Non-decay of correlations in the dimer model and phase transition in lattice permutations in $\mathbb{Z}^d$, $d > 2$, via reflection positivity

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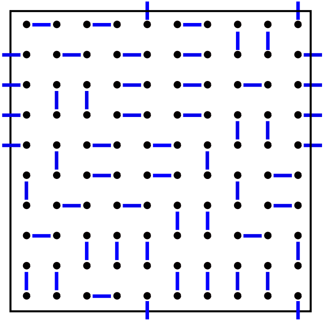
Definition (Dimer Cover)

A *dimer cover* of the graph $\mathcal{G} = (V, E)$ is a spanning sub-graph of $\mathcal{G}$ such that every vertex has degree one.

- **Exact enumeration on $\mathbb{Z}^2$**
  (*Kasteleyn, Fisher and Temperley, 1961*).
- **Correlations on planar graphs**
  (*Fisher and J. Stephenson, 1963)*
- **Connections to critical planar Ising model**
  (Kasteleyn 1961, Fisher 1966).
- **No phase transition in monomer-dimer model**
  (*Heilmann, Lieb, 1972*)
- **Arctic circle phenomenon**
  (*H. Cohn, N. Elkies, J. Propp, 1996)*
- **Scaling limits, conformal invariance** *(Kenyon, 2000 - 2014).*



What about $\mathbb{Z}^d$, $d > 2$?

- Hammersley *et al*, 1969: *'Negative Finding for the Three-Dimensional Dimer Problem'.*
- Jerrum, 1987: *'Monomer-dimer systems are computationally intractable'.*

The dimer model

Definition (Monomer-monomer correlation)

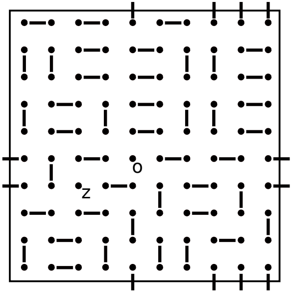
Define $\mathbb{T}_L := \mathbb{Z}^d / L\mathbb{Z}^d$ and, for any $M \subset \mathbb{T}_L$ (*set of monomers*), let $\mathcal{D}(M)$ be the set of dimer covers of $\mathbb{T}_L \setminus M$. We define the *monomer-monomer correlation*,

$$\forall z \in \mathbb{T}_L \quad \Xi_L(z) := \frac{|\mathcal{D}(\{o, z\})|}{\mathcal{D}(\emptyset)}.$$

- **Conjecture** *(Fisher and Stephenson)*:

$$\text{On } \mathbb{Z}^2 \quad \lim_{L \to \infty} \Xi_L(z) \sim \frac{1}{|z|^{\frac{1}{2}}}$$

- **Proved for:**
  - z along the cartesian axis (*Fisher, Stephenson, 1963*)
  - z along diagonals (*Hartwig, 1966*)



Let r_d be the expected number of returns of a simple random walk on $\mathbb{Z}^d$.
Define the *odd and even sub-lattices*

$$\mathbb{T}_L^e := \{x \in \mathbb{T}_L \ : \ d(o,x) \in 2\mathbb{N}\}, \quad \mathbb{T}_L^o := \{x \in \mathbb{T}_L \ : \ d(o,x) \in 2\mathbb{N}+1\},$$

Theorem (Taggi, 2019+)

Suppose that $d > 2$. For any $L \in 2\mathbb{N}$, we have that,

$$\frac{1}{|\mathbb{T}_L^o|} \sum_{x \in \mathbb{T}_L^o} \Xi_L(x) \geq \frac{1}{2d}(1 - \frac{r_d}{2}). \tag{1}$$

Moreover, there exists $c \in (0, \frac{1}{2})$ such that for any $L \in 2\mathbb{N}$ and any odd integer $n \in (0, c\,L)$,

$$\Xi_L(n\,\boldsymbol{e}_1) \geq \frac{1}{2d}(1 - r_d). \tag{2}$$

Remark

$r_3 \simeq 0.52$ (exact computation Watson, 1939). Moreover, $r_{d+1} \leq r_d$.

Remark

$\Xi_L(x) = 0$ if $x \in \mathbb{T}_L^e$ and $L \in 2\mathbb{N}$.

Theorem (Lees, Taggi, 2019)

Suppose that $L \in 2\mathbb{N}$, let $z \in \mathbb{T}_L$ be such that $n = z \cdot \mathbf{e}_i$ is odd for some $i \in \{1, \ldots, d\}$, suppose that $n \in (0, \frac{L}{2})$. Then,

$$\Xi_L(z) \leq \Xi_L\big(\mathbf{e}_i n\big) \leq \Xi_L\big(\mathbf{e}_i(n-2)\big) \leq \Xi_L\big(\mathbf{e}_i\big) = \frac{1}{2d}. \qquad (3)$$

Remark

Since $r_d \to 0$ as $d \to \infty$, the lower and upper bound are sharp in the limit $d \to \infty$,

$$\frac{1}{2d}(1 - \frac{r_d}{2}) \leq \frac{1}{|\mathbb{T}_L^o|} \sum_{x \in \mathbb{T}_L^o} \Xi_L(x) \leq \frac{1}{2d}.$$

Remark

The site monotonicity properties (Lees, Taggi 2019) hold for other models, e.g, Spin O(N) model with arbitrary $N \in \mathbb{N}_{>0}$, Loop O(N) model, lattice permutations, and are not limited to the inequalities in (3).

Definition

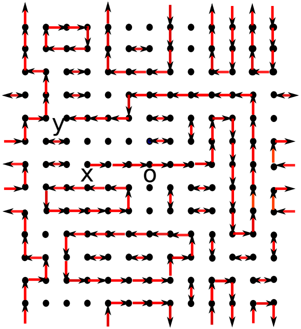
Let $\Omega_{x,y}$ be the set of *bijections* $\pi : \mathbb{T}_L \setminus \{y\} \to \mathbb{T}_L \setminus \{x\}$ such that $\forall z \in \mathbb{T}_L \setminus \{y\}$, $|\pi(z) - z|_1 \leq 1$. Define $\Omega = \cup_{x \in \mathbb{T}_L} \Omega_{o,x}$. Fix arbitrary $N, \lambda \geq 0$, define

$$\forall \pi \in \Omega \qquad \mathbb{P}_{L,N,\lambda}(\pi) := \frac{\lambda^{\mathcal{H}(\pi)} \, (N/2)^{\mathcal{L}(\pi)}}{Z_{L,\lambda,N}},$$

where $\mathcal{H}(\pi) := |\{z \in \mathbb{T}_L \ : \ \pi(z) \neq z\}|$ is the *number of (directed) edges* in the picture and $\mathcal{L}(\pi)$ is the *number of loops and double dimers* in π.

Terminology: *Loops*, *double dimers*, *monomers*, *walk*.

- Closely related to Loop O(N) model
- $\lambda = 1$, $N = 0$: uniform SAW in a box (*Duminil-Copin, Kozma, Yadin, 2014*)
- $N = 2$ related to quantum Bose gas
  (*Feynmann, 1953*)
  (*Ueltschi, Betz, 2010, 2011)*
  (*Elboim, Peled, 2017*)

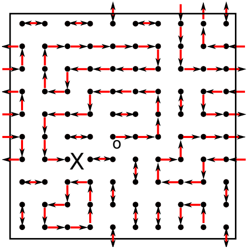


Define the *fully-packed* lattice permutation model,

$$\mathcal{P}_{L,N}(\pi) := \lim_{\lambda\to\infty} \mathbb{P}_{L,N,\lambda}(\pi)$$

We say that π is *fully-packed* if it contains no monomer.

Let $X : \Omega \to \mathbb{T}_L$ be the last point of the walk (*target point*).



Theorem (Taggi, 2019+)

In any dimension $d > 2$, for any integer N such that $0 < N < \frac{4}{r_d}$, the following holds for any $L \in 2\mathbb{N}$:

$$\forall A \subset \mathbb{T}_L, \quad \mathcal{P}_{L,N}\big(X \in A\big) \leq \frac{1}{1 - \frac{N r_d}{4}} \frac{|A|}{L^d}.$$

- For example plug in $A = \mathbb{T}_{\epsilon L}$ for small enough ϵ,
- When N is large, *exponential decay* for all λ:
  - **Intersecting loops on $\mathbb{Z}^d$**: Chayes, Pryadko, Shtengel, 1999.
  - **Loop $O(N)$ on honeycomb:** Duminil-Copin, Peled, Samotij, Spinka, 2014.

Definition (Two point function)

Let Ω^ℓ be the set of *permutations* $\pi : \mathbb{T}_L \to \mathbb{T}_L$ such that, for any $z \in \mathbb{T}_L$, $|\pi(z) - z|_1 \leq 1$.

$$Z^\ell_{L,N,\lambda} := \sum_{\pi \in \Omega^\ell} \lambda^{\mathcal{H}(\pi)} (N/2)^{\mathcal{L}(\pi)},$$

and, for any $x, y \in \mathbb{T}_L$, we define

$$Z_{L,N,\lambda}(x,y) := \sum_{\pi \in \Omega_{x,y}} \lambda^{\mathcal{H}(\pi)} (N/2)^{\mathcal{L}(\pi)},$$

Finally, we define the *two point function*,

$$G_{L,N,\lambda}(x,y) := \frac{\lambda \, Z_{L,N,\lambda}(x,y)}{Z^\ell_{L,N,\lambda}},$$

and note that, in the limit $\lambda \to \infty$, it collects only the contribution of fully packed configurations,

$$G_{L,N,\infty}(x,y) := \lim_{\lambda \to \infty} G_{L,N,\lambda}(x,y) = \frac{\sum\limits_{\substack{\pi \in \Omega_{x,y}: \\ \pi \text{ is f.p.}}} \left(\frac{N}{2}\right)^{\mathcal{L}(\pi)}}{\sum\limits_{\substack{\pi \in \Omega^\ell: \\ \pi \text{ is f.p.}}} \left(\frac{N}{2}\right)^{\mathcal{L}(\pi)}}.$$

Theorem (Taggi, 2019+)

Suppose that $d > 2$. For any integer N such that $0 < N < \frac{4}{r_d}$, and $L \in 2\mathbb{N}$, we have that,

$$\frac{1}{|\mathbb{T}_L^o|} \sum_{x \in \mathbb{T}_L^o} G_{L,N,\infty}(o,x) \geq \frac{1}{2d}\left(\frac{2}{N} - \frac{r_d}{2}\right).$$

Moreover, there exists $c \in (0, \frac{1}{2})$ such that for any $L \in 2\mathbb{N}$ and any odd integer $n \in (0, c\,L)$,

$$G_{L,N,\infty}(o, n\,\boldsymbol{e}_1) \geq \frac{1}{2d}\left(\frac{2}{N} - r_d\right).$$

Remark

From the monotonicity properties (Lees, Taggi 2019) and the fact that $r_d \to 0$ as $d \to \infty$, we deduce that the lower and upper bound are sharp in the limit $d \to \infty$,

$$\frac{1}{2d}\left(\frac{2}{N} - \frac{r_d}{2}\right) \leq \frac{1}{|\mathbb{T}_L^o|} \sum_{x \in \mathbb{T}_L^o} G_{L,N,\infty}(x) \leq \frac{1}{2d}\,\frac{2}{N}.$$

Relation between lattice permutations and dimers

Lemma

$$G_{L,2,\infty}(x,y) = \Xi_L(x,y).$$

Proof.

There exist two bijections Π^1, Π^2,

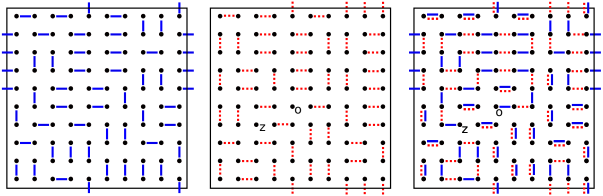
$$\Pi^1 : \mathcal{D}(\emptyset) \times \mathcal{D}(\{x,y\}) \mapsto \tilde{\Omega}_{x,y} := \{\pi \in \Omega_{x,y} \ : \pi \text{ is f.p.}\}$$

$$\Pi^2 : \mathcal{D}(\emptyset) \times \mathcal{D}(\emptyset) \mapsto \tilde{\Omega}^{\ell} := \{\pi \in \Omega^{\ell} \ : \pi \text{ is f.p.}\}$$

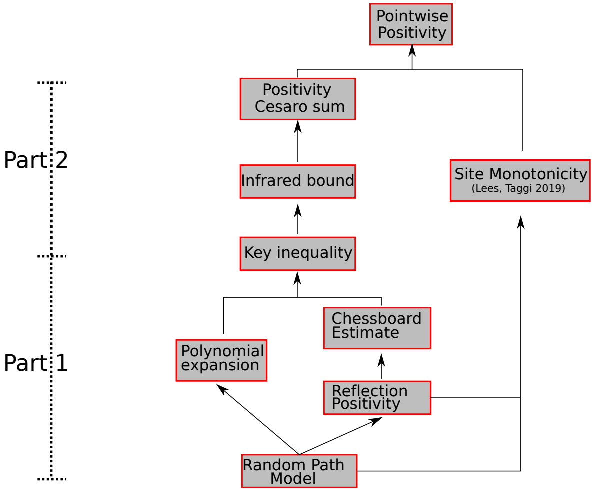
Hence,

$$G_{L,2,\infty}(x,y) = \frac{|\tilde{\Omega}_{x,y}|}{|\tilde{\Omega}^{\ell}|} = \frac{|\mathcal{D}(\emptyset)|\ |\mathcal{D}(\{x,y\})|}{|\mathcal{D}(\emptyset)|^2} = \frac{|\mathcal{D}(\{x,y\})|}{|\mathcal{D}(\emptyset)|} = \Xi_L(x,y).$$

□



Proof overview



Comment: Inspired by the famous proof of *Fröhlich, Simon, Spencer 1976* for the spin O(N) model

Method overviews: *Biskup, Friedli and Velenik, Spinka and Peled, Ueltschi.*

Dual torus, $\mathbb{T}_L^* := \left\{ \frac{2\pi}{L}(k_1, \dots, k_d) \in \mathbb{R}^d \;:\; k_i \in (-\frac{L}{2}, \frac{L}{2}] \cap \mathbb{Z} \right\}$. For $f \in \ell^2(\mathbb{T}_L)$,

$$\forall k \in \mathbb{T}_L^*, \quad \hat{f}(k) := \sum_{x \in \mathbb{T}_L} e^{-ik\cdot x} f(x).$$

$$\forall x \in \mathbb{T}_L, \quad f(x) = \frac{1}{|\mathbb{T}_L|} \sum_{k \in \mathbb{T}_L^*} e^{ik\cdot x} \hat{f}(k).$$

Put $G_{L,N,\infty}(x) := G_{L,N,\infty}(o, x)$.

Lemma

Define the Fourier modes $p := (\pi, \dots, \pi), o := (0, \dots, 0) \in \mathbb{T}_L^*$. *We have that,*

$$\frac{2}{|\mathbb{T}_L|} \sum_{x \in \mathbb{T}_L} G_{L,N,\infty}(x) = G_{L,N,\infty}(\boldsymbol{e}_1) - \frac{1}{|\mathbb{T}_L|} \sum_{k \in \mathbb{T}_L^* \setminus \{o,p\}} e^{ik\cdot \boldsymbol{e}_1} \hat{G}_{L,N,\infty}(k).$$

Proof: From the inverse Fourier transform formula:

$$G_{L,N,\infty}(\boldsymbol{e}_1) = \frac{1}{|\mathbb{T}_L|} \hat{G}_{L,N,\infty}(o) + \frac{e^{ip\cdot \boldsymbol{e}_1}}{|\mathbb{T}_L|} \hat{G}_{L,N,\infty}(p) + \frac{1}{|\mathbb{T}_L|} \sum_{k \in \mathbb{T}_L^* \setminus \{o,p\}} e^{ik\cdot \boldsymbol{e}_1} \hat{G}_{L,N,\infty}(k)$$

the fact that $\hat{G}_{L,N,\infty}(p) = -\hat{G}_{L,N,\infty}(o)$ since we are in the **fully packed regime** ($\lambda = \infty$) and from the Fourier transform formula: $\hat{G}_{L,N,\infty}(o) = \sum_{x \in \mathbb{T}_L} G_{L,N,\infty}(x)$.

Dual torus, $\mathbb{T}_L^* := \{ \frac{2\pi}{L}(k_1, \ldots, k_d) \in \mathbb{R}^d \ : \ k_i \in (-\frac{L}{2}, \frac{L}{2}] \cap \mathbb{Z} \}$. For $f \in \ell^2(\mathbb{T}_L)$,

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Note: $\quad G_{L,N,\infty}(\boldsymbol{e}_1) = \frac{1}{2d} \frac{2}{N}$

Goal: bound $\frac{1}{|\mathbb{T}_L|} \sum_{k \in \mathbb{T}_L^* \setminus \{o,p\}} e^{ik\cdot \boldsymbol{e}_1} \hat{G}_{L,N,\infty}(k)$ **away from** $\frac{1}{2d} \frac{2}{N}$ **uniformly!!**

Theorem (Key inequality)

For any $N \in \mathbb{N}_{>0}$, $\lambda \in \mathbb{R}_{>0} \cup \{\infty\}$, $L \in 2\mathbb{N}_{>0}$, *any real vector* $\mathbf{h} = (h_x)_{x \in \mathbb{T}_L}$,

$$\sum_{x,y \in \mathbb{T}_L} G_{L,N,\lambda}(x,y)(\triangle h)_x (\triangle h)_y \leq \sum_{\{x,y\} \in \mathbb{E}_L} (h_y - h_x)^2,$$

where $(\triangle h)_x := \sum_{y \sim x} (h_y - h_x)$.

- Case of *Fröhlich, Simon and Spencer*: $< S_x \cdot S_y >$ in place of $G(x,y)$ and factor $\frac{1}{\beta}$ in the RHS

Application of Key Inequality with $h_x := cos(k \cdot x)$ *(Fröhlich, Simon, Spencer 1976)*

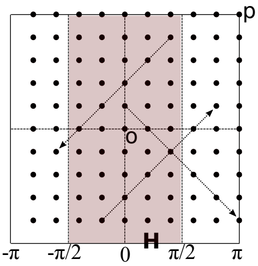
For any $k = (k_1, \ldots, k_d) \in \mathbb{T}_L^*$, define $\varepsilon(k) := \frac{1}{2\sum_{i=1}^d (1 - cos(k_i))}$,

$$k \in \mathbb{T}_L^* \setminus \{o\}, \qquad \hat{G}(k) \leq \frac{1}{\varepsilon(k)}.$$

Note: $\hat{G}(k)$ is real.

$$\frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} e^{ik\cdot e_1}\hat{G}_\infty(k) = \frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} Re\left(e^{ik\cdot e_1}\hat{G}_\infty(k)\right) = \frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} \cos(k\cdot \boldsymbol{e}_1)\hat{G}_\infty(k)$$

- Goal: bound red expression away from $\frac{1}{2d}\frac{2}{N}$ (uniformly in L) to conclude.
- Apply: $\hat{G}_\infty(k) \le \frac{1}{\epsilon(k)}$ (derived from Key Inequality)
- Define: $\mathbb{H} := \left\{k \in \mathbb{T}_L^* \;:\; k_1 \in (-\frac{\pi}{2}, \frac{\pi}{2}]\right\}$,
- Define **bijection** $\Psi : \mathbb{H}\setminus\{o\} \mapsto \mathbb{H}^c\setminus\{p\}$ such that $\Psi(k) = k + (\pm\pi, \dots, \pm\pi)$.
- Note: $\hat{G}_\infty(k + (\pm\pi, \dots, \pm\pi)) = -\hat{G}_\infty(k)$ since $G_\infty(x) = 0$ at even sites (**f.p. regime**!)



Positivity Cesaro sum given Key Inequality

$$\frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} e^{ik\cdot\mathbf{e}_1}\hat{G}_\infty(k) = \frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} Re\Big(e^{ik\cdot\mathbf{e}_1}\hat{G}_\infty(k)\Big) = \frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} \cos(k\cdot\mathbf{e}_1)\hat{G}_\infty(k)$$

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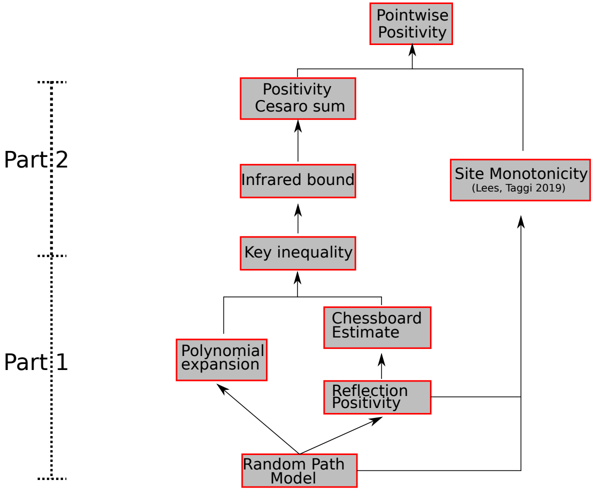
$$\frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} \cos(k\cdot\mathbf{e}_1)\hat{G}(k) = \frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{H}\setminus\{o\}} \Big(\cos(k\cdot\mathbf{e}_1)\,\hat{G}(k) + \cos(\Psi(k)\cdot\mathbf{e}_1)\hat{G}\big(\Psi(k)\big)\Big)$$

$$= \frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{H}\setminus\{o\}} 2\cos(k\cdot\mathbf{e}_1)\hat{G}(k) \le \frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{H}\setminus\{o\}} \frac{2\cos(k\cdot\mathbf{e}_1)}{\epsilon(k)}$$

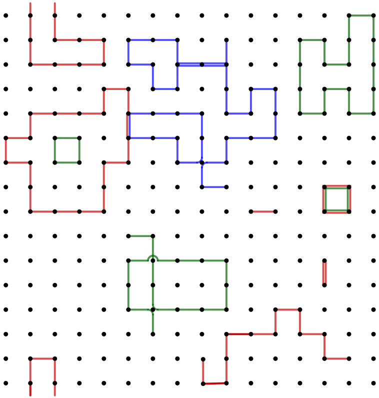
$$= \frac{1}{2d}\frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{H}\setminus\{o\}} \frac{2\cos(k\cdot\mathbf{e}_1)}{1-\frac{1}{d}\sum_{i=1}^d \cos(k\cdot\mathbf{e}_1)} \longrightarrow \frac{1}{2d}\frac{1}{2}\frac{1}{(2\pi)^d}\int_H dk \frac{2\cos(k\cdot\mathbf{e}_1)}{1-\frac{1}{d}\sum_{i=1}^d \cos(k\cdot\mathbf{e}_1)}$$

$$= \frac{1}{4d} r_d, \qquad \text{where } H := [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times [-\pi,\pi] \ldots \times [-\pi,\pi].$$

Proof overview



The random path model



1. arbitrary number of undirected loops, double dimers and walks,
2. such objects are allowed to 'use' the same edge multiple times,
3. it can be used to **represent** different models by choosing the *weight function* appropriately: e.g. spin O(N) model, loop O(N) model, random permutations, dimer model.
4. it will be possible for the walks to enter 'from the top'

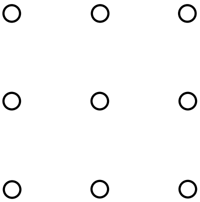
The random path model

Definition (Set of configurations)

- **Undirected finite graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- **Link cardinalities** $m \in \mathcal{M}_{\mathcal{G}} := \mathbb{N}^{\mathcal{E}}$. More specifically

$$m = (m_e)_{e \in \mathcal{E}},$$

  where $m_e \in \mathbb{N}$ represents the **number of links on the edge** e.
- A **pairing** $\boldsymbol{\pi} = (\boldsymbol{\pi}_x)_{x \in \mathcal{V}}$ **for** $\boldsymbol{m} \in \mathcal{M}_{\mathcal{G}}$ is such that $\boldsymbol{\pi}_x$ pairs links incident to x so that
  1. any link incident to x is *paired at* x to at most an other link incident to x
  2. any link incident to x might be *unpaired* at x
- $\boldsymbol{\mathcal{W}}_{\mathcal{G}}$ **set of configurations** $w = (m, \pi)$ such that $m \in \mathcal{M}_{\mathcal{G}}$ and π is a pairing for m.



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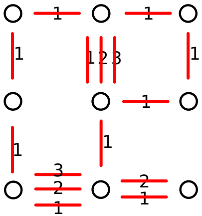

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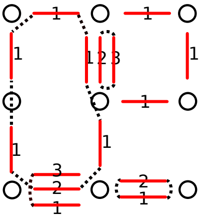
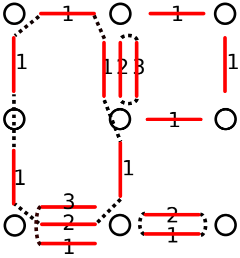


Definition (Measure)

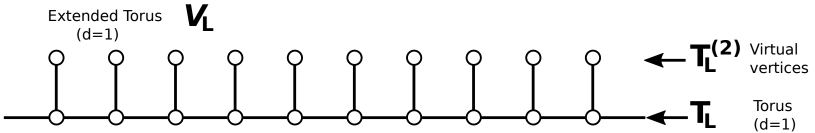
For any $w \in \mathcal{W}_{\mathcal{G}}$, define the (not normalised, possibly signed) measure,

$$\forall w = (m, \pi) \quad \mu_{\mathcal{G},N,\lambda}(w) = \left(\prod_{e \in \mathcal{E}} \frac{\lambda^{m_e}}{m_e!} \right) \left(\prod_{x \in \mathcal{V}} U_x(w) \right) N^{\mathcal{L}(w)}$$

where $U = (U_x)_{x \in \mathcal{V}}$ are the **weight functions**, U_x has **domain** $\{x\}$ and $\mathcal{L}(w)$ is the number of *link-connected components of* w.




$\mathbb{T}_L$
Torus
(d=1)

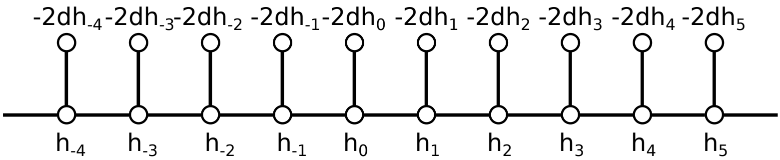

Extended Torus
(d=1)
V_L
$T_L^{(2)}$ Virtual vertices
T_L Torus (d=1)

Definition

Let $\boldsymbol{h} = (h_x)_{x \in \mathbb{T}_L}$ be a real vector, define

$$\mathcal{Z}_{L,N,U}(\boldsymbol{h}) := \mu_{N,\lambda,U}\Big(\prod_{x \in \mathbb{T}_L} h_x^{u_x} \, (-2\, d\, h_x)^{u_{H(x)}} \Big),$$

where u_y is the number of links **unpaired** at $y \in \mathcal{V}_L$ and for any $x \in \mathbb{T}_L$ (original torus), $H(x)$ is the vertex on top of x.



Definition

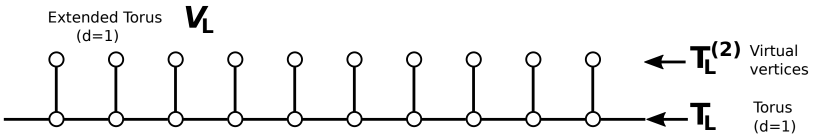
Let n_x be the number of pairings at at x. We define $U = (U_x)_{x \in \mathcal{V}_L}$:

$$\forall x \in \mathbb{T}_L \qquad U_x := \begin{cases} 1 & \text{if } n_x \leq 1 \text{ and no link on } \{x, H(x)\} \text{ is unpaired at } x, \\ \frac{1}{2} & \text{if } n_x \leq 1 \text{ and } \geq 1 \text{ links on } \{x, H(x)\} \text{ are unpaired at } x, \\ 0 & \text{if } n_x > 1. \end{cases}$$

$$\forall x \in \mathbb{T}_L^{(2)} \qquad U_x := \mathbb{1}_{\{n_x = 0\}}$$

$H(x)$ is the vertex "placed on top" of $x \in \mathbb{T}_L$, i.e, $H(x) \in \mathbb{T}_L^{(2)}$

Loops and double dimers are vertex-self-avoiding and cannot touch virtual vertices, walks are 'not entirely' vertex-self-avoiding and can end on virtual vertices.



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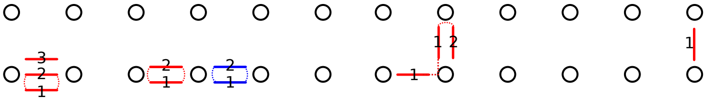

Figure: A configuration w such that $\mu(w) = 0$.

Theorem (Polynomial expansion)

We have,

$$\mathcal{Z}_{L,N,\lambda,U}(\varphi\, \boldsymbol{h}) = Z^{\ell}_{L,N,\lambda} + \varphi^2 \mathcal{Z}^{(2)}_{L,N,\lambda}(\boldsymbol{h}) + o(\varphi^2),$$

in the limit as $\varphi \to 0$*, where*

$$\mathcal{Z}^{(2)}_{L,N,\lambda}(\boldsymbol{h}) := -\sum_{\{x,y\}\in\mathbb{E}_L} (h_y - h_x)^2 \frac{N\lambda}{2}\, Z^{\ell}_{L,N,\lambda} + \\ + \frac{N\lambda^2}{2} \sum_{x,y\in\mathbb{T}_L} Z_{L,N,\lambda}(x,y)\, (\triangle h)_x\, (\triangle h)_y$$

Note: the **Key Inequality** is: $\mathcal{Z}^{(2)}_{L,N,\lambda}(\boldsymbol{h}) \leq 0$!!!

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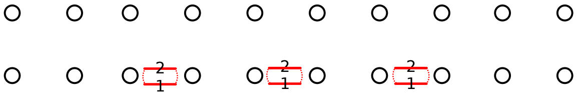

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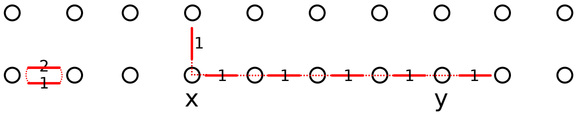

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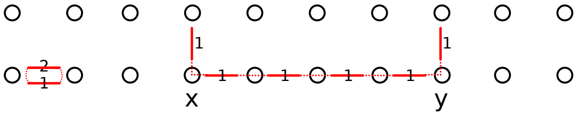

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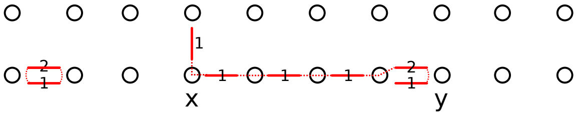


Chessboard estimate

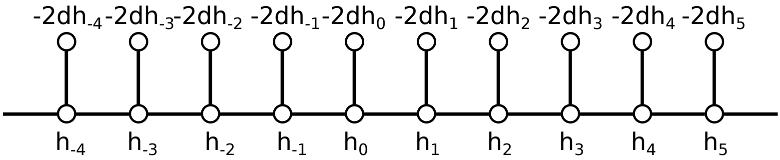
Theorem (Chessboard estimate)

For any $\boldsymbol{h} = (h_z)_{z\in\mathbb{T}_L}$, *define* $\boldsymbol{h}^x = (h^x_z)_{z\in\mathbb{T}_L}$ *as the vector which is obtained from* $\boldsymbol{h}$ *as follows:*

$$h^x_z := h_x \text{ for every } z \in \mathbb{T}_L.$$

Then,

$$\mathcal{Z}(\boldsymbol{h}) \leq \Big(\prod_{x\in\mathbb{T}_L} \mathcal{Z}(\boldsymbol{h}^x)\Big)^{\frac{1}{|\mathbb{T}_L|}}$$



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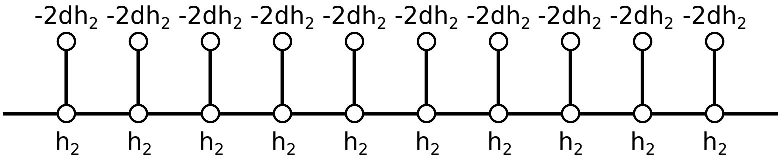


Derivation Key Inequality from Chessboard estimate and Polynomial expansion

Let $\boldsymbol{h} = (h_z)_{z \in \mathbb{T}_L}$ be arbitrary, we have:

$$\forall x \in \mathbb{T}_L \quad \mathcal{Z}^{(2)}_{L,N,\lambda,U}(\boldsymbol{h}^x) = 0$$

Thus,

$$\begin{aligned}
\mathcal{Z}_{L,N,\lambda,U}(\varphi \boldsymbol{h}) &= Z^{\ell}_{L,N,\lambda} + \varphi^2 \mathcal{Z}^{(2)}_{L,N,\lambda,U}(\boldsymbol{h}) + o(\varphi^2) \\
&\leq \Big(\prod_{x \in \mathbb{T}_L} \mathcal{Z}_{L,N,\lambda,U}((\varphi \boldsymbol{h}^x)) \Big)^{\frac{1}{|\mathbb{T}_L|}} \\
&= \Big(\prod_{x \in \mathbb{T}_L} (Z^{\ell}_{L,N,\lambda} + o(\varphi^2)) \Big)^{\frac{1}{|\mathbb{T}_L|}} \\
&= Z^{\ell}_{L,N,\lambda} + o(\varphi^2),
\end{aligned}$$

We conclude that,

$$\mathcal{Z}^{(2)}_{L,N,\lambda,U}(\boldsymbol{h}) \leq 0.$$

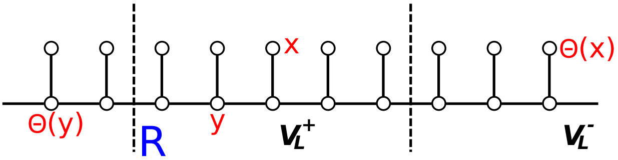
This gives the **Key Inequality.**

Definition (Reflections)

- R reflection plane through edges, orthogonal to $\boldsymbol{e}_i$ for some $i \in \{1, \ldots, d\}$,
- $\Theta : \mathcal{V}_L \mapsto \mathcal{V}_L$ reflection with respect to R,
- $\mathcal{V}_L^+, \mathcal{V}_L^- \subset \mathcal{V}_L$ subsets such that $\Theta(\mathcal{V}_L^\pm) = \mathcal{V}_L^\mp$,
- $\Theta : \mathcal{W}_L \mapsto \mathcal{W}_L$ reflects $w \in \mathcal{W}_L$ with respect to R (see Figure)
- Given $f : \mathcal{W}_L \mapsto \mathbb{R}$, define the function Θf as

$$\Theta f(w) := f(\Theta(w)).$$

- Let $\mathcal{A}^\pm$ be the class of functions with domain in $\mathcal{V}_L^\pm$.



Reflection positivity

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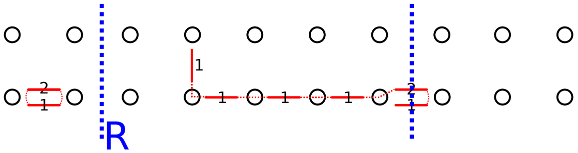

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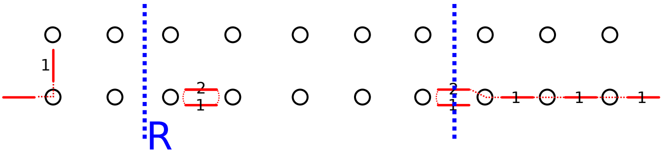

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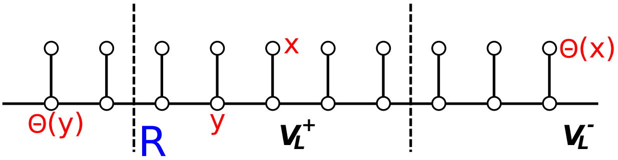


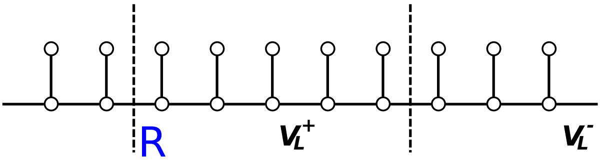
Theorem (Reflection positivity)

For any pair of functions $f, g \in \mathcal{A}^+$, we have that,

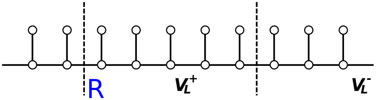
1. $\mu_{L,N,\lambda,U}(f\Theta g) = \mu_{L,N,\lambda,U}(g\Theta f)$,
2. $\mu_{L,N,\lambda,U}(f\Theta f) \geq 0$,

from which we deduce that $\mu_{L,N,\lambda,U}$ is **reflection positive,** *namely:*

$$\mu_{L,N,\lambda,U}\big(f\,\Theta g\big) \leq \mu_{L,N,\lambda,U}\big(f\,\Theta f\big)^{\frac{1}{2}}\;\mu_{L,N,\lambda,U}\big(g\,\Theta g\big)^{\frac{1}{2}}.$$



Proof of $\mu_{L,N,\lambda,U}(f\Theta f) \geq 0$ **when** $N = 1$:



- $\mathcal{E}^R$:= edges with one end-point in $\mathcal{V}_L^+$ and in $\mathcal{V}_L^-$,
- $\mu^R(w) := \prod_{e\in\mathcal{E}^R} \frac{\lambda^{m_e}}{m_e!}$
- $\mathcal{E}^\pm$:= edges with **at least** one end-point in $\mathcal{V}_L^\pm$,
- $\mu^\pm(w) := \Big(\prod_{x\in\mathcal{V}^\pm} U_x(w)\Big)\Big(\prod_{e\in\mathcal{E}_L^\pm\setminus\mathcal{E}_L^R} \frac{\lambda^{m_e}}{m_e!}\Big)$
- $\mathcal{W}^R$:= configurations with **links only above** $\mathcal{E}^R$ and all of them **unpaired**
- $w^\pm$ is the **restriction** of w to $\mathcal{V}_L^\pm$ (keep links incident to sites in $\mathcal{V}_L^\pm$),

$$\mu(f\Theta f) = \sum_{w'\in\mathcal{W}^R}\sum_{\substack{w\in\mathcal{W}\\ P_R(w)=w'}} f(w)\Theta f(w)\mu(w) = \sum_{w'\in\mathcal{W}^R}\sum_{\substack{w\in\mathcal{W}\\ P_R(w)=w'}} f(w^+)\Theta f(w^-)\mu^R(w')\mu^+(w^+)\mu^-(w^-)$$

$$= \sum_{w'\in\mathcal{W}^R}\mu^R(w')\Big(\sum_{\substack{w\in\mathcal{W}\\ P_R(w)=w'}} f(w^+)\mu^+(w)\Big)\Big(\sum_{\substack{w\in\mathcal{W}\\ P_R(w)=w'}} \Theta f(w^-)\mu^-(w)\Big) =$$

$$= \sum_{w'\in\mathcal{W}^R}\mu^R(w')\Big(\sum_{\substack{w\in\mathcal{W}\\ P_R(w)=w'}} f(w^+)\mu^+(w)\Big)^2 \geq 0.$$

Lemma

For arbitrary $\boldsymbol{h}$, *define* $\boldsymbol{h}^{\pm}$ *as follows:*

$$\forall x \in \mathbb{T}_L \qquad h^{\pm}_x := \begin{cases} h_x & \text{if } x \in \mathbb{T}^+_L \\ h_{\Theta(x)} & \text{if } x \in \mathbb{T}^-_L \end{cases}.$$

We have that,

$$\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}) \leq \sqrt{\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}^+)\ \mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}^-)}$$

Proof. Note that

$$\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}) = \mu\Big(\prod_{x \in \mathbb{T}^+_L} \big(h_x^{u_x}(-2dh_x)^{u_{H(x)}}\big) \prod_{x \in \mathbb{T}^-_L} \big(h_x^{u_x}(-2dh_x)^{u_{H(x)}}\big)\Big)$$

and apply R.P.

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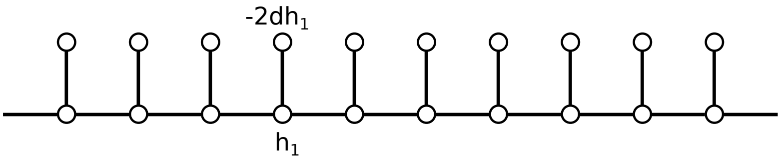

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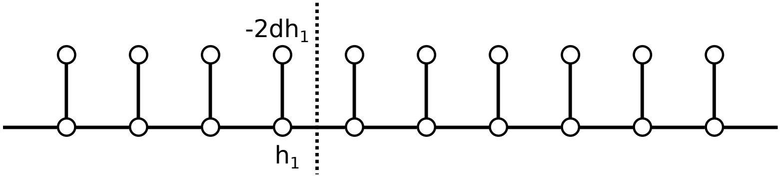

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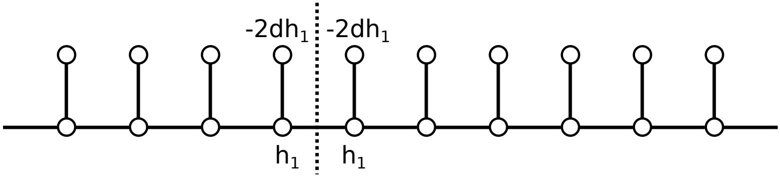

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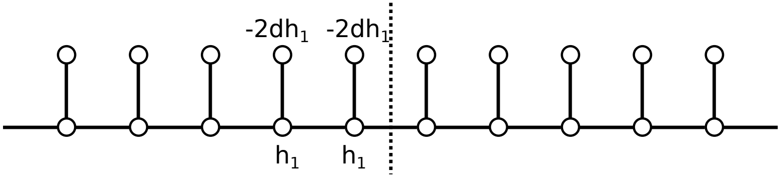

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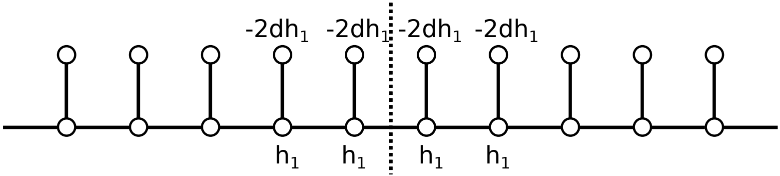

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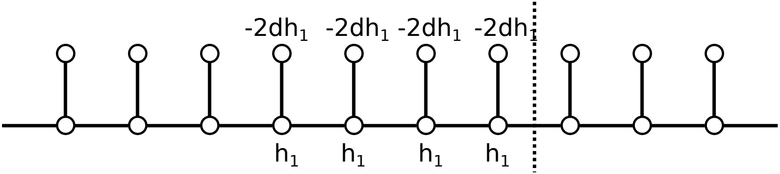

Lemma

For arbitrary $\boldsymbol{h}$*, define* $\boldsymbol{h}^{\pm}$ *as follows:*

$$\forall x \in \mathbb{T}_L \qquad h^{\pm}_x := \begin{cases} h_x & \text{if } x \in \mathbb{T}^{+}_L \\ h_{\Theta(x)} & \text{if } x \in \mathbb{T}^{-}_L \end{cases}.$$

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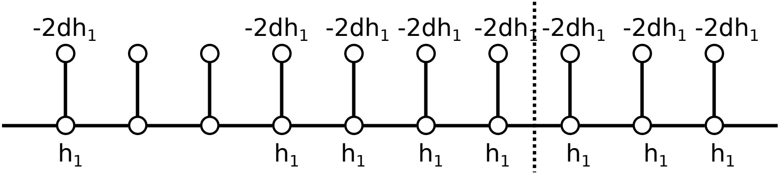

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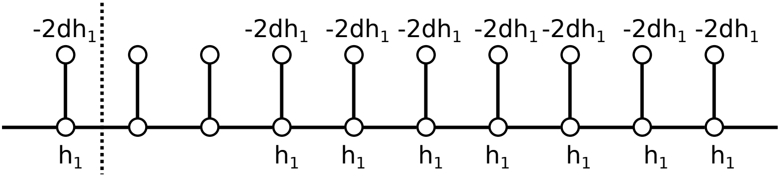

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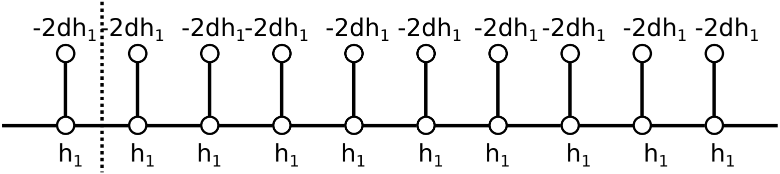
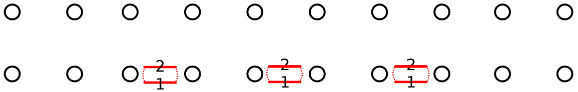


Justification of **polynomial expansion:**

$$\mathcal{Z}_{L,N,\lambda,U}(\varphi\,\boldsymbol{h}) = Z^{\ell}_{L,N,\lambda} + \varphi^2 \mathcal{Z}^{(2)}_{L,N,\lambda}(\boldsymbol{h}) + o(\varphi^2),$$

$Z^{\ell}_{L,N,\lambda}$ is the *contribution from random path configurations with no unpaired links*:



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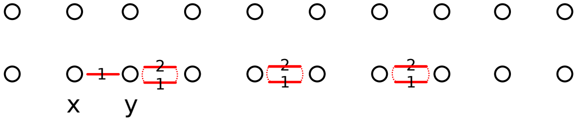
Contribution from *random path configurations with a link unpaired at its end-points* x *and* y such that $\{x,y\} \in \mathbb{E}_L$:

$$N\,\lambda\,h_x\,h_y\,Z^{\ell}_{L,N,\lambda}$$



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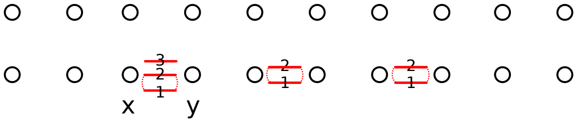
$$N\lambda \sum_{\{x,y\} \in \mathbb{E}_L} h_x h_y Z^{\ell}_{L,N,\lambda}$$



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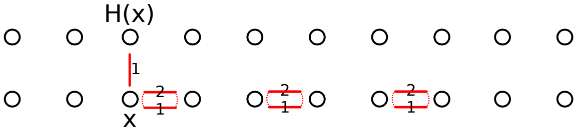
Contribution from random path configurations with a link unpaired at its end-points $x \in \mathbb{T}_L$ and y *with* y *on the top of* x:

$$-\frac{1}{2}\,N\,\lambda\,(2\,d\,h_x^2) Z^{\ell}_{L,N,\lambda}$$



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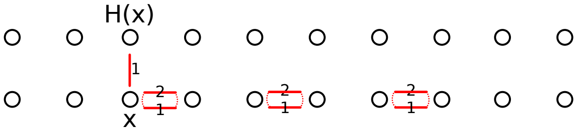
Summing contributions with a link unpaired at both its end-points:

$$N\lambda\Big(\sum_{\{x,y\}\in\mathbb{E}_L} h_x h_y \,-\, \sum_{x\in\mathbb{T}_L} d h_x^2 \Big) Z^{\ell}_{L,N,\lambda} \;=\; -\sum_{\{x,y\}\in\mathbb{E}_L} (h_y \,-\, h_x)^2\, \frac{N\lambda}{2}\; Z^{\ell}_{L,N,\lambda}. \qquad (4)$$



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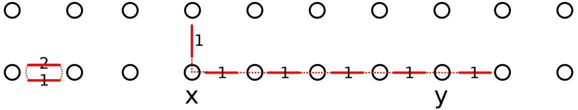
Contribution from random path configurations with a walk having x **and** y **as second-last** points:

$$N\lambda^2(\triangle h)_x(\triangle h)_y Z_{L,N,\lambda}(x,y)$$



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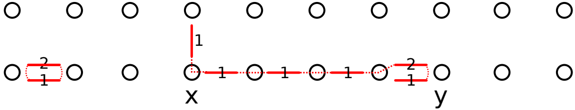

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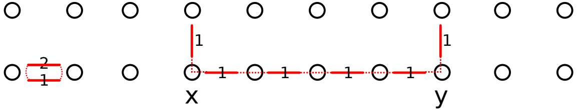

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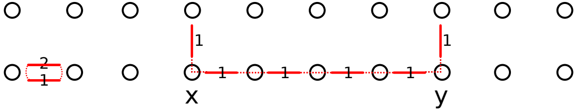

Questions

- Other Applications of the **key inequality** (e.g. Merming-Wagner or polynomial decay of correlations in $d = 2$?)

$$\sum_{x,y \in \mathbb{T}_L} G_{L,N,\lambda}(x,y)(\triangle h)_x \, (\triangle h)_y \leq \sum_{\{x,y\} \in \mathbb{E}_L} \left(h_y - h_x\right)^2.$$

- Implementation of the method for the (loop representation of) **Quantum bose gas** or quantum **Heisenberg model**?