Counting planar maps with a height function

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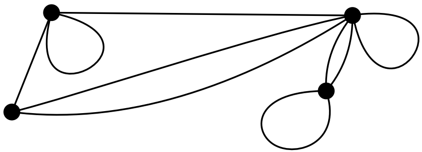
Joint work with Mireille Bousquet-Mélou

Université de Bordeaux, France

02/09/2019

PLANAR MAPS



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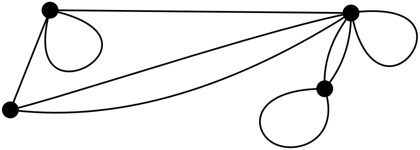
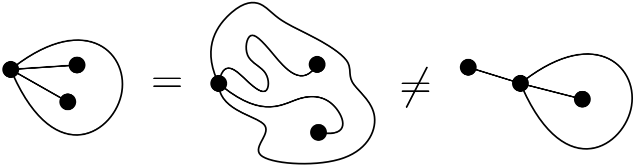
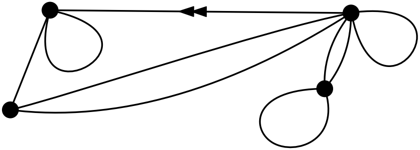
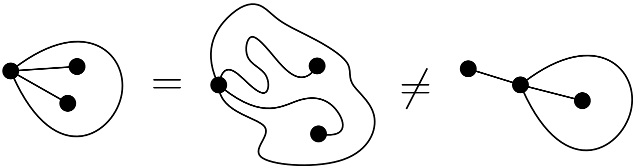


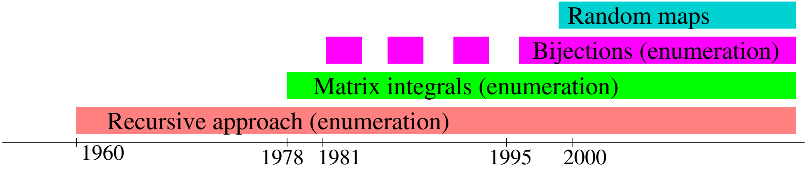


Rooted planar maps





A CHRONOLOGY OF PLANAR MAPS



- **Recursive approach:** Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Wanless...
- **Matrix integrals:** Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Kostov, Zinn-Justin, Boulatov, Kazakov, Mehta, Bouttier, Di Francesco, Guitter, Eynard...
- **Bijections:** Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, Bousquet-Mélou, Chapuy...
- **Geometric properties of random maps:** Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Jean-François Le Gall, Miermont, Curien, Albenque, Bettinelli, Ménard, Angel, Sheffield, Miller, Gwynne...

Maps equipped with an additional structure

- **How many maps equipped with...**
  - a spanning tree [Mullin 67, Bernardi]
  - a spanning forest? [Bouttier et al., Sportiello et al., Bousquet-Mélou & Courtiel]
  - a self-avoiding walk? [Duplantier & Kostov; Gwynne & Miller]
  - a proper q-colouring? [Tutte 74-83, Bouttier et al.]
  - a bipolar orientation? [Kenyon, Miller, Sheffield, Wilson, Fusy, Bousquet-Mélou...]
- **What is the expected partition function of...**
  - the Ising model? [Boulatov, Kazakov, Bousquet-Mélou, Schaeffer, Chen, Turunen, Bouttier et al., Albenque, Ménard...]
  - the hard-particle model? [Bousquet-Mélou, Schaeffer, Jehanne, Bouttier et al.]
  - the Potts model? [Eynard-Bonnet, Baxter, Bousquet-Mélou & Bernardi, Guionnet et al., Borot et al., ...]

OUR ADDITIONAL STRUCTURES

We will consider the following objects:

- **Weakly height-labelled maps** (maps decorated by a weak height function)
- **Height-labelled quadrangulations** (quadrangulations decorated by a height function)
- **4-valent Eulerian orientations** (the six vertex model)

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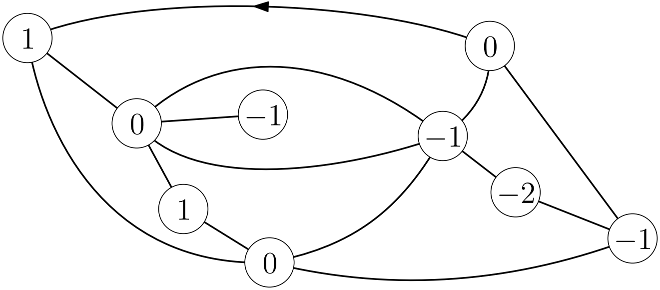
These are all in bijection with each other.

HEIGHT-LABELLED QUADRANGULATIONS

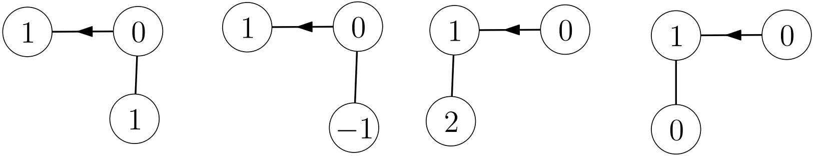
- Each face has degree 4
- Adjacent labels differ by 1
- Root edge labelled from 0 to 1



Height-labelled quadrangulations

- Each face has degree 4
- Adjacent labels differ by 1
- Root edge labelled from 0 to 1

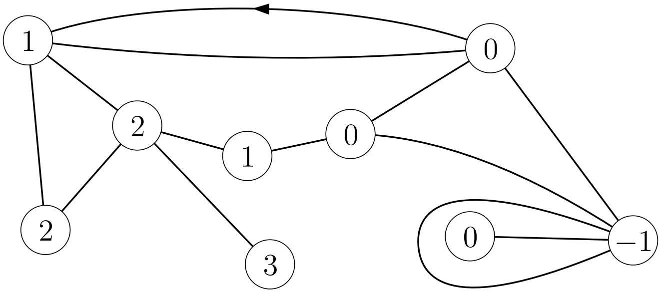
Aim: determine the generating function $\mathsf{Q}(t) = 4t + 35t^2 + \ldots$ that counts height-labelled quadrangulations by faces.



WEAKLY HEIGHT-LABELLED MAPS

- Adjacent labels differ by at most 1
- Root edge points to vertex labelled 1

We will see that $\mathsf{Q}(t) = 4t + 35t^2 + \ldots$ counts weakly height-labelled maps by edges.



Height-labelled quadrangulations background

In 2017, EP and Guttmann:

- Computed the number q_n of height-labelled quadrangulations for $n < 100$.
- Predicted that

$$q_n \sim \kappa_q \frac{(4\sqrt{3}\pi)^n}{n^2(\log n)^2}.$$

This led us to conjecture the exact solution

PREVIEW: EXACT SOLUTION

Let $\mathsf{R}(t)$ be the unique series satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}(t)^{n+1}.$$

Theorem: The generating function of height-labelled quadrangulations is given by

$$\mathsf{Q}(t) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}(t)).$$

Asymptotically,

$$q_n \sim \kappa \frac{\mu^{n+2}}{n^2(\log n)^2},$$

where $\kappa = 1/18$ and $\mu = 4\sqrt{3}\pi$.

TALK OUTLINE

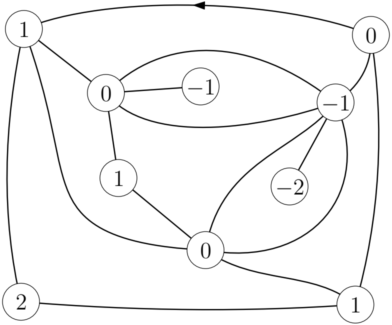
- **Part 1:** Bijection between height-labelled quadrangulations and weakly height-labelled maps (Miermont/Ambjørn and Budd)
- **Part 2:** Enumeration of height-labelled quadrangulations. (Bousquet-Mélou and EP)
- **Part 3:** Bijection to the ice model (EP and Guttmann)
- **Part 4:** Six vertex model solution (Kostov/EP and Zinn-Justin)
- **Part 5:** Height distribution (Bousquet-Mélou and EP)

Part 1: Bijection between height-labelled quadrangulations and weakly height-labelled maps

(Miermont (2009)/Ambjørn and Budd (2013)).

QUADRANGULATIONS TO MAPS

Start with a height-labelled quadrangulation.



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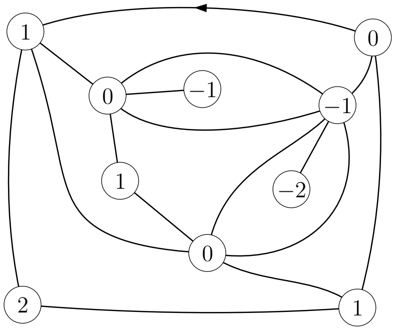
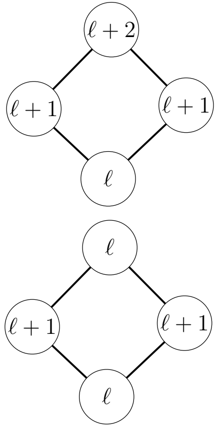
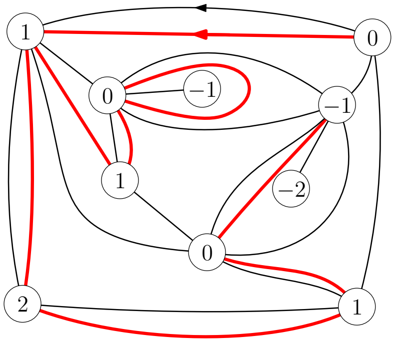
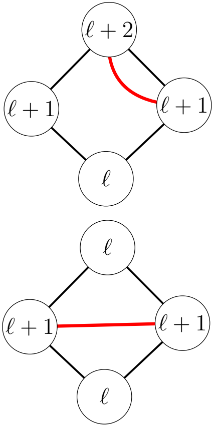




Quadrangulations to maps

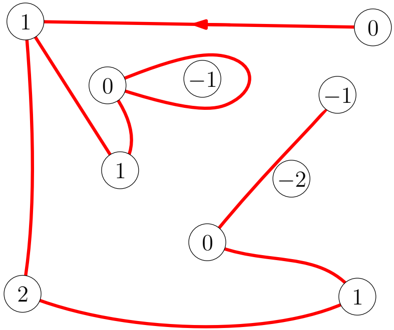
Draw a red edge in each face according to the rule.





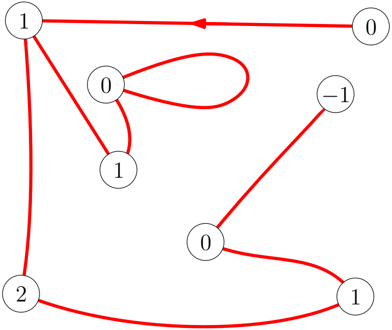
Quadrangulations to maps

Remove all of the original edges.



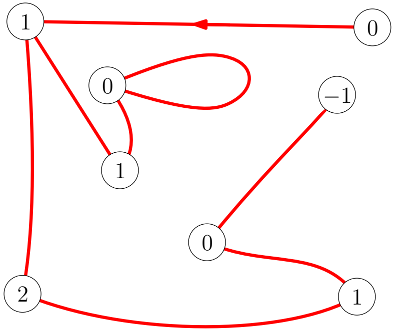
QUADRANGULATIONS TO MAPS

Remove any isolated vertices.



QUADRANGULATIONS TO MAPS

The new map is a weakly height-labelled map (adjacent labels differ by *at most* 1).



Part 2: Exact solution for height-labelled quadrangulations

(Bousquet-Mélou and EP (2018)).

Counting height-labelled quadrangulations

By generalising the problem, we deduce a system of functional equations which defines $\mathsf{Q}(t)$:

Counting height-labelled quadrangulations

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$$\begin{aligned}
\mathsf{Q}(t) &= [y]\mathsf{P}(t,y)\\
\mathsf{P}(t,y) &= \frac{1}{y}[x^1]\mathsf{C}(t,x,y)\\
\mathsf{D}(t,x,y) &= \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)}\\
\mathsf{D}(t,x,y) &= 1 + y\mathsf{D}(t,x,y)[y^1]\mathsf{D}(t,x,y) + y[x^{\geq 0}]\frac{1}{x}\mathsf{P}\left(t,\frac{1}{x}\right)\mathsf{D}(t,x,y)\\
[y^1]\mathsf{D}(t,x,y) &= \frac{1}{1-x}(1+2t[y^2]\mathsf{D}(t,x,y) - t([y^1]\mathsf{D}(t,x,y))^2).
\end{aligned}$$

Counting height-labelled quadrangulations

By generalising the problem, we deduce a system of functional equations which defines $\mathsf{Q}(t)$:

$$\mathsf{Q}(t) = [y]\mathsf{P}(t,y)$$

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I will show one element of the proof.

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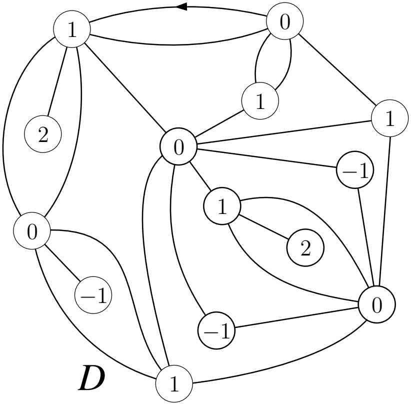
D-PATCHES

D-patch: Digons are allowed next to the root vertex and the outer face may have any degree.

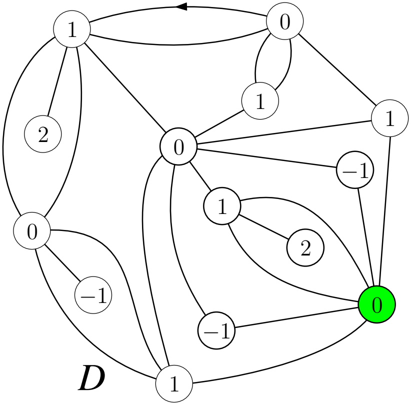


Restrictions:
- outer labels must be 0 or 1.
- vertices adjacent to the root must be labelled 1.

In $\mathsf{D}(t, x, y)$:
- t counts quadrangles.
- x counts digons.
- y counts the degree of the outer face (halved).

DECOMPOSITION OF D-PATCHES

Colour the vertex two places clockwise from the root vertex around the outer face.



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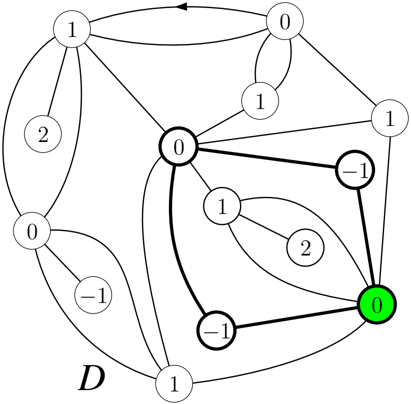
Decomposition of D-patches

Highlight the maximal connected subgraph of nonpositive labels, containing the coloured vertex.



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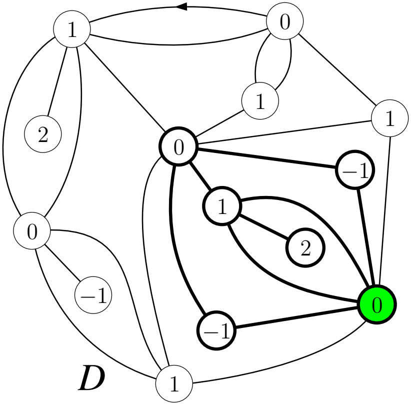
Decomposition of D-patches

Add to the subgraph all vertices and edges contained in its inner face(s).



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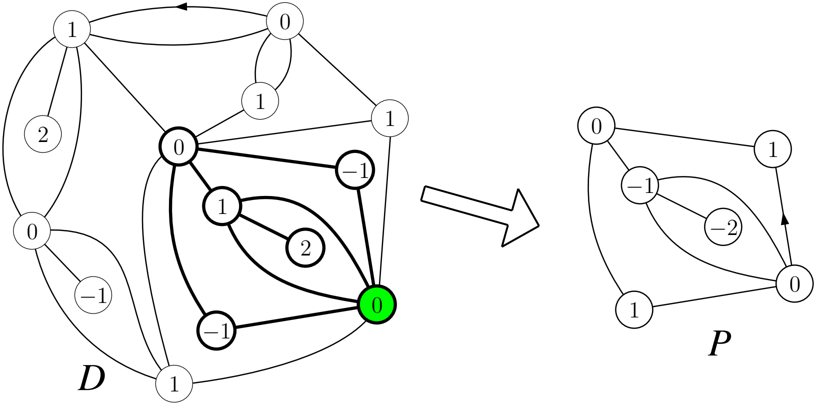
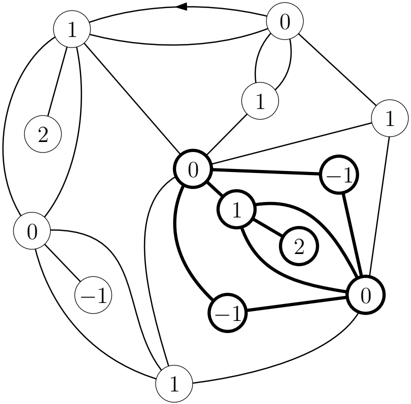
Decomposition of D-patches

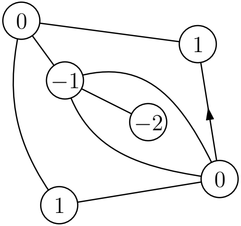
Record the subgraph with inverted labels.



DECOMPOSITION OF D-PATCHES

Contract the highlighted map to a single vertex (labelled 0).





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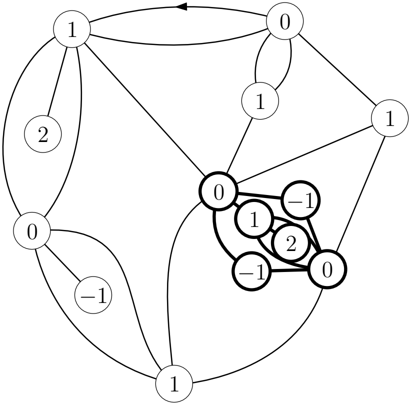

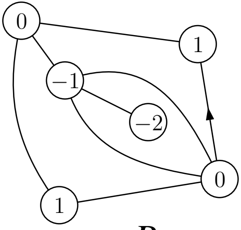

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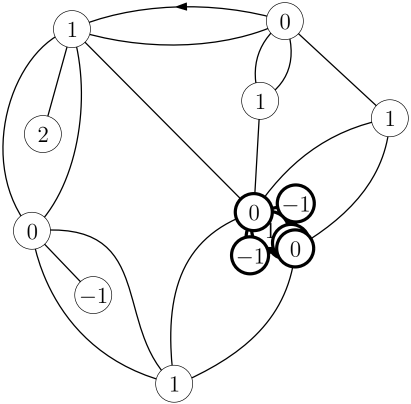

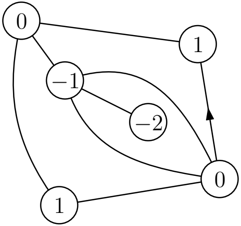

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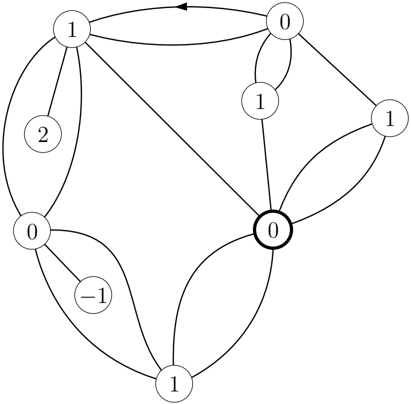

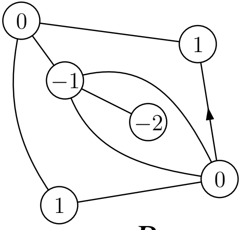

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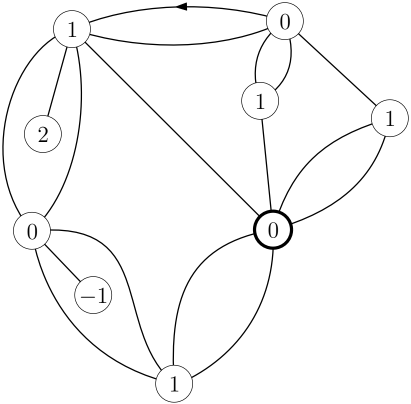
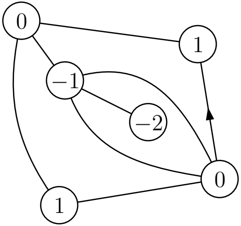
DECOMPOSITION OF D-PATCHES

Contract the highlighted map to a single vertex (labelled 0). The new vertex may be adjacent to digons.





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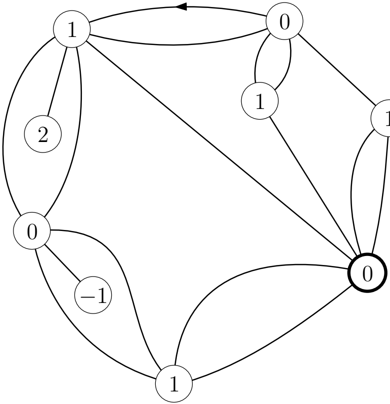
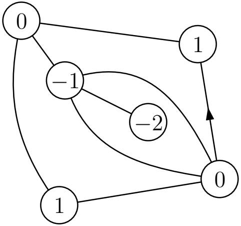
Decomposition of D-patches

Merge the new vertex with the root vertex.





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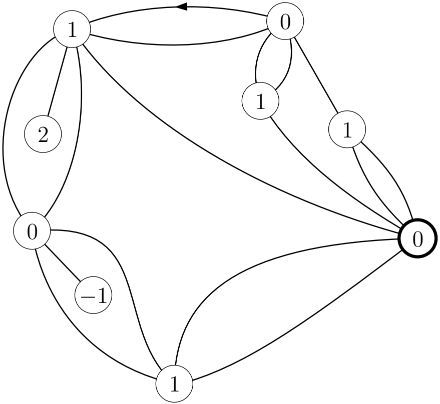

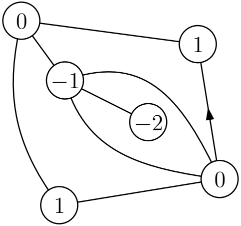

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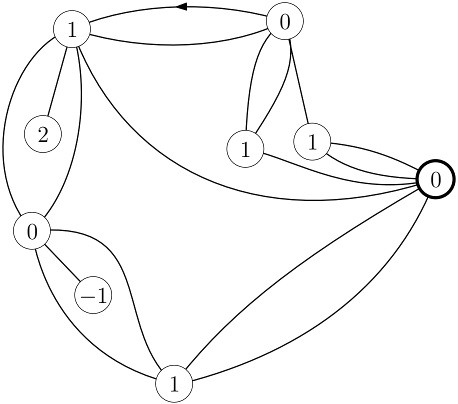

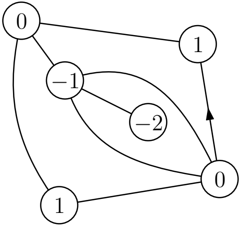

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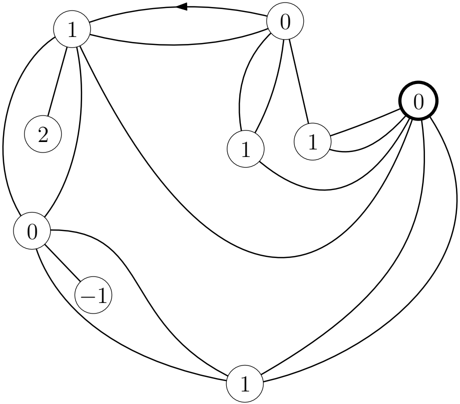

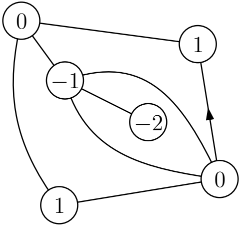

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DECOMPOSITION OF D-PATCHES

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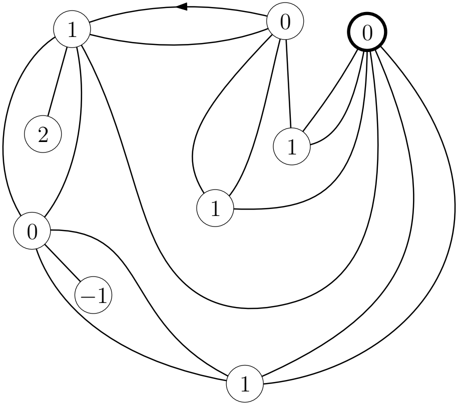
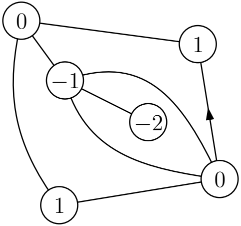
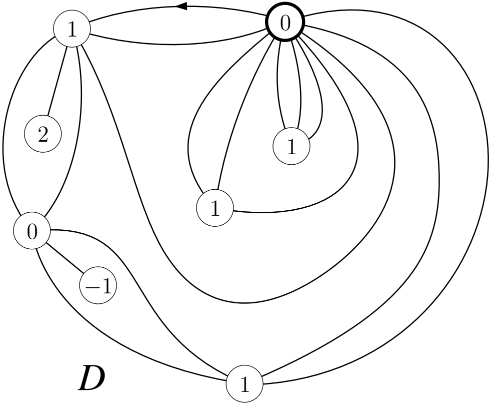




P

Decomposition of D-patches

Merge the new vertex with the root vertex.



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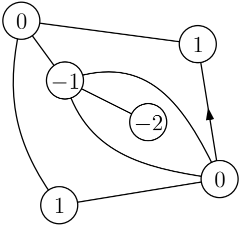

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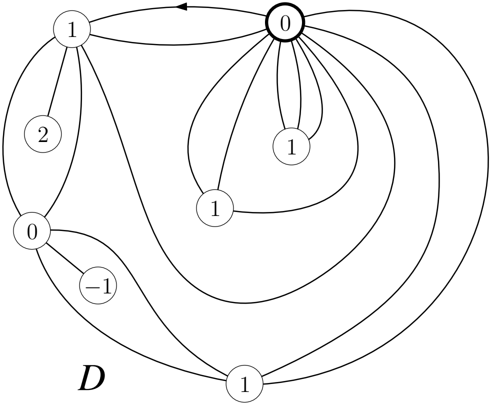
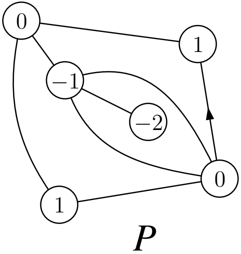
Decomposition of D-patches

Merge the new vertex with the root vertex. This new map is a D-patch!





Equations for labelled quadrangulations

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$$\mathsf{P}(t,y) = \frac{1}{y}[x^1]\mathsf{C}(t,x,y)$$

$$\mathsf{D}(t,x,y) = \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)}$$

$$\mathsf{D}(t,x,y) = 1 + y\mathsf{D}(t,x,y)[y^1]\mathsf{D}(t,x,y) + y[x^{\geq 0}]\frac{1}{x}\mathsf{P}\left(t,\frac{1}{x}\right)\mathsf{D}(t,x,y)$$

$$[y^1]\mathsf{D}(t,x,y) = \frac{1}{1-x}(1 + 2t[y^2]\mathsf{D}(t,x,y) - t([y^1]\mathsf{D}(t,x,y))^2).$$

Solving the equations

- At this point we just needed to guess the values of the series P, C and D and verify that the guesses satisfy the equations.

SOLVING THE EQUATIONS

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- Bref, we did.

Solution for labelled quadrangulations

Let $\mathsf{R}(t)$ be the unique series satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}(t)^{n+1}.$$

Then the series $\mathsf{P}(t, y)$, $\mathsf{C}(t, x, y)$ and $\mathsf{D}(t, x, y)$ are given by:

$$t\mathsf{P}(t, ty) = \sum_{n\geq 0} \sum_{j=0}^{n} \frac{1}{n+1} \binom{2n-j}{n} \binom{3n-j}{n} y^j \mathsf{R}^{n+1},$$

$$\mathsf{C}(t, x, ty) = 1 - \exp\left(- \sum_{n\geq 0} \sum_{j=0}^{n} \sum_{i=0}^{2n-j} \frac{1}{n+1} \binom{2n-j}{n} \binom{3n-i-j}{n} x^{i+1} y^{j+1} \mathsf{R}^{n+1} \right),$$

$$\mathsf{D}(t, x, ty) = \exp\left(\sum_{n\geq 0} \sum_{j=0}^{n} \sum_{i\geq 0} \frac{1}{n+1} \binom{2n-j}{n} \binom{3n+i-j+1}{2n-j} x^i y^{j+1} \mathsf{R}^{n+1} \right).$$

SOLUTION FOR LABELLED QUADRANGULATIONS

Let $\mathsf{R}(t)$ be the unique series satisfying

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Theorem: The generating function of labelled quadrangulations is given by

$$\mathsf{Q}(t) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}(t)).$$

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Theorem: The generating function of labelled quadrangulations is given by

$$\mathsf{Q}(t) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}(t)).$$

Asymptotically,

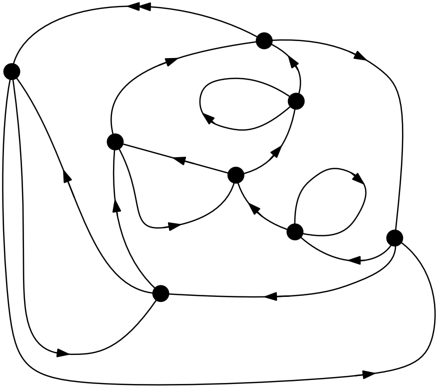
$$q_n \sim \kappa \frac{\mu^{n+2}}{n^2(\log n)^2},$$

where $\kappa = 1/18$ and $\mu = 4\sqrt{3}\pi$.

Part 3: Bijection to the Ice model

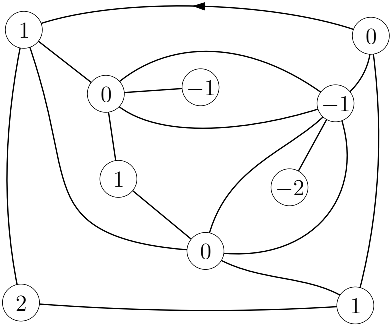
(EP and Gutmann (2017)).

Ice model: each vertex has two incoming and two outgoing edges.
Counted by vertices



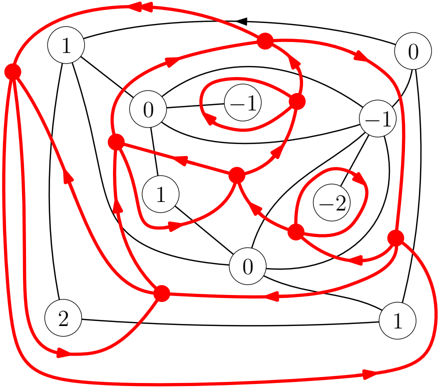
Bijection to the ice model

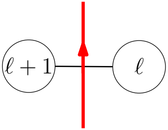
Start with a height-labelled quadrangulation.



Bijection to the ice model

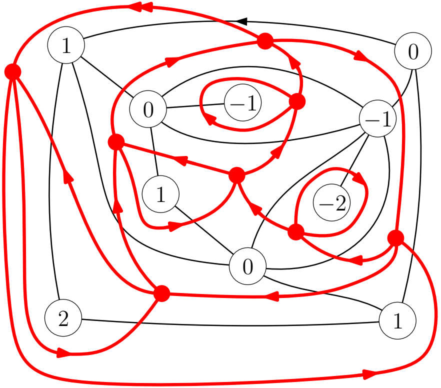
Draw the dual with edges oriented according to the rule.

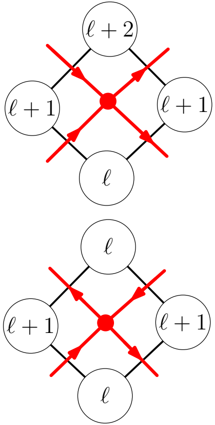




Bijection to the ice model

Each red vertex has two incoming and two outgoing edges.





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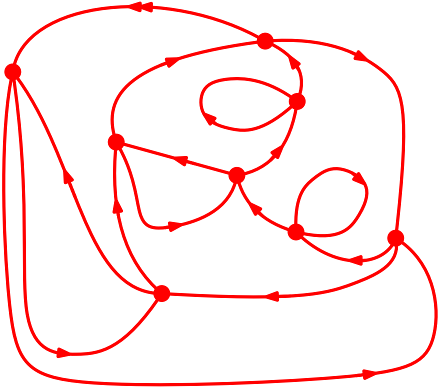

BIJECTION TO THE ICE MODEL

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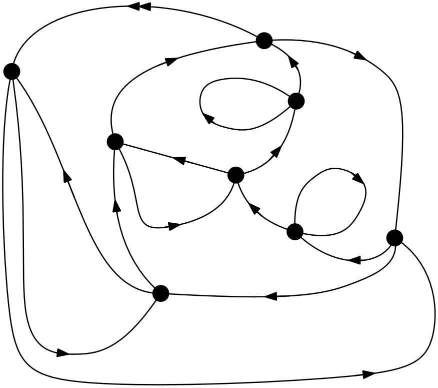
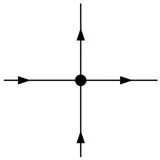


Part 4: Six vertex model

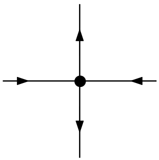
(Kostov (2000)/EP and Zinn-Justin (2019)).

Six vertex model

Six vertex model: weight γ per alternating vertex.
Generating function: $\mathsf{Q}(t,\gamma)$.



Non-alternating
(weight t)



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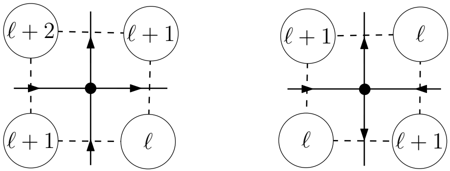

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The weight γ counts:

- Alternating faces in height-labelled quadrangulations

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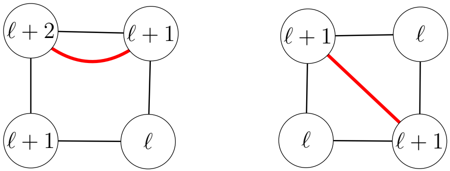

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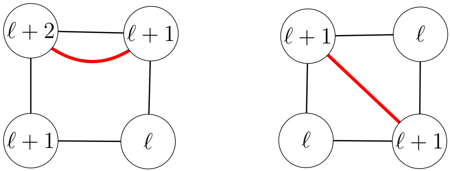

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Alternating (weight $t\gamma$)

The weight γ counts:

- Alternating faces in height-labelled quadrangulations
- Edges joining equal labels in weakly height-labelled maps

Note: $\mathsf{Q}(t, 0)$ counts height-labelled maps.

Solution for height-labelled maps

Let $\mathsf{R}_0(t)$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1},$$

Then the generating function of height labelled maps counted by edges is

$$\mathsf{G}(t) = \mathsf{Q}(t, 0) = \frac{1}{2t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

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Asymptotically, the coefficients behave as

$$g_n \sim \kappa \frac{\mu^{n+2}}{n^2(\log n)^2},$$

where $\kappa = 1/8$ and $\mu = 4\pi$.

Six vertex model background (from physics)

- Solved at criticality by Zinn-Justin in 2000.
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- Solution was not completely rigorous... and we didn't understand it

Last year we discussed it with Paul Zinn-Justin:

- He corrected a mistake and simplified the solution.
- Together, we *proved* the result.

RECALL: SOLUTIONS AT $\gamma = 0, 1$

The generating function $\mathsf{Q}(t, 0)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1},$$

$$\mathsf{Q}(t, 0) = \frac{1}{2t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

The generating function $\mathsf{Q}(t, 1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}_1(t)^{n+1},$$

$$\mathsf{Q}(t, 1) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}_1(t)).$$

SOLUTION FOR $\mathsf{Q}(t,\gamma)$

Define

$$\vartheta(z,q) = \sum_{n=0}^{\infty} (-1)^n \left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right) q^{(2n+1)^2/8}.$$

Let $q = q(t,\alpha)$ be the unique series satisfying

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2} + \frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

Part 5: Height distribution

(Bousquet-Mélou and EP (2019+)).

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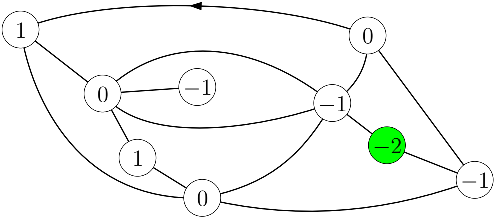
Disclaimer: This is work in progress; these “results” are not completely proven yet.

We now count height-labelled quadrangulations with a highlighted vertex v which gets weight $\delta^{\text{height of } v}$.

New generating function: $\hat{\mathsf{Q}}(t, \gamma, \delta)$.



This example contributes $t^7\gamma^2\delta^{-2}$ to $\hat{\mathsf{Q}}(t, \gamma, \delta)$

We have now found the exact form of $\hat{\mathsf{Q}}(t, \gamma, \delta)$, using theta functions.

HEIGHT DISTRIBUTION

From $\hat{\mathsf{Q}}(t, 1, \delta)$ we get the exact distribution of vertex heights in height-labelled quadrangulations with n faces.

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- The variance V_n grows like

$$V_n \sim \frac{3}{2\pi^2} \log(n)^2.$$

Height distribution

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$$V_n \sim \frac{3}{2\pi^2} \log(n)^2.$$

- After rescaling by dividing each height by $3\log(n)/\pi$, the limiting distribution has kth moment

$$m_k = |(k-1)B_k|,$$

  where B_k is the kth Bernoulli number.

FURTHER QUESTIONS

- Can we count height-labelled maps and/or weakly height-labelled maps by edges *and vertices*?

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- Are there any phase transitions?
- Do these have limiting objects? What are they?
- Random generation?

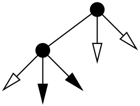
SOMETHING TO TAKE HOME

Recall: The GF $\mathsf{Q}(t)$ of height-labelled quadrangulations is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}(t)^{n+1},$$

$$\mathsf{Q}(t) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}(t)).$$

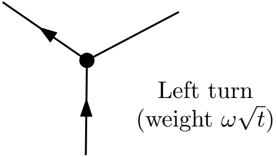
Another interpretation (Bousquet-Mélou and Courtiel 15):
Consider ternary trees with black and white leaves. Define the charge at a node to be the number of white leaves minus the number of black leaves in the associated subtree. Then $(t-R)/t = 3t(1+Q)$ counts (by nodes) trees in which the root is the only vertex of charge 1.

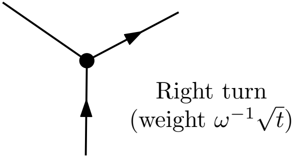


Bijection?

Thank you!

Let $\mathsf{C}(t,\omega)$ be the generating function for partially oriented cubic maps in which each vertex is one of the following types.





Bonus slide: bijection to a loop model

Let $\mathsf{C}(t,\omega)$ be the generating function for partially oriented cubic maps in which each vertex is one of the following types.

Theorem: $\mathsf{Q}(t,\omega^2+\omega^{-2}) = \mathsf{C}(t,\omega)$.

Bonus slide: bijection to a loop model

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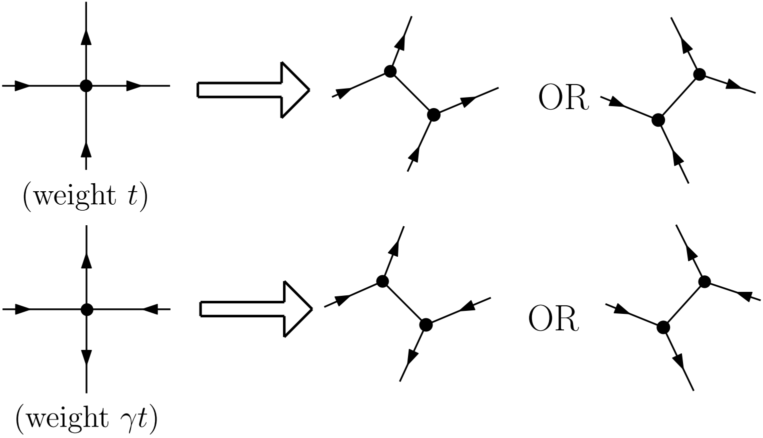

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