Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$

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Abstract

We prove that the q-state Potts model and the random-cluster model with cluster weight $q > 4$ undergo a discontinuous phase transition on the square lattice. More precisely, we show

1. Existence of multiple infinite-volume measures for the critical Potts and random-cluster models,
2. Ordering for the measures with monochromatic (resp. wired) boundary conditions for the critical Potts model (resp. random-cluster model), and
3. Exponential decay of correlations for the measure with free boundary conditions for both the critical Potts and random-cluster models.

The proof is based on a rigorous computation of the Perron-Frobenius eigenvalues of the diagonal blocks of the transfer matrix of the six-vertex model, whose ratios are then related to the correlation length of the random-cluster model.

As a byproduct, we rigorously compute the correlation lengths of the critical random-cluster and Potts models, and show that they behave as $\exp(\pi^2/\sqrt{q-4})$ as q tends to 4.

1 Introduction

1.1 Motivation

Lattice spin models were introduced to describe specific experiments; they were later found to be illustrative of a large variety of physical phenomena. Depending on a parameter (most commonly temperature), they exhibit different macroscopic behaviours (also called phases), and phase transitions between them. Phase transitions may be continuous or discontinuous, and determining their type is one of the first steps towards a deeper understanding of the model.

In recent years, the Potts and random-cluster models have been the object of a revived interest after new rigorous results were proved. In [3], the critical points of the models were determined for any $q \geq 1$. In [10], the models were proved to undergo a continuous phase transition for $1 \leq q \leq 4$, thus proving half of a famous prediction by Baxter. The object of this paper is to prove the second half of his prediction - namely, that the phase transition is discontinuous when $q > 4$.

1.2 Results for the Potts model

The Potts model was introduced by Potts [18] following a suggestion of his adviser Domb. While the model received little attention early on, it became the object of great interest in the last 50 years. Since then, mathematicians and physicists have been studying it intensively, and much

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is known about its rich behaviour, especially in two dimensions. For a review of the physics results, see [21].

In this paper, we will focus on the case of the square lattice $\mathbb{Z}^2$ composed of vertices $x = (x_1, x_2) \in \mathbb{Z}^2$, and edges between nearest neighbours. In the q-state ferromagnetic Potts model (where q is a positive integer larger than or equal to 2), each vertex of a graph receives a *spin* taking value in $\{1, \dots, q\}$. The energy of a configuration is then proportional to the number of neighbouring vertices of the graph having different spins. Formally, the Potts measure on a finite subgraph $G = (V, E)$ of the square lattice, at inverse temperature $\beta > 0$ and boundary conditions $i \in \{0, 1, \dots, q\}$, is defined for every $\sigma_0 \in \{1, \dots, q\}^V$ by the formula

$$\mu^i_{G,\beta}[\sigma_0] := \frac{\exp[-\beta \mathbf{H}^i_G(\sigma_0)]}{\sum\limits_{\sigma \in \{1,\dots,q\}^V} \exp[-\beta \mathbf{H}^i_G(\sigma)]}, \tag{1.1}$$

where

$$\mathbf{H}^i_G(\sigma) := -\sum_{\{x,y\} \in E} \mathbf{1}[\sigma_x = \sigma_y] - \sum_{x \in \partial V} \mathbf{1}[\sigma_x = i].$$

Above, $\mathbf{1}[\cdot]$ denotes the indicator function and ∂V is the set of vertices of G with at least one neighbour (in $\mathbb{Z}^2$) outside of G. Note that when $i = 0$, the second sum is zero for all σ.

For any boundary conditions i, the family of measures $\mu^i_{G,\beta}$ converges as G tends to the whole square lattice. The resulting measure μ^i_β defined on the square lattice is called the Gibbs measure with *free* boundary condition if $i = 0$ (respectively, *monochromatic* boundary condition equal to i if $i \in \{1, \dots, q\}$).

The Potts model undergoes an order/disorder phase transition, meaning that there exists a *critical inverse temperature* $\beta_c = \beta_c(q) \in (0, \infty)$) such that:

- For $\beta < \beta_c$, the measures μ^i_β, $i = 0, \dots, q$, are all equal.
- For $\beta > \beta_c$, the measures μ^i_β, $i = 0, \dots, q$, are all distinct.

Baxter [1] conjectured that the phase transition is continuous if $q \le 4$ and discontinuous if $q > 4$, meaning that all the measures $\mu^i_{\beta_c}$ with $i = 0, \dots, q$ are equal if and only if $q \le 4$. It was shown in [3] that $\beta_c = \log(1 + \sqrt{q})$; moreover, when $q \le 4$, it was proved in [10] that the phase transition is indeed continuous, along with more detailed properties of the unique critical measure μ_{β_c}. The goal of this article is to complete the proof of Baxter's conjecture by proving the following theorem.

Theorem 1.1. *Consider the q-state Potts model on the square lattice with $q > 4$. Then,*

1. *all the measures $\mu^i_{\beta_c}$ for $i = 0, \dots, q$ are distinct and ergodic (in particular, $\mu^0_{\beta_c}$ is not equal to the average of the $\mu^i_{\beta_c}$ with $i \in \{1, \dots, q\}$);*
2. *for any $i \in \{1, \dots, q\}$, $\mu^i_{\beta_c}[\sigma_0 = i] > \frac{1}{q}$.*
3. *Let $\lambda > 0$ satisfy $\cosh(\lambda) = \sqrt{q}/2$ and x_n denote the site of $\mathbb{Z}^2$ with both coordinates equal to $\lfloor n/2 \rfloor$. Then*

$$\lim_{n\to\infty} -\tfrac{1}{n} \log\left(\mu^0_{\beta_c}[\sigma_0 = \sigma_{x_n}] - \tfrac{1}{q}\right) = \lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda).$$

Furthermore, the quantity above is strictly positive.

The limit computed in the final item above is the inverse correlation length of the critical Potts model in the diagonal direction. This theorem follows directly from Theorem 1.2 below via the standard coupling between the Potts and random-cluster models (see Section 3.4 for details).

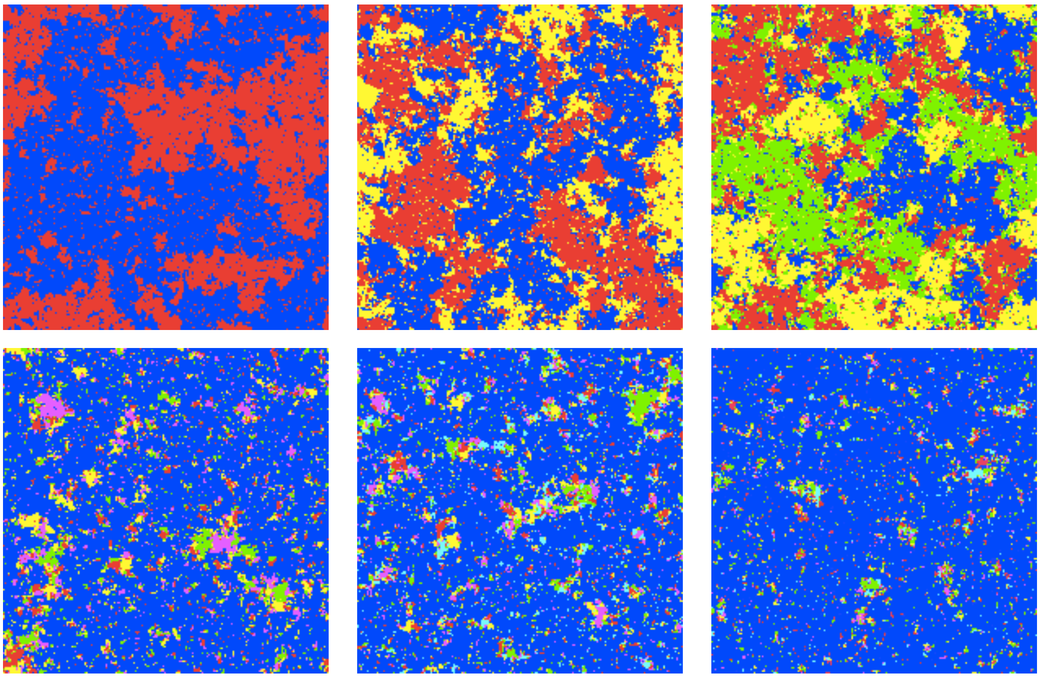

Figure 1: Simulations (due to Vincent Beffara) of the critical planar Potts model $\mu^1_{\beta_c}$ (the spin 1 is depicted in blue) with q equal to 2, 3, 4, 5, 6 and 9 respectively. The behaviour for $q \le 4$ is clearly different from the behaviour for $q > 4$. In the first three pictures, each spin seems to play the same role, while in the last three, the blue spin dominates the other ones.

1.3 Results for the random-cluster model

The random-cluster model (also called Fortuin-Kasteleyn percolation) was introduced by Fortuin and Kasteleyn around 1970 (see [12] and [13]) as a class of models satisfying specific series and parallel laws. It is related to many other models of statistical mechanics, including the Potts model. For background on the random-cluster model and the results mentioned below, we direct the reader to the monographs [16] and [6].

Consider a finite subgraph $G = (V, E)$ of the square lattice. A percolation configuration ω is an element of $\{0,1\}^E$. An edge e is said to be *open* (in ω) if $\omega(e) = 1$, otherwise it is *closed*. A configuration ω can be seen as a subgraph of G with vertex set V and edge-set $\{e \in E : \omega(e) = 1\}$. When speaking of connections in ω, we view ω as a graph. A *cluster* is a connected component of ω. Let $o(\omega)$ and $c(\omega)$ denote the number of open edges and closed edges in ω respectively. Let $k_0(\omega)$ denote the number of clusters of ω, and $k_1(\omega)$ the number of clusters of ω when all clusters intersecting ∂V are counted as a single one – as before, ∂V is the set of vertices of G adjacent to a vertex of $\mathbb{Z}^2$ not contained in G.

For $i \in \{0,1\}$, the random-cluster measure with parameters $p \in [0,1]$, $q > 0$ and boundary conditions i is given by

$$\phi^i_{G,p,q}(\omega) = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k_i(\omega)}}{Z^i(G,p,q)},$$

where $Z^i(G,p,q)$ is a normalizing constant called the partition function. When $i = 0$ and $i = 1$, we speak of free and wired boundary conditions respectively.

The family of measures $\phi^i_{G,p,q}$ converges weakly as G tends to the whole square lattice. The limiting measures are denoted by $\phi^i_{p,q}$ and are called *infinite-volume* random-cluster measures with free and wired boundary conditions (for i equal to 0 and 1 respectively).

For $q \ge 1$, the random-cluster model undergoes a phase transition at the critical parameter

$p_c = p_c(q) = \sqrt{q}/(1+\sqrt{q})$ (see [3] or [8, 9] for alternative proofs), in the following sense:

- if $p > p_c(q)$, $\phi^0_{p,q} = \phi^1_{p,q}$ and the probability of having an infinite cluster in ω is 1.
- if $p < p_c(q)$, $\phi^0_{p,q} = \phi^1_{p,q}$ and the probability of having an infinite cluster in ω is 0.

As before, one may ask whether the phase transition is continuous or not; which comes down to whether there exists a single critical measure or multiple ones. In [10], it was proved that for $1 \le q \le 4$, $\phi^0_{p_c,q} = \phi^1_{p_c,q}$ and the probability of having an infinite cluster under this measure is 0. In this article, we complement this result by proving the following theorem. Recall that, in this model, q is not necessarily an integer.

Theorem 1.2. *Consider the random-cluster model on the square lattice with $q > 4$. Then*

1. $\phi^1_{p_c,q} \neq \phi^0_{p_c,q}$;
2. $\phi^1_{p_c,q}[$*there exists an infinite cluster*$] = 1$;
3. *if $\lambda > 0$ satisfies $\cosh(\lambda) = \sqrt{q}/2$, then*

$$\lim_{n\to\infty} -\tfrac{1}{n}\log \phi^0_{p_c,q}[0 \textit{ is connected to (graph) distance } n] = \lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k}\tanh(k\lambda). \quad (1.2)$$

Furthermore, the quantity on the right-hand side is positive and as $q \searrow 4$,

$$\lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k}\tanh(k\lambda) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\sinh\left(\frac{\pi^2(2k+1)}{2\lambda}\right)} \sim 8\exp\left(-\frac{\pi^2}{\sqrt{q-4}}\right). \quad (1.3)$$

As in the Potts model, the quantity on the left-hand side of (1.2) corresponds to the inverse correlation length in the diagonal direction.

The proof of this theorem relies on the connection between the random-cluster model and the six-vertex model defined below. At the level of partition functions, this connection was made explicit by Temperley and Lieb in [20]. Here, we will explore the connection further to derive the inverse correlation length; see Section 3.3 for more details.

1.4 Results for the six-vertex model

The six-vertex model was initially proposed by Pauling in 1931 for the study of the thermodynamic properties of ice. While we are mainly interested in it for its connection to the previously discussed models, the six-vertex model is a major object of study on its own right. We do not attempt to give here an overview of the six-vertex model; we refer to [19] and Chapter 8 of [2] (and references therein) for a bibliography on the subject and to the companion paper [7] for details specifically used below.

Fix N and M even numbers, and consider the torus $\mathbb{T}_{N,M} := \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ as a graph with edge-set denoted $E(\mathbb{T}_{N,M})$. An *arrow configuration* $\vec{\omega}$ is a map attributing to each edge $e = \{x,y\} \in E(\mathbb{T}_{N,M})$ one of the two oriented edges (x,y) and (y,x). We say that an arrow configuration satisfies the *ice rule* if each vertex of $\mathbb{T}_{N,M}$ is incident to two edges pointing towards it (and therefore to two edges pointing outwards from it). The ice rule leaves six possible configurations at each vertex, depicted in Fig. 1.4, whence the name of the model. Each arrow configuration $\vec{\omega}$ receives a weight

$$w(\vec{\omega}) := \begin{cases} a^{n_1+n_2}\cdot b^{n_3+n_4}\cdot c^{n_5+n_6} & \text{if } \vec{\omega} \text{ satisfies the ice rule,} \\ 0 & \text{otherwise,} \end{cases} \quad (1.4)$$

where a, b, c are three positive numbers, and n_i denotes the number of vertices with configuration $i \in \{1,\dots,6\}$ in $\vec{\omega}$. In this article, we will focus on the case $a = b = 1$ and $c > 2$, and will therefore only consider such weights from now on. This choice of parameters is such that the six-vertex model is related to the critical random cluster model with cluster weight $q > 4$ on a tilted square lattice, as explained in Section 3.3.

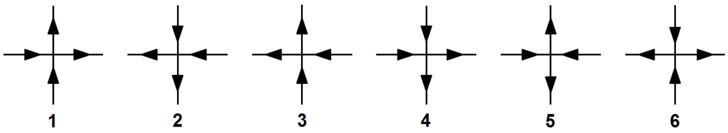

Figure 2: The 6 possibilities for vertices in the six-vertex model. Each possibility comes with a weight a, b or c.

In the context of this paper, the interest of the six-vertex model stems from its solvability using the transfer-matrix formalism. More precisely, the partition function of a toroidal six-vertex model may be expressed as the trace of the M-th power of a matrix V called the *transfer matrix*, which we define next. For more details, see [7].

Set $\vec{x} = (x_1, \dots, x_n)$ to be a set of ordered integers $1 \le x_1 < \cdots < x_n \le N$ with $0 \le n \le N$. Let $\Omega = \{-1, 1\}^{\otimes N}$ be the 2^N-dimensional real vector space spanned by the vectors $\Psi_{\vec{x}} \in \{\pm 1\}^N$ given by $\Psi_{\vec{x}}(i) = 1$ if $i \in \{x_1, \dots, x_n\}$, and -1 otherwise. The matrix V is defined by the formula

$$V(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = \begin{cases} 2 & \text{if } \Psi_{\vec{x}} = \Psi_{\vec{y}}, \\ c^{|\{i:\, \Psi_{\vec{x}}(i) \neq \Psi_{\vec{y}}(i)\}|} & \text{if } \Psi_{\vec{x}} \neq \Psi_{\vec{y}} \text{ and } \Psi_{\vec{x}} \text{ and } \Psi_{\vec{y}} \text{ are interlaced,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.5}$$

where $\vec{x}$ and $\vec{y}$ are *interlaced* if they have same number of entries n and $x_1 \le y_1 \le x_2 \le \cdots \le x_n \le y_n$ or $y_1 \le x_1 \le y_2 \le \cdots \le y_n \le x_n$. It is immediate that V is a symmetric matrix; in particular all its eigenvalues are real. Furthermore, it is made up of diagonal-blocks $V^{[n]}$ corresponding to its action on the vector spaces

$$\Omega_n := \mathrm{Vect}(\Psi_{\vec{x}} : \vec{x} \text{ has } n \text{ entries}) \qquad 0 \le n \le N.$$

As discussed in [7], each block $V^{[n]}$ satisfies the assumption of the Perron-Frobenius theorem[(1)], and thus has one dominant, positive, simple eigenvalue. For an integer $0 \le r \le N/2$, let $\Lambda_r(N)$ be the Perron-Frobenius eigenvalue of the block $V^{[N/2-r]}$, where we emphasize the dependence of Λ_r on N (recall that N is even). The main result dealing with the six-vertex model is the following asymptotic for the afore-mentioned eigenvalues.

Theorem 1.3. *For $c > 2$ and $r > 0$ integer, fix $\lambda > 0$ to satisfy $\cosh(\lambda) = \frac{c^2-2}{2}$. Then,*

$$\lim_{\substack{N\to\infty \\ N \in 4\mathbb{N}}} \frac{1}{N} \log \Lambda_0(N) = \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda} \tanh(k\lambda)}{k} \tag{1.6}$$

$$\lim_{\substack{N\to\infty \\ N \in 4\mathbb{N}}} \frac{\Lambda_r(N)}{\Lambda_0(N)} = \exp\Big[-|r| \Big(\lambda + 2 \sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda) \Big) \Big]. \tag{1.7}$$

The limit of $\frac{\Lambda_1(N)}{\Lambda_0(N)}$ is sometimes interpreted as twice the *surface tension* of the six-vertex model, and the second equation is effectively a computation of this quantity. The first identity may be reformulated in terms of the free energy, which defines the asymptotic behaviour of the partition function, as described below.

Corollary 1.4. *Fix $c > 2$ and $\lambda > 0$ such that $\cosh(\lambda) = \frac{c^2-2}{2}$. Then the free-energy $f(1,1,c)$ of the six-vertex model satisfies*

$$f(1,1,c) := \lim_{N,M\to\infty} \frac{1}{NM} \log \Big(\sum_{\vec{\omega} \text{ on } \mathbb{T}_{N,M}} w(\vec{\omega}) \Big) = \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda} \tanh(k\lambda)}{k}.$$

(1) More precisely, the entries of $V^{[n]}$ are non-negative and there exists an integer k such that $\big(V^{[n]}\big)^k$ has only positive entries. We will henceforth call a matrix with these properties a Perron-Frobenius matrix.

The previous corollary follows trivially from Theorem 1.3 once observed that the free energy does exist, and that the leading eigenvalue of V is the Perron-Frobenius eigenvalue of $V^{[N/2]}$ (see Section 3.2 for details).

Theorem 1.3 above will be obtained by applying the coordinate Bethe Ansatz to the blocks $V^{[n]}$ of the transfer matrix. This ansatz, aimed at finding eigenvalues of certain types of matrices, was introduced by Bethe [4] in 1931 for the Hamiltonian of the XXZ model. It has since been widely studied and developed, with applications in various circumstances, such as the one at hand. Its formulation for the six-vertex model is described in detail in [7]. For completeness, let us briefly discuss this technique again.

The idea is to try to express the eigenvalues of $V_N^{[n]}$ as explicit functions (see Theorem 3.1 below) of an n-uplet $\mathbf{p} = (p_1, \dots, p_n) \in (-\pi, \pi)^n$ satisfying the n equations

$$Np_j = 2\pi I_j - \sum_{k=1}^{n} \Theta(p_j, p_k) \qquad \forall j \in \{1, \dots, n\}, \tag{BE$_\Delta$}$$

where the I_j are integers or half-integers (depending on whether n is odd or even) between $-N/2$ and $N/2$, and $\Theta : \mathbb{R}^2 \to \mathbb{R}$ is the unique continuous function[(2)] satisfying $\Theta(0,0) = 0$ and

$$\exp(-i\Theta(x,y)) = e^{i(x-y)} \cdot \frac{e^{-ix} + e^{iy} - 2\Delta}{e^{ix} + e^{-iy} - 2\Delta}, \tag{1.8}$$

where $\Delta = (2 - c^2)/2$. This parametrization of the six-vertex model will be used throughout the paper. Depending on the choice of the I_j, the eigenvalue obtained may be different. It is also *a priori* unclear whether all eigenvalues of $V_N^{[n]}$ can be obtained via this procedure.

The asymptotic behaviour of $\Lambda_0(N)$ was computed in [22] using the coordinate Bethe Ansatz. The argument of [22] assumed that $\Lambda_0(N)$ is produced by a solution $\mathbf{p}(N) = (p_1, \dots, p_{N/2})$ to (BE$_\Delta$) with $n = N/2$ and the special choice $I_j = j - (n+1)/2$. An asymptotic analysis of the distribution of $p_1, \dots, p_{N/2}$ on $[-\pi, \pi]$ was then used to derive the asymptotic behaviour of $\Lambda_0(N)$. To our best understanding, certain gaps prevent this derivation from being completely rigorous in this first paper. Among them are the existence of solutions to (BE$_\Delta$), the fact that the associated eigenvector constructed by the Bethe Ansatz is non-zero, and the justification of the weak convergence of the point measure of $\mathbf{p}$ to an explicit continuous distribution. The recent work of Goldbaum [15] in the context of the Hubbard model (see below for more details) may in all likelihood be adapted to the six-vertex model to fill in these gaps, and provide a completely rigorous proof of (1.6).

The more refined asymptotic (1.7) requires further justification. For $r = 1$ (or equivalently -1), the limit was claimed to be derived in [2] and [5]. The two results do not seem to be coherent with each other (numerically, they do not match, even when allowing for a multiplicative constant). Baxter's result [2] is based on computations involving a more sophisticated version of the Bethe Ansatz and the eight-vertex model, which generalizes the six-vertex model. To our best understanding, both computations require assumptions which are difficult to rigorously justify. We are not aware of any computation of (1.7) for $|r| \geq 2$.

In light of this, we chose to write a fully rigorous, self-contained derivation of both (1.6) and (1.7). Moreover, we only use elementary tools, so as to render it accessible to a more diverse audience, less accustomed to the mathematical physics literature. The computations of the two limits in Theorem 1.3 will be used in a crucial way in the proof of Theorem 1.2.

1.5 Organization of the paper

Section 2: Study of the Bethe Equation. This step consists in the study of (BE$_\Delta$) with the choice

$$I_j := j - \frac{n+1}{2} \qquad \text{for } j \in \{1, \dots, n\}. \tag{1.9}$$

[(2)] The fact that Θ is well-defined, real-valued and analytic can be checked easily (see [7] for a full derivation).

This section does not involve any reference to the Bethe Ansatz or the six-vertex model. It is divided in three steps:

1. We first study two functional equations that we call the *continuous Bethe Equation* and the *continuous Offset Equation*, respectively, via Fourier analysis.

2. We then construct solutions to (BE_Δ) with prescribed properties. This represents an alternative proof (to that of [15]) of the existence of solutions to the Bethe Equation. The advantage of this approach (in addition to the fact that it is elementary) is that it provides good control on the increments $p_{j+1}-p_j$ of the solution. This will be crucial when analysing the asymptotic of $\Lambda_r(N)/\Lambda_0(N)$. It also provides tools for proving that the eigenvectors built via the Bethe Ansatz are non-zero and correspond to Perron-Frobenius eigenvalues.

3. Finally, we study the asymptotic behaviour of the solutions of the discrete Bethe equations using the continuous Bethe Equation. Furthermore, we compare solutions with different values of n using the continuous Offset Equation.

Section 3: From the Bethe Equation to the different models. This part contains the proofs of the main theorems. It is divided in two steps.

1. We use the Bethe Ansatz to relate the Bethe Equation to the eigenvalues of the transfer matrix of the six-vertex model. Then we study the asymptotic behaviour of the Perron-Frobenius eigenvalues of the different blocks of the transfer matrix using the asymptotic behaviour of the solutions to the Bethe Equation derived in the previous section (see the proof of Theorem 1.3).

2. We relate the six-vertex model to the random-cluster and Potts models via classical couplings. These relations, together with new results on the random-cluster model, enable us to prove Theorems 1.2 and 1.1.

Section 4: Fourier computations. The study will require certain computations using Fourier decompositions. While these computations are elementary, they may be lengthly, and would break the pace of the proofs. We therefore defer all of them to Section 4.

Notation. Most functions hereafter depend on the parameter $\Delta = \frac{2-c^2}{2} < -1$. For ease of notation, we will generally drop the dependency in Δ, and recall it only when it is relevant. We write ∂_i for the partial derivative in the i^{th} coordinate.

2 Study of the Bethe Equation

2.1 The continuous Bethe and Offset Equations

This section studies the following continuous functional equations for $\Delta < -1$

$$2\pi\rho(x) = 1 + \int_{-\pi}^{\pi} \partial_1\Theta(x,y)\rho(y)dy \qquad \forall x \in [-\pi,\pi], \qquad (\mathrm{cBE}_\Delta)$$

$$2\pi\tau(x) = \frac{\Theta(x,-\pi)+\Theta(x,\pi)}{2} - \int_{-\pi}^{\pi} \partial_2\Theta(x,y)\tau(y)dy \qquad \forall x \in [-\pi,\pi]. \qquad (\mathrm{cOE}_\Delta)$$

The first equation naturally arises as a continuous version of the Bethe Equation, while the second one will be useful when studying the displacement between solutions of the Bethe Equation for different values of n.

The main object of the section is the following proposition. For $\Delta < -1$, let k be the unique continuous function[3] from $[-\pi,\pi]$ to itself satisfying

$$e^{ik(\alpha)} = \frac{e^{\lambda} - e^{-i\alpha}}{e^{\lambda-i\alpha} - 1},$$

where $\lambda > 0$ is such that $\cosh(\lambda) = -\Delta$.

Proposition 2.1. *For $\Delta < -1$, the functions ρ and τ defined by*[4]

$$\rho(k(\alpha)) = \frac{1}{4\lambda k'(\alpha)} \sum_{j\in\mathbb{Z}} \frac{1}{\cosh[\pi(2\pi j + \alpha)/(2\lambda)]}, \tag{2.1}$$
$$\tau(k(\alpha)) = \sum_{m>0} \frac{(-1)^m}{\pi m} \tanh(\lambda m)\sin(m\alpha),$$

are the only solutions in $L^2([-\pi,\pi])$ of (cBE$_\Delta$) *and* (cOE$_\Delta$) *respectively. In particular, the function $\rho : (\Delta, x) \mapsto \rho(x)$ is strictly positive and analytic in $\Delta < -1$ and $x \in [-\pi,\pi]$.*

We prove this result by making a change of variables $x = k(\alpha)$ to obtain equations involving a convolution operator, and then use Fourier analysis to compute ρ and τ (and therefore deduce their uniqueness).

Proof In this proof, we fix $\Delta < -1$ and drop it from the notation. Set $R(\alpha) = 2\pi\rho(k(\alpha))k'(\alpha)$ and $T(\alpha) = 2\pi\tau(k(\alpha))$. The change of variables $x = k(\alpha)$ transforms (cBE$_\Delta$) and (cOE$_\Delta$) into[5]

$$R(\alpha) = \Xi_\lambda(\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha-\beta)R(\beta)d\beta \qquad \forall\alpha\in[-\pi,\pi], \tag{cBE$'_\Delta$}$$
$$T(\alpha) = \Psi(\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha-\beta)T(\beta)d\beta \qquad \forall\alpha\in[-\pi,\pi], \tag{cOE$'_\Delta$}$$

where, for $\mu \in \mathbb{R}$ and $\alpha \in [-\pi,\pi]$,

$$\Xi_\mu(\alpha) := \frac{\sinh(\mu)}{\cosh(\mu) - \cos(\alpha)} \qquad \text{and} \qquad \Psi(\alpha) = \frac{\Theta(k(\alpha),-\pi) + \Theta(k(\alpha),\pi)}{2}.$$

For any function $f \in L^2([-\pi,\pi])$, denote by $(\hat f(m))_{m\in\mathbb{Z}}$ its Fourier coefficients, defined as $\hat f(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\alpha}f(\alpha)d\alpha$. Then, (cBE$'_\Delta$) and (cOE$'_\Delta$) may be rewritten as

$$\hat R(m) = \hat\Xi_\lambda(m) - \hat\Xi_{2\lambda}(m)\hat R(m) \qquad \text{and} \qquad \hat T(m) = \hat\Psi(m) - \hat\Xi_{2\lambda}(m)\hat T(m) \qquad \forall m\in\mathbb{Z}. \tag{2.2}$$

The end of the proof is a simple computation which we resume next; details are given in Section 4. The residue theorem shows that $\hat\Xi_\mu(m) = \exp(-\mu|m|)$. In addition, a simple computation implies that $\hat\Psi(m) = \frac{(-1)^m}{im}\left(1 - \hat\Xi_{2\lambda}(m)\right)$ for $m \neq 0$ and $\hat\Psi(0) = 0$. Substituting these in (2.2), we deduce that, for all $m\in\mathbb{Z}$ ($m \neq 0$ for the second equality),

$$\hat R(m) = \frac{\hat\Xi_\lambda(m)}{1+\hat\Xi_{2\lambda}(m)} = \frac{1}{2\cosh(\lambda m)} \qquad \text{and} \qquad \hat T(m) = \frac{\hat\Psi(m)}{1+\hat\Xi_{2\lambda}(m)} = \frac{(-1)^m}{im}\tanh(\lambda|m|).$$

The conclusion follows by checking that functions given in (2.1) have the Fourier coefficients above; details are given in Section 4. The properties of positivity and analyticity of ρ follow

[3] The existence of k follows by taking the complex logarithm and fixing $k(\pm\pi) = \pm\pi$.

[4] In the definition of τ, the series is not absolutely summable; it stands for the limit of the partial sums.

[5] A series of algebraic manipulations is necessary for this step. The key is to observe that $k'(\alpha) = \Xi_\lambda(\alpha)$ and $\Xi_{2\lambda}(\alpha-\beta) = -\frac{d}{d\alpha}\Theta(k(\alpha),k(\beta)) = \frac{d}{d\beta}\Theta(k(\alpha),k(\beta))$.

directly from its explicit expression (observe that the terms of the sum in (2.1) are positive and converge exponentially fast to 0). □

Before turning to the discrete equations, let us provide an alternative proof of the uniqueness of the solution to (cBE$_\Delta$) based on a fixed-point theorem. While this proof does not give an explicit formula for ρ (a formula which will be useful later on), it highlights the importance of a particular norm which will play a central role in the next section.

Fix $\Delta < -1$. Consider the map T_c from the set $\mathscr{H}$ of bounded functions $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ with $\int_{-\pi}^{\pi} f(x)dx = 1/2$ to itself[(6)] defined by

$$2\pi \mathsf{T}_c(f)(x) = 1 + \int_{-\pi}^{\pi} \partial_1 \Theta(x,y) f(y) dy \qquad \forall x \in [-\pi, \pi].$$

We claim that this map is contractive for the norm defined by

$$\|f\| := \sup\left\{ \left|k'(k^{-1}(x)) f(x)\right|, x \in [-\pi,\pi] \right\} = \sup\left\{ \left|k'(\alpha) f(k(\alpha))\right|, \alpha \in [-\pi,\pi] \right\}. \tag{2.3}$$

Note that k' is bounded away from 0 and infinity, so that the norm above is equivalent to the supremum norm $\|\cdot\|_\infty$ (with constants depending on $\Delta < -1$).

Indeed, let f and g be two functions in $\mathscr{H}$. Set $F = k' \cdot (f \circ k)$, $G = k' \cdot (g \circ k)$, $\tilde{F} = k' \cdot (\mathsf{T}_c(f) \circ k)$ and $\tilde{G} = k' \cdot (\mathsf{T}_c(f) \circ k)$, and notice that all these functions integrate to 1/2 on $[-\pi, \pi]$. Letting $m_\Xi = \min\{\Xi_{2\lambda}(x) : x \in [-\pi,\pi]\} > 0$, we find that, for any $\alpha \in [-\pi, \pi]$,

$$\begin{aligned}
|\tilde{F}(\alpha) - \tilde{G}(\alpha)| &= \tfrac{1}{2\pi} \Big| \int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha - \beta)(F(\beta) - G(\beta)) d\beta \Big| \\
&= \tfrac{1}{2\pi} \Big| \int_{-\pi}^{\pi} \big(\Xi_{2\lambda}(\alpha - \beta) - m_\Xi\big)(F(\beta) - G(\beta)) d\beta \Big| \\
&\le \tfrac{1}{2\pi} \|F - G\|_\infty \int_{-\pi}^{\pi} \big(\Xi_{2\lambda}(\beta) - m_\Xi\big) d\beta \\
&\le (1 - m_\Xi) \|F - G\|_\infty,
\end{aligned} \tag{2.4}$$

where we used the fact that $\Xi_{2\lambda}(\beta)$ integrates to 2π (since $\hat{\Xi}_{2\lambda}(0) = 1$) in the final line. Observing that $\|F - G\|_\infty = \|f - g\|$ and $\|\tilde{F} - \tilde{G}\|_\infty = \|\mathsf{T}_c(f) - \mathsf{T}_c(g)\|$, we conclude that T_c is contracting.

2.2 The discrete Bethe Equation

The main object of this section is to prove existence and regularity of solutions to the Bethe Equation recalled below

$$N p_j = 2\pi I_j - \sum_{k=1}^{n} \Theta(p_j, p_k), \qquad \forall j \in \{1, \dots, n\}, \tag{BE$_\Delta$}$$

with the choice (1.9) for the I_j, namely $I_j = j - \frac{n+1}{2}$ for $1 \le j \le n$. We will be looking for solutions $\mathbf{p}$ with additional symmetry (which takes into account the symmetry of the I_j). More precisely, we will be looking for solutions in

$$\mathscr{S}_n := \Big\{ \mathbf{p} = (p_1, \dots, p_n) : -\pi < p_1 < p_2 < \dots < p_n < \pi \text{ and } p_{n+1-j} = -p_j, \forall j \Big\}.$$

For any vector $\mathbf{p} = (p_1, \dots, p_n)$, we set $p_0 = p_n - 2\pi$ and $p_{n+1} = p_1 + 2\pi$. Hereafter, N will always denote an even integer.

(6) That $\mathsf{T}_c(\mathscr{H}) \subset \mathscr{H}$ follows from Fubini's theorem and the fact that $\Theta(\pi, y) - \Theta(-\pi, y) = -2\pi$ for all $y \in [-\pi, \pi]$.

Maybe the most natural approach to proving the existence of solutions to (BE_Δ) (for fixed Δ, N and n) is to apply the Brouwer Fixed-Point Theorem[7] to the map $\mathsf{T} : \mathscr{S}_n \to \mathscr{S}_n$ defined by

$$\mathsf{T}(p_1, \dots, p_n) = \left(\frac{2\pi I_j}{N} - \frac{1}{N} \sum_{k=1}^{n} \Theta(p_j, p_k) \right)_{1 \le j \le n}.$$

Indeed, $\mathbf{p}$ being a fixed point of T is equivalent to it satisfying (BE_Δ). The fact that T maps $\mathscr{S}_n$ to itself follows directly from the monotonicity and anti-symmetry of Θ, and from the fact that

$$-2\pi \le \Theta(x,y) + \Theta(x,-y) \le 2\pi \qquad \forall x, y \in [-\pi, \pi].$$

The Brouwer Fixed-Point Theorem indeed applies to T, and solutions to (BE_Δ) may thus be shown to exist for any $\Delta < -1$. Having said that, it will be important that the solutions vary continuously as functions of Δ, which does not follow from such arguments. Such a continuity statement was proved by Pedro Goldbaum in [15] for the 1D Hubbard model. His argument entails applying an Index theorem to a well-chosen field, and thus deducing that the solutions form families of continuous curves. This argument may be adapted to our context to prove that there exists a continuous curve of solutions to (BE_Δ) in the set $[-\infty, -1) \times [-\pi, \pi]^n$, extending over the whole range of Δ.

However, we wish to prove a slightly stronger statement: we would like the solutions to have some regularity, in that they should be close to ρ, the solution we explicitly computed in (2.1), in some appropriately chosen sense. This will be important when comparing solutions for different values n to compute the limit of $\Lambda_r(N)/\Lambda_0(N)$.

We therefore chose another path to prove existence of solutions, based on the Implicit Function Theorem. Our approach has the further advantage of being short and elementary, and of proving that the obtained solution is close to the continuous one (which renders the asymptotic analysis of $\Lambda_0(N)$ essentially trivial). Furthermore, we will also prove that the map $\Delta \mapsto \mathbf{p}_\Delta$ is not only continuous but analytic, a fact which will be useful in proving that the eigenvalue associated to $\mathbf{p}_\Delta$ is the Perron-Frobenius one. The downside is that it only yields a solution on an interval $[-\infty, \Delta_N]$ with $\Delta_N < -1$, tending to -1 as N tends to infinity (which will be sufficient for the application we have in mind).

Before stating the theorem, let us explain how we will compare a solution $\mathbf{p}$ of (BE_Δ) to the continuous solution ρ of (cBE_Δ). For $\mathbf{p} \in \mathscr{S}_n$, introduce $\rho_\mathbf{p} : [-\pi, \pi] \to \mathbb{R}$ defined by

$$\rho_\mathbf{p}(t) = \frac{I_{j+1} - I_j}{N(p_{j+1} - p_j)} \qquad \text{if } t \in [p_j, p_{j+1}), \tag{2.5}$$

where I_{n+1} and I_0 are defined by $I_{n+1} - I_1 = I_n - I_0 = N/2$. We measure the distance from $\mathbf{p}$ to the continuous solution using $\|\rho_\mathbf{p} - \rho\|$, where $\|\cdot\|$ is the norm introduced in (2.3). This norm appears naturally in this context since the map T_c – which may be viewed as a continuous version of T – is contractive for $\|\cdot\|$.

Remark 2.2. *We chose to write $I_{j+1} - I_j$ in the numerators, since this would be the natural quantity would the I_j take arbitrary values. In our case, $I_{j+1} - I_j$ is equal to 1 for any $1 \le j < n$, and to $r+1$ for $j = 0$ and n (recall that $n = N/2 - r$).*

We are now in a position to state the main theorem of this section.

Theorem 2.3. *Fix $r \ge 0$ and $\Delta_0 < -1$. There exists $K > 0$ and N_0 such that, for any $N \ge N_0$, there exists a family of solutions $(\mathbf{p}_\Delta)_{\Delta \le \Delta_0}$ to the Bethe Equation* (BE_Δ) *with $n = N/2 - r$ satisfying*

[7] For sufficiently small values of Δ, the Brouwer Fixed-Point Theorem is not even necessary, since one may show that T is contractive for the ℓ^1 norm; this is not true for Δ close to -1.

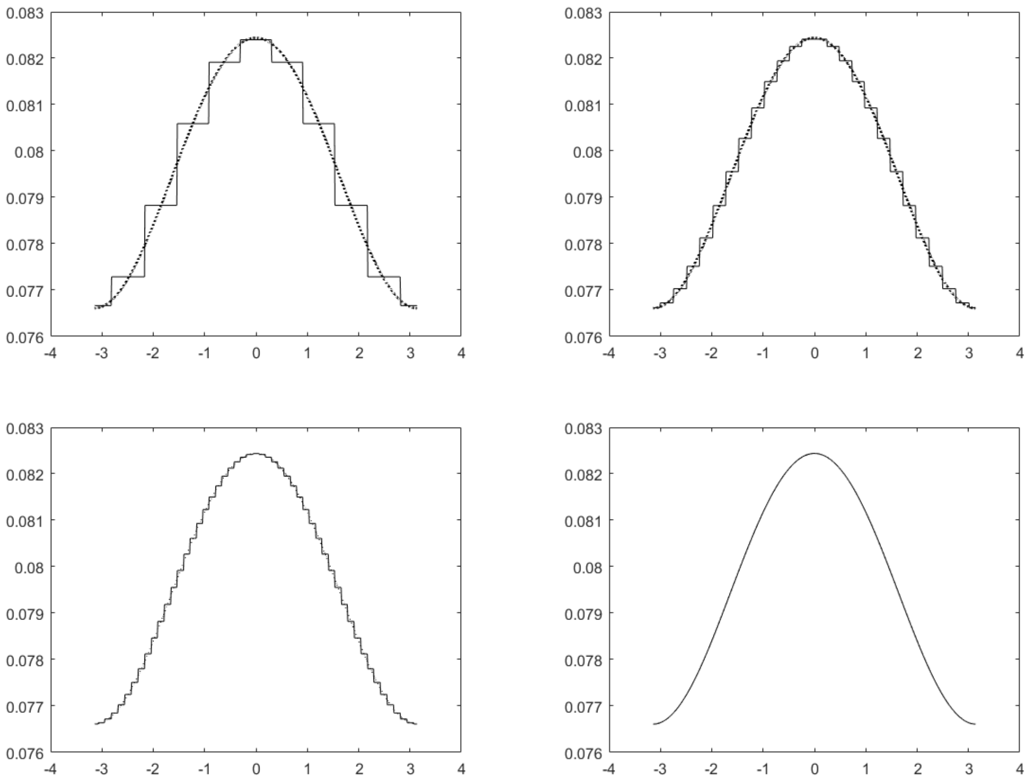

Figure 3: Numerical simulations of $\rho_{\mathbf{p}}$ for $\lambda = 4$, $N = 20, 50, 100$ and $n = N/2$. The limit distribution ρ is shown with the dotted line on each plot and on the last plot.

(i) $\Delta \mapsto \mathbf{p}_\Delta$ *is analytic on* $[-\infty, \Delta_0]$,

(ii) $\|\rho_{\mathbf{p}_\Delta} - \rho\| \le \frac{K}{N}$ *for all* $\Delta \le \Delta_0$.

Property (ii) should be understood as a regularity statement. It implies in particular that, for all $0 \le j \le n$,

$$p_{j+1} - p_j - \frac{I_{j+1} - I_j}{\rho(p_j)N} \;=\; O\Big(\frac{1}{N^2}\Big), \tag{2.6}$$

where $O(\cdot)$ depends on Δ_0 only[8]. As an important consequence for us, the previous expression implies that, for $N > N_0$ large enough and $0 \le j \le n$,

$$p_{j+1} - p_j \;\le\; \frac{2(I_{j+1} - I_j)}{m_\rho N}, \tag{2.7}$$

where $m_\rho > 0$ is the infimum of ρ over $x \in [-\pi, \pi]$ and $\Delta \le \Delta_0$. It will be crucial to us that the bound (2.7) above does not depend on the quantity K of Theorem 2.3 (even though N_0 may depend on K). Also notice that (ii) implicitly shows that $\mathbf{p}_\Delta$ is strictly inside $\mathscr{S}_n$ for all $\Delta \le \Delta_0$, provided N is large enough.

The rest of this section is dedicated to proving Theorem 2.3.

As we already mentioned, our strategy is based on the Implicit Function Theorem, which will be applied to $\mathbb{I} - \mathsf{T}$ (seen as a function of Δ and $\mathbf{p}$), where $\mathbb{I}$ denotes the identity function:

$$\mathbb{I}(\Delta, \mathbf{p}) = \mathbf{p}, \qquad \forall \mathbf{p} \in \mathscr{S}_n \text{ and } \Delta < -1.$$

[8] The notation $O(N^\alpha)$ is used to indicate a quantity that is bounded by CN^α for all N, where C is a constant. We say $O(\cdot)$ is uniform in certain parameters, to mean that C may be chosen independently of those parameters.

As noticed in [15], there is a priori no reason to be able to apply the Implicit Function Theorem at any zero of $\mathbb{I} - \mathsf{T}$, as the differential is not guaranteed to be invertible. Nonetheless, we will show that we may construct a family of such zeros that remains close to the continuous solution, and that this ensures that the differential of $\mathbb{I} - \mathsf{T}$ is invertible. The key of this argument is the following stability lemma.

Lemma 2.4. *Fix $r \geq 0$ and $\Delta_0 < -1$. Then, for $K > 0$ and N_0 large enough, for any $\Delta \leq \Delta_0$ and $N \geq N_0$, there exists no solution $\mathbf{p} \in \mathscr{S}_n$ of* (BE_Δ) *with $n = N/2 - r$ and*

$$\frac{K}{2N} \leq \|\rho_{\mathbf{p}} - \rho\| \leq \frac{K}{N}.$$

This lemma should not appear as a surprise. Indeed, as mentioned above, T is somewhat a discrete version of T_c (which is contractive and has fixed point ρ), at least in a vicinity of ρ. We did not manage to prove directly that T is contractive, but the lemma above is sufficient for our use.

Let us assume this lemma for now. Write $\mathbb{R}^n_{\text{sym}}$ for the $\lfloor n/2 \rfloor$-dimensional subspace of $\mathbb{R}^n$ of symmetric vectors:

$$\mathbb{R}^n_{\text{sym}} = \{(q_1, \dots, q_n) \in \mathbb{R}^n : q_j = -q_{n+1-j}, \ \forall j\}.$$

The map $\mathbb{I} - \mathsf{T}$ leaves this space stable (as can be seen by the symmetry properties of Θ). Therefore, we may apply the Implicit Function Theorem to $\mathbb{I} - \mathsf{T}$ as a function from $[-\infty, \Delta_0] \times \mathscr{S}_n$ to $\mathbb{R}^n_{\text{sym}}$ (recall that $\mathscr{S}_n \subset \mathbb{R}^n_{\text{sym}}$). Write $d(\mathbb{I} - \mathsf{T})$ for (the restriction of) the differential of $\mathbb{I} - \mathsf{T}$ in $\mathbf{p}$ as an automorphism of $\mathbb{R}^n_{\text{sym}}$. To apply the Implicit Function Theorem at some point $(\Delta, \mathbf{p})$ one needs to ensure that $d(\mathbb{I} - \mathsf{T})(\Delta, \mathbf{p})$ is invertible. This is done via the lemma below; its proof is deferred to the end of the section.

Lemma 2.5. *Fix $r \geq 0$, $\Delta_0 < 1$ and $K > 0$. Then there exists N_0 such that, for any $\Delta \leq \Delta_0$ and $N \geq N_0$, $d(\mathbb{I} - \mathsf{T})(\Delta, \mathbf{p})$ is invertible for any solution $\mathbf{p} \in \mathscr{S}_n$ of* (BE_Δ) *with $n = N/2 - r$ and such that $\|\rho_{\mathbf{p}} - \rho\| \leq K/N$.*

Proof of Theorem 2.3 Fix $r \geq 0$ and $\Delta_0 < -1$; $K \geq 2r$ and N will be assumed large enough for Lemmas 2.4 and 2.5 to apply, further conditions on N will appear in the proof.

For $\Delta = -\infty$ we have $\Theta(x, y) = y - x$, and the Bethe Equation has a unique solution $\mathbf{p}_{-\infty}$ with $p_j = 2\pi I_j/(N - n)$ for $1 \leq j \leq n$. This solution satisfies $\mathbf{p} \in \mathscr{S}_n$ and $\|\rho_{\mathbf{p}} - \rho\| \leq K/N$ (ρ is the constant function $1/(4\pi)$ when $\Delta = -\infty$ and we assumed $K \geq 2r$).

Due to Lemma 2.5, the Implicit Function Theorem may be applied repeatedly to extend the solution from $\mathbf{p}_{-\infty}$ to an analytic function $\Delta \mapsto \mathbf{p}_\Delta$, as long as $\|\rho_{\mathbf{p}_\Delta} - \rho\| \leq K/N$ and $\mathbf{p}_\Delta \in \mathscr{S}_n$. The latter condition is implied by the former when N is large enough; we may therefore ignore it. Lemma 2.4 shows that $\mathbf{p}_\Delta$, being continuous in Δ, may never exit the ball of radius $K/(2N)$ around ρ for the $\|\cdot\|$-norm. Thus $\Delta \mapsto \mathbf{p}_\Delta$ is defined for all $\Delta \leq \Delta_0$, analytic and such that $\|\rho_{\mathbf{p}_\Delta} - \rho\| \leq K/N$ for all $\Delta \leq \Delta_0$. $\square$

To close the section, we prove Lemmas 2.4 and 2.5.

Proof of Lemma 2.4 Let $r \geq 0$ and $\Delta_0 < -1$; bounds on K and N_0 will appear throughout the proof. Consider $\Delta \leq \Delta_0$, $N \geq N_0$ and $\mathbf{p} = (p_1, \dots, p_n)$ a solution of (BE_Δ) with $n = N/2 - r$ and $\|\rho_{\mathbf{p}} - \rho\| \leq K/N$.

In this proof, $O(\cdot)$ is uniform in K and $j = 1, \dots, N/2 - r$ (but may depend on r). In particular, by (2.7), we may write that $p_{j+1} - p_j = O(1/N)$, provided N_0 is large enough. For further reference, note that the derivatives of the functions ρ, k, Θ, etc, are all bounded uniformly in $\Delta < \Delta_0$.

Define the smooth function $f_{\mathbf{p}} : \mathbb{R} \to \mathbb{R}$ by

$$f_{\mathbf{p}}(x) := \frac{1}{2\pi}\Big(x + \frac{1}{N}\sum_{k=1}^{n} \Theta(x, p_k)\Big). \tag{2.8}$$

For any $t \in [p_j, p_{j+1})$, apply the Mean Value Theorem to construct $\xi_j \in (p_j, p_{j+1})$ such that

$$\rho_{\mathbf{p}}(t) \overset{(2.5)}{=} \frac{I_{j+1} - I_j}{N(p_{j+1} - p_j)} = \frac{f_{\mathbf{p}}(p_{j+1}) - f_{\mathbf{p}}(p_j)}{p_{j+1} - p_j} = f'_{\mathbf{p}}(\xi_j). \tag{2.9}$$

In the second identity, we used that $\mathbf{p}$ is a fixed point for T and therefore satisfies $f_{\mathbf{p}}(p_j) = I_j/N$.

Since $p_{j+1} - p_j = O(1/N)$, for any $t \in [p_j, p_{j+1})$, we may approximate $\rho(t)$ by $\rho(\xi_j)$ and $k'(k^{-1}(t))$ by $k'(k^{-1}(\xi_j))$ to deduce that

$$\begin{aligned} k'(k^{-1}(t))|\rho_{\mathbf{p}}(t) - \rho(t)| &\le (1 + O(\tfrac{1}{N}))\, k'(k^{-1}(\xi_j))|f'_{\mathbf{p}}(\xi_j) - \rho(\xi_j)| + O(\tfrac{1}{N}) \\ &\le (1 + O(\tfrac{1}{N}))\|f'_{\mathbf{p}} - \rho\| + O(\tfrac{1}{N}). \end{aligned}$$

Therefore, the lemma follows readily from the following inequality, which we prove below:

$$\|f'_{\mathbf{p}} - \rho\| \le (1 - m_\Xi)\|\rho_{\mathbf{p}} - \rho\| + O(\tfrac{1}{N}), \tag{2.10}$$

where $m_\Xi = \inf\{\Xi_{2\lambda}(x) : x \in [-\pi, \pi] \text{ and } \Delta \le \Delta_0\} > 0$. Indeed, assuming (2.10) holds, the previous computation shows

$$\|\rho_{\mathbf{p}} - \rho\| \le (1 - m_\Xi)\|\rho_{\mathbf{p}} - \rho\| + O(\tfrac{1}{N}),$$

which implies the result for K large enough (recall that the constant in $O(1/N)$ above does not depend on K).

Hence, we only need to prove (2.10) to finish the proof of the lemma. Set $R_{\mathbf{p}}(\alpha) := \rho_{\mathbf{p}}(k(\alpha))k'(\alpha)$. Fix $x = k(\alpha)$. With this definition, the change of variable explained in the previous section implies that

$$2\pi f'_{\mathbf{p}}(x) \;=\; 1 + \frac{1}{N}\sum_{k=1}^{n} \partial_1\Theta(x, p_k) \;=\; \int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha - \beta)R_{\mathbf{p}}(\beta)d\beta \;+\; O(\tfrac{1}{N}),$$

where we used again that $\max\{p_{j+1} - p_j\} = O(\frac{1}{N})$ and that $\partial_2\partial_1\Theta$ is bounded uniformly to approximate $\partial_1\Theta(x, p_k)$ by $\partial_1\Theta(x, k(\beta))$. Thus,

$$\begin{aligned} k'(k^{-1}(x))|f'_{\mathbf{p}}(x) - \rho(x)| \;&=\; \Big|\tfrac{1}{2\pi}\int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha - \beta)(R_{\mathbf{p}}(\beta) - R(\beta))d\beta\Big| \;+\; O(\tfrac{1}{N}) \\ &\overset{(2.4)}{\le}\; (1 - m_\Xi)\|\rho_{\mathbf{p}} - \rho\| \;+\; O(\tfrac{1}{N}), \end{aligned}$$

where in the last inequality, we can apply (2.4) since $\int_{-\pi}^{\pi} R_{\mathbf{p}}(\alpha)d\alpha = \int_{-\pi}^{\pi} \rho_{\mathbf{p}}(x)dx = \frac{1}{2}$. □

Proof of Lemma 2.5 Let r, Δ_0 and K be as in the lemma; N_0 will be chosen later in the proof. Fix $\Delta \le \Delta_0$, $N \ge N_0$ and $\mathbf{p} \in \mathscr{S}_n$ satisfying (BE_Δ) with $n = N/2 - r$ and such that $\|\rho_{\mathbf{p}} - \rho\| \le K/N$.

Note that for $\Delta = -\infty$, T is equal to $\mathbb{I}/2$, and the result is trivial. We may therefore assume $\Delta \in (-\infty, \Delta_0]$.

Write A for $d(\mathbb{I} - \mathsf{T})(\Delta, \mathbf{p})$, the differential of $\mathbb{I} - \mathsf{T}$ in $\mathbf{p}$ at the point $(\Delta, \mathbf{p})$ fixed above. Recall that we see A as an automorphism of $\mathbb{R}^n_{\mathrm{sym}}$. We will regard it as a square matrix of size

$\lfloor n/2 \rfloor$, when written in the basis $(e_j - e_{n+1-j})_{1\le j\le n/2}$ of $\mathbb{R}^n_{\mathrm{sym}}$, where $(e_j)_{1\le j\le n}$ is the canonical basis of $\mathbb{R}^n$. We may write A explicitly:

$$A_{jk} = \frac{\partial\big[(\mathbb{I}-\mathsf{T})(\Delta,\mathbf{p})\big]_j}{\partial p_k} - \frac{\partial\big[(\mathbb{I}-\mathsf{T})(\Delta,\mathbf{p})\big]_j}{\partial p_{n+1-k}} = \begin{cases} 1 + \frac{1}{N}\sum\limits_{k\neq j}\partial_1\Theta(p_j,p_k) - \frac{1}{N}\partial_2\Theta(p_j,-p_j) & \text{if } j = k, \\ \frac{1}{N}\big[\partial_2\Theta(p_j,p_k) - \partial_2\Theta(p_j,-p_k)\big] & \text{if } j \neq k, \end{cases}$$

for $1 \le j,k \le n/2$. For the second equality, we have used $p_{n+1-k} = -p_k$.

Also, write B for the diagonal matrix of size n, with entries $N(p_{j+1}-p_j) = \rho_{\mathbf{p}}(p_j)^{-1}$ on the diagonal. Rather than proving that A is invertible, we will prove that $\tilde{A} = AB$ is invertible, by showing that it is diagonally dominated – i.e. $\tilde{A}_{ii} > \sum_{j\neq i}\tilde{A}_{ij}$ for every i.

Below, the notation $O(\cdot)$ is considered uniform in $\Delta < \Delta_0$ and j, but may depend on the fixed constants K and r. Due to the condition $\|\rho_{\mathbf{p}} - \rho\| \le K/N$, we may write $p_{j+1} - p_j = O(1/N)$. Finally, we will use that the functions ρ, $\rho_{\mathbf{p}}^{-1}$, Θ and their derivatives are uniformly bounded for $\Delta \le \Delta_0$ (provided that N is large enough).

The diagonal terms of $\tilde{A}$ are

$$\begin{aligned}\tilde{A}_{jj} &= \frac{1}{\rho_{\mathbf{p}}(p_j)}\Big(1 + \frac{1}{N}\sum_{k\neq j}\partial_1\Theta(p_j,p_k)\Big) + O\big(\tfrac{1}{N}\big) \qquad (2.11)\\ &= \frac{1}{\rho_{\mathbf{p}}(p_j)}\Big(1 + \int_{-\pi}^{\pi}\partial_1\Theta(x,y)\rho(y)\Big) + O\big(\tfrac{1}{N}\big) \overset{(\mathrm{cBE}_\Delta)}{=} \frac{2\pi\rho(p_j)}{\rho_{\mathbf{p}}(p_j)} + O\big(\tfrac{1}{N}\big) = 2\pi + O\big(\tfrac{1}{N}\big).\end{aligned}$$

For the second equality, [9] we used $\|\rho_{\mathbf{p}} - \rho\| \le K/N$. We further note that the final equality follows thanks to the fact that $m_\rho > 0$.

For $j \neq k$, since $\Theta(x,-y) = -\Theta(-x,y)$ and Θ is increasing in the second variable,

$$\tilde{A}_{jk} = (p_{k+1} - p_k)\big[\partial_2\Theta(p_j,p_k) + \partial_2\Theta(-p_j,p_k)\big] \ge 0.$$

Therefore, for any fixed $1 \le j \le n/2$,

$$\begin{aligned}\sum_{k\neq j}|\tilde{A}_{jk}| = \sum_{k\neq j}\tilde{A}_{jk} &= \sum_{k=1}^{\lfloor n/2\rfloor}(p_{k+1}-p_k)\big[\partial_2\Theta(p_j,p_k) - \partial_2\Theta(-p_j,p_k)\big] + O\big(\tfrac{1}{N}\big)\\ &= \Theta(p_j,0) - \Theta(p_j,-\pi) - \Theta(-p_j,0) + \Theta(-p_j,-\pi) + O\big(\tfrac{1}{N}\big).\end{aligned}$$

Define $G(x,y) := \Theta(x,y) - \Theta(-x,y)$ to obtain

$$\sum_{k\neq j}|\tilde{A}_{jk}| = G(p_j,0) - G(p_j,-\pi) + O\big(\tfrac{1}{N}\big). \qquad (2.12)$$

A straightforward calculus exercise can show that, for any $\Delta < \Delta_0$, the function $G(x,0) - G(x,-\pi)$ satisfies

$$G(x,0) - G(x,-\pi) \le 4\arctan\Big(\frac{1}{4\Delta_0\sqrt{\Delta_0^2-1}}\Big) < 2\pi, \qquad \forall x \in [-\pi,0].$$

In conclusion, (2.11) and (2.12) show that for N large enough (depending on Δ_0, r and K only), $\tilde{A}$ is diagonal dominant and therefore invertible. □

2.3 The asymptotic behaviour of the solutions to the Bethe Equation

This section is devoted to two results that control the asymptotic behaviour of solutions to the Bethe Equation when $\rho_{\mathbf{p}}$ is close to ρ. The first deals with the "first order" asymptotics of solutions to (BE_Δ) with $n = N/2 - r$, for fixed r.

Theorem 2.6. *Fix $\Delta < -1$ and $r \ge 0$. Consider a sequence $\{\mathbf{p}(N) \in \mathscr{S}_n : N \ge 2r\}$ (for different dimensions N and with $n = N/2 - r$) satisfying $\|\rho_{\mathbf{p}(N)} - \rho\| \longrightarrow 0$. Then, $\mu_N := \frac{1}{N}\sum_{i=1}^n \delta_{p_i(N)}$ converges weakly to $\rho(x)dx$, where dx is Lebesgue's measure on $[-\pi,\pi]$.*

[9] The equality is obtained by a simple computation similar to that of the proof of Theorem 2.6 below. We omit the details here.

Proof Fix $\Delta < -1$ and $r \geq 0$. For any continuous function g on $[-\pi,\pi]$ and $N \geq 2r$, define $g_{\mathbf{p}(N)} : [-\pi,\pi] \to \mathbb{R}$ by $g_{\mathbf{p}(N)}(t) := g(p_j)$ if $t \in [p_j, p_{j+1})$ for some $0 \leq j \leq n$. Then

$$\int_{-\pi}^{\pi} g(x) d\mu_N(x) = \frac{1}{N} \sum_{j=1}^{n} g(p_j) = \int_{-\pi}^{\pi} g_{\mathbf{p}(N)}(x) \rho_{\mathbf{p}(N)}(x) dx - \frac{r}{N} g(p_n),$$

and we find

$$\begin{aligned} &\int_{-\pi}^{\pi} g(x)\rho(x)dx - \int_{-\pi}^{\pi} g(x)d\mu_N(x) \\ &= \int_{-\pi}^{\pi} g(x)\big[\rho(x) - \rho_{\mathbf{p}(N)}(x)\big]dx + \int_{-\pi}^{\pi} \big[g(x) - g_{\mathbf{p}(N)}(x)\big]\rho_{\mathbf{p}(N)}(x)dx + \frac{r}{N} g(p_n). \end{aligned}$$

Then (2.6) and (2.7) imply that each integral above converges to 0, and the result follows. □

The second result deals with the displacement of the solutions to the Bethe Equation with N and $n = N/2 - r$ with respect to the solution with N and $n = N/2$. Fix $r > 0$ and write henceforth $n = N/2 - r$. For $\mathbf{p} = (p_1, \ldots, p_{N/2}) \in \mathscr{S}_{N/2}$ and $\tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_n) \in \mathscr{S}_n$, introduce the *offset displacement* $\varepsilon = \varepsilon(\mathbf{p}, \tilde{\mathbf{p}}) \in \mathbb{R}^n$ defined for $1 \leq j \leq n$ by

$$\varepsilon_j = \begin{cases} N\big(\tilde{p}_j - p_{j+r/2}\big) & \text{if } r \text{ is even,} \\ N\Big(\tilde{p}_j - \dfrac{p_{j-(r-1)/2} + p_{j-(r+1)/2}}{2}\Big) & \text{if } r \text{ is odd} \end{cases} \tag{2.13}$$

and the *offset function* $f_{\mathbf{p},\tilde{\mathbf{p}}}(t) := \varepsilon_j$ if $t \in [\tilde{p}_j, \tilde{p}_{j+1})$ for some $0 \leq j \leq n$.

Remark 2.7. *The difference of index in* (2.13) *between* $\tilde{\mathbf{p}}$ *and* $\mathbf{p}$ *is made in such a way that the indices coincide when "starting from the middle of the interval* $[-\pi,\pi]$*".*

Remark 2.8. *Consider* $\mathbf{p}$ *and* $\tilde{\mathbf{p}}$ *given by Theorem 2.3 for* r *and* $r+1$*. Then, the solutions may be proved to be interlaced*[(10)]*, in the sense that* $p_j < \tilde{p}_j < p_{j+1}$ *for any* $1 \leq j < n$*. We will not use this property later, but this may be useful in subsequent works.*

While the asymptotic behaviour of individual solutions $\mathbf{p}$ is described by the continuous Bethe Equation, that of the offset displacement is governed by the Offset Equation, as shown in the next theorem.

Theorem 2.9. *Fix* $\Delta < -1$ *and* $r \geq 0$*. Consider two sequences* $\mathbf{p}(N) \in \mathscr{S}_{N/2}$ *and* $\tilde{\mathbf{p}}(N) \in \mathscr{S}_n$ *of solutions to the Bethe Equation with parameters* Δ*,* N *and respectively* $n = N/2$ *and* $N/2 - r$*. If* $\|\rho_{\mathbf{p}(N)} - \rho\| = O(\frac{1}{N})$ *and* $\|\rho_{\tilde{\mathbf{p}}(N)} - \rho\| = O(\frac{1}{N})$*, then*

1. $\rho \cdot f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}$ *converges uniformly on* $[-\pi,\pi]$ *to* $r \cdot \tau$*.*
2. *There exists* $C > 0$ *such that* $|f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}(x)| \leq C|x| + O(1/N)$ *for all* N *and* $x \in [-\pi,\pi]$*.*

Proof We drop N and $n = N/2 - r$ from the notation in the computations, except that we set $f_N = f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}$. We treat the case r even and odd separately. Below, all quantities $O(\cdot)$ may depend on Δ and r but are uniform in $j = 1, \ldots, n$ and $x \in [-\pi,\pi]$.

---

(10) The strategy is to show that the property of being interlaced is true for $\Delta = -\infty$ (this is a straightforward computation) and that this property does not cease to be true when increasing Δ continuously. Namely, one can prove that for any $\Delta < -1$, it is not possible that $p_j \leq \tilde{p}_j \leq p_{j+1}$ for every $1 \leq j < n$ and $\tilde{p}_k$ be equal to p_k or p_{k+1} for some $1 \leq k < n$. This is based on the fact that $\Theta(x,0) \in (-\pi,\pi)$ for any $x \in (-\pi,\pi)$, and that $\Theta(x,y) + \Theta(x,-y)$ defined on $[-\pi,0]^2$ is increasing in the first variable and decreasing in the second one. The continuity of $\Delta \mapsto \mathbf{p}, \tilde{\mathbf{p}}$ is then used to conclude. We leave the details of the computation to the reader.

Case r even. First, we bound the increments of f_N and show that f_N is almost equal to 0 at the origin, so as to prove the second property. Equation (2.6) and the bound (2.7) on the increments of $\mathbf{p}$ and $\tilde{\mathbf{p}}$ (both valid due to our assumptions) imply that

$$|\varepsilon_{j+1} - \varepsilon_j| \overset{(2.6)}{\leq} \Big|\frac{1}{\rho(p_{j+r/2+1})} - \frac{1}{\rho(p_{j+r/2})}\Big| + \Big|\frac{1}{\rho(\tilde{p}_{j+1})} - \frac{1}{\rho(\tilde{p}_j)}\Big| + O\big(\tfrac{1}{N}\big) \overset{(2.7)}{=} O\big(\tfrac{1}{N}\big). \tag{2.14}$$

Now, by symmetry, $p_{N/2} = p_{N/2+1}$ and $\tilde{p}_{n/2} = -\tilde{p}_{n/2+1}$ so

$$\varepsilon_{n/2} = -\varepsilon_{n/2+1} = O\big(\tfrac{1}{N}\big). \tag{2.15}$$

Finally, observe that $\rho_{\tilde{p}}$ is bounded uniformly in N (since it converges to ρ in the norm $\|\cdot\|$, it also does in the uniform norm), and therefore $\tilde{p}_{j+1} - \tilde{p}_j > c/N$ for all N and j, where $c > 0$ is some constant independent of N and j. This implies the existence of $C > 0$, independent of N and j, such that

$$|f_N(x)| \leq C|x| + O\big(\tfrac{1}{N}\big) \qquad \text{for all } x \in [-\pi, \pi]. \tag{2.16}$$

Let us now prove the first statement, that is the convergence of ρf_N. In light of (2.14), we may apply the Arzela-Ascoli theorem to the sequence (f_N) to extract a sub-sequential limit f. It suffices to show that $\rho f = r\tau$ to conclude.

For N and $1 \leq j \leq n$, the Bethe Equation applied to $p_{j+r/2}$ and $\tilde{p}_j$ implies

$$\varepsilon_j = \sum_{k=1}^{N/2} \Theta(p_{j+r/2}, p_k) - \sum_{k=1}^{n} \Theta(\tilde{p}_j, \tilde{p}_k).$$

In the first sum, we Taylor expand the terms $\Theta(p_{j+r/2}, p_{k+r/2})$ at $(\tilde{p}_j, \tilde{p}_k)$ for any $1 \leq k \leq n$ (while leaving the remaining terms as they are). This gives

$$\varepsilon_j = \underbrace{\sum_{k=1}^{r/2} \Theta(p_{j+r/2}, p_k) + \Theta(p_{j+r/2}, p_{n+1-k})}_{(1)} - \underbrace{\frac{1}{N}\sum_{k=1}^{n} \partial_1\Theta(\tilde{p}_j, \tilde{p}_k)\varepsilon_j}_{(2)} - \underbrace{\frac{1}{N}\sum_{k=1}^{n} \partial_2\Theta(\tilde{p}_j, \tilde{p}_k)\varepsilon_k}_{(3)} + O\big(\tfrac{1}{N}\big).$$

The final term is due to the second order errors in the Taylor expansion; it is indeed $O\big(\frac{1}{N}\big)$, since it contains $O(N)$ terms of order $O\big(\frac{1}{N^2}\big)$.

Fix $x \in [-\pi, \pi]$ and for each N (along the subsequence for which f_N tends to f) pick $\tilde{p}_j$ so that $x \in [\tilde{p}_j, \tilde{p}_{j+1})$. Then the equation displayed above offers an expression for $f_N(x)$. Taking N to ∞, we find that (1) converges to $\frac{r}{2}(\Theta(x, -\pi) + \Theta(x, \pi))$, and (2) and (3) converge to $(1 - 2\pi\rho(x))f(x)$ and $\int_{-\pi}^{\pi} \partial_2\Theta(x, y) f(y)\rho(y)dy$, respectively, by the definition of f and the weak convergence of μ_N (defined in statement of Theorem 2.6). Thus,

$$2\pi f(x)\rho(x) = \tfrac{r}{2}\big(\Theta(x, -\pi) + \Theta(x, \pi)\big) - \int_{-\pi}^{\pi} \partial_2\Theta(x, y) f(y)\rho(y)dy.$$

It follows that $\frac{1}{r}\rho(x)f(x) = \tau(x)$ by the uniqueness of the solution to the Offset Equation (cOE$_\Delta$).

Case r odd. The reasoning is similar. Equation (2.14) may be obtained in the same way and (2.15) may be replaced by $\varepsilon_{(n+1)/2} = 0$, which results from the symmetry of $\mathbf{p}$ and $\tilde{\mathbf{p}}$. One then expands around $(\tilde{p}_i, \tilde{p}_k)$ the expression

$$\sum_{k=1}^{n} \Theta(\tilde{p}_j, \tilde{p}_k) - \tfrac{1}{2}\big[\Theta(p_{j+(r-1)/2}, p_k) + \Theta(p_{j+(r+1)/2}, p_k)\big]$$

to obtain the same result. □

3 Proofs of the theorems

3.1 Perron-Frobenius eigenvalues of six-vertex model via Bethe Ansatz

The goal of this section is to show that the Perron-Frobenius eigenvalue of $V^{[n]}$ is given by the Bethe Ansatz from the solution $\mathbf{p}$ of (BE$_\Delta$) given by Theorem 2.3 (recall the choice $I_j = j - \frac{n+1}{2}$ for $1 \le j \le n$ in the theorem). We start by recalling the Bethe Ansatz for the transfer matrix of the six-vertex model. A more detailed discussion (with references) and an exposition of a proof may be found in the companion paper [7].

Recall that $\Delta = (2-c^2)/2$ and that the function Θ depends implicitly on Δ. For $z \neq 1$, define

$$L(z) := 1 + \frac{c^2 z}{1-z}, \qquad M(z) := 1 - \frac{c^2}{1-z}. \tag{3.1}$$

Theorem 3.1 (Bethe Ansatz for V)**.** *Fix $n \le N/2$ and $\Delta < -1$. Let $(p_1, p_2, \ldots, p_n) \in (-\pi,\pi)^n$ be distinct and satisfy the equations*

$$\exp(iNp_j) = (-1)^{n-1} \exp\left(-i \sum_{k=1}^{n} \Theta(p_j, p_k)\right) \quad \forall j \in \{1, 2, \ldots, n\}. \tag{BE}$$

Then, $\psi = \sum_{|\vec{x}|=n} \psi(\vec{x})\, \Psi_{\vec{x}}$, given by

$$\psi(\vec{x}) := \sum_{\sigma \in \mathfrak{S}_n} A_\sigma \prod_{k=1}^{n} \exp\left(ip_{\sigma(k)} x_k\right) \quad \textit{where} \quad A_\sigma := \varepsilon(\sigma) \prod_{1 \le k < \ell \le n} e^{ip_{\sigma(k)}} \left(e^{-ip_{\sigma(k)}} + e^{ip_{\sigma(\ell)}} - 2\Delta\right),$$

(for σ an element of the symmetry group $\mathfrak{S}_n$) satisfies the equation $V\psi = \Lambda\psi$, where

$$\Lambda = \Lambda(\mathbf{p}) := \begin{cases} \displaystyle\prod_{j=1}^{n} L(e^{ip_j}) + \prod_{j=1}^{n} M(e^{ip_j}) & \textit{if } p_1, \ldots, p_n \textit{ are non zero,} \\ \displaystyle\Big[2 + c^2(N-1) + c^2 \sum_{j \neq \ell} \partial_1 \Theta(0, p_j)\Big] \cdot \prod_{j \neq \ell} M(e^{ip_j}) & \textit{if } p_\ell = 0 \textit{ for some } \ell. \end{cases}$$

It is *a priori* unclear whether Ψ is non-zero, so that the previous theorem does not trivially imply that $\Lambda(\mathbf{p})$ is an eigenvalue of V. It is also unclear whether solutions of (BE) exist. Nonetheless, any solutions of (BE$_\Delta$) do also satisfy (BE). In particular, Theorem 2.3 provides us with a family of solutions to (BE$_\Delta$), and our goal is to prove that the corresponding value Λ given by the theorem above is the Perron-Frobenius eigenvalue of $V^{[n]}$.

Below, we will view $V^{[n]}$ as a function of Δ, hence we write it $V_\Delta^{[n]}$. We begin by computing the asymptotic of the Perron-Frobenius eigenvalue of $V_\Delta^{[n]}$ when Δ tends to $-\infty$.

Lemma 3.2. *Fix $r \ge 0$ and $N > 2r$ an even integer. Set $n = N/2 - r$. Then the largest eigenvalue λ of the matrix*

$$V_\infty^{[n]} := \lim_{\Delta \to -\infty} \frac{V_\Delta^{[n]}}{(-2\Delta)^n}$$

is simple and satisfies

$$\lambda \le 2^r \prod_{j=0}^{r-1} \left[1 + \cos\left(\frac{\pi(2j+1)}{n+2r}\right)\right], \tag{3.2}$$

where the empty product is set to 1.

Remark 3.3. *The matrix $V_\infty^{[n]}$ is symmetric and thus all its eigenvalues are real; its largest eigenvalue is therefore well-defined. It is not a Perron-Frobenius matrix, and thus we cannot be sure* a priori *that the largest eigenvalue is simple and largest in absolute value. We further note that the largest eigenvalue of $V_\infty^{[n]}$ is actually equal to the RHS of* (3.2)*, as will be shown in the proof of Corollary 3.4 below.*

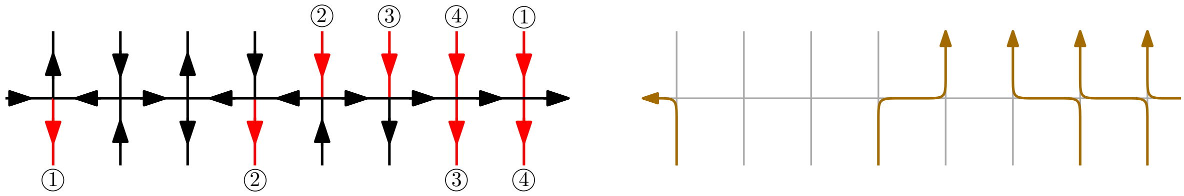

Figure 4: *Left*: Two configurations $\vec{x}$ and $\vec{y}$ with $N = 8$ and $r = 2$. The defects are marked by red arrows and are numbered. Notice that each defect has moved by one unit when going from $\Psi_{\vec{x}}$ (below) to $\Psi_{\vec{y}}$ (above), but none have exchanged places. *Right*: The $2r$ paths corresponding to the evolutions of the defects.

Proof Fix N and $n = N/2 - r$ as in the lemma. For two distinct configurations $\Psi_{\vec{x}}$ and $\Psi_{\vec{y}}$ in Ω_n, recall that $V_\Delta^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}})$ is non-zero only when $\Psi_{\vec{x}}$ and $\Psi_{\vec{x}}$ are interlacing, and in this case it is equal to

$$c^{|\{i:\,\Psi_{\vec{x}}(i)\neq\Psi_{\vec{y}}(i)\}|} = (2-2\Delta)^{\frac{1}{2}|\{i:\,\Psi_{\vec{x}}(i)\neq\Psi_{\vec{y}}(i)\}|}.$$

Since $\Psi_{\vec{x}}, \Psi_{\vec{y}} \in \Omega_n$, the number $P(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = |\{i : \Psi_{\vec{x}}(i) \neq \Psi_{\vec{y}}(i)\}|$ is at most $2n$. The normalization $(-2\Delta)^n$ is chosen to ensure that, for any pair of configurations $\vec{x}$ and $\vec{y}$ as above,

$$V_\infty^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = \begin{cases} 1 & \text{if } P(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

If $\vec{x}$ and $\vec{y}$ are configurations as above with $V_\infty^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = 1$, then $\Psi_{\vec{x}}$ has no consecutive up-arrows (and by symmetry neither does $\Psi_{\vec{y}}$). Indeed, if we suppose that $\Psi_{\vec{x}}$ has at least two consecutive up-arrows, then interlacement requires $\Psi_{\vec{y}}$ to have an up-arrow above at least one of the consecutive up-arrows of $\vec{x}$, which induces $P(\Psi_{\vec{x}}, \Psi_{\vec{y}}) < 2n$ and therefore $V_\infty^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = 0$. Thus, to study $V_\infty^{[n]}$, we may study its restriction to the set of configurations with no consecutive up-arrows.

In the case $n = N/2$, there is only one pair of such configurations: the completely staggered configurations – i.e. those with alternating up and down arrows. Hence, $V_\infty^{[N/2]}$ breaks down into a block-diagonal structure: a 2×2 block of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and a $\left[\binom{N}{N/2} - 2\right]$-dimensional block of 0's. The spectral structure of this matrix is very straightforward - there are simple eigenvalues at ± 1, and all other eigenvalues are 0, as required.

For $n = N/2 - r$, the situation is more complicated, and a direct computation of the spectrum of $V_\infty^{[n]}$ is best avoided. We do, however, have the tools to bound the dominant eigenvalue.

The set of configurations with no consecutive up-arrows can be parameterized by the location of the $2r$ "defects" – i.e. coordinates i with a down-arrow preceded by another down-arrow. By periodicity, we say that $\vec{x}$ has a defect at 1 if $\vec{x}$ has a down-arrow at 1 and N.

It is straightforward to show that a configuration with n up-arrows has no consecutive up-arrows if and only if there are exactly $2r$ defects whose parities alternate. Moreover, $\vec{x}$ and $\vec{y}$ are such that $V_\infty^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = 1$ if and only if $\Psi_{\vec{x}} \neq \Psi_{\vec{y}}$ and the locations of the defects of $\vec{y}$ may be obtained from those of $\vec{x}$, by moving each defect by precisely one unit (taken toroidally). Since the parity of the defects alternates in both states, no two defects can exchange positions between $\vec{x}$ and $\vec{y}$. See Fig. 4 for an example.

Write $\tilde{\Omega}_n$ for the sub-space of Ω_n generated by basis vectors $\Psi_{\vec{x}}$ with no two consecutive up arrows. Then, $V_\infty^{[n]}$ leaves this space stable, and we may consider its restriction to $\tilde{\Omega}_n$. A straightforward computation shows that this matrix is irreducible, in the sense that, for any $\Psi_{\vec{x}}, \Psi_{\vec{y}} \in \tilde{\Omega}_n$, there exists K such that $[V_\infty^{[n]}]^K(\Psi_{\vec{x}}, \Psi_{\vec{y}}) > 0$. As any symmetric irreducible matrix, it is either aperiodic or of period 2; a precise analysis shows that the latter occurs. Thus, the Perron-Frobenius theorem for irreducible (but not aperiodic) matrices guarantees that the largest eigenvalue is simple and maximizes the absolute value; the *smallest* eigenvalue actually has the same absolute value as the largest, unlike for true Perron-Frobenius matrices.

To determine λ, the largest eigenvalue, consider the following related construction. Let M be an even integer and $(a_1,\dots a_{2r})$ be an ordered set of integers between 1 and N of alternating parity. Consider families of $2r$ paths on $\mathbb{Z}/N\mathbb{Z}$ denoted $\{X_j(t) : 0 \le t \le M;\ j = 1,\dots,2r\}$ such that, for each j, $X_j(0) = X_j(M) = a_j$ and $|X_j(t+1) - X_j(t)| = 1$ for $1 \le t < M$. Additionally, impose that the paths $X_1,\dots,X_n$ are *non-intersecting*, in the sense that no pair of adjacent paths ever exchange position. Let $Z(M; a_1,\dots a_{2r})$ be the number of such paths, and $Z(M)$ the sum of $Z(M; a_1,\dots a_{2r})$ over all admissible $(a_1,\dots a_{2r})$. The discussion above indicates that

$$Z(M) = \mathrm{Tr}\left([V_\infty^{[n]}]^M\right),$$

which in turn implies that the largest eigenvalue (in absolute value) of $V_\infty^{[n]}$ is given by

$$\lambda = \lim_{M\to\infty} Z(M)^{1/M}.$$

Families of non-intersecting paths as those appearing in the definition of $Z(M)$ have been studied before, in particular in the work of Fulmek [14], which enables us to compute the asymptotic of $Z(M)$ directly. Fulmek enumerates the number of vertex-avoiding paths (i.e. families of paths as above, but such that no two ever hit the same vertex, rather than not intersecting). Luckily, the two are closely related: consider the transformation of a set of paths $\{X_j(t) : t, j\}$ as above to the set of paths $\{\tilde{X}_j(t) : t, j\}$ on $\mathbb{Z}/(N+2r)\mathbb{Z}$, where

$$\tilde{X}_j(t) = X_j(t) + j, \qquad \forall 1 \le t \le M,\ 1 \le j \le 2r.$$

On may check that this transformations induces a bijection between the set of non-intersecting paths starting and ending at $(a_1,\dots a_{2r})$ on $\mathbb{Z}/N\mathbb{Z}$ and that of vertex-avoiding paths starting and ending at $(a_1+1,\dots a_{2r}+2r)$ on $\mathbb{Z}/(N+2r)\mathbb{Z}$. Note that, while vertex-avoiding paths are generally allowed to intersect, the equal parity of their starting positions prevents them from doing so in this case (more precisely, observe that $\tilde{X}_{j+1}(t) - \tilde{X}_j(t)$ is even for all t and j).

Since we may get from any admissible starting position (that is, with even spacing between the starting points) to the position $(2,4,\dots,4r)$ in at most N steps, the limit of interest to us may be computed as

$$\lim_{M\to\infty} Z(M)^{1/M} = \lim_{M\to\infty} Z(M; 2,4,\dots,2r)^{1/M}.$$

We now state Corollary 7 of [14], which provides an exact expression of $Z(M; 2,4,\dots,2r)$ as the determinant of a matrix of size $2r$:

$$Z(M;2,4,\dots,2r) = (N+2r)^{-2r} \det\left[\xi^{i-j}\sum_{\ell=0}^{N+2r-1} \xi^{2(i-j)\ell}\left(2\cos\left(\frac{\pi(2\ell+1)}{N+2r}\right)\right)^M\right]_{1\le i,j\le 2r},$$

where we set $\xi = e^{i\frac{2\pi}{N+2r}}$. Since we are only interested in $\lim_{M\to\infty} Z(M;2,4,\dots,2r)^{1/M}$, we can simply study the dominating terms as $M \to \infty$. However, the apparently maximal terms cancel out in the computation of the determinant, and some care is needed.

To start, observe that the entries of the matrix above may be rewritten by grouping the terms ℓ and $\ell + N/2 + r$ (which are equal) as

$$2\xi^{i-j}\sum_{\ell=0}^{n+2r-1} \xi^{2(i-j)\ell}\Big[2\cos\left(\frac{\pi(2\ell+1)}{N+2r}\right)\Big]^M.$$

Then, we write the determinant out as

$$(N+2r)^{2r}2^{-2r(M+1)}Z(M;2,4,\dots,2r) = \sum_{\substack{\sigma\in\mathfrak{S}_{2r}\\ 0\le \ell_1,\dots,\ell_{2r}<n+2r}} \varepsilon(\sigma)\prod_{i=1}^{2r}\xi^{(i-\sigma(i))(2\ell_i+1)}\left[\cos\left(\frac{\pi(2\ell_i+1)}{N+2r}\right)\right]^M.$$

In the above, note that the term taken to the power M does not depend on σ. We conclude that

$$\lim_{M\to\infty} Z(M)^{1/M} = 2^{2r}\prod_{i=1}^{2r}\left|\cos\left(\frac{\pi(2\ell_i+1)}{N+2r}\right)\right|, \tag{3.3}$$

where $\ell_1,\dots,\ell_{2r}\in\{0,\dots,n+2r-1\}$ is the one that maximises the product above and is such that

$$\sum_{\sigma\in\mathfrak{S}_{2r}}\varepsilon(\sigma)\prod_{i=1}^{2r}\xi^{(i-\sigma(i))(2\ell_i+1)}\neq 0. \tag{3.4}$$

Consider $\ell_1,\dots,\ell_{2r}$ as above with $\ell_j=\ell_{j'}$ for some $j\neq j'$ and a permutation σ. Write $\tau_{j,j'}$ for the transposition of j and j'. The sum of the terms corresponding to $\ell_1,\dots,\ell_{2r}$ with σ and $\sigma\circ\tau_{j,j'}$ sum up to 0, and we find that the term in (3.4) is zero.

Thus, we may limit ourselves to terms with $\ell_1,\dots,\ell_{2r}$ all distinct. Among such sets, one maximising the product in (3.3) is given by $\ell_i=i-1$ for $i\le r$ and $\ell_i=n+2r-i$ for $i>r$. For this set, we find that the term in (3.3) is equal to

$$2^{2r}\prod_{i=1}^{r}\left[\cos\left(\frac{\pi(2\ell_i+1)}{N+2r}\right)\right]^2 = 2^r\prod_{j=0}^{r-1}\left[1+\cos\left(\frac{\pi(2j+1)}{n+2r}\right)\right].$$

This does not prove that $\lim_{M\to\infty}Z(M)^{1/M}$ is equal to the above, since (3.4) may not be satisfied. It does however show the claimed inequality. □

Corollary 3.4. *Fix $\Delta_0<-1$ and $r\ge 0$. Then, for N large enough, the Perron-Frobenius eigenvalue of $V_N^{[N/2-r]}$ for Δ_0 is given by $\Lambda(\mathbf{p}_{\Delta_0})$, where $(\mathbf{p}_\Delta)_{\Delta\le\Delta_0}$ is the family given by Theorem 2.3 applied to Δ_0 and r.*

Proof Fix $\Delta_0<-1$, $r\ge 0$ and let N be large enough for Theorem 2.3 to apply. Write $n=N/2-r$. Since N is fixed, we drop it from the notation.

The dependency of the Perron-Frobenius eigenvalue of $V_\Delta^{[n]}$ on Δ will be important, and we therefore denote it by $\Lambda_r(\Delta)$. Also, write $\psi(\mathbf{p}_\Delta)$ for the vector given by Theorem 3.1 for the solution $\mathbf{p}_\Delta$ to (BE$_\Delta$). We wish to prove that $\Lambda_r(\Delta)=\Lambda(\mathbf{p}_\Delta)$ for $\Delta=\Delta_0$. We will prove more generally that this is true for all $\Delta\le\Delta_0$.

First, observe that the Perron-Frobenius eigenvalue of a family of irreducible symmetric matrices varying analytically in a parameter (here Δ) varies analytically in this parameter as well (since it is an isolated zero of the characteristic polynomial). Therefore, $\Lambda_r(\Delta)$ is an analytic function. Since $\Delta\mapsto\mathbf{p}_\Delta$ is analytic, we deduce that $\Delta\mapsto\Lambda(\mathbf{p}_\Delta)$ also is, so that it is sufficient to show that $\Lambda(\mathbf{p}_\Delta)=\Lambda_r(\Delta)$ for Δ small enough in order to conclude that the two are equal for all $\Delta\le\Delta_0$. To do this, we shall prove two facts, namely that

- $\psi(\mathbf{p}_\Delta)$ is non-zero for Δ small enough (which implies that $\Lambda(\mathbf{p}_\Delta)$ is an eigenvalue of $V_\Delta^{[n]}$ for the corresponding values of Δ),
- $\lim_{\Delta\to-\infty}\frac{1}{(-2\Delta)^n}\Lambda(\mathbf{p}_\Delta)$ is the largest eigenvalue of $V_\infty^{[n]}$ (defined in Lemma 3.2).

These two facts indeed prove the result: since the largest eigenvalue of $V_\infty^{[n]}$ is simple, by continuity of $\Delta\mapsto\Lambda(\mathbf{p}_\Delta)$ and $\Delta\mapsto V_\Delta^{[n]}$, we deduce that $\Lambda(\mathbf{p}_\Delta)$ is the largest eigenvalue of $V_\Delta^{[n]}$

for Δ small enough. However, for finite Δ, $V_\Delta^{[n]}$ is a Perron Frobenius matrix, and $\Lambda(\mathbf{p}_\Delta)$ is then its Perron Frobenius eigenvalue. The observation of the previous paragraph is then sufficient to conclude.

The rest of the proof is dedicated to the two facts listed above. Recall that, at $\Delta = -\infty$, we have a simple formula for $\mathbf{p}$, namely

$$p_j = \frac{2\pi I_j}{N-n} \qquad \text{for all } 1 \le j \le n.$$

For the rest of the proof, write $\zeta = e^{2\pi i/(N-n)}$.

We start with the study of $\psi(\mathbf{p}_\Delta)$. Set $\psi_\infty := \lim_{\Delta\to-\infty}(-2\Delta)^{-\frac{n(n-1)}{2}}\psi(\mathbf{p}_\Delta)$. It suffices then to prove that ψ_∞ has at least one non-zero coordinate, and we shall do so for the coordinate $\psi_\infty(2,4,\ldots,2n)$. First we need to study the asymptotics of the coefficients A_σ appearing in the definition of ψ. For $\sigma \in \mathfrak{S}_n$, as $\Delta \to -\infty$,

$$A_\sigma = \varepsilon(\sigma)\prod_{j<k}\left[-2\Delta\zeta^{\sigma(j)-\frac{n+1}{2}}\right] + o(\Delta^{\frac{n(n-1)}{2}}).$$

By injecting this into the definition of ψ, we find that,

$$\begin{aligned}
\psi_\infty(2,\ldots,2n) &= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{k=1}^{n}\exp\left(ip_{\sigma(k)}\cdot 2k\right) \\
&= \sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\prod_{k=1}^{n}\zeta^{(2\sigma(k)-n+1)k}\prod_{1\le j<k\le n}\left(\zeta^{\sigma(j)-\frac{n+1}{2}}\right) \\
&= \zeta^{-\frac{1}{4}(n+1)^2 n}\sum_{\sigma}\varepsilon(\sigma)\zeta^{\sum_{j=1}^{n}\sigma(j)j}.
\end{aligned}$$

In the sum above, we recognise the determinant of the matrix $\left(\zeta^{j\cdot k}\right)_{1\le j,k\le n}$. This is the Vandermonde matrix corresponding to the values $\zeta,\zeta^2,\ldots,\zeta^n$, which are all distinct. Thus,

$$\psi_\infty(2,\ldots,2n) = \lim_{\Delta\to-\infty}(-2\Delta)^{-\frac{n(n-1)}{2}}\psi_\Delta(2,\ldots,2n) \neq 0.$$

We now turn to the study of $\lim_{n\to\infty}(-2\Delta)^{-n}\Lambda(\mathbf{p}_\Delta)$ (we show below that this limit exists). Before starting, we mention that, since $\psi_\infty \neq 0$, the above limit is an eigenvalue of $V_\infty^{[n]}$. With this and Lemma 3.2 in mind, it suffices to prove that it is equal to the RHS of (3.2) to deduce that it is the largest eigenvalue of $V_\infty^{[n]}$. We do this below.

The functions L and M defined in (3.1) depend on Δ and degenerate when $\Delta \to -\infty$. However, we have

$$\frac{1}{-2\Delta}L(z) \xrightarrow[\Delta\to-\infty]{} \frac{z}{1-z} \quad \text{and} \quad \frac{1}{-2\Delta}M(z) \xrightarrow[\Delta\to-\infty]{} \frac{-1}{1-z} \qquad \forall z \in [-\pi,\pi]\smallsetminus\{0\}.$$

Therefore, we find that

$$\frac{\Lambda(\mathbf{p}_\Delta)}{(-2\Delta)^n} \xrightarrow[\Delta\to-\infty]{} \begin{cases} \frac{2}{\prod_{j=1}^{n}(1-\zeta^{j-(n+1)/2})} & \text{if } n \text{ is even,} \\ (N-n)\times\prod_{j=1,j\neq(n+1)/2}^{n}\left(\frac{1}{1-\zeta^{j-(n+1)/2}}\right) & \text{if } n \text{ is odd.} \end{cases} \tag{3.5}$$

When n is an even number, the decomposition of the polynomial $x^{N-n}-1$ reads

$$x^{N-n}-1 = \prod_{j=1}^{N-n}\left(x-\zeta^{j-n/2}\right).$$

Thus, if we multiply the numerator and denominator in (3.5) by the terms corresponding to $j = n+1$ to $N-n$ and apply the above to $x = \zeta^{1/2}$, we find that

$$\begin{aligned}
\frac{2}{\prod_{j=1}^{n}(1-\zeta^{j-(n+1)/2})} &= \frac{2 \times \prod_{j=n+1}^{N-n}\left(1-\zeta^{j-(n+1)/2}\right)}{\zeta^{-(N-n)/2}\left(\zeta^{(N-n)/2}-1\right)} \\
&= \prod_{j=0}^{2r-1}\left(1-\zeta^{j+(n+1)/2}\right) \\
&= \prod_{j=0}^{r-1}\left[2-2\cos\left(\frac{\pi(2j+n+1)}{N-n}\right)\right] \\
&= 2^r \prod_{j=0}^{r-1}\left[1+\cos\left(\frac{\pi(2j+1)}{N-n}\right)\right],
\end{aligned}$$

where we grouped the $j = k$ and $j = 2r-1-k$ terms in the penultimate equality and used repeatedly that $N-n = n+2r$. This matches the expression in (3.2), as required.

We use a similar strategy when n is odd. Noting that

$$\prod_{\substack{1\le j\le N-n \\ j\ne (n+1)/2}}\left(1-\zeta^{j-(n+1)/2}\right) = \lim_{x\to 1}\frac{x^{N-n}-1}{x-1} = N-n,$$

we may perform a similar computation to obtain again

$$(N-n)\times \prod_{j=1, j\ne(n+1)/2}^{n}\left(\frac{1}{1-\zeta^{j-(n+1)/2}}\right) = 2^r \prod_{j=0}^{r-1}\left[1+\cos\left(\frac{\pi(2j+1)}{N-n}\right)\right].$$

□

Remark 3.5. *The analyticity of $\Delta \mapsto \mathbf{p}_\Delta$ allows us to avoid using a highly non-trivial fact (which would be necessary would we have continuity only), namely that for each Δ, N and n, the vector obtained by the Bethe Ansatz from the solution $\mathbf{p}_\Delta$ to* (BE$_\Delta$) *is non-zero. This is necessary to deduce that the associated value $\Lambda(\mathbf{p}_\Delta)$ is indeed an eigenvalue of the transfer-matrix.*

3.2 From the Bethe Equation to the six-vertex model: proof of Theorem 1.3

The goal of this section is the proof of Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3 We divide the proof in three steps. We first treat relation (1.6). We then focus on (1.7) with $r > 0$, even, and finally treat the case of (1.7) with $r > 0$, odd. Note that (1.7) with $r < 0$ follows directly from $r > 0$ since the transfer matrix V is invariant under global arrow flip, and therefore, the spectrums of V on Ω_n and Ω_{N-n} are identical.

Fix $c > 2$ and recall that $\Delta = \frac{2-c^2}{2} < -1$. Generically, in this proof $\mathbf{p} = \mathbf{p}_\Delta(N)$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}_\Delta(N)$ are given by Theorem 2.3 applied to $\Delta_0 = \Delta$ and $n = N/2$ and $N/2 - r$ respectively. We will always assume N to be a multiple of 4. For clarity, we will drop N and Δ from the notation and write $n = N/2 - r$.

Proof of (1.6)**.** The Bethe Ansatz and Corollary 3.4 imply that

$$\Lambda_0(N) := 2\prod_{j=1}^{n}|M(e^{ip_j})|,$$

where we used above that $\mathbf{p}$ is symmetric with respect to the origin and that $L(z) = \overline{M(\overline{z})}$ for $|z| = 1$ to deduce both products in the expression in Theorem 3.1 are equal to the product of the $|M|$. By Theorem 2.6, we deduce[11] that

$$\lim_{N\to\infty} \frac{1}{N} \log \Lambda_0(N) = \int_{-\pi}^{\pi} \log \left|M\left(e^{ix}\right)\right| \rho(x) dx. \tag{3.6}$$

The explicit form of ρ enables us to compute explicitly this integral via Fourier analysis (see Section 4 for details) to obtain the result.

Proof of (1.7)**, case** $r > 0$ **even.** In this case, both $N/2$ and n are even, so that the Bethe Ansatz together with Corollary 3.4 imply that

$$\frac{\Lambda_r(N)}{\Lambda_0(N)} = \underbrace{\prod_{j=1}^{n} \frac{|M\left(e^{i\tilde{p}_j}\right)|}{|M\left(e^{ip_{j+r/2}}\right)|}}_{(1)} \cdot \underbrace{\Big(\prod_{j=1}^{r/2} |M\left(e^{ip_j}\right)|\Big)^{-2}}_{(2)}, \tag{3.7}$$

where again we used that $\mathbf{p}$ is symmetric with respect to the origin, to group the two products into a single one.

We study the two terms separately. The term (2) converges to $|\Delta|^{-r}$, since M is continuous, $M(-1) = \Delta$ and the first $r/2$ coordinates of $\mathbf{p}$ converge to $-\pi$ as $N \to \infty$. As for the first term, by taking the logarithm and using that μ_N converges weakly (by Theorem 2.6) and $f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}$ converges uniformly (by Theorem 2.9), we deduce that (1) converges to

$$\exp\Big(r \int_{-\pi}^{\pi} \ell'(x)\tau(x) dx\Big), \tag{3.8}$$

where $\ell(x) := \log |M(e^{ix})|$. Note that $\ell'(x)$ behaves like $1/|x|$ near the origin. Nonetheless, this does not raise any issue here since by Theorem 2.9, $f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}(x) \leq C|x|$ uniformly in N; thus, $\ell'(x)\tau(x)$ is uniformly bounded, and the weak convergence applies.

The explicit forms of τ and ℓ lead to the expression in the statement of Theorem 1.3, thus concluding the proof. The relevant computation is based on Fourier analysis and is deferred to Section 4.

Proof of (1.7)**, case** $r > 0$ **odd.** In this case $N/2$ is even and n is odd. The Bethe Ansatz and Corollary 3.4 imply that

$$\frac{\Lambda_r(N)}{\Lambda_0(N)} = \underbrace{\frac{2 + c^2(N-1) + c^2 \sum\limits_{j\neq(n+1)/2} \partial_1\Theta(0,\tilde{p}_j)}{2}}_{(A)} \underbrace{\frac{\prod\limits_{j=1:j\neq(n+1)/2}^{n} |M\left(e^{i\tilde{p}_j}\right)|}{\prod\limits_{j=1}^{N/2} |M\left(e^{ip_j}\right)|}}_{(B)}.$$

(The 2 in the denominator of the first fraction comes from the fact that $\Lambda_0(N)$ involves two products, whereas $\Lambda_r(N)$ only contains one.) First, observe that the weak convergence of $\tilde{p}_j$ and (cBE$_\Delta$) imply that

$$(A) = \frac{Nc^2}{2}\Big(1 + \int_{-\pi}^{\pi} \partial_1\Theta(0,x)\rho(x)dx + o(1)\Big) = \frac{c^2}{2} 2\pi\rho(0)N + o(N).$$

[11] One should be wary of the log singularity at 0 of $\log|M|$. However, since $\log|M|$ is in $L^1[(-\pi,\pi)]$, standard truncation techniques are sufficient to show the convergence of the sum to the integral above. In particular one uses that the p_j's are well-separated – that is that $p_{j+1} - p_j \geq 2\pi/N$, which follows from (BE$_\Delta$) and the monotonicity of Δ – to ensure that there are not too many p_j's near the origin.

We now focus on (B) and divide it into four terms

$$(B) = \underbrace{\prod_{\substack{j=1\\ j\neq(n+1)/2}}^{n} \frac{|M\left(e^{i\tilde{p}_j}\right)|}{|M\left(e^{i\hat{p}_j}\right)|}}_{(1)} \cdot \underbrace{\Big(|M(e^{ip_{(r+1)/2}})| \prod_{j=1}^{(r-1)/2} |M\left(e^{ip_j}\right)|^2\Big)^{-1}}_{(2)} \cdot \underbrace{|M(e^{ip_{N/2}})|^{-1}}_{(3)}$$

$$\cdot \underbrace{\prod_{j=1}^{n} \frac{|M\left(e^{i\hat{p}_j}\right)|}{\sqrt{|M\left(e^{ip_{j+(r-1)/2}}\right)||M\left(e^{ip_{j+(r+1)/2}}\right)|}}}_{(4)}$$

where $\hat{p}_j = \frac{1}{2}(p_{j+(r-1)/2} + p_{j+(r+1)/2})$. To obtain the terms (2) and (3), we have used that $p_{N/2+1-j} = -p_j$. The same arguments as in the previous case imply that (1) converges to $\exp(r\int_{-\pi}^{\pi}\ell'(x)\tau(x)dx)$ and (2) to Δ^r. Furthermore, symmetry and (2.6) imply[(12)] that for each fixed k,

$$p_{N/2+k} = \frac{k-1/2}{\rho(0)N} + O\Big(\frac{k^2}{N^2}\Big),$$

where $O(\cdot)$ is uniform in k and N. Since $|M(e^{ip})| = c^2/|p| + o(1)$ for p close to 0, we deduce that

$$(3) = \frac{1}{2\rho(0)c^2N} + o\Big(\tfrac{1}{N}\Big),$$
$$(4) = \prod_{k=N/2+1}^{\infty} \frac{4p_kp_{k+1}}{(p_k+p_{k+1})^2} + o(1) = \prod_{k=1}^{\infty}\Big(1-\frac{1}{4k^2}\Big) + o(1) = \frac{2}{\pi} + o(1).$$

(In approximating (4), we used (2.6) to control $p_{N/2+k+1} - p_{N/2+k}$ for $k \geq N^{1/2}$.) Combining the estimates above and appealing to the computation of $\int_{-\pi}^{\pi}\ell'(x)\tau(x)dx$ in Section 4, we obtain the expected result. □

We now prove Corollary 1.4. The proof consists in two steps. We first prove that the free energy exists and that it is related to the sum of weighted configurations that are "balanced". We then relate the latter to the rate of growth of the Perron-Frobenius eigenvalue of $V_N^{[N/2]}$.

The proof of the existence of the free energy is slightly tedious due to the fact that the six-vertex model does not enjoy the finite-energy property. Nevertheless, it is close in spirit to corresponding proofs for other models. The next section enables to deduce the existence of the limit along N and M even using the connection to the random-cluster model. Nonetheless, we believe that a direct proof is of value.

The proof below is not connected to other arguments in this paper except through its result. We encourage the reader mostly interested in Theorems 1.1 and 1.2 to skip this proof.

Proof of Corollary 1.4 Step 1: Existence of free energy. For $N, M \in \mathbb{N}$, let $\mathsf{R}_{N,M}$ be the subgraph of $\mathbb{Z}^2$ with vertex set $V(\mathsf{R}_{N,M}) = \{1,\ldots,N\}\times\{1,\ldots,M\}$ and edge-set $E(\mathsf{R}_{N,M})$ formed of all edges of $\mathbb{Z}^2$ with both endpoints in $V(\mathsf{R}_{N,M})$. Define the edge-boundary of $\mathsf{R}_{N,M}$ as the set $\partial_e\mathsf{R}_{N,M}$ of edges of $\mathbb{Z}^2$ with exactly one endpoint in $V(\mathsf{R}_{N,M})$.

A six-vertex configuration on $\mathsf{R}_{N,M}$ is an assignment of directions to each edge of $E(\mathsf{R}_{N,M}) \cup \partial_e\mathsf{R}_{N,M}$. For such a configuration $\vec{\omega}$, the weight is computed as on the torus:

$$w(\vec{\omega}) = a^{n_1+n_2}\cdot b^{n_3+n_4}\cdot c^{n_5+n_6},$$

(12) We used that $p_{N/2+1} = \frac{1}{2}(p_{N/2+1} - p_{N/2}) = \dfrac{1}{2\rho(0)N} + O\Big(\dfrac{1}{N^2}\Big)$ and $p_{N/2+k+1} - p_{N/2+k} = \dfrac{1}{\rho(0)N} + O\Big(\dfrac{k}{N^2}\Big)$.

where $n_1, \dots, n_6$ are the numbers of vertices of $\mathsf{R}_{N,M}$ of types $1, \dots, 6$ respectively (as on the torus, we implicitly assign weight 0 to configurations not obeying the ice rule). As in the rest of the paper, we fix $a = b = 1$ and $c > 0$.

A *boundary condition* ξ for $\mathsf{R}_{N,M}$ is an assignment of directions to each boundary edge of $\mathsf{R}_{N,M}$. Let

$$Z^{\xi}_{N,M} = \sum_{\vec{\omega}} w(\vec{\omega}) \mathbf{1}_{\{\vec{\omega}(e)=\xi(e)\ \forall e \in \partial_e E(\mathsf{R}_{N,M})\}}.$$

Here, we are effectively summing only over configurations which agree with ξ on the boundary edges. Observe that configurations obeying the ice-rule and consistent with ξ exist only when $\mathsf{R}_{N,M}$ has as many incoming as outgoing edges in ξ.

A boundary condition ξ is called *toroidal* if $\xi(e) = \xi(f)$ for any boundary edges e and f of $\mathsf{R}_{N,M}$ such that f is a translate of e by $(0, M)$ or $(N, 0)$. When N is even, a toroidal boundary condition ξ is called *balanced* if it contains exactly $N/2$ up arrows on the lower row of $\partial_e E(\mathsf{R}_{N,M})$. Using this notation, the partition function of the six-vertex model on $\mathbb{T}_{N,M}$ may be expressed as

$$Z_{N,M} = \sum_{\vec{\omega} \in \mathbb{T}_{N,M}} w(\vec{\omega}) = \sum_{\xi:\, \text{toroidal}} Z^{\xi}_{N,M},$$

where the second sum is over all toroidal boundary conditions ξ on $\mathsf{R}_{N,M}$. Moreover, set

$$Z^{(\text{bal})}_{N,M} = \sum_{\xi:\, \text{balanced}} Z^{\xi}_{N,M},$$

the sum now being only over toroidal and balanced boundary conditions. Our goal is to prove that the following limits exist:

$$\lim_{N,M \to \infty} \frac{1}{MN} \log Z_{N,M} = \lim_{\substack{N,M \to \infty \\ N \text{ even}}} \frac{1}{MN} \log Z^{(\text{bal})}_{N,M}. \tag{3.9}$$

Above, the limits can be taken in the order that we want. We leave it as a simple exercise to the reader to check that (3.9) can be easily deduced from the following lemma.

This lemma may also be used to show that the free energy of the six vertex model with "free boundary conditions" (i.e. with partition function $\sum_{\xi} Z^{\xi}_{N,M}$ with sum over all boundary conditions) is equal to $f(1, 1, c)$.

Lemma 3.6. *(i) The following inequality holds:*

$$Z_{2N,2M} \geq Z^{(\text{bal})}_{2N,2M} \geq (\tfrac{1}{2})^{M+N} \big(Z_{N,M} \big)^4.$$

(ii) There exists $C > 0$ such that for all integers $n > N$ and $m > M$ with n and N even,

$$\frac{1}{nm} \log Z^{(\text{bal})}_{n,m} \geq \frac{1}{MN} \log Z^{(\text{bal})}_{N,M} - C\Big(\frac{N}{n} + \frac{M}{m} \Big).$$

Proof of Lemma 3.6 (i) Before proceeding to the proof, observe that the weight of a configuration is invariant under horizontal and vertical reflections and rotations by π of the configuration, as well as under inversion of all arrows. It follows that if ξ is a boundary condition on some rectangle $\mathsf{R}_{N,M}$ and ξ' is a boundary condition obtained from ξ via one of the operations mentioned above, then

$$Z^{\xi'}_{N,M} = Z^{\xi}_{N,M}.$$

Let N, M be integers and ξ be a boundary condition on $\mathsf{R}_{N,M}$. The construction below is described in Fig. 3.2. Let ξ_1 be the boundary condition on $\mathsf{R}_{N,M}$ obtained from ξ by horizontal reflection and arrow reversal, ξ_2 be obtained from ξ by vertical reflection and arrow reversal and ξ_3 be obtained from ξ by rotation by π. Let ζ be the toroidal boundary condition on $\mathsf{R}_{2N,2M}$ composed as follows:

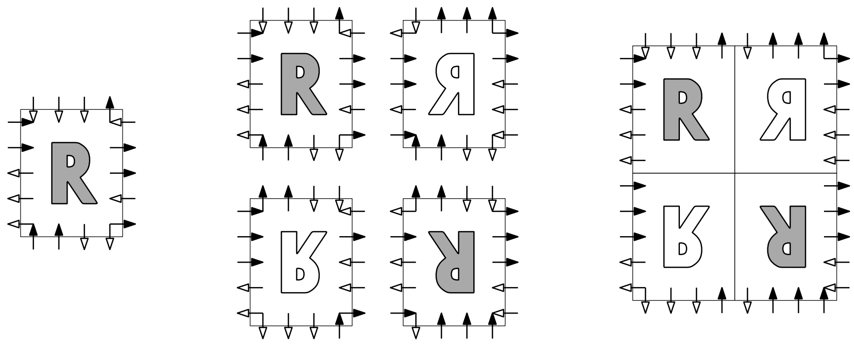

Figure 5: The passage from a boundary condition ξ to a balanced toroidal boundary condition ζ. The letter R inside the rectangles is only to indicate the performed transformations (rotations, reflections and reversal of all arrows).

- the top half of the left side agrees with ξ,
- the bottom half of the left side agrees with ξ_2,
- the left half of the top side agrees with ξ and
- the right half of the top side agrees with ξ_1.

Note that ζ is balanced so that

$$Z_{2N,2M} \geq Z^{(\mathrm{bal})}_{2N,2M} \geq Z^{\zeta}_{2N,2M}.$$

Upon inspection of Fig. 3.2, one may easily deduce that

$$Z^{\zeta}_{2N,2M} \geq Z^{\xi}_{N,M} Z^{\xi_1}_{N,M} Z^{\xi_2}_{N,M} Z^{\xi_3}_{N,M} = \left(Z^{\xi}_{N,M}\right)^4.$$

By summing over the 2^{N+M} toroidal boundary conditions ξ, the result follows.

(ii) Write $n = aN + r$ and $m = bM + q$ with $0 \leq r < N$ and $0 \leq q < M$. Fix a balanced toroidal boundary condition ξ on $\mathsf{R}_{N,M}$. The construction that follows is described in Fig. 3.2.

Let ξ_1 be the toroidal boundary condition on $\mathsf{R}_{aN,bM}$ obtained by repeating a times each horizontal side and b times each vertical side of ξ on the corresponding sides of ξ_1.

Let ξ_2 be the toroidal boundary condition on $\mathsf{R}_{r,bM}$, equal to ξ_1 on the vertical sides, with $r/2$ down arrows amassed to the left of the bottom (and top) side, completed by $r/2$ up arrows at the right of the bottom (and top) side.

Finally, define ξ_3 to be the toroidal boundary condition on $\mathsf{R}_{n,q}$ with only left-pointing arrows on the vertical sides, equal to the top of ξ_1 for the left-most aN arrows of both the top and bottom sides and equal to top of ξ_2 for the remaining r right-most arrows of the top and bottom sides.

Set ζ to be the boundary condition obtained from the gluing of ξ_1, ξ_2 and ξ_3, that is:

- the top and bottom sides of ζ are equal to those of ξ_3,
- the bottom bM arrows of the left- and right-sides of ζ are equal to those of ξ_1,
- the top q arrows of the left- and right-sides of ζ are pointing left-wards.

We thus easily deduce that

$$Z^{\zeta}_{n,m} \geq Z^{\xi_1}_{aN,bM} Z^{\xi_2}_{r,bM} Z^{\xi_3}_{n,q} \geq \left(Z^{\xi}_{N,M}\right)^{ab} Z^{\xi_2}_{r,bM} Z^{\xi_3}_{n,q}.$$

It remains to prove a lower bound on the last two terms. Observe that there exists at least one configuration $\vec{\omega}_3$ on $\mathsf{R}_{aN+r,q}$, agreeing with the boundary condition ξ_3 and having non-zero

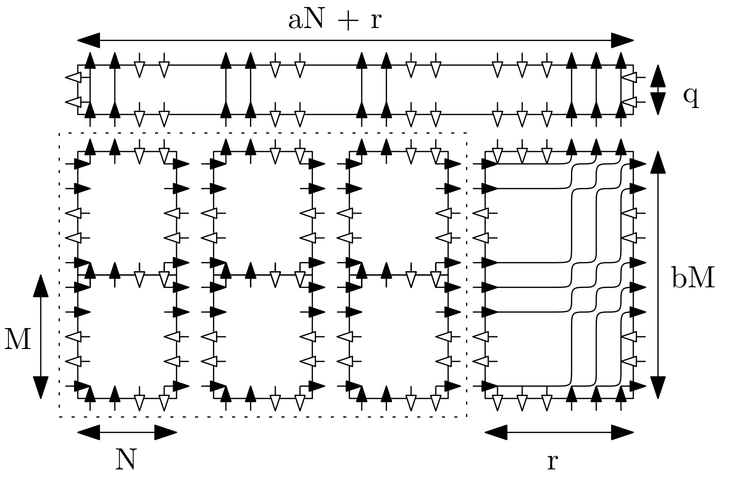

Figure 6: The block of size $\mathsf{R}_{aN,bM}$ with balanced toroidal boundary conditions ξ_1 is encircled; on its right is the block $\mathsf{R}_{r,bM}$ with boundary conditions ξ_2 and above is the block $\mathsf{R}_{aN+r,q}$ with boundary conditions ξ_3. In the two latter blocks, examples of configurations with positive weight are given (only the up and right-pointing edges are drawn in the interior of the blocks).

weight. It is obtained by setting all horizontal edges pointing left and all rows of vertical edges being identical to the top of ξ_3. This proves that

$$Z^{\xi_3}_{aN+r,q} \geq w(\vec{\omega}_3) \geq \min\{1,c\}^{(aN+r)q}.$$

A slightly more involved construction is necessary to exhibit a configuration $\vec{\omega}_2$ on $\mathsf{R}_{r,bM}$, consistent with ξ_2 and with non-zero weight. We represent it in Fig. 3.2 and leave it to the meticulous reader to check the details of its construction. It follows that

$$Z^{\xi_2}_{r,bM} \geq w(\vec{\omega}_2) \geq \min\{1,c\}^{bMr}.$$

We conclude that

$$Z^{(\mathrm{bal})}_{n,m} \geq Z^{\zeta}_{n,m} \geq \big(Z^{\xi}_{N,M}\big)^{ab} \min\{1,c\}^{bMr+aNq+rq}.$$

By choosing ξ maximizing $Z^{\xi}_{N,M}$, we deduce that

$$Z^{(\mathrm{bal})}_{n,m} \geq \big(Z^{(\mathrm{bal})}_{N,M}\big)^{ab} \min\{1,c\}^{bMr+aNq+rq}\big(\tfrac{1}{2}\big)^{ab(M+N)}.$$

The result follows by taking the logarithm

□

Calculation of the free energy. Recall Proposition 2.1 of [7] and more specifically equation (3.1), that expresses $Z_{N,M}$ as the trace of V^M. A straightforward adaptation shows that, for all N multiple of 4 and even,

$$Z^{(\mathrm{bal})}_{N,M} = \mathrm{Tr}\Big[\big(V^{[N/2]}\big)^M\Big] = \lambda_0^M + \lambda_1^M + \dots,$$

where $\lambda_0, \lambda_1, \dots$ are the $\binom{N}{N/2}$ eigenvalues of the diagonalizable matrix $V^{[N/2]}$, listed with multiplicity and indexed such that $|\lambda_0| \geq |\lambda_1| \geq \dots$. Since $V^{[N/2]}$ is a Perron-Frobenius matrix, $|\lambda_0| > |\lambda_1|$ and $\lambda_0 = \Lambda_0(N)$ (the eigenvalue computed in Theorem 1.3), so that

$$\lim_{M\to\infty} \frac{1}{M} \log \mathrm{Tr}\, \big(V^{[N/2]}\big)^M = \log \Lambda_0(N).$$

In light of (3.9), the limit defining $f(1,1,c)$ may be taken with $M\to\infty$ first, then $N\to\infty$ along multiples of 4. Thus we find,

$$f(1,1,c)=\lim_{\substack{N\to\infty\\ N\in 4\mathbb{N}}}\frac{1}{N}\log\Lambda_0(N)\overset{(1.6)}{=}\frac{\lambda}{2}+\sum_{m=1}^{\infty}\frac{e^{-m\lambda}\tanh(m\lambda)}{m}.$$

□

Remark 3.7. *We have shown that the free energy for the torus is the same as that for "free" boundary conditions. However, it is possible to construct boundary conditions on rectangles that lead to nonzero, but strictly smaller free energy. One prominent example of this is the Domain Wall boundary conditions, which have been studied extensively due to their relations to combinatorial objects, such as Young diagrams. Under these boundary conditions, the six-vertex model partition functions satisfy recursion relations that make it possible to exactly compute them for finite lattices (see [23] for more detail). This technique gives a formula for the free energy of this model (see[17]), which is different from the one we showed above.*

3.3 From the six-vertex to the random-cluster model: proof of Theorem 1.2

The proof is split into two main steps. First, we present a classical correspondence between the six-vertex and random cluster models using a series of intermediate representations (this correspondence may be found in [2]). Then, certain estimates on the random cluster model are provided, that are used to relate its correlation length to quantities obtained via the six-vertex model.

3.3.1 Correspondence between the random-cluster and six-vertex models

Fix two integers M,N, both even and $q>4$. Notice that the torus $\mathbb{T}_{N,M}$ is then a bipartite graph. Let $V_\circ(\mathbb{T}_{N,M})$ and $V_\bullet(\mathbb{T}_{N,M})$ be a partition of the vertices of the graph $\mathbb{T}_{N,M}+(\frac{1}{2},\frac{1}{2})$ (that is, $\mathbb{T}_{N,M}$ translated by $(\frac{1}{2},\frac{1}{2})$), each containing no adjacent vertices. Define the graphs $\mathbb{T}^\diamond_{N,M}$ and $(\mathbb{T}^\diamond_{N,M})^*$ as having vertex set $V_\bullet(\mathbb{T}_{N,M})$ and $V_\circ(\mathbb{T}_{N,M})$, respectively, and having an edge between vertices u and v if u is a translation of v by $(1,1)$ or $(-1,1)$ (see Fig. 7). By construction, $(\mathbb{T}^\diamond_{N,M})^*$ is the dual graph of $\mathbb{T}^\diamond_{N,M}$.

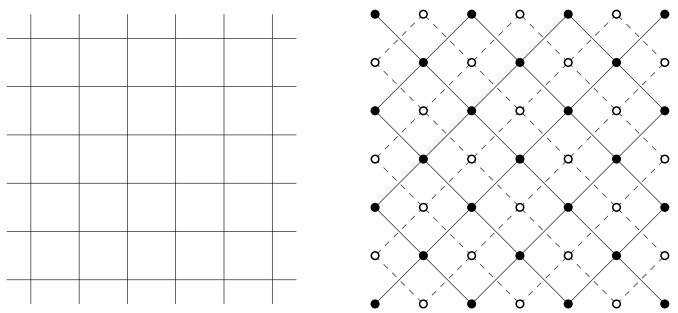

Figure 7: Left: the lattice $\mathbb{T}_{N,M}$ used for the six-vertex model. Right: the corresponding lattice for the Potts model, $\mathbb{T}^\diamond_{N,M}$ (in solid lines), and its dual (with dotted lines).

Let Ω_{RC} be the set of random cluster configurations on $\mathbb{T}^\diamond_{N,M}$ and $\Omega_{6\mathrm{V}}$ be the set of six-vertex configurations on $\mathbb{T}_{N,M}$. We will exhibit a correspondence between Ω_{RC} and $\Omega_{6\mathrm{V}}$ that will allow us to relate the free energy and correlation length of the two models. The correspondence consists of several intermediate steps embodied by Lemmas 3.8 – 3.11; the whole process is depicted in Fig. 8. The ultimate goal of this part is Corollary 3.12; which will alone be used to prove Theorem 1.2.

In linking the random-cluster and six-vertex models, we will use another type of configurations called *loop configurations*. An oriented loop on $\mathbb{T}_{N,M}$ is a cycle on $\mathbb{T}_{N,M}$ which is edge-disjoint and non-self-intersecting. We may view oriented loops as ordered collections of edges of $E(\mathbb{T}_{N,M})$, quotiented by cyclic permutations of the indices. Un-oriented loops (or simply loops) are oriented loops considered up to reversal of the indices. A (oriented) loop configuration on $\mathbb{T}_{N,M}$ is a partition of $E(\mathbb{T}_{N,M})$ into (oriented) loops.

To each $\omega \in \Omega_{\mathrm{RC}}$ we associate a loop configuration $\omega^{(\ell)}$ as in Fig. 1.4. That is, $\omega^{(\ell)}$ is the loop configuration on $\mathbb{T}_{N,M}$ created by loops that do not cross ω or its dual ω^*. It is easy to see that $\omega \mapsto \omega^{(\ell)}$ is a bijection between Ω_{RC} and the set of all loop configurations.

Call $\ell(\omega)$ the number of different loops of $\omega^{(\ell)}$, and $\ell_0(\omega)$ the number of such loops that are not retractible (on the torus) to a point. Call $\ell_c(\omega) := \ell(\omega) - \ell_0(\omega)$, the number of contractible loops. We say that ω has a *net* if it has a component that winds around $\mathbb{T}^{\diamond}_{N,M}$ in both directions. Set

$$s(\omega) = \begin{cases} 0 & \text{if } \omega \text{ has no net,} \\ 1 & \text{if } \omega \text{ has a net.} \end{cases}$$

Fix $q > 4$. For $\omega \in \Omega_{\mathrm{RC}}$, define the weight of ω in the critical random cluster model as

$$w_{\mathrm{RC}}(\omega) = p_c^{o(\omega)}(1-p_c)^{c(\omega)} q^{k(\omega)},$$

where we recall that $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$ (see [3]).

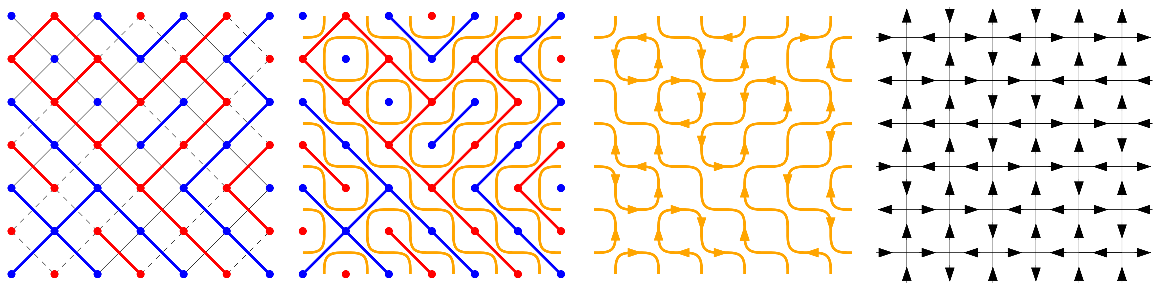

Figure 8: The different steps in the correspondence between the random-cluster and six-vertex models on a torus. From left to right: A random cluster configuration and its dual, the corresponding loop configuration, an orientation of the loop configuration, the resulting six-vertex configuration. Note that in the first picture, there exist both a primal and dual component winding vertically around the torus; this leads to two loops that wind vertically (see second picture); if these loops are oriented in the same direction (as in the third picture) then the number of up arrows on every row of the six-vertex configuration is equal to $\frac{N}{2} \pm 1$.

Lemma 3.8. *For all $\omega \in \Omega_{\mathrm{RC}}$,*

$$w_{\mathrm{RC}}(\omega) = C\sqrt{q}^{\ell(\omega)+2s(\omega)},$$

where $C = q^{\frac{MN}{4}}(1+\sqrt{q})^{-MN}$ is a constant not depending on ω.

Proof Fix $\omega \in \Omega_{\mathrm{RC}}$. Observe that, due to the Euler formula,

$$2k(\omega) = \ell(\omega) - o(\omega) + 2s(\omega) + |V|.$$

This relation offers us an alternative way of writing the random cluster weight of a configuration:

$$w_{\mathrm{RC}}(\omega) = (1-p_c)^{|E|}\left(\frac{p_c}{1-p_c}\right)^{o(\omega)}\sqrt{q}^{\ell(\omega)-o(\omega)+2s(\omega)+|V|}.$$

Since $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$, the above becomes

$$w_{\mathrm{RC}}(\omega) = \left(\frac{1}{1+\sqrt{q}}\right)^{|E|}\sqrt{q}^{|V|}\sqrt{q}^{\ell(\omega)+2s(\omega)} = C\sqrt{q}^{\ell(\omega)+2s(\omega)},$$

where we have used that $|E(\mathbb{T}^\diamond_{N,M})| = MN$ and $|V(\mathbb{T}^\diamond_{N,M})| = |V_\bullet(\mathbb{T}_{N,M})| = MN/2$. $\square$

Write ω° for oriented loop configurations, $\ell_0(\omega^\circ)$ for the number of non-retractible loops of ω° and $\ell_-(\omega^\circ)$ and $\ell_+(\omega^\circ)$ for the number of retractible loops of ω° which are oriented clockwise and counter-clockwise, respectively. We introduce $\lambda > 0$ defined by

$$e^\lambda + e^{-\lambda} = \sqrt{q}. \tag{3.10}$$

For an oriented loop configuration ω°, write

$$w_\circ(\omega^\circ) = e^{\lambda\ell_+(\omega^\circ)}\,e^{-\lambda\ell_-(\omega^\circ)}.$$

Lemma 3.9. *For any $\omega \in \Omega_{\mathrm{RC}}$,*

$$w_{\mathrm{RC}}(\omega) = C\left(\frac{\sqrt{q}}{2}\right)^{\ell_0(\omega)} q^{s(\omega)}\sum_{\omega^\circ} w_\circ(\omega^\circ),$$

where the sum is over the $2^{\ell(\omega)}$ oriented loop configurations ω° obtained by orienting each loop of $\omega^{(\ell)}$ in one of two possible ways.

Proof Fix $\omega \in \Omega_{\mathrm{RC}}$ and consider its associated loop configuration $\omega^{(\ell)}$. In summing the $2^{\ell(\omega)}$ oriented loop configurations ω° associated with $\omega^{(\ell)}$, each loop appears with both orientations. Thus,

$$\sum_{\omega^\circ} w_\circ(\omega^\circ) = \left(1+1\right)^{\ell_0(\omega)}\left(e^\lambda + e^{-\lambda}\right)^{\ell_c(\omega)} = 2^{\ell_0(\omega)}\sqrt{q}^{\ell_c(\omega)} = \frac{1}{C}\left(\frac{2}{\sqrt{q}}\right)^{\ell_0(\omega)} q^{-s(\omega)} w_{\mathrm{RC}}(\omega).$$

$\square$

Notice now that an oriented loop configuration gives rise to 8 different configurations at each vertex. These are depicted in Fig. 9. For an oriented loop configuration ω°, write $n_i(\omega^\circ)$ for the number of vertices of type i in ω°, with $i = 1, 2, 3, 4, 5A, 5B, 6A, 6B$.

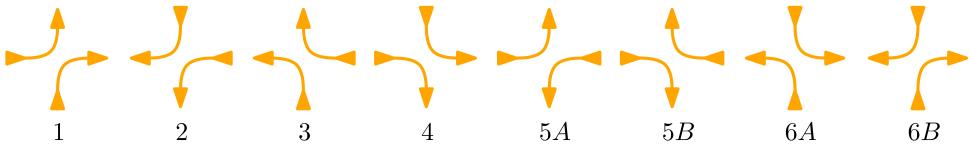

Figure 9: The 8 different types of vertices encountered in an oriented loop configuration.

Lemma 3.10. *For any oriented loop configuration ω°,*

$$w_\circ(\omega^\circ) = e^{\frac{\lambda}{2}[n_{5A}(\omega^\circ)+n_{6A}(\omega^\circ)]}\,e^{-\frac{\lambda}{2}[n_{5B}(\omega^\circ)+n_{6B}(\omega^\circ)]}.$$

Proof Fix an oriented loop configuration $\omega^{\circlearrowleft}$. Notice that the retractible loops of $\omega^{\circlearrowleft}$ which are oriented clockwise have total winding -2π, while those oriented counter-clockwise have winding 2π. Loops which are not retractible have total winding 0. Write $W(\ell)$ for the winding of a loop $\ell \in \omega^{\circlearrowleft}$. Then

$$w_{\circlearrowleft}(\omega^{\circlearrowleft}) = \exp\Big(\frac{\lambda}{2\pi}\sum_{\ell\in\omega^{\circlearrowleft}} W(\ell)\Big), \tag{3.11}$$

where the sum is over all loops ℓ of $\omega^{\circlearrowleft}$. The winding of each loop may be computed by summing the winding of every turn along the loop. The compounded winding of the two pieces of paths appearing in the different diagrams of Fig. 9 are

- vertices of type $1,\dots,4$: total winding 0;
- vertices of type $5A$ and $6A$: total winding π;
- vertices of type $5B$ and $6B$: total winding $-\pi$.

The total winding of all loops may therefore be expressed as

$$\sum_{\ell\in\omega^{\circlearrowleft}} W(\ell) = \pi\,\big[n_{5A}(\omega^{\circlearrowleft}) + n_{6A}(\omega^{\circlearrowleft}) - n_{5B}(\omega^{\circlearrowleft}) - n_{6B}(\omega^{\circlearrowleft})\big].$$

The lemma follows from the above and (3.11). □

For the final step of the correspondence, notice that each diagram in Fig. 9 corresponds to a six-vertex local configuration (as those depicted in Fig. 1.4). Indeed, configurations $5A$ and $5B$ correspond to configuration 5 in Fig. 1.4 and configurations $6A$ and $6B$ correspond to configuration 6 in Fig. 1.4. The first four configurations of Fig. 9 correspond to the first four in Fig. 1.4, respectively.

Thus, to each oriented loop configuration $\omega^{\circlearrowleft}$ is associated a six vertex configuration $\vec{\omega}$. Note that the map associating $\vec{\omega}$ to $\omega^{\circlearrowleft}$ is not injective since there are $2^{n_5(\vec{\omega})+n_6(\vec{\omega})}$ oriented loop configurations corresponding to each $\vec{\omega}$.

Define the parameter c of the six-vertex model by

$$c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}} = \sqrt{2+\sqrt{q}}. \tag{3.12}$$

(The latter equality is obtained from (3.10) by straightforward computation.) As in the rest of the paper, $a = b = 1$ are fixed. Write $w_{6V}(\vec{\omega})$ instead of simply $w(\vec{\omega})$ for the weight of a six-vertex configuration $\vec{\omega}$ as defined in (1.4).

Lemma 3.11. *For all six-vertex configurations $\vec{\omega}$ (that is configurations obeying the ice rule),*

$$w_{6V}(\vec{\omega}) = \sum_{\omega^{\circlearrowleft}} w_{\circlearrowleft}(\omega^{\circlearrowleft}),$$

where the sum is over all oriented loop configurations $\omega^{\circlearrowleft}$ corresponding to $\vec{\omega}$.

Proof Fix a six-vertex configuration $\vec{\omega}$. Let $N_{5,6}(\vec{\omega})$ be the set of vertices of type 5 and 6 in $\vec{\omega}$. Then, due to the choice of c,

$$w_{6V}(\vec{\omega}) = \prod_{u\in N_{5,6}(\vec{\omega})}\big(e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}}\big) = \sum_{\varepsilon\in\{\pm1\}^{N_{5,6}(\vec{\omega})}}\ \prod_{u\in N_{5,6}(\vec{\omega})} e^{\frac{\lambda}{2}\varepsilon(u)} = \sum_{\omega^{\circlearrowleft}} w_{\circlearrowleft}(\omega^{\circlearrowleft}).$$

For the last equality above, notice that each choice of $\varepsilon \in \{\pm1\}^{N_{5,6}(\vec{\omega})}$ corresponds to a choice of type A or B for every vertex of $N_{5,6}(\vec{\omega})$, and hence to one of the $2^{n_5(\vec{\omega})+n_6(\vec{\omega})}$ oriented loop configurations corresponding to $\vec{\omega}$. □

For a six-vertex configuration $\vec{\omega}$ on $\mathbb{T}_{N,M}$, write $|\vec{\omega}|$ for the number of up arrows on each row (recall that this number is the same on all rows). The notation obviously extends to oriented loop configurations. Moreover, for $r \geq 0$, set

$$Z_{6V}^{(r)}(N,M) = \sum_{\vec{\omega}:\,|\vec{\omega}|=\frac{N}{2}-r} w_{6V}(\vec{\omega}).$$

For $\omega \in \Omega_{\mathrm{RC}}$, let $2U(\omega)$ be the total number of times loops of $\omega^{(\ell)}$ wind vertically around $\mathbb{T}_{N,M}^{\diamond}$ (due to periodicity, this number is necessarily even).

Corollary 3.12. *Let $q > 4$ and set $c = \sqrt{2+\sqrt{q}}$. Fix $r \geq 1$. For N, M even, set $C = q^{\frac{MN}{4}}(1+\sqrt{q})^{-MN}$. Then*

$$\sum_{\omega\in\Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C\ Z_{6V}(N,M); \tag{i}$$

$$\sum_{\omega\in\Omega_{\mathrm{RC}}:\,U(\omega)=1} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} \leq 4C\ Z_{6V}^{(1)}(N,M); \tag{ii}$$

$$\sum_{\omega\in\Omega_{\mathrm{RC}}:\,U(\omega)\geq r} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} \geq C\ Z_{6V}^{(r)}(N,M). \tag{iii}$$

Proof Let us start by proving (i). Due to Lemmas 3.9 and 3.11, we have

$$\sum_{\omega\in\Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C\sum_{\omega^{\circ}} w_{\circ}(\omega^{\circ}) = C\sum_{\vec{\omega}} w_{6V}(\vec{\omega}) = Z_{6V}(N,M),$$

where the sums in the second and third terms run over all oriented loop configurations and six-vertex configurations, respectively.

Let us now prove (ii). We restrict ourselves to random-cluster configurations with $U(\omega) = 1$. For such configuration ω, $\omega^{(\ell)}$ has two loops winding vertically around $\mathbb{T}$. Moreover, for any oriented loop configuration ω° which is compatible with $\omega^{(\ell)}$, we may consider the oriented loop configuration $\tilde{\omega}^{\circ}$, obtained from ω° by orienting the two vertically-winding loops downwards. Then $w_{\circ}(\omega^{\circ}) = w_{\circ}(\tilde{\omega}^{\circ})$ and there are at four oriented loop configurations corresponding to any $\tilde{\omega}^{\circ}$. Thus,

$$w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = 4C\sum_{\omega^{\circ}} w_{\circ}(\omega^{\circ}),$$

where the sum in the right-hand side is over oriented loop configurations corresponding to ω in which the two vertically-winding loops are oriented downwards. Since all other loops do not wind vertically around $\mathbb{T}$, the total number of up arrows on any given row of such an oriented loop configuration is $N/2-1$. Thus

$$\sum_{\omega\in\Omega_{\mathrm{RC}}:\,U(\omega)=1} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} \leq 4C \sum_{\omega^{\circ}:\,|\omega^{\circ}|=N/2-1} w_{\circ}(\omega^{\circ}) = 4C\ Z_{6V}^{(1)}(N,M).$$

Finally we show (iii). If ω° is an oriented loop configuration with $|\omega^{\circ}| = N/2-r$, then, by the same up-arrow counting argument as above, the corresponding random-cluster configuration ω has $U(\omega) \geq r$. Thus,

$$CZ_{6V}^{(r)}(N,M) = C\sum_{\omega^{\circ}:\,|\omega^{\circ}|=N/2-r} w_{\circ}(\omega^{\circ}) \leq \sum_{\omega\in\Omega_{\mathrm{RC}}:\,U(\omega)\geq r} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)}.$$

$\square$

3.3.2 Random-cluster computations

The two following lemmas will be used to prove Theorem 1.2.

Lemma 3.13. *For all $q \geq 1$,*

$$\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{M}\log\phi_{p_c,q,\mathbb{T}^\diamond_{N,M}}\Big[\big(\tfrac{2}{\sqrt{q}}\big)^{\ell_0(\omega)}q^{-s(\omega)}\Big]=0. \tag{3.13}$$

Write $\xi(q)$ for the correlation length of the critical random cluster model defined by

$$\xi(q)^{-1}=\lim_{n\to\infty}-\tfrac{1}{n}\log\phi^0_{p_c,q}[0\text{ is connected to graph distance }n]. \tag{3.14}$$

The limit may be shown to exist by sub-additivity arguments.

Lemma 3.14. *For all $q \geq 1$ and $r \geq 1$, we have that*

$$\liminf_{N\to\infty}\liminf_{M\to\infty}\frac{1}{M}\log\phi_{p_c,q,\mathbb{T}^\diamond_{N,M}}(U(\omega)=1)\geq-\xi(q)^{-1}, \tag{3.15}$$

$$\limsup_{N\to\infty}\limsup_{M\to\infty}\frac{1}{M}\log\phi_{p_c,q,\mathbb{T}^\diamond_{N,M}}(U(\omega)\geq r)\leq-(r-1)\xi(q)^{-1}. \tag{3.16}$$

Unlike the rest of the paper, both lemmas above are based on probabilistic estimates specific to the random-cluster model. We will apply repeatedly classical facts about the random-cluster model, such as the finite-energy property, the domain Markov property and the comparison between boundary conditions. We refer the reader to [16] and [6] for background.

In both proofs below, $q \geq 1$ and $p = p_c(q)$ are fixed, and we drop them from the notation of the random cluster measure. In particular, ϕ^0 denotes the infinite-volume measure with free boundary conditions. We will view $\mathbb{T}^\diamond_{N,M}$ as having vertices (i, j) with i, j integers of even sum, taken modulo N and M respectively.

Proof of Lemma 3.13 Fix $q \geq 1$. Since $q^{-s(\omega)} \geq q^{-1}$, it is sufficient to prove that

$$\lim_{N\to\infty}\lim_{M\to\infty}\tfrac{1}{M}\log\phi_{\mathbb{T}^\diamond_{N,M}}\Big[\big(\tfrac{2}{\sqrt{q}}\big)^{\ell_0(\omega)}\Big]=0.$$

Fix $\delta > 0$. To start, we will bound $\phi_{\mathbb{T}^\diamond_{N,M}}(\ell_0(\omega) \geq \delta M)$. Observe that, due to the finite-energy property, there exists a constant $c_0 > 0$ independent of N, M and δ such that

$$\phi_{\mathbb{T}^\diamond_{N,M}}(\ell_0(\omega)\geq\delta M)\leq c_0^{M+N}\phi^0_{\mathrm{R}^\diamond_{N,M}}(\exists n\text{ disjoint clusters crossing }\mathbb{T}^\diamond_{N,M}\text{ horizontally}), \tag{3.17}$$

where $n = \delta M - N$ and $\phi^0_{\mathrm{R}^\diamond_{N,M}}$ denotes the random-cluster measure obtained from $\phi_{\mathbb{T}^\diamond_{N,M}}$ by conditioning on all edges intersecting $\mathbb{R}\times\{-\frac{1}{2}\}$ and $\{-\frac{1}{2}\}\times\mathbb{R}$ to be closed. It may be seen as the random-cluster measure with free boundary conditions on the rectangle $\mathrm{R}^\diamond_{N,M} = [0, N-1] \times [0, M-1]$ of the lattice $\sqrt{2}\mathbb{Z}^2$ rotated by $\pi/4$. The appearance of $-N$ in the definition of n is due to the fact that $\ell_0(\omega)$ may contain up to N loops intersecting the horizontal line $\mathbb{R}\times\{\frac{1}{2}\}$.

If ω is a configuration contributing to the right-hand side of (3.17), then there exists $0 \leq k_1 \leq \cdots \leq k_n \leq M$ such that all the points $x_j := (0, k_j)$ are connected to the right side of $\mathrm{R}^\diamond_{N,M}$ by open paths, and $x_1, \dots, x_n$ are all in different clusters. For any fixed set $x_1, \dots, x_n$ of points on the left side of $\mathrm{R}^\diamond_{N,M}$, conditionally on $x_1, \dots, x_j$ being connected to the right side of $\mathrm{R}^\diamond_{N,M}$, the probability of x_{j+1} being connected to the right side but not to $x_1, \dots, x_j$ is bounded by the probability for x_{j+1} to be connected to distance N for the infinite-volume measure ϕ^0 with free boundary conditions[13]. See Fig. 10 for an illustration of this argument.

[13]This follows from a very standard argument: condition on the clusters of x_1 to x_j. Then the measure in the complement S of these clusters is dominated by the random-cluster measure with free boundary conditions at infinity thanks to the domain Markov property and the comparison between boundary conditions (see [6] for more details on this type of reasoning). Since the event of having an open path from x_{j+1} to the right of $\mathrm{R}^\diamond_{N,M}$ in S is included in the event of having an open path from x_{j+1} to distance N, the claim follows.

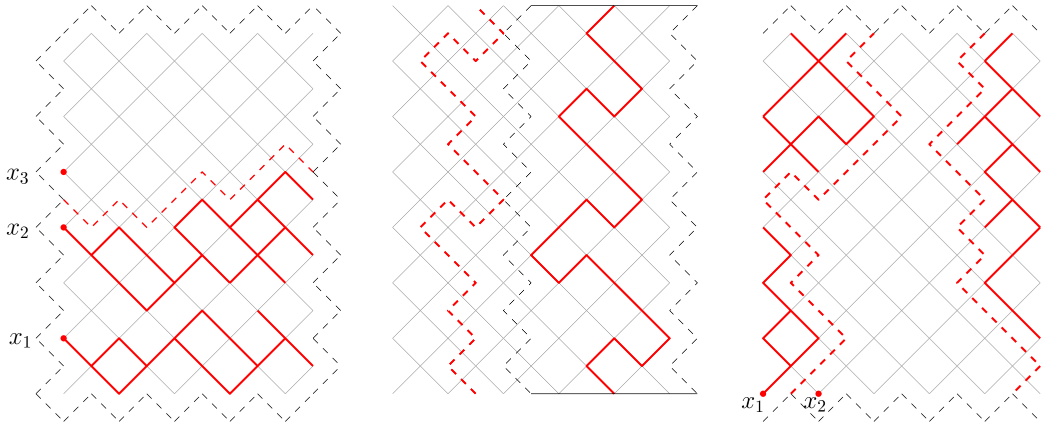

Figure 10: *Left:* Exploring one by one the disjoint, horizontally crossing clusters contributing to (3.17). Each new cluster (for instance the one of x_3) is surrounded by free boundary conditions. *Middle:* To create ω with $U(\omega) \ge 1$, it suffices to ensure H^* occurs (dotted red line) and that conditionally on H^*, H also occurs. The latter is more likely than the occurrence of a top-bottom crossing in the black rectangle with free boundary conditions on the lateral sides. *Right:* In exploring $H(x_1, \dots, x_r)$, every cluster crossing vertically the torus (except the first) is surrounded by free boundary conditions.

By using this observation, after taking the union over all possible $0 \le k_1 \le \dots \le k_n \le M$, we deduce from (3.17) that

$$\phi_{\mathbb{T}^\diamond_{N,M}}(\ell_0(\omega) \ge \delta M) \le c^{M+N}\binom{M}{n}\phi^0\Big(0 \text{ is connected to distance } N\Big)^n,$$

Now, it is classical [16, Theorem 6.17] that $\phi^0(0 \text{ is connected to distance } N)$ tends to 0 as N tends to infinity so that for N large enough,

$$\lim_{M\to\infty} \phi_{\mathbb{T}^\diamond_{N,M}}(\ell_0(\omega) \ge \delta M) = 0. \tag{3.18}$$

This implies that for any $\delta > 0$, provided that N is large enough,

$$\liminf_{M\to\infty} \tfrac{1}{M} \log \phi_{\mathbb{T}^\diamond_{N,M}}\Big[\big(\tfrac{2}{\sqrt{q}}\big)^{\ell_0(\omega)}\Big] \ge -\delta, \tag{3.19}$$

which concludes the proof by letting δ tend to 0. □

Proof of Lemma 3.14

Proof of (3.15)**.** A standard duality argument (see [3, Lemma 6]) implies that there exists a constant $c > 0$ such that for any n, N, M with $n \le \min\{N, M\}$,

$$\phi_{\mathbb{T}^\diamond_{N,M}}([0,n]^2 \text{ contains a vertical crossing}) \ge c.$$

By standard reflection arguments, we deduce that for any $x \in [0,n] \times \{0\}$, for all M and N large enough,

$$\phi_{\mathbb{T}^\diamond_{N,M}}\Big(x \xleftrightarrow{[0,n]\times[0,2n]} x + (0,2n)\Big) \ge \frac{c^2}{n^4}.$$

In the above, $\overset{\mathscr{D}}{\longleftrightarrow}$ stands for "connected by an open path contained in the region $\mathscr{D}$". Let H^* be the event that there exists a dually-open loop contained in $[0, N/2] \times [0, M]$, winding vertically around $\mathbb{T}^\diamond_{N,M}$. Suppose henceforth that $N > 2n$. If M is a multiple of $2n$, by the FKG inequality, self-duality, and the estimate above,

$$\phi_{\mathbb{T}^\diamond_{N,M}}(H^*) \geq \Big(\frac{c^2}{n^4}\Big)^{M/2n}. \tag{3.20}$$

For $M = 2nk + r$, due to the finite-energy property of $\phi_{N,M}$,

$$\phi_{\mathbb{T}^\diamond_{N,M}}(H^*) \geq \Big(\frac{c^2}{n^4}\Big)^{k} c_0^r,$$

for a constant c_0 independent of M, N, n and r. In conclusion, for all M,

$$\phi_{\mathbb{T}^\diamond_{N,M}}(H^*) \geq \Big(\frac{c^2}{n^4}\Big)^{M/2n} c_0^n.$$

Let H be the event that $[N/2, N] \times [0, M]$ contains an open primal loop winding vertically around $\mathbb{T}^\diamond_{N,M}$. The measure $\phi_{\mathbb{T}^\diamond_{N,M}}(.|H^*)$ dominates the random cluster measure on $[N/2, N] \times [0, M]$ with periodic boundary conditions on the top and bottom and free boundary conditions on the lateral sides; write $\phi^{0\|0}_{[N/2,N]\times[0,M]}$ for this measure. Fix $\varepsilon > 0$ and let $y = (3N/4, 0)$. Then, for all N and M large enough,

$$\begin{aligned}
\phi_{\mathbb{T}^\diamond_{N,M}}\Big(y \xleftrightarrow{[N/2,N]\times[0,M]} y + (0, 2n)\Big|H^*\Big) &\geq \frac{1}{2}\liminf_{N,M\to\infty} \phi^{0\|0}_{[N/2,N]\times[0,M]}\Big(y \xleftrightarrow{[N/2,N]\times[0,M]} y + (0, 2n)\Big) \\
&\geq \frac{1}{2}\phi^0\Big(0 \leftrightarrow (0, 2n)\Big) \\
&\geq \frac{1}{2}\exp\Big(-\frac{2n}{\xi(q) - \varepsilon}\Big),
\end{aligned}$$

whenever n is large enough for the last inequality to hold[14] (recall that ϕ^0 denotes the infinite-volume measure with free boundary conditions). Using the above bound and the FKG property repeatedly (we also use the finite-energy property when M is not an exact multiple of n), we deduce that

$$\phi_{\mathbb{T}^\diamond_{N,M}}\big(H\big|H^*\big) \geq c_0^n \phi_{\mathbb{T}^\diamond_{N,M}}\Big(0 \xleftrightarrow{[N/2,N]\times[0,M]} (0, 2n)\Big|H^*\Big)^{\frac{M}{2n}} \geq \frac{c_0^n}{2^{M/2n}}\exp\Big(-\frac{M}{\xi(q) - \varepsilon}\Big).$$

Finally, observe that, using the finite energy property again[15],

$$\phi_{\mathbb{T}^\diamond_{N,M}}(U(\omega) = 1) \geq c_0^N \phi_{\mathbb{T}^\diamond_{N,M}}(H \cap H^*).$$

The lower bound (3.20) on $\phi_{\mathbb{T}^\diamond_{N,M}}(H^*)$ and the above bound on $\phi_{N,M}(H|H^*)$ imply

$$\lim_{M\to\infty} \frac{1}{M}\log \phi_{\mathbb{T}^\diamond_{N,M}}(U(\omega) = 1) \geq -\frac{1}{\xi(q) - \varepsilon} - \frac{\log 2}{n}.$$

Since ε is arbitrary and n may be taken as large as desired (provided N is also large enough), the above yields (3.15).

---

[14] The last inequality is based on a classical alternative definition of $\xi(q)$ given (and discussed) in (3.24) below.

[15] More precisely, $H \cap H^*$ implies $U(\omega) \geq 1$. If we further impose that all but one edge intersecting $\mathbb{R} \times \{-\frac{1}{2}\}$ are open, we deduce that $U(\omega) = 1$.

Proof of (3.16)**.** Fix $r \geq 1$ and consider $N \geq 2r$ an even integer. Denote by $x_i = (2i, 0)$ (for $i = 1, \dots, N/2$) the points on the lower side of the torus $\mathbb{T}^\diamond_{N,M}$ and by $\phi^{0=}_{\mathbb{T}^\diamond_{N,M}}$ the measure $\phi_{\mathbb{T}^\diamond_{N,M}}$ conditioned on that all edges intersecting $\mathbb{R} \times \{-\frac{1}{2}\}$ be closed; it may be viewed as a random cluster measure on a cylinder of height M with free boundary conditions on the top and bottom. Moreover, let $H(x_1, \dots, x_r)$ be the event that x_j is connected by an open path to $y_j = x_j + (1, M-1)$ for $j = 1, \dots, r$ and that the clusters of $x_1, \dots, x_r$ are all distinct. Due to the finite energy property

$$\limsup_{M\to\infty} \frac{1}{M} \log \phi_{\mathbb{T}^\diamond_{N,M}}(U(\omega) \geq r) = \limsup_{M\to\infty} \frac{1}{M} \log \phi^{0=}_{\mathbb{T}^\diamond_{N,M}}\big[H(x_1, \dots, x_r)\big]. \tag{3.21}$$

Write C_{x_j} for the cluster of the point x_j. Then, for any $j \geq 1$, by a standard exploration argument involving the domain Markov property and the comparison between boundary conditions, similar to that of Lemma 3.13, we deduce

$$\begin{aligned}\phi^{0=}_{\mathbb{T}^\diamond_{N,M}}\big[y_{j+1} \in C_{x_{j+1}} \text{ and } x_1, \dots, x_j \notin C_{x_{j+1}} \big| H(x_1, \dots, x_j)\big] &\leq \phi^0(y_{j+1} \in C_{x_{j+1}}) \\ &\leq \phi^0(0 \text{ connected to distance } M).\end{aligned}$$

Applying this $r-1$ times yields

$$\phi^{0=}_{\mathbb{T}^\diamond_{N,M}}\big[H(x_1, \dots, x_r)\big] \leq \phi^0(0 \text{ connected to graph distance } M)^{r-1}.$$

The conclusion follows from (3.21) and the definition of $\xi(q)$. $\square$

3.3.3 Proof of Theorem 1.2

Fix $q > 4$. By [10], for points 1 and 2 it is sufficient to show that $\xi(q) < \infty$. We therefore focus on point 3, that is we compute $\xi(q)^{-1}$ explicitly and show that it is equal to

$$R(q) := \lambda + 2 \sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda) > 0,$$

where $\lambda > 0$ satisfies $e^\lambda + e^{-\lambda} = \sqrt{q}$. We will show that this quantity is positive and analyze its asymptotics in Section 4.

We will refer to the associated six-vertex model, with $c = \sqrt{2 + \sqrt{q}}$. Write $Z_{\mathrm{RC}}(N, M)$ for the partition function of the random cluster model with parameters p_c, q on $\mathbb{T}^\diamond_{N,M}$, that is

$$Z_{\mathrm{RC}}(N, M) := \sum_{\omega \in \Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega).$$

Lower bound on the inverse correlation length Equation (3.15) may be rewritten as

$$\xi(q)^{-1} \geq -\liminf_{N\to\infty} \liminf_{M\to\infty} \frac{1}{M} \log \frac{\sum_{\omega: U(\omega)=1} w_{\mathrm{RC}}(\omega)}{Z_{\mathrm{RC}}(N, M)}.$$

Since all configurations with $U(\omega) = 1$ have exactly two non-retractible loops and no net, Corollary 3.12 (ii) implies that the numerator above is smaller than

$$\frac{\sqrt{q}}{2} q^{\frac{MN}{4}} (1 + \sqrt{q})^{-MN} Z^{(1)}_{6V}(N, M).$$

Furthermore, in light of Corollary 3.12 (i), Lemma 3.13 may be rewritten as

$$\lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{M} \log \frac{q^{\frac{MN}{4}} (1 + \sqrt{q})^{-MN} \; Z_{6V}(N, M)}{Z_{\mathrm{RC}}(N, M)} = 1. \tag{3.22}$$

Therefore, we may write

$$\xi(q)^{-1} \geq -\liminf_{N\to\infty} \liminf_{M\to\infty} \frac{1}{M} \log \frac{Z^{(1)}_{6V}(N, M)}{Z_{6V}(N, M)} = -\liminf_{N\to\infty} \log \frac{\Lambda_1(N)}{\Lambda_0(N)} \overset{(1.7)}{=} R(q). \tag{3.23}$$

Upper bound on the inverse correlation length. For all $r \geq 2$, (3.16) may be written as

$$(r-1)\xi(q)^{-1} \leq -\limsup_{N\to\infty}\limsup_{M\to\infty}\frac{1}{M}\log\frac{\sum_{\omega:U(\omega)\geq r} w_{\mathrm{RC}}(\omega)}{Z_{\mathrm{RC}}(N,M)}.$$

Using Corollary 3.12 (iii) and (3.22) again, we find

$$(r-1)\xi(q)^{-1} \leq -\limsup_{N\to\infty}\limsup_{M\to\infty}\ \frac{1}{M}\log\frac{Z_{6V}^{(r)}(N,M)}{Z_{6V}(N,M)} = -\limsup_{N\to\infty}\ \log\frac{\Lambda_r(N)}{\Lambda_0(N)} \overset{(1.7)}{=} rR(q).$$

The bound above being valid for all $r \geq 2$, one may divide by $r-1$ and take r to infinity. The resulting upper bound on $\xi(q)^{-1}$ matches the lower bound of (3.23), and the theorem is proved.

Remark 3.15. *Inequality* (3.23) *should, in fact, be an equality. This would allow us to compute $\xi(q)^{-1}$ using nothing but the asymptotics of $\Lambda_0(N)$ and $\Lambda_1(N)$, and require no control of $\Lambda_r(N)$ for $r \geq 2$. Unfortunately, we did not manage to derive the reversed inequality of* (3.23) *using random-cluster model only, and used our control of $\Lambda_r(N), r \geq 2$ as an indirect route to the desired bound.*

3.4 From the random-cluster to the Potts model: proof of Theorem 1.1

The results for the Potts model can be obtained from those for the random-cluster model via a classical coupling, see [11, 16]. We describe consequences of this coupling in the theorem below; for a proof, see the references.

In the next statement, the operation of *attributing a spin $s \in \{1,\dots,q\}$ to a set S of vertices* means that we fix $\sigma_x = s$ for every $x \in S$. Below, we consider the Potts and random-cluster models on the standard lattice $\mathbb{Z}^2$; no reference to the rotated lattice is used.

Theorem 3.16. *Fix $\beta > 0$ and an integer $q \geq 2$. Set $p = 1 - e^{-\beta}$.*

- *Consider ω with law $\phi^0_{p,q}$. Then, the law of $\sigma \in \{1,\dots,q\}^{\mathbb{Z}^2}$ obtained by attributing independently and uniformly a spin in $\{1,\dots,q\}$ to each cluster of ω is μ^0_β.*
- *Fix $i \in \{1,\dots,q\}$ and consider ω with law $\phi^1_{p,q}$. Then, the law of $\sigma \in \{1,\dots,q\}^{\mathbb{Z}^2}$ obtained by attributing independently and uniformly a spin in $\{1,\dots,q\}$ to each* finite *cluster of ω, and spin i to the* infinite *clusters of ω is μ^i_β.*

Theorem 3.16 implies immediately the following facts.

1. The critical inverse-temperature of the Potts model and the critical parameter of random-cluster model are related by the formula $p_c = 1 - e^{\beta_c}$.
2. For any $i \in \{1,\dots,q\}$,
$$\mu^i_\beta[\sigma_0 = i] = \tfrac{1}{q} + \phi^1_{p,q}[0 \text{ is in an infinite cluster}].$$
3. For any $x, y \in \mathbb{Z}^2$,
$$\mu^0_\beta[\sigma_x = \sigma_y] = \tfrac{1}{q} + \phi^0_{p,q}[x \text{ and } y \text{ are in the same cluster}].$$

Altogether, these properties show that Theorem 1.1 follows from Theorem 1.2, as described below.

Theorem 1.1 (2) follows directly from items 1. and 2. above combined with (2) of Theorem 1.2. For Theorem 1.1 (1), it is well-known (see for instance results in [16]) that a Gibbs measure is extremal if and only if it is ergodic. Thus, the measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are ergodic for any value of $p \in [0,1]$. Since there exists no infinite cluster $\phi^0_{p_c,q}$-almost surely (by (3) of Theorem 1.2), the construction of $\mu^0_{\beta_c}$ from $\phi^0_{p_c,q}$ described in Theorem 3.16 implies that $\mu^0_{\beta_c}$ is ergodic as well. In the same way, each measure $\mu^i_{\beta_c}$, $i = 1,\dots,q$, may be shown to be ergodic (here the existence

of an infinite cluster is not problematic, since it is attributed the fixed spin i). By Theorem 1.1 (2), the measures $\mu^i_{\beta_c}$ induce different distributions for the spin of any given vertex, hence they are all distinct.
Theorem 1.1 (3): Recall the notation $x_n = (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor) \in \mathbb{Z}^2$. The union bound implies that there exists x at distance n of the origin such that

$$\phi^0_{p,q}[0 \text{ and } x \text{ are connected}] \geq \frac{1}{4n} \phi^0_{p,q}[0 \text{ is connected to graph distance } n].$$

Using the FKG inequality and the symmetry under reflection, we deduce that

$$\phi^0_{p,q}[0 \text{ and } x_{2n} \text{ are connected}] \geq \Big(\frac{1}{4n} \phi^0_{p,q}[0 \text{ is connected to graph distance } n] \Big)^2.$$

This shows that

$$\lim_{n\to\infty} -\frac{1}{n} \log \phi^0_{p,q}[0 \text{ and } x_n \text{ are connected}] = \frac{1}{\xi(q)} \tag{3.24}$$

with $\xi(q)$ defined in (3.14). Point 3. above and the expression for $\xi(q)$ obtained in Theorem 1.2 lead to the desired result.

4 Fourier computations

In this section, we gather the computations of certain Fourier-analytic identities used throughout the paper.

Evaluation of the Fourier coefficients of Ξ_λ and R. Let $m \geq 0$ and consider the contour integral

$$\frac{1}{2\pi} \int_{C_N} \frac{\sinh(\lambda) e^{-imz}}{\cosh(\lambda) - \cos(z)} dz,$$

where C_N is the boundary of $[-\pi, \pi] + i[-N, 0]$, oriented clockwise. As N goes to infinity, this integral goes to $\hat{\Xi}_\lambda(m)$. Since the only residues of the integrand in the interior of C_N occur at $-i\lambda$, we conclude that

$$\hat{\Xi}_\lambda(m) = e^{-\lambda m} \qquad m \geq 0.$$

If $m < 0$, we integrate around C'_N, the boundary of $[-\pi, \pi] + i[0, N]$, oriented counter-clockwise. The residue will now be at $i\lambda$, and

$$\hat{\Xi}_\lambda(m) = e^{\lambda m} \qquad m < 0.$$

Via (2.2), this implies

$$\hat{R}(m) = \frac{e^{-\lambda|m|}}{1 + e^{-2\lambda|m|}} = \frac{1}{2\cosh(\lambda m)}. \tag{4.1}$$

Evaluation of the Fourier coefficients of Ψ and T. To evaluate $\hat{\Psi}$, we first note that $k(\alpha)$ is an odd function, and Θ is anti-symmetric, meaning Ψ is an odd function and $\hat{\Psi}(0) = 0$. As a consequence, (2.2) implies $\hat{T}(0) = 0$.

For $m \neq 0$, we first replace $\Theta(k(\alpha),\pi)+\Theta(k(\alpha),-\pi)$ with the equivalent expression $2[\Theta(k(\alpha),\pi)-\pi]$ (using the fact that $\Theta(x,\pi)=\Theta(x,-\pi)+2\pi$). Then, using integration by parts, we find

$$\begin{aligned}\hat{\Psi}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[\Theta(k(\alpha),\pi)-\pi]e^{-im\alpha}d\alpha \\ &= \frac{[\Theta(\pi,\pi)-\Theta(-\pi,\pi)](-1)^m}{-2\pi im} + \frac{1}{2\pi im}\int_{-\pi}^{\pi}\frac{d}{d\alpha}\Theta(k(\alpha),\pi)e^{-im\alpha}d\alpha \\ &= \frac{(-1)^m}{im} - \frac{1}{2\pi im}\int_{-\pi}^{\pi}\Xi_{2\lambda}(\alpha-\pi)e^{-im\alpha}d\alpha \\ &= \frac{(-1)^m}{im}\left(1-\hat{\Xi}_{2\lambda}(m)\right),\end{aligned}$$

where we used $\Theta(\pi,\pi)-\Theta(-\pi,\pi)=-2\pi$ and the change of variable $u=\alpha-\pi$ and the periodicity of $\Xi_{2\lambda}$ to show that the integral in the penultimate line is equal to $2\pi(-1)^m\hat{\Xi}_{2\lambda}(m)$. Thus,

$$\hat{T}(m) = \frac{(-1)^m\left(1-e^{-2\lambda|m|}\right)}{im\left(1+e^{-2\lambda|m|}\right)} = \frac{(-1)^m}{im}\tanh(\lambda|m|).$$

Computations of R and T. We start with T. Pairing the terms for $\pm m$, we find[16]

$$T(\alpha) = 2\sum_{m>0}\frac{(-1)^m}{m}\tanh(\lambda m)\left(\frac{e^{im\alpha}-e^{-im\alpha}}{2i}\right) = 2\sum_{m>0}\frac{(-1)^m}{m}\tanh(\lambda m)\sin(m\alpha).$$

We now turn to R. We will show that it is equal to the sum

$$\mathscr{R}(\alpha) := \frac{\pi}{2\lambda}\sum_{r\in\mathbb{Z}}\frac{1}{\cosh[\pi(2\pi r+\alpha)/(2\lambda)]}$$

by showing that the two have the same Fourier coefficients. By direct computation and the Dominated Convergence Theorem,

$$\begin{aligned}\hat{\mathscr{R}}(m) &= \frac{1}{4\lambda}\sum_{r\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{\pi e^{-im\alpha}d\alpha}{2\lambda\cosh[\pi(2\pi r+\alpha)/(2\lambda)]} \\ &= \frac{1}{4\lambda}\int_{-\infty}^{\infty}\frac{e^{-im\alpha}d\alpha}{\cosh(\pi\alpha/2\lambda)}\end{aligned}$$

using the 2π periodicity of the numerator. Observe that the hyperbolic secant function can be written as a *continuous* Fourier transform:

$$\frac{1}{\cosh(\lambda m)} = \frac{1}{2\lambda}\int_{-\infty}^{\infty}\frac{e^{-im\alpha}d\alpha}{\cosh(\pi\alpha/2\lambda)}.$$

This concludes the proof.

Computation of the integral on the right-hand side of (3.6). The change of variable $x=k(\alpha)$ and some elementary algebraic manipulations give

$$\int_{-\pi}^{\pi}\log\left|M\left(e^{ix}\right)\right|\rho(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}P(\alpha)R(\alpha)d\alpha,$$

with

$$P(\alpha) := \log|M(e^{ik(\alpha)})| = \tfrac{1}{2}\log\left(\frac{\cosh(2\lambda)-\cos(\alpha)}{1-\cos(\alpha)}\right) = \int_0^{\lambda}\Xi_{2t}(\alpha)dt.$$

[16] In the formula, the series is not absolutely convergent, however $\sum_{m=1}^{N}\tanh(\lambda m)\sin(m\alpha)$ converges as $N\to\infty$, and we will consider this as the limit.

The final equality may be checked by noticing that the two sides have equal derivatives and are both equal to 0 when $\lambda = 0$. We note that, even though $P(\alpha)$ is not a bounded function, its singularity at $\alpha = 0$ is logarithmic, and hence it is in $L^2([-\pi,\pi])$. Thus, we can use Fubini's Theorem to deduce that

$$\hat{P}(m) = \int_0^{\lambda} e^{-2t|m|}dt = \begin{cases} \lambda & \text{if } m = 0, \\ \frac{1-\exp(-2\lambda|m|)}{2|m|} & \text{if } m \neq 0. \end{cases} \tag{4.2}$$

Finally, Parseval's Theorem implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(\alpha)R(\alpha)d\alpha = \sum_{m\in\mathbb{Z}} \hat{P}(m)\hat{R}(-m) = \frac{\lambda}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\lambda}\tanh(m\lambda)}{m}$$

using (4.1) in the final equality.

Computation of the integral on the right-hand side of (3.7) **and** (3.8)**.** We begin our analysis of the second integral by recalling (2.16), which implies the existence of C such that $|\tau(x)| < C|x|$ for all $x \in [-\pi,\pi]$. Thus, although $\ell'(x)$ grows as $1/|x|$ near the origin, the integrand is uniformly bounded. Using the Dominated Convergence Theorem[17] and the explicit computation of τ in Proposition 2.1, we find

$$\int_{-\pi}^{\pi} \ell'(x)\tau(x)dx = \int_{-\pi}^{\pi} P'(\alpha)\tau(k(\alpha))d\alpha = \sum_{m>0} \frac{(-1)^m\tanh(\lambda m)}{m}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} P'(\alpha)\sin(m\alpha)d\alpha\right].$$

Calculating the integrals in the right-hand side is a simple case of integration by parts:

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} P'(\alpha)\sin(m\alpha)d\alpha &= \frac{P(\alpha)\sin(m\alpha)}{\pi}\bigg|_{-\pi}^{\pi} - \frac{m}{\pi}\int_{-\pi}^{\pi} P(\alpha)\cos(m\alpha)d\alpha \\ &= -m[\hat{P}(m) + \hat{P}(-m)] \\ &= e^{-2\lambda m} - 1,\end{aligned}$$

where we use our earlier computation (4.2) for the final line. Substituting this in (3.7) yields

$$\lim_{N\to\infty} \log\frac{\Lambda_r(N)}{\Lambda_0(N)} = -r\cdot\Big[\log|\Delta| - \sum_{m>0}\frac{(-1)^m}{m}\tanh(\lambda m)(e^{-2\lambda m} - 1)\Big].$$

By expanding $\log|\Delta| = \log\cosh(\lambda)$ in powers of $e^{-\lambda}$ and manipulating the result algebraically, we find that

$$\log|\Delta| = \lambda - \sum_{m>0}\frac{(-1)^m(e^{-2\lambda m} - 1)}{m}.$$

This directly implies

$$\log|\Delta| - \sum_{m>0}\frac{(-1)^m}{m}\tanh(\lambda m)(e^{-2\lambda m} - 1) = \lambda + 2\sum_{m>0}\frac{(-1)^m}{m}\tanh(m\lambda). \tag{4.3}$$

[17] In the formula below, the series in the right-hand side is not absolutely convergent. However, if terms are paired (each odd term with the succeeding even one) the resulting series becomes absolutely convergent. This observation is used here and below.

Proof of (1.3)**.** We wish to show that

$$\lambda + 2\sum_{m\geq 1}\frac{(-1)^m}{m}\tanh(m\lambda) = \sum_{m\geq 0}\frac{4}{(2m+1)\sinh\left[\pi^2(2m+1)/(2\lambda)\right]}. \tag{4.4}$$

Let C_N be the boundary of the rectangle $[-(2N+1)/2,(2N+1)/2]+i[-\pi N/\lambda,\pi N/\lambda]$, oriented counter-clockwise, and consider

$$\mathscr{I}_N := \int_{C_N}\frac{\pi\tanh(\lambda z)dz}{z\sin(\pi z)}.$$

The integrand has a simple pole at every integer m and at $i\pi(2r+1)/(2\lambda)$ for every integer r. A straightforward computation shows that the residues of the integrand at the natural numbers are:

$$\mathrm{Res}\left(\frac{\pi\tanh(\lambda z)}{z\sin(\pi z)},m\right) = \begin{cases}\frac{\tanh(\lambda m)}{\cosh(\pi m)m} & m\neq 0,\\ \lambda & m=0.\end{cases}$$

Summing over $m\in[-N,N]\cap\mathbb{Z}$ gives the partial sums of the right-hand side of (4.4). Meanwhile,

$$\mathrm{Res}\left(\frac{\pi\tanh(\lambda z)}{z\sin(\pi z)},i\pi(2m+1)/(2\lambda)\right) = \frac{-2}{(2m+1)\sinh[\pi^2(2m+1)/(2\lambda)]}.$$

The hyperbolic tangent being bounded away for its poles (and therefore on C_N), we deduce that, for some uniform constant c_0,

$$|\mathscr{I}_N| \leq \frac{c_0}{N}\left[\int_{-\pi N/\lambda}^{\pi N/\lambda}\frac{dt}{\cosh(\pi t)} + \int_{-(2N+1)/2}^{(2N+1)/2}\frac{dt}{|\sin(i\pi^2/\lambda+t)|}\right].$$

Both integrals are uniformly finite in N, hence $\mathscr{I}_N$ converges to zero. As a consequence, the sum of residues of the integrand converges to zero. Using the residues computed above, this implies[18] (4.4).

Upon inspection of the right-hand side of (4.4), we observe that the quantity in the equation is strictly positive whenever $\lambda > 0$. The asymptotic behaviour of (4.4) as Δ tends to -1 (corresponding to $2\lambda\sim\sqrt{q-4}$ tending to 0) is governed by the first term or the right-hand side, namely $\frac{4}{\sinh(\pi^2/(2\lambda))}\sim 8e^{-\pi^2/(2\lambda)}$.

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References

[1] R. J. Baxter, *Solvable eight-vertex model on an arbitrary planar lattice*, Philos. Trans. Roy. Soc. London Ser. A, 289 (1978), pp. 315–346.

[2] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989. Reprint of the 1982 original.

[3] V. Beffara and H. Duminil-Copin, *The self-dual point of the two-dimensional random-cluster model is critical for $q\geq 1$*, Probab. Theory Related Fields, 153 (2012), pp. 511–542.

(18) We obtain explicitly $\lambda+2\sum_{m=1}^N\frac{(-1)^m}{m}\tanh(m\lambda)-\sum_{m=0}^N\frac{4}{(2m+1)\sinh[\pi^2(2m+1)/(2\lambda)]}\to 0$ as $N\to\infty$.

[4] H. Bethe, *Zur Theorie der Metalle I. Eigenwerte und Eigenfunktionen der Hnearen Atomkette*, Zeitschrift für Physik, 71 (1931), pp. 205–226.

[5] E. Buffenoir and S. Wallon, *The correlation length of the Potts model at the first-order transition point*, Journal of Physics A: Mathematical and General, 26 (1993), p. 3045.

[6] H. Duminil-Copin, *Parafermionic observables and their applications to planar statistical physics models*, vol. 25 of Ensaios Matematicos, Brazilian Mathematical Society, 2013.

[7] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu, and V. Tassion, *The Bethe ansatz for the six-vertex and XXZ models: an exposition.* preprint, 2016.

[8] H. Duminil-Copin and I. Manolescu, *The phase transitions of the planar random-cluster and Potts models with $q \geq 1$ are sharp*, Probability Theory and Related Fields, 164 (2016), pp. 865–892. arXiv:1409.3748.

[9] H. Duminil-Copin, A. Raoufi, and V. Tassion, *A new computation of the critical point for the planar random-cluster model with $q \geq 1$*, preprint arXiv:1604.03702, (2016).

[10] H. Duminil-Copin, V. Sidoravicius, and V. Tassion, *Continuity of the phase transition for planar Potts models with $1 \leq q \leq 4$*, Communications in Mathematical Physics, (2016), pp. 1–61.

[11] R. G. Edwards and A. D. Sokal, *Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm*, Phys. Rev. D (3), 38 (1988), pp. 2009–2012.

[12] C. M. Fortuin, *On the Random-Cluster model*, doctoral thesis, University of Leiden, 1971.

[13] C. M. Fortuin and P. W. Kasteleyn, *On the random-cluster model. I. Introduction and relation to other models*, Physica, 57 (1972), pp. 536–564.

[14] M. Fulmek, *Nonintersecting lattice paths on the cylinder*, Séminaire Lotharingien de Combinatoire, 52 (2004). Article B52b.

[15] P. S. Goldbaum, *Existence of solutions to the Bethe ansatz equations for the 1D Hubbard model: Finite lattice and thermodynamic limit*, Communications in Mathematical Physics, 258 (2005), pp. 317–337.

[16] G. Grimmett, *The random-cluster model*, vol. 333 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2006.

[17] V. Korepin and P. Zinn-Justin, *Thermodynamic limit of the six-vertex model with domain wall boundary conditions*, Journal of Physics A: Mathematical and General, 33 (2000), p. 7053.

[18] R. B. Potts, *Some generalized order-disorder transformations*, in Proceedings of the Cambridge Philosophical Society, vol. 48, Cambridge Univ Press, 1952, pp. 106–109.

[19] N. Reshetikhin, *Lectures on the integrability of the 6-vertex model*, arXiv1010.5031, (2010).

[20] H. N. V. Temperley and E. H. Lieb, *Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the 'percolation' problem*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 322 (1971), pp. 251–280.

[21] F. Y. Wu, *The Potts model*, Rev. Modern Phys., 54 (1982), pp. 235–268.

[22] C. N. Yang and C. P. Yang, *One-dimensional chain of anisotropic spin-spin interactions. I. Proof of Bethe's hypothesis for ground state in a finite system*, Phys. Rev., 150 (1966), pp. 321–327.

[23] P. Zinn-Justin, *Six-vertex model with domain wall boundary conditions and one-matrix model*, Phys. Rev. E, 62 (2000), pp. 3411–3418.