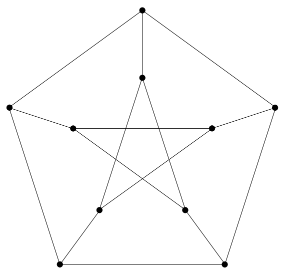
# Mathematical Methods for Computer Science I

Fall 2020

Series 4 – Hand in before Monday, 19.10.2020 - 12.00

1. The graph shown below is called the *Petersen graph*.



Show that it is isomorphic to the following graph:

$$V = \{\text{all two-element subsets of } \{1,2,3,4,5\}\},$$
$$E = \{\{A,B\} : A \cap B = \varnothing\}.$$

2. Can you give an example of
   a) a 4-regular graph with 7 vertices?
   b) a 3-regular graph with 8 vertices?
   c) a 3-regular graph with 7 vertices?

3. a) Let $G$ be a graph with 15 vertices such that one of the vertices has degree 14. Show that this graph is connected.
   b) Let $G$ be a graph with 15 vertices such that every vertex has degree at least 7. Show that this graph is connected.

4. Show that if a connected graph $G = (V,E)$ contains no cycles of odd length, then it is bipartite.

   (*Hint*: Fix a vertex $v \in V$ of the graph and consider the function $f : V \to \{0,1\}$ sending a vertex $w$ to 0 if the shortest path from $v$ to $w$ is of even length, and to 1 if it is of odd length.)

5. Let $A$ be the adjacency matrix of a graph $G = (V,E)$, where $V = \{v_1, v_2, \ldots, v_n\}$.
   a) Let $b_{ij}$ be the $(i,j)$-entry of the matrix $A^2$. Show that:
      i) $b_{ii}$ is equal to the degree of the vertex $v_i$,
      ii) $b_{ij}$ equals the number of paths of length two from $v_i$ to $v_j$, for $i \neq j$.
   b) Let $a_{ij}^{(k)}$ be the $(i,j)$-entry of the matrix $A^k$, for a fixed $k \geq 1$. Show that $a_{ij}^{(k)}$ equals the number of walks of length $k$ from $v_i$ to $v_j$.

   (*Hint*: Induction on $k$ and the definition of matrix multiplication.)