Logarithmic Variance for the Height Function of Square Ice

Matan Harel

Joint work with: H. Duminil-Copin (IHES, UniGe), B. Laslier (Diderot),

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Tel Aviv University

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The typical behavior of h under this setup is shockingly unamenable to analysis by general techniques of random surfaces, due to hardcore constraints.

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Such dichotomy theorems have been shown by (DCST '17) for FK-percolation on $\mathbb{Z}^2$, and in greater generality by (DCT '19).

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	0	1	2	3	4	5	4	3	2	
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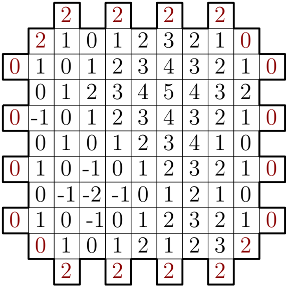

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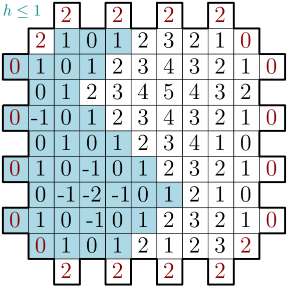

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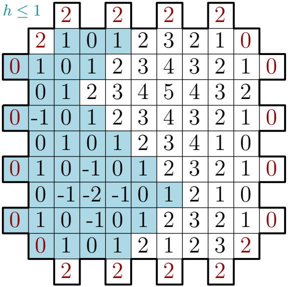

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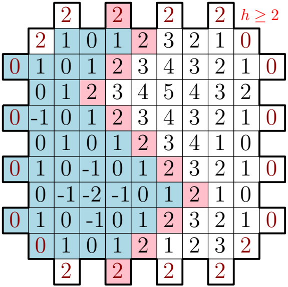

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- $\phi^0_{\Lambda_n}[h_0 > r] < e^{-kr^{\alpha}}$, for some $k, \alpha > 0$, or
- there exists $c(k, r, \rho)$ such that, for any r, $k > (2 + \rho)$, and n large enough,

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- A renormalization argument, which will use the generalized RSW estimate above to prove that

$$\phi^0_{\Lambda_{20n}}[\exists \times\text{-circuit of } h \geq 2 \text{ in } \Lambda_{20n} \setminus \Lambda_{10n}] \leq C \cdot \phi^0_{\Lambda_{2n}}[\exists \times\text{-circuit of } h \geq 2 \text{ in } \Lambda_{2n} \setminus \Lambda_n]^2.$$

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- h has the $\times$-Domain Markov Property.
- Under 'good' boundary conditions, there are several equivalent ways to express crossing events:

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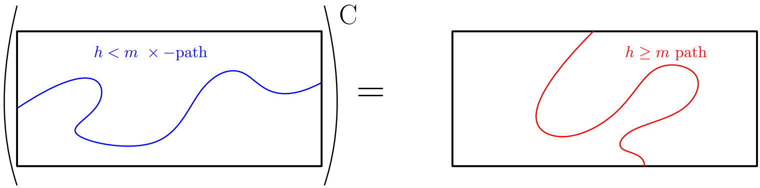
$$\mathcal{H}^{\times}_{h<m}(R)^c = \mathcal{V}_{h\geq m}(R)$$



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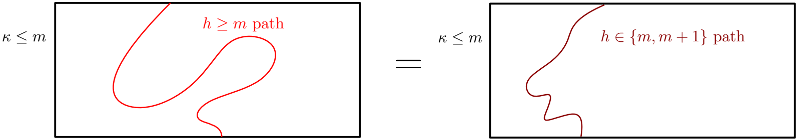

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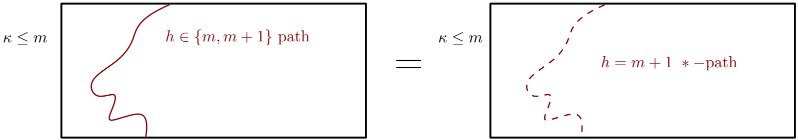

where $*$-paths connect vertices at ℓ^1-distance 2.

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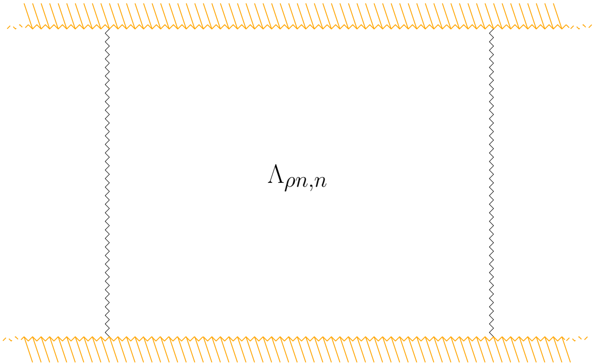

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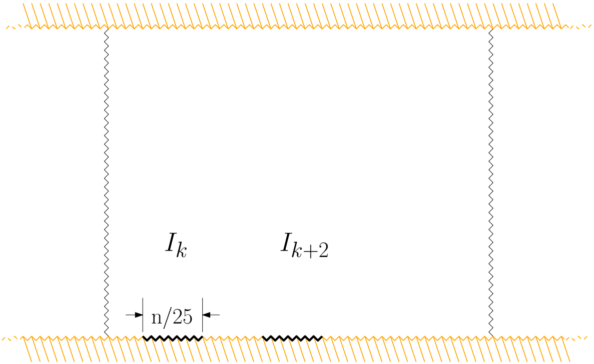

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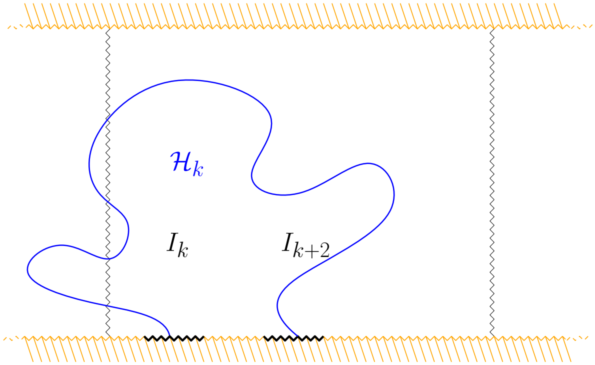

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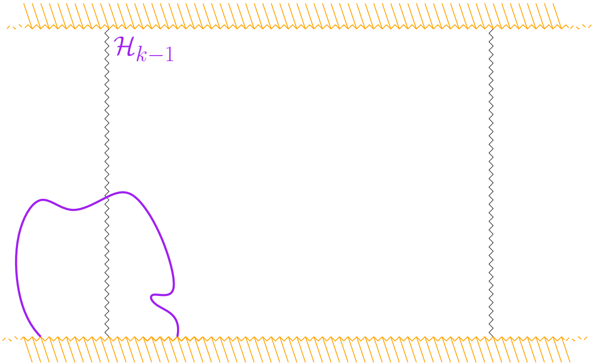

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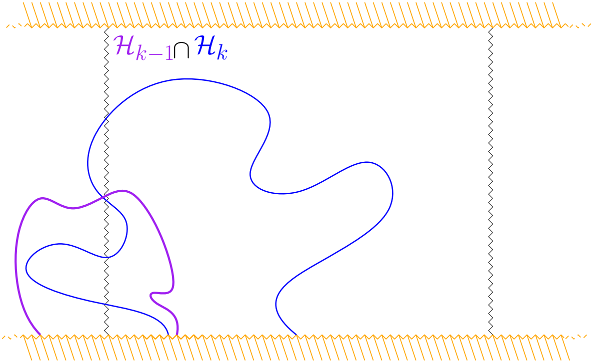

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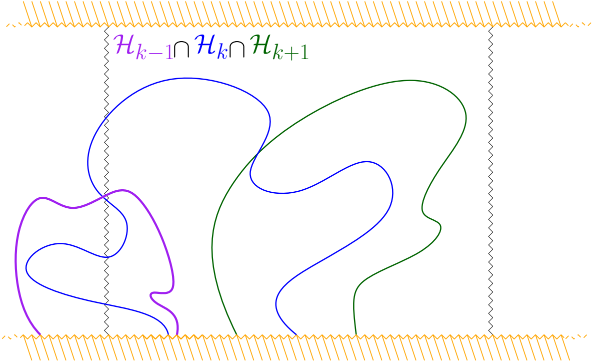

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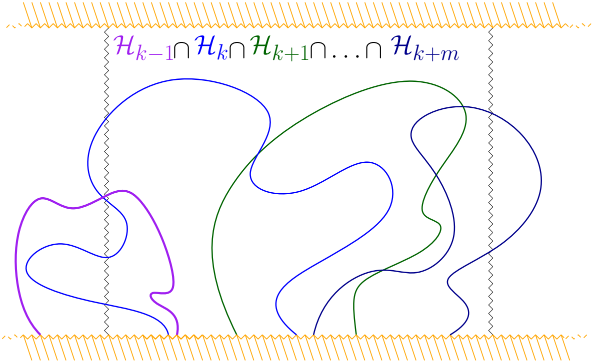

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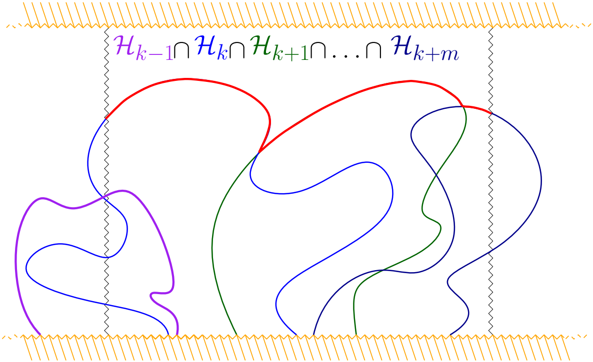
Let $\mathcal{H}_k$ be the event that I_k and I_{k+2} are connected by a $\times$-path of $h \geq 2$.

The intersection of (at most) $(25\rho + 1)$ $\mathcal{H}_i$'s implies the existence of a horizontal crossing of $\Lambda_{\rho n,n}$.



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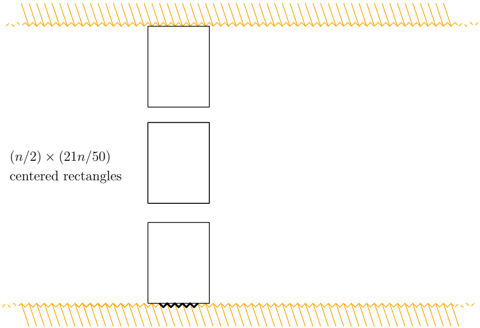

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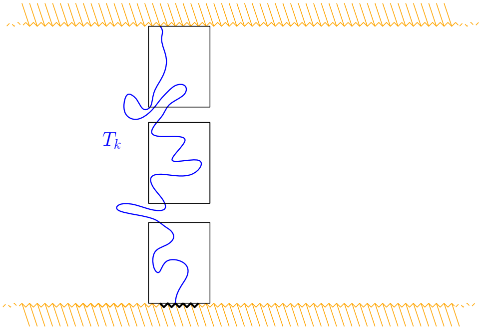

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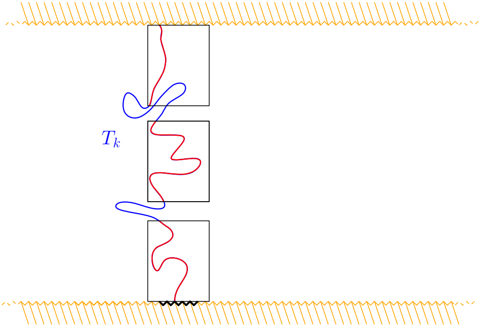

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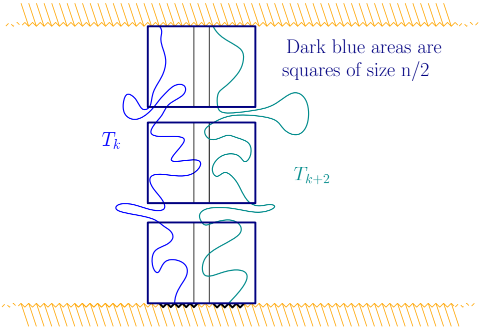
When T_k and T_{k+2} occur simultaneously, we have three squares that are doubly crossed by $\times$-paths of $h \geq 2$.



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We now make a (rather major) assumption:

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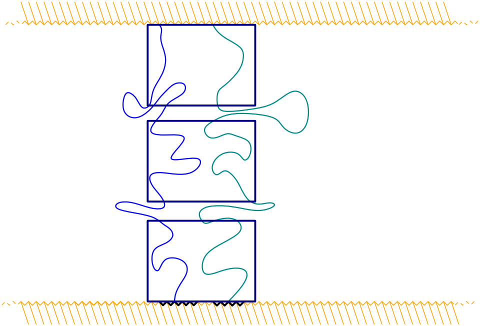

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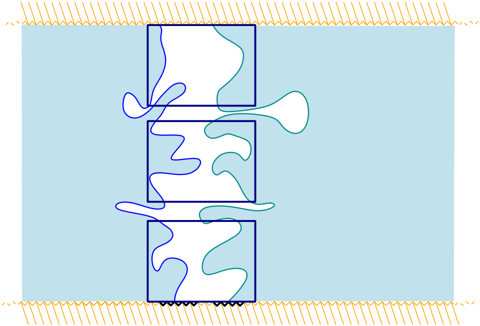
Condition on the value of h to the left of the leftmost path satisfying T_k, and to the right of the rightmost path satisfying T_{k+2}.



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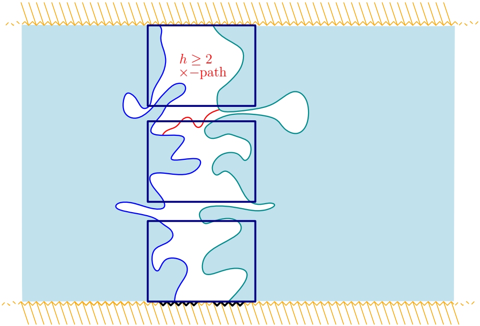
It will be sufficient to prove that probability of crossing the white region horizontally is bounded below by a constant.



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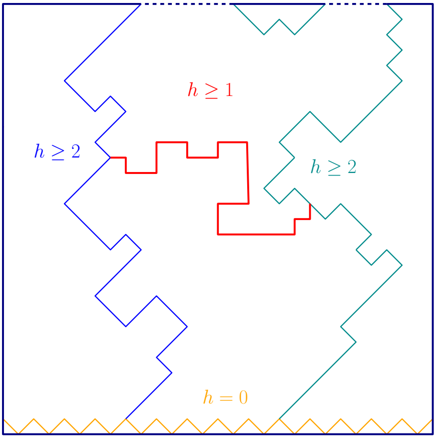
We zoom in on the bottom square S^{-}, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .



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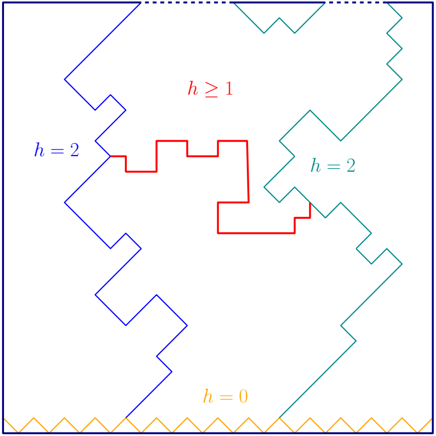

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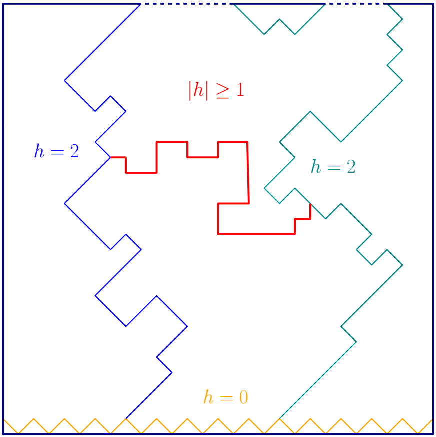

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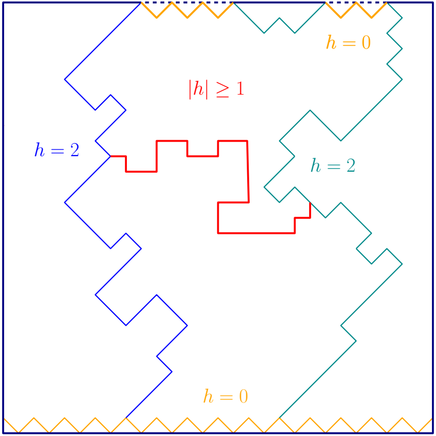

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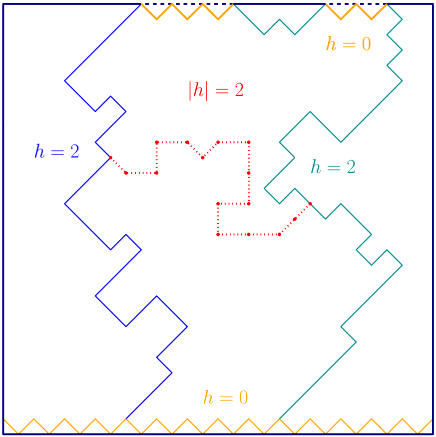
We zoom in on the bottom square S^{-}, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .



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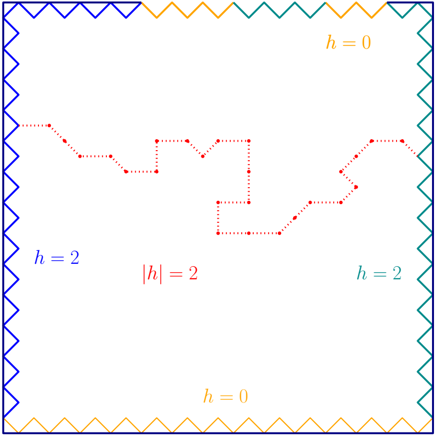

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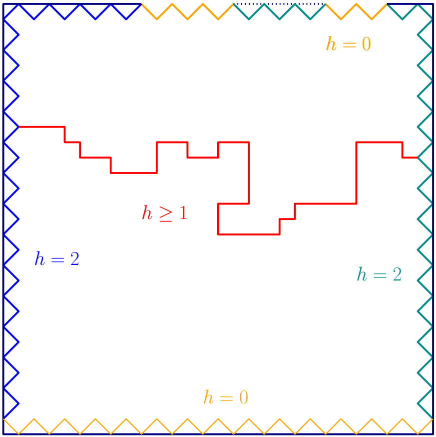

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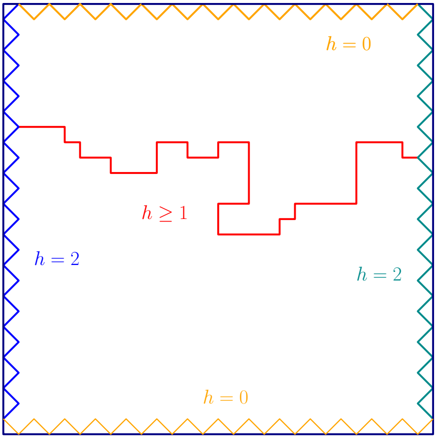
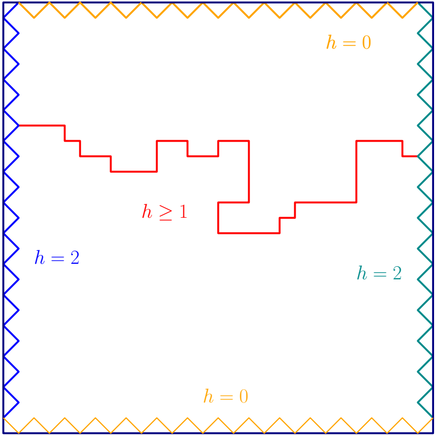


RSW Proof: Step 1

Thus, we deduce that the probability of $\bar{H}$ is bounded below by

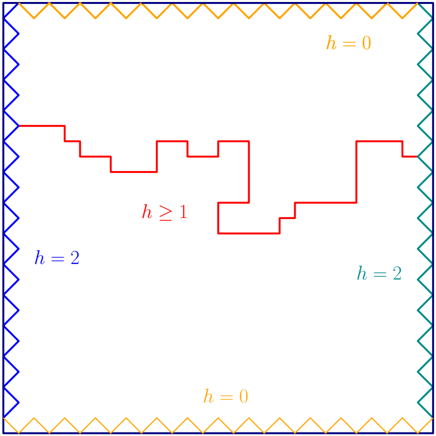
$$\phi_{S^-}^{0/2}[\mathcal{H}_{h\geq 1}(S^-)] \qquad .$$



RSW Proof: Step 1

Thus, we deduce that the probability of $\bar{H}$ is bounded below by

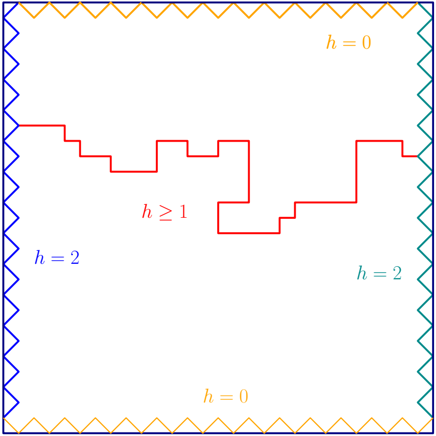
$$\phi^{0/2}_{S^-}[\mathcal{H}_{h\geq 1}(S^-)] = 1 - \phi^{0/2}_{S^-}[\mathcal{V}^{\times}_{h\leq 0}(S^-)] \qquad .$$



RSW Proof: Step 1

Thus, we deduce that the probability of $\bar{H}$ is bounded below by

$$\phi_{S^-}^{0/2}[\mathcal{H}_{h\geq 1}(S^-)] = 1 - \phi_{S^-}^{0/2}[\mathcal{V}_{h\leq 0}^{\times}(S^-)] \geq 1 - \phi_{S^-}^{0/2}[\mathcal{V}_{h\leq 1}(S^-)].$$



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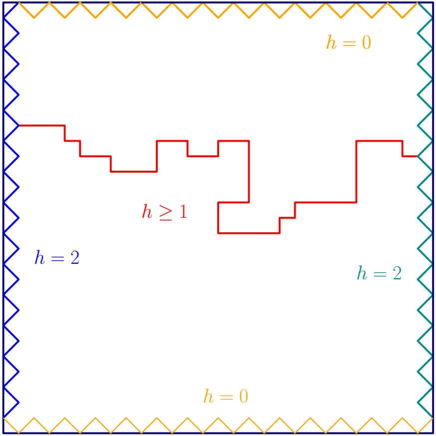
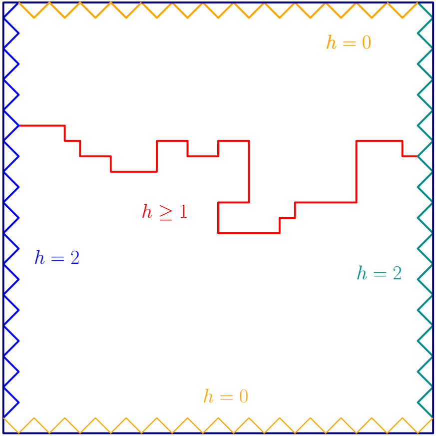


RSW Proof: Step 1

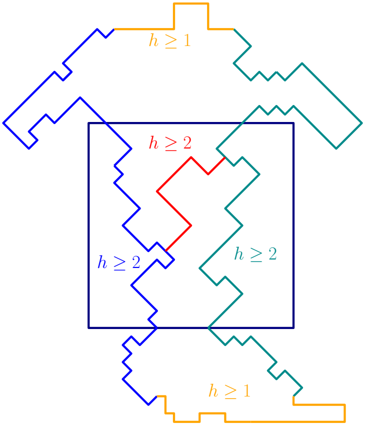
Thus, we deduce that the probability of $\bar{H}$ is bounded below by

$$\phi_{S^-}^{0/2}[\mathcal{H}_{h\geq 1}(S^-)] \geq 1/2$$



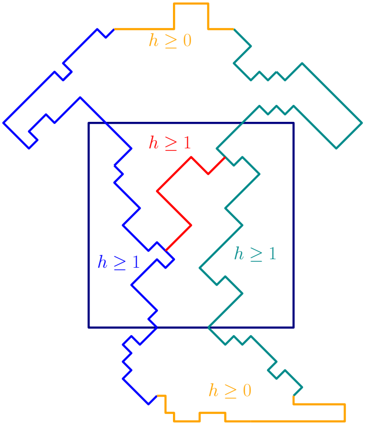
RSW Proof: Step 2

We zoom in on the middle square S, and look for a $h \geq 2$ $\times$-crossing.



RSW Proof: Step 2

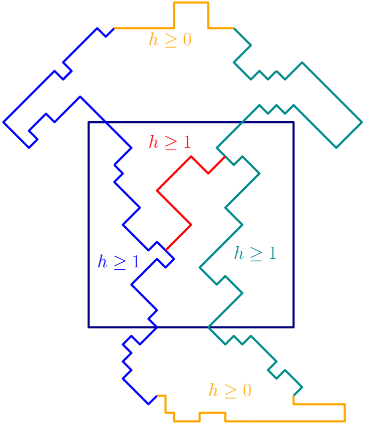
We zoom in on the middle square S, and look for a $h \geq 1$ $\times$-crossing.



RSW Proof: Step 2

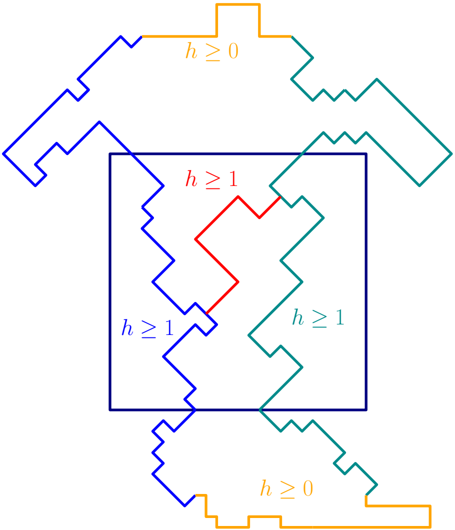
We zoom in on the middle square S, and look for a $h \geq 1$ $\times$-crossing.

Unlike before, we cannot push boundary conditions of $h = 0$ in, because $h \geq 1$ is *not* the same as $|h| \geq 1$!



RSW Proof: Step 2

We look for a symmetric domain in other ways:



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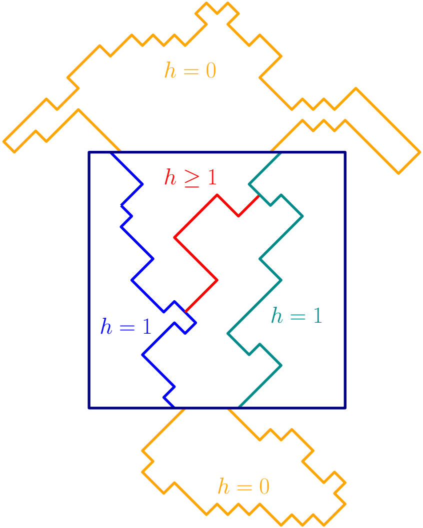

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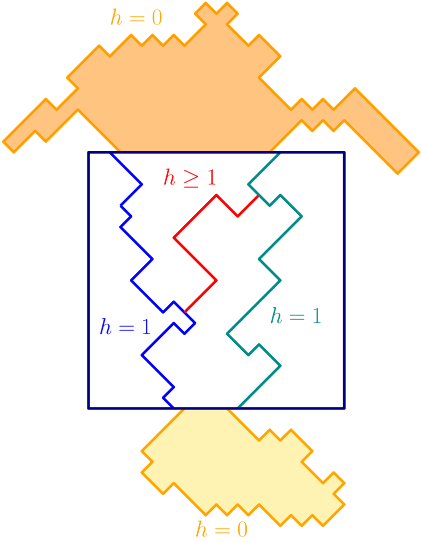

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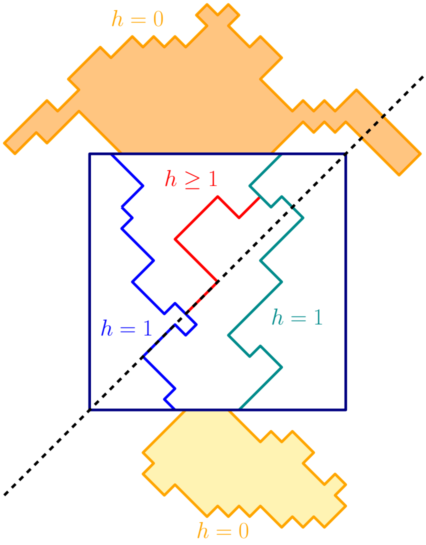

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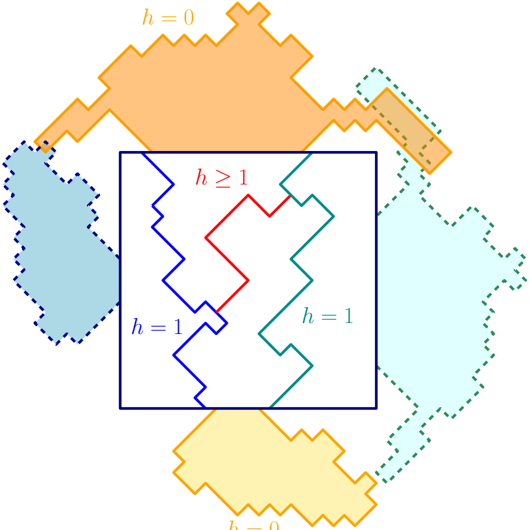

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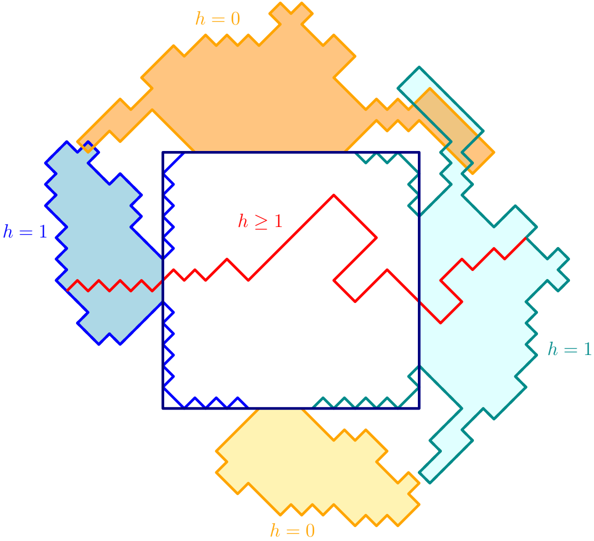

Thank you!