Mathematical Methods for Computer Science I

Fall 2020

Series 11 – Hand in before Monday, 7.12.2020 - 12.00

1. Consider the signature

$$\mathcal{F} : f(\cdot)$$
$$\mathcal{P} : P(\cdot,\cdot),\ Q(\cdot)$$

   a) Which of the following expressions are predicate *formulas* based on the above signature? For those which are not, explain why not.

   i) $P(f(x), x)$ ii) $f(f(x))$ iii) $P(Q(x), y)$ iv) $\exists x(Q(x, f(x)))$

   b) Write a closed formula involving all functions and predicates of the above signature.

2. a) For the formula you have written in Problem 1b), find one satisfying and one falsifying structure. (In the unlikely case that your formula is unsatisfiable or a tautology, take another one.) You may use universes from real life, from mathematics, or artificially constructed small ones.

   b) Rectify the formula

$$\forall x P(x, y) \vee \exists y(Q(y) \wedge \forall x P(x, y))$$

   and form its universal closure.

3. Let P and Q be binary predicate symbols. Consider the formulas

$$\phi = \forall x \exists y(P(x, y) \to Q(x, y))$$
$$\psi = \forall x \exists y(Q(x, y) \to P(x, y))$$

   and the structure $M = (U, I)$ given by

$$U = \{a, b, c\}$$
$$I(P) = \{(a, b), (b, c)\}$$
$$I(Q) = \{(a, b), (b, a), (c, a), (c, b), (c, c)\}$$

   For each of the formulas ϕ and ψ determine if they are satisfied by the structure M. Justify your answer.

4. Consider the signature consisting of just one binary predicate P and two structures

$$M_1 = (\mathbb{N}, \leq), \quad M_2 = (\mathbb{Z}, \leq).$$

   (the universes are the sets of positive integers and, respectively, integers, the predicate is each time interpreted as $I(P)(x, y)$ is true if and only if $x \leq y$).

   a) Write a formula which is satisfied by both structures.

   b) Write a formula which is satisfied by M_1 but not satisfied by M_2.

   c) Write a formula which is satisfied by M_2 but not satisfied by M_1.

5. Let ϕ be the formula

$$\forall x \neg R(x, x) \wedge \forall x \forall y \forall z(R(x, y) \wedge R(y, z) \to R(x, z)) \wedge \forall x \exists y R(x, y).$$

   a) Give an infinite model for ϕ.

   b) Show that ϕ has no finite model.