Convergence of linear barycentric rational interpolation for analytic functions

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Joint work with Stefan Güttel from the University of Oxford
Outline

1. Linear barycentric rational interpolation
2. Polynomial interpolation of analytic functions
3. Barycentric rational interpolation of analytic functions
Linear barycentric rational interpolation

Polynomial interpolation of analytic functions

Barycentric rational interpolation of analytic functions

Floater–Hormann interpolation

Condition: Lebesgue constant

Convergence of LBRI for analytic functions
Given:

\[ a \leq x_0 < x_1 < \ldots < x_n \leq b, \quad n + 1 \text{ distinct nodes and corresponding values.} \]

We study functions \( g \) from a finite-dimensional linear subspace of \( (C[a, b], \| \cdot \|_\infty) \) which interpolate \( f \) between the nodes,

\[ g(x_i) = f(x_i) = f_i, \quad i = 0, \ldots, n. \]
Construction presented by Floater and Hormann

- Given \( n \), choose an integer \( d \in \{0, 1, \ldots, n\} \), the “blending parameter”.
- for \( i = 0, \ldots, n - d \), define \( p_i(x) \), the polynomial of degree \( \leq d \) interpolating \( f_i, f_{i+1}, \ldots, f_{i+d} \).

The \( d \)-th interpolant of the family is a “blend” of the \( p_i \),

\[
r_n(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \quad \text{with} \quad \lambda_i(x) = \frac{(-1)^i}{(x - x_i) \ldots (x - x_{i+d})}.
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Note that for \( d = n \), \( r_n \) simplifies to \( p_n \).
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Barycentric form

For its evaluation, we write $r_n$ in **barycentric form**

$$r_n(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{\sum_{i=0}^{n} \frac{w_i}{x - x_i} f_i}{\sum_{i=0}^{n} \frac{w_i}{x - x_i}}.$$

The barycentric weights $w_i$ can be given explicitly.
Properties of Floater–Hormann interpolation

<table>
<thead>
<tr>
<th>Theorem (Floater–Hormann (2007))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $0 \leq d \leq n$ and $f \in C^{d+2}[a, b]$, $h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i)$, then</td>
</tr>
<tr>
<td>- the rational function $r_n$ has no poles in $\mathbb{R}$,</td>
</tr>
<tr>
<td>- if $d \geq 1$,</td>
</tr>
<tr>
<td>$|f - r_n|<em>\infty = \max</em>{a \leq x \leq b}</td>
</tr>
<tr>
<td>- if $d = 0$,</td>
</tr>
<tr>
<td>$|f - r_n|_\infty \leq K \beta h$,</td>
</tr>
<tr>
<td>where $\beta$ is a mesh ratio and $K$ is a constant, independent of $n$.</td>
</tr>
</tbody>
</table>
Interpolation of $1/(1 + 25x^2)$ in $[-1, 1]$

**Figure:** Errors for $f(x) = 1/(1 + 25x^2)$ with $11 \leq n \leq 1000$ equispaced nodes in $[-1, 1]$ and $d = 1, 3, 5$. Algebraic convergence, no Runge phenomenon.
The **Lebesgue constant** associated with linear barycentric interpolation,

\[ \Lambda_n = \max_{a \leq x \leq b} \left\| \sum_{i=0}^{n} \frac{w_i}{x - x_i} \right\|, \]

is the condition number of the interpolation scheme.

**Theorem (Bos–De Marchi–Hormann–K. (2012))**

Let \(0 \leq d \leq n\) and the nodes \(x_i, i = 0, \ldots, n\), be equispaced. Then

\[ \frac{2^{d-2}}{d+1} \log \left( \frac{n}{d} - 1 \right) \leq \Lambda_n \leq 2^{d-1}(2 + \log n). \]
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Lebesgue constant

Figure: The Lebesgue function associated with Floater–Hormann interpolation with equispaced nodes grows logarithmically with $n$ (left) and exponentially with $d$ (right), as is to be expected [Platte–Trefethen–Kuijlaars].
Convergence/divergence of polynomial interpolation for analytic functions
Let the nodes $x_i$ be distributed according to a **node measure** $\mu$ with support $[a, b]$ and positive piecewise continuous **node density**

$$\phi(x) = \frac{d\mu}{dx}(x) > 0, \quad \text{for } x \in [a, b].$$

Associated with $\mu$ is a logarithmic potential

$$U^\mu(z) := -\int_a^b \log |z - x| \, d\mu(x) = -\int_a^b \phi(x) \log |z - x| \, dx.$$
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$$U^\mu(z) := -\int_a^b \log |z - x| \, d\mu(x) = -\int_a^b \phi(x) \log |z - x| \, dx.$$
For a given node measure $\mu$ and the associated potential $U^\mu$, let $f$ be analytic inside $C_s$, the level line of $U^\mu$ which passes through a singularity $s$ of $f$. The polynomial interpolant $p_n$ of $f$ then converges to $f$ inside $C_s$, diverges outside and

$$\lim_{n \to \infty} |f(z) - p_n(z)|^{1/n} = \exp(U^\mu(s) - U^\mu(z)).$$
Polynomial convergence/divergence rates

Figure: Level lines of $\exp(U^\mu(s) - \min_{-1 \leq x \leq 1} U^\mu(x))$ for polynomial interpolation with equispaced nodes (left) and Chebyshev points (right).
Polynomial interpolation of \( \frac{1}{1 + 25x^2} \) in \([-1, 1]\)

**Figure:** Polynomial interpolation of \( f(x) = \frac{1}{1 + 25x^2} \) with \( n = 11 \) equispaced nodes in \([-1, 1]\); \( f \) has a singularity at \( s = \pm i/5 \).
Convergence/divergence of linear barycentric rational interpolation for analytic functions
Aim: We generalize the potential theory from polynomial interpolation to linear rational interpolation.

From now on, the blending parameter $d$ is a variable nonnegative integer $d(n)$ such that

$$d(n)/n \to C, \quad n \to \infty,$$

for $C \in (0, 1]$ fixed. In practice, e.g., $d(n) = \text{round}(Cn)$.
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**Variable blending parameter**

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Asymptotic setting

We suppose that \( j(n) \) is a sequence of indices such that \( j(n) \leq n - d(n) \) and \( x_{j(n)} \to \alpha \) for some \( \alpha \in [a, b] \).

One can show that the nodes \( x_{j(n)}, \ldots, x_{j(n)+d(n)} \) of \( p_{j(n)}(x) \), are then asymptotically contained in an interval \( [\alpha, \beta] \subseteq [a, b] \), and distributed according to the node density \( \phi(x) \), restricted and normalized to that interval, and a node measure \( \nu_\alpha \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( d )</th>
<th>Nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
<td><img src="image1.png" alt="Graph1" /></td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td><img src="image2.png" alt="Graph2" /></td>
</tr>
<tr>
<td>40</td>
<td>8</td>
<td><img src="image3.png" alt="Graph3" /></td>
</tr>
<tr>
<td>80</td>
<td>16</td>
<td><img src="image4.png" alt="Graph4" /></td>
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Two important bounds

Recall that

\[ r_n(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x)p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \quad \lambda_i(x) = \frac{(-1)^i}{(x - x_i)\ldots(x - x_{i+d})}. \]

**Lemma**

For any \( C \in (0, 1], z \in \mathbb{C} \setminus [a, b] \) and \( x \in [a, b] \), we have

\[
\limsup_{n \to \infty} \left| \sum_{i=0}^{n-d(n)} \lambda_i(z) \right|^{1/(n+1)} \leq \max_{\alpha} \exp(CU^{\nu}\alpha(z))
\]

and

\[
\liminf_{n \to \infty} \left| \sum_{i=0}^{n-d(n)} \lambda_i(x) \right|^{1/(n+1)} \geq \max_{\alpha} \exp(CU^{\nu}\alpha(x)).
\]
Hermite-type error formula

If $f$ is analytic inside a simple, closed and rectifiable curve $C$, which is contained in a closed simply connected region around the nodes, then the interpolation error may be written as

$$f(x) - r_n(x) = \frac{1}{2\pi i} \int_C \frac{f(s)}{x - s} \cdot \frac{\sum_{i=0}^{n-d} \lambda_i(s)}{\sum_{i=0}^{n-d} \lambda_i(x)} \, ds,$$

which is a **Hermite-type error formula**.

We define the new “potential function”

$$V^{C,\mu}(z) := \max_{\alpha} C U^{\nu\alpha}(z),$$

and the contours

$$C_R := \left\{ z \in \mathbb{C} : \frac{\exp(V^{C,\mu}(z))}{\min_{x \in [a,b]} \exp(V^{C,\mu}(x))} = R \right\}.$$
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Main convergence/divergence theorem

Theorem (Güttel–K. (2012))

Let $f$ be a function analytic in an open neighbourhood of $[a, b]$ and let $R > 0$ be the smallest number such that $f$ is analytic in the interior of $C_R$, then

$$\limsup_{n \to \infty} \|f - r_n\|_\infty^{1/n} \leq R.$$ 

In the case of equispaced nodes, further simplifications occur in $\sum_{i=0}^{n-d} \lambda_i(z)$ due to symmetries.
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In the case of equispaced nodes, further simplifications occur in $\sum_{i=0}^{n-d} \lambda_i(z)$ due to symmetries.
Effects of the symmetry in $\sum \lambda_i$

Figure: Levels of $|\sum_{i=0}^{n-d} \lambda_i(z)|^{1/(n+1)}$ with $d = 20$ for $n = 100$ equispaced nodes (left) and perturbed equispaced nodes (right) on a log$_{10}$ scale.
Convergence/divergence behaviour

Figure: Level lines of convergence for barycentric rational interpolation for $C = 0.2$ with equispaced nodes (left) and (right) relative error curves for the interpolation of $1/(x - 0.3i)$ with both node sequences, asymptotic relative error bound and upper bound on $\epsilon \Lambda_n$. 
Runge’s phenomenon

For \( n \) large enough and \( C \) constant, in exact arithmetic,

\[
\|f - r_n\|_\infty \lesssim DR^n.
\]

For \( d \) fixed, \( \|f - r_n\|_\infty \leq Kh^{d+1} \), excluding Runge’s phenomenon!

*Intuitively*, the interpolation error

\[
f(x) - r_n(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x)(f(x) - p_i(x))}{\sum_{i=0}^{n-d} \lambda_i(x)}
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The choice of $C$ and $d$

observed error($C, n$)

$\approx$ interpolation error in e.a. + imprecision $\times$ condition number

$\leq DR^n + \varepsilon \|f\|_{\infty} \Lambda_n$

$\leq D \left( \exp \left( V^{C,\mu}(s) - C \right) \left( C(b-a) \right)^C \right)^n$

$+ \varepsilon \|f\|_{\infty} \cdot 2^{Cn-1}(2 + \log n)$

$=: \text{predicted error}(C, n)$.

Aim: given $n$ and the closest singularity $s$ of $f$, determine $C \in (0, 1]$ such that the predicted error is minimal.
The choice of $C$ and $d$

observed error$(C, n)$

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\[ \log(1.2 - x)/(x^2 + 2) \text{ in } [-1, 1] \]

**Figure:** Relative errors for \( f(x) = \log(1.2 - x)/(x^2 + 2) \) with \( 2 \leq n \leq 250 \) equispaced nodes in \([-1, 1]\) with \( d = \text{round}(Cn) \), asymptotic convergence rates and nearly optimal values of \( C \) and \( d \).
$\log(1.2 - x)/(x^2 + 2)$ in $[-1, 1]$ 

**Figure:** Relative errors for $f(x) = \log(1.2 - x)/(x^2 + 2)$ with $2 \leq n \leq 250$ equispaced nodes in $[-1, 1]$ with $d = \text{round}(Cn)$, asymptotic convergence rates and nearly optimal values of $C$ and $d$. 
**arctan(\(\pi x\)) in \([-1, 1]\)**

**Figure:** Relative errors for \(f(x) = \arctan(\pi x)\) with \(2 \leq n \leq 250\) equispaced nodes in \([-1, 1]\) with \(d = \text{round}(Cn)\) and nearly optimal values of \(C\) and \(d\) (\(s = \pm i/\pi\)), and asymptotic convergence rates.
\( \Gamma(x + 1.1) \) in \([-1, 1]\)

**Figure:** Relative errors for \( f(x) = \Gamma(x + 1.1) \) with \( 2 \leq n \leq 250 \) equispaced nodes in \([-1, 1]\) with \( d = \text{round}(Cn) \) and nearly optimal values of \( C \) and \( d \) (\( s = -1.1 \)), and asymptotic convergence rates.
**sin(x) in [-5, 5]**

**Figure:** Relative errors for \( f(x) = \sin(x) \) with \( 2 \leq n \leq 1000 \) equispaced nodes in \([-5, 5]\) with \( d = \text{round}(Cn) \) and nearly optimal values of \( C \) and \( d \) (taking \( s = 10 \)), and asymptotic convergence rates.
Thank you!