Université de Fribourg — Département de Mathématiques
Universität Freiburg — Mathematatik Departement

Mathematical Methods for Computer Science II

Spring 2021

Series 3 – Hand in before Monday, 22.03.2021 - 12.00

1. Prove that

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = 1+x+x^2+x^3+\cdots$$

   a) by using the existence and the uniqueness of the binary representation of integers;
   b) by algebraic operations with formal power series.

2. The following formal power series have both positive and negative coefficients. Describe the meanings of the absolute value and of the sign of the coefficient at x^n.

(a)
$$(1-x)(1-x^2)(1-x^4)(1-x^8)\cdots$$

(b)
$$\frac{1}{(1+x)(1+x^3)(1+x^5)(1+x^7)\cdots}$$

3. We introduce the following notation:

$$(2k-1)!! = 1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1), \quad (2k)!! = 2\cdot 4\cdot \ldots \cdot (2k).$$

   We also use the convention $(-1)!! = 1$.
   a) Show that for all $k \geq 0$ we have

$$\binom{-\frac{1}{2}}{k} = (-1)^k \frac{(2k-1)!!}{(2k)!!}.$$

   b) Give an explicit formula for the coefficients of the formal power series

$$\sqrt{\frac{1+x}{1-x}} = a_0 + a_1 x + a_2 x^2 + \cdots .$$

   (Hint: use the identity $\sqrt{\frac{1+x}{1-x}} = (1+x)\sqrt{\frac{1}{1-x^2}}$.)

4. Let p_n be the number of partitions of n. Show that

$$p_n \leq 2^{n-1}.$$

   (Hint: compare the number of partitions with the number of compositions.)

5. Denote by a_n the number of partitions of n without parts of size 1. (For example, there are only two partitions of 5 with this property: $5 = 5$ and $5 = 3+2$, so that $a_5 = 2$.)
   a) Prove that $a_n = p_n - p_{n-1}$ by first proving the identity

$$\sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{\infty} p_n x^n.$$

   b) Give a bijective proof of $a_n = p_n - p_{n-1}$. (Hint: if a partition of n has a part of size 1, then by removing this part one obtains a partition of $n-1$.)