CHARACTERISATION OF THE 2D GFF

Joint work with Nathanaël Berestycki and Gourab Ray

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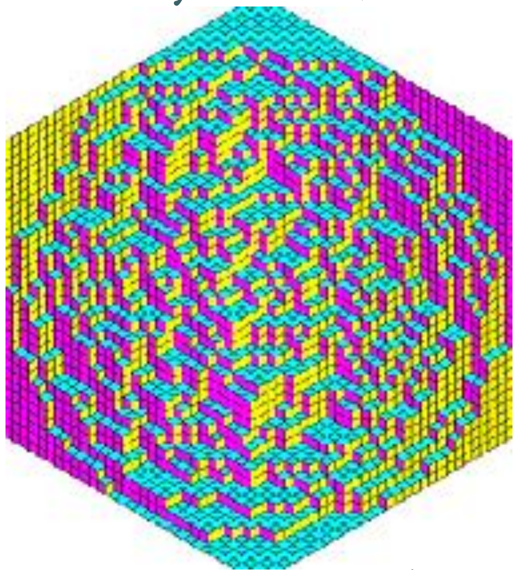
Ellen Powell (Durham University)

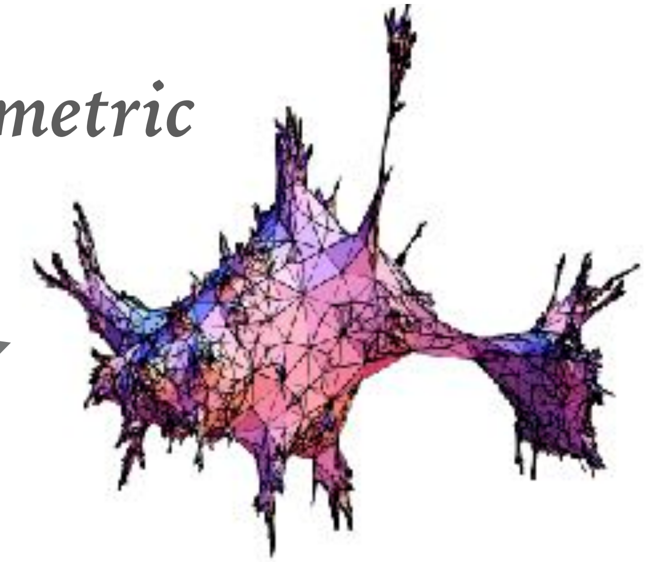
GFF AS A SCALING LIMIT

Kenyon ‘01, …



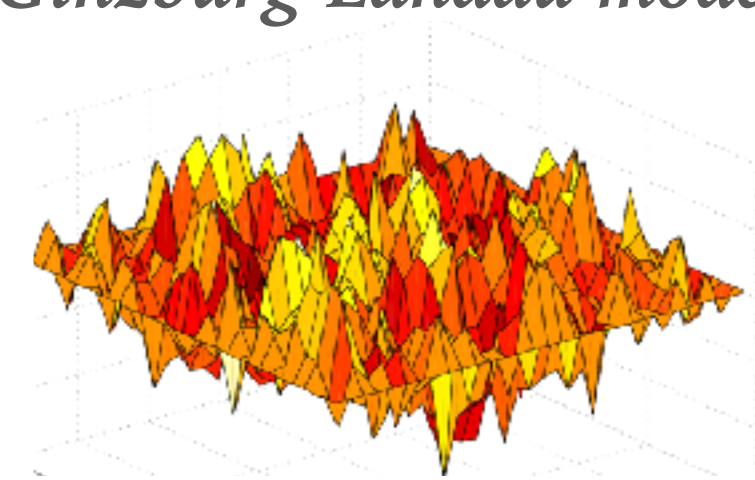
Dimer model
Height function

Random metric spaces



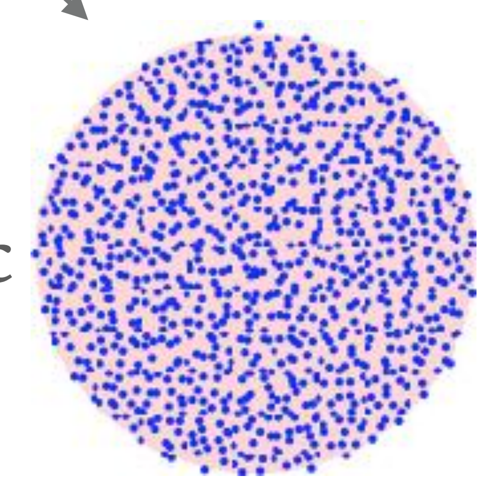
Le Gall ‘13, Miermont ‘13, Miller-Sheffield ‘15+, …

Ginzburg-Landau model



Giacomin-Olla-Spohn ‘01, Miller ‘11 …

Random matrices
Log characteristic polynomial



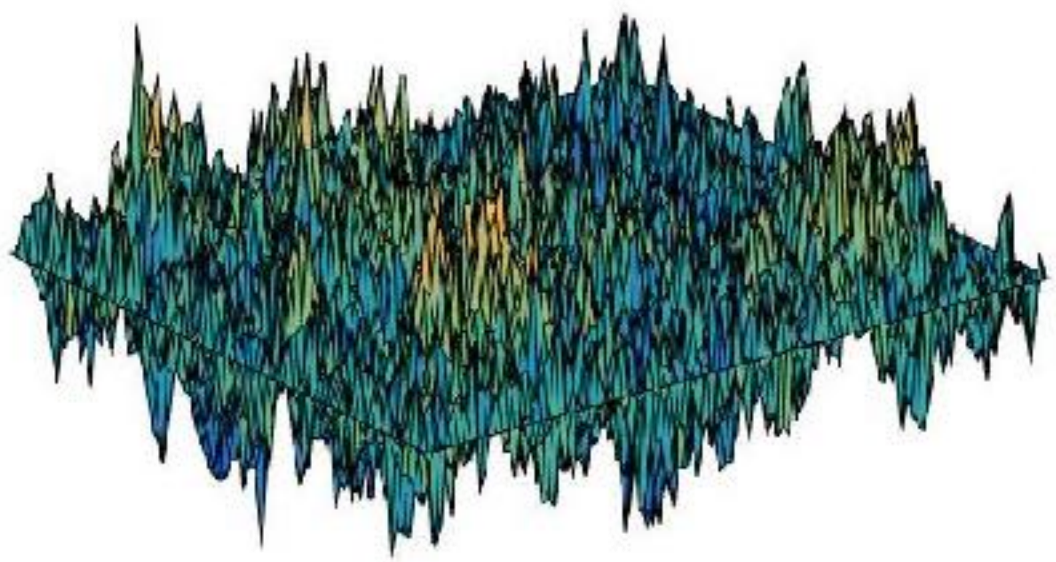
Fyodorov-Khoruzhenko-Simm ‘01, Rider-Virag ‘07, Hughes-Keating-O’Connell ‘01

2D GFF: DEFINITION

Random Gaussian "function" on $D \subset \mathbb{C}$

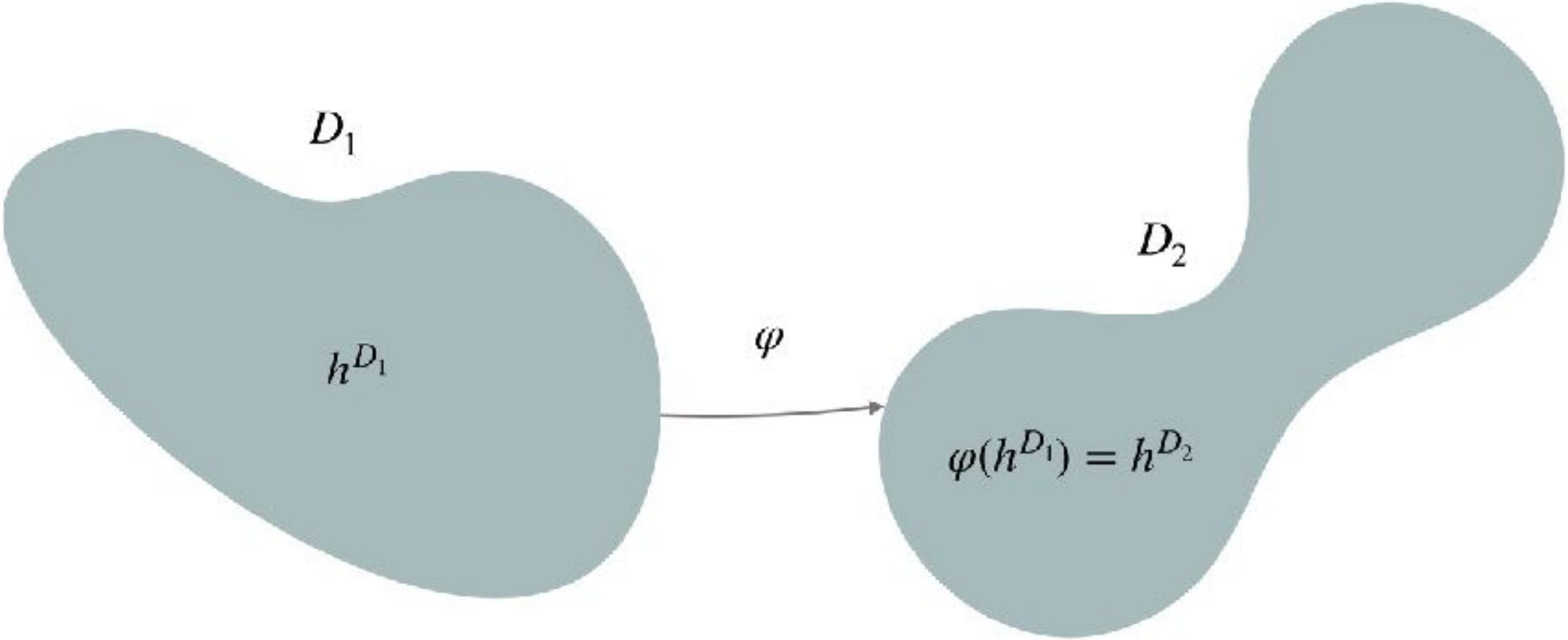
Mean 0, Variance = Green's function

$$G_D(x, y) \sim \log(|x - y|^{-1}) \ as \ |x - y| \to 0$$



- Gaussian function with this covariance does not exist!
- GFF lives in the space of generalised functions
- Not defined point-wise but can test against smooth functions

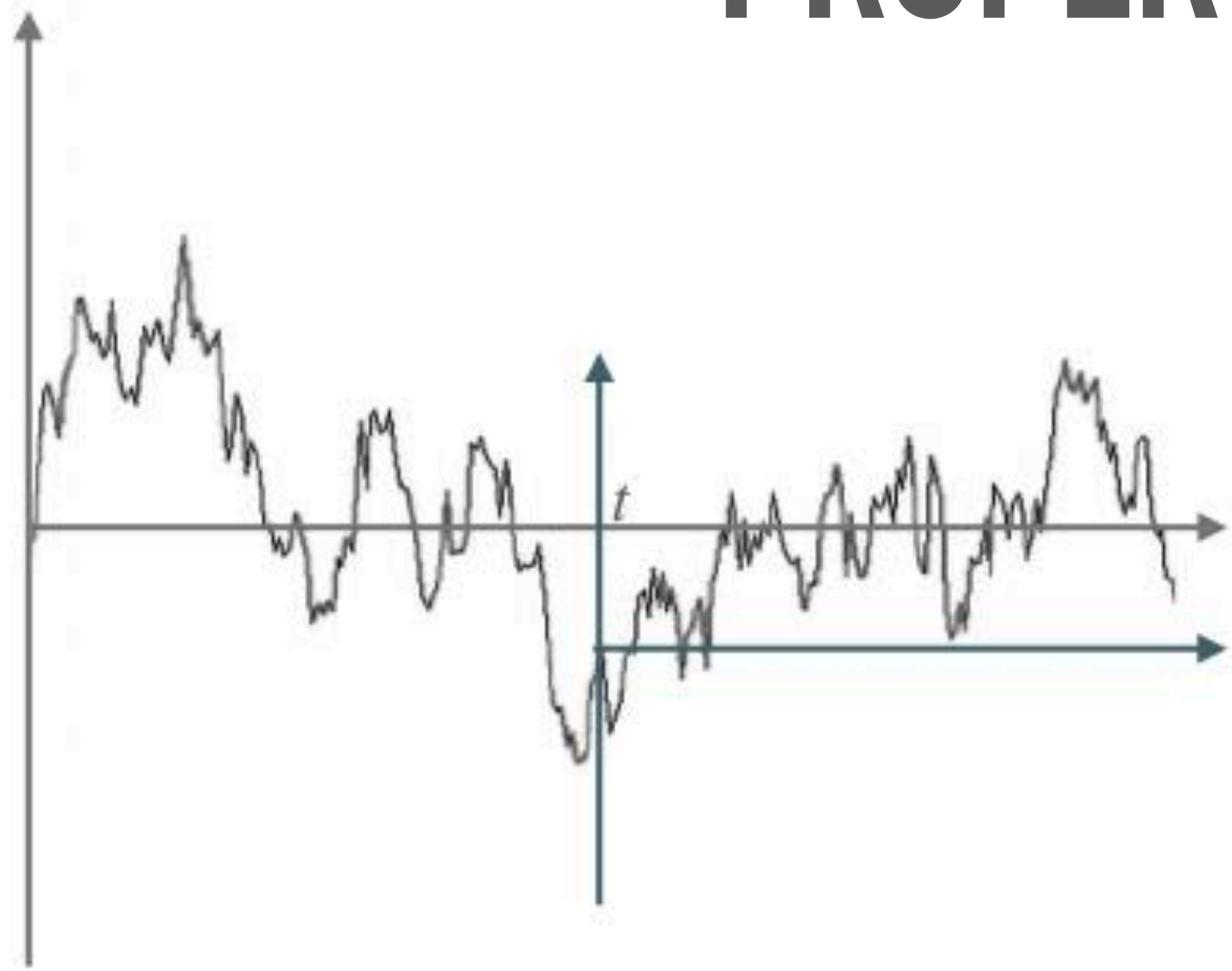
PROPERTIES

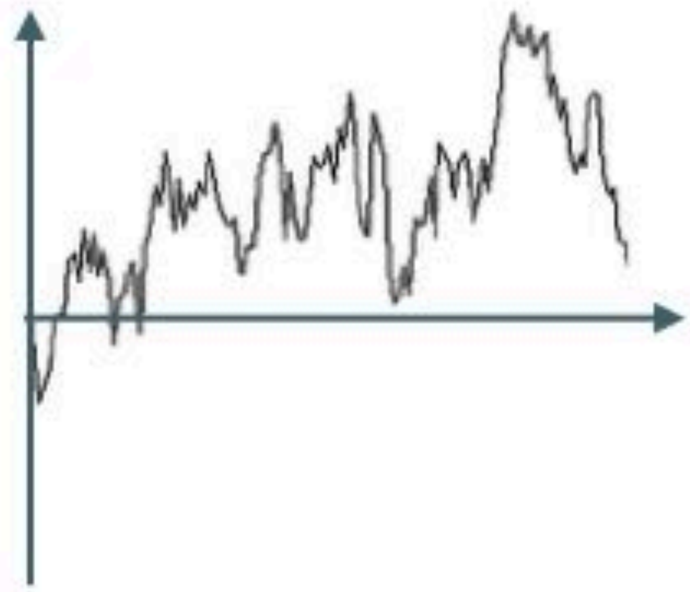


CONFORMAL INVARIANCE

Image of a GFF under a conformal map has the law of a GFF

PROPERTIES

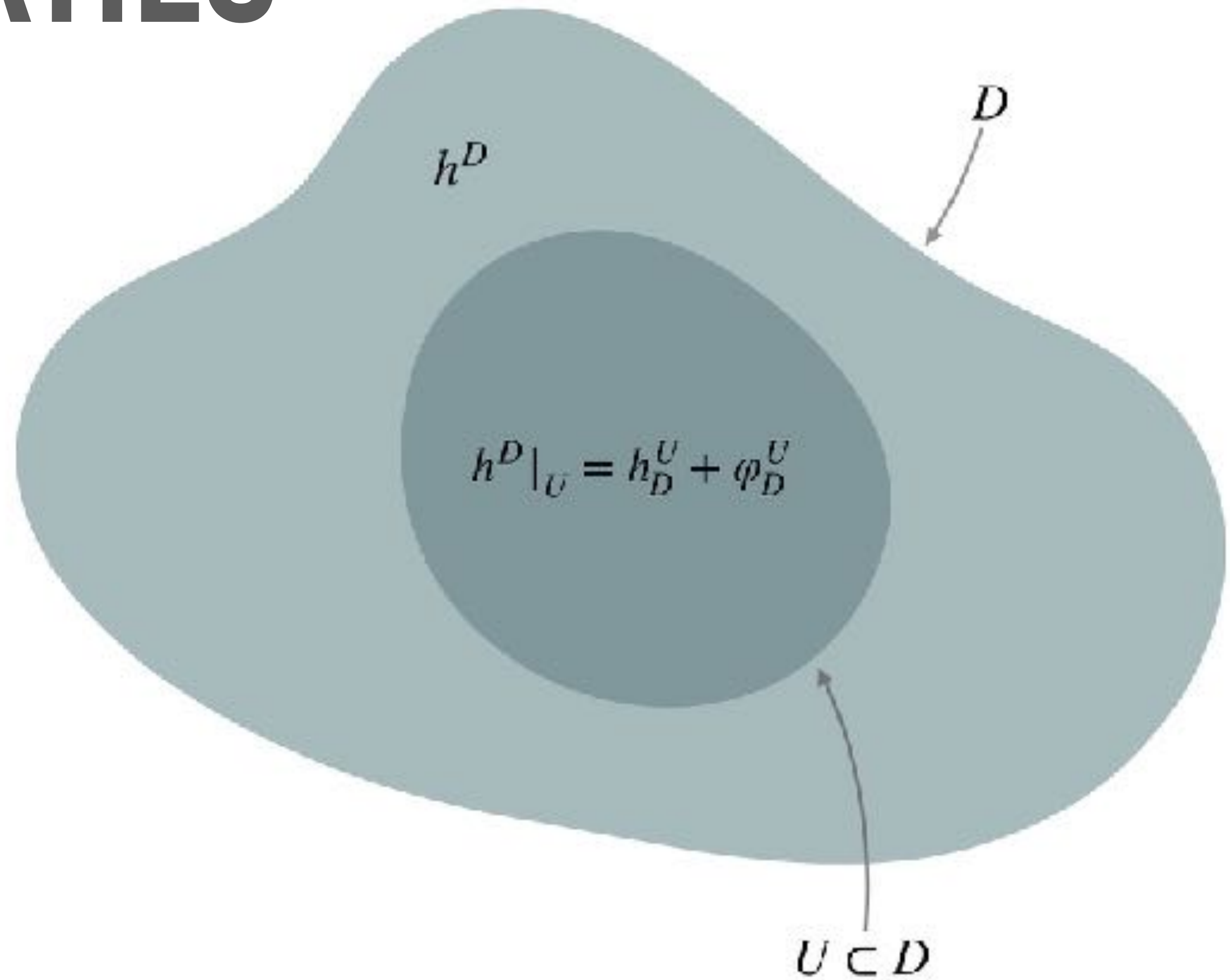




MARKOV PROPERTY (BROWNIAN MOTION)

Brownian motion from a given time onwards is equal to: the position at that time plus an independent Brownian motion

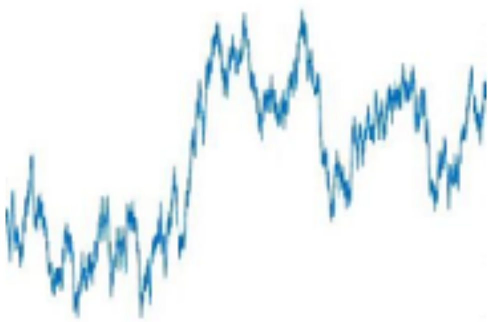
PROPERTIES

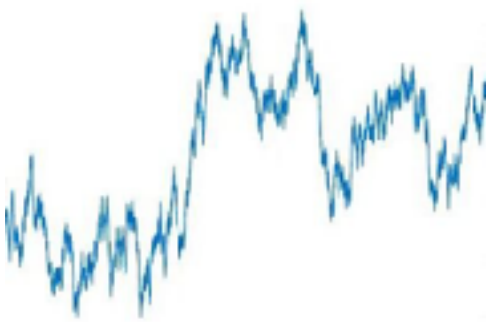


DOMAIN MARKOV PROPERTY (GFF)

GFF restricted to a subdomain is equal to: a harmonic function plus an independent GFF in the subdomain

CHARACTERISATION OF BROWNIAN MOTION





"Brownian motion is the only random continuous process with stationary, centred, and independent increments"

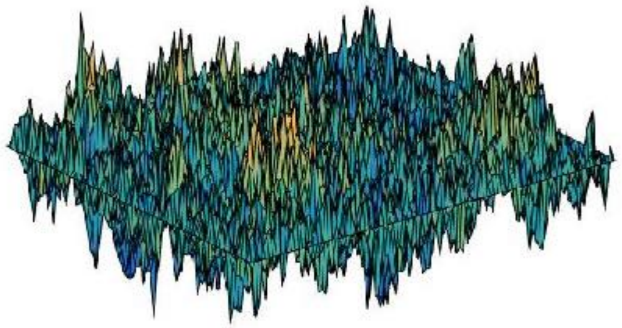
- $(B_t)_{t \geq 0}$ *is almost surely continuous*
- $B_t - B_s$ *is equal in law to* B_{t-s} *for any* $0 \leq s \leq t$
- $(B_{t_2} - B_{t_1}), (B_{t_3} - B_{t_2}), \ldots, (B_{t_n} - B_{t_{n-1}})$
are independent for any $0 \leq t_1 \leq t_2 \cdots \leq t_n$

CHARACTERISATION OF THE GFF

Theorem (Berestycki-P-Ray '18)

"The GFF is the only* random field indexed by two-dimensional domains that satisfies conformal invariance and the domain Markov property."

- *We also have a moment assumption on the field
- This could allow us to characterise scaling limits



MAIN THEOREM

Assume that for each $D \subset \mathbb{C}$ simply connected we are given the law of

$$(h^D, f)_{f \in C_c^\infty(D)}$$

a linear stochastic process indexed by $f \in C_c^\infty(D)$

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ASSUMPTIONS:

Conformal invariance:

if $f : D \to D'$ is conformal, then we have

$$h^{D'} = h^D \circ f^{-1}$$

in law as stochastic processes

MAIN THEOREM

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ASSUMPTIONS:

Domain Markov property:

if $U \subset D$ is simply connected, we can write $h^D = h_D^U + \varphi_D^U$, where:

- *the two summands are independent;*
- *h_D^U has the law of h^U when restricted to U and is zero outside of U;*
- *φ_D^U is harmonic when restricted to U*

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ASSUMPTIONS:

Dirichlet boundary condition:

$$\mathbb{E}[(h^D, f)] = 0 \quad \forall f \in C_c^\infty(D)$$

Moreover, for any sequence $(f_n)_n \in C_c^\infty(D)$ radially symmetric with bounded mass, and with support eventually contained outside any $M \Subset D$

$$var(h^D, f_n) \to 0 \text{ as } n \to \infty$$

MAIN THEOREM

Assume that for each $D \subset \mathbb{C}$ simply connected we are given the law of

$$(h^D, f)_{f \in C_c^\infty(D)}$$

a linear stochastic process indexed by $f \in C_c^\infty(D)$

ASSUMPTIONS:

Moments/stochastic continuity:

$$\mathbb{E}[(h^D, f)^4] < \infty \textit{ for every } f \in C_c^\infty(D)$$

$$\textit{and } (f, g) \mapsto K^D(f, g) := \mathbb{E}[(h^D, f)(h^D, g)]$$

is a continuous bilinear form on $C_c^\infty(D)$

MAIN THEOREM

Theorem (Berestycki-P-Ray '18)

If these assumptions hold, then for some $a > 0$, h^D is equal to a times a GFF in D for every D

REMARKS

- Also works in 1d, giving a new characterisation of the Brownian bridge
- CI, DMP and Dirichlet BCs seems indispensable, but moments… (more later!)

QUESTIONS

- Can we relax the moment assumption?

Yes! We can obtain the result assuming moments of order $1 + \varepsilon$ for $\varepsilon > 0$ (forthcoming work...)

- Do we have a similar characterisation in higher dimensions?
- What about different boundary conditions?
- Can we characterise the Gaussian free field on different 2d surfaces?
- Does there exist a “stable” version of the free field?

THANKS!