$;
- taking the same diamonds as $\mathbb{L}$ on the part of t_{i-1} and t_i on the right of x_k;
- taking the same diamonds as $\mathbb{L}'$ on the part of t_{i-1} and t_i on the left of x_k;
- adding a diamond above x_k to complete the gap.

Note that the properties above determine all the diamonds in $\mathbb{L}_k$, and that there is only one diamond in $\mathbb{L}_k$ which does not belong to either $\mathbb{L}$ or $\mathbb{L}'$. Denote this diamond by $\lozenge$. We now define an operator sending configurations on $\mathbb{L}$ to configurations on $\mathbb{L}'$, that gives a formal meaning to the intuitive idea of inserting $\lozenge$ at the position $+\infty$ and using the star-triangle transformation to exchange the tracks by moving $\lozenge$ step by step to $-\infty$.

Let ω be some configuration on $\mathbb{L}$ and define for every $k \in \mathbb{Z}$ the configuration $\tilde{\omega}_k$ on $\mathbb{L}_k$ coinciding with ω on the diamonds common to $\mathbb{L}_k$ and $\mathbb{L}$ (i.e. outside t_{i-1}, t_i and on the left of x_k), and defined arbitrarily otherwise. Denote $\tilde{\omega}_k^k := \tilde{\omega}_k$ and for every $j < k$, define inductively $\tilde{\omega}_k^j$ to be the result of the star-triangle transformation mapping a configuration on $\mathbb{L}_{j+1}$ to a configuration on $\mathbb{L}_j$, applied to $\tilde{\omega}_k^{j+1}$. Define $\omega_k := \lim_{j \to -\infty} \tilde{\omega}_k^j$, which is a configuration on $\mathbb{L}'$. Now remark the important fact that if we have three integers $k, k' \geq j$ such that $\tilde{\omega}_k^j$ and $\tilde{\omega}_{k'}^j$ coincide on $\lozenge$, then the (local) outcome of the star-triangle transformation from $\tilde{\omega}_k^j$ and $\tilde{\omega}_{k'}^j$ will be the same (as long as it uses the same external randomness). More generally, applying all the subsequent steps we see that ω_k and $\omega_{k'}$ coincide on the part of $t_{i-1} \cup t_i$ that is to the left of x_j. Finally, notice that some configurations on the two diamonds left of $\lozenge$ in $\mathbb{L}_k$ fix deterministically the state of $\lozenge$ in $\mathbb{L}_{k-1}$ after a star-triangle transformation (e.g. see Figure 12). Denote by F_k this event. If F_k occurs for ω, then for all $k', k'' > k$, it also does by definition for $\omega_{k'}$ and $\omega_{k''}$, and therefore $\omega_{k'}$ and $\omega_{k''}$ coincide left of x_k. This leads to the following definition.

Definition 3.11 (Track exchange by star-triangle transformation)**.** If $\alpha_{i-1} > \alpha_i$, and ω is a percolation configuration on $\mathbb{L}$ such that $\omega \in F_k$ occurs for an infinite number of indices $k > 0$, define the track-exchange operator $\mathbf{T}_i$ by $\mathbf{T}_i(\omega) = \lim_{k \to -\infty} \omega_k$, where ω_k is defined as in the previous paragraph.

We will only work with measures (random cluster measures, IIC measures) that verify some finite energy property so that F_k occurs for an infinite number of $k < 0, k > 0$ almost surely. Hence the operator $\mathbf{T}_i$ is well defined on almost all configurations ω.

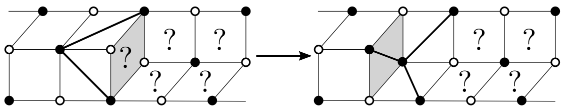

Figure 12: An example where the configuration on the two diamonds left of the grey diamond $\Diamond$ determines the configuration on $\Diamond$ after the star-triangle transformation, irrespectively of the configuration on or right of $\Diamond$.

If $\alpha_i > \alpha_{i-1}$, we construct $\mathbf{T}_i$ similarly by inverting the left and the right, and $-\infty$ and $+\infty$.

It should be noted that the mixing properties of the random-cluster model implies that the random-cluster measure on $\mathbb{L}$ is the limit of the random-cluster measures on $\mathbb{L}_k$ and therefore, if ω is distributed according to $\phi_{\mathbb{L}}$, then $\mathbf{T}_i(\omega)$ has law $\phi_{\mathbb{L}'}$. Let us also insist on the fact that $\mathbf{T}_i$ is not a deterministic map, as at each step where a star-triangle operator is used, there is extra randomness in the outcome of the transformation.

We finish this section by an important proposition.

Proposition 3.12. *If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy $\alpha_i = \beta_i$ for $a \le i \le b$, the law of ω restricted to the strip between t_a^- and t_b^+ as well as the law of the homotopy classes of loops in ω with respect to points in this strip is the same in $\phi_{\mathbb{L}(\boldsymbol{\alpha})}$ and $\phi_{\mathbb{L}(\boldsymbol{\beta})}$.*

Proof. As a sequence of star-triangle transformations, the track-exchange operator preserves the connection properties of the vertices that are not on the tracks which are exchanged. From this, one may deduce that for every event A involving only edges inside the strip, or only the homotopy classes mentioned above,

$$\phi_{\mathbb{L}(\boldsymbol{\alpha})}[A] = \lim_{R\to\infty} \phi_{\mathbb{L}(\boldsymbol{\alpha}(R))}[A] = \lim_{R\to\infty} \phi_{\mathbb{L}(\boldsymbol{\beta}(R))}[A] = \phi_{\mathbb{L}(\boldsymbol{\beta})}[A],$$

where

$$\boldsymbol{\alpha}(R) := \begin{cases} \alpha_i & \text{if } |i| \le R, \\ \beta_{i-R+b} & \text{if } i > R, \\ \beta_{R-i+a} & \text{if } i < -R, \end{cases} \quad \text{and} \quad \boldsymbol{\beta}(R) := \begin{cases} \beta_i & \text{if } |i| \le R, \\ \alpha_{i-R+b} & \text{if } R < i < 2R-b, \\ \alpha_{R-i+a} & \text{if } -2R+a < i < -R, \\ \beta_i & \text{otherwise.} \end{cases}$$

(In the first and last inequalities, we use the measurability and the uniqueness of the infinite-volume measure, and in the second one the track-exchange operator.) $\square$

4 Probabilities in 2-rooted IIC: proof of Theorem 2.4

The goal of this section is to prove Theorem 2.4. As mentioned in the introduction, the main steps will be Propositions 2.5 and 2.6. We prove these two statements in Sections 4.1 and 4.2 respectively. The proof of Theorem 2.4 is postponed to Section 4.3 (recall that it relies on Theorem 2.7, which was obtained in [17]).

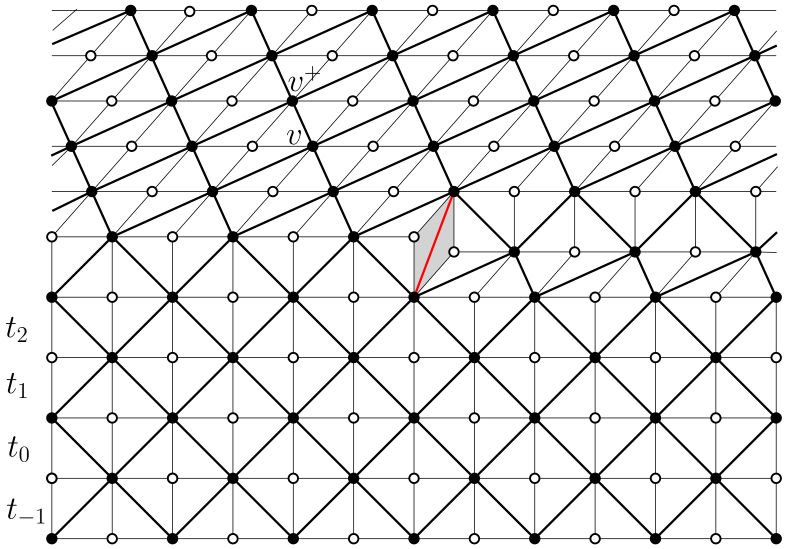

Figure 13: An example of a graph $\mathbb{L}_k$ for $k = 4$. What happens between tracks t_2 and t_5 is a mixture of the isoradial lattice $\mathbb{L}$ with angles $\pi/2$ for $i \leq 3$ and α for $i \geq 4$, and $\mathbb{L}'$ is obtained by exchanging the tracks 4 and 5. The only diamond that does not belong to $\mathbb{L}$ or $\mathbb{L}'$ is in gray.

4.1 Harvesting exact integrability: Proof of Proposition 2.5

Below, for an event E and i, M, N, introduce the convenient notation

$$Z_{\mathbb{T}_i(N,M)}[E] := Z^{\xi}_{\mathrm{RCM}}(\mathbb{T}_i(N,M), q)\phi^{\xi}_{\mathbb{T}_i(N,M),q}[E].$$

We divide the proof in two lemmata. The first one uses an aspect of the commutation of transfer matrices. To be more precise, we will use a result of [50, Theorem 1.3] which is written for Bernoulli percolation but works with almost no change for the random cluster model, proving the existence of a (track-exchange) map $\widetilde{\mathbf{T}}_i : \Omega_{\mathbb{T}_i(N,M)} \to \Omega_{\mathbb{T}_{i-1}(N,M)}$ (slightly different from our track-exchange maps) between percolation configurations on the tori, such that,

(a) For any $\omega \in \Omega_{\mathbb{T}_i(N,M)}$, $\widetilde{\mathbf{T}}_i(\omega)$ and ω coincide outside of $t_{i-1} \cup t_i$;
(b) For any $x, y \notin t_i^-$, x and y are connected in ω if and only if they are in $\widetilde{\mathbf{T}}_i(\omega)$;
(c) For every event E, $Z_{\mathbb{T}_i(N,M)}[\widetilde{\mathbf{T}}_i^{-1}(E)] = Z_{\mathbb{T}_{i-1}(N,M)}[E]$.

Let us mention that with a little bit of work one can also simply use the star-triangle transformation, or the commutation of transfer matrices to produce a more abstract proof.

Recall the definition of $E_j(k)$ from the introduction.

Lemma 4.1. *For every k, N, M, we have that*

(i) For fixed j, $i \mapsto Z_{\mathbb{T}_i(N,M)}[E_j(k)]$ is constant for $i > j$ and similarly for $i < j$,

(ii) $Z_{\mathbb{T}_1(N,M)}[E_0(k)\cup E_1(k)] = Z_{\mathbb{T}_0(N,M)}[E_0(k)\cup E_1(k)]$.

Proof. We use the track-exchange map mentioned above. The connectivity preservation (a) and (b) imply that ω belongs to $E_j(k)\cup E_{j+1}(k)$ (resp. $E_j(k)$ for $j\neq i$) if and only if $\widetilde{\mathbf{T}}_i(\omega)$ does. Therefore, (c) implies the lemma. □

The second lemma harvests the transfer matrix formalism to get an explicit formula for the probability of events $E_j(k)$ in terms of eigenvalues of the transfer matrix. Below, we use the following connection between the eigenvalues of the transfer matrix of the six-vertex model and the partition function of the random-cluster model obtained via the Baxter-Kelland-Wu coupling [4] (see also [20, Section 3.3.] for details). Let $\mathbb{T}(N,M)$ be the N by $2M$ torus with tracks of angle β only and introduce the notation $Z_{\mathbb{T}(N,M)}[E]$ in the same way as for $\mathbb{T}_i(N,M)$. Consider the event $G(k)$ that there exist exactly k disjoint clusters wrapping around the torus in the vertical direction. Then[6]

$$Z_{\mathbb{T}(N,M)}[G(k)] = C(q,N,M)(1+o_M(1))\cdot(q/4)^k\cdot\lambda_N^{(k)}(\beta)^{2M}, \tag{19}$$

$$Z_{\mathbb{T}_i(N,M)}[G(k)] = C(q,N,M)(1+o_M(1))\cdot(q/4)^k\cdot\lambda_N^{(k)}(\alpha)\lambda_N^{(k)}(\beta)^{2M-1}, \tag{20}$$

where $C(q,N,M) := q^{NM/2}/(1+\sqrt{q})^{2MN}$ and $o_M(1)$ is a quantity tending to 0 as M tends to infinity.

Lemma 4.2. *For every $k,N,\alpha,\beta\in(0,\pi)$, there exists $C_N^{(k)}(\alpha,\beta)\in(0,\infty)$ such that*

$$\lim_{M\to\infty}\frac{Z_{\mathbb{T}_i(N,M)}[E_j(k)]}{Z_{\mathbb{T}_0(N,M)}[E_0(k)]} = \Big[\frac{\lambda_N^{(k+3)}(\beta)}{\lambda_N^{(k)}(\beta)}\Big]^j \times \begin{cases} C_N^{(k)}(\alpha,\beta)\frac{\lambda_N^{(k)}(\alpha)}{\lambda_N^{(k)}(\beta)} & \text{if } i>j,\\ 1 & \text{if } i=j,\\ C_N^{(k)}(\alpha,\beta)\frac{\lambda_N^{(k+3)}(\alpha)}{\lambda_N^{(k+3)}(\beta)} & \text{if } i<j.\end{cases} \tag{21}$$

Proof. We start with the case $i=j$. For $1\le m\le M/3$ with $M-j-m$ even, let $E_j(k,m)\subset E_j(k)$ be the event (see Figure 14) that

[6] To be precise, [20] proves an inequality only, but an equality is easily derived. Indeed, to explain the first formula, recall from [20] that the weight of each random cluster configuration ω may be written as the sum over all orientations of its loop configuration of the weight of the ensuing oriented loop configuration. The latter is the product over each oriented loop of $e^{+i\zeta}$, $e^{-i\zeta}$ or $\sqrt{q}/2$ depending whether the oriented loop is retractable and oriented counter-clockwise, clockwise or non-retractable, respectively, with $\zeta=\arccos\sqrt{q/2}$. Notice now that for $\omega\in G(k)$, there exist at least $2k$ non-retractable loops winding vertically around the torus; all but an exponentially small proportion of $Z_{\mathbb{T}(N,M)}[G(k)]$ actually comes from configuration with exactly $2k$ non-retractable loops, and we will ignore all other contributions as they can be incorporated in the $o_M(1)$. For each such configuration, rather than orienting all loops in one of two directions, consider the two possible orientations only for retractable loops and orient all vertically-winding loops upwards. When summing the weights of resulting oriented loop configurations, we obtain the partition function of the six vertex model on the torus with exactly $N/2+k$ up-arrows on each row (up to the multiplicative factor $C(q,N,M)$). This may be written using the transfer matrix as $\lambda_N^{(k)}(\beta)^{2M}(1+o_M(1))$. The factor $(2/\sqrt{q})^{2k}$ in the formula for $Z_{\mathbb{T}(N,M)}[G(k)]$ accounts for the arbitrary choice of orientation of the vertically-winding loops. The same explanation applies for the second formula, with the only difference coming from the computation of the partition function in the six-vertex model.

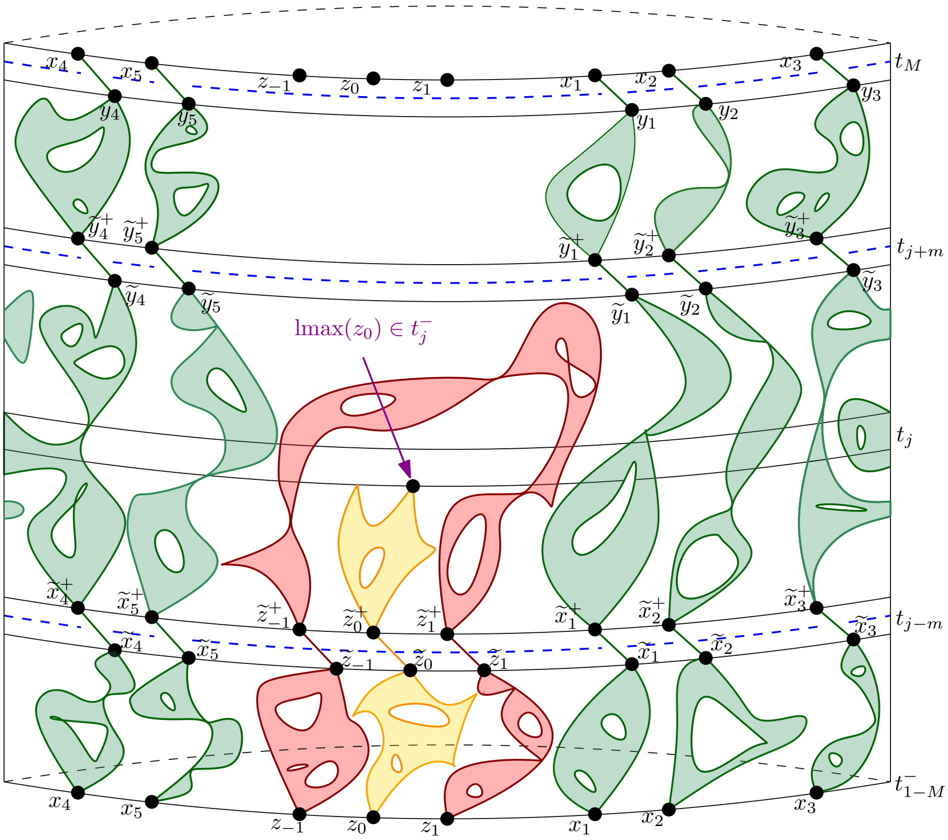

Figure 14: A picture of the event $E_j(k,m)$ (except that the minimality of m is not really explicitly depicted). Compared to Figure 5, additional conditions are forced on the tracks t_{j-m} and t_{j+m} and what happens in between.

- all the edges of t_{j+m} are closed except the edges from $\widetilde{y}_1, \dots, \widetilde{y}_k$ to $\widetilde{y}_1^+, \dots, \widetilde{y}_k^+$, where the former are the vertical translates on t_{j+m}^- of the vertices $y_1, \dots, y_k$ used in the definition of $E(k)$,
- all the edges of t_{j-m} are closed except the edges from $\widetilde{x}_1, \dots, \widetilde{x}_k, \widetilde{z}_{-1}, \widetilde{z}_0, \widetilde{z}_1$ to $\widetilde{x}_1^+, \dots, \widetilde{x}_k^+, \widetilde{z}_{-1}^+, \widetilde{z}_0^+, \widetilde{z}_1^+$, where the latter are the vertical translates on t_{j-m}^- of the vertices $x_1, \dots, x_k, z_{-1}, z_0, z_1$ used in the definition of $E(k)$,
- $E_j(k)$ occurs and $\widetilde{x}_i$ and $\widetilde{y}_i$ are connected to x_i (and therefore y_i) for $1 \le i \le k$, and $\widetilde{z}_i$ to z_i for $-1 \le i \le 1$,
- m is the smallest integer satisfying the three first properties.

With this definition, we can now proceed as follows. On the one hand, let $Z_j(k,m) = Z(k,m)$ (it does not depend on j by vertical translation invariance) be the sum of the random-cluster weights (counted as there would be free boundary conditions) of configurations ω on $t_{j-m+1} \cup \dots \cup t_{j+m-1}$ that are compatible with the occurrence of $E_j(k,m)$.

Then, if $M^+ := \frac{1}{2}(M-j-m)$ and $M^- := \frac{1}{2}(M+j-m)$, we find that

$$Z_{\mathbb{T}_j(N,M)}[E_j(k,m)] = Z(k,m)Z_{\mathbb{T}(N,M^+)}[F(k)]Z_{\mathbb{T}(N,M^-)}[F(k+3)],$$

where $F(\ell)$ is the event that the conditions (i) and (ii) of the definition of $E(\ell)$ occur. Now, the existence of a thermodynamical limit (as M tends to infinity) implies that for fixed ℓ,

$$Z_{\mathbb{T}(N,M^\pm)}[F(\ell)] = C_N^{(\ell)}(\beta)(1+o_M(1))Z_{\mathbb{T}(N,M^\pm)}[G(\ell)].$$

Therefore, (19) and the two previous displayed equations give that uniformly in $1 \le m \le M/3$,

$$\lim_{M\to\infty} \frac{Z_{\mathbb{T}_j(N,M)}[E_j(k,m)]}{Z_{\mathbb{T}_0(N,M)}[E_0(k,m)]} = \Big[\frac{\lambda_N^{(k+3)}(\beta)}{\lambda_N^{(k)}(\beta)}\Big]^j.$$

The claim follows by summing over m and observing that the finite-energy property implies the existence of $c = c(N) > 0$ such that for every m,

$$\phi_{\mathbb{T}_j(N,M)}\Big[\bigcup_{1\le m'\le m} E_j(k,m')\Big|E_j(k)\Big] \ge 1-\exp[-cm]. \tag{22}$$

We now turn to the case $i > j$. We first use Lemma 4.1(i) to "push the track of angle α up to macroscopic distance", meaning that we observe that for $M/2 > j$,

$$Z_{\mathbb{T}_i(N,M)}[E_j(k)] = Z_{\mathbb{T}_{M/2}(N,M)}[E_j(k)].$$

As before, we can fix m and run the same argument to get that for some constant $Z'(k,m)$ and with $i' := M/2 - j - m$,

$$\begin{aligned} &Z_{\mathbb{T}_{M/2}(N,M)}[E_j(k,m)] \\ &\qquad = (1+o_M(1))C_N^{(k)}(\beta)C_N^{(k+3)}(\beta)Z'(k,m)Z_{\mathbb{T}_{i'}(N,M^+)}[G(k)]Z_{\mathbb{T}(N,M^-)}[G(k+3)] \\ &\qquad = (1+o_M(1))\frac{Z'(k,m)}{Z(k,m)}\frac{Z_{\mathbb{T}_{i'}(N,M^+)}[G(k)]}{Z_{\mathbb{T}(N,M^+)}[G(k)]}Z_{\mathbb{T}_0(N,M)}[E_0(k,m)]. \end{aligned}$$

We wish to highlight the fact that the constants $C_N^{(\ell)}(\beta)$ involved in the previous equation are the same as for $i = j$, as the track of angle α is at a distance larger than $M/2 - m$ of the m-th track (this quantity tends to infinity as M tends to infinity), but that the constant $Z'(k,m)$ is a priori different from $Z(k,m)$ (it is a sum on the same configurations but the track t_j has an angle of α instead of β, hence some edge-weights are different).

Using (20) instead of (19) to estimate $Z_{\mathbb{T}_{i'}(N,M^+)}[F(k)]$, we infer that the second ratio converges to $\lambda_N^{(k)}(\alpha)/\lambda_N^k(\beta)$. We obtain the result by summing over m. Indeed, we may use again a uniform bound that is similar to (22) and observe that the case $i = j$ immediately implies that $\phi_{\mathbb{T}_0(N,M)}[E_0(k,m)|E_0(k)]$ converges as M tends to infinity. Note that

$$C_N^{(k)}(\alpha,\beta) := \sum_m \frac{Z'(k,m)}{Z(k,m)} \lim_{M\to\infty} \phi_{\mathbb{T}_0(N,M)}[E_0(k,m)|E_0(k)].$$

Using this definition of the constant and applying the same reasoning for $i < j$ concludes the proof. $\square$

We are now in a position to prove Proposition 2.5.

Proof of Proposition 2.5. Lemma 4.2 (for $i, j = 0, 1$) and Lemma 4.1(ii) imply that

$$C_N^{(k)}(\alpha,\beta)\frac{\lambda_N^{(k)}(\alpha)}{\lambda_N^{(k)}(\beta)} + \frac{\lambda_N^{(k+3)}(\beta)}{\lambda_N^{(k)}(\beta)} = 1 + C_N^{(k)}(\alpha,\beta)\frac{\lambda_N^{(k+3)}(\alpha)}{\lambda_N^{(k)}(\beta)},$$

which gives

$$C_N^{(k)}(\alpha,\beta) = \frac{\lambda_N^{(k)}(\beta) - \lambda_N^{(k+3)}(\beta)}{\lambda_N^{(k)}(\alpha) - \lambda_N^{(k+3)}(\alpha)}.$$

Plugging this formula into Lemma 4.2 gives

$$\lim_{M\to\infty}\frac{\phi_{\mathbb{T}_1}[E_1(k)]}{\phi_{\mathbb{T}_1}[E_0(k)]} = \frac{\lambda_N^{(k)}(\beta)}{\lambda_N^{(k+3)}(\beta)} \times \frac{1}{C_N^{(k)}(\alpha,\beta)\frac{\lambda_N^{(k)}(\alpha)}{\lambda_N^{(k)}(\beta)}} = \frac{\lambda_N^{(k)}(\beta)}{\lambda_N^{(k+3)}(\beta)} \times \frac{1-\lambda_N^{(k+3)}(\alpha)/\lambda_N^{(k)}(\alpha)}{1-\lambda_N^{(k+3)}(\beta)/\lambda_N^{(k)}(\beta)}.$$

Similarly,

$$\lim_{M\to\infty}\frac{\phi_{\mathbb{T}_0}[E_1(k)]}{\phi_{\mathbb{T}_0}[E_0(k)]} = \frac{\lambda_N^{(k+3)}(\alpha)}{\lambda_N^{(k)}(\beta)} \times C_N^{(k)}(\alpha,\beta) = \frac{\lambda_N^{(k+3)}(\alpha)}{\lambda_N^{(k)}(\alpha)} \times \frac{1-\lambda_N^{(k+3)}(\beta)/\lambda_N^{(k)}(\beta)}{1-\lambda_N^{(k+3)}(\alpha)/\lambda_N^{(k)}(\alpha)}.$$

$\square$

4.2 Separation of interfaces: Proof of Proposition 2.6

In this section, $\mathbf{\Gamma}$ is the left-most boundary of the cluster of z_1. For $r > 0$ and $a \le b$ in $\mathbb{Z}$, introduce two events

$$\mathrm{Iso}(r) := \{\mathbf{\Gamma} \cap \Lambda_r(\mathrm{lmax}(z_0)) = \emptyset\},$$
$$E_{[a,b]}(k) := \bigcup_{a\le j\le b} E_j(k).$$

The following proposition states a form of typical isolation of clusters.

Proposition 4.3 (Isolation of the top of the cluster of z_0)**.** *For every $\varepsilon > 0$, there exist $C, \eta > 0$ such that for every $\alpha, \beta \in (\varepsilon, \pi - \varepsilon)$, $i, j \in \mathbb{Z}$, $k \le N$, and $13r \le s \le N$, we have that for M large enough,*

$$\phi_{\mathbb{T}_i(N,M)}[E_j(k) \setminus \mathrm{Iso}(r) | E_{[j-s,j]}(k)] \le \frac{Cr^\eta}{s^{1+\eta}}. \tag{23}$$

Before focusing on this proposition, let us explain how it combines with the mixing of the 2-rooted IIC (Proposition 3.9) to imply Proposition 2.6. The key observation is that when $\mathrm{Iso}(r)$ occurs, one may sample everything but the cluster of z_0, and then sample the cluster of z_0 in such a way that near its maximum, the configuration looks like a 2-rooted IIC (since this maximum is far from the other clusters).

Proof of Proposition 2.6. The result is trivial when $\lambda_N^{(k)}(\beta)/\lambda_N^{(k+3)}(\beta)-1$ is large as we may choose C in such a way that the right-hand side is larger than 1. We will therefore assume in the proof that it is small. We will also omit integer approximations. For a vertex v, define the event

$$E_v(k) := E(k)\cap\{\text{lmax}(z_0)=v\}.$$

We can write

$$\phi_{\mathbb{T}_i(N,M)}[E_1(k)|E_{[0,1]}(k),\text{Iso}(r)] = \frac{\sum_{v\in t_0^-}\phi_{\mathbb{T}_i(N,M)}[E_{v^+}(k)|\text{Iso}(r)]}{\sum_{v\in t_0^-}\phi_{\mathbb{T}_i(N,M)}[E_{v^+}(k)\cup E_v(k)|\text{Iso}(r)]}. \tag{24}$$

Fix some $v\in t_0^-$. On the event $\text{Iso}(r)$, one may explore the clusters of the x_i's and $\mathbf{\Gamma}$ and use the spatial Markov property to sample the cluster of z_0. The mixing property of the 2-rooted IIC given by Proposition 3.9 thus implies that

$$|\phi_{\mathbb{T}_i(N,M)}[E_{v^+}(k)|E_v(k)\cup E_{v^+}(k),\text{Iso}(r)] - \Phi_i^2[\text{lmax}(\infty)=0^+]| \le Cr^{-\eta}. \tag{25}$$

Proposition 4.3 gives that for every choice of s, for M large enough,

$$\begin{aligned}\phi_{\mathbb{T}_i(N,M)}[\text{Iso}(r)^c|E_{[0,1]}(k)] &\le \phi_{\mathbb{T}_i(N,M)}[\text{Iso}(r)^c|E_1(k)] + \phi_{\mathbb{T}_i(N,M)}[\text{Iso}(r)^c|E_0(k)]\\ &\le \frac{Cr^\eta}{s^{1+\eta}}\Big(\frac{\phi_{\mathbb{T}_i(N,M)}[E_{[1-s,1]}(k)]}{\phi_{\mathbb{T}_i(N,M)}[E_1(k)]} + \frac{\phi_{\mathbb{T}_i(N,M)}[E_{[-s,0]}(k)]}{\phi_{\mathbb{T}_i(N,M)}[E_0(k)]}\Big).\end{aligned}$$

Using Lemma 4.2 and taking the limsup as M tends to infinity implies that

$$\limsup_{M\to\infty}\phi_{\mathbb{T}_i(N,M)}[\text{Iso}(r)^c|E_{[0,1]}(k)] \le \frac{2Cr^\eta}{s^{1+\eta}}\sum_{u=0}^{s}\Big(\frac{\lambda_N^{(k)}(\beta)}{\lambda_N^{(k+3)}(\beta)}\Big)^u \le \frac{2Cr^\eta}{s^{1+\eta}}\frac{\Big(\frac{\lambda_N^{(k)}(\beta)}{\lambda_N^{(k+3)}(\beta)}\Big)^{s+1}}{\frac{\lambda_N^{(k)}(\beta)}{\lambda_N^{(k+3)}(\beta)}-1}. \tag{26}$$

Combining (24), (25), and (26) for

$$s := \Big\lfloor\frac{1}{\log[\lambda_N^{(k)}(\beta)/\lambda_N^{(k+3)}(\beta)]}\Big\rfloor \qquad\text{and}\qquad r := \lfloor\sqrt{s}\rfloor$$

(one has $r\le s/13$ since we assume the ratio of eigenvalues is close to 1) gives the result by possibly changing the value of C. □

We now focus on Proposition 4.3. In the rest of this section, we fix i,j, and $k\le N\le M$ as well as $13r\le s\le N$. We drop the dependency in these parameters when it cannot lead to any confusion. In particular, we write

$$E := E_{[j-s,j]}(k)$$

and

$$\phi := \phi_{\mathbb{T}_i(N,M)}[\,\cdot\,|\text{edges of } t_M \text{ that are open are exactly the } \{x_i, y_i\} \text{ for } 1 \le i \le k].$$

We first prove, in Lemma 4.4, a bound for the probability that a vertex x is equal to $\mathrm{lmax}(z_0)$ while being not isolated, conditionally on the event that the cluster of z_0 intersects a box of size s centered around x (in fact around a vertex y near x). Then we prove, in Lemma 4.5, that conditionally on $E_{[j-s,j]}(k)$, the number of disjoint boxes of size s centered on a vertex of t_j^- intersected by the cluster of z_0 is bounded in expectation. The proposition then follows by combining the two lemmata (see below for a formal proof).

Lemma 4.4. *There exist uniform constants $c, \eta > 0$ such that for every two vertices $x, y \in t_j^-$ such that $d(x,y) \le s/4$,*

$$\phi[x = \mathrm{lmax}(z_0), \mathrm{Iso}(r)^c | E, z_0 \leftrightarrow \Lambda_s(y)] \le \frac{Cr^\eta}{s^{2+\eta}}. \tag{27}$$

Proof. Fix $x, y \in t_j^-$ as in the statement. Let $\mathbf{C}$ be the union of the clusters of $x_1, \ldots, x_k$, and $\mathbf{C}_0$ be the cluster of z_0 in $\mathbb{T}_i \setminus \Lambda_s(y)$. Introduce the events

$$F := E \cap \{z_0 \longleftrightarrow \Lambda_s(y)\},$$
$$\mathrm{Risk}_x := \{d(x, \mathbf{\Gamma}) \le r\} \cap \{x \text{ is below } \mathbf{\Gamma}\},$$

where by "below" we mean in the connected component of z_0 in the graph $\mathbb{T}_i \setminus (\mathbf{\Gamma} \cup t_M^-)$.

We have that

$$\begin{aligned}\phi[x = \mathrm{lmax}(z_0), \mathrm{Iso}(r)^c | F] &= \phi[x = \mathrm{lmax}(z_0), \mathrm{Risk}_x | F] \\ &= \phi[x = \mathrm{lmax}(z_0) | \mathrm{Risk}_x, F] \phi[\mathrm{Risk}_x | F]. \end{aligned} \tag{28}$$

We now bound separately the two probabilities of the last product.

Claim 1. *There exists $C_0 > 0$ independent of everything such that*

$$\phi[x = \mathrm{lmax}(z_0) | \mathrm{Risk}_x, F] \le C_0 s^{-2}. \tag{29}$$

Proof. Let $\mathbf{C}_{\mathbf{\Gamma}}$ be the union (see Figure 15) of the clusters intersecting $\mathbf{\Gamma}$ in $\omega \setminus \Lambda_{s/2}(y)$. Introduce the random variable $\mathcal{A} := (\mathbf{C}, \mathbf{\Gamma}, \mathbf{C}_0, \mathbf{C}_{\mathbf{\Gamma}})$. The following inequality will imply (29): for every $A = (\mathcal{C}, \Gamma, \mathcal{C}_0, \mathcal{C}_\Gamma)$,

$$\phi[x = \mathrm{lmax}(z_0), \mathrm{Risk}_x, F | \mathcal{A} = A] \le C_0 s^{-2} \phi[\mathrm{Risk}_x, F | \mathcal{A} = A]. \tag{30}$$

Indeed, it suffices to sum the above over all possible realizations A of $\mathcal{A}$. We now prove (30).

Below, we set Ω to be the set of edges below $\mathbf{\Gamma}$ whose state is not deterministically fixed on the event $\{\mathcal{A} = A\}$. We may assume without loss of generality that the probability

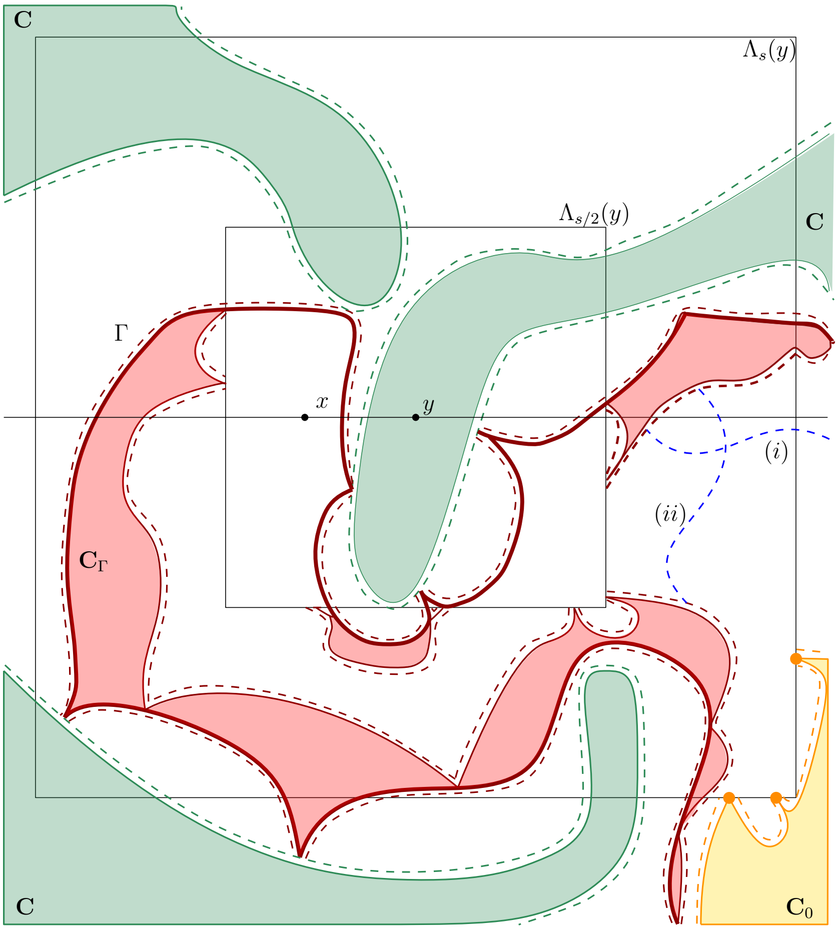

Figure 15: The conditioning on $(\mathbf{C}, \mathbf{\Gamma}, \mathbf{C_0}, \mathbf{C_\Gamma})$. The plain lines correspond to open paths, and the dashed ones to closed ones, or more precisely to open paths in the dual configuration ω^*. We kept the same color code as in Figure 14. Green clusters are the clusters of $x_1, \ldots, x_k$ and depict $\mathbf{C}$. The red ones depict $\mathbf{\Gamma}$ (in bold) and $\mathbf{C_\Gamma}$ (they are connected to z_{-1} and z_1). These induce wired boundary conditions on the part of $\Omega \cap \Lambda_{s/2}(y)$ below Γ, and free boundary conditions outside of $\Lambda_{s/2}(y)$. Finally, the yellow part is $\mathbf{C}_0$, i.e. the cluster of z_0 outside of $\Lambda_s(y)$. We also depicted (i) and (ii) in blue.

on the left is strictly positive otherwise there is nothing to prove. Under this condition, $\mathrm{Risk}_x \cap \{z_0 \longleftrightarrow \Lambda_s(y)\}$ already happens (since it is measurable in terms of $(\Gamma, \mathcal{C}_0)$). As a consequence, the following two conditions are sufficient (but not necessary) for E to happen:

(i) z_0 is not connected to t_j^+ in $\mathbb{T}_i \setminus \Lambda_{s/2}(y)$,
(ii) $\partial\Lambda_s(y)$ is not connected to $\partial\Lambda_{s/2}(y)$ in Ω.

Indeed, we must guarantee that z_0 is not connected to $z_{\pm 1}$ (or equivalently to Γ), and that the highest-most vertex of the cluster of z_0 is strictly below t_j^+. The conditioning on $\mathcal{A} = A$ implies that the only way for z_0 to be connected to Γ is via a path intersecting $\partial\Lambda_{s/2}(y)$. Since we only conditioned on $\mathbf{C}_0 = \mathcal{C}_0$, i.e. on the cluster of z_0 *outside* $\Lambda_s(y)$, the condition (ii) is sufficient to prevent the occurrence of a connection between z_0 and Γ. Moreover, (ii) ensures that z_0 is not connected to $\Lambda_{s/2}(y)$. Once this is guaranteed, (i) suffices to ensure that z_0 is disconnected from t_j^+.

Since the boundary conditions induced by $\mathcal{A} = A$ are free on the part of $\partial\Omega$ strictly inside $\Lambda_s(y) \setminus \Lambda_{s/2}(y)$, Proposition 3.6 shows that (ii) happens with probability larger than some universal constant $c_0 > 0$. Since both events in (i) and (ii) are decreasing, the FKG inequality implies that

$$\phi[\mathrm{Risk}_x, F | \mathcal{A} = A] \geq \phi[(i), (ii) | \mathcal{A} = A] \geq c_0 \phi[(i) | \mathcal{A} = A]. \tag{31}$$

Conversely, on $\{\mathcal{A} = A\}$, for $x = \mathrm{lmax}(z_0)$ to occur, (i) must occur together with

(iii) The half-plane three-arm event with type 010 in Ω to distance $s/4$ of x.

This gives

$$\begin{aligned} \phi[x = \mathrm{lmax}(z_0), \mathrm{Risk}_x, F | \mathcal{A} = A] &\leq \phi[(iii), (i) | \mathcal{A} = A] \\ &= \phi[(iii) | (i), \mathcal{A} = A] \phi[(i) | \mathcal{A} = A] \\ &\leq \tfrac{1}{c_0} \phi[(iii) | (i), \mathcal{A} = A] \phi[\mathrm{Risk}_x, F | \mathcal{A} = A] \end{aligned} \tag{32}$$

(in the last line we used (31)). Thus, to prove (30) it suffices to show that

$$\phi[(iii) | (i), \mathcal{A} = A] \leq C_1 s^{-2}.$$

In order to see that, we claim the following. For every n, every Ω' containing 0, and every boundary conditions ξ for which all the vertices of $\partial\Omega' \cap \Lambda_n$ are wired together,

$$\phi^{\xi}_{\Omega'}[A^{\mathrm{T}}_{010}(0, n)] \leq C_2 \phi_{\mathbb{L}(\beta)}[A^{\mathrm{T}}_{010}(0, n/2)] \leq C_3 n^{-2} \tag{33}$$

(the last inequality follows from Proposition 3.4). Note that this inequality would imply the result. Indeed, if $\mathcal{A} = A$ and (i) occurs, the remaining unexplored edges in $\Lambda_{s/2}(y)$ that are in the connected component of x are bordered, strictly inside $\Lambda_{s/2}(y)$, by Γ only, which is wired by definition.

We will now conclude the proof of the claim by showing (33). Notice that this is a general property independent of the setting used in the claim. Fix Ω' and some boundary conditions ξ. By spatial Markov property, we may condition on everything outside Λ_n and assume without loss of generality that $\Omega' \subset \Lambda_n$. To get (33), let $\mathbf{C}(0)$ be the cluster of 0 in Ω' (without considering the connection due to boundary conditions). Let ψ be the boundary conditions on Λ_n corresponding to ξ on $(\partial\Omega') \setminus \Lambda_{n-1}$, and wired for all the other vertices of $\partial\Lambda_n$. For every realization $\mathcal{C}(0) \subset \Omega'$ of $\mathbf{C}(0)$ for which $A^{\mathrm{T}}_{010}(0,n)$ occurs, the spatial Markov property and the FKG inequality imply

$$\begin{aligned}\phi^{\xi}_{\Omega'}[\mathbf{C}(0) = \mathcal{C}(0)] &= \phi^{\psi}_{\Lambda_n}[\mathbf{C}(0) = \mathcal{C}(0) | \omega_{|\Lambda_n \setminus \Omega'} = 1] \\ &= \frac{\phi^{\psi}_{\Lambda_n}[\omega_{|\Lambda_n \setminus \Omega'} = 1 | \mathbf{C}(0) = \mathcal{C}(0)]}{\phi^{\psi}_{\Lambda_n}[\omega_{|\Lambda_n \setminus \Omega'} = 1]} \phi^{\psi}_{\Lambda_n}[\mathbf{C}(0) = \mathcal{C}(0)] \\ &= \frac{\phi^{\psi}_{\Lambda_n}[\omega_{|\Lambda_n \setminus \Omega'} = 1 | \omega_{|\partial_e \mathcal{C}(0)} = 0]}{\phi^{\psi}_{\Lambda_n}[\omega_{|\Lambda_n \setminus \Omega'} = 1]} \phi^{\psi}_{\Lambda_n}[\mathbf{C}(0) = \mathcal{C}(0)] \le \phi^{\psi}_{\Lambda_n}[\mathbf{C}(0) = \mathcal{C}(0)],\end{aligned}$$

where $\partial_e \mathcal{C}(0)$ is the edge-boundary composed of the edges in Λ_n with one endpoint in and the other outside $\mathcal{C}(0)$. Summing over the $\mathcal{C}(0)$ included in Ω', we obtain that

$$\phi^{\xi}_{\Omega'}[A^{\mathrm{T}}_{010}(0,n)] \le \phi^{\psi}_{\Lambda_n}[A^{\mathrm{T}}_{010}(0,n)].$$

Comparing the later to the full space is now a simple use of the mixing property (Proposition 3.5):

$$\phi^{\psi}_{\Lambda_n}[A^{\mathrm{T}}_{010}(0,n)] \le \phi^{\psi}_{\Lambda_n}[A^{\mathrm{T}}_{010}(0,n/2)] \le C_{\mathrm{mix}} \phi^{0}_{\mathbb{L}(\beta)}[A^{\mathrm{T}}_{010}(0,n/2)].$$

The previous inequalities imply (33) and therefore conclude the proof. □

We now turn to the bound on the second term of (28). Introduce

$$G := F \cap \{z_0 \not\longleftrightarrow \Lambda_{s/2}(y)\} \cap \{z_1 \longleftrightarrow z_{-1} \text{ in } \mathbb{T} \setminus \Lambda_{s/3}(y)\}.$$

Claim 2. *There exists $C_1 > 0$ independent of everything such that*

$$\phi[\mathrm{Risk}_x | F] \le C_1 \phi[\mathrm{Risk}_x | G]. \tag{34}$$

Proof. We reuse the notation from Claim 1. We only need to prove that for every A,

$$\phi[\mathrm{Risk}_x, F | \mathcal{A} = A] \le C \phi[\mathrm{Risk}_x, G | \mathcal{A} = A]. \tag{35}$$

Fix A and note that we may focus on A for which the left-hand side is strictly positive. Now, for such values of A, $\mathrm{Risk}_x \cap G$ occurs if the following sufficient conditions do:

(*i*) z_0 is not connected to t^+_j in $\mathbb{T}_i \setminus \Lambda_{s/2}(y)$;
(*ii*) $\partial\Lambda_s(y)$ is not connected to $\partial\Lambda_{s/2}(y)$ in $\omega \cap \Omega$;
(*iv*) $\partial\Lambda_{s/2}(y)$ and $\partial\Lambda_{s/3}(y)$ are not connected in $\omega^* \cap \Omega$.

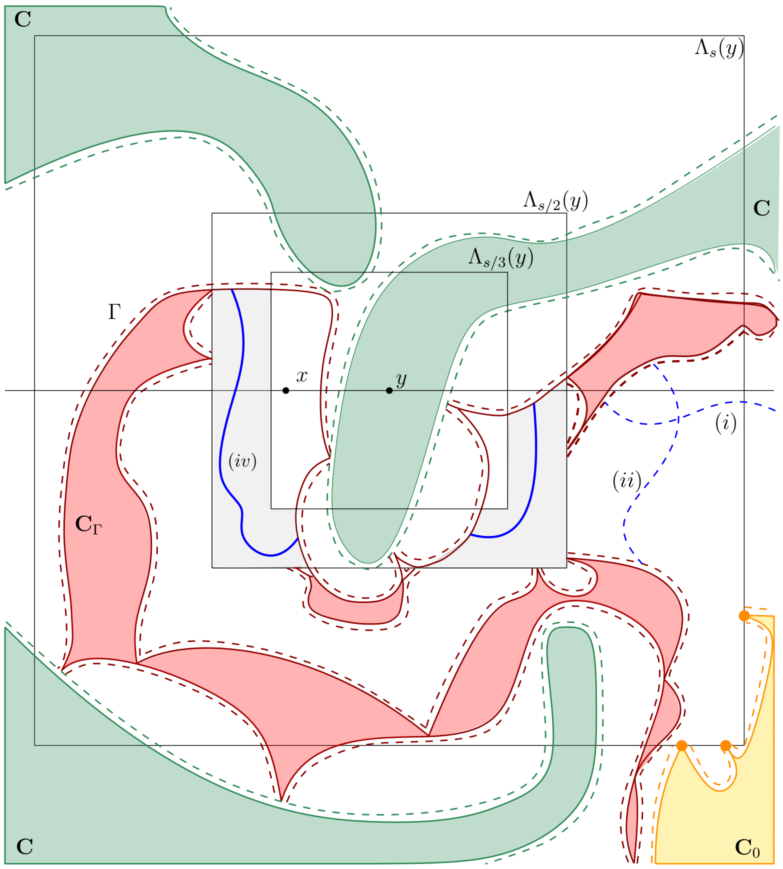

Figure 16: The picture is almost the same as in the previous one, except we depicted (iv) instead of (iii).

Conditions (i) and (ii) are the same as in Claim 1. They ensure that F and $z_0 \not\leftrightarrow \partial\Lambda_{s/2}(y) \cup t_j^+$ occur. Condition (iv) ensures that, when $\mathbf{\Gamma}$ visits $\Lambda_{s/3}(x)$, there exists a path between z_{-1} and z_1 that by-passes $\Lambda_{s/3}(x)$.

Using the previous claim, we already know that

$$\phi[(i),(ii)|\mathcal{A}=A] \geq c_0\phi[(i)|\mathcal{A}=A].$$

Also, since the boundary conditions induced by $\{\mathcal{A}=A\}\cap(ii)$ on vertices in $\partial\Omega\cap\Lambda_{s/2}(y)$ are wired (see Figure 16), we deduce from Proposition 3.6 applied to the dual model that

$$\phi[(iv)|(i),(ii),\mathcal{A}=A] \geq c_0.$$

Combining the previous two displayed inequalities implies that

$$\phi[\text{Risk}_x, G|\mathcal{A}=A] \geq \phi[(i),(ii),(iv)|\mathcal{A}=A] \geq c_0^2\phi[(i)|\mathcal{A}=A].$$

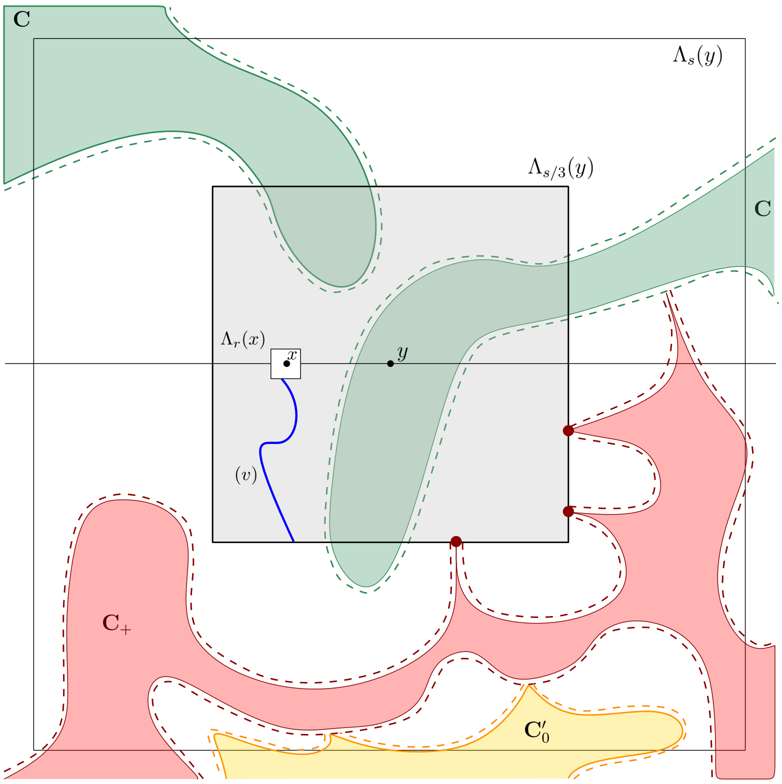

Figure 17: The conditioning on $(\mathbf{C}, \mathbf{C}'_0, \mathbf{C}_+)$. Note that the boundary conditions in the remaining set Ω are free within $\Lambda_{s/3}(y)$. In fact, the only points that are wired on $\partial\Omega$ are the vertices of $\mathbf{C}_+$ on the boundary of $\partial\Lambda_{s/3}(y)$ (the red bullets on the picture). In blue, the path that must exist for (v) to occur. Note that since $r < s/13$, the distance between $\Lambda_r(x)$ and $\partial\Lambda_{s/3}(y)$ is larger than $s/3 - |x-y| - r \geq s/156$.

Since (i) is needed for F to occur, we find

$$\phi[(i)|\mathcal{A} = A] \geq \phi[\text{Risk}_x, F|\mathcal{A} = A]$$

and therefore (35) follows from the last two displayed equations. This concludes the proof of the claim. □

Claim 3. *There exist $c_2, C_2 > 0$ independent of everything such that for every $r \leq s/13$ and every M large enough,*

$$\phi[\text{Risk}_x|G] \leq C_2(r/s)^{c_2}. \tag{36}$$

Proof. Recall the definition of $\mathbf{C}$ and introduce the clusters $\mathbf{C}'_0$ of z_0 in $\mathbb{T}_i$ and $\mathbf{C}_+$ of z_1 in $\mathbb{T}_i \setminus \Lambda_{s/3}(y)$. Consider the random variable $\mathcal{B} := (\mathbf{C}, \mathbf{C}'_0, \mathbf{C}_+)$. Since G is $\mathcal{B}$-measurable, it suffices to show that for every $B = (\mathcal{C}, \mathcal{C}'_0, \mathcal{C}_+)$ for which G occurs,

$$\phi[\mathrm{Risk}_x, G | \mathcal{B} = B] \le C_2 (r/s)^{c_2} \tag{37}$$

and to sum the previous inequality over all possible B.

Fix B as above. Below, we set Ω to be the set of edges whose states are not deterministically fixed on the event $\{\mathcal{B} = B\}$. Note that for Risk_x to occur, there must exist (see Figure 17)

(v) an open path in Ω from $\partial\Lambda_{s/3}(y)$ to $\partial\Lambda_r(x)$.

Since the boundary conditions induced by $\{\mathcal{B} = B\}$ on the part of $\partial\Omega$ strictly inside $\Lambda_{s/3}(y)$ are free, Proposition 3.6 implies that

$$\phi[\mathrm{Risk}_x | \mathcal{B} = B] \le \phi[(v) | \mathcal{B} = B] \le C_2 (\tfrac{r}{s})^{c_2}.$$

This concludes the proof of (37) and of the claim. □

Plugging Claims 1, 2 and 3 into (28) concludes the proof of the lemma. □

We now deal with the second lemma, which states a bound on the probability that two boxes of size s centered on vertices in t_j^- are connected to z_0 in terms of the probability that one of them is. Below, $|\cdot|$ denotes the Euclidean distance.

Lemma 4.5. *There exists a uniform constant $C > 0$ such that for every $x, y \in t_j^-$,*

$$\phi[z_0 \leftrightarrow \Lambda_s(x), z_0 \leftrightarrow \Lambda_s(y), E] \le C (\tfrac{s}{|y-x|})^2 \max_{z \in \{x,y\}} \phi[z_0 \leftrightarrow \Lambda_s(z), E].$$

Proof. Assume that $|x - y| \ge 13s$ otherwise one may simply set $C = 16$ to guarantee the inequality. Set $L := \lfloor |x-y|/3 \rfloor$. Let $\mathbf{C}$ and $\mathbf{\Gamma}$ be defined as in the proof of Lemma 4.4, and write $\mathcal{D} := (\mathbf{C}, \mathbf{\Gamma})$. We will prove that for every possible realization $D = (\mathcal{C}, \Gamma)$ of $\mathcal{D}$,

$$\begin{aligned}\phi[z_0 \leftrightarrow \Lambda_s(x), & z_0 \leftrightarrow \Lambda_s(y), E | \mathcal{D} = D] \\ &\le C (\tfrac{s}{|y-x|})^2 \Big(\phi[z_0 \leftrightarrow \Lambda_s(x), E | \mathcal{D} = D] + \phi[z_0 \leftrightarrow \Lambda_s(y), E | \mathcal{D} = D] \Big), \end{aligned} \tag{38}$$

which implies the claim by summing over all possible D.

From now on, fix a realization D for which the left-hand side is strictly positive. Let Ω be the set below $\mathbf{\Gamma}$. Consider the family of arcs $\ell_{x,i}$ (indexed by i) of $\Omega \cap \partial\Lambda_L(x)$ disconnecting z_0 in Ω from at least one vertex in $\Lambda_s(x)$. Since $\mathbf{\Gamma}$ is a path, any $z \in \Lambda_s(x) \cap \Omega$ is separated from z_0 by at least one such arc. Let $P_{x,i}$ be the region of $\Omega \setminus \ell_{x,i}$ separated from z_0, see Figure 18. Introduce the events

$$H_{x,i} := \{\exists z \in \Lambda_s(x) \cap P_{x,i} : z \longleftrightarrow z_0\} \cap \{\exists z' \in \Lambda_s(y) \setminus P_{x,i} : z' \longleftrightarrow z_0\} \cap E$$

and $H_{y,j}$ defined in a similar fashion by considering arcs of $\Omega \cap \partial\Lambda_L(y)$, with the roles of x and y exchanged.

We claim that

$$\{z_0 \longleftrightarrow \Lambda_s(x), z_0 \longleftrightarrow \Lambda_s(y), E\} \subset \Big(\bigcup_i H_{x,i}\Big) \cup \Big(\bigcup_j H_{y,j}\Big). \tag{39}$$

Indeed, assume that the event on the left holds true and consider the regions $P_{u,k}$ with $u \in \{x, y\}$ for which there exists $z \in \Lambda_s(u) \cap P_{u,k}$ with $z \longleftrightarrow z_0$. There exists at least one $P_{u,k}$ with this property. Fix a region $P_{u,k}$ with the property above, and which is minimal among such regions for the inclusion. For simplicity, assume $u = x$. The first condition is ensured by the choice of $P_{x,k}$, and E occurs by assumption. The only way for $H_{x,k}$ to fail is if the second condition does, which implies the existence of $z' \in \Lambda_s(y) \cap P_{x,k}$ which is connected to z_0. Now, if we take $P_{y,j}$ to be the minimal (for the inclusion again) region containing z', we have $P_{y,j} \subsetneq P_{x,k}$, which contradicts the minimality of $P_{x,k}$.

We now prove the following claim.

Claim. *There exists a universal constant $C_0 > 0$ such that*

$$\phi[H_{x,i}|\mathcal{D} = D] \le C_0 \phi_{\mathbb{Z}^2}[E_{x,i}]\phi[z_0 \nleftrightarrow \Lambda_s(y), E|\mathcal{D} = D], \tag{40}$$

where $E_{x,i}$ is the event that $\Lambda_s(x) \cap P_{x,i}$ contains a vertex z which is connected to $\partial\Lambda_{L/4}(x)$ but not to $\partial P_{x,i} \cup t_j^+$.

Proof. Introduce the event $H'_{x,i}$ that $H_{x,i}$ occurs and there exists $z' \in \Lambda_s(y) \setminus P_{x,i}$ which is connected to z_0 *outside* of $P_{x,i} \cap \Lambda_{L/2}(x)$. There exists $c_0 > 0$ such that

$$\phi[H'_{x,i}|\mathcal{D} = D] \ge c_0 \phi[H_{x,i}|\mathcal{D} = D]. \tag{41}$$

Indeed, consider the outer boundary $\mathbf{\Gamma}_0$ of the cluster of z_0. By definition on $H_{x,i}$, $\mathbf{\Gamma}_0$ must contain a vertex $z \in \Lambda_s(x) \cap P_{x,i}$ and a vertex $z' \in \Lambda_s(y) \setminus P_{x,i}$. As a consequence, $H'_{x,i}$ occurs as soon as there is no crossing in ω^* from $\partial\Lambda_{L/2}(x)$ to $\partial\Lambda_L(x)$ in the interior of $\mathbf{\Gamma}_0$, see Figure 18 and its caption for more details. Since conditioning on $\mathbf{\Gamma}_0$ induces wired boundary conditions on its interior, the probability of this event is bounded from below by some universal constant $c_0 > 0$ by Proposition 3.6.

Now, following a reasoning similar to Claim 1 of the previous lemma – here $\mathbf{C}_0$ becomes the cluster of z_0 outside $P_{x,i} \cap \Lambda_{L/2}(x)$ (which, on the event $H'_{x,i}$, necessarily contains a vertex in $\Lambda_s(y)$), and $\mathbf{C}_{\mathbf{\Gamma}}$ the union of the clusters of $\mathbf{\Gamma}$ outside of $P_{x,i} \cap \Lambda_{L/4}(x)$ – we obtain that

$$\phi[H'_{x,i}|\mathcal{D} = D, z_0 \text{ connected to } \Lambda_s(y) \text{ but not to } t_j^+ \text{ in } \Omega \setminus \Lambda_{L/2}(x)] \le C_1 \phi_{\mathbb{Z}^2}[E_{x,i}]. \tag{42}$$

The two inequalities together give the result. □

We are ready to conclude. Write $\mathbf{N}$ for the number of disjoint clusters (in $\Lambda_{L/4}(x)$) from $\Lambda_s(x)$ to $\partial\Lambda_{L/4}(x)$ that are contained in the lower half-plane. By exploring the clusters one by one we obtain easily from (RSW) and the comparison between boundary conditions that there exist $c_2, C_2 \in (0, \infty)$ such that for every $k \ge 0$,

$$\phi_{\mathbb{Z}^2}[\mathbf{N} > k] \le C_2 (s/L)^{2+c_2 k}.$$

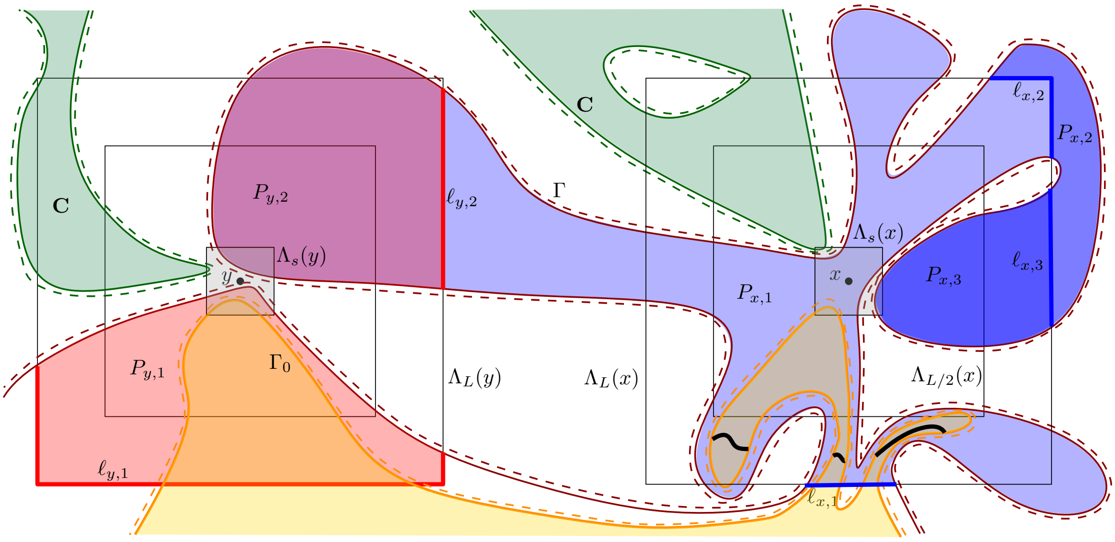

Figure 18: We depicted in green (for $\mathbf{C}$) and dark red (for $\mathbf{\Gamma}$) the event $\mathcal{D} = D$. We also listed the arcs $\ell_{x,i}$ and $\ell_{y,i}$, as well as the domains $P_{x,i}$ enclosed by them. Note that the domains can be included into each other: here $P_{y,2}$ and $P_{x,2}$ are included in $P_{x,1}$, and $P_{x,3}$ is itself included in $P_{x,2}$. In yellow, the path Γ_0 contains a vertex in $\Lambda_s(y)$ and a vertex in $\Lambda_s(x)$. In the picture, $H_{x,1}$ occurs. Then, if the bold black paths are open, z_0 is connected to $\Lambda_s(y)$ outside of $P_{x,1} \cap \Lambda_{s/2}(x)$ by following $\mathbf{\Gamma}_0$ and shortcutting any visit of $\mathbf{\Gamma}_0$ to $\Lambda_{L/2}(x)$ via the black paths.

We deduce that

$$\sum_i \phi_{\mathbb{Z}^2}[E_{x,i}] \leq \phi[\mathbf{N}] \leq C_3(s/L)^2.$$

Summing over i the estimate provided by the claim and using the previous inequality gives

$$\phi\Big[\bigcup_i H_{x,i}\Big|\mathcal{D} = D\Big] \leq C_4(s/L)^2 \phi[z_0 \leftrightarrow \Lambda_s(y), E|\mathcal{D} = D]. \tag{43}$$

A similar estimate holds for the union of the $H_{y,j}$. Together with (39), this gives (38) and therefore the claim. □

We are now in a position to prove Proposition 4.3.

Proof of Proposition 4.3. Without loss of generality, we may assume that 4 divides s which divides N (otherwise the proof can be trivially adapted). Let Y be a set of vertices $y \in t_j^-$ at a distance $s/4$ of each other. For any vertex $x \in t_j^-$, define $[x]$ to be the vertex $y \in Y$ closest to x.

On the one hand, using the inclusion of events, we get that

$$\begin{aligned}\phi[E_0(k),\mathrm{Iso}(r)^c|E] &\le \sum_{x\in t_j^-}\phi[x=\mathrm{lmax}(z_0), z_0\leftrightarrow \Lambda_s([x]),\mathrm{Iso}(r)^c|E]\\ &= \sum_{y\in Y}\sum_{\substack{x\in t_j^-\\ [x]=y}}\phi[x=\mathrm{lmax}(z_0), z_0\leftrightarrow \Lambda_s(y),\mathrm{Iso}(r)^c|E].\end{aligned}$$

Now, Lemma 4.4 and the fact that there are at most s vertices x satisfying $[x]=y$ for every fixed $y\in Y$ give

$$\sum_{\substack{x\in t_j^-\\ [x]=y}}\phi[x=\mathrm{lmax}(z_0), z_0\leftrightarrow \Lambda_s(y),\mathrm{Iso}(r)^c|E]\le \frac{Cr^\eta}{s^{1+\eta}}\phi[z_0\leftrightarrow\Lambda_s(y)|E].$$

It remains to bound the sum over $y\in Y$ of $\phi[z_0\leftrightarrow\Lambda_s(y)|E]$ by a uniform constant. To do that, observe that the Cauchy-Schwarz inequality and Lemma 4.5 give

$$\begin{aligned}\Big(\sum_{y\in Y}\phi[z_0\leftrightarrow\Lambda_s(y)|E]\Big)^2 &\le \sum_{y,z\in Y}\phi[z_0\leftrightarrow\Lambda_s(y), z_0\leftrightarrow\Lambda_s(z)|E]\\ &\le C\sum_{y\in Y}\phi[z_0\leftrightarrow\Lambda_s(y)|E]\\ &\quad + C\sum_{y\ne z\in Y}(\tfrac{s}{|z-y|})^2\Big[\phi[z_0\leftrightarrow\Lambda_s(y)|E]+\phi[z_0\leftrightarrow\Lambda_s(z)|E]\Big]\\ &\le C'\sum_{y\in Y}\phi[z_0\leftrightarrow\Lambda_s(y)|E],\end{aligned}$$

which implies that the sum is at most C', and therefore concludes the proof. □

4.3 Proof of Theorem 2.4

We prove the first identity (the second follows from the same argument). It suffices to show that

$$\frac{Ф_1^2[\mathrm{lmax}(\infty)=0^+]}{Ф_1^2[\mathrm{lmax}(\infty)=0]}=\frac{\sin\alpha}{\sin\beta}.$$

First of all, as seen in Theorem 3.8, for every $\varepsilon>0$, there exists $R>0$ such that for every $\alpha,\beta\in(\varepsilon,\pi-\varepsilon)$ and $q\in[1,4]$, for every event A,

$$|Ф_1^2[A]-\phi^0_{\mathbb{L}_1}[A|\{0\leftrightarrow\partial\Lambda_R,\mathrm{lmax}(0)=0\}\cup\{0^+\leftrightarrow\partial\Lambda_R,\mathrm{lmax}(0^+)=0^+\}]|\le\varepsilon.$$

The convergence in Theorem 3.8 is uniform in $q\in[1,4]$ and $\alpha,\beta\in(\varepsilon,\pi-\varepsilon)$ (as shown by Proposition 3.9). Also, the eigenvalues $\lambda_N^{(k)}(\theta)$ are continuous in θ and q. As a consequence, we only need to prove the claim for $\alpha,\beta\ne\pi/2$ and $1\le q<4$.

We now focus on this case. Applying Propositions 2.5 and 2.6 gives that

$$\left|\frac{\Phi_1^2[\mathrm{lmax}(\infty)=0^+]}{\Phi_1^2[\mathrm{lmax}(\infty)=0]}-\frac{\lambda^{(k)}(\beta)}{\lambda^{(k+3)}(\beta)}\times\frac{1-\lambda_N^{(k+3)}(\alpha)/\lambda_N^{(k)}(\alpha)}{1-\lambda_N^{(k+3)}(\beta)/\lambda_N^{(k)}(\beta)}\right|\le C\Big(\frac{\lambda_N^{(k)}(\beta)}{\lambda_N^{(k+3)}(\beta)}-1\Big)^{\eta}. \quad (44)$$

Below, $o(1)$ denotes a quantity tending to 0 as N tends to infinity. Theorem 2.7 implies that for $k\in[N^{1/2},2N^{1/2}]$,

$$\frac{1-\lambda_N^{(k+3)}(\alpha)/\lambda_N^{(k)}(\alpha)}{1-\lambda_N^{(k+3)}(\beta)/\lambda_N^{(k)}(\beta)}=\frac{\log\lambda_N^{(k)}(\alpha)-\log\lambda_N^{(k+3)}(\alpha)}{\log\lambda_N^{(k)}(\beta)-\log\lambda_N^{(k+3)}(\beta)}+o(1)$$

and

$$\frac{\lambda_N^{(k)}(\beta)}{\lambda_N^{(k+3)}(\beta)}=1+o(1)$$

so (44) implies that

$$\frac{\Phi_1^2[\mathrm{lmax}(\infty)=0^+]}{\Phi_1^2[\mathrm{lmax}(\infty)=0]}=\frac{\log\lambda_N^{(k)}(\alpha)-\log\lambda_N^{(k+3)}(\alpha)}{\log\lambda_N^{(k)}(\beta)-\log\lambda_N^{(k+3)}(\beta)}+o(1). \quad (45)$$

At this stage, we use Theorem 2.7 one more time to notice that

$$\frac{\log\lambda_N^{(N^{1/2})}(\alpha)-\log\lambda_N^{(2N^{1/2})}(\alpha)}{\log\lambda_N^{(N^{1/2})}(\beta)-\log\lambda_N^{(2N^{1/2})}(\beta)}=\frac{\sin\alpha}{\sin\beta}+o(1).$$

We deduce that there exists k_- between $N^{1/2}$ and $2N^{1/2}$ such that

$$\frac{\log\lambda_N^{(k_-)}(\alpha)-\log\lambda_N^{(k_-+3)}(\alpha)}{\log\lambda_N^{(k_-)}(\beta)-\log\lambda_N^{(k_-+3)}(\beta)}\ge\frac{\sin\alpha}{\sin\beta}-o(1)$$

and similarly k_+ such that

$$\frac{\log\lambda_N^{(k_+)}(\alpha)-\log\lambda_N^{(k_++3)}(\alpha)}{\log\lambda_N^{(k_+)}(\beta)-\log\lambda_N^{(k_++3)}(\beta)}\le\frac{\sin\alpha}{\sin\beta}+o(1).$$

Applying (45) to k_+ and k_- and letting N tend to infinity concludes the proof.

5 Homotopy distance: proof of Theorem 2.2

5.1 Encoding of homotopy classes

In the introduction, we were not precise in the way we compute homotopy classes. We now remedy this approximation. Recall that $\mathbb{B}_\eta:=\eta\mathbb{Z}^2\cap[-1/\eta,1/\eta]^2$. Consider the set of *oriented edges* $\vec{E}$:

$$\vec{E}:=\{(x,y):x,y\in\mathbb{B}_\eta:\|x-y\|_2=\eta\}.$$

Below, we see an oriented edge (x, y) as a segment joining the endpoints x and y with an orientation from x to y.

Let $\mathcal{W}$ be the set of finite words on the alphabet $\vec{E}$ and denote the empty word by $\emptyset$. Define the "cyclic" equivalence relation $\sim$ on $\mathcal{W}$ by $(u_i)_{1\leq i\leq p} \sim (v_j)_{1\leq j\leq q}$ if and only if $p = q$ and there exists $k \in [1, p]$ such that $u_1 \dots u_p = v_k \dots v_p v_1 \dots v_{k-1}$. Define the set of *cyclic words* as the quotient $\mathcal{CW} := \mathcal{W}/\sim$.

We also wish to work with a reduced representation of cyclic words. Let $\preceq$ be the (smallest) order relation on $\mathcal{CW}$ such that for any word $u = u_1 \cdots u_p$, any integer $1 \leq k < p$ (resp. $1 \leq k \leq p$) such that $u_{k+1} = (x, y)$ and $u_k = (y, x)$, and $v = u_1 \cdots u_{k-1} u_{k+2} \cdots u_p$, we have $v \preceq u$. It is straightforward to check that there exists a smallest $\underline{u} \preceq u$ for every u. We call this the *reduced word of* u.

Definition 5.1 (Homotopy classes)**.** For a non-self-intersecting smooth loop $\gamma \subset \mathbb{R}^2 \setminus \mathbb{B}_\eta$, let $u = \mathbf{u}(\gamma)$ be the *word associated to* γ defined as follows: orient the loop counterclockwise, fix $a \in \gamma$ not on an oriented edge[7], and let $\mathbf{u}(\gamma) = u_1 \cdots u_k$, where u_i is the i-th (when going counter-clockwise along the curve starting from a) oriented edge (x, y) crossed by γ in such a way that x is on the left of the crossing and y on the right. Then, the *homotopy class* $[\gamma]_\eta$ of γ is the reduced word $\underline{u}(\gamma)$.

Remark 5.2. The previous definitions are sufficient for the proof of Theorem 2.2. In preparation for the proof of Theorem 2.3 in the next section, we explain how homotopy classes in other spaces considered in this paper are encoded. Consider a collection of non-self-intersecting loops $(\ell_x : x \in \mathbb{B}_\eta)$ satisfying that the right-most point $z(\ell_x)$ in ℓ_x (if there is more than one such point, consider the lowest one) belongs to $B(x, \frac{1}{4}\eta)$ for every $x \in \mathbb{B}_\eta$. We wish to compute homotopy classes in the space $\mathbb{R}^2 \setminus \cup_{x\in\mathbb{B}_\eta} \ell_x$ in a way that is consistent with the definition of $[\cdot]$ above. Define the segment (x, y) for x and y neighbors to be the oriented segment from $z(\ell_x)$ to $z(\ell_y)$. Encode the homotopy classes of loops in $\mathbb{R}^2 \setminus \cup_{x\in\mathbb{B}_\eta} \ell_x$ using reduced words in the same way as above with segments between $z(\ell_x)$ and $z(\ell_y)$ playing the role of the segment between x and y in the case of $\mathbb{R}^2 \setminus \mathbb{B}_\eta$.

5.2 Proof of Theorem 2.2

Proof of Theorem 2.2. We show the result for the Camia-Newman distance. The version of the result for the Schramm-Smirnov distance follows readily from known implications between the former and the latter (see e.g. [11, Theorem 7]). In this proof, fix $\kappa, \eta, \delta > 0$ such that

$$\kappa > 12\sqrt{2}\eta \geq 1000\delta.$$

Below, a word $v = v_1 \cdots v_\ell \in \mathcal{W}$ is a *subword* of $u \in \mathcal{CW}$ if there exists k such that $v_i = u_{k+i}$ for all $1 \leq i \leq \ell$. We extend the order relation to non-cyclic words and can therefore talk of $\underline{v}$. Also, we call $\mathrm{diam}(v)$ the maximal Euclidean distance between two centers of edges in $\{v_1, \dots, v_k\}$ and say that v *intersects* $B(0, 1/\kappa)$ if it contains a letter

[7] Changing a will correspond to a rerooting of the loop and will lead to the same cyclic word.

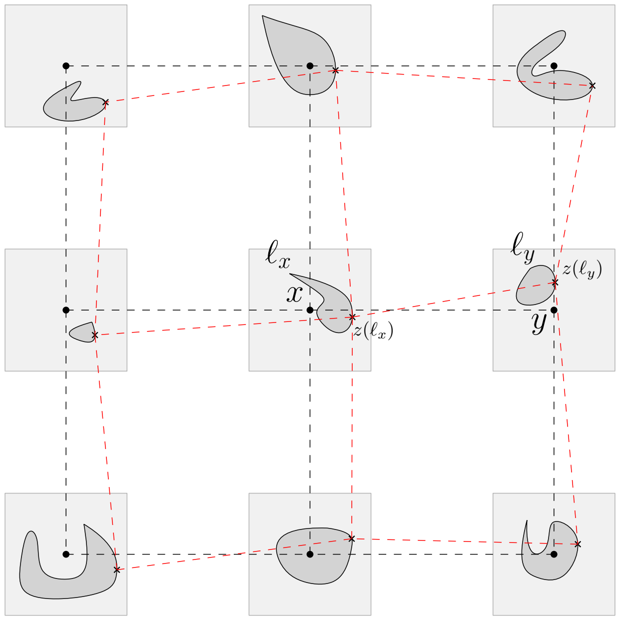

Figure 19: In dashed black, the segments corresponding to the oriented edges in $\mathbb{B}_\eta$. Also, the crosses correspond to the points $z(\ell_x)$ for the loops. Finally, the associated segments are depicted in dashed red.

which is incident to a point of $\mathbb{B}_\eta \cap B(0, 1/\kappa)$. With these definitions, introduce the events

$$\text{Normal}(\kappa, \eta) := \left\{ \begin{array}{c} \nexists \text{ a loop } \gamma \text{ such that } \mathbf{u}(\gamma) \text{ contains a subword } v \\ \text{intersecting } B(0, 1/\kappa) \text{ with } \underline{v} = \emptyset \text{ and } \text{diam}(v) \geq \frac{1}{3}\kappa \end{array} \right\},$$

$$\text{Dense}(\eta) := \{\text{every face of } \eta\mathbb{Z}^2 \cap B(0, 1/\eta) \text{ contains a loop in } \mathcal{F}_0 \text{ and one in } \mathcal{F}_1\}.$$

Now, consider $\omega_\delta \sim \phi_{\delta\mathbb{L}(\alpha)}$ and $\overline{\omega}'_\delta$ of $\omega_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$ and let $\mathcal{F}_i(\omega_\delta)$ and $\mathcal{F}_i(\omega'_\delta)$ be the collections of loops (primal or dual depending on i) in $\overline{\omega}_\delta$ and $\overline{\omega}'_\delta$ respectively.

Claim 1. *For every $\kappa > 12\sqrt{2}\eta$, one has*

$$\{d_{\mathbf{H}}(\omega_\delta, \omega'_\delta) \leq \eta\} \cap \{\omega_\delta, \omega'_\delta \in \text{Normal}(\kappa, \eta) \cap \text{Dense}(\eta, \delta)\} \subset \{d_{\mathbf{CN}}(\omega_\delta, \omega'_\delta) \leq \kappa\}. \quad (46)$$

Proof. Consider a loop $\gamma \in \mathcal{F}_i(\omega_\delta)$ that is included in $B(0, 1/\kappa)$, we need to prove that there exists $\gamma' \in \mathcal{F}_i(\omega'_\delta)$ such that $d(\gamma', \gamma) \leq \kappa$. Since the same can be done for $\mathcal{F}_i(\omega'_\delta)$, this will conclude the proof of (46).

Write $\boldsymbol{u}(\gamma) = \underline{u}_1 v^1 \underline{u}_2 v^2 \cdots \underline{u}_k v^k$, where $\underline{u}_i$ are the letters of $\underline{u} = [\gamma]_\eta$ and $v^1, \ldots, v^k$ are words satisfying $\underline{v}^1 = \cdots = \underline{v}^k = \emptyset$ (such a decomposition exists but may not be unique). We justify in the next paragraph that $\omega_\delta \in \text{Normal}(\kappa, \eta)$ implies that $d(\gamma, \boldsymbol{\gamma}) \leq \kappa/2$ for *any* non-self-crossing smooth curve $\boldsymbol{\gamma}$ satisfying $\mathbf{u}(\boldsymbol{\gamma}) = [\gamma]_\eta = \underline{u}$.

To prove that $d(\gamma,\boldsymbol{\gamma}) \le \kappa/2$, consider a parametrization of γ on $[0,1]$ and let t_i be the first time $t \ge t_{i-1}$ such that $\gamma(t_i) \in \underline{u}_i$ (where we consider $t_0 = 0$) and parametrize $\boldsymbol{\gamma}$ on $[0,1]$ in such a way that $\boldsymbol{\gamma}(t_i) \in \underline{u}_i$. Then, we claim that for every $t \in [0,1]$, $|\gamma(t) - \boldsymbol{\gamma}(t)| \le \kappa/2$. Indeed, we know that for $t_i \le t < t_{i+1}$, $\boldsymbol{\gamma}(t)$ belongs to the face of $\eta\mathbb{Z}^2$ that contains $\underline{u}_i$ and $\underline{u}_{i+1}$ and $\gamma(t)$ belongs to one of the faces bordering $\underline{u}_i$, $\underline{u}_{i+1}$, or one of the letters in v^i. Now, the diameter of v^i is smaller than $\kappa/3$, and we therefore deduce that $\gamma(t)$ is within distance $\kappa/3 + 2\sqrt{2}\eta \le \kappa/2$ of $\boldsymbol{\gamma}(t)$, hence $d(\gamma,\boldsymbol{\gamma}) \le \kappa/2$.

We are now ready to conclude. Assume first that γ surrounds at most one point $x \in \mathbb{B}_\eta$. Then, $[\gamma]_\eta$ is either the empty word, or a word made of four letters corresponding to edges incident to x. As a consequence, we may choose $\boldsymbol{\gamma}$ with a diameter which is smaller than $2\sqrt{2}\eta$ and such that $u(\boldsymbol{\gamma}) = [\gamma]_\eta$. Also, since $\omega'_\delta \in \mathrm{Dense}(\eta)$, there exists a loop $\gamma' \in \mathcal{F}_i(\omega'_\delta)$ included in one of the faces intersected by $\boldsymbol{\gamma}$. We obtain immediately that $d(\boldsymbol{\gamma},\gamma') \le 2\sqrt{2}\eta$ and therefore $d(\gamma,\gamma') \le \kappa/2 + 2\sqrt{2}\eta \le \kappa$.

Let us now assume that γ surrounds at least two points in $\mathbb{B}_\eta$. Being included in $B(0,1/\kappa)$, γ cannot surround all the points in $\mathbb{B}_\eta$. The fact that $d_{\mathbf{H}}(\omega_\delta,\omega'_\delta) \le \eta$ thus implies the existence of $\gamma' \in \mathcal{F}'_i$ such that $[\gamma]_\eta = [\gamma']_\eta$. Since γ' must intersect $B(0,1/\kappa)$ as well, and ω_δ and ω'_δ are in $\mathrm{Normal}(\kappa,\eta)$, we obtain that $d(\gamma,\boldsymbol{\gamma}) \le \kappa/2$ and $d(\gamma',\boldsymbol{\gamma}) \le \kappa/2$ for every $\boldsymbol{\gamma}$ with $u(\boldsymbol{\gamma}) = [\gamma]_\eta = [\gamma']_\eta$. The triangular inequality gives that $d(\gamma',\gamma) \le \kappa$. This concludes the proof. □

We now turn to another claim.

Claim 2. *There exist $c, C \in (0,\infty)$ such that for every β,*

$$\phi_{\delta\mathbb{L}(\beta)}[\mathrm{Normal}(\kappa,\eta)] \ge 1 - \tfrac{C}{\eta^2\kappa^2}\exp[-c\kappa/\eta]. \tag{47}$$

Proof. Let $A(\kappa,\eta)$ be the event that there exists a crossing of the rectangle $R := [0,\kappa/3] \times [0,\kappa/12]$ whose cluster in the strip $\mathbb{R} \times [0,\kappa/12]$ surrounds no vertex in $\eta\mathbb{Z}^2$. We claim the existence of $c > 0$ such that

$$\phi_{\delta\mathbb{L}(\beta)}[A(\kappa,\eta)] \le C\exp[-c\kappa/\eta]. \tag{48}$$

To prove (48), let $\mathbf{N}$ be the number of clusters that contain a vertical crossing of R and for $i \le \mathbf{N}$, call Γ_i the right-boundary of the i-th cluster $\mathcal{C}_i$ in R crossing R when starting counting clusters from the right. Let Ω_i be the set of vertices in $\mathbb{R} \times [0,\kappa/12]$ on the left of (and including) Γ_i (see Figure 20 for a picture). Note that Γ_i is measurable in terms of the edges on Γ_i or on its right, and that it induces wired boundary conditions on Γ_i for the measure in Ω_i. Let X_i be a maximal set of vertices in $\Omega_i \cap \eta\mathbb{Z}^2$ that are at a distance at least 4η of each other[8], but at a distance at most η of Γ_i. Note that one may easily construct such a set of cardinality at least $\lfloor \kappa/(48\eta) \rfloor - 1$. Then, for every $x \in X_i$, by (RSW) there exists an open path disconnecting $\Lambda_\eta(x)$ from $\partial\Lambda_{2\eta}(x)$ in Ω_i

[8]This is a technical statement enabling to use the comparison between boundary conditions "independently" in each of the boxes $\Lambda_{2\eta}(x)$.

with probability larger than $c_0 > 0$, even when we enforce free boundary conditions on $\partial\Lambda_{2\eta}(x)$. The comparison between boundary conditions thus implies that

$$\phi_{\delta\mathbb{L}(\beta)}[\nexists x \in X_i \text{ surrounded by } \mathcal{C}_i | \Gamma_i] \leq (1-c_0)^{|X_i|} \leq (1-c_0)^{\lfloor \kappa/(48\eta) \rfloor - 1}.$$

It remains to sum over i and use that $\phi_{\delta\mathbb{L}(\beta)}[\mathbf{N}] \leq C_0$ (Proposition 3.7) to get (48):

$$\phi_{\delta\mathbb{L}(\beta)}[A(\kappa,\eta)] \leq (1-c_0)^{\lfloor \kappa/(48\eta) \rfloor - 1} \phi_{\delta\mathbb{L}(\beta)}[\mathbf{N}] \leq C_0 (1-c_0)^{\lfloor \kappa/(48\eta) \rfloor - 1}.$$

We are now in a position to conclude. Consider the square box $B_1 := [-\kappa/24, \kappa/24]^2$ as well as the rectangle $R_1^R := [\kappa/12, \kappa/6] \times [-\kappa/6, \kappa/6]$ and its rotations R_1^T, R_1^L, and R_1^B by angles $\pi/2$, π, and $3\pi/2$, respectively. Also, consider a collection of translates $(B_i, R_i^R, R_i^T, R_i^L, R_i^B)$ of $(B_1, R_1^R, R_1^T, R_1^L, R_1^B)$ such that the boxes B_i cover $B(0, 1/\eta - \kappa/3)$. Note that the probability that a translate/rotation/dual of $A(\kappa,\eta)$ occurs for some $R_i^{\#}$ is bounded by the right-hand side of (48). Write A_{global} for the event that the rotation/translation of A or its dual occurs for some $R_i^{\#}$. The union bound implies that

$$\phi_{\delta\mathbb{L}(\beta)}[A_{\text{global}}(\kappa,\eta)] \leq \frac{C_1}{\eta^2\kappa^2}(1-c_0)^{\lfloor \kappa/(48\eta) \rfloor - 1}.$$

We next prove that $\text{Normal}(\kappa,\eta)$ occurs as soon as $A_{\text{global}}(\kappa,\eta)$ does.

To see this, assume $\text{Normal}(\kappa,\eta)$ fails and consider a loop γ and a subword v of $u(\gamma)$ with $\underline{v} = \emptyset$ of maximal diameter among the subwords intersecting $B(0, 1/\kappa)$. The first and last letters of v must necessarily border the same face f. Consider two times s and t such that $\gamma(s), \gamma(t) \in f$ and $\gamma([s,t])$ has v as an encoding word, and close $\gamma([s,t])$ into a non-self-crossing loop $\ell(\gamma)$ by going from $\gamma(s)$ to $\gamma(t)$ inside the face f. Note that f must intersect a box B_i, and that $\ell(\gamma)$ must cross one of the four rectangles $R_i^{\#}$ around it. Now, outside of f, $\ell(\gamma)$ is identical to γ so it has either primal edges of ω_δ bordering it on its interior or dual edges. In the former case, the non occurrence of $A(\kappa,\eta)$ for the rectangle mentioned above implies that $\ell(\gamma)$ must necessarily surround a point in $\mathbb{B}_\eta$, which contradicts the fact that $\underline{v} = \emptyset$. In the latter case, the non occurrence of the dual of $A(\kappa,\eta)$ implies the same claim. □

We are now in a position to conclude the proof of the theorem. Claim 1 implies that

$$\begin{aligned}\mathbf{P}[d_{\mathbf{H}}(\omega_\delta, \omega'_\delta) \leq \eta, d_{\mathbf{CN}}(\omega_\delta, \omega'_\delta) > \kappa] \leq \phi_{\delta\mathbb{L}(\alpha)}[\text{Normal}(\kappa,\eta)^c] + \phi_{\delta\mathbb{L}(\pi/2)}[\text{Normal}(\kappa,\eta)^c] \\ + \phi_{\delta\mathbb{L}(\alpha)}[\text{Dense}(\eta)^c] + \phi_{\delta\mathbb{L}(\pi/2)}[\text{Dense}(\eta)^c].\end{aligned}$$

Now, Claim 2 applied to β equal to α or $\frac{\pi}{2}$ gives

$$\phi_{\delta\mathbb{L}(\alpha)}[\text{Normal}(\kappa,\eta)^c] + \phi_{\delta\mathbb{L}(\pi/2)}[\text{Normal}(\kappa,\eta)^c] \leq \tfrac{1}{2}\kappa,$$

provided $\eta = \eta(\kappa) > 0$ is chosen small enough.

Finally, since any vertex with four closed edges incident to it gives rise to a small loop in $\mathcal{F}_1$, and similarly for $\mathcal{F}_0$ when considering the dual graph, the finite-energy property immediately implies that

$$\phi_{\delta\mathbb{L}(\alpha)}[\text{Dense}(\eta)^c] + \phi_{\delta\mathbb{L}(\pi/2)}[\text{Dense}(\eta)^c] \leq \frac{2C}{\eta^2}\exp[-c(\eta/\delta)^2] \leq \tfrac{1}{2}\kappa,$$

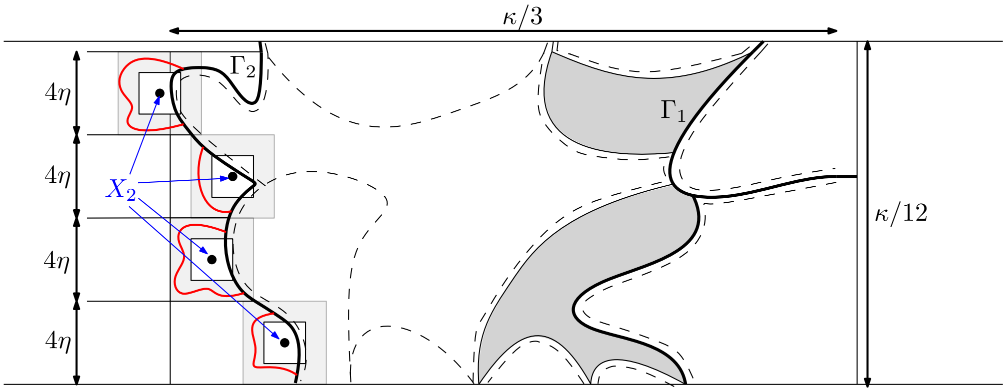

Figure 20: An example with two clusters crossing. Note that the set X_i does not have to be included in the rectangle R. A way to construct a large set of points with the properties of X_i is to choose for each $1 \leq j < \lfloor \kappa/48\eta \rfloor - 1$, on each line $\{(x, y) \in \mathbb{R}^2 : y = (2j+1)\eta\}$, a vertex of Ω_i which is at a distance smaller than η of Γ_i.

provided $\delta = \delta(\eta)$ small enough. The last three displayed inequalities conclude the proof of the theorem. □

6 Universality in isoradial rectangular graphs: proof of Theorem 2.3

The section is divided in six subsections. In the first one, we recall the setting of the proof and introduce some convenient notation. In the second one, we define the notion of nails, give a precise definition of $\mathcal{H}$, and introduce the formal definition of our coupling. The third one explains how one can couple the increments of the maximal coordinates of nails with independent increments that have the law of increments in a track-exchange on the IIC. In the fourth subsection, we will explore the combination of several increments into so-called compounded steps corresponding to bringing down one track from its starting to its ending position. The fifth subsection shows that the speed that can be associated to the evolution of a compounded step is approximately zero. Finally, the last subsection contains the proof of the theorem.

6.1 Setting of the proof

Below, fix $\alpha \in (0, \pi/2)$ (the case $\alpha > \pi/2$ can be obtained by a global reflection with respect to the y-axis). We further assume, except when otherwise stated, that

$$\cos\alpha \notin \mathbb{Q}. \tag{49}$$

This assumption plays an implicit role in the definition of the coupling, and is essential in the proof of Proposition 6.14, see Remark 6.15.

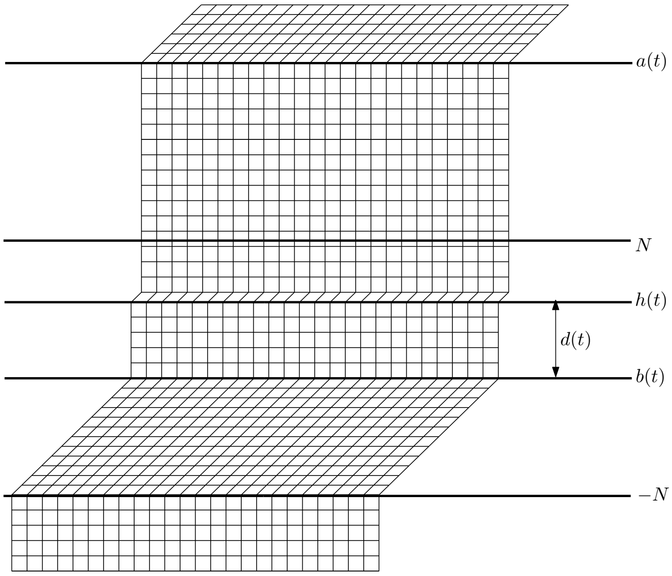

Figure 21: The quantities $h(t)$, $a(t)$, $b(t)$ and $d(t)$. One may deduce from the picture that $\lfloor t/(2N)\rfloor = 12$ as there are 12 tracks of angle α stacked above level $-N$.

Also, let $0 < \varepsilon \ll \eta \ll 1$ be fixed along the whole section (they will be chosen appropriately in the proof of the theorem at the end of the section). Set

$$\mathbb{B}_\eta(N) := \eta N\mathbb{Z}^2 \cap [-N, N]^2$$

(note that it is not quite the blow up by a factor N of $\mathbb{B}_\eta$) and

$$T := 2N \times \lceil 2N/\sin\alpha\rceil.$$

As in Section 2, $\mathbb{L}^{(0)}$ is the isoradial lattice with angles

$$\alpha_j = \alpha_j(\alpha, N) := \begin{cases} \alpha & \text{if } j \geq N, \\ \frac{\pi}{2} & \text{if } j < N. \end{cases}$$

Recall the successive transformations $\mathbf{T}_{j(t)}$ of the lattice described in Section 2: at time $0 \leq t < T$, the track to be descended is $t_{j(t)}$ with

$$j(t) := N + (2N+1)\lfloor t/(2N)\rfloor - t.$$

and the graph $\mathbb{L}^{(t)}$ obtained from the graph $\mathbb{L}^{(0)}$ by applying successively the maps $\mathbf{T}_{j(s)}$ for $0 \leq s < t$.

We use the following four convenient quantities (see Figure 21):

$$\begin{aligned}
h(t) &:= \text{the second coordinate of the horizontal line } t^-_{j(t)} \text{ in } \mathbb{L}^{(t)},\\
a(t) &:= h(2N\lfloor t/(2N)\rfloor) + \sin\alpha,\\
b(t) &:= h(2N\lfloor t/(2N)\rfloor - 1) + \sin\alpha - 1 \text{ if } t > 2N \text{ and } := -N \text{ if } t \leq 2N,\\
d(t) &:= \min\{h(t) - b(t), a(t) - h(t)\}.
\end{aligned}$$

Note that $b(t)$ is the top of the track of angle α below $t_{j(t)}$ (except when $t < 2N$ in which case we set it to be $-N$ by convention), and $a(t)$ is the bottom of the track of angle α above $t_{j(t)}$.

6.2 Definition of nails, marked nails, and the coupling P

Recall the definition, for a (primal) cluster $\mathcal{C}$ in a configuration ω, of $\mathrm{T}(\mathcal{C})$, $\mathrm{B}(\mathcal{C})$ and $\mathrm{R}(\mathcal{C})$, which are respectively the maximal second, minimal second and maximal first coordinates of a vertex in $\mathcal{C}$. Define $\mathrm{Vspan}(\mathcal{C}) := \mathrm{T}(\mathcal{C}) - \mathrm{B}(\mathcal{C})$.

Definition 6.1 (Nail)**.** For $x = (x_1, x_2) \in \mathbb{B}_\eta(N)$ and a configuration ω, call a (primal) cluster $\mathcal{C}$ of ω on some $\mathbb{L}^{(t)}$ a *nail (near x)* if

$$\mathrm{Vspan}(\mathcal{C}) \geq \varepsilon N \qquad \text{and} \qquad \max\{|\mathrm{T}(\mathcal{C}) - x_2|, |\mathrm{B}(\mathcal{C}) - x_2|, |\mathrm{R}(\mathcal{C}) - x_1|\} \leq \sqrt{\eta\varepsilon} N.$$

We now define the coupling of the measures $\phi_{\mathbb{L}^{(t)}}$ for $0 \leq t \leq T$.

Step 0 of the coupling P. Sample $\omega^{(0)} \sim \phi_{\mathbb{L}^{(0)}}$.

Index the nails near all points in $\mathbb{B}_\eta$ in $\omega^{(0)}$ by integers $1, \ldots, M = M(\omega^{(0)})$, and let $\mathcal{C}(\omega^{(0)}, i)$ be the nail indexed by i. Define $I^{(0)} := \{1, \ldots, M\}$. Write $\mathrm{T}^{(0)}$, $\mathrm{B}^{(0)}$ and $\mathrm{R}^{(0)}$ for the functions from $I^{(0)}$ into $\mathbb{R}$ giving, for every $i \in I^{(0)}$ and $A \in \{\mathrm{T}, \mathrm{B}, \mathrm{R}\}$, $A^{(0)}(i) := A(\mathcal{C}(\omega^{(0)}, i))$. Also set $\mathrm{Vspan}(i) := \mathrm{Vspan}(\mathcal{C}(\omega^{(0)}, i))$.

For each $x = (x_1, x_2) \in \mathbb{B}_\eta(N)$, choose, if it exists, $i_x \in I^{(0)}$ such that

$$\mathrm{Vspan}(i_x) \geq 2\varepsilon N \qquad \text{and} \qquad \max\{|\mathrm{T}(i_x) - x_2|, |\mathrm{B}(i_x) - x_2|, |\mathrm{R}(i_x) - x_1|\} \leq (\sqrt{\eta\varepsilon} - \varepsilon)N \tag{50}$$

(if there is more than one, pick the smallest such integer). Call $\mathcal{C}(\omega^{(0)}, i_x)$ the *marked nail near x*. Let $I_\bullet \subset I^{(0)}$ be the indexes corresponding to the marked nails near each $x \in \mathbb{B}_\eta(N)$, with the understanding that there may be some x for which there is no such marked nail (we will see later in this section that, with a very large probability, there is a marked nail near every $x \in \mathbb{B}_\eta(N)$).

Finally, if there exists a marked nail near every $x \in \mathbb{B}_\eta$, introduce the two multisets[9] $[\cdot]^{(0)}_{\bullet,0}$ and $[\cdot]^{(0)}_{\bullet,1}$ gathering the homotopy classes (in the sense of Remark 5.2) in the full

[9]Formally, these are functions from the set of homotopy classes, or in our case of reduced words, into non-negative integers.

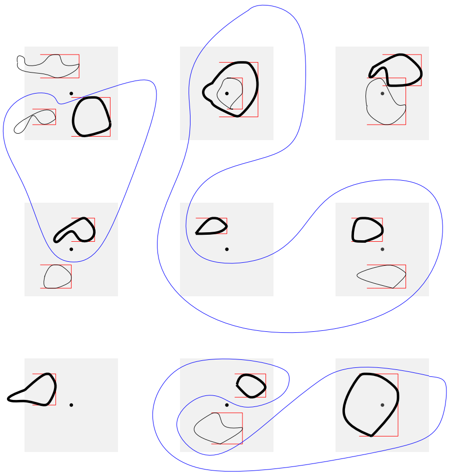

Figure 22: In black, the nails (i.e. the elements indexed by $I(\omega)$) and in bold black the marked nails. The red segments depict the information provided by $(\mathrm{T}^{(t)}, \mathrm{B}^{(t)}, \mathrm{R}^{(t)})$. The blue loops are the loops surrounding at least two marked nails, i.e. the loops contributing to $[\cdot]_{\bullet,0}$ and $[\cdot]_{\bullet,1}$. The grey area are the boxes $\Lambda_{\sqrt{\varepsilon\eta}N}(x)$, which one should think of potentially much bigger than the minimal size $O(\varepsilon N)$ of nails, but much smaller than the minimal distance ηN between vertices of $\mathbb{B}_\eta(N)$.

plane minus the marked nails $\mathbb{R}^2 \setminus \{\mathcal{C}(\omega, i) : i \in I_\bullet\}$ of the loops in $\mathcal{F}_0(\omega^{(0)})$ and $\mathcal{F}_1(\omega^{(0)})$ that surround at least two but not all marked nails. At this stage, we insist on the fact that $[\cdot]^{(0)}_{\bullet,0}$ and $[\cdot]^{(0)}_{\bullet,1}$ are multisets as there may be more than one loop in $\omega^{(0)}$ of a given homotopy class. If there exists $x \in \mathbb{B}_\eta$ that does not have a marked nail near it, simply set $[\cdot]^{(0)}_{\bullet,0} = [\cdot]^{(0)}_{\bullet,1} = \emptyset$.

To lighten the notation, we write

$$\mathcal{H}^{(0)} := (I^{(0)}, \mathrm{T}^{(0)}, \mathrm{B}^{(0)}, \mathrm{R}^{(0)}, [\cdot]^{(0)}_{\bullet,0}, [\cdot]^{(0)}_{\bullet,1}).$$

Fix now $0 \le t < T$ and assume that $\omega^{(t)}$ and $\mathcal{H}^{(t)} = (I^{(t)}, \mathrm{T}^{(t)}, \mathrm{B}^{(t)}, \mathrm{R}^{(t)}, [\cdot]^{(t)}_{\bullet,0}, [\cdot]^{(t)}_{\bullet,1})$ have been constructed (for the latter, the way it is given in terms of $\omega^{(t)}$ is explained for $t+1$ below), where

- $I^{(t)}$ is a subset of $\mathbb{Z}_{>0}$,
- $\mathrm{T}^{(t)}, \mathrm{B}^{(t)}, \mathrm{R}^{(t)}$ are functions from $I^{(t)}$ to $\mathbb{R}$,
- $[\cdot]^{(t)}_{\bullet,0}$ and $[\cdot]^{(t)}_{\bullet,1}$ are multisets with elements in homotopy classes in $\mathbb{R}^2 \setminus \{\mathcal{C}(\omega^{(t)}, i) : i \in I_\bullet\}$ if $I_\bullet \subset I^{(t)}$, and equal to $\emptyset$ otherwise.

Step t to $t+\frac{1}{2}$ of the coupling P. Sample $\omega^{(t+1/2)} \sim \phi_{\mathbb{L}^{(t)}}[\,\cdot\,|\mathcal{H} = \mathcal{H}^{(t)}]$,

where $\omega \in \{\mathcal{H} = H\}$ (with $H = (I, \mathrm{T}, \mathrm{B}, \mathrm{R}, [\cdot]_{\bullet,0}, [\cdot]_{\bullet,1})$ a possible realization of $\mathcal{H}^{(t)}$) denotes the event that

(i) there exists an indexation of the nails in ω by I (call $\mathcal{C}(\omega, i)$ the nail indexed by $i \in I$);
(ii) $A(\mathcal{C}(\omega, i)) = A(i)$ for every $i \in I$ and $A \in \{\mathrm{T}, \mathrm{B}, \mathrm{R}\}$;
(iii) if $I_\bullet \subset I$, the further requirement that $[\cdot]_{\bullet,0}$ and $[\cdot]_{\bullet,1}$ are giving the homotopy classes of the loops of ω that surround at least two but not all marked nails.

Step $t+\frac{1}{2}$ to $t+1$ of the coupling P. Set $\omega^{(t+1)} := \mathbf{T}_{j(t)}(\omega^{(t+1/2)})$ (remember that $\mathbf{T}_{j(t)}$ is a map sending a configuration to a *random* configuration).

Due to the previous step, $\omega^{(t+1/2)}$ necessarily satisfies the event $\{\mathcal{H} = \mathcal{H}^{(t)}\}$. Consider the indexation of the nails of $\omega^{(t+1/2)}$ by $I^{(t)}$ given by (i) and write $\mathcal{C}(\omega^{(t+1/2)}, i)$ for the nail indexed by i.

Since the track-exchange map $\mathbf{T}_{j(t)}$ is obtained as a sequence of star-triangle transformations, one can check that a nail $\mathcal{C}(\omega^{(t+1/2)}, i)$ is transformed into a cluster $\mathcal{C}$ in $\omega^{(t+1)}$. If $\mathcal{C}$ is still a nail in $\omega^{(t+1)}$, include i in $I^{(t+1)}$ and define $A^{(t+1)}(i) = A(\mathcal{C})$ for $A \in \{\mathrm{T}, \mathrm{B}, \mathrm{R}\}$. If this is not the case, then do not include i in $I^{(t+1)}$. Finally, for each "new" nail $\mathcal{C}'$ in $\omega^{(t+1)}$, i.e. a nail that was not in $\omega^{(t+1/2)}$, pick an integer i that was not used in any of the $I^{(s)}$ for $s \le t$ and include it in $I^{(t+1)}$. As before, let $A^{(t+1)}(i) = A(\mathcal{C}')$ for $A \in \{\mathrm{T}, \mathrm{B}, \mathrm{R}\}$. Note that nails may appear or disappear when applying $\mathbf{T}_{j(t)}$ since extrema of clusters may move during the star-triangle transformations, which may alter the validity of the conditions of being a nail.

If $I_\bullet \subset I^{(t+1)}$, define $[\cdot]^{(t+1)}_{\bullet,0}$ and $[\cdot]^{(t+1)}_{\bullet,1}$ to be the multisets giving the homotopy classes in $\mathbb{R}^2 \setminus \{\mathcal{C}(\omega^{(t+1)}, i) : i \in I_\bullet \cap I^{(t+1)}\}$ of the loops surrounding at least two but not all marked nails. Otherwise, set $[\cdot]^{(t+1)}_{\bullet,0} = [\cdot]^{(t+1)}_{\bullet,1} = \emptyset$.

Set
$$\mathcal{H}^{(t+1)} := (I^{(t+1)}, \mathrm{T}^{(t+1)}, \mathrm{B}^{(t+1)}, \mathrm{R}^{(t+1)}, [\cdot]^{(t+1)}_{\bullet,0}, [\cdot]^{(t+1)}_{\bullet,1}).$$

From now on, call **P** the coupling thus obtained.

Remark 6.2. Let us mention that a marked nail can disappear at some time t, meaning that some $i \in I_\bullet$ can be in $I^{(t)}$ but not in $I^{(t+1)}$, but no new marked nail may appear. The disappearance of a marked nail affects significantly the notion of homotopy and we stop keeping track of it (hence the convention of denoting $[\cdot]^{(t)}_{\bullet,0} = [\cdot]^{(t)}_{\bullet,1} = \emptyset$ and to consider it as an empty condition in (iii) of the definition of $\mathcal{H} = \mathcal{H}^{(t)}$). We will see that the condition (50) on our marked nails guarantees a posteriori that the marked nails do not disappear during the whole process with high probability. We will also see that $[\cdot]^{(t)}_{\bullet,0}$ and $[\cdot]^{(t)}_{\bullet,1}$ are preserved, and therefore equal to their values at time 0.

6.3 Controlling one single time step using IIC increments

In this section, we wish to connect our configurations $(\omega^{(t)} : 0 \le t < T)$ with IIC measures in order to be able to control the displacements of the nails during the process. We therefore "decorate" our coupling by enhancing it with additional configurations sampled according to IIC measures.

Recall the lattice $\mathbb{L}^{(i)}$ with horizontal tracks of angle β except for t_i which has angle α. In this section, β is fixed to be $\pi/2$. Recall also the IIC measures $\rotatebox{45}{$\Psi$}_i$, Ψ_i and $\rotatebox{90}{$\Psi$}$ defined on $\mathbb{L}^{(i)}$ and $\mathbb{L}^{(0)}$ respectively.

Let us give ourselves i.i.d. families of random variables $\omega^{(t)}_{A,j}$ indexed by $0 \le t < T$, $j \in \mathbb{Z}$, and $A \in \{\mathrm{T}, \mathrm{B}, \mathrm{R}\}$, with laws $\rotatebox{45}{$\Psi$}_j$, Ψ_j, and $\rotatebox{90}{$\Psi$}$ if $A = \mathrm{T}, \mathrm{B}, \mathrm{R}$ respectively (when $A = \mathrm{R}$ there is no need for a subscript j but we will use this convenient "unified" notation).

We also give ourselves independent $\{0, 1\}$-valued random variables $X^{(t)}_H(i)$, indexed by all possible values $H = (I, \mathrm{T}, \mathrm{B}, \mathrm{R}, [\cdot]_{\bullet,0}, [\cdot]_{\bullet,1})$ of $\mathcal{H}^{(t)}$ and $i \in I$, satisfying

$$\mathbf{P}[X^{(t)}_H(i) = 1] = \phi_{\mathbb{L}^{(t)}}[(\mathrm{R}(i), h(t)) \in \mathcal{C}(\omega, i) \,|\, \mathcal{H} = H]$$

(it is possible that $(\mathrm{R}(i), h(t))$ is not a vertex of $\mathbb{L}^{(t)}$, in which case $X^{(t)}_H(i)$ is 0 almost surely). To get an intuition on these variables, in the coupling below, the fact that $X^{(t)}_{\mathcal{H}^{(t)}}(i)$ is equal to 1 will detect whether the cluster $\mathcal{C}(\omega, i)$ has a right extremum on $t^-_{j(t)}$.

We are now ready to define the coupling of the process $(\omega^{(t)} : 0 \le t < T)$ with the random variables introduced above. Below, the steps 0 and $t + 1/2$ to $t + 1$ are done exactly as in the previous section. We therefore focus on the steps t to $t + 1/2$. We will sample $\omega^{(t+1/2)}$ from $\omega^{(t)}$ in a few steps, coupled to the variables introduced in the two last paragraphs. Nevertheless, notice that the law of $\omega^{(t+1/2)}$ given $\omega^{(t)}$ is the same as in the previous section. For this reason, we keep denoting this bigger coupling $\mathbf{P}$.

For $0 \le t < T$ and a nail $\mathcal{C}$, call a vertex $x \in \mathbb{L}^{(t)}$ a

- *Top t-extremum* (of $\mathcal{C}$) if $x \in t^-_{j(t)} \cup t^-_{j(t)-1}$ and its second coordinate equals $\mathrm{T}(\mathcal{C})$,
- *Bottom t-extremum* (of $\mathcal{C}$) if $x \in t^-_{j(t)} \cup t^-_{j(t)+1}$ and its second coordinate equals $\mathrm{B}(\mathcal{C})$,
- *Right t-extremum* (of $\mathcal{C}$) if $x \in t^-_{j(t)}$ and its first coordinate equals $\mathrm{R}(\mathcal{C})$,
- *Fake right t-extremum* (of $\mathcal{C}$) if $x \in t^-_{j(t)}$, its first coordinate is strictly larger than $\mathrm{R}(\mathcal{C}) - \cos\alpha$, *and* the vertex of $\mathcal{C}$ with maximal first coordinate is *below* $b(t)$.

We also use *vertical t-extremum* to denote a top or bottom t-extremum.

Remark 6.3. Note that for $A(\mathcal{C})$ to be possibly modified by the track-exchange, there must exist a A t-extremum or fake right t-extremum in the case $A = \mathrm{R}$.

Complete description of the coupling $\mathbf{P}$ from time t to $t+1/2$

Fix $0 \le t < T$ and assume that $I_\bullet$, $\omega^{(t)}$, and $\mathcal{H}^{(t)}$ have been defined. We divide the construction of $\omega^{(t+1/2)}$ in four cases; which case applies is determined by $\mathcal{H}^{(t)}$.

Case 0: Two distinct nails contain vertical t-extrema. In such case, sample $\omega^{(t+1/2)}$ as in previous section independently of the variables $\omega^{(t)}_{A,j}$ and $X^{(t)}_H$.

Case 1: A unique nail contains a vertical t-extremum and it is a top one. Let i be the index of this nail. Proceed as follows:

Step 1. Sample the random variable $\mathbf{x}$ in such a way that for every $x \in \mathbb{L}^{(t)}$,

$$\mathbf{P}[\mathbf{x} = x] = \phi_{\mathbb{L}^{(t)}}[\mathrm{lmax}(\mathcal{C}(\omega, i)) = x | \mathcal{H} = \mathcal{H}^{(t)}].$$

Step 2. Sample ω according to

$$\phi_{\mathbb{L}^{(t)}}[\,\cdot\,| \mathrm{lmax}(\mathcal{C}(\omega, i)) = \mathbf{x}, \mathcal{H} = \mathcal{H}^{(t)}]$$

and sample $\omega^{(t+1/2)} = \omega$ on the set $\Omega(\mathbf{x}, \omega)$ of edges outside of $\Lambda_{d(t)^{1/3}}(\mathbf{x})$ that are connected in ω to the complement of $\Lambda_{d(t)^{1/2}}(\mathbf{x})$ (see Figure 26).

Step 3. Sample $(\omega^{(t+1/2)}, \omega^{\mathrm{T}}_{h(t)-T^{(t)}(i)})$ (for the first one we only need to sample the remaining edges) using the coupling between

$$\phi_{\mathbb{L}^{(t)}}[\,\cdot\,| \mathrm{lmax}(\mathcal{C}(\omega^{(t+1/2)}, i)) = \mathbf{x}, \mathcal{H} = \mathcal{H}^{(t)}, \omega^{(t+1/2)}_{|\Omega(\mathbf{x}, \omega^{(t+1/2)})}] \quad \text{and} \quad \Phi_{j(t)-T^{(t)}(i)}$$

which is maximizing the probability that $\omega^{(t+1/2)}$ and the translate of $\omega^{(t)}_{\mathrm{T}, j(t)-T^{(t)}(i)}$ by $\mathbf{x}$ coincide on $\Lambda_{d(t)^{1/4}}(\mathbf{x})$.

Case 2: A unique nail contains a vertical t-extremum and it is a bottom one. Proceed exactly as in the previous step with B instead of T and Ψ instead of Φ.

Case 3: No nail contains a vertical t-extremum. Proceed as follows,

Step -1. Couple[a] in the best possible way the random variables

$$X^{(t)} := (X^{(t)}_{\mathcal{H}^{(t)}}(i) : i \in I^{(t)}) \qquad \text{and} \qquad \widetilde{X}^{(t)} = (\widetilde{X}^{(t)}(i) : i \in I^{(t)}),$$

where $\widetilde{X}^{(t)}$ has the law of the random variable $(\mathbf{1}[(\mathrm{R}(i), h(t)) \in \mathcal{C}(\omega, i)] : i \in I^{(t)})$ with $\omega \sim \phi_{\mathbb{L}^{(t)}}[\,\cdot\,| \mathcal{H} = \mathcal{H}^{(t)}]$.

Step 0. If $X^{(t)} \neq \widetilde{X}^{(t)}$ or $\sum_{i\in I^{(t)}} \widetilde{X}^{(t)}(i) \neq 1$, sample independently the random variables $\omega_{A,j}^{(t)}$ and the random variable

$$\omega^{(t+1/2)} \sim \phi_{\mathbb{L}^{(t)}}[\,\cdot\,|\mathcal{H} = \mathcal{H}^{(t)}, X = \widetilde{X}^{(t)}],$$

where the event $X = \widetilde{X}^{(t)}$ means that $(\mathrm{R}(i), h(t))$ belongs to the nail indexed by i if and only if $\widetilde{X}^{(t)}(i) = 1$.

Step 1. Otherwise, set $\mathbf{x} := (\mathrm{R}^{(t)}(i), h(t))$ with i the integer such that $\widetilde{X}^{(t)}(i) = 1$.

Step 2–3. Proceed as in Case 1 with R instead of T and $\rotatebox[origin=c]{90}{Ψ}$ instead of $\rotatebox[origin=c]{180}{Ψ}$, except that we further condition at each step on $X = \widetilde{X}^{(t)}$.

[a] Note that $X^{(t)}$ and $\widetilde{X}^{(t)}$ have the same marginal laws, but that in the former the random variables $X_{\mathcal{H}^{(t)}}^{(t)}(i)$ are independent, while in the latter they are not.

Remark 6.4. We will see in the next section that Case 0 occurs very rarely, hence we do not bother coupling efficiently the true configuration with an IIC configuration in this case. In Step 3 of Case 1, it could be that the best coupling is terrible due to the fact that $\omega^{(t+1/2)}$ does something strange on $\Omega(\mathbf{x}, \omega^{(t+1/2)})$. Yet, this will be shown to occur with only small probability and the best coupling guarantees equality of the two configurations with very large probability. Finally, in Step 0 of Case 3, the best coupling (which depends on $\mathcal{H}^{(t)}$) is typically making the two random variables equal. We will see that in this case there is typically a single i for which $X^{(t)}(i) = 1$.

We now turn to an important proposition describing the increments $A^{(t+1)}(i) - A^{(t)}(i)$ for the nails $i \in I^{(t)}$ in terms of increments of IIC variables, called the *IIC displacement random variables.*

Definition 6.5 (IIC displacements)**.** Sample a configuration according to $\rotatebox[origin=c]{180}{Ψ}_j$, apply $\mathbf{T}_j$, and call $\delta_j^{\mathrm{IIC}}\mathrm{T}$ the maximal y-coordinate of a vertex of the incipient infinite cluster after the transformation. Similarly, sample a configuration according to Ψ_j, apply $\mathbf{T}_j$, and call $\delta_j^{\mathrm{IIC}}\mathrm{B}$ the maximal y-coordinate of a vertex of the incipient infinite cluster after the transformation. Finally, sample a configuration according to $\rotatebox[origin=c]{90}{Ψ}$, apply $\mathbf{T}_0$, and call $\delta^{\mathrm{IIC}}\mathrm{R}$ the maximal x-coordinate of a vertex of the incipient infinite cluster after the transformation.

Remark 6.6. The effect of a track exchange on the top and right of a cluster is described in Figures 23 and 24. Notice that $\delta_1^{\mathrm{IIC}}\mathrm{T} \in \{0, \sin\alpha\}$, $\delta_0^{\mathrm{IIC}}\mathrm{T} \in \{-1, \sin\alpha - 1\}$, and $\delta_j^{\mathrm{IIC}}\mathrm{T} = 0$ for other values of j. Similarly, $\delta_0^{\mathrm{IIC}}\mathrm{B} \in \{\sin\alpha, \sin\alpha - 1\}$, $\delta_{-1}^{\mathrm{IIC}}\mathrm{B} \in \{0, -1\}$, and $\delta_j^{\mathrm{IIC}}\mathrm{B} = 0$ for other values of j. For the right, note that $\delta^{\mathrm{IIC}}\mathrm{R} \in \{0, -1, \cos\alpha - 1, \cos\alpha\}$.

Remark 6.7. The effect of a track exchange implies that $\mathrm{T}^{(t+1)}(i) - \mathrm{T}^{(t)}(i)$ and $\mathrm{B}^{(t+1)}(i) - \mathrm{B}^{(t)}(i)$ belong to $\{-1, \sin\alpha - 1, 0, \sin\alpha\}$. For $\mathrm{R}^{(t+1)}(i) - \mathrm{R}^{(t)}(i)$, the situation is more complicated since there may be a fake right t-extremum, whose coordinate is therefore not equal to $\mathrm{R}(\mathcal{C}(\omega^{(t+1/2)}, i))$, that jumps right and after the transformation has a first coordinate equal to $\mathrm{R}(\mathcal{C}(\omega^{(t+1)}, i))$. Nevertheless, one always has $|\mathrm{R}^{(t+1)}(i) - \mathrm{R}^{(t)}(i)| \leq 1$.

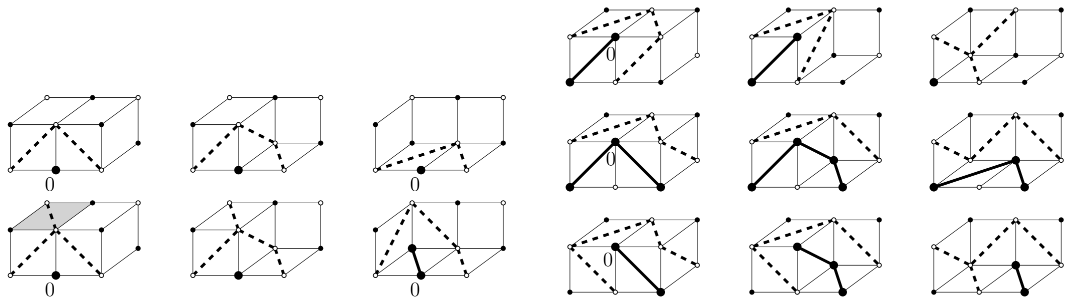

Figure 23: Different environments around 0 picked according to the incipient infinite cluster measures with a track of angle α at height 1 (left) and 0 (right). The bold edges and points are part of the incipient infinite clusters. The different outcomes of the transformations give different top-most points for the infinite cluster.
Left two diagrams: Two possible outcomes of the track-exchange $\mathbf{T}_1$ corresponding to $\delta_1^{\mathrm{IIC}}\mathrm{T} = 0$ and $\delta_1^{\mathrm{IIC}}\mathrm{T} = \sin\alpha$. The first outcome occurs certainly when the gray rhombus contains a primal edge, and with positive probability when it contains a dual one; in the latter case, the second outcome is also possible.
Right three diagrams: Three possible outcomes of the track-exchange $\mathbf{T}_0$ corresponding to $\delta_0^{\mathrm{IIC}}\mathrm{T} = -1$ and $\delta_0^{\mathrm{IIC}}\mathrm{T} = \sin\alpha - 1$, respectively. The first outcome occurs only when the edge below and to the left of 0 is the unique open edge adjacent to 0.

Let $\delta_j^{\mathrm{IIC}}A^{(t)}$ be the displacement constructed out of the IIC configuration $\omega_{A,j}^{(t)}$ by applying $\mathbf{T}_{j(t)}$. Note that the $\delta_j^{\mathrm{IIC}}A^{(t)}$ form families of i.i.d. random variables with law $\delta_j^{\mathrm{IIC}}A$. For $i \in I^{(t)}$, define the random variables

$$\begin{aligned}
\delta^{\mathrm{err}}\mathrm{T}^{(t)}(i) &:= \mathrm{T}^{(t+1)}(i) - \mathrm{T}^{(t)}(i) - \delta_{h(t)-T^{(t)}(i)}^{\mathrm{IIC}}\mathrm{T}^{(t)}(i), \\
\delta^{\mathrm{err}}\mathrm{B}^{(t)}(i) &:= \mathrm{B}^{(t+1)}(i) - \mathrm{B}^{(t)}(i) - \delta_{h(t)-B^{(t)}(i)}^{\mathrm{IIC}}\mathrm{B}^{(t)}(i), \\
\delta^{\mathrm{err}}\mathrm{R}^{(t)}(i) &:= \mathrm{R}^{(t+1)}(i) - \mathrm{R}^{(t)}(i) - X_{\mathcal{H}^{(t)}}^{(t)}(i)\,\delta^{\mathrm{IIC}}\mathrm{R}^{(t)}(i).
\end{aligned}$$

We are now ready to present the main statement of this section.

Proposition 6.8 (Properties of the coupling)**.** *For $\varepsilon, \eta > 0$, the coupling $\mathbf{P}$ satisfies the following properties:*

(o) for every $0 \le t \le T$, $\omega^{(t)} \sim \phi_{\mathbb{L}^{(t)}}$;

(i) $([\cdot]_{\bullet,0}^{(t)}, [\cdot]_{\bullet,1}^{(t)}) = ([\cdot]_{\bullet,0}^{(0)}, [\cdot]_{\bullet,1}^{(0)})$ for every $t < \tau$, where $\tau := \inf\{t > 0 : I_\bullet \not\subset I^{(t)}\}$;

(ii) for every $t \ge 0$, the variables $(\delta_j^{\mathrm{err}}A^{(s)}(i), \delta_j^{\mathrm{IIC}}A^{(s)}(i), X_H^{(s)}(i) : A, i, j, H, s < t)$ are independent of the $(\delta_j^{\mathrm{IIC}}A^{(t)}(i), X_H^{(t)}(i) : A, i, j, H)$;

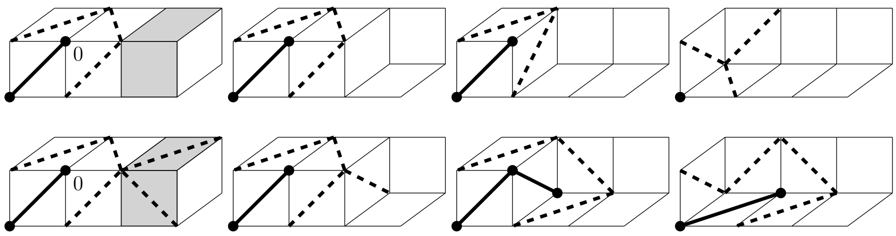

Figure 24: When performing a track exchange between t_0 and t_{-1}, the vertex 0 is modified locally so that the coordinate of the right-most point in $t_0^- \cup t_{-1}^-$ moves by either -1 (first line) or $\cos\alpha$ (second line). The first outcome occurs certainly when both gray rhombi contain primal edges, and with positive probability otherwise; the second outcome may only occur when at least one of the two gray rhombi contains a dual edge. When the second outcome occurs, $\delta^{\mathrm{IIC}}\mathrm{R} = \cos\alpha$. For the first outcome, $\delta^{\mathrm{IIC}}\mathrm{R}$ may take values 0, $\cos\alpha - 1$, or -1. The first two values appear if the incipient infinite cluster contains a vertex below 0 with first coordinate 0, or one above 0, with first coordinate $\cos\alpha - 1$, respectively. The same outcomes occur for any environment in t_0 and t_{-1} to the left of 0.

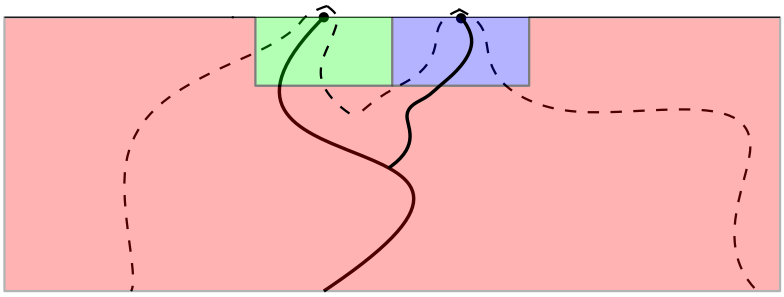

Figure 25: We depicted the example of a nail having two top t-extrema.

(iii) there exist $C, c \in (0,\infty)$ such that for every $t \geq 0$,

$$\mathbf{E}[\mathrm{Err}^{(t)}] \leq \frac{C}{d(t)N^c},$$

where $\mathrm{Err}^{(t)} := \sum_{i \in I^{(t)}} M^{(t)}(i)$ *with* $M^{(t)}(i) := |\delta^{\mathrm{err}}\mathrm{T}^{(t)}(i)| + |\delta^{\mathrm{err}}\mathrm{B}^{(t)}(i)| + |\delta^{\mathrm{err}}\mathrm{R}^{(t)}(i)|.$

Remark 6.9. Let us mention that we expect the bound of (iii) to be valid with N^{1+c} instead of $d(t)N^c$ in the denominator. The reason for the appearance of $d(t)$ is due to the fact that we do not, at this stage, know how to prove (15) for generic sequences of angles $\boldsymbol{\alpha}$. In retrospect, our rotation invariance result shows that (15) does hold for arbitrary sequences, but only post factum.

The rest of the subsection is dedicated solely to proving Proposition 6.8. This proof is tedious, but does not involve particularly innovative ideas (the heavy lifting was done when properly defining the coupling). In the first reading, one may skip the proof and focus on the next sections first.

Before diving into the proof of this proposition, let us start with a lemma. Define

$$\begin{aligned}
\mathrm{BAD}_1(t) &:= \{\text{a nail contains a vertical and a right } t\text{-extremum}\},\\
\mathrm{BAD}_2(t) &:= \{\text{a nail contains two vertical } t\text{-extrema } x,y \text{ satisfying } |x-y| \geq d(t)^{1/5}\},\\
\mathrm{BAD}_3(t) &:= \{\text{two nails contain a } t\text{-extremum}\},\\
\mathrm{BAD}_4(t) &:= \{\text{a nail contains a fake right } t\text{-extremum}\},\\
\mathrm{BAD}_5(t) &:= \left\{ \begin{array}{c} \text{a nail } i \text{ contains a right } t\text{-extremum and a vertex } (x_1,x_2) \\ \text{with } \mathrm{R}^{(t)}(i)-1 < x_1 < \mathrm{R}^{(t)} \text{ and } |x_2 - h(t)| \geq d(t)^{1/5} \end{array} \right\}.
\end{aligned}$$

Lemma 6.10. *There exist $C, c \in (0,\infty)$ such that for every $0 \leq t < T$ and $1 \leq i \leq 4$,*

$$\phi_{\mathbb{L}^{(t)}}[\mathrm{BAD}_i(t)] \leq \frac{C}{d(t)N^c}.$$

Proof. We divide into the different events $\mathrm{BAD}_i(t)$.

Bound on the probability of $\mathrm{BAD}_1(t)$. Assume that the vertical t-extremum is a top t-extremum (the bottom case is treated similarly) and let x be the right t-extremum of the nail. In this case, there must be a three-arm event in the bottom-left quarterplane translated by x, and going from $\Lambda_1(x)$ to $\partial\Lambda_{\varepsilon N}(x)$. We therefore deduce from (13) that the probability of this is bounded by $C/(\varepsilon N)^{2+c}$. Summing over $O(N)$ possible values of x – recall that since it is a right t-extremum, x is within a distance $\sqrt{\varepsilon\eta}N$ of one of the points in $\mathbb{B}_\eta(N)$ – gives the required bound.

Bound on the probability of $\mathrm{BAD}_2(t)$. In this case, the two t-extrema are either both in the top direction, or both in the bottom one (since $\mathrm{Vspan} \geq \varepsilon N > 1 + \sin\alpha$ for N large enough). Let us assume it is the former that happens and let x and y be the two t-extrema. We assume that x is on the left of y. Also, note that they have to be exactly at the same height as they belong to the same nail. The following must therefore occur (see Figure 25 for a picture):

- a 3-arm event in the half-plane below x from x to $\partial\Lambda_{|x-y|/2}(x)$;
- a 3-arm event in the half-plane below y from y to $\partial\Lambda_{|x-y|/2}(y)$;
- a 3-arm event in the half-plane below x from $\Lambda_{2|x-y|}(x)$ to $\partial\Lambda_{\varepsilon N}(x)$ (this may be an empty condition if $|x-y| \geq \frac{1}{2}\varepsilon N$);
- a 1-arm event from $\Lambda_{\varepsilon N}(x)$ to Λ_N in $\mathbb{Z}\times[-N,N]$ (this last condition is only relevant in case $x \notin \Lambda_N$, which may occur since nails may be very long in the left direction, and have maxima far on the left of Λ_N – obviously, this is atypical, but should be taken care of nonetheless).

We deduce from (10) that

$$\phi_{\mathbb{L}^{(t)}}[x, y \text{ top } t\text{-extrema of the same nail}] \le \frac{C}{|x-y|^4}\Big(\frac{|x-y|}{\varepsilon N}\Big)^2 \exp\Big(-\frac{c|x|}{N}\Big). \tag{51}$$

Summing over x and y at a distance $d(t)^{1/4}$ of each other (with y on the right of x and left of the right-side of Λ_N) gives that

$$\phi_{\mathbb{L}^{(t)}}[\mathrm{BAD}_2(t)] \le \frac{C'}{\varepsilon^2 N d(t)^{1/4}}. \tag{52}$$

Bound on the probability of $\mathrm{BAD}_3(t)$**.** Assume that x and y are the right t-extrema of two different nails. The cases of the top or bottom t-extrema are actually simpler to handle (and they give better bounds).

First, assume that $|x-y| \le 2d(t)$ and that y is on the right of x. In this case, for x and y to be right t-extrema of their respective nails, there must be

- a 3-arm event in the half-plane on the left of x from x to $\partial\Lambda_{|x-y|/2}(x)$;
- a 3-arm event in the half-plane on the left of y from y to $\partial\Lambda_{|x-y|/2}(y)$;
- a 5-arm event in the half-plane on the left of y from $\Lambda_{2|x-y|}(y)$ to $\partial\Lambda_{d(t)}(y)$ (this may be an empty condition if $|x-y| \ge d(t)/2$);
- a 3-arm event[10] in the half-plane on the left of y from $\Lambda_{d(t)}(y)$ to $\partial\Lambda_{\varepsilon N/2}(y)$.

Using (15) (twice), (16), and (12), we deduce that

$$\phi_{\mathbb{L}^{(t)}}[x, y \text{ right } t\text{-extrema of distinct nails}] \le \frac{C}{|x-y|^4}\Big(\frac{|x-y|}{d(t)}\Big)^{2+c_0}\Big(\frac{2d(t)}{\varepsilon N}\Big)^{1+c_1}. \tag{53}$$

Now, assume $|x-y| > 2d(t)$ and assume that y is right of x. In this case, for x and y to be at the right-most ends of their respective nails, there must be

- a 3-arm event in the half-plane on the left of x from x to $\partial\Lambda_{d(t)}(x)$;
- a 3-arm event in the half-plane on the left of y from y to $\partial\Lambda_{d(t)}(y)$;
- a 3-arm event in the half-plane on the left of x from $\Lambda_{d(t)}(x)$ to $\partial\Lambda_{|x-y|/2}(x)$;
- a 3-arm event in the half-plane on the left of y from $\Lambda_{d(t)}(y)$ to $\partial\Lambda_{|x-y|/2}(y)$;
- a 3-arm event[11] in the half-plane on the left of y from $\Lambda_{2|x-y|}(y)$ to $\partial\Lambda_{\varepsilon N/2}(y)$ (this condition is empty if $|x-y| \ge \varepsilon N/4$).

Using (15) (twice) and (12) (three times), we deduce that

$$\phi_{\mathbb{L}^{(t)}}[x, y \text{ right } t\text{-extrema of distinct nails}] \le \frac{C}{d(t)^4}\Big(\frac{d(t)}{|x-y|}\Big)^{2+2c_1}\Big(\frac{2|x-y|}{\varepsilon N}\Big)^{1+c_1}. \tag{54}$$

[10] In fact even a 5-arm event occurs up to $\partial\Lambda_{\varepsilon N/2-|x-y|}(y)$. Also, once this is observed, one may wonder why we distinguish between the third and fourth bullets since in both cases a 5-arm event occurs. The reason comes from the fact that the estimate used in both cases is not quite the same, since one occurs in an area with all but one track having a transverse angle equal to $\frac{\pi}{2}$, while the second occurs in a "mixed" lattice.

[11] Again, there is even a 5-arm event up to $\partial\Lambda_{\varepsilon N/2-|x-y|}(y)$.

Let us now sum on y and then on x to obtain

$$\begin{aligned}
&\phi_{\mathbb{L}^{(t)}}[\text{two distinct nails in } I^{(t)} \text{ contain a right } t\text{-extremum}]\\
&\quad\le 2CN\Big[\sum_{k=1}^{2d(t)}\frac{1}{k^4}\Big(\frac{k}{d(t)}\Big)^{2+c_0}\Big(\frac{d(t)}{\varepsilon N}\Big)^{1+c_1}+\sum_{k=2d(t)}^{2N}\frac{1}{d(t)^4}\Big(\frac{d(t)}{k}\Big)^{2+2c_1}\Big(\frac{k}{\varepsilon N}\Big)^{1+c_1}\Big]\\
&\quad\le \frac{C_1}{d(t)^{1+c_0}}\Big(\frac{d(t)}{\varepsilon N}\Big)^{c_1}+\frac{C_2}{d(t)^2}\Big(\frac{d(t)}{\varepsilon N}\Big)^{c_1}\le \frac{C_3}{d(t)(\varepsilon N)^{c_3}},
\end{aligned}$$

where we choose $c_3 := \min\{c_0, c_1\}$.

Doing the same with other types of t-extrema implies the bound for $\mathrm{BAD}_3(t)$.

Bound on the probability of $\mathrm{BAD}_4(t)$**.** For x to be a fake right t-extremum of a nail $\mathcal{C}$, there must exist $y = (y_1, y_2) \in \mathcal{C}$ with $y_2 \le b(t)$ and $y_1 = \mathrm{R}(\mathcal{C})$. As a consequence, there must be

- a 3-arm event in the half-plane on the left of x from x to $\partial\Lambda_{d(t)/2}(x)$;
- a 3-arm event in the half-plane on the left of x from $\Lambda_{d(t)/2}(x)$ to $\partial\Lambda_{|x-y|/2}(x)$;
- a 3-arm event in the half-plane on the left of y from y to $\partial\Lambda_{|x-y|/2}(y)$;
- a 3-arm event in the half-plane on the left of y from $\Lambda_{2|x-y|}(y)$ to $\partial\Lambda_{\varepsilon N/2}(x)$.

If we denote $E(x, y)$ the previous event, we deduce that

$$\phi_{\mathbb{L}^{(t)}}[E(x,y)] \le \frac{C}{d(t)^2}\Big(\frac{d(t)}{|x-y|}\Big)^{1+c_1}\Big(\frac{2}{|x-y|}\Big)^{1+c_1}\Big(\frac{4|x-y|}{\varepsilon N}\Big)^{1+c_1}. \tag{55}$$

Summing over y and then over x gives,

$$\phi_{\mathbb{L}^{(t)}}[\mathrm{BAD}_4(t)] \le \sum_{x,y}\phi_{\mathbb{L}^{(t)}}[E(x,y)] \le \sum_x \frac{C}{d(t)}\frac{C_2}{N^{1+c_1}} \le \frac{C_3}{d(t)N^{c_1}}. \tag{56}$$

Bound on the probability of $\mathrm{BAD}_5(t)$**.** The bound follows from a combination of the arguments for $\mathrm{BAD}_2(t)$ and $\mathrm{BAD}_4(t)$. We leave it to the reader. □

Proof of Proposition 6.8. By construction of the coupling, (o) and (ii) are trivial.

Property (i) follows from the locality of the star-triangle operations and the common way of measuring $([\cdot]^{(t)}_{\bullet,0}, [\cdot]^{(t)}_{\bullet,1})$. Indeed, since $[\cdot]^{(t-1)}_{\bullet,0} = [\cdot]^{(t-1/2)}_{\bullet,0}$ by definition of $\omega^{(t-1/2)}$, it suffices to check that $[\cdot]^{(t-1/2)}_{\bullet,0} = [\cdot]^{(t)}_{\bullet,0}$. Now, $I_\bullet \subset I^{(s)}$ for every $s \le t$ so there are nails near every $x \in \mathbb{B}_\eta(N)$. Recall the definition of oriented edges (x, y) from Section 5.1. Since the star-triangle transformations modify the lowest right-most point of marked nails of $\omega^{(t-1/2)}$, but do so only locally, while preserving the connections outside of $t^-_{j(t)}$, we immediately get that the reduced words are not modified by the track-exchange, and therefore $[\cdot]^{(t-1/2)}_{\bullet,0} = [\cdot]^{(t)}_{\bullet,0}$.

We therefore focus on proving (iii). We divide the analysis of $\mathrm{Err}^{(t)}$ depending on the case used for the coupling. Below, constants c_i, C_i are universal and independent of everything else except ε and η.

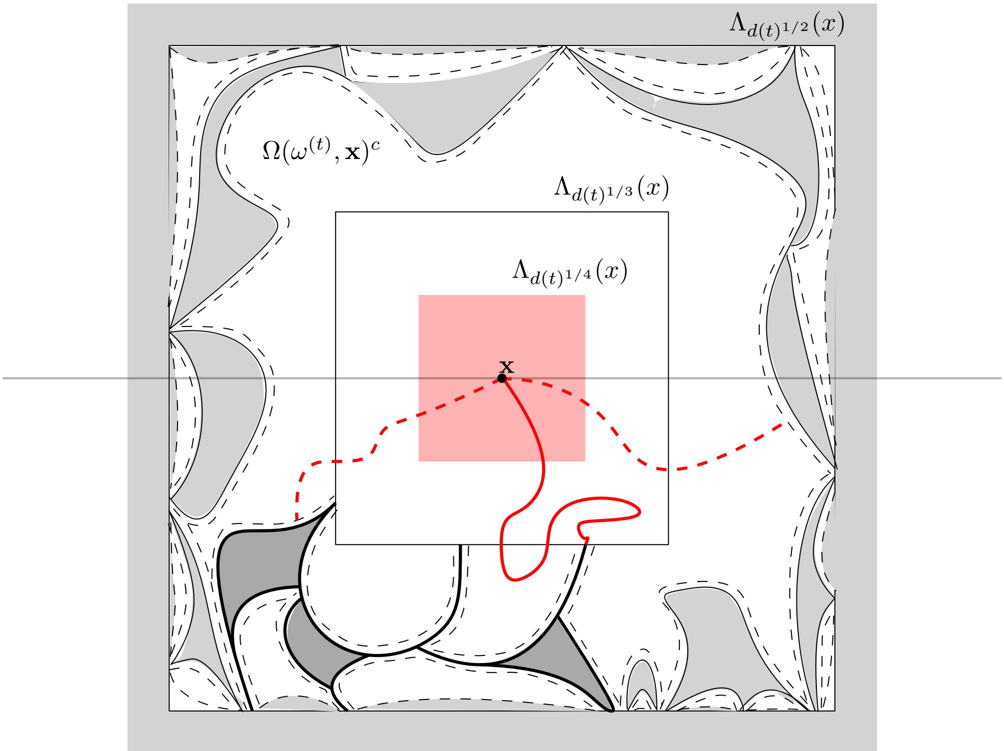

Figure 26: A picture of the set $\Omega(\omega^{(t+1/2)},x)$ (in grey). The dark grey cluster is the unique cluster of the annulus crossing from inside to outside. Everything outside $\Lambda_{d(t)^{1/2}}(x)$ is included in $\Omega(\omega^{(t+1/2)},x)$. The red part is the place where we try to couple as best as possible $\omega^{(t+1/2)}$ and $\mathbf{x}+\omega^{(t)}_{\mathrm{T},j(t)-T^{(t)}(i)}$. In the proof of Proposition 6.8, note that when applied to $\omega^{(t+1/2)}\in E(x)$ (which means that there exists a unique cluster crossing the annulus), all the conditions on the t-extrema of other nails and homotopy classes of loops are not impacted by what happens inside $\Omega(\omega^{(t+1/2)},\mathbf{x})^c$. As a consequence, there is a "screening" property and the conditioning inside is simply the existence of the red arms, meaning that $\mathbf{x}$ is equal to the left-most top-most vertex in the cluster that is crossing the annulus.

Error in Case 0 If Case 0 holds, then $\omega^{(t)} \in \mathrm{BAD}_3(t)$. Due to the coupling generated by the track-exchange, the displacement of any t-extremum is at most 1 so all variables $\delta^{\mathrm{err}} A^{(t)}(i)$ are deterministically bounded by 2. Thus, Markov's inequality implies that for every $\lambda \geq 0$,

$$\mathbf{E}[\mathrm{Err}^{(t)} \mathbf{1}_{\mathrm{Case}\ 0}] \leq 6\mathbf{E}[|I^{(t)}| \mathbf{1}_{\omega^{(t)} \in \mathrm{BAD}_3(t)}] \leq 6\lambda \mathbf{P}[\omega^{(t)} \in \mathrm{BAD}_3(t)] + 6\mathbf{P}[|I^{(t)}| > \lambda].$$

Lemma 6.10 and Proposition 3.7 imply that by choosing λ to be a large multiple of $\log N$, we obtain

$$\mathbf{E}[\mathrm{Err}^{(t)} \mathbf{1}_{\mathrm{Case}\ 0}] \leq \frac{C_2}{d(t) N^{c_2}}.$$

Error in Cases 1 and 2 We deal with Case 1 as Case 2 can be treated in the same way. Since $\mathbf{x}$ can take values $x \in S^{(t)}$ only, and that by (10),

$$\phi_{\mathbb{L}^{(t)}}[\mathrm{lmax}(\mathcal{C}(\omega, i)) = x, \mathbf{x} = x, \mathcal{H} = \mathcal{H}^{(t)}] \leq \frac{C_3}{N^2} \exp[-c_3 \tfrac{|x|}{N}], \tag{57}$$

(as in the bound of the probability of $\mathrm{BAD}_2(t)$, we need to account for the possibility that x is far on the left of Λ_N), it suffices to show that

$$\mathbf{P}[\mathrm{Err}^{(t)} \neq 0 | \omega^{(t+1/2)} \in \{\mathrm{lmax}(\mathcal{C}(\omega, i)) = x, \mathbf{x} = x, \mathcal{H} = \mathcal{H}^{(t)}\}]$$

is small and to plug it in (57) above. Then, summing over x and applying a manipulation similar to Case 0 will conclude the proof.

For (A, i) in $\{(\mathrm{B}, i), i\} \cup \{(\mathrm{T}, i), i \neq \mathbf{i}\}$, with $\mathbf{i}$ the unique integer such that $\mathcal{C}(\omega, \mathbf{i})$ contains a top t-extremum, we immediately find that

$$\delta^{\mathrm{err}} A^{(t)}(i) = A^{(t+1)}(i) - A^{(t)}(i) = \delta^{\mathrm{IIC}} A^{(t)}(i) = 0.$$

Therefore, the errors can only come from the evolution of $\mathrm{T}^{(t)}(\mathbf{i})$ and the $\mathrm{R}^{(t)}(i)$ for $i \in I^{(t)}$. We treat the case $A = \mathrm{T}$ and $A = \mathrm{R}$ separately.

Below, we fix x and set $\omega^{\mathrm{T}} := \omega^{(\mathrm{T})}_{h(t) - \mathrm{T}^{(t)}(i)}$.

Error from the top t-extremum We start with $|\delta^{\mathrm{err}} \mathrm{T}^{(t)}(\mathbf{i})|$, which can come from a number of facts (see Figure 27):

(i) $\omega^{(t+1/2)}$ and $x + \omega^{\mathrm{T}}$ are not equal on $\Lambda_{d(t)^{1/4}}(x)$;
(ii) there is another top t-extremum in $\mathcal{C}(\omega^{(t+1/2)}, \mathbf{i})$ at a distance at least $d(t)^{1/5}$ of x;
(iii) there are two top t-extrema in ω^{T} at a distance at least $d(t)^{1/5}$ of each other.
(iv) $\omega^{(t+1/2)}$ and $x + \omega^{\mathrm{T}}$ are equal on $\Lambda_{d(t)^{1/4}}(x)$ but the track-exchange operators outputs on $\Lambda_{d(t)^{1/5}}(x)$ are different in $\omega^{(t+1/2)}$ and ω^{T};

Indeed, if none of (i)–(iv) occurs, then (i) gives that $\omega^{(t+1/2)}$ and $x + \omega^{\mathrm{T}}$ coincide on $\Lambda_{d(t)^{1/4}}(x)$, (iv) guarantees that the result of the track-exchange output is the same in $\Lambda_{d(t)^{1/5}}(x)$. Finally, the absence of other top t-extrema in either $\omega^{(t+1/2)}$ and ω^T guarantees that the change of height is measured by what happens within $\Lambda_{d(t)^{1/5}}(x)$.

Subcase (i). Let $E(x)$ be the event that there is a unique cluster in $\omega^{(t+1/2)}$ crossing the annulus $\Lambda_{d(t)^{1/2}}(x)\setminus\Lambda_{d(t)^{1/3}}(x)$ from outside to inside. Note that this event is measurable in terms of $\omega^{(t+1/2)}$ restricted to $\Omega(x,\omega^{(t+1/2)})$. Furthermore, observe that this event has a "screening effect" (see Figure 26) implying

$$\phi_{\mathbb{L}^{(t)}}[\,\cdot\,|\mathrm{lmax}(\mathcal{C}(\omega,\mathbf{i}))=x,\mathbf{x}=x,\mathcal{H}=\mathcal{H}^{(t)},\omega^{(t+1/2)}_{|\Omega(x,\omega^{(t+1/2)})}]=\phi^{\xi}_{\Omega(x,\omega^{(t+1/2)})^c}[\,\cdot\,|\mathrm{lmax}(\mathcal{C})=x],$$

where ξ are the boundary conditions induced by $\omega^{(t+1/2)}$ on the graph $\mathbb{L}^{(t)}\setminus\Omega(x,\omega^{(t+1/2)})$, and $\mathcal{C}$ is the unique cluster crossing the annulus $\Lambda_{d(t)^{1/2}}(x)\setminus\Lambda_{d(t)^{1/3}}(x)$.

Therefore, the mixing property of the IIC given by Proposition 3.9 implies that on $\omega^{(t+1/2)}\in E(x)$, the coupling does not give equality with probability at most $C_3 d(t)^{-c_3}$. Combined with (57) (and using $|\delta^{\mathrm{err}}\mathrm{T}^{(t)}|\le 2$), we deduce that

$$\begin{aligned}
&\mathbf{E}[|\delta^{\mathrm{err}}\mathrm{T}^{(t)}(\mathbf{i})|\mathbf{1}_{(i),\mathrm{lmax}(\mathcal{C}(\omega^{(t+1/2)},\mathbf{i}))=x}]\\
&\le \frac{2C_3}{d(t)^{c_3}}\mathbf{P}[\mathrm{lmax}(\mathcal{C}(\omega^{(t+1/2)},\mathbf{i}))=x]+2\mathbf{P}[\mathrm{lmax}(\mathcal{C}(\omega^{(t+1/2)},\mathbf{i}))=x,\omega^{(t+1/2)}\notin E(x)]\\
&\le \frac{C_4}{N^2 d(t)^{c_4}}\exp[-c|x|/N],
\end{aligned}$$

where in the last inequality we used (10) and (14).

Subcase (ii). In this case, $\omega^{(t+1/2)}$ belongs to $\mathrm{BAD}_2(t)$ so Lemma 6.10 gives

$$\mathbf{E}[|\delta^{\mathrm{err}}\mathrm{T}^{(t)}(\mathbf{i})|\mathbf{1}_{(ii)}]\le 2\phi_{\mathbb{L}^{(t)}}[\mathrm{BAD}_2(t)]\le\frac{C_5}{d(t)N^{c_5}}$$

(we directly provided the estimate summed over x in this case as it follows from the statement of Lemma 6.10).

Subcase (iii). First, since by construction ω^{T} is independent of the event $\omega^{(t+1/2)}\in\{\mathrm{lmax}(\mathcal{C}(\omega,i))=x,\mathbf{x}=x,\mathcal{H}=\mathcal{H}^{(t)}\}$, it suffices to prove that

$$\mathbf{P}[\omega^{\mathrm{T}}\text{ contains a top }t\text{-extremum outside }\Lambda_{d(t)^{1/5}}]\le\frac{C_6}{d(t)^{1/4}}.$$

To see this, simply sum over $y\in\mathbb{Z}\setminus\Lambda_{d(t)^{1/4}}$ the probability of y being a top t-extremum, which can easily be proved to be of order $C_7/|y|^2$. We conclude that

$$\mathbf{P}[|\delta^{\mathrm{err}}\mathrm{T}^{(t)}(\mathbf{i})|\mathbf{1}_{(iii),\mathrm{lmax}(\mathcal{C}(\omega^{(t+1/2)},\mathbf{i}))=x}]\le\frac{C_8}{N^2 d(t)^{1/4}}\exp[-c|x|/N].$$

Subcase (iv). To be in this case, it must be that in the intersection of the annulus $\Lambda_{d(t)^{1/4}}(x)\setminus\Lambda_{d(t)^{1/5}}(x)$ with $t^-_{j(t)}\cup t^-_{j(t)-1}$ on the left and the right of x, there is no pair of closed edges on top of each other since the existence of such edges decouple

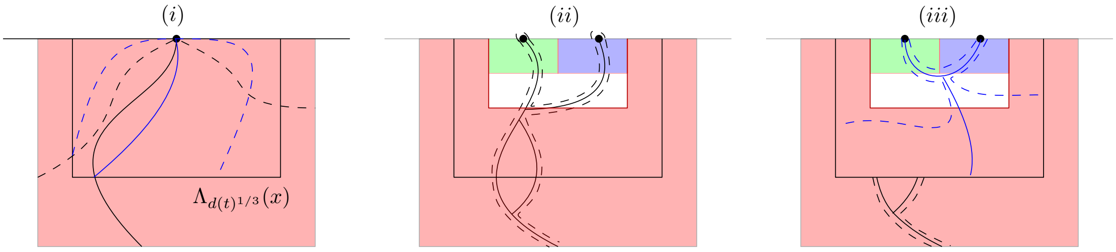

Figure 27: The different cases zoomed at a distance $d(t)^{1/2}$ around $\mathbf{x}$. The configuration $\omega^{(t+1/2)}$ is depicted in black, and $\mathbf{x}+\omega^{(t)}_{\mathrm{T},j}$ in blue.

the star-triangle transformations on their left and right as seen in the paragraph above Definition 3.11. We deduce that

$$\mathbf{E}[|\delta^{\mathrm{err}}\mathrm{T}^{(t)}(\mathbf{i})|\mathbf{1}_{(iv),\mathrm{lmax}(\mathbf{i})=x}] \le \frac{C_8\exp[-c_8 d(t)^{1/4}]}{N^2}\exp[-c|x|/N].$$

Error from the right t-extrema On the one hand, there can exist $i \in I^{(t)}$ such that $\mathrm{R}^{(t+1)}(i) \neq \mathrm{R}^{(t)}(i)$. Yet, this can occur only when some $\mathcal{C}(\omega^{(t+1/2)}, i)$ contains a right t-extremum, i.e. when $\omega^{(t+1/2)} \in \mathrm{BAD}_1(t)\cup\mathrm{BAD}_3(t)$. On the other hand, there can exist i such that $X^{(t)}_{\mathcal{H}^{(t)}}(i) = 1$ and $\delta^{\mathrm{IIC}}\mathrm{R}^{(t)}(i) \neq 0$. Yet, the probability that $X^{(t)}_{\mathcal{H}^{(t)}}(i) = 1$ is such that

$$\mathbf{P}[X^{(t)}_{\mathcal{H}^{(t)}}(i) = 1|\mathcal{H}^{(t)}] \le \mathbf{P}[\omega^{(t+1/2)} \in \mathrm{BAD}_1(t)\cup\mathrm{BAD}_3(t)|\mathcal{H}^{(t)}].$$

By proceeding in the same way as in Case 0, and using Lemma 6.10, we deduce that

$$\mathbf{E}\Big[\Big(\sum_{i\in I^{(t)}}|\delta^{\mathrm{err}}\mathrm{R}^{(t)}(i)|\Big)\mathbf{1}_{\mathrm{Case\ 1}}\Big] \le 2\mathbf{E}[|I^{(t)}|\mathbf{1}_{\omega^{(t+1/2)}\in\mathrm{BAD}_1(t)\cup\mathrm{BAD}_3(t)}] \le \frac{C_9}{d(t)N^{c_9}}$$

(again here we directly give the summed error as it is provided by Lemma 6.10).

Error in Case 3 In Case 3, no error is made for T and B, and we only need to control the error due to movements of $\mathrm{R}^{(t)}(i)$. Also, the error strictly after Step 0 can be treated in exactly the same way as in Case 1. Indeed, any such error either implies the occurrence of $\mathrm{BAD}_5(t)$ or is generated by the configuration in the box of size $d(t)$ around $\mathbf{x}$, in which case we use (15) instead of (10) as for Case 1.

The only new type of errors we need to control are those in Case 0, and they are of three types:

(i) $X^{(t)}$ and $\widetilde{X}^{(t)}$ do not couple,
(ii) $X^{(t)} = \widetilde{X}^{(t)}$ but there is no i with $\widetilde{X}^{(t)}(i) = 1$,
(iii) $X^{(t)} = \widetilde{X}^{(t)}$ but there are two i with $\widetilde{X}^{(t)}(i) = 1$.

We divide our analysis between the different cases.

Subcase (i). For every $i \in I^{(t)}$, $X^{(t)}(i)$ and $\widetilde{X}^{(t)}(i)$ have the same law. Using the inclusion-exclusion principle, we see that for the best coupling between the two random variables, we have that

$$\mathbf{P}[X^{(t)}(i) \neq \widetilde{X}^{(t)}(i)|\mathcal{H}^{(t)}] \leq C_{13}\mathbf{P}[|\widetilde{X}^{(t)}| \geq 2|\mathcal{H}^{(t)}].$$

Yet, using an argument similar to Case 2, we find that

$$\begin{aligned}\mathbf{E}\Big[\mathrm{Err}^{(t)}\mathbf{1}_{\text{Case } 2, X^{(t)}\neq \widetilde{X}^{(t)}}\Big] &\leq 2\mathbf{E}[|I^{(t)}|\mathbf{1}(X^{(t)} \neq \widetilde{X}^{(t)})] \\ &\leq 2C_{13}\mathbf{E}[|I^{(t)}|\mathbf{1}(|\widetilde{X}^{(t)}| \geq 2)] \\ &= 2C_{13}\mathbf{E}[|I^{(t)}|\mathbf{1}(\omega^{(t+1/2)} \in \mathrm{BAD}_3(t))]\end{aligned}$$

(the last equality is due to the fact that $\omega^{(t+1/2)} \in \{X = \widetilde{X}^{(t)}\}$). Then, we conclude using Lemma 6.10 as before.

Subcase (ii). In this case, $\delta^{\mathrm{IIC}}\mathrm{R}^{(t)}(i) = 0$ for every $i \in I^{(t)}$. Yet, for $\mathrm{R}^{(t+1)}(i)$ to be different from $\mathrm{R}^{(t)}(i)$, it must be that $\mathcal{C}(\omega^{(t+1/2)}, i)$ contains a fake right t-extremum (as it does not contain a right t-extremum on the event $X = \widetilde{X}^{(t)}$). Therefore, $\omega^{(t+1/2)}$ must contain a fake right t-extremum, i.e. that $\omega^{(t+1/2)} \in \mathrm{BAD}_4(t)$. We deduce the result from Lemma 6.10.

Subcase (iii). In this case, $\omega^{(t+1/2)}$ must contain two right t-extrema. Therefore, $\omega^{(t+1/2)} \in \mathrm{BAD}_3(t)$ again and the proof follows from Lemma 6.10 and an argument similar to Case 0. □

6.4 Compounded time steps

We now group steps into so-called *compounded time steps* corresponding to the action of a single track going down from its initial position to its final one. More precisely, for $0 \leq k < \lceil 2N/\sin\alpha \rceil$ we study the steps $t \in [\tau_k, \tau_{k+1})$, where $\tau_k := 2kN$ (it will be important that the time steps correspond to the action of the same track of angle α, here the $(k+1)$-st one) to be pushed down.

First, introduce the *speeds* of the IIC in each direction, a notion which will be useful in the next sections. Note that the definition below does not immediately seem to be connected to the speed of a process. We will see later that it will in fact correspond to the speed (or "drift") of extrema of nails when bringing tracks down.

Definition 6.11 (Speed in each direction)**.** Define

$$v_{\mathrm{T}} := \sin\alpha - \frac{\mathbf{P}[\delta_1^{\mathrm{IIC}}\mathrm{T} = 0]}{\mathbf{P}[\delta_0^{\mathrm{IIC}}\mathrm{T} = \sin\alpha - 1]} \qquad v_{\mathrm{B}} := \sin\alpha - \frac{\mathbf{P}[\delta_0^{\mathrm{IIC}}\mathrm{B} = \sin\alpha - 1]}{\mathbb{P}[\delta_{-1}^{\mathrm{IIC}}\mathrm{B} = 0]},$$

$$v_{\mathrm{R}} := \frac{\mathbf{E}[\delta^{\mathrm{IIC}}\mathrm{R}]}{\mathbf{P}[\delta^{\mathrm{IIC}}\mathrm{R} \in \{0, -1\}]}.$$

For $i \in I^{(\tau_k)}$, introduce the random time at which i ceases to be the index of a nail:

$$\tau_{\text{end}}(i) := \min\{s : i \notin I^{(s)}\}$$

(it is equal to T if such an s does not exist). Let $\mathcal{F}_k$ be the σ-algebra containing all the variables

$$(\omega^{(s)} : s \le \tau_k)\ ,\ (\omega^{(s+1/2)} : s < \tau_k)\ ,\ (X_H^{(s)}(i) : s < \tau_k)\ ,\ (\delta_j^{\text{IIC}} A^{(s)}(i) : s < \tau_k).$$

Recall the definition of $M^{(s)}(i)$ from the previous section.

Proposition 6.12 (Compounded time step for $A = \text{T}$ or B)**.** *There exist $c, C \in (0,\infty)$ such that for $A \in \{\text{T}, \text{B}\}$, $i \in \mathbb{Z}_{>0}$, $0 \le k < \lceil 2N/\sin\alpha \rceil$, there exist random variables $\Delta^{\text{IIC}} A^{(k)}(i)$ and $\Delta^{\text{err}} A^{(k)}(i)$ such that a.s. for every $i \in I^{(\tau_k)}$,*

(i) $A^{(\tau_{\text{end}} \wedge \tau_{k+1})}(i) - A^{(\tau_k)}(i) = \Delta^{\text{IIC}} A^{(k)}(i) + \Delta^{\text{err}} A^{(k)}(i)$;

(ii) $\mathbf{E}[\exp(c|\Delta^{\text{IIC}} A^{(k)}(i)|)|\mathcal{F}_k] \le C$;

(iii) $\mathbf{E}[|\Delta^{\text{err}} A^{(k)}(i)|\,|\mathcal{F}_k] \le C\,\mathbf{E}\Big[\sum_{s=\tau_k}^{\tau_{\text{end}} \wedge \tau_{k+1}} M^{(s)}(i)\Big|\mathcal{F}_k\Big]$;

(iv) $\mathbf{E}[\Delta^{\text{IIC}} A^{(k)}(i)|\mathcal{F}_k] = \begin{cases} 0 & \text{if } A^{(\tau_k)}(i) \le b(\tau_k), \\ v_A + O(e^{-c|A^{(\tau_k)}(i) - b(\tau_k)|} + \mathbf{P}[\tau_{\text{end}} < \tau_{k+1}|\mathcal{F}_k]) & \text{otherwise.} \end{cases}$

Remark 6.13. The $O(\cdot)$ quantity in (iv) comes from the fact that there are two types of errors: the first term comes from cases where $A^{(\tau_k)}(i)$ is close to the bottom $b(\tau_k)$ in which case the process described in the next proof could be stopped because the track reaches its final position, and the second from the fact that the nail i can cease to be indexed during the interval $[\tau_k, \tau_{k+1})$.

Proof. We treat the case of $A = \text{T}$. The case of B is similar. In the whole proof, fix k and $i \in I^{(\tau_k)}$. To lighten the notation we omit i in the notation.

Case 1 $\text{T}^{(\tau_k)} \le b(\tau_k)$**.** In this case, the coupling $\mathbf{P}$ is such that $\text{T}^{(\tau_{\text{end}} \wedge \tau_{k+1})} - \text{T}^{(\tau_k)} = 0$ so we may define

$$\Delta^{\text{IIC}} \text{T}^{(k)} = \Delta^{\text{err}} \text{T}^{(k)} := 0.$$

Case 2 $\text{T}^{(\tau_k)} > b(\tau_k)$**.** We first describe how a track-exchange at time $\tau_k \le s < \tau_{k+1}$ modifies the value of $\text{T}^{(s)}$. There are three possibilities:

- $\text{T}^{(s)} \notin \{h(s) - 1, h(s)\}$, in which case $\text{T}^{(s)}$ is not altered;
- $\text{T}^{(s)} = h(s) - 1$ which happens exactly once. In this case, T may either stay put or increase by $\sin\alpha$. In the former case, T is not altered by subsequent steps and in the latter $\text{T}^{(s+1)} = h(s+1)$;

- $\mathrm{T}^{(s)} = h(s)$, which implies that either $T^{(s-1)} = h(s-1)$ and the track-exchange "dragged down" the top of the nail, or $\mathrm{T}^{(s-1)} = h(s-1)-1$ but the track-exchange failed to increase the top. In this case, $\mathrm{T}^{(s)}$ may either increase by $\sin\alpha - 1$ in which case it will not move at subsequent steps, or decrease by -1, in which case $\mathrm{T}^{(s+1)} = h(s+1)$.

From the previous discussion, we find

$$\mathrm{T}^{(\tau_{\mathrm{end}}\wedge\tau_{k+1})} - \mathrm{T}^{(\tau_k)} = \sin\alpha - (\sigma - \tau), \tag{58}$$

where τ is the first (and unique) time for which $\mathrm{T}^{(\tau)} = h(\tau)-1$ and σ is the time defined by

$$\sigma := \begin{cases} \tau & \text{if } \mathrm{T}^{(\tau+1)} = \mathrm{T}^{(\tau)} + \sin\alpha, \\ \inf\{s \in (\tau, \tau_{\mathrm{end}} \wedge \tau_{k+1}) : \mathrm{T}^{(s+1)} - \mathrm{T}^{(s)} \neq -1\} & \text{if } \mathrm{T}^{(\tau+1)} = \mathrm{T}^{(\tau)} \text{ and } s \text{ exists}, \\ \tau_{\mathrm{end}} \wedge \tau_{k+1} & \text{otherwise.} \end{cases}$$

To define $\Delta^{\mathrm{IIC}}\mathrm{T}$, we use a similar formula except that we consider the random variables $\delta_j^{\mathrm{IIC}}\mathrm{T}^{(s)}$ instead of the true increments:

$$\Delta^{\mathrm{IIC}}\mathrm{T}^{(k)} := \sin\alpha - (\sigma^{\mathrm{IIC}} - \tau), \tag{59}$$

where

$$\sigma^{\mathrm{IIC}} := \begin{cases} \tau & \text{if } \delta_1^{\mathrm{IIC}}\mathrm{T}^{(\tau)} = \sin\alpha, \\ \inf\{s \in (\tau, \tau_{\mathrm{end}} \wedge \tau_{k+1}) : \delta_0^{\mathrm{IIC}}\mathrm{T}^{(s)} \neq -1\} & \text{if } \delta_1^{\mathrm{IIC}}\mathrm{T}^{(\tau)} = 0 \text{ and such an } s \text{ exists}, \\ \tau_{\mathrm{end}} \wedge \tau_{k+1} & \text{otherwise} \end{cases}$$

(note that τ_{end} is still a function of the true increments).

Finally, we set

$$\Delta^{\mathrm{err}}\mathrm{T}^{(k)} := \sigma^{\mathrm{IIC}} - \sigma. \tag{60}$$

We are now in a position to derive our proposition. First, (i) is satisfied by construction and (58)–(60). The definition of σ^{IIC} from independent "trial" events immediately leads to (ii). For (iv), we have that

$$\begin{aligned} \mathbf{E}[\Delta^{\mathrm{IIC}}\mathrm{T}^{(k)}|\mathcal{F}_k] &= \sin\alpha - \mathbf{E}[\sigma^{\mathrm{IIC}} - \tau|\mathcal{F}_k] \\ &= \sin\alpha - \frac{\mathbf{P}[\delta_1^{\mathrm{IIC}}\mathrm{T} = 0]}{1 - \mathbf{P}[\delta_0^{\mathrm{IIC}}\mathrm{T} = -1]} + O(e^{-c|\mathrm{T}^{(t)} - b(t)|} + \mathbf{P}[\tau_{\mathrm{end}} < \tau_{k+1}|\mathcal{F}_k]), \end{aligned}$$

where the error term comes from the fact that σ^{IIC} can be equal to $\tau_{\mathrm{end}} \wedge \tau_{k+1}$. More precisely, when $\tau_{\mathrm{end}} \geq \tau_{k+1}$, we obtain the first error since $\tau_{k+1} - \tau \geq \mathrm{T}^{(t)} - b(t)$ and the difference is a geometric random variable, and when $\tau_{\mathrm{end}} < \tau_{k+1}$, we obtain the second term in the $O(\cdot)$.

It only remains to prove (iii), i.e. to bound $\mathbf{E}[|\sigma^{\mathrm{IIC}} - \sigma|\,|\mathcal{F}_k]$. In order to do it, introduce further random times defined recursively by $\sigma_0^{\mathrm{IIC}} = \sigma^{\mathrm{IIC}}$ and

$$\sigma_{\ell+1}^{\mathrm{IIC}} := \inf\{s \in (\sigma_\ell^{\mathrm{IIC}}, \tau_{\mathrm{end}}) : \delta_0^{\mathrm{IIC}}\mathrm{T}^{(s)} \neq -1\}$$

when s exists and $\sigma_{\ell+1}^{\mathrm{IIC}} = \tau_{\mathrm{end}} \wedge \tau_{k+1}$ otherwise (note that for ℓ large enough, the sequence becomes stationary at $\tau_{\mathrm{end}} \wedge \tau_{k+1}$, which is compatible with the formula below). We have

$$\sigma^{\mathrm{IIC}} - \sigma = \sum_{\tau \le s \le \sigma^{\mathrm{IIC}}} \mathbf{1}_{s>\sigma} - \sum_{\ell \ge 0} \sum_{s=\sigma_\ell^{\mathrm{IIC}}+1}^{\sigma_{\ell+1}^{\mathrm{IIC}}} \mathbf{1}_{s \le \sigma}.$$

Now, let $\mathbf{X}$ denote the sum of the $|\delta^{\mathrm{err}}\mathrm{T}^{(s)}|$ for $s \in [\tau, \tau_{\mathrm{end}} \wedge \tau_{k+1}]$. On the one hand, for $\tau \le s \le \sigma^{\mathrm{IIC}}$ to satisfy $s > \sigma$, it must be that $\delta^{\mathrm{err}}\mathrm{T}^{(r)} \neq 0$ for some $r \in [\tau, s)$ and that $\delta_0^{\mathrm{IIC}}\mathrm{T}^{(r')} = -1$ for every $r' \in (r, s)$. Independence provided by Proposition 6.8(ii) implies that

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{\tau \le s \le \sigma^{\mathrm{IIC}}} \mathbf{1}_{s>\sigma}\Big|\mathcal{F}_k\Big] \\
&\quad \le \sum_{s \ge r \ge \tau} \mathbf{P}[\forall r' \in (\tau, r), \delta^{\mathrm{err}}\mathrm{T}^{(r')} = 0; \delta^{\mathrm{err}}\mathrm{T}^{(r)} \neq 0; \forall s' \in (r, s), \delta_0^{\mathrm{IIC}}\mathrm{T}^{(s')} = -1|\mathcal{F}_k] \\
&\quad \le \frac{\mathbf{P}[\mathbf{X} \ge \sin\alpha|\mathcal{F}_k]}{\mathbf{P}[\delta_0^{\mathrm{IIC}}\mathrm{T} = \sin\alpha - 1]}.
\end{aligned}$$

On the other hand, for $\sigma_\ell^{\mathrm{IIC}} < s \le \sigma_{\ell+1}^{\mathrm{IIC}}$ to be smaller than σ, it must be that $\delta^{\mathrm{err}}\mathrm{T}^{(\sigma_l^{\mathrm{IIC}})} = -\sin\alpha$ for every $0 \le l \le \ell$, and that $\delta_0^{\mathrm{IIC}}\mathrm{T}^{(r)} = -1$ for every $r \in [\sigma_\ell^{\mathrm{IIC}}, s]$ so that by independence of the variables $\delta_0^{\mathrm{IIC}}\mathrm{T}^{(r)}$ for $r > \sigma_\ell^{\mathrm{IIC}}$ and $\delta^{\mathrm{err}}\mathrm{T}^{(\sigma_l^{\mathrm{IIC}})}$ for $l \le \ell$, we get in a fairly similar fashion to the previous displayed equation that

$$\begin{aligned}
\mathbf{P}\Big[\sum_{s=\sigma_\ell^{\mathrm{IIC}}+1}^{\sigma_{\ell+1}^{\mathrm{IIC}}} \mathbf{1}_{s \le \sigma}\Big|\mathcal{F}_k\Big] &\le \sum_{j \ge 0} \mathbf{P}[\delta_0^{\mathrm{IIC}}\mathrm{T} = -1]^j \mathbf{P}[\mathbf{X} \ge \ell \sin\alpha|\mathcal{F}_k] \\
&\le C\,\mathbf{P}[\mathbf{X} \ge \ell\sin\alpha|\mathcal{F}_k].
\end{aligned}$$

The claim follows by summing over ℓ. □

We now treat the impact of compounded steps on R.

Proposition 6.14 (Compounded time step for $A = \mathrm{R}$)**.** *There exist $c, C \in (0, \infty)$ such that for $i \in \mathbb{Z}_{>0}$ and $0 \le k < \lceil 2N/\sin\alpha \rceil$, there exist random variables $\Delta^{\mathrm{IIC}}\mathrm{R}^{(k)}(i)$ and $\Delta^{\mathrm{err}}\mathrm{R}^{(k)}(i)$ such that a.s. for every $i \in I^{(\tau_k)}$,*

- *(i)* $\mathrm{R}^{(\tau_{\mathrm{end}} \wedge \tau_{k+1})}(i) - \mathrm{R}^{(\tau_k)}(i) = \Delta^{\mathrm{IIC}}\mathrm{R}^{(k)}(i) + \Delta^{\mathrm{err}}\mathrm{R}^{(k)}(i)$;
- *(ii)* $\mathbf{E}[\exp(c|\Delta^{\mathrm{IIC}}\mathrm{R}^{(k)}(i)|)|\mathcal{F}_k] \le C$;

(iii) $\mathbf{E}[|\Delta^{\mathrm{err}}\mathrm{R}^{(k)}(i)|\,|\mathcal{F}_k] \le C\mathbf{E}\Big[\sum_{s=t}^{\tau_{\mathrm{end}}\wedge\tau_{k+1}} M^{(s)}(i)\Big|\mathcal{F}_k\Big]$;

(iv) $\mathbf{E}[\Delta^{\mathrm{IIC}}\mathrm{R}^{(k)}(i)|\mathcal{F}_k]$ *is equal to*

$$\begin{cases} 0 & \text{if } \mathrm{R}^{(\tau_k)}(i) \notin k\cos\alpha + \mathbb{Z}, \\ v_{\mathrm{R}} + O\Big(\mathbf{E}\Big[\sum_{s=\tau_k}^{\tau_{\mathrm{end}}\wedge\tau_{k+1}} M^{(s)}(i)\Big|\mathcal{F}_k\Big] + \mathbf{P}[\tau_{\mathrm{end}} < \tau_{k+1}|\mathcal{F}_k]\Big) & \text{otherwise.} \end{cases}$$

Remark 6.15. When $\cos\alpha \notin \mathbb{Q}$, the condition $\mathrm{R}^{(\tau_k)} \notin k\cos\alpha + \mathbb{Z}$ implies that no right-most vertex can belong to the area below the $(k+1)$-st track. The information on R therefore gives more than simply the first-coordinate of the right-most point, it also provides information on its vertical position. This is not necessary true for rational values of $\cos\alpha$. In this case, one should therefore record this information more explicitly. We chose to restrict ourselves to α with $\cos\alpha$ irrational as we will see it is sufficient to get our result.

Proof. Again, we fix k and i and drop i from the notation. We first describe how a track-exchange for $\tau_k \le s < \tau_{k+1}$ modifies the value of $\mathrm{R}^{(s)}$. There are three possibilities:

- $\mathrm{R}^{(s)} \in k\cos\alpha + \mathbb{Z}$ and $(\mathrm{R}^{(s)}, h(s))$ does not belong to the nail $\mathcal{C}(\omega^{(s)}, i)$, in such case $\mathrm{R}^{(s+1)} = \mathrm{R}^{(s)}$,
- $\mathrm{R}^{(s)} \in k\cos\alpha + \mathbb{Z}$ and $(\mathrm{R}^{(s)}, h(s))$ belongs to the cluster. In such case, the track-exchange creates a change of $\cos\alpha$, $\cos\alpha - 1$, or -1 (see Figure 24). In the former case, $\mathrm{R}^{(s+1)} = \mathrm{R}^{(s)} + \cos\alpha$ and the next track-exchanges will not impact the maximum. In the latter, $\mathrm{R}^{(s)}$ can change by values in $[-1, 0] \cap (\mathbb{Z} + \{0, \dots, k-2, k-1, k+1\}\cos\alpha)$ due to the possible existence of other vertices that are not affected by the track-exchange but had almost-maximal first coordinate[12].
- $\mathrm{R}^{(s)} \notin k\cos\alpha + \mathbb{Z}$. In such a case, there is only one possibility for $\mathrm{R}^{(s+1)}$ not to be equal to $\mathrm{R}^{(s)}$, which is that there exists a fake right s-extremum and that the track-exchange implies an increase of $\cos\alpha$ locally, which leads to $\mathrm{R}^{(s+1)} = x_1 + \cos\alpha$ and no further change can occur.

Now, recall the definition of $X_H^{(s)} = X_H^{(s)}(i)$ from the previous section, and introduce

$$\Delta^{\mathrm{IIC}}\mathrm{R}^{(k)} := \sum_{\tau_k \le s \le \sigma^{\mathrm{IIC}}} X_{\mathcal{H}^{(s)}}^{(s)} \delta^{\mathrm{IIC}}\mathrm{R}^{(s)},$$

[12] In fact, essentially the only other two values that are possible are 0 if there is another extremum of the cluster in the square region below $t_{j(s)}^-$, or $\cos\alpha - 1$ if there is a "near" extremum in the square region above height $h(s)$ that becomes the right-most point after the transformation. For the increment to be different from 0, $\cos\alpha - 1$ or -1, it must be that $\mathcal{C}(\omega^{(s)}, i)$ contains a vertex below $b(t)$ with first coordinate in $(\mathrm{R}^{(s)} - 1, \mathrm{R}^{(s)})$, which has small probability as shown in Lemma 6.10.

where

$$\sigma^{\rm IIC} := \begin{cases} \min\{s \in [\tau_k, \tau_{\rm end} \wedge \tau_{k+1}) : \delta^{\rm IIC}\mathrm{R}^{(s)} \in \{\cos\alpha, \cos\alpha - 1\} \text{ and } X^{(s)}_{\mathcal{H}^{(s)}} = 1\} & \text{if } s \text{ exists,} \\ \tau_{\rm end} \wedge \tau_{k+1} & \text{otherwise,} \end{cases}$$

and

$$\Delta^{\rm err}\mathrm{R}^{(k)} := \mathrm{R}^{(\tau_{\rm end}\wedge\tau_{k+1})} - \mathrm{R}^{(\tau_k)} - \Delta^{\rm IIC}\mathrm{R}^{(k)} .$$

By definition, (i) and (ii) are satisfied. The proof of (iii) follows the same steps as the proof of (iii) in Proposition 6.12 (we leave the details to the reader), except in the case corresponding to the third bullet above, i.e. that $\mathrm{R}^{(s)} \notin k\cos\alpha + \mathbb{Z}$ but $\mathrm{R}^{(s+1)} \neq \mathrm{R}^{(s)}$. Yet, in this case $M^{(s)} \neq 0$ and no further error is made at later times.

For (iv), note that if $R^{(\tau_k)} \notin k\cos\alpha + \mathbb{Z}$, then $X^{(s)}_{\mathcal{H}^{(s)}}$ is always equal to 0 and $\Delta^{\rm IIC}\mathrm{R} = 0$. If, on the contrary, $R^{(\tau_k)} \in k\cos\alpha + \mathbb{Z}$, define

$$\widetilde{\sigma}^{\rm IIC} := \min\{s \geq \tau_k : \delta^{\rm IIC}\mathrm{R}^{(s)} \in \{\cos\alpha, \cos\alpha - 1\} \text{ and } Y^{(s)} = 1\},$$

where $Y^{(s)} = X^{(s)}_{\mathcal{H}^{(s)}}$ for $s \leq \tau_{\rm end} \wedge \tau_{k+1}$ and 1 for $s > \tau_{\rm end} \wedge \tau_{k+1}$. Also define $\widetilde{\Delta}^{\rm IIC}\mathrm{R}^{(k)}$ using the same formulas as for $\Delta^{\rm IIC}\mathrm{R}^{(k)}$ but with $\widetilde{\sigma}^{\rm IIC}$ instead of $\sigma^{\rm IIC}$. Then, a direct computation gives

$$\mathbf{E}[\widetilde{\Delta}^{\rm IIC}\mathrm{R}^{(k)}|\mathcal{F}_k] = v_{\rm R}.$$

Moreover,

$$\begin{aligned} |\mathbf{E}[\Delta^{\rm IIC}\mathrm{R}^{(k)}|\mathcal{F}_k] - v_{\rm R}| &\leq \mathbf{E}[\widetilde{\sigma}^{\rm IIC} - \sigma^{\rm IIC}|\mathcal{F}_k] \leq C_0\mathbf{P}[\widetilde{\sigma}^{\rm IIC} \neq \sigma^{\rm IIC}|\mathcal{F}_k] \\ &\leq C_0\mathbf{P}[\forall s \leq \tau_{\rm end} \wedge \tau_{k+1} : X^{(s)}_{\mathcal{H}^{(s)}} = 1, \delta^{\rm IIC}R^{(s)} = -1|\mathcal{F}_k]. \end{aligned}$$

To estimate the probability on the right, observe that if $\tau_{\rm end} \geq \tau_{k+1}$ and $\delta^{\rm IIC}R^{(s)} = -1$ for every s such that $X^{(s)}_{\mathcal{H}^{(s)}} = 1$, it must be that $M^{(s)} \neq 0$ for at least one s since otherwise $\mathrm{R}^{(\tau_{k+1})} \in \mathrm{R}^{(\tau_k)} + \mathbb{Z}$, which is impossible. We therefore obtain that

$$\begin{aligned} &\mathbf{P}[\forall s \leq \tau_{\rm end} \wedge \tau_{k+1} : X^{(s)}_{\mathcal{H}^{(s)}} = 1, \delta^{\rm IIC}R^{(s)} = -1|\mathcal{F}_k] \\ &\qquad\qquad \leq \mathbf{P}[\tau_{\rm end} < \tau_{k+1}|\mathcal{F}_k] + \mathbf{E}\Big[\sum_{s=\tau_k}^{\tau_{\rm end}\wedge\tau_{k+1}} M^{(s)}\Big|\mathcal{F}_k\Big]. \end{aligned}$$

This concludes the proof of the proposition. □

6.5 Speed of the drift

In this section, we compute v_A for $A = \mathrm{T}, \mathrm{B}, \mathrm{R}$. We start with the first two.

Proposition 6.16. *We have* $v_{\rm T} = v_{\rm B} = 0$.

Proof. We treat the case of v_{T} (the case of v_{B} is the same). Introduce $\mathbf{l}_N := \mathrm{lmax}((0,-N))$ (i.e. the left-most highest vertex in the cluster of $(0,-N)$) and let $\mathbf{E}$ be the coupling between $\omega_1 \sim \phi_{\mathbb{L}_1}$ and $\omega_0 \sim \phi_{\mathbb{L}_0}$ obtained by setting $\omega_0 = \mathbf{T}_1(\omega_1)$. Also, let $\Delta\mathrm{T}$ be the difference between the top height of the cluster of $(0,-N)$ in ω_1 and in ω_0.

We find that

$$\begin{aligned}\phi_{\mathbb{L}_0}[\mathbf{l}_N \in t_1^-] &= \mathbf{P}[\mathbf{l}_N \in t_1^- \text{ in } \omega_0] \qquad (61)\\ &= \mathbf{P}[\mathbf{l}_N \in t_1^- \text{ in } \omega_1 \text{ and } \Delta\mathrm{T} = \sin\alpha - 1] + \mathbf{P}[\mathbf{l}_N \in t_0^- \text{ in } \omega_1 \text{ and } \Delta\mathrm{T} = \sin\alpha]\\ &= \phi_{\mathbb{L}_1}[\mathbf{l}_N \in t_1^-]\mathbf{P}[\delta_0^{\mathrm{IIC}} = \sin\alpha - 1] + \phi_{\mathbb{L}_1}[\mathbf{l}_N \in t_0^-]\mathbf{P}[\delta_1^{\mathrm{IIC}} = \sin\alpha] + o_N(1),\end{aligned}$$

where in the second step we used that we may couple the increment $\Delta\mathrm{T}$ with an IIC increment exactly as we did in the previous section (to estimate the error, one needs to perform a reasoning similar to the error in the top extremum in Case 1 of the coupling).

Using the same coupling, we also see that $\mathbf{l}_N \in t_0^- \cup t_1^-$ in ω_0 if and only if it does in ω_1, so we get that

$$\phi_{\mathbb{L}_0}[\mathbf{l}_N \in t_0^- \cup t_1^-] = \phi_{\mathbb{L}_1}[\mathbf{l}_N \in t_0^- \cup t_1^-]. \qquad (62)$$

Decomposing on the possible values of $\mathbf{l}_N$ (like in the proof of Proposition 2.6) and using the mixing of the IIC (Proposition 3.9), we also find that for $i = 0, 1$,

$$\phi_{\mathbb{L}_i}[\mathbf{l}_N \in t_1^- | \mathbf{l}_N \in t_0^- \cup t_1^-] = \mathbb{\Phi}_i^2[\mathrm{lmax}(\infty) = 0^+] + o_N(1). \qquad (63)$$

Dividing (61) by (62) and plugging (63) into it, we find that

$$\begin{aligned}&\mathbb{\Phi}_0^2[\mathrm{lmax}(\infty) = 0^+] \qquad (64)\\ &\quad = \mathbb{\Phi}_1^2[\mathrm{lmax}(\infty) = 0^+]\mathbf{P}[\delta_0^{\mathrm{IIC}} = \sin\alpha - 1] + \mathbb{\Phi}_1^2[\mathrm{lmax}(\infty) = 0]\mathbf{P}[\delta_1^{\mathrm{IIC}} = \sin\alpha].\end{aligned}$$

Theorem 2.4 applied to $\beta = \frac{\pi}{2}$ gives

$$\mathbb{\Phi}_0^2[\mathrm{lmax}(\infty) = 0^+] = \mathbb{\Phi}_1^2[\mathrm{lmax}(\infty) = 0] = 1 - \mathbb{\Phi}_1^2[\mathrm{lmax}(\infty) = 0^+] = \frac{1}{1 + \sin\alpha},$$

which, when inserted in (64) and multiplied by $1 + \sin\alpha$, gives $v_{\mathrm{T}} = 0$. □

Next, we turn to the lateral speed v_{R}, whose value is deduced from the one of v_{T}.

Proposition 6.17. *We have $v_{\mathrm{R}} = 0$.*

The idea of the proof is to obtain the right displacement of the cluster as the top displacement in a rotated version of the process. Below, we mention not only horizontal tracks but also vertical tracks and their track-exchanges. We believe that at this point the reader may easily make sense of these transformations so we omit the details of the definitions. Also, we refer to [18] for more information.

Proof. We refer to Figure 28 for an illustration. Consider M and N two integers satisfying $M = N^2$. Consider the graph $\mathbb{B}^{(0)}$ formed of $2M+2$ "horizontal" tracks $t_{-M}, \dots, t_M, t_\alpha$ of transverse angles $\pi/2$ for the first $2M+1$ and transverse angle α for t_α and M "vertical"

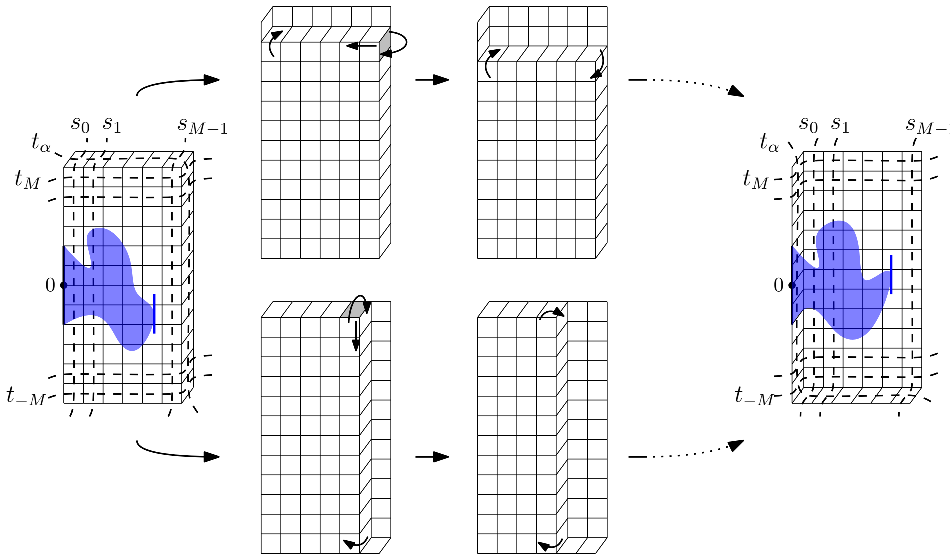

Figure 28: The initial and final graphs $\mathbb{B}^{(0)}$ and $\mathbb{B}^{(2M+1)}$ of the two processes are the same, but the intermediate graphs $(\mathbb{B}^{(t)})_{0<t\leq 2M}$ (top) and $(\widetilde{\mathbb{B}}^{(t)})_{0<t<M}$ (bottom) are different (the figure depicts the diamond graphs). In the top process, horizontal tracks are exchanged successively, by pushing the gray rhombus from right to left; in the bottom process, the rhombus is pushed downwards, effectively exchanging vertical tracks. Throughout the two processes, we record the right-most coordinate of the union of all the clusters intersecting the base.

tracks $s_0, \dots, s_{M-1}$ of transverse angle 0. In addition to the intersection between the vertical and horizontal tracks, $\mathbb{B}^{(0)}$ contains also the intersection of tracks $t_{-M}, \dots, t_M$ with t_α; these occur at the right side of the graph (note that t_α is not straight and does a sharp turn at the top-right corner of the rectangle). Translate $\mathbb{B}^{(0)}$ so that 0 is the vertex left of s_0 and below t_0.

We perform track-exchanges via star-triangle transformations applied to the graph $\mathbb{B}^{(0)}$. Contrary to the other parts of the paper, where exchanged tracks change name, here the tracks will conserve their indexing during track-exchanges. Define recursively $\mathbb{B}^{(t+1)}$ for $0 \leq t \leq 2M$ as obtained from $\mathbb{B}^{(t)}$ by performing the track-exchange $\mathbf{T}_{M-t}$ between t_{M-t} and t_α which is a composition of M star-triangle transformations.

We follow the extrema of the set $\mathcal{C}_{\text{base}}$ obtained as the union of the primal clusters intersecting the *base* $\{0\} \times [-N, N]$. Let $(\omega^{(t)}, \mathrm{R}^{(t)})_{0\leq t\leq 2M+1}$ be obtained as follows:

- the initial step is defined by sampling $\omega^{(0)}$ according to $\phi^0_{\mathbb{B}^{(0)}}$ and setting $\mathrm{R}^{(0)}$ to be the maximal first coordinate of vertices in $\mathcal{C}_{\text{base}}$;
- at time $0 \leq t \leq 2M$, sample a configuration

$$\omega^{(t+1/2)} \sim \phi^0_{\mathbb{B}^{(t)}}[\,\cdot\,|\mathrm{R}(\mathcal{C}_{\text{base}}) = \mathrm{R}^{(t)}],$$

let $\omega^{(t+1)} := \mathbf{T}_{M-t}(\omega^{(t)})$, and set $\mathrm{R}^{(t+1)}$ to be the maximal first coordinate of a vertex in $\mathcal{C}_{\mathrm{base}}$.

Following the proofs of the previous sections (with some additional simplifications in this context, for instance τ_{end} does not need to be introduced) we find that

$$\mathrm{R}^{(2M+1)} - \mathrm{R}^{(0)} = \Delta^{\mathrm{IIC}}\mathrm{R} + \Delta^{\mathrm{err}}\mathrm{R}, \tag{65}$$

with the equivalent of (iii) and (iv) of Proposition 6.14 being that a.s.,

$$\mathbf{E}[|\Delta^{\mathrm{err}}\mathrm{R}|\,|\mathcal{F}_0] \le C\,\mathbf{E}\Big[\sum_{t=0}^{2M}\mathrm{Err}^{(t)}\Big|\mathcal{F}_0\Big],$$
$$\mathbf{E}[\Delta^{\mathrm{IIC}}\mathrm{R}|\mathcal{F}_0] = v_{\mathrm{R}} + O\Big(\mathbf{E}\Big[\sum_{t=0}^{2M}\mathrm{Err}^{(t)}\Big|\mathcal{F}_0\Big]\Big),$$

where $\mathrm{Err}^{(t)}$ is defined in a similar fashion to Proposition 6.14, but with $\mathcal{C}_{\mathrm{base}}$ playing the role of the union of the nails now.

Let $h(t)$ be the height of the bottom of t_α at time t, $d(t) := \min\{M-h(t), h(t)+M\}$. Following an argument similar to the proof of Proposition 6.8, the mistake contributing to $\mathrm{Err}^{(t)}$ can be of three types:

- (i) $(\mathrm{R}^{(t)}, h(t)) \in \mathcal{C}_{\mathrm{base}}$ and $\mathrm{R}^{(t)} \le \sqrt{N}$,
- (ii) $(\mathrm{R}^{(t)}, h(t)) \in \mathcal{C}_{\mathrm{base}}$ and $\mathrm{R}^{(t)} \ge M - \sqrt{N}$,
- (iii) $(\mathrm{R}^{(t)}, h(t)) \in \mathcal{C}_{\mathrm{base}}$ and $\sqrt{N} \le \mathrm{R}^{(t)} \le M - \sqrt{N}$, but the true configuration and the IIC configurations are not coupled in the box of radius $\min\{d(t), N\}^{1/4}$ around $(\mathrm{R}^{(t)}, h(t))$,
- (iv) $(\mathrm{R}^{(t)}, h(t)) \in \mathcal{C}_{\mathrm{base}}$ and there is a vertex $x = (x_1, x_2) \in \mathcal{C}_{\mathrm{base}}$ with $x_1 \in (\mathrm{R}^{(t)}-1, \mathrm{R}^{(t)}]$ and $|x_2 - h(t)| \ge d(t)^{1/5}$.

Recalling that the error is deterministically bounded by 2, we therefore have that

$$\mathbf{E}[\mathrm{Err}^{(t)}] \le 2(\mathbf{P}[(i)] + \mathbf{P}[(ii)] + \mathbf{P}[(iii)] + \mathbf{P}[(iv)]). \tag{66}$$

We now bound the probabilities of the events (i), (ii), and (iii) separately. For (i), (RSW) immediately implies the existence of $c > 0$ such that for every $0 \le t \le 2M$,

$$\mathbf{P}[(i)] = \phi^0_{\mathbb{B}^{(t)}}[(i)] \le \phi^0_{\mathbb{B}^{(t)}}[\mathrm{R}^{(t)} \le \sqrt{N}] \le \exp[-c\sqrt{N}].$$

To estimate (ii) and (iii), let us first estimate, for $x \in t^+_{M-t}$, the probability of the event $E(x)$ that $(\mathrm{R}^{(t)}, h(t)) = x$ and base $\longleftrightarrow x$. Let $s(x)$ be the distance between x and $\partial\mathbb{B}^{(t)}$. We have that

$$\phi^0_{\mathbb{B}^{(t)}}[E(x)] \le \phi_{\mathbb{B}^{(t)}}\Big[\{\mathrm{base} \longleftrightarrow \partial\Lambda_{d(t)}(x)\} \cap A^{\mathrm{R}}_{010,x}(s(x), \tfrac{d(t)}{2}) \cap A^{\mathrm{R}}_{010,x}(0, \tfrac{s(x)}{2})\Big], \tag{67}$$

where $A^{\mathrm{R}}_{010,x}(r,R)$ is the translate of $A^{\mathrm{R}}_{010}(r,R)$ by x. Using the mixing property, (RSW) for the first event on the right-hand side, an argument similar to (12) for the third, and (15) for the fourth, we obtain that

$$\phi^0_{\mathbb{B}^{(t)}}[E(x)] \le C\Big(\frac{N}{\max\{N,|x|\}}\Big)^c \times \Big(\frac{d(t)}{|x|}\Big)^c \times \Big(\frac{s(x)}{d(t)}\Big)^{1+c} \times \Big(\frac{1}{s(x)}\Big)^2.$$

Summing over the $x \in t^+_{M-t}$ that are at a distance at most $\sqrt{N}$ from the right-hand side of $\mathbb{B}^{(t)}$, we obtain that

$$\mathbf{P}[(ii)] = \phi^0_{\mathbb{B}^{(t)}}[(ii)] \le C\Big(\frac{N}{M}\Big)^{2c} \times \frac{1}{d(t)}. \tag{68}$$

For (iii), using the same argument as in Proposition 6.8 in the first step and summing over x in the second

$$\mathbf{P}[(iii)] \le \frac{C}{\min\{d(t),N\}^c} \sum_{x\in t^+_{M-t}} \phi^0_{\mathbb{B}^{(t)}}[E(x)] \le \frac{C}{\min\{d(t),N\}^c} \times \frac{1}{M^c d(t)^{1-c}}. \tag{69}$$

The bound on (iv) can be obtained as in Lemma 6.10:

$$\mathbf{P}[(iv)] \le \frac{C}{d(t)M^c}. \tag{70}$$

Plugging (67)–(70) into (66) gives

$$\mathbf{E}[\mathrm{Err}^{(t)}] = O\Big(\exp(-c\sqrt{N}) + \frac{1}{d(t)}\Big(\frac{N}{M}\Big)^{2c} + \frac{1}{\min\{d(t),N\}^c d(t)^{1-c} M^c} + \frac{1}{d(t)M^c}\Big). \tag{71}$$

When summing over t and using that $M = N^2$, we deduce that

$$\mathbf{E}\Big[\sum_t \mathrm{Err}^{(t)}\Big] = O(N^{-c}).$$

Overall, we find that

$$\mathbf{E}[\mathrm{R}^{(2M+1)}] - \mathbf{E}[\mathrm{R}^{(0)}] = \mathbf{E}[\Delta^{\mathrm{err}}\mathrm{R}] + \mathbf{E}[\Delta^{\mathrm{IIC}}\mathrm{R}] = v_{\mathrm{R}} + O(N^{-c}).$$

Now, define a similar sequence of graphs $\widetilde{\mathbb{B}}^{(t)}$ for $0 \le t \le M$ by setting $\widetilde{\mathbb{B}}^{(0)} = \mathbb{B}^{(0)}$ and obtaining $\widetilde{\mathbb{B}}^{(t+1)}$ from $\widetilde{\mathbb{B}}^{(t)}$ by performing the track-exchange between s_{M-1-t} and t_α. Also, define a Markov chain $(\widetilde{\mathrm{R}}^{(t)})_{0\le t\le M}$ as before. Following again the same reasoning as in the previous sections, and observing that the behaviour of $\widetilde{\mathrm{R}}^{(t)}$ under the track-exchange of vertical tracks is the same as the behaviour of the top of a cluster when exchanging horizontal tracks, we obtain using a reasoning similar to Propositions 6.12 and 6.8 (with the same adaptation as above) that

$$\mathbf{E}[\widetilde{\mathrm{R}}^{(M)}] - \mathbf{E}[\widetilde{\mathrm{R}}^{(0)}] = v_{\mathrm{T}} + O(N^{-c}) + O(\phi^0_{\mathbb{B}^{(0)}}[e^{-c|\mathrm{R}^{(0)}|}]) = v_{\mathrm{T}} + O(N^{-c})$$

(in the second equality we used (67)). Here $v_{\rm T}$ refers to passing a track with transverse angle $\alpha+\pi/2$, or equivalently $\pi/2-\alpha$ by symmetry.

By definition, $\mathrm{R}^{(0)}$ and $\widetilde{\mathrm{R}}^{(0)}$ have the same law. Observe that $\mathbb{B}^{(2M+1)}=\widetilde{\mathbb{B}}^{(M)}$ and since our transformations ensure that the random-cluster law is preserved, $\widetilde{\mathrm{R}}^{(M)}$ has the same law as $\mathrm{R}^{(2M+1)}$. Thus,

$$v_{\rm R}=v_{\rm T}+O(N^{-c})=O(N^{-c}),$$

where in the last equality we used Proposition 6.16. Letting N go to infinity concludes the proof. □

6.6 Proof of Theorem 2.3

We start with a lemma gathering the estimates obtained on the increments of the extrema.

Lemma 6.18 (Nails do not move). *There exist $c_0,C_0\in(0,\infty)$ such that for every N,*

$$\mathbf{P}[\exists i\in I^{(0)},\exists t\le\tau_{\rm end}(i),\exists A\in\{\mathrm{T},\mathrm{B},\mathrm{R}\},|A^{(t)}(i)-A^{(0)}(i)|\ge N^{1-c_0}]\le\frac{C_0}{N^{c_0}}.$$

Proof. First of all, observe that it is sufficient to control the increments at compounded steps τ_k since $A^{(t)}(i)$ is between $A^{(\tau_k)}(i)$ and $A^{(\tau_{k+1}\wedge\tau_{\rm end}(i))}(i)$ for every $t\in[\tau_k,\tau_{k+1}]$. For this reason, we only focus on compounded steps and introduce the time $\tau'_{\rm end}(i)$ denoting the integer k such that $\tau_k\le\tau_{\rm end}(i)<\tau_{k+1}$. For each $i\in I^{(0)}$, introduce the processes indexed by integer times $0\le K<\lceil 2N/\sin\alpha\rceil$,

$$\begin{aligned}\mathbf{\Sigma}_{A,i}(K)&:=\sum_{k=0}^{K\wedge\tau'_{\rm end}(i)}\mathbf{E}[\Delta^{\rm IIC}A^{(k)}(i)|\mathcal{F}_k],\\ \mathbf{M}_{A,i}(K)&:=\sum_{k=0}^{K\wedge\tau'_{\rm end}(i)}\Delta^{\rm IIC}A^{(k)}(i)-\mathbf{\Sigma}_{A,i}(K),\\ \mathbf{\Delta}^{\rm err}A(i,K)&:=\sum_{k=0}^{K\wedge\tau'_{\rm end}(i)}|\Delta^{\rm err}A^{(k)}(i)|.\end{aligned}$$

We now bound the probability that each one of these processes is large, which by (i) of Propositions 6.12–6.14 will bound the probability that $|A^{(\tau_{K+1})}-A^{(0)}|$ is large.

Below, the constants $c,C\in(0,\infty)$ are introduced to satisfy Propositions 3.4, 6.8, 6.12, and 6.14. They are fixed all along the proof. The other constants c_i,C_i are independent of everything and should be thought of as being respectively much smaller than c and much larger than C.

We start with the easiest process, which is the last one. Note that the process is increasing and non-negative. Markov's inequality and Propositions 6.12–6.14(iii) imply that for every $i\in I^{(0)}$,

$$\begin{aligned}\mathbf{P}[\mathbf{\Delta}^{\rm err}A(i,\lceil 2N/\sin\alpha\rceil-1)\ge N^{1-c_0}|\mathcal{F}_0]&\le\frac{1}{N^{1-c_0}}\mathbf{E}[\mathbf{\Delta}^{\rm err}A(i,\lceil 2N/\sin\alpha\rceil-1)|\mathcal{F}_0]\\&\le\frac{C}{N^{1-c_0}}\mathbf{E}\Big[\sum_{0\le t<T}M^{(t)}(i)\Big|\mathcal{F}_0\Big].\end{aligned}$$

Summing over on $i \in I^{(0)}$, averaging on $\mathcal{F}_0$ gives that

$$\mathbf{P}\Big[\exists A, \exists i \in I^{(0)}, \exists K : \mathbf{\Delta}^{\mathrm{err}} A(i,K) \geq N^{1-c_0}\Big] \leq \frac{C}{N^{1-c_0}} \mathbf{E}\Big[\sum_{0\leq t<T} \mathrm{Err}^{(t)}\Big] \leq \frac{C_1 \log N}{N^{c-c_0}}, \quad (72)$$

where in the second inequality we used Proposition 6.8(iii).

Let us now turn to the second process, which is a martingale with increments that have uniform exponential moments because of Propositions 6.12–6.14(ii). We deduce from a trivial modification of the Azuma-Hoeffding inequality (to accommodate the unbounded increments, simply truncate the martingale increments at N^{c_1} and bound the error by the probability that there exists a single increment larger than N^{c_1}) that for every $i \in I^{(0)}$,

$$\mathbf{P}\Big[\exists A : \max_K |\mathbf{M}_{A,i}(K)| > N^{3/4} \,\Big|\, \mathcal{F}_0\Big] \leq \exp[-c_2 N^{c_2}].$$

By averaging on $\mathcal{F}_0$ and using that $|I^{(0)}|$ has uniformly bounded expectation (by Proposition 3.7), we deduce that

$$\mathbf{P}\Big[\exists A, \exists i \in I^{(0)} : \max_K |\mathbf{M}_{A,i}(K)| > N^{3/4}\Big] \leq \exp[-c_3 N^{c_3}].$$

It only remains to prove the following inequality:

$$\mathbf{P}[\exists A, \exists i \in I^{(0)}, \max_K |\mathbf{\Sigma}_{A,i}(K)| \geq \tfrac{1}{2} N^{1-c_0}] \leq N^{-c_4}. \quad (73)$$

In order to prove this, let $\mathbf{N}_1$ be the number k such that there exists $i \in I^{(\tau_k)}$ and $A \in \{\mathrm{T}, \mathrm{B}\}$ such that $|A^{(\tau_k)}(i) - b(\tau_k)| \leq N^{c_5}$, and $\mathbf{N}_2$ the number of k such that $I^{(\tau_k)} \not\subset I^{(\tau_k+1)}$.

Propositions 6.12–6.14(iv) give that for every A, i, and K,

$$|\mathbf{\Sigma}_{A,i}(K)| \leq N \exp(-cN^{c_5}) + C_1 \mathbf{N}_1 + C_1 \sum_{0\leq k<\lceil 2N/\sin\alpha\rceil} \mathbf{P}[\tau'_{\mathrm{end}} = k | \mathcal{F}_k].$$

Since the last term on the right has an expectation which is bounded by the expectation of $\mathbf{N}_2$, the Markov property implies that for $c_0 < c$ and N large enough,

$$\mathbf{P}[\exists A, \exists i \in I^{(0)}, \max_K |\mathbf{\Sigma}_{A,i}(K)| \geq \tfrac{1}{2} N^{1-c_0}] \leq \frac{2C_1 \mathbf{E}[\mathbf{N}_1 + \mathbf{N}_2]}{N^{1-c_0}} + 2N^{c_0} \exp(-cN^{c_5}). \quad (74)$$

Yet, for each time t it is a direct consequence of Propositions 3.4 that for c_5 sufficiently small,

$$\mathbf{P}[\exists i \in I^{(t)}, \exists A \in \{\mathrm{T}, \mathrm{B}\} : |A^{(t)}(i) - b(t)| \leq N^{c_5}] \leq C_2 N^{c_5 - c}$$

so $\mathbf{E}[\mathbf{N}_1] \leq C_3 N^{1+c_5-c}$.

Now, pick $c_6 < c/2$. To have $I^{(\tau_k)} \not\subset I^{(\tau_{k+1})}$, it must be that one of the following three things occurs:

- there exists $i \in I^{(\tau_k)}$ with $\mathrm{Vspan}^{(\tau_k)}(i) \leq \varepsilon N + 2N^{1-c_6}$ or $\max\{|\mathrm{T}^{(\tau_k)}(i) - x_2|, |\mathrm{B}^{(\tau_k)} - x_2|, |\mathrm{R}^{(\tau_k)}(i) - x_1|\} \geq \sqrt{\eta\varepsilon} N - 2N^{1-c_6}$,

- $\Delta^{\rm err} A^{(k)}(i) \geq N^{1-c_6}$ for some A and $i \in I^{(\tau_k)}$,
- $\Delta^{\rm IIC} A^{(k)}(i) \geq N^{1-c_6}$ for some A and $i \in I^{(\tau_k)}$.

Using Propositions 3.4 again, the first item occurs with probability $O(N^{c_6-c})$. The second item occurs with probability $O(N^{c_6-c})$ by the same computation as (72). The last item occurs with probability $O(\exp(-cN^{1-c_6}))$ by (ii) of Propositions 6.12 and 6.14. The bound $c_6 < c/2$ gives

$$\mathbf{E}[\mathbf{N}_2] \leq C_4 N^{1-c_6}.$$

Moreover, by picking $c_0 \ll c_i$ small enough and plugging the two expectation estimates into (74) implies (73). This concludes the proof. $\square$

We now turn to a second lemma stating that with large probability, marked nails exist near every $x \in \mathbb{B}_\eta(N)$ at time 0, or in other words when defining $I_\bullet$ at the first step of the coupling, we get $|I_\bullet| = |\mathbb{B}_\eta(N)|$.

Lemma 6.19 (Nails exist). *There exist $c, C \in (0,\infty)$ such that for every $0 < \varepsilon \ll \eta$,*

$$\mathbf{P}[|I_\bullet| = |\mathbb{B}_\eta(N)|] \geq 1 - \frac{C}{\eta^2}\Big(\frac{\varepsilon}{\eta}\Big)^c.$$

Proof. Set $\kappa := (\eta\varepsilon^3)^{1/4}$. There are $O(1/\eta^2)$ elements in $\mathbb{B}_\eta(N)$. Furthermore, for fixed $x \in \mathbb{B}_\eta(N)$, the non-existence of a "markable" nail near x requires the existence of a dual path from $\Lambda_{2\varepsilon N}(x)$ to $\Lambda_{\kappa N}(x)$ or a primal path from $\Lambda_{\kappa N}(x)$ to $\Lambda_{((\eta\varepsilon)^{1/2}-\varepsilon)N}(x)$ (otherwise there exists a primal circuit in the annulus $\Lambda_{\kappa N}(x) \setminus \Lambda_{2\varepsilon N}(x)$ that is not connected to $\partial\Lambda_{((\eta\varepsilon)^{1/2}-\varepsilon)N}(x)$ and therefore constitutes a nail at x that we may mark. Using (9) and the assumption that $\varepsilon \ll \eta$ concludes the proof. $\square$

Proof of Theorem 2.3. We start by assuming that $\cos\alpha \notin \mathbb{Q}$. Consider $1 \gg \eta \gg \varepsilon > 0$ and assume in particular that $C\varepsilon^c/\eta^{2+c} \leq \eta/2$, where c and C are the constants of Lemma 6.19. Also, we assume N is large enough that $C_0/N^{c_0} \leq \eta/2$, where c_0 and C_0 are the constants of Lemma 6.18.

The two previous lemmata imply immediately that provided that ε is sufficiently small with probability $1-\eta$, marked nails exist near all points in $\mathbb{B}_\eta(N)$ and belong to $I^{(t)}$ for every $0 \leq t < T$. By Proposition 6.8(i), we deduce that $([\cdot]^{(T)}_{\bullet,0}, [\cdot]^{(T)}_{\bullet,1}) = ([\cdot]^{(0)}_{\bullet,0}, [\cdot]^{(0)}_{\bullet,1})$.

Now, the homotopy classes with respect to $\mathbb{B}_\eta(N)$ and with respect to marked nails are equal for any loop that remains at a distance $\sqrt{\eta\varepsilon}N$ of $\mathbb{B}_\eta(N)$. Since all loops (in $\overline{\omega}^{(0)}$ and $\overline{\omega}^{(T)}$) surrounding at least two but not all points in $\mathbb{B}_\eta(N)$ have a diameter which is larger than ηN, (RSW) immediately implies that they satisfy the previous property with probability larger than $1-\eta$ provided $\varepsilon = \varepsilon(\eta) > 0$ is chosen small enough.

In particular, when setting $\eta_0 = \sqrt{\eta}$ and assuming that $2\eta \leq \eta_0$, we obtain that the rescaled configurations $\omega_\delta^{(0)}$ and $\omega_\delta^{(T)}$ satisfy

$$\mathbf{P}[d_{\mathbf{H}}(\omega_\delta^{(0)}, \omega_\delta^{(T)}) \leq \eta_0] \leq 2\eta \leq \eta_0.$$

It remains to observe that thanks to properties of the track-exchange operators (see Remark 3.12), the law of the homotopy classes around $\mathbb{B}_\eta(N)$ is the same under $\phi_{\mathbb{L}(\pi/2)}$

and $\phi_{\mathbb{L}^{(0)}}$ (and similarly under $\phi_{\mathbb{L}(\alpha)}$ and $\phi_{\mathbb{L}^{(T)}}$). As a consequence, we may construct a coupling between $\widetilde{\omega}_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$ and $\widetilde{\omega}'_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$ by first using Remark 3.12 to couple ω_δ and $\omega_\delta^{(0)}$ in such a way that the homotopy classes of loops surrounding one but not all points in $\mathbb{B}_\eta$ are the same, then use the coupling constructed above, and finally couple $\omega_\delta^{(T)}$ with ω'_δ using Remark 3.12 again. Overall, we exactly proved that the rescaled version of $\mathbf{P}$ satisfies the properties of the statement of our theorem for η_0, so the proof is finished.

To get the result for $\cos\alpha$ rational, simply take the coupling obtained as the weak limit of couplings with α_n satisfying $\cos\alpha_n \notin \mathbb{Q}$ and tending to α. One easily checks that the limit makes sense and satisfies all the requested properties as the bounds are continuous in α (note that one may also directly define the coupling in this setting, being careful with the vertical position of right-most points, see Remark 6.15 again). We insist that this limit should be taken at N (or equivalently $\delta > 0$) fixed.

□

7 Proofs of the main theorems

7.1 Proofs of the results for the random-cluster model

Proof of Theorem 1.2. We prove the result for the Schramm-Smirnov topology but a similar proof works for the Camia-Newman one. By Theorem 2.2, it suffices to construct a coupling of $(\omega_\delta, \omega'_\delta)$ with $\omega, \omega' \sim \phi_{\delta\mathbb{L}(\pi/2)}$ for which the distance $d_{\mathbf{H}}(\omega_\delta, e^{i\alpha}\omega'_\delta)$ is typically small.

Case of $\Omega = \mathbb{R}^2$ We start with a coupling on the full space $\delta\mathbb{L}(\pi/2)$. Let σ_u be the reflection with respect to the line $e^{iu}\mathbb{R}$.

Fix $\varepsilon > 0$ and choose $\eta \le \varepsilon/4$ so that Theorem 2.2 implies that for every coupling of $\omega_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$ and $\omega'_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$, we have

$$\mathbf{P}[d_{\mathbf{H}}(\omega_\delta, e^{i\alpha}\omega'_\delta) \le \eta, d_{\mathbf{SS}}(\omega_\delta, e^{i\alpha}\omega'_\delta) > \tfrac{\varepsilon}{2}] \le \tfrac{\varepsilon}{4}. \tag{75}$$

Now, construct an explicit coupling $\mathbb{P}$ between $\omega_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$ and $\omega'_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$ as follows: sample $\omega'_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$ and couple $\sigma_0\omega'_\delta$ with $\omega_\delta^\alpha \sim \phi_{\delta\mathbb{L}(\alpha)}$ using Theorems 2.2 and 2.3 (this is doable since $\sigma_0\omega'_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$) and (75) in such a way that

$$\mathbb{P}[d_{\mathbf{SS}}(\sigma_0\omega'_\delta, \omega_\delta^\alpha) \ge \tfrac{\varepsilon}{2}] \le \tfrac{\varepsilon}{2}, \tag{76}$$

then, couple $\sigma_{\alpha/2}\omega_\delta^\alpha$ with $\omega_\delta \sim \phi_{\delta\mathbb{L}(\pi/2)}$ by Theorems 2.2 and 2.3 (this is doable since $\sigma_{\alpha/2}\omega_\delta^\alpha \sim \phi_{\delta\mathbb{L}(\alpha)}$) in such a way that

$$\mathbb{P}[d_{\mathbf{SS}}(\omega_\delta, \sigma_{\alpha/2}\omega_\delta^\alpha) \ge \tfrac{\varepsilon}{2}] \le \tfrac{\varepsilon}{2}. \tag{77}$$

Since

$$\begin{aligned} d_{\mathbf{SS}}(\omega_\delta, e^{i\alpha}\omega'_\delta) &= d_{\mathbf{SS}}(\omega_\delta, \sigma_{\alpha/2}\sigma_0\omega'_\delta) = d_{\mathbf{SS}}(\sigma_{\alpha/2}\omega_\delta, \sigma_0\omega'_\delta) \\ &\le d_{\mathbf{SS}}(\sigma_{\alpha/2}\omega_\delta, \omega_\delta^\alpha) + d_{\mathbf{SS}}(\omega_\delta^\alpha, \sigma_0\omega'_\delta) = d_{\mathbf{SS}}(\omega_\delta, \sigma_{\alpha/2}\omega_\delta^\alpha) + d_{\mathbf{SS}}(\omega_\delta^\alpha, \sigma_0\omega'_\delta). \end{aligned} \tag{78}$$

The result then follows by combining (76)–(78).

Case of a bounded simply connected domain Ω with C^1-smooth boundary To obtain the result in a finite domain, we use the domain Markov property and the fact that one may approximate $\phi^0_{\Omega_\delta}$ by asking that there exists a loop Γ within distance η of $\partial\Omega$ in the infinite-volume measure. More precisely, let $A(\Omega,\eta)$ be the event that there exists a loop $\mathbf{\Gamma}\in\mathcal{F}_0(\omega_\delta)$ which is included in Ω and such that $d(\mathbf{\Gamma},\partial\Omega)\le\eta$ (d is the distance between loops defined in the introduction). Note that whether $A(\Omega,\eta)$ occurs or not can be measured in the Schramm-Smirnov topology (we leave this as an exercise).

Now, fix $\varepsilon_0>0$. We use the characterization of the Schramm-Smirnov distance provided in [32, Proposition 3.9]. There exists a family of non-degenerate quads $Q_1,\dots,Q_n$ in Ω such that if the sets of quads in $Q_1,\dots,Q_n$ that are crossed are the same in ω_δ and ω'_δ, then $d_{\mathbf{SS}}(\omega_\delta,\omega'_\delta)\le\varepsilon_0$. In particular, we deduce that if $H_{\vec Q}(I)$ denotes the event that Q_i is crossed if and only if $i\in I$, then there exists a coupling $\mathbf{P}$ between $\omega_\delta\sim\phi^0_{\Omega_\delta}$ and $\omega'_\delta\sim\phi^0_{e^{i\alpha}\Omega_\delta}$ such that

$$\mathbf{P}[d_{\mathbf{SS}}(\omega_\delta,\omega'_\delta)\ge\varepsilon_0]\le\varepsilon_0$$

if and only if for every $I\subset\{1,\dots,n\}$,

$$|\phi^0_{\Omega_\delta}[H_{\vec Q}(I)]-\phi^0_{e^{i\alpha}\Omega_\delta}[H_{e^{i\alpha}\vec Q}(I)]|\le\varepsilon_0/2^n=:\varepsilon. \tag{79}$$

Now, the infinite volume result above implies that for every $\delta<\delta_0(\Omega,\eta,\varepsilon)$,

$$|\phi_{\delta\mathbb{Z}^2}[H_{\vec Q}(I)|A(\Omega,\eta)]-\phi_{\delta\mathbb{Z}^2}[H_{e^{i\alpha}\vec Q}(I)|A(e^{i\alpha}\Omega,\eta)]|\le\tfrac12\varepsilon. \tag{80}$$

We therefore wish to prove that

$$|\phi^0_{\Omega_\delta}[H_{\vec Q}(I)]-\phi_{\delta\mathbb{Z}^2}[H_{\vec Q}(I)|A(\Omega,\eta)]|\le\tfrac12\varepsilon. \tag{81}$$

The same can be done for the rotated version, so that the previous displayed equations imply (79) and conclude the proof.

To get (81), let $\mathbf{\Omega}_\delta$ be the interior of the outer-most loop in $\mathcal{F}_0(\omega)$ satisfying the conditions of $A(\Omega,\eta)$. Using the spatial Markov property, it suffices to show that

$$|\phi^0_{\Omega_\delta}[H_{\vec Q}(I)]-\phi^0_{\mathbf{\Omega}_\delta}[H_{\vec Q}(I))]|\le\tfrac12\varepsilon. \tag{82}$$

Note that there is a clear increasing coupling between $\boldsymbol{\omega}_\delta\sim\phi^0_{\mathbf{\Omega}_\delta}$ and $\omega_\delta\sim\phi^0_{\Omega_\delta}$ ($\boldsymbol{\omega}_\delta\le\omega_\delta$ because of $\mathbf{\Omega}_\delta\subset\Omega_\delta$), so that for ω_δ to belong to $H_{\vec Q}(I)$ but not $\boldsymbol{\omega}_\delta$ or vice versa, it must be that one of the quads Q_i must be crossed in one but not in the other. We deduce that it suffices to show that for every possible realization of $\mathbf{\Omega}_\delta$,

$$\phi^0_{\Omega_\delta}[\mathcal{C}(Q_i)]-\phi^0_{\mathbf{\Omega}_\delta}[\mathcal{C}(Q_i)]\le\tfrac{1}{2n}\varepsilon. \tag{83}$$

Therefore, the result boils down to the following.

Claim *For every $\epsilon > 0$, every bounded simply connected domain Ω with C^1-smooth boundary, and every quad Q inside Ω, there exists $\eta = \eta(\Omega, Q, \epsilon) > 0$ such that for every $\Omega' \subset \Omega$ with $d(\partial\Omega', \partial\Omega) \le \eta$,*

$$\phi^0_{\Omega_\delta}[\mathcal{C}(Q)] \le \phi^0_{\Omega'_\delta}[\mathcal{C}(Q)] + \epsilon$$

for δ small enough.

Proof. We only sketch the proof. Consider first the "epigraph" domains indexed by continuous functions f from $[-2, 2]$ to $\mathbb{R}$ given by

$$\Omega(f) := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-2, 2), f(x_1) < x_2 < 2\}$$

(see Figure 29). Define $\Lambda := [-1, 1]^2$. For $\alpha > 0$, a straightforward yet quite lengthy application of the techniques developed[13] in [24, Lemma 5.3] implies that for every $f \le -2$ and $1 \le k \le \frac{1}{2}\lfloor 1/\alpha \rfloor =: K$,

$$\phi^0_{\Omega(f)_\delta}[\mathcal{C}(\Lambda)] - \phi^0_{\Omega(f+\alpha)_\delta}[\mathcal{C}(\Lambda)] \le C\big(\phi^0_{\Omega(f+k\alpha)_\delta}[\mathcal{C}(\Lambda)] - \phi^0_{\Omega(f+(k+1)\alpha)_\delta}[\mathcal{C}(\Lambda)]\big).$$

Summing over $1 \le k \le K$, we deduce that

$$\phi^0_{\Omega(f)_\delta}[\mathcal{C}(\Lambda)] - \phi^0_{\Omega(f+\alpha)_\delta}[\mathcal{C}(\Lambda)] \le \frac{C}{K}\big(\phi^0_{\Omega(f)_\delta}[\mathcal{C}(\Lambda)] - \phi^0_{\Omega(f+K\alpha)_\delta}[\mathcal{C}(\Lambda)]\big) \le \frac{C}{K} \le 4C\alpha. \quad (84)$$

Note that a similar argument works for any rotation, translate, or rescaling of the domains above.

We now use our assumption that $\partial\Omega$ is C^1-smooth. Since $\partial\Omega$ is given by a curve γ which is C^1 and has non-vanishing differential, one may find (see Figure 29) constants $\kappa = \kappa(\Omega) > 0$ and $C = C(\Omega) > 0$, functions $f_s : [-2, 2] \to (-\infty, -2]$ and $T_s : \mathbb{R}^2 \to \mathbb{R}^2$ for $1 \le s \le S$, where S depends on Ω (through the modulus of continuity of the derivative for the function parametrizing $\partial\Omega$) but not on η, satisfying the following properties:

- T_s is the composition of a rotation, a translation, and the multiplication by κ;
- $T_s(\Omega(f_s))$ is included in Ω for every s;
- for all η small enough, $\{x \in \Omega : d(x, \Omega^c) \le \eta\}$ is included in the union of the sets

$$A_s := T_s(\{x = (x_1, x_2) : x_1 \in [-1, 1], f(x_1) < x_2 < f(x_1) + C\eta\}).$$

Introducing the domains $\Omega_s := \Omega \setminus \bigcup_{t=1}^s A_t$, and using again [24] for the first and second inequalities, one can prove the existence of $C_i = C_i(\Omega, Q, \kappa) > 0$ such that

$$\begin{aligned}\phi^0_{\Omega_{s-1}}[\mathcal{C}(Q)] - \phi^0_{\Omega_s}[\mathcal{C}(Q)] &\le C_1\big(\phi^0_{\Omega_{s-1}}[\mathcal{C}(T_s(\Lambda))] - \phi^0_{\Omega_s}[\mathcal{C}(T_s(\Lambda))]\big) \\ &\le C_2\big(\phi^0_{\Omega(f_s)}[\mathcal{C}(\Lambda)] - \phi^0_{\Omega(f_s+C\eta)}[\mathcal{C}(\Lambda)]\big) \\ &\le C_3\eta,\end{aligned} \quad (85)$$

[13] The whole fo Section 4 of [24] should be adapted to finite domains and considering the covariance of crossing events with edges on the boundary of the domain.

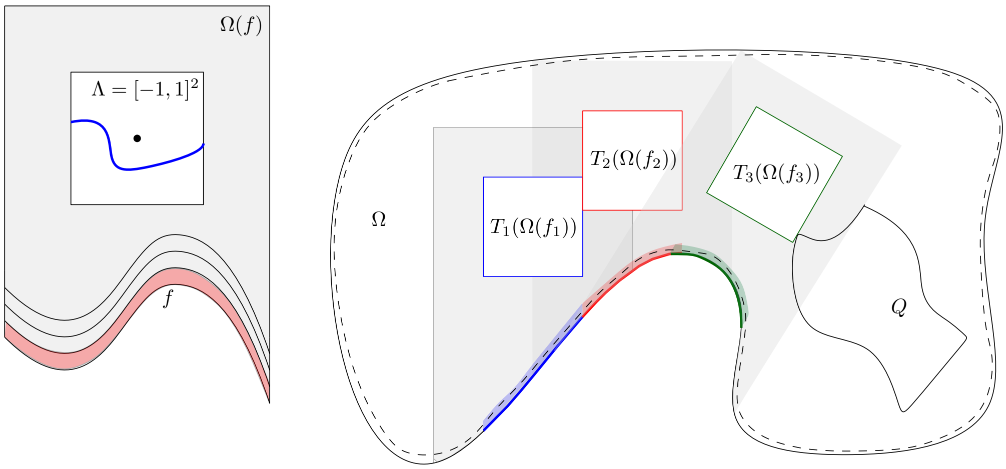

Figure 29: On the left, an example of a domain $\Omega(f)$. Note that the sets $\Omega(f + k\alpha)$ have a nested structure (the red part denotes $\Omega(f) \setminus \Omega(f + \alpha)$). On the right, the impact of changing the boundary is compared with the impact of changing the boundary in a family of subdomains which are images by simple transformations of domains $\Omega(f)$ (with potentially different functions f). The existence of such a decomposition is made possible by the fact that the boundary of Ω is C^1.

where the last line is due to (84) applied to $\alpha = C\eta$.

Choose $\eta = \eta(\Omega, \epsilon, S) > 0$ small enough. Summing (85) over s gives

$$\phi^0_{\Omega_\delta}[\mathcal{C}(Q)] - \phi^0_{\Omega'_\delta}[\mathcal{C}(Q)] \leq \sum_{s=0}^{S-1} \phi^0_{\Omega_{s-1}}[\mathcal{C}(Q)] - \phi^0_{\Omega_s}[\mathcal{C}(Q)] \leq \epsilon.$$

□

Case of a (possibly unbounded) simply connected domain Ω with C^1-smooth boundary For every $\varepsilon > 0$, to determine the Schramm-Smirnov distance up to a precision of $\varepsilon > 0$, only quads in $B(0, 1/\varepsilon)$ need to be considered. Consider a bounded domain $\Omega^{(\varepsilon)}$ that coincides with Ω on $B(0, 1/\varepsilon^C)$. By the mixing property, one has that for every $\delta > 0$ and every event E depending on edges in $\delta\mathbb{Z}^2 \cap B(0, 1/\varepsilon)$ only,

$$|\phi_{\Omega^{(\varepsilon)}_\delta}[E] - \phi_{\Omega_\delta}[E]| \leq C_{\text{mix}} \varepsilon^{c_{\text{mix}}(C-1)} \phi_{\Omega_\delta}[E].$$

Now, take the domain $\Omega^{(\varepsilon)}$ very large but finite, equal to Ω up to large distance. Using the invariance by rotation in $\Omega^{(\varepsilon)}_\delta$ and taking δ to 0 then ε to 0 concludes the proof. □

Proof of Corollary 1.3. When one considers a quad Q that remains at a distance at least ε of the boundary of Ω, the result follows directly from Theorem 1.2 and the measurability of $\mathcal{C}(Q)$ in the Schramm-Smirnov topology (note that the event gets rewritten as $Q \in \omega$ when ω is seen as an element of $\mathcal{H}$).

Now, when $1 \leq q < 4$, to get the result without any assumption on the distance to the boundary, note that for a quad Q, there exists a quad Q' that is such that its distance to $\partial\Omega$ is at least ε, and which is in Hausdorff distance at a distance at most 2ε from Q. Using the strong version of crossing estimates from [25], we obtain easily (this type of reasoning is now classical, see for instance [24, Lemma 3.12] for an example) that

$$|\phi_{\Omega_\delta}[\mathcal{C}(Q)] - \phi_{\Omega_\delta}[\mathcal{C}(Q')]| \leq C\varepsilon^c$$

for two constants $C > 0$ and $c > 0$. The result follows readily by first choosing ε small enough and then letting δ tend to zero and use the rotational invariance result for Q'. □

Proof of Corollary 1.4. We use a conditional mixing argument due to Garban, Pete, and Schramm [34, Section 3] in the case of Bernoulli percolation and that can be extended to the random-cluster model using crossing estimates. Consider the *Euclidean* ball B_n of radius n, and its boundary ∂B_n. Introduce the quantities

$$\epsilon(n, N) := \phi^0_{\mathbb{Z}^2}[0 \longleftrightarrow B_N^c | B_n \longleftrightarrow B_N^c] \quad \text{and} \quad \epsilon(n) := \lim_{N\to\infty} \epsilon(n, N).$$

The statement of conditional mixing from [34] implies the following claim (in [34] it is stated for the four-arm event, but a similar – in fact simpler – argument can be performed for the one-arm event, see e.g. Proposition 5.3 of the same paper). For every $\beta, \varepsilon > 0$, there exists $\eta = \eta(\beta, \varepsilon) > 0$ such that for every Ω and every $x_1, \ldots, x_n$ at a distance ε of each other and of the boundary, and every partition P of $(x_1, \ldots, x_n)$,

$$\Big|\phi^0_{\Omega_\delta}[\mathcal{E}(P, x_1, \ldots, x_n)] - \epsilon(\tfrac{\eta}{\delta})^n \phi^0_{\Omega_\delta}[\mathcal{E}(P, B_{\eta/\delta}(x_1), \ldots, B_{\eta/\delta}(x_n))]\Big| \leq \beta\phi^0_{\Omega_\delta}[\mathcal{E}(P, x_1, \ldots, x_n)],$$

where $\mathcal{E}(P, B_{\eta/\delta}(x_1), \ldots, B_{\eta/\delta}(x_n))$ is the event that the balls $B_{\eta/\delta}(x_i)$ are connected to each other if and only if they belong to the same element of the partition P. The same formula applies in the rotated measure.

We conclude, by observing that $e^{i\alpha} B_{\eta/\delta}(x_i)$ and $B_{\eta/\delta}(e^{i\alpha} x_i)$ are equal, and that the event $\mathcal{E}(P, B_{\eta/\delta}(x_1), \ldots, B_{\eta/\delta}(x_n))$ is measurable in the Schramm-Smirnov topology, so that its probability or the probability of its rotation by an angle of α are close to each other by Theorem 1.2. □

7.2 Proofs of the theorems for the other models

Proof of Corollary 1.6. Fix $\tau_1, \ldots, \tau_n \in \mathbb{T}_q$. Let $I_i \subset \{x_1, \ldots, x_n\}$ be the sets of x_j such that $\tau_j = i$ and call a partition $P = (P_1, \ldots, P_k)$ of $\{x_1, \ldots, x_n\}$ *compatible with* τ if each P_j is included in one of the I_i. Also, let $|P| = k$ be the number of elements in the partition. The Edwards-Sokal coupling implies that

$$\mu_{\Omega_\delta}[\sigma_{x_i} = \tau_i, 1 \leq i \leq n] = \sum_{\text{compatible } P} q^{-|P|} \phi^0_{\Omega_\delta}[\mathcal{E}(P, x_1, \ldots, x_n)].$$

We deduce the corollary by using Corollary 1.4. □

Proof of Corollary 1.8. Fix $x_1, \dots, x_{2n} \in (\delta\mathbb{Z}^2)^*$ and let $\Gamma_1, \dots, \Gamma_{2n}$ be the exterior-most loops in $\overline{\omega}_\delta$ that surround one x_i but not the others. Also, let Γ be the smallest loop surrounding all the x_i. Finally, let $\overline{\mathcal{F}}_\delta = \overline{\mathcal{F}}_\delta(x_1, \dots, x_{2n})$ be the set of loops in $\overline{\omega}_\delta$ that surround at least one of the Γ_i and are surrounded by Γ (including Γ and all Γ_i).

Also, let $\mathbf{N}$ be the number of loops surrounding the origin and introduce, for a curve γ and $\delta > 0$,

$$c_\delta(\gamma) := \phi^0_{\overline{\gamma}_\delta}[\mathbf{N}] - \phi^0_{B_\delta}[\mathbf{N}],$$

where $\overline{\gamma}$ is the domain surrounded by γ.

It is shown in [22] that there exits a function F taking possible realizations of $\overline{\mathcal{F}}_\delta$ as arguments and outputting a complex number such that

- $\mathbb{E}^{6V}_{\delta\mathbb{Z}^2}\big[\prod_{i=1}^n (h_{x_{2i}} - h_{x_{2i-1}})\big] = \phi_{\delta\mathbb{Z}^2}[F(\overline{\mathcal{F}}_\delta)]$,
- $F(\overline{\mathcal{F}}_\delta) = G(\overline{\mathcal{F}}_\delta, c_\delta(\Gamma_1), \dots, c_\delta(\Gamma_{2n}))$ where $G(\overline{\mathcal{F}}_\delta, c_1, \dots, c_{2n})$ is a function that depends on $\overline{\mathcal{F}}_\delta$ only through the inclusions between the different loops.
- $|F(\overline{\mathcal{F}}_\delta)| \le C(\log|\overline{\mathcal{F}}_\delta|)^n$ where $|\overline{\mathcal{F}}_\delta|$ is the number of loops in $\overline{\mathcal{F}}_\delta$.

Now, the conditional mixing from [34] easily implies that $c_\delta(\Gamma) - c_\delta(e^{i\alpha}\Gamma)$ tends to 0 as δ tends to 0. So we deduce the result from Theorem 1.2 for the Camia-Newman distance using dominated convergence. One may be slightly worried about the fact that $c_\delta(\Gamma)$ is not bounded from below and can be very negative, but this is treated by the third point: for $|F(\overline{\mathcal{F}}_\delta)|$ to exceed $\log k$, there must be at least k loops in $\overline{\mathcal{F}}_\delta$, an event which occurs with probability smaller than $C_n k^{-c}$ for some constant $c > 0$ independent of everything. □

Remark 7.1. The previous connection between the six-vertex model on the medial lattice and the random-cluster model on the primal one extends to generic weights a and b. The outcome is a random-cluster model with isoradial weights. When choosing the corresponding isoradial embedding for the primal lattice, and defining the six-vertex model on the associated medial lattice, one obtains a rotational invariance for every a, b, c with $\Delta := (a^2 + b^2 - c^2)/2ab \in [-1, -1/2]$.

Acknowledgments The first author was funded by the ERC CriBLaM. The third and fourth authors were funded by the Swiss FNS. The first, third, fourth and fifth authors were partially funded by the NCCR SwissMap and the Swiss FNS.

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