Université de Fribourg
Universität Freiburg

Département de Mathématiques
Mathematik Departement

# Mathematical Methods for Computer Science I

Fall 2020

Series 12 – Hand in before Monday, 14.12.2020 - 12.00

1. Prove the validity of the following predicate formulas by constructing a proof tree for each of them.
   a) $(\exists x \forall y P(x, y)) \to (\forall y \exists x P(x, y))$
   b) $(\forall x P(x) \wedge \exists y Q(y)) \to (P(f(v)) \wedge \exists z Q(z))$

2. Construct finite counterexample trees for the following predicate formulas. Describe the corresponding structures (universe plus predicate interpretation) explicitly.
   a) $\exists x P(x) \to \forall x P(x)$
   b) $\exists x P(x) \wedge \exists x Q(x) \to \exists x (P(x) \wedge Q(x))$

3. Apply the algorithm from the lecture to construct an infinite deduction tree for the formula
$$(\forall x \exists y P(x, y)) \to (\exists y \forall x P(x, y)).$$
   Describe explicitly the infinite counterexample generated by this tree.

4. Let $A$ be a propositional formula with propositional symbols $p_1, \dots, p_n$, and let $\phi_1, \dots, \phi_n$ be any closed predicate formulas. Define $A[\phi_1/p_1, \dots, \phi_n/p_n]$ to be the result of substitution of $\phi_i$ for $p_i$ in $A$. (Clearly, the result is a well-formed predicate formula.)
   a) Prove that if $A$ is a tautology, then $A[\phi_1/p_1, \dots, \phi_n/p_n]$ is valid.
   b) Prove that if $A$ is not satisfiable, then so is $A[\phi_1/p_1, \dots, \phi_n/p_n]$.
   c) Show that if $A$ is satisfiable, then $A[\phi_1/p_1, \dots, \phi_n/p_n]$ is not necessarily satisfiable.

5. Consider a signature that consists of $n$ unary predicates only (no functions, no $k$-ary predicates with $k > 1$). Show that if a formula in this signature is satisfiable, then it is satisfied by some structure with a finite universe of at most $2^n$ elements.