

Exponential decay in the loop $O(n)$ model: $n > 1$, $x < \frac{1}{\sqrt{3}} + \varepsilon(n)$

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Abstract

We show that the loop $O(n)$ model exhibits exponential decay of loop sizes whenever $n \geq 1$ and $x < \frac{1}{\sqrt{3}} + \varepsilon(n)$, for some suitable choice of $\varepsilon(n) > 0$.

It is expected that, for $n \leq 2$, the model exhibits a phase transition in terms of x , that separates regimes of polynomial and exponential decay of loop sizes. In this paradigm, our result implies that the phase transition for $n \in (1, 2]$ occurs at some critical parameter $x_c(n)$ strictly greater than that $x_c(1) = 1/\sqrt{3}$. The value of the latter is known since the loop $O(1)$ model on the hexagonal lattice represents the contours of spin-clusters of the Ising model on the triangular lattice.

The proof is based on developing n as $1 + (n - 1)$ and exploiting the fact that, when $x < \frac{1}{\sqrt{3}}$, the Ising model exhibits exponential decay on any (possibly non simply-connected) domain. The latter follows from the positive association of the FK-Ising representation.

1 Introduction

The loop $O(n)$ model was introduced in [7] as a graphical model expected to be in the same universality class as the spin $O(n)$ model. The latter is a generalisation of the seminal Ising model [16] that incorporates spins contained on the n -dimensional sphere. See [19] for a survey of both $O(n)$ models. For integers $n > 1$, the connection between the loop and the spin $O(n)$ models remains purely heuristic. Nevertheless, the loop $O(n)$ became an object of study in its own right; it is predicted to have a rich phase diagram [3] in the two real parameters $n, x > 0$. For $n = 0, 1, 2$ the loop $O(n)$ model is closely related to self-avoiding walk, the Ising model, and a certain random height model, respectively.

Let \mathbb{H} denote the hexagonal lattice. A *domain* is a subgraph $\mathcal{D} = (V_{\mathcal{D}}, E_{\mathcal{D}})$ of \mathbb{H} formed of the edges contained inside or along some simple cycle $\partial\mathcal{D} \subset E(\mathbb{H})$ (hereafter called a *loop*), and all endpoints of such edges. Write $F_{\mathcal{D}}$ for the set of faces of \mathbb{H} delimited by edges of \mathcal{D} only. A *loop configuration* is any element of $\{0, 1\}^{E_{\mathcal{D}}}$ that is even, which is to say that the degree of any vertex is 0 or 2 when ω is seen as a spanning subgraph of \mathcal{D} containing only edges e with $\omega(e) = 1$. As such ω may be seen as a set of disjoint

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loops on \mathcal{D} . Loops are allowed to run along the boundary edges, but may not terminate at boundary points.

For real $n, x > 0$, let $\mathbf{Loop}_{\mathcal{D}, n, x}$ be the measure on loop configurations given by

$$\mathbf{Loop}_{\mathcal{D}, n, x}(\omega) = \frac{1}{Z_{\text{loop}}(\mathcal{D}, n, x)} \cdot x^{|\omega|} n^{\ell(\omega)},$$

where $|\omega|$ is the number of edges in ω , $\ell(\omega)$ is the number of loops in ω and $Z_{\text{loop}}(\mathcal{D}, n, x)$ is a constant called the partition function, chosen so that $\mathbf{Loop}_{\mathcal{D}, n, x}$ is a probability measure.

We will consider that the origin 0 is a vertex of the hexagonal lattice and will always consider domains \mathcal{D} containing 0. We say that the loop $O(n)$ model with edge-weight x exhibits exponential decay of loop lengths if there exists $c > 0$ such that for any $k \geq 1$ and any domain \mathcal{D} ,

$$\mathbf{Loop}_{\mathcal{D}, n, x}[\mathbf{R} \geq k] \leq \exp(-ck), \quad (1.1)$$

where \mathbf{R} stands for the length of the biggest loop surrounding 0.

According to physics predictions [17, 3], the loop $O(n)$ model exhibits macroscopic loops (that is loops on every scale) when $n \in [1, 2]$ and $x \geq x_c(n) = \frac{1}{\sqrt{2+\sqrt{2-n}}}$. For all other values of n and x , the model is expected to exhibit exponential decay. Moreover, it was conjectured (see e.g. [15, Section 5.6]) that in the macroscopic-loops phase, the model has a conformally invariant scaling limit given by the Conformal Loop Ensemble (CLE) of parameter κ , where:

$$\kappa = \begin{cases} \frac{4\pi}{2\pi - \arccos(-n/2)} \in [\frac{8}{3}, 4] & \text{if } x = x_c(n), \\ \frac{4\pi}{\arccos(-n/2)} \in [4, 8] & \text{if } x > x_c(n). \end{cases}$$

In the present paper, we establish exponential decay for all $n > 1$ and $x < 1/\sqrt{3} + \varepsilon(n)$, where $\varepsilon(n)$ is a certain strictly positive function. This is in agreement with the predicted phase diagram.

Theorem 1.1. *For any $n > 1$, there exists $\varepsilon(n) > 0$ such that the loop $O(n)$ model exhibits exponential decay (1.1) for all $x < \frac{1}{\sqrt{3}} + \varepsilon(n)$.*

We note that prior to our work, the best known bound on the regime of exponential decay for $n > 1$ was $x < \frac{1}{\sqrt{2+\sqrt{2}}} + \varepsilon(n)$ [22], where $\sqrt{2+\sqrt{2}}$ is the connective constant of the hexagonal lattice computed in [10]. Also, in [9] it was shown that when n is large enough the model exhibits exponential decay for any value of $x > 0$.

Apart from the improved result, our paper provides a method of relating (some form of) monotonicity in x and n . See Section 5 for more details.

Existence of macroscopic loops was shown for $n \in [1, 2]$ and $x = x_c(n) = \frac{1}{\sqrt{2+\sqrt{2-n}}}$ in [8], for $n = 2$ and $x = 1$ in [12], and for $n \in [1, 1 + \varepsilon]$ and $x \in [1 - \varepsilon, \frac{1}{\sqrt{n}}]$ in [6]. Additionally, for $n = 1$ and $x \in [1, \sqrt{3}]$ (which corresponds to the antiferromagnetic Ising model) as well as for $n \in [1, 2]$ and $x = 1$, a partial result in the same direction was shown in [6]. Indeed, it was proved that in this range of parameters, at least one loop of length comparable to the size of the domain exists with positive probability (thus excluding the exponential decay). All results appear on the phase diagram of Figure 1.

We finish the introduction by providing a sketch of our proof. There are three main steps in it. Fix $n > 1$. First, we develop the partition function in $n = (n - 1) + 1$ (aka Chayes–Machta [4]), so that it takes the form of the loop $O(1)$ model sampled on the

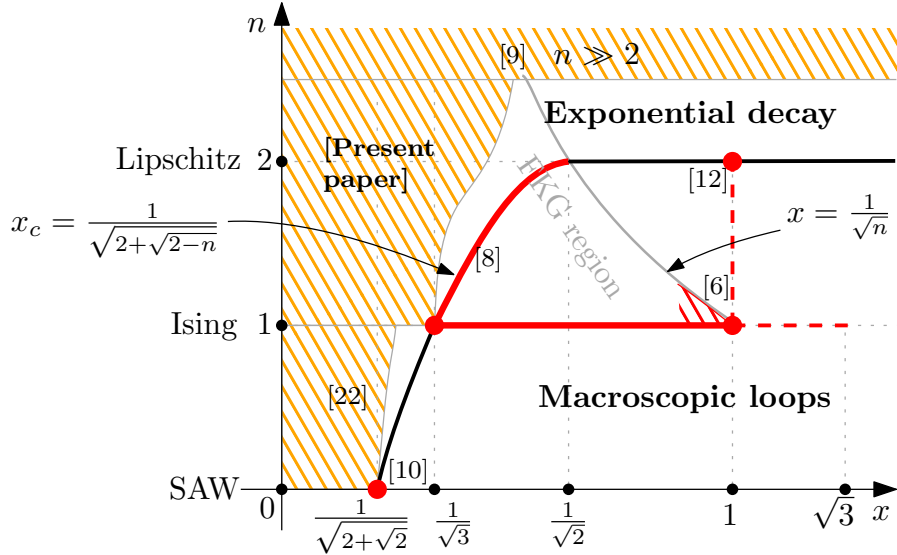


Figure 1: The phase diagram of the loop $O(n)$ model. It is expected that above and to the left of the curve $x_c(n) = \frac{1}{\sqrt{2+\sqrt{2-n}}}$ (in black) the model exhibits exponential decay of loop lengths; below and on the curve, it is expected to have macroscopic loops and converge to $\text{CLE}(\kappa)$ in the scaling limit. Conformal invariance was established only at $n = 1, x = \frac{1}{\sqrt{3}}$ (critical Ising model [21, 5]) and $n = x = 1$ (site percolation on \mathbb{T} at $p_c = \frac{1}{2}$ [20]1). Regions where the behaviour was confirmed by recent results are marked in orange (for exponential decay) and red (for macroscopic loops). The relevant references are also marked.

vacant space of a weighted loop $O(n-1)$ model. Second, we use that the loop $O(1)$ model is the representation of the Ising model on the faces of \mathbb{H} ; the latter exhibiting exponential decay of correlations for all $x < 1/\sqrt{3}$. Via the FK-Ising representation, this statement may be extended when the Ising model is sampled in the random domain given by a loop $O(n-1)$ configuration. At this stage we will have shown that the loop $O(n)$ model exhibits exponential decay when $x < 1/\sqrt{3}$. Finally, using enhancement techniques, we show that the presence of the loop $O(n-1)$ configuration strictly increases the critical parameter of the Ising model, thus allowing to extend our result to all $x < 1/\sqrt{3} + \varepsilon(n)$.

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2 The Ising connection

In this section we formalise a well-known connection between the Ising model (and its FK-representation) and the loop $O(1)$ model (see for instance [11, Sec. 3.10.1]). It will be useful to work with inhomogeneous measures in both models.

Fix a domain $\mathcal{D} = (V, E)$; we will omit it from notation when not necessary. Let $\mathbf{x} =$

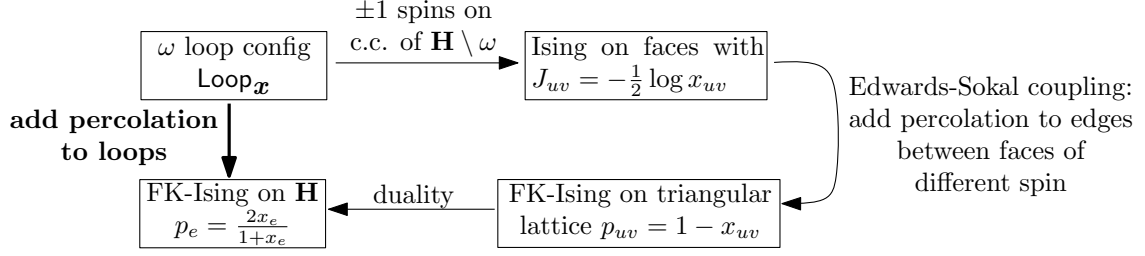


Figure 2: The coupling of Proposition 2.1 via the spin-Ising representation.

$(x_e)_{e \in E} \in [0, 1]^E$ be a family of parameters. The loop $O(1)$ measure with parameters \mathbf{x} is given by

$$\text{Loop}_{\mathcal{D}, 1, \mathbf{x}}(\omega) = \text{Loop}_{\mathbf{x}}(\omega) = \frac{1}{Z_{\text{loop}}(\mathcal{D}, 1, \mathbf{x})} \left(\prod_{e \in \omega} x_e \right) \cdot \mathbf{1}_{\{\omega \text{ loop config.}\}} \quad \text{for all } \omega \in \{0, 1\}^E.$$

The percolation measure $\text{Perco}_{\mathbf{x}}$ of parameters \mathbf{x} consists of choosing the state of every edge independently, open with probability x_e for each edge $e \in E$:

$$\text{Perco}_{\mathbf{x}}(\omega) = \left(\prod_{e \in \omega} x_e \right) \left(\prod_{e \in E \setminus \omega} (1 - x_e) \right), \quad \text{for all } \omega \in \{0, 1\}^E.$$

Finally, associate to the parameters \mathbf{x} the parameters $\mathbf{p} = (p_e)_{e \in E} \in [0, 1]^E$ defined by

$$p_e = p(x_e) = \frac{2x_e}{1 + x_e}, \quad \text{for all } e \in E.$$

Define the FK-Ising measure on \mathcal{D} by

$$\Phi_{\mathbf{x}}(\omega) = \frac{1}{Z_{FK}(\mathbf{x})} \left(\prod_{e \in \omega} p_e \right) \left(\prod_{e \in E \setminus \omega} (1 - p_e) \right) 2^{k(\omega)}, \quad \text{for all } \omega \in \{0, 1\}^E.$$

where $k(\omega)$ is the number of connected components of ω and $Z_{FK}(\mathbf{x})$ is a constant chosen so that $\Phi_{\mathbf{x}}$ is a probability measure and called the partition function.

When \mathbf{x} is constant equal to some $x \in [0, 1]$, write x instead of \mathbf{x} . For $D \subset E$, write $\Phi_{D, x}$ and $\text{Loop}_{D, 1, x}$ for the FK-Ising and loop $O(1)$ measures, respectively, on \mathcal{D} with inhomogeneous weights $(x \mathbf{1}_{e \in D})_{e \in E}$. These are simply the measures $\Phi_{\mathcal{D}, x}$ and $\text{Loop}_{\mathcal{D}, 1, x}$ conditioned on $\omega \cap D^c = \emptyset$.

Proposition 2.1. *Fix $\mathbf{x} = (x_e)_{e \in E} \in [0, 1]^E$ and let $\omega, \pi \in \{0, 1\}^E$ be two independent configurations chosen according to $\text{Loop}_{\mathbf{x}}$ and $\text{Perco}_{\mathbf{x}}$, respectively. Then the configuration $\omega \vee \pi$ defined by $(\omega \vee \pi)(e) = \max\{\omega(e), \pi(e)\}$ has law $\Phi_{\mathbf{x}}$. In particular*

$$\text{Loop}_{\mathbf{x}} \leq_{st} \Phi_{\mathbf{x}}. \quad (2.1)$$

We give a short proof below. The reader familiar with the Ising model may consult the diagram of Figure 2 for a more intricate but more natural proof.

Proof Write $\text{Loop}_{\mathbf{x}} \otimes \text{Perco}_{\mathbf{x}}$ for the measure sampling ω and π independently. Fix $\eta \in \{0, 1\}^E$ and let us calculate

$$\begin{aligned}
\text{Loop}_{\mathbf{x}} \otimes \text{Perco}_{\mathbf{x}}(\omega \vee \pi = \eta) &= \sum_{\substack{\omega \subset \eta \\ \omega \text{ loop config}}} \text{Loop}_{\mathcal{D}, \mathbf{x}}(\omega) \cdot \text{Perco}_{\mathcal{D} \setminus \omega, \mathbf{x}}(\eta \setminus \omega) \\
&= \sum_{\substack{\omega \subset \eta \\ \omega \text{ loop config}}} \frac{1}{Z_{\text{loop}}(\mathcal{D}, 1, \mathbf{x})} \left(\prod_{e \in \omega} x_e \right) \left(\prod_{e \in \eta \setminus \omega} x_e \right) \left(\prod_{e \in E \setminus \eta} (1 - x_e) \right) \\
&= \frac{1}{Z_{\text{loop}}(\mathcal{D}, 1, \mathbf{x})} \left(\prod_{e \in \eta} x_e \right) \left(\prod_{e \in E \setminus \eta} (1 - x_e) \right) \sum_{\substack{\omega \subset \eta \\ \omega \text{ loop config}}} 1. \quad (2.2)
\end{aligned}$$

Next we estimate the number of loop configurations ω contained in η . Consider η as a planar graph and let $F(\eta)$ be the set of connected components of $\mathbb{R}^2 \setminus \eta$; these are the faces of η . The set of loop configurations ω contained in η is in bijection with the set of assignments of spins ± 1 to the faces of η , with the only constraint that the infinite face has spin $+1$. Indeed, given a loop configuration $\omega \subset \eta$, assign spin -1 to the faces of η surrounded by an odd number of loops of ω , and $+1$ to all others. The inverse map is obtained by considering the edges separating faces of distinct spin.

The Euler formula applied to the graph η reads $|V| - |\eta| + |F(\eta)| = 1 + k(\eta)$. Hence, the number of loop configurations contained in η is

$$\sum_{\substack{\omega \subset \eta \\ \omega \text{ loop config}}} 1 = 2^{F(\eta)-1} = 2^{k(\eta)+|\eta|-|V|}.$$

Inserting this in (2.2), we find

$$\begin{aligned}
\text{Loop}_{\mathbf{x}} \otimes \text{Perco}_{\mathbf{x}}(\omega \vee \pi = \eta) &= \frac{2^{-|V|}}{Z_{\text{loop}}(\mathcal{D}, 1, \mathbf{x})} \left(\prod_{e \in \eta} 2x_e \right) \left(\prod_{e \in E \setminus \eta} (1 - x_e) \right) 2^{k(\eta)} \\
&= \frac{2^{-|V|}(1 + x_e)^{|E|}}{Z_{\text{loop}}(\mathcal{D}, 1, \mathbf{x})} \left(\prod_{e \in \eta} \frac{2x_e}{1+x_e} \right) \left(\prod_{e \in E \setminus \eta} \left(1 - \frac{2x_e}{1+x_e} \right) \right) 2^{k(\eta)}.
\end{aligned}$$

Since $\text{Loop}_{\mathbf{x}} \otimes \text{Perco}_{\mathbf{x}}$ is a probability measure, we deduce that it is equal to $\Phi_{\mathbf{x}}$ and that the normalising constants are equal, namely

$$\frac{Z_{\text{loop}}(\mathcal{D}, 1, \mathbf{x})}{2^{-|V|}(1 + x_e)^{|E|}} = Z_{\text{FK}}(\mathcal{D}, \mathbf{x}). \quad (2.3)$$

□

While the loop model has no apparent monotonicity, the FK-Ising model does. This will be of particular importance.

Proposition 2.2 (Thm. 3.21 [13]). *Let $\mathbf{x} = (x_e)_{e \in E} \in [0, 1]^E$ and $\tilde{\mathbf{x}} = (\tilde{x}_e)_{e \in E} \in [0, 1]^E$ be two sets of parameters with $x_e \leq \tilde{x}_e$ for all $e \in E$. Then $\Phi_{\mathbf{x}} \leq_{st} \Phi_{\tilde{\mathbf{x}}}$ in the sense that*

$$\Phi_{\mathbf{x}}(A) \leq \Phi_{\tilde{\mathbf{x}}}(A) \quad \text{for any increasing event } A.$$

The version above is slightly different from [13, Thm3.21], as it deals with inhomogeneous measures; adapting the proof is straightforward.

Finally, it is well known that the FK-Ising model on the hexagonal lattice exhibits a sharp phase transition at $p_c = \frac{2}{\sqrt{3+1}}$ — the critical point for the Ising model was computed

by Onsager [18] (see [2] for the explicit formula on the triangular lattice) sharpness was shown in [1]. For $p = p(x)$ strictly below this value, which is to say $x < \frac{1}{\sqrt{3}}$, the model exhibits exponential decay of cluster volumes. Indeed, this may be easily deduced using [Thm. 5.86 [13]].

Theorem 2.3. *For $x < \frac{1}{\sqrt{3}}$ there exist $c = c(x) > 0$ and $C > 0$ such that, for any domain \mathcal{D} ,*

$$\Phi_{\mathcal{D},x}(|\mathcal{C}_0| \geq k) \leq C e^{-ck},$$

where \mathcal{C}_0 denotes the cluster containing 0 and $|\mathcal{C}_0|$ its number of vertices.

3 $\mathbf{n} = (\mathbf{n-1}) + \mathbf{1}$

Fix a domain $\mathcal{D} = (V, E)$ and a value $n > 1$. Choose ω according to $\mathbf{Loop}_{\mathcal{D},n,x}$. Colour each loop of ω in blue with probability $1 - \frac{1}{n}$ and red with probability $\frac{1}{n}$. Let ω_b and ω_r be the configurations formed only of the blue and red loops, respectively; extend $\mathbf{Loop}_{\mathcal{D},n,x}$ to incorporate this additional randomness.

Proposition 3.1. *For any two non-intersecting loop configurations ω_b and ω_r , we have*

$$\begin{aligned} \mathbf{Loop}_{\mathcal{D},n,x}(\omega_r | \omega_b) &= \mathbf{Loop}_{\mathcal{D} \setminus \omega_b, 1, x}(\omega_r) && \text{and} \\ \mathbf{Loop}_{\mathcal{D},n,x}(\omega_b) &= \frac{Z_{\text{loop}}(\mathcal{D} \setminus \omega_b, 1, x)}{Z_{\text{loop}}(\mathcal{D}, n, x)} (n-1)^{\ell(\omega_b)} x^{|\omega_b|}. \end{aligned}$$

Proof For two non-intersecting loop configurations ω_b and ω_r , if we write $\omega = \omega_b \vee \omega_r$, we have

$$\begin{aligned} \mathbf{Loop}_{\mathcal{D},n,x}(\omega_b, \omega_r) &= \left(\frac{n-1}{n}\right)^{\ell(\omega_b)} \left(\frac{1}{n}\right)^{\ell(\omega_r)} \mathbf{Loop}_{\mathcal{D},n,x}(\omega) \\ &= \frac{1}{Z_{\text{loop}}(\mathcal{D}, n, x)} (n-1)^{\ell(\omega_b)} x^{|\omega_b| + |\omega_r|} \\ &= \frac{Z_{\text{loop}}(\mathcal{D} \setminus \omega_b, 1, x)}{Z_{\text{loop}}(\mathcal{D}, n, x)} (n-1)^{\ell(\omega_b)} x^{|\omega_b|} \cdot \frac{x^{|\omega_r|}}{Z_{\text{loop}}(\mathcal{D} \setminus \omega_b, 1, x)}. \end{aligned}$$

Notice that ω_r only appears in the last fraction. Moreover, if we sum this fraction over all loop configurations ω_r not intersecting ω_b , we obtain 1. This proves both assertions of the proposition. \square

Recall that for a percolation configuration, \mathcal{C}_0 denotes the connected component containing 0. If ω is a loop configuration ω , then $\mathcal{C}_0(\omega)$ is simply the loop in ω that passes through 0 (with $\mathcal{C}_0(\omega) := \{0\}$ if no such loop exists).

Corollary 3.2. *Let $n \geq 1$ and $x < 1/\sqrt{3}$. Then $\mathbf{Loop}_{\mathcal{D},n,x}$ exhibits exponential decay.*

Proof For any domain \mathcal{D} and $k \geq 1$ we have

$$\begin{aligned} \frac{1}{n} \mathbf{Loop}_{\mathcal{D},n,x}(|\mathcal{C}_0(\omega)| \geq k) &= \mathbf{Loop}_{\mathcal{D},n,x}(|\mathcal{C}_0(\omega_r)| \geq k) \\ &\leq \mathbf{Loop}_{\mathcal{D},n,x}[\Phi_{\mathcal{D} \setminus \omega_b, x}(|\mathcal{C}_0| \geq k)] && \text{by Prop. 2.1} \\ &\leq \Phi_{\mathcal{D},x}(|\mathcal{C}_0| \geq k) && \text{by Prop. 2.2} \\ &\leq C e^{-ck} && \text{by Thm. 2.3,} \end{aligned}$$

where $c = c(x) > 0$ and $C > 0$ are given by Theorem 2.3. Thus, the length of the loop of 0 has exponential tail, uniformly in the domain \mathcal{D} . In particular, if \mathcal{D} is fixed, the above bound also applies to any translates of \mathcal{D} , hence to the loop of any given point in \mathcal{D} .

Let $v_0, v_1, v_2 \dots$ be the vertices of \mathcal{D} on the horizontal line to the right of 0, ordered from left to right, starting with $v_0 = 0$. If $R \geq k$, then the largest loop surrounding 0 either passes through one of the points v_0, \dots, v_{k-1} and has length at least k , or it passes through some v_j avec $j \geq k$, and has length at least j , so as to manage to surround 0. Thus, using the bound derived above, we find

$$\text{Loop}_{\mathcal{D},n,x}(R \geq k) \leq n C k e^{-ck} + \sum_{j \geq k} n C e^{-cj} \leq C' e^{-c'k},$$

for some altered constants $c' > 0$ and C' that depend on c, C and n but not on k . \square

4 The little extra juice: enhancement

Fix some domain $\mathcal{D} = (V, E)$ for the whole of this section. Let ω_b be a blue loop configuration. Associate to it the spin configuration $\sigma_b \in \{-1, +1\}^{F_{\mathcal{D}}}$ obtained by awarding spins -1 to all faces of \mathcal{D} that are surrounded by an odd number of loops, and spins $+1$ to all other faces. Write $D_+ = D_+(\sigma_b)$ (and $D_- = D_-(\sigma_b)$, respectively) for the set of edges of \mathcal{D} that have σ_b -spin $+1$ (and -1 , respectively) on either side. All faces outside of \mathcal{D} are considered to have spin $+1$ in this definition. Equivalently, D_- is the set of edges of $\mathcal{D} \setminus \omega_b$ surrounded by an odd number of loops of ω_b and $D_+ = \mathcal{D} \setminus (\omega_b \cup D_-)$. Both D_+ and D_- will also be regarded as spanning subgraphs of \mathcal{D} with edge-sets D_+ and D_- , respectively.

Since no edge of D_+ is adjacent to any edge of D_- , a sample of the loop $O(1)$ measure $\text{Loop}_{\mathcal{D} \setminus \omega_b, 1, x}$ may be obtained by the superposition of two independent samples from $\text{Loop}_{D_+, 1, x}$ and $\text{Loop}_{D_-, 1, x}$, respectively. In particular, using (2.1),

$$\begin{aligned} \text{Loop}_{\mathcal{D} \setminus \omega_b, 1, x}(|\mathcal{C}_0(\omega_r)| \geq k) &= \text{Loop}_{D_+, 1, x}(|\mathcal{C}_0(\omega_r)| \geq k) + \text{Loop}_{D_-, 1, x}(|\mathcal{C}_0(\omega_r)| \geq k) \\ &\leq \Phi_{D_+, x}(|\mathcal{C}_0| \geq k) + \Phi_{D_-, x}(|\mathcal{C}_0| \geq k). \end{aligned} \quad (4.1)$$

Actually, depending on ω_b , at most one of the terms on the RHS above is non-zero. We nevertheless keep both terms as we will later average on ω_b . The two following lemmas will be helpful in proving Theorem 1.1.

Lemma 4.1. *Let $x < 1$ and set*

$$\alpha = \left[\frac{\max\{(n-1)^2, (n-1)^{-2}\}}{(x/2)^6 + \max\{(n-1)^2, (n-1)^{-2}\}} \right]^{1/6} < 1. \quad (4.2)$$

If ω_b has the law of the blue loop configuration of $\text{Loop}_{\mathcal{D}, n, x}$, then both laws of D_+ and D_- are stochastically dominated by Perco_{α} .

Lemma 4.2. *Fix $x \in (0, 1)$ and $\alpha < 1$. Let $\tilde{x} < x$ be such that*

$$\frac{\tilde{x}}{1 - \tilde{x}} = \frac{x}{1 - x} \cdot \left(1 + \frac{1 + x}{2(1 - x)} \cdot \frac{1 - \alpha}{\alpha} \right)^{-1}. \quad (4.3)$$

Write $\text{Perco}_{\alpha}(\Phi_{D,x}(\cdot))$ for the law of η chosen using the following two step procedure: choose D according to Perco_{α} , then choose η according to $\Phi_{D,x}$. Then

$$\text{Perco}_{\alpha}(\Phi_{D,x}(\cdot)) \leq_{st} \Phi_{\mathcal{D}, \tilde{x}}.$$

Before proving the two lemmas above, let us show that they imply the main result.

Proof of of Theorem 1.1 Fix $n > 1$. An elementary computation proves the existence of some $\varepsilon = \varepsilon(n) > 0$ such that, if $x < \frac{1}{\sqrt{3}} + \varepsilon(n)$ and α and \tilde{x} are defined in terms of n and x via (4.2) and (4.3), respectively, then $\tilde{x} < \frac{1}{\sqrt{3}}$ ⁽ⁱ⁾.

Fix $x < \frac{1}{\sqrt{3}} + \varepsilon(n)$ along with the resulting values α and $\tilde{x} < \frac{1}{\sqrt{3}}$. Then, for any domain \mathcal{D} and $k \geq 1$ we have

$$\begin{aligned} \frac{1}{n} \text{Loop}_{\mathcal{D},n,x}(|\mathcal{C}_0(\omega)| \geq k) &= \text{Loop}_{\mathcal{D},n,x}(|\mathcal{C}_0(\omega_r)| \geq k) \\ &\leq \text{Loop}_{\mathcal{D},n,x}[\Phi_{D_+,x}(|\mathcal{C}_0| \geq k) + \Phi_{D_-,x}(|\mathcal{C}_0| \geq k)] && \text{by (4.1)} \\ &\leq 2 \text{Perco}_\alpha[\Phi_{D,x}(|\mathcal{C}_0| \geq k)] && \text{by Lemma 4.1} \\ &\leq 2 \Phi_{\mathcal{D},\tilde{x}}(|\mathcal{C}_0| \geq k) && \text{by Lemma 4.2} \\ &\leq 2C e^{-ck} && \text{by Thm. 2.3.} \end{aligned}$$

In the third line, we have used Lemma 4.1 and the stochastic monotonicity of Φ in terms of the domain. Indeed, Lemma 4.1 implies that $\text{Loop}_{\mathcal{D},n,x}$ and Perco_α may be coupled so that the sample D_+ obtained from the former is included in the sample D obtained from the latter. Thus $\Phi_{D_+,x} \leq \Phi_{D,x}$. The same applies separately for D_- .

To conclude (1.1), continue in the same way as in the proof of Corollary 3.2. \square

The following computation will be useful for the proofs of both lemmas. Let $D \subset E$ and $e \in E \setminus D$. We will also regard D as a spanning subgraph of \mathcal{D} with edge-set D . Recall that $Z_{FK}(D, x)$ is the partition function of the FK-Ising measure $\Phi_{D,x}$ on D . Then

$$\begin{aligned} Z_{FK}(D, x) &= \sum_{\eta \subset D} p^{|\eta|} (1-p)^{|D|-|\eta|} 2^{k(\eta)} \\ &= \sum_{\eta \subset D} p^{|\eta|} (1-p)^{|D \cup \{e\}| - |\eta|} 2^{k(\eta)} + p^{|\eta \cup \{e\}|} (1-p)^{|D|-|\eta|} 2^{k(\eta)} \\ &\geq \sum_{\eta \subset D \cup \{e\}} p^{|\eta|} (1-p)^{|D \cup \{e\}| - |\eta|} 2^{k(\eta)} = Z_{FK}(D \cup \{e\}, x), \end{aligned} \quad (4.4)$$

since $k(\eta) \geq k(\eta \cup \{e\})$. Conversely, $k(\eta) \leq k(\eta \cup \{e\}) + 1$, which implies

$$Z_{FK}(D, x) \leq 2 Z_{FK}(D \cup \{e\}, x). \quad (4.5)$$

Proof of of Lemma 4.1 For $\beta \in (0, 1)$ let P_β be the percolation on the faces of \mathcal{D} of parameter β :

$$P_\beta(\sigma) = \beta^{\#\{u: \sigma(u)=+1\}} (1-\beta)^{\#\{u: \sigma(u)=-1\}} \quad \text{for all } \sigma \in \{-1, +1\}^{F_{\mathcal{D}}}.$$

To start, we will prove that the law induced on σ_b by $\text{Loop}_{\mathcal{D},n,x}$ is dominated by P_β for some β sufficiently close to 1. Both measures are positive, and Holley's inequality [14] states that the stochastic ordering is implied by

$$\frac{\text{Loop}_{\mathcal{D},n,x}(\sigma_b = \varsigma_1)}{\text{Loop}_{\mathcal{D},n,x}(\sigma_b = \varsigma_1 \wedge \varsigma_2)} \leq \frac{P_\beta(\varsigma_1 \vee \varsigma_2)}{P_\beta(\varsigma_2)} = \left(\frac{\beta}{1-\beta} \right)^{\#\{u: \varsigma_1(u)=+1, \varsigma_2(u)=-1\}} \quad \text{for all } \varsigma_1, \varsigma_2 \in \{\pm 1\}^{F_{\mathcal{D}}}.$$

The RHS above only depends on $|\varsigma_1 \setminus \varsigma_2|$. It is then elementary to check that the general inequality above is implied by the restricted case where ς_1 differs at exactly one face u from ς_2 , and $\varsigma_1(u) = +1$ but $\varsigma_2(u) = -1$.

⁽ⁱ⁾When $n \searrow 1$, we have $\varepsilon(n) \sim C(n-1)^2$, where $C = \frac{1+\sqrt{3}}{12^4 \sqrt{3}}$.

Fix two such configurations ς_1, ς_2 ; write ω_1 and ω_2 for their associated loop configurations. Then, by Lemma 3.1,

$$\frac{\text{Loop}_{\mathcal{D},n,x}(\sigma_b = \varsigma_1)}{\text{Loop}_{\mathcal{D},n,x}(\sigma_b = \varsigma_2)} = \frac{Z_{\text{loop}}(\mathcal{D} \setminus \omega_1, 1, x)}{Z_{\text{loop}}(\mathcal{D} \setminus \omega_2, 1, x)} (n-1)^{\ell(\omega_1) - \ell(\omega_2)} x^{|\omega_1| - |\omega_2|}.$$

Since ς_1 and ς_2 only differ by one face, ω_1 and ω_2 differ only in the states of the edges surrounding that face. In particular $\|\omega_1\| - \|\omega_2\| \leq 6$ and $|\ell(\omega_1) - \ell(\omega_2)| \leq 2$. Finally, using (4.4) and (4.5), we find

$$\frac{Z_{\text{loop}}(\mathcal{D} \setminus \omega_1, 1, x)}{Z_{\text{loop}}(\mathcal{D} \setminus \omega_2, 1, x)} \leq \frac{Z_{\text{loop}}(\mathcal{D} \setminus (\omega_1 \wedge \omega_2), 1, x)}{Z_{\text{loop}}(\mathcal{D} \setminus \omega_2, 1, x)} \leq 2^{|\omega_2| - |\omega_1 \wedge \omega_2|} \leq 2^6.$$

In conclusion

$$\frac{\text{Loop}_{\mathcal{D},n,x}(\sigma_b = \varsigma_1)}{\text{Loop}_{\mathcal{D},n,x}(\sigma_b = \varsigma_2)} \leq \left(\frac{2}{x}\right)^6 \cdot \max\{(n-1)^2, (n-1)^{-2}\}.$$

Then, if we set

$$\beta = \frac{\left(\frac{2}{x}\right)^6 \cdot \max\{(n-1)^2, (n-1)^{-2}\}}{1 + \left(\frac{2}{x}\right)^6 \cdot \max\{(n-1)^2, (n-1)^{-2}\}},$$

we indeed obtain the desired domination of σ_b by P_β ⁽ⁱⁱ⁾. The same proof shows that $-\sigma_b$ is also dominated by P_β .

Next, let us prove the domination of D_+ by a percolation measure. Set $\alpha = \beta^{1/6}$ ⁽ⁱⁱⁱ⁾. Let η_L and η_R be two percolation configurations chosen independently according to Perco_α . Also choose an orientation for every edge of E ; for boundary edges, orient them such that the face of \mathcal{D} adjacent to them is on their left.

Define $\tilde{\sigma} \in \{\pm 1\}^{F_\mathcal{D}}$ as follows. Consider some face u . For an edge e adjacent to u , u is either on the left of e or on its right, according to the orientation chosen for e . If it is on the left, retain the number $\eta_L(e)$, otherwise retain $\eta_R(e)$. Consider that u has spin $+1$ under $\tilde{\sigma}$ if and only if all the six numbers retained above are 1. Formally, for each $u \in F_\mathcal{D}$, set $\tilde{\sigma}(u) = +1$ if and only if

$$\prod_{e \text{ adjacent to } u} (\eta_L(e) \mathbf{1}_{\{u \text{ is left of } e\}} + \eta_R(e) \mathbf{1}_{\{u \text{ is right of } e\}}) = 1.$$

As a consequence, for an edge e in the interior of \mathcal{D} to be in $D_+(\tilde{\sigma})$, the faces on either side of e need to have $\tilde{\sigma}$ -spin $+1$, hence $\eta_L(e) = \eta_R(e) = 1$ is required. For boundary edges e to be in $D_+(\tilde{\sigma})$, only the restriction $\eta_L(e) = 1$ remains. In conclusion $\eta_L \geq D_+$.

Let us analyse the law of $\tilde{\sigma}$. Each value $\eta_L(e)$ and $\eta_R(e)$ appears in the definition of one $\tilde{\sigma}(u)$. As a consequence, the variables $(\tilde{\sigma}(u))_{u \in F}$ are independent. Moreover, $\tilde{\sigma}(u) = 1$ if and only if all the six edges around e are open in one particular configuration η_L or η_R , which occurs with probability $\alpha^6 = \beta$. As a consequence $\tilde{\sigma}$ has law P_β .

⁽ⁱⁱ⁾This domination is of special interest as $n \searrow 1$ and for $x \geq 1/\sqrt{3}$. Then we may simplify the value of β as $\beta = \frac{(2\sqrt{3})^6}{(n-1)^2 + (2\sqrt{3})^6} \sim 1 - \frac{1}{(2\sqrt{3})^6} (n-1)^2$.

⁽ⁱⁱⁱ⁾As $n \searrow 1$ and $x \geq 1/\sqrt{3}$, we may assume that $\alpha \sim 1 - \frac{1}{6(2\sqrt{3})^6} (n-1)^2$.

By the previously proved domination, $\text{Loop}_{\mathcal{D},n,x}$ may be coupled with P_β so that $\tilde{\sigma} \geq \sigma_b$. If this is the case, we have

$$\eta_L \geq D_+(\tilde{\sigma}) \geq D_+(\sigma_b).$$

Thus, η_L indeed dominates $D_+(\sigma_b)$, as required.

The same proof shows that η_L dominates $D_-(\sigma_b)$. For clarity, we mention that this does not imply that η_L dominates $D_+(\sigma_b)$ and $D_-(\sigma_b)$ simultaneously, which would translate to η_L dominating $\mathcal{D} \setminus \omega_b$. \square

Proof of Lemma 4.2 The statement of Holley's inequality applied to our case may easily be reduced to

$$\frac{\text{Perco}_\alpha[\Phi_{D,x}(\eta \cup \{e\})]}{\text{Perco}_\alpha[\Phi_{D,x}(\eta)]} \leq \frac{\Phi_{\mathcal{D},\tilde{x}}(\tilde{\eta} \cup \{e\})}{\Phi_{\mathcal{D},\tilde{x}}(\tilde{\eta})} \quad \text{for all } \eta \leq \tilde{\eta} \text{ and } e \notin \tilde{\eta}. \quad (4.6)$$

Fix $\eta, \tilde{\eta}$ and $e = (uv)$ as above. For $D \subset E$ with $e \in D$, a standard computation yields

$$\varphi_x(e|\eta) := \frac{\Phi_{D,x}(\eta \cup \{e\})}{\Phi_{D,x}(\eta)} = \begin{cases} \frac{2x}{1-x} & \text{if } u \overset{\eta}{\leftrightarrow} v \text{ and} \\ \frac{x}{1-x} & \text{otherwise.} \end{cases}$$

The same quantity may be defined for \tilde{x} instead of x and $\tilde{\eta}$ instead of η ; it is increasing in both η and x . Moreover $\varphi_x(e|\eta)$ does not depend on D , as long as $e \in D$ and $\eta \subset D$. If the first condition fails, then the numerator is 0; if the second fails then the denominator is null and the ratio is not defined.

Let us perform a helpful computation before proving (4.6). Fix D with $e \in D$. By (4.5),

$$\frac{\Phi_{D \setminus \{e\},x}(\eta)}{\Phi_{D,x}(\eta)} = \frac{Z_{FK}(D,x)}{Z_{FK}(D \setminus \{e\},x)} \cdot \frac{1+x}{1-x} \geq \frac{1+x}{2(1-x)}.$$

The factor $\left(\frac{1+x}{1-x}\right)^{-1}$ comes from the fact that the weights of η under $\Phi_{D \setminus \{e\},x}$ and $\Phi_{D,x}$ differ by the contribution of the closed edge e . It follows that

$$\left(1 + \frac{1+x}{2(1-x)} \cdot \frac{1-\alpha}{\alpha}\right) \Phi_{D,x}(\eta) \leq \Phi_{D,x}(\eta) + \frac{1-\alpha}{\alpha} \Phi_{D \setminus \{e\},x}(\eta).$$

The choice of \tilde{x} is such that

$$\frac{\varphi_x(e|\eta)}{\varphi_{\tilde{x}}(e|\tilde{\eta})} = \frac{x}{1-x} \cdot \frac{1-\tilde{x}}{\tilde{x}} = 1 + \frac{1+x}{2(1-x)} \cdot \frac{1-\alpha}{\alpha}.$$

Using the last two displayed equations, we find

$$\begin{aligned} \text{Perco}_\alpha[\Phi_{D,x}(\eta \cup \{e\})] &= \sum_{D \subset E} \alpha^{|D|} (1-\alpha)^{|E|-|D|} \Phi_{D,x}(\eta \cup \{e\}) \\ &= \sum_{\substack{D \subset E \\ \text{with } e \in D}} \alpha^{|D|} (1-\alpha)^{|E|-|D|} \varphi_x(e|\eta) \Phi_{D,x}(\eta) \\ &\leq \left(1 + \frac{1+x}{2(1-x)} \cdot \frac{1-\alpha}{\alpha}\right)^{-1} \varphi_x(e|\eta) \sum_{\substack{D \subset E \\ \text{with } e \in D}} \alpha^{|D|} (1-\alpha)^{|E|-|D|} \left[\Phi_{D,x}(\eta) + \frac{1-\alpha}{\alpha} \Phi_{D \setminus \{e\},x}(\eta) \right] \\ &= \varphi_{\tilde{x}}(e|\tilde{\eta}) \sum_{D \subset E} \alpha^{|D|} (1-\alpha)^{|E|-|D|} \Phi_{D,x}(\eta) \\ &= \varphi_{\tilde{x}}(e|\tilde{\eta}) \text{Perco}_\alpha[\Phi_{D,x}(\eta)]. \end{aligned}$$

Divide by $\text{Perco}_\alpha[\Phi_{D,x}(\eta)]$ and recall the definition of $\varphi_{\tilde{x}}(e|\tilde{\eta})$ to obtain (4.6). \square

5 Open questions / perspectives

The strategy of our proof was based on the following observation. The loop $O(1)$ model, or rather its associated FK-Ising model, has a certain monotonicity in x . This translates to a monotonicity in the domain: the larger the domain, the higher the probability that a given point is contained in a large loop. This fact is used to compare the loop $O(1)$ model in a simply connected domain \mathcal{D} with that in the domain obtained from \mathcal{D} after removing certain interior parts. The latter is generally not simply connected, and it is essential that our monotonicity property can handle such domains.

Question 5.1. *Associate to the loop $O(n)$ model with edge weight x in some domain \mathcal{D} a positively associated percolation model $\Psi_{\mathcal{D},n,x}$ with the property that, if one exhibits exponential decay of connection probabilities, then so does the other.*

The percolation model $\Psi_{\mathcal{D},n,x}$ actually only needs to have some monotonicity property in the domain, sufficient for our proof to apply. Unfortunately, we only have such an associated model when $n = 1$.

Suppose that one may find such a model Ψ for some value of n and let us explain what may be deduced. Fix x such that the loop $O(n)$ model exhibits exponential decay. Then $\Psi_{\mathcal{D},n,x}$ also exhibits exponential decay for any domain \mathcal{D} . Consider now the loop $O(\tilde{n})$ model with edge-weight x for $\tilde{n} > n$ and colour each loop independently in red with probability n/\tilde{n} and in blue with probability $(\tilde{n} - n)/\tilde{n}$. Then, conditionally on the blue loop configuration ω_b , the red loop configuration has the law of the loop $O(n)$ model with edge-weight x in the domain $\mathcal{D} \setminus \omega_b$. By positive association, since $\Psi_{\mathcal{D},n,x}$ exhibits exponential decay, so does $\Psi_{\mathcal{D} \setminus \omega_b, n, x}$. Then the loop $O(\tilde{n})$ model exhibits exponential decay of lengths of red loops and hence in general of lengths of all loops.

Assume a model Ψ as in Question 5.1 may be found for all $n > 0$. Then, the discussion above shows that the critical point $x_c(n)$ (assuming it exists) is increasing in n . The same technique as in Section 4 may even prove that it is strictly increasing.

Recently it was shown in [8] that, in the regime $n \geq 1$ and $x \leq \frac{1}{\sqrt{n}}$, the loop $O(n)$ model satisfies the following dichotomy: either it exhibits macroscopic loops or exponential decay of loop lengths. Moreover, for $n \in [1, 2]$ and $x = x_c(n) = \frac{1}{\sqrt{2+\sqrt{2-n}}}$ the loop $O(n)$ model is shown to exhibit macroscopic loops. Thus, assuming that Ψ may be constructed for all $1 < n \leq 2$, we deduce that the loop $O(n)$ model with $1 \leq n \leq 2$ and $x \in [\frac{1}{\sqrt{2+\sqrt{2-n}}}, \frac{1}{\sqrt{2}}]$ exhibits macroscopic loops.

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