The non-arithmetic cusped hyperbolic 3-orbifold of minimal volume

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We show that the 1-cusped quotient of the hyperbolic space $\mathbb{H}^3$ by the tetrahedral Coxeter group $\Gamma_* = [5,3,6]$ has minimal volume among all non-arithmetic cusped hyperbolic 3-orbifolds, and as such it is uniquely determined. Furthermore, the lattice Γ_* is incommensurable to any Gromov-Piatetski-Shapiro type lattice. Our methods have their origin in the work of C. Adams [2,3]. We extend considerably this approach via the geometry of the underlying horoball configuration induced by a cusp.

Thursday 2$^{\text{nd}}$ June, 2022

1 Introduction

Let $\mathbb{H}^3$ be the hyperbolic 3-space viewed in the upper half space U^3 of Poincaré, and let Isom $\mathbb{H}^3$ be its isometry group. A cusped hyperbolic 3-orbifold V is the quotient of $\mathbb{H}^3$ by a non-cocompact lattice $\Gamma \subset \text{Isom}\,\mathbb{H}^3$, that is, by a discrete group of hyperbolic isometries with a non-compact fundamental polyhedron of finite volume. In particular, Γ contains a non-trivial parabolic subgroup whose elements fix a point q on the boundary $\partial\mathbb{H}^3$. The stabiliser Γ_q gives rise to a maximal horoball B_q touching some of its Γ-images, called full-sized horoballs, and which project to a maximal cusp $C \subset V$ of finite volume in V. Hence, the number of non-conjugate parabolic subgroups of Γ equals the number of maximal cusps in V. The orthogonal projection of the full-sized horoballs to the horosphere $\partial B_q = H_q$ yields a horoball-packing with a characteristic horoball diagram on H_q. In this way, a combination of results from crystallography and a density result of K. Böröczky [7, Theorem 4] allows one to deduce lower volume bounds.

Using this picture, R. Meyerhoff [23] identified the 1-cusped (non-orientable) quotient space $\mathbb{H}^3/[3,3,6]$ as the minimal volume cusped hyperbolic 3-orbifold. The Coxeter group with symbol $[3,3,6]$ generates the group of symmetries of an ideal regular tetrahedron S_{reg}^∞ of dihedral angle $\frac{\pi}{3}$. Furthermore, it is an arithmetic group commensurable to the Eisenstein modular group $\text{PSL}(2,\mathbb{Z}[\omega])$ where $\omega = (-1 + \sqrt{-3})/2$ is a primitive cubic root of unity. The orbifold $\mathbb{H}^3/[3,3,6]$ is covered by the well-known (non-orientable) 1-cusped Gieseking manifold G which, by a result of Adams [1], is of minimal volume among all cusped hyperbolic 3-manifolds.

The result of Meyerhoff was considerably extended by Adams [3, Theorem 6.1, Corollary 6.2] who identified the six cusped (orientable and non-orientable) hyperbolic 3-orbifolds of smallest volume. Crucial in Adams' proof was the assumption of the strict upper volume bound of $\frac{1}{4}\text{vol}(S_{reg}^{\infty}) \approx 0.253735$ in the case of orientable orbifolds, implying that such an orbifold has only one cusp and a single orbit of full-sized horoballs modulo the action of the corresponding stabiliser. In [25], W. Neumann and A. Reid characterised Adams' small volume orbifolds and showed that they are all arithmetically defined.

In this work, we consider *non-arithmetic* cusped hyperbolic 3-orbifolds and prove the following result.

Main Theorem *Among all non-arithmetic cusped hyperbolic 3-orbifolds, the 1-cusped (non-orientable) quotient space V_* of $\mathbb{H}^3$ by the tetrahedral Coxeter group $[5,3,6]$ has minimal volume. As such the orbifold V_* is unique, and its volume $v_* \approx 0.171502$ is given explicitly by* (2–7).

The Coxeter group $[5,3,6]$ gives rise to the group of symmetries of an ideal regular dodecahedron $D_{reg}^{\infty} \subset \mathbb{H}^3$ of dihedral angle $\frac{\pi}{3}$. By applying different face identifications to D_{reg}^{∞}, the orbifold V_* admits several non-isometric non-arithmetic cover manifolds; see [10].

Notice that the non-arithmetic orbifold $V_* = \mathbb{H}^3/[5,3,6]$ does not relate to a Gromov–Piatetski-Shapiro construction. In fact, the Coxeter tetrahedron associated to $[5,3,6]$ is not *splittable* in the sense of [11, Section 6.2, Example 6.10]. Therefore, by [11, Lemma 6.9], the group Γ_* is incommensurable to any Gromov–Piatetski-Shapiro type lattice.

Consider the smooth case of non-arithmetic cusped hyperbolic 3-manifolds. There are many such manifolds. For a list containing small volume examples, we refer to [12] and [22, Section 13.6]. Here, we focus on *Coxeter manifolds*, that is, manifolds whose fundamental groups are commensurable with hyperbolic Coxeter groups. Their existence is guaranteed by Selberg's Lemma. The Main Theorem allows us to deduce the following result in terms of the non-arithmetic tetrahedral Coxeter group $[(3^3,6)]$.

Proposition I *The fundamental group of a non-arithmetic cusped hyperbolic Coxeter 3-manifold M_* of minimal volume is incommensurable to the Coxeter group $[5,3,6]$; the volume of M_* is smaller than or equal to $24 \cdot \text{covol}([(3^3,6)]) \approx 8.738570$.*

Furthermore and as a by-product of the geometric methods used to prove the Main Theorem, we obtain the following two-dimensional analogue which has possibly been overlooked so far.

Proposition II *Among all non-arithmetic cusped hyperbolic 2-orbifolds, the 1-cusped quotient space V_* of $\mathbb{H}^2$ by the triangle Coxeter group $[5,\infty]$ has minimal area. As such the orbifold V_* is unique, and its area is given by $\frac{3\pi}{10}$.*

This work is structured as follows. In Section 2 we present the necessary background about cusped hyperbolic orbifolds, their (non-)arithmeticity and the realisation as quotients by hyperbolic Coxeter groups. We provide a short overview about volume computations for hyperbolic (truncated) tetrahedra and finish by presenting the necessary

information about horoball packings and cusp densities. Section 3 contains the proof of the Main Theorem which consists of several steps. We first show that a non-arithmetic cusped hyperbolic 3-orbifold $\mathbb{H}^3/\Gamma$ of minimal volume has exactly one cusp C, and then, that C is a rigid cusp of type $\{2,3,6\}$ or $\{2,4,4\}$. We study these two cases separately and have also to distinguish – in contrast to Adams' work [3] – whether there is one or more equivalence classes of a full-sized horoball B covering C with respect to the stabiliser Γ_q. An essential aspect is the view of the Γ_q-periodic horoball packing induced by B by means of its horoball diagram. The possible configurations of full-sized horoballs and their hierarchy allow us to estimate minimal distances of their centres and to derive lower volume bounds. In all cases, we are able to identify fundamental polyhedra for Γ or to exclude groups when they are arithmetic or of a too big covolume. In Section 4, we discuss briefly the related results given by the two propositions above, both due to the first author [9, Chapter IV and Chapter V]. In the parts A and B of the Appendix, we describe certain small volume orientable orbifolds with precisely one cusp of type $\{2,3,6\}$ and $\{2,4,4\}$, respectively, whose associated parabolic groups give rise to only one equivalence class of full-sized horoballs. In the case of $\{2,3,6\}$, we correct the corresponding construction and result of Adams [3, p. 10].

Acknowledgements

2 Non-arithmetic cusped hyperbolic 3-orbifolds

2.1 Cusps of hyperbolic orbifolds

Consider the hyperbolic space $\mathbb{H}^3$ in the upper half space $U^3 = \{(x,y,t) \in \mathbb{R}^3 \mid t > 0\}$ equipped with the metric $ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$. Points on the boundary $\partial U^3 = \{(x,y,0) = (u,0) \in \mathbb{R}^3\} \cup \{\infty\} = \mathbb{R}^2 \cup \{\infty\}$ are called *ideal* points, and points in $\mathbb{R}^3$ with $t < 0$ are called *ultraideal*.

Let $\Gamma \subset \text{Isom}\mathbb{H}^3$ be a non-cocompact lattice, that is, Γ is a discrete group with a non-compact fundamental polyhedron $P \subset \mathbb{H}^3$ of finite volume. Then, the quotient space $V = \mathbb{H}^3/\Gamma$ is a *cusped* hyperbolic 3-orbifold of finite volume which is a smooth manifold if the group Γ has no torsion elements. By Selberg's lemma (see [26, Theorem 7.5.7], for example), Γ always has a finite index subgroup Λ which is torsion-free. In particular, Γ and Λ are *commensurable* groups in $\text{Isom}\mathbb{H}^3$, that is, the intersection of Γ with some conjugate of Λ in $\text{Isom}\mathbb{H}^3$ is of finite index in both groups. Recall that commensurability is an equivalence relation preserving properties such as cocompactness, finite covolume and arithmeticity.

Each cusp (or cusp neighborhood) C of V is of the form B_q/Γ_q where $B_q \subset \mathbb{H}^3$ is a horoball based at an ideal point $q \in \partial\mathbb{H}^3$ where $\Gamma_q \subset \Gamma$ is the (non-trivial) stabiliser of q in Γ. Enlarge C so that it touches either itself or another cusp of V. Such a

cusp is called a *maximal* cusp of V. After a possible conjugation, we will assume that a maximal cusp of V is covered by the horoball $B_\infty = \{(x,y,t) \in \mathbb{R}^3 \mid t > 1\}$ based at ∞, with distance 1 from the ground space $\{t = 0\}$, and bounded by the horosphere $H_\infty = \{(x,y,t) \in \mathbb{R}^3 \mid t = 1\}$. Recall that the induced metric $ds^2\mid_{t=1}$ on H_∞ coincides with the Euclidean metric with distance function denoted by d_0. Since the cusp C is maximal, there are horoball images $\gamma(B_\infty)$ with $\gamma \in \Gamma$ not fixing ∞ whose closures touch B_∞. These images are called *full-sized* horoballs. By looking at their orthogonal projections onto the horosphere H_∞, we get horodisks whose centres coincide with the touching points of the corresponding full-sized horoballs. It will be convenient to identify full-sized horoballs and their base points with their horodisks and centres and vice versa. We always suppose, without loss of generality, that one full-sized horoball is based at the origin 0 of $\{t = 0\}$.

The group Γ_∞ is a *crystallographic* group acting cocompactly by Euclidean isometries on $\{t = 0\}$ (and on H_∞). As such it contains a *translation lattice* $L \subset \Gamma_\infty$ of rank two, with minimal translation length $\tau \geq 1$, and a finite subgroup $\phi \subset O(2)$ fixing the origin, called the *point group*, which is a subgroup of the automorphism group $\mathrm{Aut}(L)$. The latter group consists of all Euclidean isometries fixing the origin and mapping L onto itself.

The orthogonal projection of the full-sized horoballs onto the horosphere H_∞ provides a sphere packing by balls of diameter 1 of the Euclidean plane $\{t = 1\}$. Recall that the densest packing of the Euclidean plane is achieved by the hexagonal lattice packing where each ball is surrounded by six balls. In the sequel, planar Euclidean and spatial horoball packings of large local densities will play an important role.

The ideal fixed point ∞ can be seen as an ideal vertex of a fundamental polyhedron P of Γ whose vertex link $P \cap H_\infty$ is a fundamental polygon for the crystallographic group Γ_∞. In particular, $P \cap H_\infty$ is given either by a triangle $\Delta = \{p,q,r\}$ with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ satisfying $(p,q,r) = (2,3,6), (2,4,4)$ or $(3,3,3)$ or by a parallelogram.
For a cusp C with a triangular description Δ as above, we say that C is *rigid* and *of type* $\{p,q,r\}$, according to the (rotational) orders p,q and r of the singular axes going directly out the cusp.
The terminology is taken over from the orientable setting where an orientable cusp is termed rigid if Dehn filling cannot be performed, and non-rigid otherwise. In [2], Adams showed that an orientable cusp is rigid if and only if there are singular axes of order different from 2 going directly out of the cusp; see also Section 3.1 and Section 3.2.

The (non-)arithmeticity of a discrete group $\Gamma \subset \mathrm{Isom}\mathbb{H}^3$ of finite covolume can be characterised by the following fundamental property due to Margulis (see [22, Theorem 10.3.5], for example). Consider the *commensurator*

$$\mathrm{Comm}(\Gamma) = \{\, \gamma \in \mathrm{Isom}\mathbb{H}^3 \mid \Gamma \text{ and } \gamma\Gamma\gamma^{-1} \text{ are commensurable}\,\}$$

of Γ in $\mathrm{Isom}\mathbb{H}^3$. Then, the group $\mathrm{Comm}(\Gamma)$ is a discrete subgroup in $\mathrm{Isom}\mathbb{H}^3$ containing Γ with finite index if and only if Γ is non-arithmetic. In particular, for Γ non-arithmetic, the commensurator $\mathrm{Comm}(\Gamma)$ is the maximal element in the commensurability class of Γ so that all non-arithmetic hyperbolic orbifolds and manifolds with fundamental groups commensurable to Γ cover a smallest common quotient.

2.2 Hyperbolic Coxeter orbifolds

Arithmeticity can be described in a nice way for the class of *hyperbolic Coxeter groups* and the associated quotients called *Coxeter orbifolds*. Consider first a *Coxeter polyhedron* in $\mathbb{H}^3$ which is a convex polyhedron $P_C \subset \mathbb{H}^3$ of finite volume all of whose dihedral angles are integral submultiples of π. A hyperbolic Coxeter group $\Gamma_C \subset \mathrm{Isom}\mathbb{H}^3$ (with fundamental polyhedron P_C) is the discrete group generated by the (finitely many) reflections in the facets of P_C. Since the dihedral angles of P_C are non-obtuse, Andreev's theorem (see [5], [6], [27]) provides necessary and sufficient conditions for its existence. In particular, there are infinitely many non-isometric Coxeter polyhedra in $\mathbb{H}^3$ but only finitely many Coxeter tetrahedra. In fact, there are precisely 9 compact Coxeter tetrahedra and 23 non-compact ones (for a list with their volumes, see [17, pp. 347-348]). By Vinberg's seminal work [28], [29] about hyperbolic Coxeter polyhedra in any dimension, many of their properties can be read off from their Coxeter graphs and Gram matrices. Let $P_C \subset \mathbb{H}^3$ be a hyperbolic Coxeter polyhedron bounded by $N \geq 4$ geodesic planes $H_1, \ldots, H_N$. Consider the $N \times N$ Gram matrix $G = G(P_C) = (g_{ij})$ of P_C which is a real symmetric matrix with $g_{ii} = 1$ and, for $i \neq j$,

$$(2\text{--}1) \qquad -g_{ij} = \begin{cases} \cos\frac{\pi}{m_{ij}} & \text{if } H_i, H_j \text{ intersect at the angle } \frac{\pi}{m_{ij}} \text{ in } \mathbb{H}^3, \\ 1 & \text{if } H_i, H_j \text{ meet at } \partial\mathbb{H}^3, \\ \cosh l_{ij} & \text{if } H_i, H_j \text{ are at distance } l_{ij} \text{ in } \mathbb{H}^3. \end{cases}$$

In case of many orthogonal bounding planes and small N it is convenient to represent P_C by means of its *Coxeter graph* which is a (weighted) graph $\Sigma = \Sigma(P_C)$ of order N defined as follows. To each bounding plane H of P_C we associate a node ν in Σ. Two different nodes ν_i, ν_j are connected by an edge with a weight if the planes H_i, H_j are not orthogonal. The weight equals m_{ij} if $g_{ij} = -\cos\frac{\pi}{m_{ij}}$. In the special (and frequent) case $m_{ij} = 3$, however, the edge carries no label. An edge will be decorated by the symbol ∞ if $g_{ij} = -1$. Edges related to disjoint planes with $g_{ij} < -1$ are replaced by dotted edges, and the weights are usually omitted. In order to describe a Coxeter graph in an abbreviated way, we use the *Coxeter symbol*. In particular, $[p, q, r]$ with integral components is associated to a linear Coxeter graph describing a *hyperbolic Coxeter orthoscheme* with dihedral angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$, and the Coxeter symbol $[(p^r, q^s)]$ describes a polyhedron with cyclic Coxeter graph having $r \geq 1$ consecutive weights p followed by s consecutive weights q (see [17, Appendix], for example). In the sequel, we often represent a Coxeter group by quoting the Coxeter symbol of its Coxeter polyhedron.

For the arithmeticity of hyperbolic Coxeter groups, there is a very efficient criterion due to Vinberg (see [29, pp. 226-227]). We quote it in the special case of a non-compact Coxeter polyhedron $P_C \subset \mathbb{H}^3$ with Gram matrix $G = (g_{ij})$ and with associated reflection group $\Gamma_C \subset \mathrm{Isom}\mathbb{H}^3$. Write $2G =: (h_{ij})$ and form *cycles (of length k)* of the form

$$(2\text{--}2) \qquad\qquad h_{i_1 i_2} h_{i_2 i_3} \cdot \ldots \cdot h_{i_{k-1} i_k} h_{i_k i_1},$$

with distinct indices i_j in $2G$. Then, Γ_C is arithmetic with field of definition $\mathbb{Q}$ if and only if all the cycles of $2G$ are rational integers. Furthermore, if the Coxeter graph $\Sigma(P_C)$ contains no dotted edges, then, by a result of Guglielmetti [13, Proposition 1.13], Γ_C is arithmetic if and only if all weights of $\Sigma(P_C)$ lie in $\{\infty, 2, 3, 4, 6\}$, and each cycle of length at least 3 in $2G$ lies in $\mathbb{Z}$.

Examples

1. Among all non-compact Coxeter orthoschemes $[p,q,r]$ in $\mathbb{H}^3$, only $[5,3,6]$ defines a non-arithmetic reflection group, which we denote by Γ_*. The associated Coxeter orbifold V_* has one cusp.

2. The Coxeter tetrahedron $P_\circ$ with the cyclic graph $[(3^3,6)]$ yields a non-arithmetic reflection group, denoted $\Gamma_\circ$. The associated Coxeter orbifold $V_\circ$ has 2 cusps.

Remark 1 *By [18, Theorem 3], the groups $[5,3,6]$ and $[(3^3,6)]$ are not commensurable; in particular, their invariant trace fields are different (see [22, Section 13.2]). Due to the graph symmetry $[(3^3,6)]$, the Coxeter tetrahedron $P_\circ$ has a symmetry plane H_r along which it can be dissected into 2 isometric tetrahedra (of non-Coxeter type), each with one cusp. The group extension $\Gamma_\circ^r := [(3^3,6)] * C_r$ by the cyclic group C_r generated by the half-turn r with respect to H_r is a non-arithmetic discrete group containing $[(3^3,6)]$ with index 2 and giving rise to the 1-cusped quotient space $V_\circ^r$.*

2.3   Volumes of non-compact hyperbolic tetrahedra

Consider a finite volume orthoscheme $R = R(\alpha,\beta) \subset \mathbb{H}^3$ with one ideal vertex q and dihedral angles $\alpha,\beta,\beta' = \frac{\pi}{2} - \beta$ such that $\beta' \leq \alpha < \frac{\pi}{2}$ (see Figure 1). More precisely, R is a tetrahedron bounded by geodesic planes $H_1,\ldots,H_4$ with opposite vertices $p_1 = q, p_2, p_3, p_4$ such that the Gram matrix $G(R) = (g_{ij})$ is given by

$$G(R) = \begin{pmatrix} 1 & -\cos\alpha & 0 & 0 \\ -\cos\alpha & 1 & -\cos\beta & 0 \\ 0 & -\cos\beta & 1 & -\cos\beta' \\ 0 & 0 & -\cos\beta' & 1 \end{pmatrix}.$$

The matrix $G(R)$ is of signature $(3,1)$ and has – beside positive definite principal submatrices – exactly one positive semi-definite principal submatrix of rank 2 (at the lower right) characterising the ideal vertex q of R.
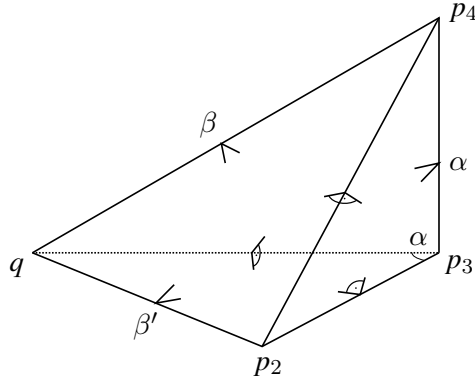

Figure 1: An orthoscheme $R(\alpha,\beta)$ with ideal vertex q

It is well-known that the scissors congruence group $\mathcal{P}(\mathbb{H}^3)$ is generated by the classes of orthoschemes $R(\alpha,\beta)$ (see [21, Chapter 3], for example). Hence, the volume of any polyhedron $P \subset \mathbb{H}^3$ is a linear combination of volumes of orthoschemes of type $R(\alpha,\beta)$.

The volume of $R(\alpha,\beta)$ is given by the expression (see [19], for example)

$$(2\text{--}3) \qquad \mathrm{vol}(R(\alpha,\beta)) = \frac{1}{4}\left\{\text{Ж}(\frac{\pi}{2}+\alpha-\beta) - \text{Ж}(\frac{\pi}{2}+\alpha+\beta) + 2\,\text{Ж}(\beta)\right\},$$

where $\text{Ж}(x) = \frac{1}{2}\sum_{r=1}^{\infty}\frac{\sin(2rx)}{r^2} = -\int_0^x \log|2\,\sin t|\,dt\,, x \in \mathbb{R},$ is Lobachevsky's function. The function $\text{Ж}(x)$ is odd, π-periodic and satisfies the distribution formula (see [17, Appendix], for example)

$$\frac{1}{k}\,\text{Ж}(kx) = \sum_{r=0}^{k-1}\text{Ж}\left(x+\frac{r\pi}{k}\right),\; k \in \mathbb{N}.$$

As a consequence, for the volume of an ideal tetrahedron $T(\alpha,\beta,\gamma)$ with dihedral angles α,β,γ along edges emanating from an ideal vertex such that $\alpha+\beta+\gamma=\pi$, one deduces that

$$(2\text{--}4) \qquad \mathrm{vol}(T(\alpha,\beta,\gamma)) = \text{Ж}(\alpha) + \text{Ж}(\beta) + \text{Ж}(\gamma).$$

As a special case, an ideal regular tetrahedron S_{reg}^{∞} (of dihedral angle $\frac{\pi}{3}$) is of volume $\mu_3 = \mathrm{vol}(S_{reg}^{\infty}) = 24\,\mathrm{vol}(R(\frac{\pi}{3},\frac{\pi}{3}))$. An ideal regular octahedron O_{reg}^{∞} (of dihedral angle $\frac{\pi}{2}$) can be dissected into 48 orthoschemes $R(\frac{\pi}{3},\frac{\pi}{4})$ so that the volume ω_3 of O_{reg}^{∞} is given by $\omega_3 = 48\,\mathrm{vol}(R(\frac{\pi}{3},\frac{\pi}{4}))$. Finally, an ideal regular dodecahedron D_{reg}^{∞} (of dihedral angle $\frac{\pi}{3}$) can be dissected into 120 orthoschemes $R(\frac{\pi}{5},\frac{\pi}{3})$ so that the volume δ_3 of D_{reg}^{∞} is given by $\delta_3 = 120\,\mathrm{vol}(R(\frac{\pi}{5},\frac{\pi}{3}))$.

Examples
The volume of the (arithmetic) Coxeter orthoscheme $R(\frac{\pi}{3},\frac{\pi}{3}) = [3,3,6]$ equals

$$(2\text{--}5) \qquad \frac{1}{8}\,\text{Ж}(\frac{\pi}{3}) \approx 0.042289.$$

In particular, we deduce that $\mu_3 = \mathrm{vol}(S_{reg}^{\infty}) = 3\,\text{Ж}(\frac{\pi}{3}) \approx 1.014942.$

The volume of the (arithmetic) Coxeter orthoscheme $R(\frac{\pi}{3},\frac{\pi}{4}) = [3,4,4]$ equals

$$(2\text{--}6) \qquad \frac{1}{6}\,\text{Ж}(\frac{\pi}{4}) \approx 0.076330.$$

As a consequence, we have that $\omega_3 = vol(O_{reg}^{\infty}) = 8\,\text{Ж}(\frac{\pi}{4}) \approx 3.663862.$

The volume of the (non-arithmetic) Coxeter orthoscheme $R(\frac{\pi}{5},\frac{\pi}{3}) = [5,3,6]$ equals

$$(2\text{--}7) \qquad \frac{1}{2}\,\text{Ж}(\frac{\pi}{3}) + \frac{1}{4}\left\{\text{Ж}(\frac{\pi}{6}+\frac{\pi}{5}) + \text{Ж}(\frac{\pi}{6}-\frac{\pi}{5})\right\} \approx 0.171502.$$

It follows that $\delta_3 = vol(D_{reg}^{\infty}) = 120\,vol([5,3,6]) \approx 20.580240.$

By dissection, the volume of the four Coxeter tetrahedra with cyclic Coxeter symbol $[(3,6,3,m)]\,, 3 \leq m \leq 6$, can be determined quite easily. In particular, the volume of the non-arithmetic Coxeter tetrahedron $[(3^3,6)]$ equals

$$(2\text{--}8) \qquad \frac{5}{8}\,\text{Ж}(\frac{\pi}{3}) + \frac{1}{3}\,\text{Ж}(\frac{\pi}{4}) \approx 0.364107,$$

which is the smallest one among the four.

For the volumes of *all* hyperbolic Coxeter tetrahedra, we refer to [17, Appendix].

A generalisation of the volume formula (2–3) for orthoschemes $R(\alpha,\beta)$ with one ideal vertex can be obtained by allowing that $\alpha+\beta < \frac{\pi}{2}$. This condition is equivalent to the assumption that the vertex p_4 as depicted in Figure 1 is ultraideal, that is, the face planes H_q, H_2 and H_3 opposite to q, p_2 and p_3 intersect (in p_4) outside of $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. In this situation, $R(\alpha,\beta)$ is of infinite volume. However, there is a unique hyperbolic plane H intersecting orthogonally H_q, H_2 and H_3. In fact, in the projective model of $\mathbb{H}^3$, the plane H is the *polar plane* associated to p_4. By truncating $R(\alpha,\beta)$ by means of H, we obtain a *simply truncated orthoscheme* denoted by $R_t(\alpha,\beta)$. By [19, (34)], the volume of $R_t(\alpha,\beta)$ is given analytically by the same formula (2–3), however under the constraint $\alpha+\beta < \frac{\pi}{2}$. This fact combined with suitable dissection procedures can be applied to determine the volumes of various families of truncated polyhedra in $\mathbb{H}^3$.

Example

For $k,l \in \mathbb{N}$ with $\frac{1}{k} + \frac{1}{l} < \frac{1}{2}$, consider a simply truncated Coxeter orthoscheme $R_t(\frac{\pi}{k}, \frac{\pi}{l})$ with one ideal vertex q. Its Coxeter graph is given by

$$\bullet\overset{k}{\rule{1cm}{0.4pt}}\bullet\overset{l}{\rule{1cm}{0.4pt}}\bullet\overset{\frac{2l}{l-2}}{\rule{1cm}{0.4pt}}\bullet\cdots\bullet \ .$$

In particular, the volume of the simply truncated Coxeter orthoscheme $R_t(\frac{\pi}{k}, \frac{\pi}{3})$, $k \geq 7$, is given by

$$(2\text{–}9) \qquad \frac{1}{2}\, \text{Ш}(\frac{\pi}{3}) + \frac{1}{4}\left\{ \text{Ш}(\frac{\pi}{6} + \frac{\pi}{k}) + \text{Ш}(\frac{\pi}{6} - \frac{\pi}{k}) \right\} \ ,$$

while the volumes of the Coxeter family $R_t(\frac{\pi}{k}, \frac{\pi}{6})$, $k \geq 4$, are equal to

$$(2\text{–}10) \qquad \frac{1}{2}\, \text{Ш}(\frac{\pi}{6}) + \frac{1}{4}\left\{ \text{Ш}(\frac{\pi}{3} + \frac{\pi}{k}) + \text{Ш}(\frac{\pi}{3} - \frac{\pi}{k}) \right\} \ .$$

In both cases, for $k \to \infty$, the limiting Coxeter polyhedron is a pyramid with two ideal vertices and Coxeter symbol $[\infty, 3, 6, \infty]$ and has volume $\frac{5}{4}\, \text{Ш}(\frac{\pi}{3}) \approx 0.42289$.

Remark 2 *Notice that for the infinite families of polyhedra $R(\alpha,\beta)$ and $R_t(\alpha,\beta)$ with fixed angle β, the volume is strictly increasing when the dihedral angle α decreases. This is a direct consequence of Schläfli's formula for the volume differential; see [19, Section 2].*

2.4   Some horoball geometry

Let $V = \mathbb{H}^3/\Gamma$ be a finite volume hyperbolic 3-orbifold with a set of disjoint cusps $\mathcal{C} = \{C_1, \ldots, C_m\}$, $m \geq 1$. Let $C \in \mathcal{C}$ be a maximal cusp such that $C = B_\infty/\Gamma_\infty$. Consider the image horoballs $\gamma(B_\infty)$ with $\gamma \notin \Gamma_\infty$ and project them orthogonally to the horosphere $H_\infty = \{t = 1\}$. The full-sized horoballs project onto Euclidean balls of diameter 1 and yield a periodic packing of the Euclidean plane by equal balls. Any minimal configuration of them providing the entire information (about translational and finite order symmetries) of the crystallographic group Γ_∞ yields a *horoball diagram* or *cusp diagram* $D \subset H_\infty$ (see [3, Figure 1 or Figure 6], for example, and Section 3.3).

Denote by $F \subset \mathbb{R}^2$ a fundamental polygon for the action of Γ_∞ on the horosphere H_∞, and let $\mathrm{vol}_0(F)$ be its Euclidean area. Then, the volume of C can be expressed by (see [8, Section 5])

$$\text{(2–11)} \qquad \mathrm{vol}(C) = \frac{1}{2}\,\mathrm{vol}_0(F)\,.$$

Notice that if Γ_∞ is a reflection group, its index two subgroup Γ_∞^+ of orientation preserving rotations yields twice the volume of (2–11).

The concept of local density of a horoball B covering C with respect to its Dirichlet-Voronoĭ cell $D(B)$ leads to the following lower volume bound for V in comparison with the total cusp volume $\mathrm{vol}(\mathcal{C}) = \sum_{r=1}^m \mathrm{vol}(C_r)$ (see [24] and [20, Lemma 3.2]).

$$\text{(2–12)} \qquad \mathrm{vol}(V) \geq \frac{\mathrm{vol}(\mathcal{C})}{d_3(\infty)}\,,$$

where $d_3(\infty) = \frac{\sqrt{3}}{2\mu_3} \approx 0.853276$ is the simplicial horoball density related to an ideal regular tetrahedron S_{reg}^∞ of volume $3\,\mathrm{JI}(\frac{\pi}{3})$.
Observe that the bound (2–12) is sharp if the lift of each element of $\mathcal{C}$ to $\mathbb{H}^3$ induces a *regular* horoball packing (see [20, Section 2]).

In the case of a 1-cusped orbifold with maximal cusp C, the cusp density $\delta(C) = \mathrm{vol}(C)/\mathrm{vol}(V) < 1$ is bounded from above by (see (2–12))

$$\text{(2–13)} \qquad \delta(C) \leq d_3(\infty)\,.$$

The following facts will be useful when studying horoball diagrams.

Lemma 1 [2, Lemma 4.4] *The centres of two tangent horoballs of radii r_1 and r_2 are at Euclidean distance $2\sqrt{r_1 r_2}$.*

Lemma 2 [2, Lemma 4.3], [16, Lemma 1] *Consider the horoball B_∞ and a full-sized horoball B_u based at $u \in \mathbb{R}^2$. Denote by l a geodesic with endpoints u and $v \in \mathbb{R}^2 \setminus \{u\}$, and let $\delta_0 = d_0(u,v)$ be the Euclidean distance from u to v. Put $a = (u,1)$, and let p be the intersection point of l with $H_u = \partial B_u$. Then, the induced distance $d(a,p)$ from a to p on H_u is given by $d(a,p) = \frac{1}{\delta_0}$.*

Corollary 1 [16, Lemma 2] *Consider a horoball $B_u(h)$ of diameter h in $\mathbb{H}^3$. Then, the interior of its upper hemisphere is an open disk of radius 1 with respect to the induced metric on the boundary $H_u(h)$.*

Corollary 2 [14, Corollary 7] *Let $B_u(h)$ and $B_v(k)$ be two horoballs with common touching point p and of Euclidean diameter h and k, respectively. Denote by δ_0 the Euclidean distance $d_0(u,v)$ and by $\delta = \delta_0/h$ the induced distance $d((u,h),(v,h))$ on the horosphere based at ∞ and at Euclidean height h. Then, the following identity holds.*

$$\frac{\sqrt{h}}{\sqrt{k}} = \frac{\delta_0}{k} = \frac{h}{\delta_0} = \frac{h\delta}{k} = \frac{1}{\delta}\,.$$

Corollary 3 *Consider a full-sized horoball B_u and a horoball $B_v(k)$ of diameter k based at $v \in \mathbb{R}^2$ which touches B_u. Denote by δ_0 the Euclidean distance $d_0(u,v)$. Then, the diameter k of B_v equals δ_0^2.*

Lemma 3 [2, Lemma 4.6], [3, Lemma 1.2] *Consider two horoballs $B_u(h)$ and $B_v(k)$ based at $u, v \in \mathbb{R}^2$ of diameter h and k, respectively, which cover the cusp C of V. Suppose that $B_u(h)$ and $B_v(k)$ are not tangent, and denote by $d_0(u,v) =: r$ the Euclidean distance between their base points u and v. Then, there exists a horoball of diameter $\frac{hk}{r^2}$ covering C.*

Consider the action of the crystallographic subgroup $\Gamma_\infty \subset \Gamma$ on the set of all full-sized horoballs. The following facts are fairly obvious (see [3, Lemma 3.1 and Lemma 2.1]).

Lemma 4 *Suppose that Γ_∞ identifies all full-sized horoballs. If the shortest translation length satisfies $\tau > 1$, then there cannot be a set of three full-sized horoballs which are pairwise tangent.*

Lemma 5 *Suppose that Γ_∞ identifies all full-sized horoballs. Then, every point of tangency between two Γ-equivalent horoballs lies on the axis of an order two elliptic isometry in Γ such that its axis is tangent to both horoballs.*

3 Proof of the Theorem

In this section, we provide a proof in several steps of our main result stated as follows.

Main Theorem *Among all non-arithmetic cusped hyperbolic 3-orbifolds, the 1-cusped quotient space V_* of $\mathbb{H}^3$ by the tetrahedral Coxeter group $[5,3,6]$ has minimal volume. As such the orbifold V_* is unique, and its volume v_* is given explicitly by (2–7).*

For the proof, we adapt and generalise the strategies and results of Adams as developed in [3], [2] and [4].

3.1 The non-rigid and multiply cusped cases

Let $\mu_3 = \mathrm{vol}(S_{reg}^\infty) = 3\,\mathrm{JI}(\frac{\pi}{3}) \approx 1.014942$ and $\omega_3 = \mathrm{vol}(O_{reg}^\infty) = 8\,\mathrm{JI}(\frac{\pi}{4}) \approx 3.663862$ be the volumes of an ideal regular tetrahedron and of an ideal regular octahedron, respectively (see (2–5) and (2–6)).

Denote by $V = \mathbb{H}^3/\Gamma$ a non-arithmetic cusped orbifold of minimal volume. By a result of Meyerhoff [23], the smallest volume of *any* cusped hyperbolic 3-orbifold equals $\frac{\mu_3}{24}$ and is realised in a unique way by the volume of the quotient of $\mathbb{H}^3$ by the arithmetic tetrahedral Coxeter group $[3,3,6]$. Therefore, the volume of V has to satisfy the inequalities

(3–1) $0.042289 \approx \dfrac{\mu_3}{24} < \mathrm{vol}(V) \leq \mathrm{vol}(V_*) = v_* \approx 0.171502$.

Recall from Section 2.1 that a cusp C of V is rigid if it is related to a triangle Δ and rotational axes of type $\{p,q,r\}$. We prove the following result assuming that V is non-arithmetic and of minimal volume.

Proposition 1 *A non-arithmetic cusped hyperbolic 3-orbifold V of minimal volume has precisely one cusp, and this cusp is a rigid one.*

Proof In [4], Adams considered the small volume spectrum of hyperbolic 3-orbifolds with $m \geq 2$ cusps. He proved that the four smallest (non-orientable) orbifolds of this sort have volumes equal to

$$(3\text{–}2) \qquad\qquad \frac{\mu_3}{6} \, , \, \frac{5\mu_3}{24} \, , \, \frac{\omega_3}{16} \quad \text{and} \quad \frac{\mu_3}{4}$$

(see [4, Corollary 2.6]). Comparing the four values in (3–2) with the volume bound in (3–1) shows that all values except the first one given by $\frac{\mu_3}{6}$ are strictly bigger than $\mathrm{vol}(V_*)$. Furthermore, in [4, Lemma 2.2], Adams identified the individual multiply-cusped orbifolds with small volumes and proved their uniqueness. According to his proof [4, pp. 155-156], one easily deduces that the value $\frac{\mu_3}{6}$ is the volume of the quotient space of $\mathbb{H}^3$ by the *arithmetic* tetrahedral Coxeter group $[3,6,3]$. Therefore, V can have only one cusp.

Consider an *orientable* hyperbolic 3-orbifold and suppose that it has a non-rigid cusp. By the results of Adams [2, Corollary 4.2 and Section 7] about limit volumes of orientable hyperbolic 3-orbifolds, the three smallest volumes of such orbifolds are $\frac{\omega_3}{12} \approx 0.305322$, 0.444457 and 0.457983. These values are realised in a unique way by orientable 1-cusped orbifolds which are explicitly described in the proof of [2, Lemma 7.1]. In fact, their fundamental groups are all arithmetic and related to Bianchi groups $\mathrm{PSL}(2, \mathcal{O}_d)$ where $\mathcal{O}_d$ is the ring of integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$. In particular, the fundamental group of the orbifold with volume $\frac{\omega_3}{12}$ is the Bianchi group $\mathrm{PSL}(2, \mathcal{O}_1)$ which is commensurable to the tetrahedral Coxeter group $[3,4,4]$ and to the fundamental group of the Borromean Rings complement (see also [22, Section 9.2]).

As a consequence, a non-orientable hyperbolic 3-orbifold with a single, non-rigid cusp of volume smaller than or equal to $\mathrm{vol}(V_*)$ is arithmetic. $\qquad\square$

3.2 The cusp type $\{3,3,3\}$

By Proposition 1, a non-arithmetic non-compact orbifold $V = \mathbb{H}^3/\Gamma$ of minimal volume has only one cusp, and the cusp is a rigid one of type $\{2,3,6\}$, $\{2,4,4\}$ or $\{3,3,3\}$. The next result allows us to exclude the type $\{3,3,3\}$ from further consideration.

Proposition 2 *A non-arithmetic cusped hyperbolic 3-orbifold V of minimal volume cannot have a cusp of type $\{3,3,3\}$.*

Proof We apply Adams' corresponding results in the orientable case including the related unicity statements as follows. In [3, Theorem 4.2], Adams proved that an orientable hyperbolic 3-orbifold with a cusp of type $\{3,3,3\}$ has volume either $\frac{\mu_3}{6}$, $\frac{\mu_3}{3}$, $\frac{5\mu_3}{12}$ or at least $\frac{\mu_3}{2}$. Furthermore, he showed that there are unique orbifolds whose volumes equal these first three values, and that they are the double covers of certain *unique* orientable orbifolds with one cusp of type $\{2,3,6\}$ of small volume. He provides an explicit description of the latter orbifolds as follows (see [3, p. 10]).

The unique orientable orbifold with one cusp of type $\{2,3,6\}$ and of volume $\frac{\mu_3}{12}$ is the orientable double cover of the arithmetic orbifold $\mathbb{H}^3/[3,3,6]$ implying that the orbifold with cusp of type $\{3,3,3\}$ and of volume $\frac{\mu_3}{6}$ is the quotient of $\mathbb{H}^3$ by the rotation subgroup of the arithmetic tetrahedral Coxeter group with symbol $[3,3^{[3]}]$.

The unique orientable orbifold with one cusp of type $\{2,3,6\}$ and of volume $\frac{\mu_3}{6}$ is the orientable double cover of the quotient space of $\mathbb{H}^3$ by the $\mathbb{Z}_2$-extension of the arithmetic Coxeter group $[3,6,3]$. Let us add that this orbifold as well as its double cover with one cusp of type $\{3,3,3\}$ are *not* Coxeter orbifolds anymore.

Observe that the third smallest value $\frac{5\mu_3}{12} \approx 0.422892$ in the above sequence is strictly bigger than the covolume $2\,\mathrm{vol}(V_*)$ of the rotation subgroup of $[5,3,6]$.

As a consequence, a (orientable or non-orientable) non-arithmetic cusped orbifold V of minimal volume cannot have a cusp of type $\{3,3,3\}$ (see also [3, Corollary 6.2]). □

3.3 Some notation

Before we continue with the proof, let us fix some notation. Consider a non-compact orbifold $V = \mathbb{H}^3/\Gamma$ with precisely one (maximal) cusp, denoted by C We assume that the *cusp point* (or *parabolic fixed point*) associated to C is based at ∞ and that $C = B_\infty/\Gamma_\infty$ with B_∞ the horoball at height 1 from the ground plane $\{t = 0\}$. Denote by $\tau \geq 1$ the minimal translation length induced by the translation lattice $L \subset \Gamma_\infty$.

Suppose that C is of type $\{p,q,r\}$ with $(p,q,r) = (2,3,6)$ or $(2,4,4)$ (see Section 2.1). In particular, C contains 3 singular (rotation) axes of orders $2,q,r$, respectively. By Proposition 2, these assumptions hold if V is non-arithmetic of minimal volume.

Consider a cusp diagram $D \subset H_\infty$ of Γ_∞. If C is of type $\{2,3,6\}$, then D is a Euclidean regular triangle of edge length τ, and if C is of type $\{2,4,4\}$, then D is given by a Euclidean square of edge length τ. In particular, the singular axes of C give rise to singular points a_2, a_q and a_r in D in the following way. The midpoint of each edge of D is a singular point a_2 of order 2, the centre of D is a singular point a_q of order q, and each vertex of D is a singular point a_r of order r. Since the horoballs covering C do not intersect in their interiors, the vertical axis l_s in the upper half space U^3 passing through a singular point a_s, $s \in \{2,q,r\}$, can lie in the interior of a full-sized horoball B_u only if the horoball B_u is centred at the intersection u of l_s with $\{t = 0\}$, that is, if $a_s = (u,1)$. We say that B_u is *centred at* a_s and write B omitting the index u, for short. By barycentric decomposition, D is tiled into copies of its *characteristic triangle* Δ with vertices a_s, $s \in \{2,q,r\}$, and which is either a right-angled triangle $[3,6]$ or a right-angled triangle $[4,4]$; see Figure 2. The aim is to determine or estimate in terms of τ or – more easily – by means of the smallest distance d of full-sized horoballs the cathetus length $d_0(a_2,a_p)$ or $d_0(a_2,a_q)$ of Δ. The position of the full-sized horoballs, projecting to full-sized horodisks (of radius $\frac{1}{2}$) in H_∞, will play a crucial role. These investigations will allow us to bound the volume of V by the expression (see (2–11) and (2–12))

$$(3\text{–}3) \qquad \mathrm{vol}(V) \geq \frac{\mathrm{vol}(C)}{d_3(\infty)} = \frac{\mu_3}{\sqrt{3}} \cdot \mathrm{vol}_0(\Delta) .$$
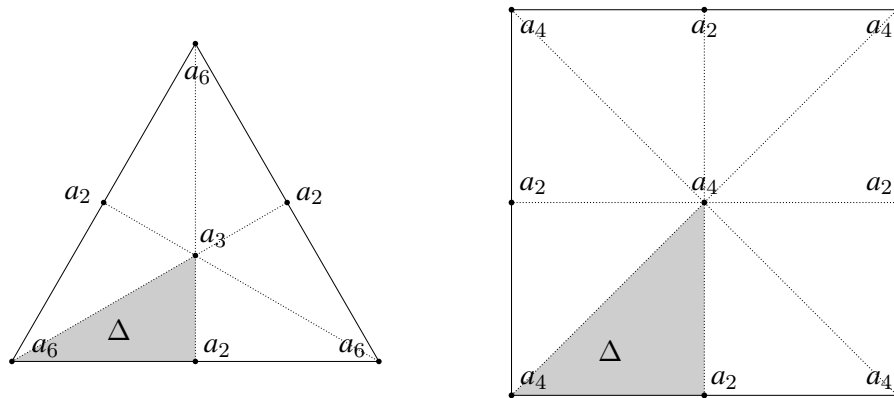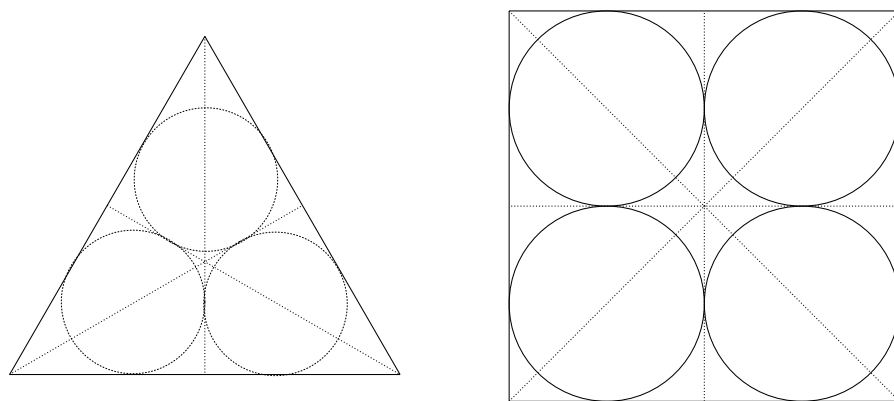
Figure 2: Tiling of D into copies of its characteristic triangle Δ

Figure 3: Full-sized horoballs not centred at singular points

3.4 One equivalence class of full-sized horoballs

Assume that the crystallographic group Γ_∞ gives rise to only one equivalence class of full-sized horoballs. If there is a full-sized horoball which is *not* centred at any of the singular points a_s, $s \in \{2, q, r\}$, in the cusp diagram D, then the least cusp volume $\mathrm{vol}(C)$ is given by (see also [3, Figure 1(a), p. 4, and Figure 6(a), p. 12] and Figure 3)

$$(3\text{–}4) \qquad \mathrm{vol}(C) = \begin{cases} \frac{\sqrt{3}}{12}\left(1 + \frac{\sqrt{3}}{2}\right) \approx 0.269338 & \text{if } C \text{ is of type } \{2,3,6\}\,, \\ \frac{1}{4} & \text{if } C \text{ is of type } \{2,4,4\}\,. \end{cases}$$

As a consequence, the volume of V is strictly bigger than $v_* = \mathrm{vol}(\mathbb{H}^3/[5,3,6])$ (compare with (3–1)).

From now on, we assume that there is a full-sized horoball centred at one of the singular points a_s, $s \in \{2, q, r\}$, in the cusp diagram D. We treat the cases $s \in \{2,3,6\}$ and $s \in \{2,4,4\}$ separately.

Some of the subsequent considerations in the context of one equivalence of full-sized horoballs overlap with Adams' work [3] but are included to fully incorporate the discussion of arithmeticity.
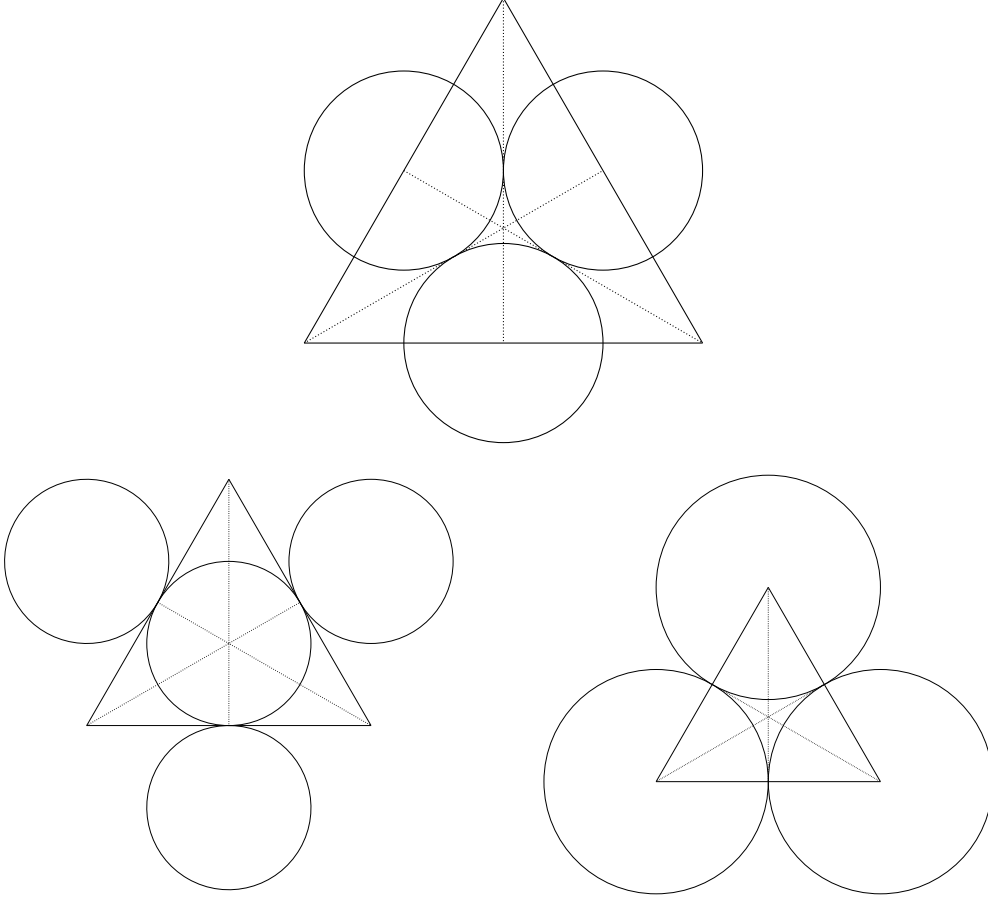
Figure 4: A $\{2,3,6\}$-cusp with full-sized horoballs centred at equivalent singular points

3.4.1 The case $\{2,3,6\}$

(i) First we assume that there are at least two full-sized horoballs which touch one another, that is, the minimal distance d of the centres of full-sized horoballs equals 1. As discussed above and in view of (3–4), in the cusp diagram D, there will be at least two full-sized disks, centred at equivalent singular points, which touch one another; see Figure 4.

Order 6. If there is a full-sized horoball (of radius $\frac{1}{2}$) centred at the singular point a_6 in Δ, then $d_0(a_2, a_6) = \frac{1}{2}$, and the cusp volume vol(C) equals $\frac{\sqrt{3}}{48}$. Since there are exactly three full-sized horoballs touching one another and the horoball B_∞, it is easy to see that the (up to isometry) unique orbifold V_0 corresponding to this configuration is given by the quotient of $\mathbb{H}^3$ modulo the *arithmetic* Coxeter group $[3,3,6]$ (see [24], [23]). The orbifold is covered by Gieseking's manifold and is known to be the cusped hyperbolic 3-orbifold of minimal volume (which equals $\frac{\mu_3}{24} = \frac{1}{8} \mathrm{JI}(\frac{\pi}{3}) \approx 0.042289$). In particular, the cusp density $\delta(C)$ satisfies vol(C)/vol(V_0) = $d_3(\infty)$ (see also (2–13)).

Order 3. Suppose that there is a full-sized horoball centred at the singular point a_3 in Δ. Then, $d_0(a_2, a_3) = \frac{1}{2}$ and $d_0(a_2, a_6) = \frac{\sqrt{3}}{2}$ so that the cusp volume equals

$\frac{\sqrt{3}}{16}$ [3, Figure 1(c), p. 4]. By Lemma 5, it is not difficult to see that there is a unique orbifold which corresponds to this configuration. Its fundamental group is given by the $\mathbb{Z}_2$-extension of the *arithmetic* Coxeter group $[3,6,3]$ (and has covolume $\frac{1}{4} \, \text{JI}(\frac{\pi}{3}) \approx 0.084579$).

Order 2. Suppose that there is a full-sized horoball centred at the vertex $a_2 \in \Delta$. Then, $d_0(a_2, a_3) = \frac{1}{\sqrt{3}}$ and $d_0(a_2, a_6) = 1$, and the cusp volume is given by $\frac{\sqrt{3}}{12} \approx 0.144338$ (see [3, Figure 1(b), p. 4]). Note that this volume is also the smallest cusp volume for the case that we have full-sized horoballs at both the singular points a_2 and a_6. There is a unique orbifold which corresponds to this configuration, and it is given by the *arithmetic* Coxeter simplex with symbol $[3^{1,1}, 6]$ and 2 cusps. By cutting along the symmetry plane of $[3^{1,1}, 6]$, we get an orbifold whose fundamental group is commensurable to $[3^{1,1}, 6]$ and isomorphic to the arithmetic Coxeter group $[4,3,6]$ of covolume $\frac{5}{16} \, \text{JI}(\frac{\pi}{3}) \approx 0.105723$ (see [17, p. 347], for example).

(ii) Suppose now that the full-sized horoballs do *not* touch another. Then, the minimal distance d of their centres satisfies $d > 1$. Assume that one full-sized horoball is centred at the singular point a_s, $s \in \{2,3,6\}$, in Δ. Since Γ_∞ identifies all full-sized horoballs, Lemma 5 yields an order two elliptic element $\gamma_s \in \Gamma$ with axis perpendicular at $a_s =: (u_s, 1)$ to the order s axis l_s. The element γ_s sends the horoball B_∞ to the full-sized horoball B_{u_s} and vice versa. Furthermore, γ_s sends the s neighboring full-sized horoballs – each at distance d from u_s – to s smaller horoballs. By the proof of Lemma 3, these smaller horoballs are of Euclidean diameter $\frac{1}{d^2}$, and by Corollary 2, their base points are at a distance $\frac{1}{d}$ from u_s. Following the terminology of Adams [3, p. 5], call each of these balls a $(\frac{1}{d})$-*ball*. By construction, the $(\frac{1}{d})$-balls are the biggest horoballs of diameter less than 1 that are tangent to full-sized horoballs.

Let $B(h)$ be a horoball of diameter h such that $\frac{1}{d^2} \le h < 1$ covering C which is *not* tangent to any larger horoball covering C. Then, in the interior of its upper hemisphere, there are no points of tangency with horoballs covering C. By Corollary 1, this upper hemisphere is an open disk of radius 1 with respect to the induced metric on its boundary. By mapping $B(h)$ to B_∞ by means of an element in Γ, we obtain a disk of radius 1 on the horosphere H_∞ which contains no point of tangency with full-sized horoballs. Such a disk is called a *disk of no tangency*. Notice that the existence of a disk of no tangency in the cusp diagram on H_∞ for C has the same effect as an extra full-sized horoball touching the centre of the disk would have.

Furthermore, by Corollary 2, one can deduce the following fact for a cusp of any type (for a proof, see [3, pp. 5–6]).

Lemma 6 *Let B be a full-sized horoball covering the cusp C, and let $B_{\frac{1}{d}}$ be a $(\frac{1}{d})$-ball touching B. If the base point x of $B_{\frac{1}{d}}$ has Euclidean distance $d_0(x,y) < 1$ from the base point y of another full-sized horoball B' or if $d_0(x,z) < \frac{1}{d}$ with z a base point of another $(\frac{1}{d})$-ball $B'_{\frac{1}{d}}$, without $B_{\frac{1}{d}}$ touching B' or $B'_{\frac{1}{d}}$, then there is a disk of no tangency in the cusp diagram of C.*

With these preliminary remarks, we investigate in detail each of the cases $s \in \{2,3,6\}$.

Order 6. Suppose that a full-sized disk $B = B_1$ is centred at the singular point $b_1 = a_6$ in D, and that B_2, B_3 are full-sized disks centred at the vertices b_2, b_3 at distance d
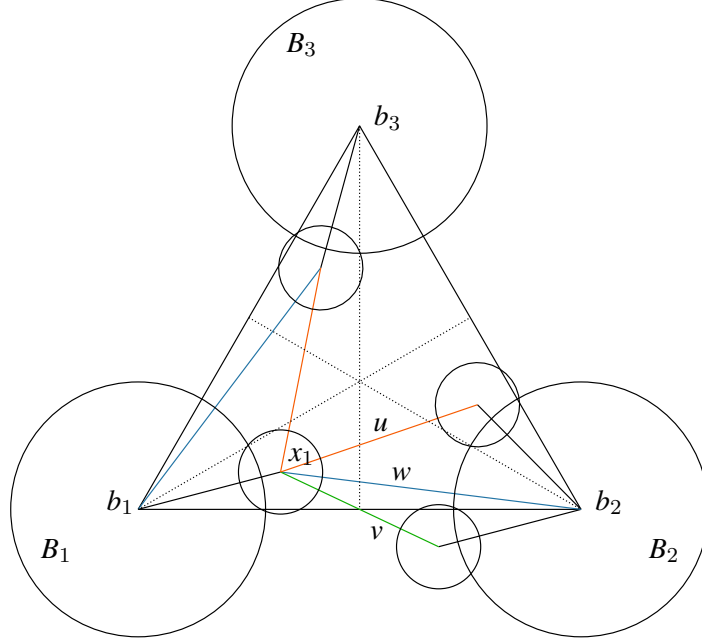
Figure 5: $\left(\frac{1}{d}\right)$-balls in the $\{2,3,6\}$-cusp triangle D

from B_1 in the cusp triangle D. For $1 \leq i \leq 3$, let x_i denote the center (at height 1) of the $(\frac{1}{d})$-ball touching B_i in D. Following Adams' notation indicated in [3, Figure 2], we similarly define the Euclidean distances $u = d_0(x_i, x_j)$, v and w as in Figure 5.

Consider the triangle $\Delta(b_1, x_1, b_2)$ with edge lengths $d_0(b_1, x_1) = \frac{1}{d}$, $d_0(x_1, b_2) = w$ and $d_0(b_1, b_2) = d$, and denote by $\theta = \measuredangle(x_1, b_2)$ the angle at $b = b_1$. By symmetry of D, we may suppose that $0 \leq \theta \leq \frac{\pi}{6}$. The elementary law of cosines allows one to deduce the following identities (see [3, (1)–(3)])

$$(3\text{--}5) \qquad u^2 = d^2 + \frac{3}{d^2} - \sqrt{3}\sin\theta - 3\cos\theta = d^2 + \frac{3}{d^2} - 2\sqrt{3}\cos(\frac{\pi}{6} - \theta)$$

$$(3\text{--}6) \qquad v^2 = d^2 + \frac{4}{d^2} - 4\cos\theta$$

$$(3\text{--}7) \qquad w^2 = d^2 + \frac{1}{d^2} - 2\cos\theta \,.$$

The strategy as developed in detail by Adams [3, pp. 6–10] consists now in analysing the different tangency possibilities of the $(\frac{1}{d})$-balls with respect to one or several full-sized horoballs and among themselves.

A crucial point will be the dichotomy of Γ_∞ preserving orientation, or not. If Γ_∞ contains a reflection, then we say that the horoball configuration in the cusp diagram D has a *mirror symmetry*.

Altogether, we will be able to determine or estimate explicitly the lengths d and τ, the associated cusp volume $\mathrm{vol}(C) = \frac{\sqrt{3}d^2}{48}$ (in case of no mirror symmetry) and to describe the corresponding orbifold V under the volume constraint $\mathrm{vol}(V) \leq \mathrm{vol}(V_*) = v_*$ (see (3–1)). Furthermore, one has to check whether the resulting orbifold V is arithmetic

or not. This is not difficult if its fundamental group is given by a hyperbolic Coxeter group (see Section 2.2) or by an explicit presentation in $\mathrm{PSL}_2(\mathbb{C})$ (see [22, Section 8]).

Remark 3 *Assuming the rough volume estimate (see (2–7), (2–12))*

$$\mathrm{vol}(V) \geq \frac{\mathrm{vol}(C)}{d_3(\infty)} = \frac{\sqrt{3}\,d^2}{48\,d_3(\infty)} > \mathrm{vol}(V_*) = v_* \approx 0.171502 \,,$$

where $d_3(\infty) \approx 0.853276$, we obtain $d > 2.013813$. Hence, we only have to consider values for $d > 1$ with $d \leq 2.013813$ or – in the case that Γ_∞ is orientation-preserving – values for d with $d \leq 1.423982$.

We start with the easy case of $(\frac{1}{d})$-balls touching at least two full-sized horoballs. This case has a complete answer showing that the resulting orbifold is arithmetic (see [3, p. 3 and p. 10] and [25]). We provide one illustrating example and suppose that a single $(\frac{1}{d})$-ball touches three full-sized horoballs giving a cusp diagram as depicted in [3, Figure 3(b)]. It easily follows that $d = \sqrt[4]{3}$ yielding a cusp volume of $\frac{1}{16}$. The full symmetry group of this horoball configuration gives rise to the unique orbifold with fundamental group given by the arithmetic Coxeter group $[3,6,3]$ (see [3, p. 10] in the corresponding oriented case).

The delicate case is when each $(\frac{1}{d})$-ball touches a unique full-sized horoball. Let us consider the mutual position of the $(\frac{1}{d})$-balls. By Lemma 4, we can exclude the case that three $(\frac{1}{d})$-balls touch each other.

Next, observe that we can assume that the distance w as depicted in Figure 5 satisfies

(3–8) $\quad w \geq 1 \quad$ implying that $\quad d \geq \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 - 4}\right) \approx 1.515464 \quad$ for $\quad \sigma = \sqrt{3 + \sqrt{3}}\,.$

In fact, if there is *no* disk of no tangency, this is a direct consequence of Lemma 6. If there *is* a disk of no tangency in D, then $d \geq \sqrt{3}$. Since $0 \leq \theta \leq \frac{\pi}{6}$, equation (3–7) for w implies that $w \geq \frac{2}{\sqrt{3}} > 1$ and the asserted estimate for d.

Consequence. If Γ_∞ is orientation-preserving, and in comparison with v_*, the bound $d \geq 1.515464$ from (3–8) yields a too big cusp estimate $\frac{\mathrm{vol}(C)}{d_3(\infty)} \geq 0.194245$ for a non-arithmetic orbifold $V = \mathbb{H}^3/\Gamma$, even for V being an orientable double cover (indeed, by [3, Theorem 6.1 and Corollary 6.2] and [25], the orientable cusped orbifolds of volume smaller than 2 are all arithmetic). Hence, we do not have to analyse in depth cusp diagrams D with no mirror symmetry.

Now, if two $(\frac{1}{d})$-balls touch one another but no further $(\frac{1}{d})$-ball touches this pair, then the point of tangency created by this pair must coincide with a 2-fold singular point a_2 in D. Since the tangency point is the midpoint of an edge e of D, one has $v = \frac{1}{d^2}$ with v given by equation (3–6). Implementing this identity into the equation (3–7) for w, the property $w \geq 1$ yields the inequality

(3–9) $$d^6 - 2d^4 - 2d^2 + 1 \geq 0\,,$$

and hence $d \geq 2\cos\frac{\pi}{5} > 1.6$. One can conclude the case by different considerations (for Adams' conclusion, see [3, pp. 7-8]). First, assume that the centres of the two $(\frac{1}{d})$-balls are *not* aligned with b_1 and b_2 on the edge e of D. Then, the group Γ_∞ consists of
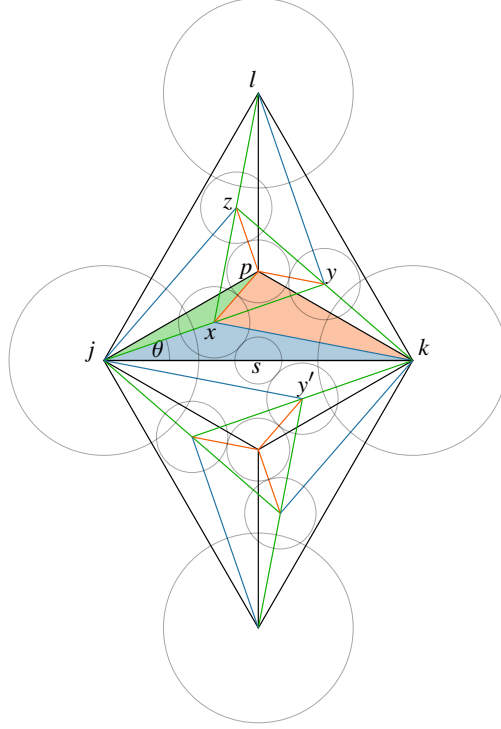
Figure 6: A single $(\frac{1}{w})$-ball in a $\{2,3,6\}$-cusp triangle D

rotations, only, and we are done.

Next, assume that the centres of the two $(\frac{1}{d})$-balls lie on the edge e. Then,

$$d = \frac{2}{d} + \frac{1}{d^2} \, ,$$

and hence, $d = 2\cos\frac{\pi}{5}$. There is a unique orbifold associated to this situation, which is based on an orthoscheme $R(\frac{\pi}{5}, \frac{\pi}{3})$ and its reflection group (see Section 2.3). Indeed, the orbifold equals V_* with non-arithmetic fundamental group $[5,3,6]$ (see [3, pp. 6-8]).

Assume now that the $(\frac{1}{d})$-balls *do not touch* each other. Following [3, p. 8], and by Lemma 5, there is an order 2 elliptic element exchanging the horoball B_∞ and a full-sized horoball B_j while sending a neighbouring full-sized horoball B_k to a $(\frac{1}{d})$-ball B_x, say (compare Figure 6). Then, the $(\frac{1}{d})$-ball B_y (at distance w from B_j) touching B_k gets sent to a ball B_s (at distance $\frac{1}{w}$ from B_j) touching B_x. Like in [3] we call this ball a $(\frac{1}{w})$-*ball*. Its diameter equals $\frac{1}{w^2 d^2}$ by Lemma 3. We study now the positions of the $(\frac{1}{w})$-balls and distinguish several cases.

• Suppose that the $(\frac{1}{w})$-balls of the three $(\frac{1}{d})$-balls in D coincide. This implies that the $(\frac{1}{w})$-ball is centred in the 3-fold singular point $p = a_3$ in D yielding $w = \frac{\sqrt{3}}{d}$. Knowing all the side lengths of the triangle $\Delta(j,k,x)$ one can calculate the angle θ with the law of cosines as $\cos\theta = \frac{5}{2\sqrt{7}}$ which yields $d = \sqrt[4]{7} > 1.6$ by means of (3–7). Furthermore, one can derive that the points j,x and y are aligned. The situation is sketched in Figure 6.

As a consequence, the horoball configuration in the cusp diagram D has no mirror

symmetry. Hence, the cusp volume equals

$$\mathrm{vol}(C) = \frac{\sqrt{21}}{24} > 0.19 > v_* \, ,$$

and we can exclude this case from our considerations.

Remark 4 *Adams considered the case above for an oriented orbifold $O = \mathbb{H}^3/\Gamma$ and calculated the corresponding cusp volume correctly as $\frac{\sqrt{21}}{24}$. However, in [3, p. 10], he gives a wrong fundamental polyhedron for Γ and a volume which is too big. In the Appendix A, we correct Adams' analysis, compute the volume and prove the arithmeticity of the orbifold O.*

- Suppose that two $(\frac{1}{w})$-balls of two $(\frac{1}{d})$-balls coincide, having center x, say, and that no further $(\frac{1}{w})$-ball touches them. If D has no mirror symmetry, then by means of the estimate $d \geq 1.515464$ given by (3–8), we obtain

$$\frac{\mathrm{vol}(C)}{d_3(\infty)} = \frac{\sqrt{3}\,d^2}{24\,d_3(\infty)} > 0.19 > v_* \, .$$

Hence, we may assume that the center x coincides with an edge center a_2 of D. It follows that $d = \frac{2}{w}$ which, by means of (3–7) and with $\theta = 0$, yields $d^4 - 2d^2 - 3 = 0$ with solution $d = \sqrt{3}$. There is a unique orbifold realising this configuration. It is arithmetic and of volume $\frac{3}{8}\mathrm{JI}(\frac{\pi}{3})$, being commensurable to the 2-cusped quotient space by the Coxeter group $[6, 3, 6]$.

- Suppose that the $(\frac{1}{w})$-balls are distinct and that D has a mirror symmetry. In general, the $(\frac{1}{w})$-balls cannot touch full-sized balls since then two $(\frac{1}{d})$-balls touch another, in contrast to our assumption above. Furthermore, three $(\frac{1}{w})$-balls cannot touch one another pairwise by Lemma 4.

Since D has a mirror symmetry, the group Γ_∞ is a reflection group. As a consequence, the centres of the $(\frac{1}{d})$-balls lie either on the edges or on the angle bisectors of D.

Assume first that the centres of the $(\frac{1}{d})$-balls lie on the edges of D. The situation is depicted in Figure 7. Consider the horoballs smaller than the full-sized horoballs in a hierarchical way as follows. For the $(\frac{1}{d})$-balls, we have the following three possibilities

(1) Each $(\frac{1}{d})$-ball touches two full-sized balls.

(2) Each $(\frac{1}{d})$-ball touches one full-sized ball and one other $(\frac{1}{d})$-ball.

(3) Each $(\frac{1}{d})$-ball touches one full-sized ball and *no* other $(\frac{1}{d})$-ball.

In the third case, there are $(\frac{1}{w})$-balls which are smaller images of the $(\frac{1}{d})$-balls under an order 2 elliptic element r which maps B_∞ to a full-sized horoball according to Lemma 5. For those $(\frac{1}{w})$-balls, there are the analogous three cases as above, and so on. This defines a sequence of horoball diagrams whose first elements are depicted in Figure 8. Note that Adams [3] discussed the depicted horoball diagrams in detail (see above). In particular, the third horoball diagram corresponds to the group $[5, 3, 6]$ while the fourth and the two first ones correspond to arithmetic groups.
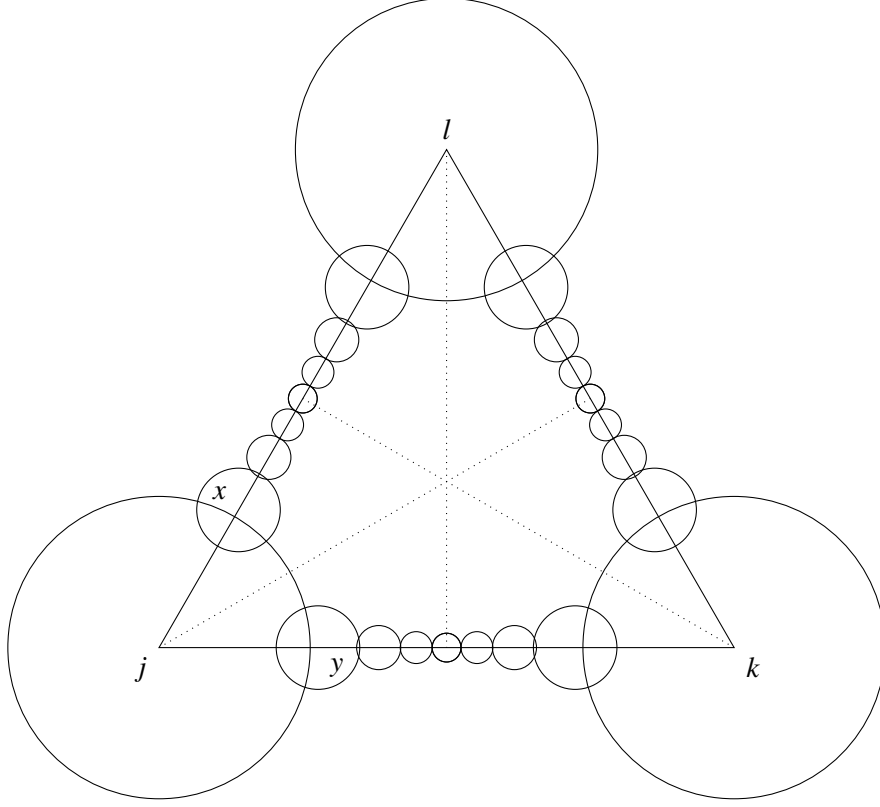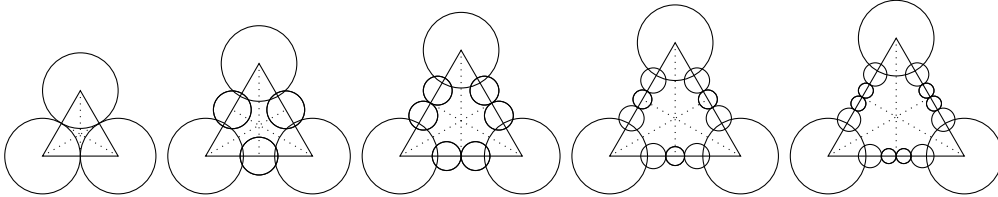
Figure 7: $(\frac{1}{d})$-balls aligned along the edges of D

For a given horoball diagram of this type, define a sequence

$$d_1 := d \ , \ d_2 := w = d - \frac{1}{d_1}, \ldots$$

in the sense that in the third case where the $(\frac{1}{d_k})$-balls do not touch, there are $(\frac{1}{d_{k+1}})$-balls which are the smaller images of the $(\frac{1}{d_k})$-balls under the isometry r. This implies the recursion

(3–10)
$$d_{k+1} = d - \frac{1}{d_k} \ , \ k \geq 1 \ .$$

Figure 8: The beginning of the sequence of horoball diagrams with aligned $(\frac{1}{d})$-balls.

Let $k_{\max} := \max \left\{ k \mid \text{the horoball diagram } D \text{ contains a } \frac{1}{d_k}\text{-ball} \right\}$. There are two cases.

If one $(\frac{1}{d_{k_{max}}})$-ball touches two $(\frac{1}{d_{k_{max}-1}})$-balls, the centre of the $(\frac{1}{d_{k_{max}}})$-ball lies in the midpoint a_2 between two neighbouring full-sized balls, leading to the equation

$$(3\text{--}11) \qquad\qquad \frac{1}{d_{k_{max}}} = \frac{d}{2}.$$

If there is a pair of touching $(\frac{1}{d_{k_{max}}})$-balls, it implies that one of those $(\frac{1}{d_{k_{max}}})$-balls is fixed by r because otherwise r sends $(\frac{1}{d_k})$-balls to $(\frac{1}{d_{k-1}})$-balls or to $(\frac{1}{d_{k+1}})$-balls but there is no $(\frac{1}{d_{k_{max}+1}})$-ball. This yields the equation

$$(3\text{--}12) \qquad\qquad 1 = d - \frac{1}{d_{k_{max}}}.$$

The recursion (3–10) together with the end conditions (3–11) or (3–12) prove that d has to be the root of a polynomial related to the Chebyshev polynomials of the first kind or of the second kind, respectively. In case (3–11), this implies

$$(3\text{--}13) \qquad\qquad d = 2\cos\left(\frac{\pi}{2k_{max}+2} \right),$$

and in case (3–12), it implies

$$(3\text{--}14) \qquad\qquad d = 2\cos\left(\frac{\pi}{2k_{max}+3} \right).$$

Details can be found in [9]. This recursion allows us to investigate the horoball diagrams and to characterise the group Γ assuming that Γ_∞ is a reflection group.

Proposition 3 *In the case of the $\{2,3,6\}$-cusp where Γ_∞ is a reflection group and the full-sized horoballs are centred in the singular points a_6, the group Γ having a horoball diagram according to Figure 7 contains the Coxeter group*

$$\bullet \!\!\overset{6}{\rule{1cm}{0.4pt}}\!\! \bullet \!\!\rule{0.6cm}{0.4pt}\!\! \bullet \!\!\overset{\alpha_k}{\rule{1cm}{0.4pt}}\!\! \bullet$$

where $d =: 2\cos\alpha_k$ is given according to (3–13) and (3–14), respectively.

Proof Since the Coxeter group $\Gamma_\infty = \bullet\!\!\overset{6}{\rule{1cm}{0.4pt}}\!\!\bullet\!\!\rule{0.6cm}{0.4pt}\!\!\bullet$ is contained in Γ, it only remains to prove the existence of the last generator in Γ. Consider an elliptic isometry r of order 2 mapping B_∞ to a full-sized horoball and vice versa.

If r is a reflection, we can see that it is the desired one: Its reflection plane R is a hemisphere orthogonal to the two reflection planes of Γ_∞ passing through a_6. The hemisphere has radius 1, and its centre has distance $\frac{d}{2}$ to the last reflection plane of Γ_∞ implying that the angle of intersection α satisfies $\cos\alpha = \frac{d}{2}$, that is, $\alpha = \alpha_k$.

In the case where r is a rotation of order 2, we can choose it such that the full-sized ball B_j is mapped to B_∞, the full-sized balls B_k is mapped to the $(\frac{1}{d})$-ball B_x, and correspondingly, B_k is mapped to B_y; see Figure 7. Then, r is the rotation in the edge ab of the orthoscheme defined by the planes mentioned above; see Figure 9. If $s \in \Gamma_\infty$ is the reflection in the face (a,b,∞), then $rs = sr$ is the reflection in the hemisphere R as above and belongs to Γ. $\qquad\square$
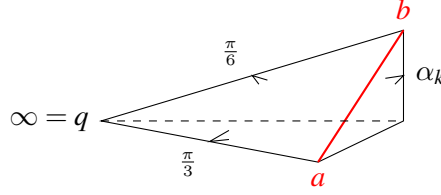
Figure 9: Fundamental orthoscheme

As a consequence of Proposition 3, the groups Γ with distances $\sqrt{2} \leq d \leq \sqrt{3}$ between full-sized horoballs are of smallest covolume if Γ coincides with (a finite index subgroup of) the arithmetic Coxeter group $[l,3,6]$ for $l = 3, 4$ or 6, or with the non-arithmetic Coxeter group $[5,3,6]$.

If the distance d between full-sized horoballs satisfies $d > \sqrt{3}$, then the hyperbolic Coxeter group $\bullet\!\!\overset{6}{-\!\!-}\!\!\bullet\!\!-\!\!\bullet\!\!\overset{\alpha_k}{-\!\!-}\!\!\bullet$ with $2\cos\alpha_k = d$ of Proposition 3 has infinite covolume since the orthoscheme $R(\alpha_k, \frac{\pi}{3})$ has an ultraideal vertex given by the graph $\bullet\!\!-\!\!\bullet\!\!\overset{\alpha_k}{-\!\!-}\!\!\bullet$ (see Section 2.3). In order to construct an orbifold with finite volume from it, its fundamental group Γ has to contain additional isometries.

Proposition 4 *In the case of Proposition 3 with $\sqrt{3} < d \leq 2$, the minimal possible orbifold volume is obtained from the truncated Coxeter orthoscheme $R_t(\alpha_k, \frac{\pi}{3})$ with Coxeter graph*

$$\bullet\!\!\overset{6}{-\!\!-}\!\!\bullet\!\!-\!\!\bullet\!\!\overset{\alpha_k}{-\!\!-}\!\!\bullet\cdots\bullet \qquad where \quad d = 2\cos\alpha_k .$$

Proof If the horoball diagram appears as in Figure 10 with $d > \sqrt{3}$, then the orthoscheme $\bullet\!\!\overset{6}{-\!\!-}\!\!\bullet\!\!-\!\!\bullet\!\!\overset{\alpha_k}{-\!\!-}\!\!\bullet$ has an ultraideal vertex v_0. Hence, there has to be an additional horoball B_+ close to the centre of D in order to obtain finite volume. In view of the realisations of the horoball B_+, Proposition 3 guarantees the existence of reflections mapping B_∞ to the full-sized balls B_j, B_k, and B_l. Denote the associated reflection planes by P_j, P_k, and P_l, respectively. The horoball B_+ cannot intersect any of those planes because otherwise it would intersect its image under the corresponding reflection. The biggest possible horoball B_+ (yielding the minimal volume orbifold) hence *touches* all three planes P_j, P_k, and P_l. Those three planes are orthogonal to the polar plane P_0 of v_0. Therefore, P_j, P_k, and P_l are left invariant under the reflection r_0 in the plane P_0. The horoballs B_∞ and B_+ are the only ones touching all three planes at once, implying that one is the image of the other under r_0. Thus, the polar plane P_0 is the bisector of B_∞ and B_+.

In our constructions, a fundamental polyhedron of Γ can be found by considering the Dirichlet-Voronoï cell of B_∞ modulo Γ_∞. The previous arguments show that a fundamental polyhedron of Γ has to include at least the truncated orthoscheme $R_t(\alpha_k, \frac{\pi}{3})$ as asserted.

$\square$

By Proposition 4, the minimal possible orbifold volume in the case $\sqrt{3} < d = 2\cos\alpha_k \leq 2$ is given by the volume of the truncated Coxeter orthoscheme $R_t(\alpha_k, \frac{\pi}{3})$. In Section 2.3, an explicit formula for $\mathrm{vol}(R_t(\alpha_k, \frac{\pi}{3}))$ is given by (2–9). Furthermore, by Remark 2,
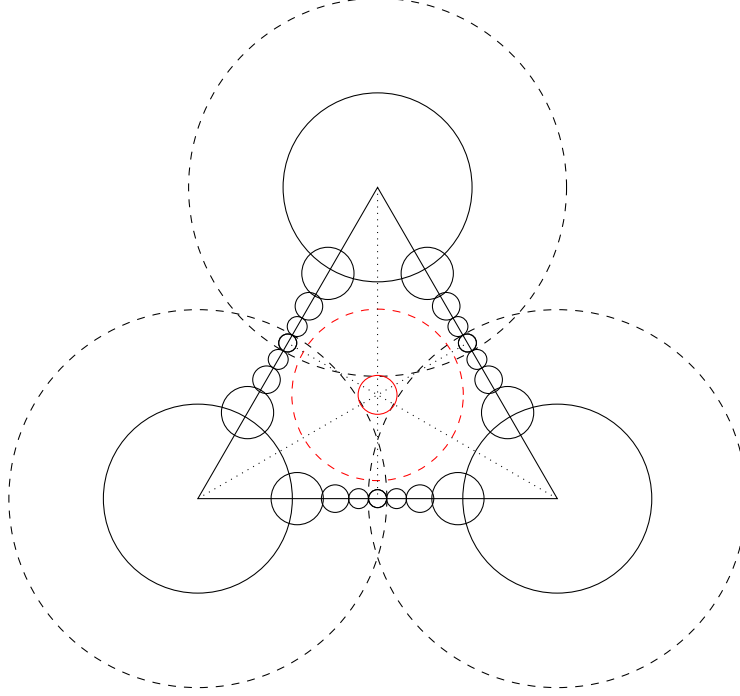
Figure 10: $(\frac{1}{d})$-balls in line

$\mathrm{vol}(R_t(\alpha_k, \frac{\pi}{3}))$ is strictly increasing when α_k decreases, that is, when d increases. In particular, for $\sqrt{3} < d \le 2$, we get the lower volume bound

$$\mathrm{vol}(R_t(\alpha_k, \frac{\pi}{3})) \ge \mathrm{vol}(R_t(\frac{\pi}{7}, \frac{\pi}{3})) > 0.317811 > v_* \,.$$

In view of Proposition 4, it remains to identify groups Γ with full-sized horoballs satisfying $2 < d \le 2.013813$ (see Remark 3). It turns out that Γ has the same basic structure as before, and truncation of its fundamental orthoscheme with the polar plane P_0 gives a polyhedron R_t of smallest volume because of the following proposition.

Proposition 5 *For any horoball on an angle bisector as depicted in Figure 12, its bisector with B_∞ intersects the polar plane P_0 exactly below the boundary of the triangle D at height 1 in the upper half space U^3. Any other cutting plane leads to a bigger volume than the volume of R_t.*

Proof The proof is basic trigonometry. If the centre of a horoball B on an angle bisector of D has distance a from the boundary of D and touches the bisector of the two closest full-sized horoballs, then its radius r satisfies

$$(3\text{–}15) \qquad\qquad (1+r)^2 = l^2 + r^2 = \frac{d^2}{4} + a^2 + r^2 \,,$$

where l is the distance between the centres of B and of the next full-sized horoball. Figure 11 depicts a vertical cut in U^3 through the centres of the two horoballs. The identity (3–15) implies

$$2r = \frac{d^2}{4} + a^2 - 1.$$

Using that the radius of the bisector is $\sqrt{2r}$, the height h of the bisector is then independent of the distance a :

$$h^2 = \left(\sqrt{2r}\right)^2 - a^2 = \frac{d^2}{4} - 1.$$
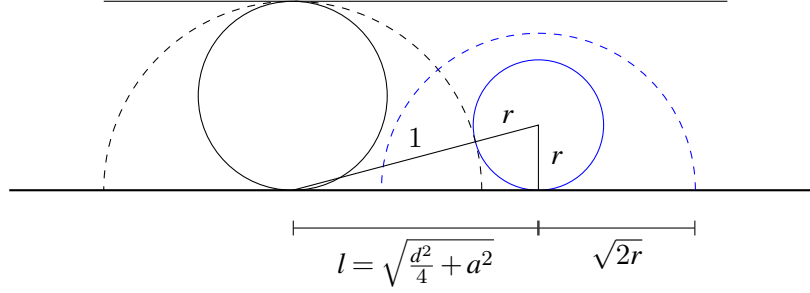
$\square$

Figure 11: Vertical cut along l through the centres of the two horoballs in U^3

By means of Proposition 5 and as a final step in the analysis of horoball diagrams D with $(\frac{1}{d_k})$-balls aligned on the edges of D according to Figure 7, we can state the following result.

Proposition 6 *Under the assumptions of Proposition 3 with $d > 2$, the minimal possible volume is obtained by an orbifold related to a truncated orthoscheme $R_t(\beta, \frac{\pi}{6})$ with Vinberg graph*

$$\bullet \overset{\beta}{\rule{1cm}{0.4pt}} \bullet \overset{6}{\rule{1cm}{0.4pt}} \bullet \rule{1cm}{0.4pt} \bullet \cdots \bullet \qquad where \quad 0 < \beta < \frac{\pi}{15} \ .$$

Proof We have to verify the bound $\beta < \frac{\pi}{15}$, only. In fact, we are assuming the upper bound $d \leq 2.013813$ according to Remark 3. By a similar trigonometrical computation in the upper half space U^3 as in the proof of Proposition 5 above, one can relate β and d according to

$$\cos\beta = \frac{d}{2\sqrt{d^2 - 3}} \ .$$

In this way, we get the estimate $\beta < \frac{\pi}{15}$. $\square$

By Proposition 6, the minimal possible orbifold volume in the case $d > 2$ is given by the volume of the truncated Coxeter orthoscheme $R_t(\beta, \frac{\pi}{6})$. By (2–10) of Section 2.3, we have an explicit formula for $\mathrm{vol}(R_t(\beta, \frac{\pi}{6}))$ which, by its strict monotonicity behavior with respect to $\beta < \frac{\pi}{15}$, yields the lower volume bound

$$\mathrm{vol}(R_t(\beta, \frac{\pi}{6})) \geq \mathrm{vol}(R_t(\frac{\pi}{15}, \frac{\pi}{6})) > 0.416491 > v_* \ .$$

These investigations complete the analysis of the case that the centres of the $(\frac{1}{d})$-balls lie on the edges of D.
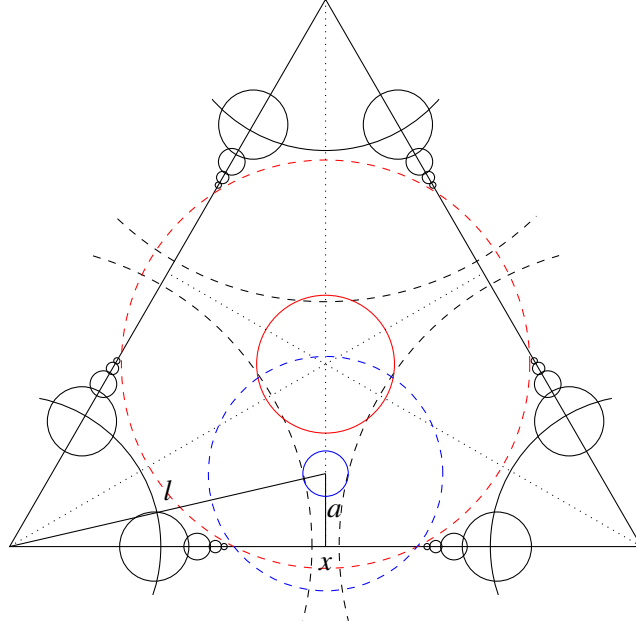
Figure 12: $(\frac{1}{d})$-balls on the edges of D with $d > 2$

In order to finish the part of full-sized horoballs centred at singular points of order 6 and of distance $d > 1$, we need to study the case when the centres of the $(\frac{1}{d})$-balls lie on the angle bisectors of D. We are still assuming that Γ_∞ is a reflection group.

The $(\frac{1}{d})$-balls cannot touch each other because then three balls would touch pairwise which is impossible because the full-sized balls do not touch. This implies the existence of $(\frac{1}{w})$-balls. For a fixed distance d, the position of the $(\frac{1}{w})$-balls is given by the similarity of the two triangles (j, z, l) and (j, x, p). See Figure 13 for the labels. This similarity and the reflection symmetry of the horoball diagram D also prove that the centre of the full-sized horoball B_j is on a common line with those of the $(\frac{1}{d})$-ball B_z and the $(\frac{1}{w})$-ball B_p.

Note that Figure 13 depicts the case where the $(\frac{1}{w})$-balls touch two $(\frac{1}{d})$-balls but the above statement about the alignment of their centres stays true if the $(\frac{1}{w})$-balls do not touch two $(\frac{1}{d})$-balls. This in turn means that each $(\frac{1}{d})$-ball touches two $(\frac{1}{w})$-balls because of the present symmetries. Hence, there are twelve $(\frac{1}{w})$-balls around a full-sized horoball.

Knowing that the centres j, p, and z are on the same line, we can calculate d by using

$$w = \frac{1}{w} + \frac{1}{wd^2}$$

and the identity $w^2 = d^2 + \frac{1}{d^2} - 2\cos\frac{\pi}{6}$ according to (3–7). This implies $d^2 = 1 + \sqrt{3}$.

The following considerations allow us to describe the orbifold associated to this horoball diagram. They are also valid if d is bigger, that is, if each of the $(\frac{1}{w})$-balls touches only one $(\frac{1}{d})$-ball. Since we assume that Γ_∞ is a reflection group, there is a reflection
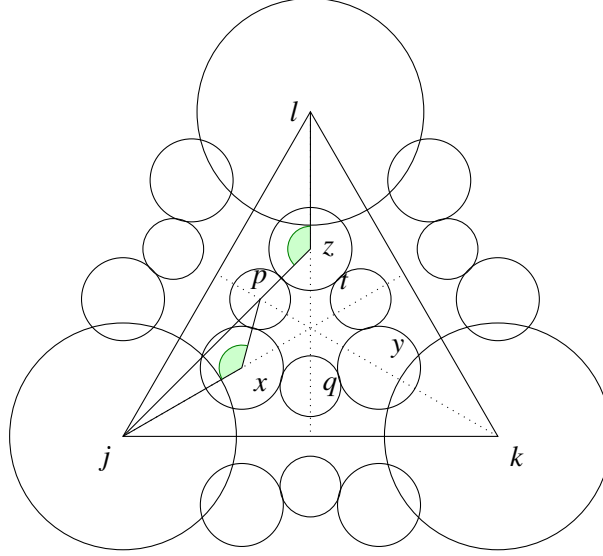
Figure 13: Horoball diagram D with $d = \sqrt{1 + \sqrt{3}}$

$s \in \Gamma_\infty$ which exchanges B_j and B_k. By Lemma 5, there is an order 2 elliptic element r in Γ which swaps

$$B_j \leftrightarrow B_\infty \qquad , \qquad B_k \leftrightarrow B_x .$$

Hence, rsr is the reflection in the bisector of B_∞ and B_x and an element of the group Γ. This proves the following result.

Proposition 7 *In the case of a $\{2,3,6\}$-cusp where Γ_∞ is a reflection group and where the $(\frac{1}{d})$-balls are centred on the angle bisectors of the horoball diagram D, there are reflections in Γ mapping B_∞ to any $(\frac{1}{d})$-ball.*

In the situation of Figure 13, the distance u between two neighbouring $(\frac{1}{d})$-balls can be calculated as $\frac{1}{d}$. Using this information, one can consider the polyhedron created from the triangular cone with apex ∞ and angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$ by cutting with the bisector/reflection plane of rsr. The bisector is orthogonal to one side of the cone and has angle $\frac{\pi}{3}$ with the other two because its centre has (Euclidean) distance $\frac{1}{2d}$ from the hyperplanes and the radius of the bisector is $\frac{1}{d}$. Thus, it can be seen that this polyhedron is the fundamental polyhedron of the (non-arithmetic) reflection group $[(3^3, 6)]$ with Coxeter graph
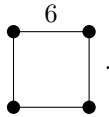
One can see that r induces the internal symmetry of that polyhedron (and its graph). Hence, Γ is equal to the group extension $\Gamma_\circ^r = [(3^3, 6)] * C_r$ of $[(3^3, 6)]$ by means of the cyclic group generated by r; see Remark 1. The volume of the tetrahedron $[(3^3, 6)]$ as given by (2–8) is bigger than 0.36 so that $\mathrm{vol}(\mathbb{H}^3/\Gamma_\circ^r) > v_*$.

If we do not assume that the $(\frac{1}{w})$-balls touch two $(\frac{1}{d})$-balls, we have basically the same case distinction as when the $(\frac{1}{d_k})$-balls are centred on the edges D.

(1)   There is a single $(\frac{1}{d_{k_{\max}}})$-ball, or

(2)   there are two touching $(\frac{1}{d_{k_{\max}}})$-balls,

where $k_{\max} := \max\left\{k \mid \text{the horoball diagram } D \text{ contains a } (\frac{1}{d_k})\text{-ball}\right\}$. See Figure 14 for a sketch of the first couple of horoball diagrams in that sequence.
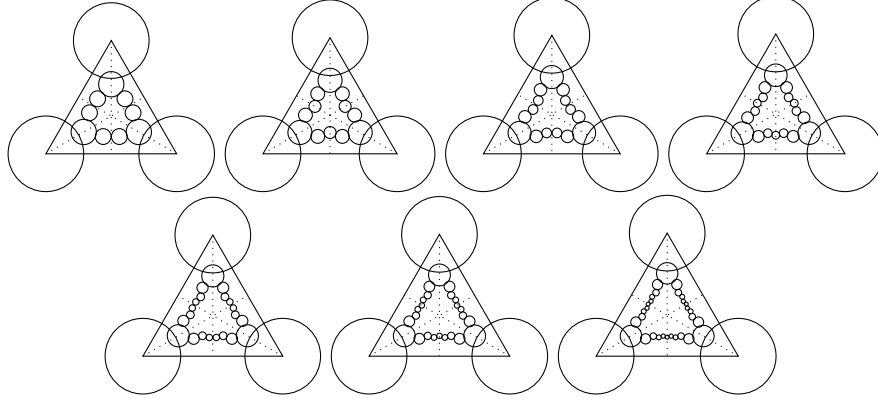

Figure 14: Horoball diagrams with angle $\theta = \frac{\pi}{6}$

In the part above, we discussed the first case with $k_{\max} = 2$, $d_2 = w$ and a horoball diagram D depicted in Figure 13. There, the isometry r maps the full-sized horoball B_k to the $(\frac{1}{d_1})$-ball B_x and the $(\frac{1}{d_1})$-ball B_y to the $(\frac{1}{d_2})$-ball B_q.

In general, there is a similar pattern as follows. A $(\frac{1}{d_k})$-ball close to the full-sized ball B_k gets mapped to a $(\frac{1}{d_{k+1}})$-ball closer to the full-sized ball B_j.

(a)   If there is a unique $(\frac{1}{d_{k_{\max}}})$-ball in the corresponding part of the horoball diagram D, then a $(\frac{1}{d_{k_{\max}-1}})$-ball on the right gets mapped to this $(\frac{1}{d_{k_{\max}}})$-ball.

(b)   If there are two $(\frac{1}{d_{k_{\max}}})$-ball in this part of the D, then a $(\frac{1}{d_{k_{\max}-1}})$-ball on the right gets mapped to the left $(\frac{1}{d_{k_{\max}}})$-ball. This implies that in the second case the right-hand $(\frac{1}{d_{k_{\max}}})$-ball is *fixed*.

In both cases, the reflection rsr then maps a $(\frac{1}{d_k})$-ball on the right to a $(\frac{1}{d_{k+2}})$-ball on the left as long is $k \le k_{\max} - 2$. In the case (a), a $(\frac{1}{d_{k_{\max}-1}})$-ball on the right is fixed by rsr because it is the only ball left. In the case (b), the $(\frac{1}{d_{k_{\max}-1}})$-ball on the right has to be mapped to the right $(\frac{1}{d_{k_{\max}}})$-ball. Refer to Figure 15 for a sketch of illustrative examples for the cases (a) and (b). Each pair of horoballs with matching colour and pattern is exchanged by rsr. Those horoballs with a solid fill are fixed.

Lemma 7 *The case (b) is impossible.*

Proof The previously discussed fact that rsr maps $(\frac{1}{d_k})$-balls to $(\frac{1}{d_{k+2}})$-balls proves that the $(\frac{1}{d_{2k+1}})$-balls are on one line with the $(\frac{1}{d_1})$-balls. We already showed that a
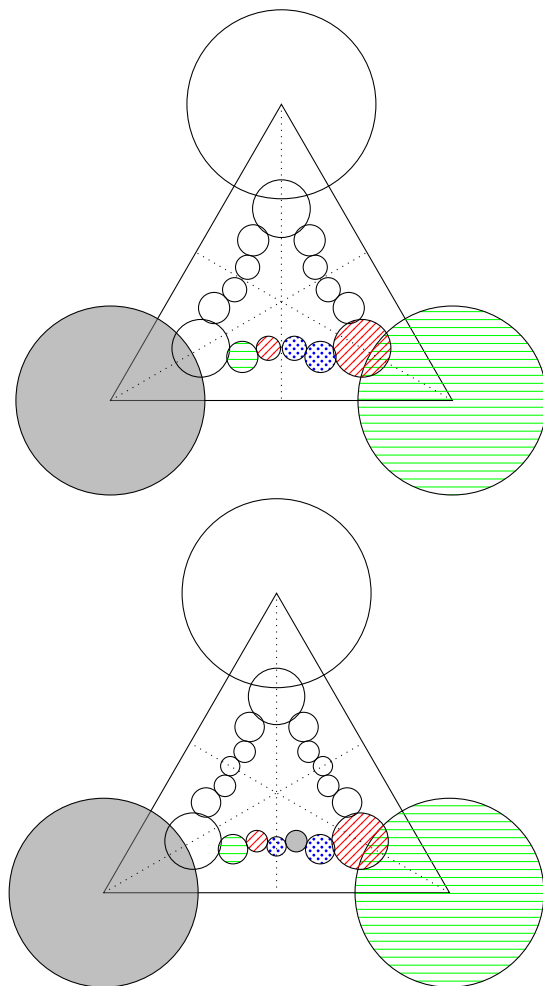
Figure 15: Action of *rsr* on some horoballs according to the cases (a) and (b)

neighboring pair of a $(\frac{1}{d_1})$-ball and a $(\frac{1}{d_2})$-ball are on a line with a full-sized horoball. This implies that the $(\frac{1}{d_{2k}})$-balls are not on the same line as the $(\frac{1}{d_{2k+1}})$-balls. However, this means that the right-hand $(\frac{1}{d_{k_{\max}}})$-ball is not on the same line as the right-hand $(\frac{1}{d_{k_{\max}-1}})$-ball. Thus it cannot be true that they get mapped to one another by rsr, proving the statement. $\qquad\square$

Hence, it remains to discuss the general case (a) with a single $(\frac{1}{d_{k_{\max}}})$-ball. It turns out to be helpful to consider the subgroup Γ' of Γ generated by Γ_∞ and rsr. In fact, this move allows us to drop the rotation r and to purge the full-sized horoballs as well as the $(\frac{1}{d_{2k}})$-balls from the horoball diagram D. Then, the $(\frac{1}{d_{2k+1}})$-balls can be increased to yield a cusp which is a maximal cusp again. More precisely, the $(\frac{1}{d})$-balls become full-sized horoballs (of diameter $\frac{1}{d}$), and then, the horoball diagram D is renormalised in order to have the horosphere $H_\infty = \partial B_\infty$ at distance 1 from ∂U^3. Figure 16 depicts the cases with small $k_{\max}$.
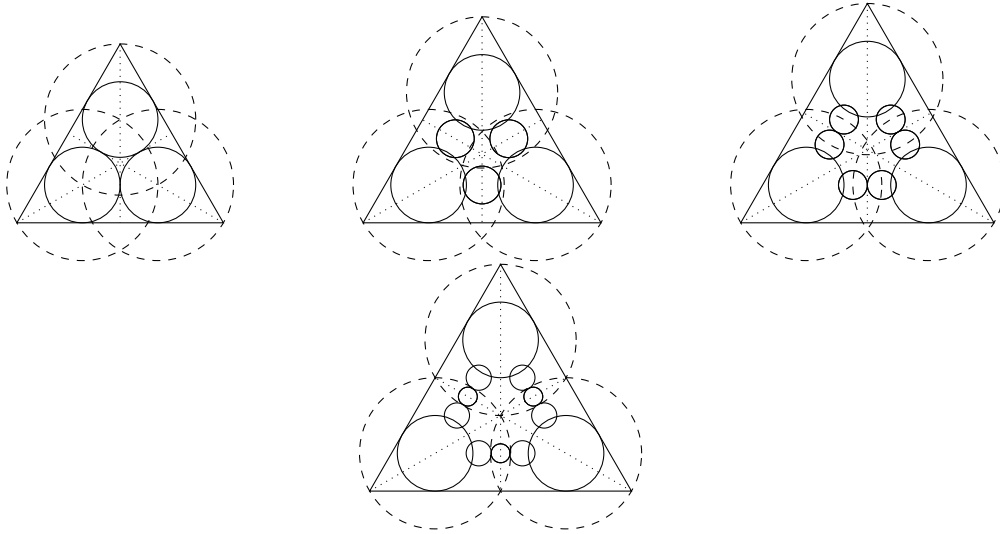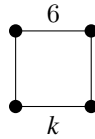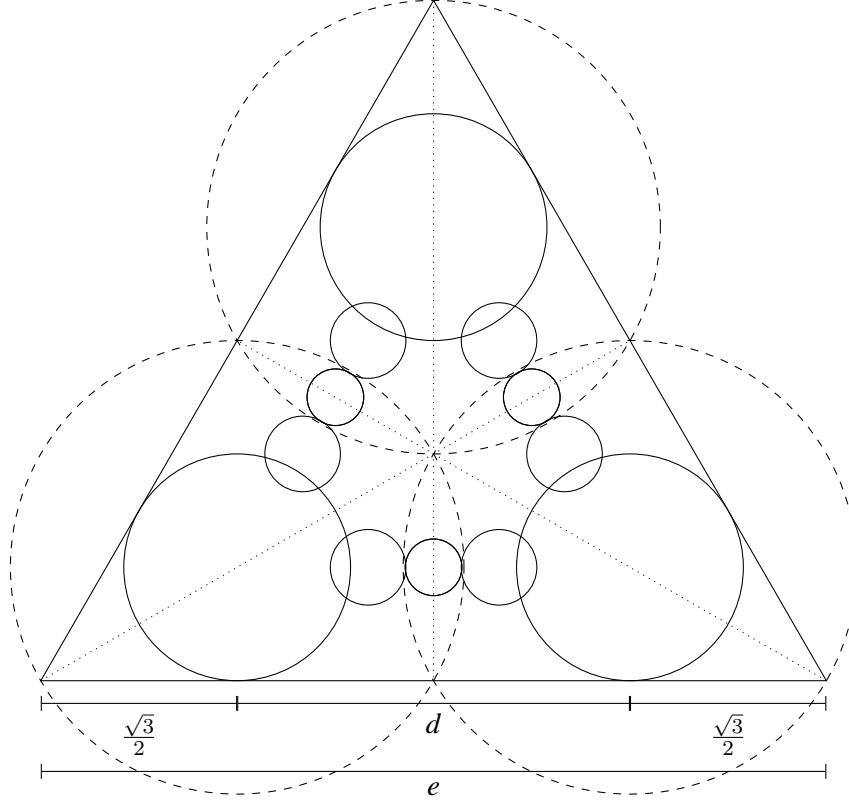

Figure 16: Horoball diagrams of the reflection subgroups generated by Γ_∞ and rsr

As a consequence, the group Γ' is a hyperbolic Coxeter group, and by the same methods as above, one can identify Γ' by means of the Coxeter graph describing a Coxeter polyhedron $P = P(\frac{\pi}{k})$, with an order 2 internal symmetry plane, according to

where k is an integer with $k \geq 3$. For $3 \leq k \leq 6$, the volume of $P(\frac{\pi}{k})$ satisfies $\mathrm{vol}(P(\frac{\pi}{k})) \geq \mathrm{vol}([3^3, 6]) > 2v_*$ by (2–8).

In the case $k = 6$, the fundamental polyhedron P of Γ' has four cusps, and for $k > 7$, there are two ultraideal vertices. We do not have to consider those cases because we can easily estimate the orbifold volume using the cusp volume. If $k = 6$, the distance

Figure 17: Horoball diagram of the reflection subgroup Γ' with $k = 6$

between two full-sized horoballs is $d = \sqrt{3}$ as calculated above in Equation (3–13). For this case and for $d \geq \sqrt{3}$, the side length of D is $e = d + \sqrt{3} \geq 2\sqrt{3}$ (compare Figure 17). The cusp volume can then be bounded by

$$(3\text{–}16) \qquad\qquad \mathrm{vol}(C) = \frac{e^2\sqrt{3}}{48} \geq \frac{\sqrt{3}}{4} > 0.43 > 2\,v_*.$$

Order 3. Suppose that a full-sized disk $B = B_1$ is centred at the singular point $b = a_3$ in the barycenter of the triangle D. Consider a full-sized horoball B_2 whose center is at distance d from b. It follows that the inradius of the cusp triangle D equals $\frac{d}{2}$ and that the edge length of D equals $\sqrt{3}d$. In particular, the cusp volume $\mathrm{vol}(C)$ is given by $\frac{\sqrt{3}d^2}{16}$. In the case that Γ_∞ is orientation-preserving, the assumption $d > 1$ yields the cusp volume estimate

$$\frac{\sqrt{3}d^2}{8} > 0.21 > v_* \,.$$

Hence, we assume that the cusp triangle D has a mirror symmetry. Since $d > 1$, there are three $(\frac{1}{d})$-balls, each of diameter $\frac{1}{d^2}$, touching B in D in a symmetrically arranged way. Denote by x the centre of one of these $(\frac{1}{d})$-balls. The distance $d_0(b,x)$ equals $\frac{1}{d}$.

• If x lies on the edge e_{23} defined by $b = a_3$ and the midpoint a_2 of an edge of D, then the bound
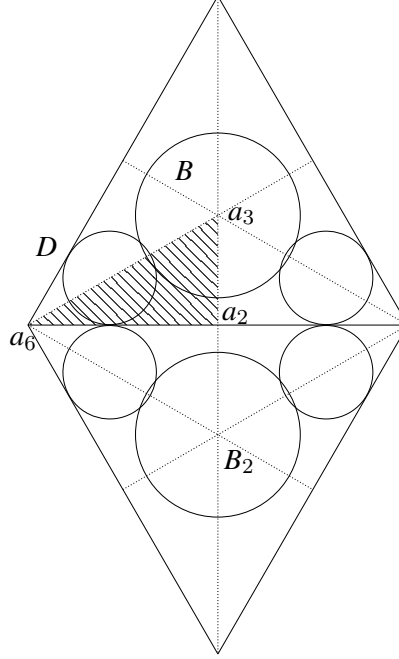
$$\frac{d}{2} = d_0(b,a_2) \geq d_0(b,x) = \frac{1}{d}$$

Figure 18: A $\{2,3,6\}$-cusp triangle D with a full-sized horoball B centred in a_3

yields $d \geq \sqrt{2}$ and, hence, $\mathrm{vol}(C) \geq \frac{\sqrt{3}}{8} \simeq 0.216506$, which is too big in comparison with $v_* = \mathrm{vol}(\mathbb{H}^3/[5,3,6]) \approx 0.171502$ (see (3–1)).

• If $x \notin e_{23}$, then x lies on the edge segment e_{36} of Δ defined by $b = a_3$ and a vertex a_6 of D. Since $d > 1$, the center x is different from a_6. Furthermore, it is easy to see that d and hence $\mathrm{vol}(C)$ take minimal values if the $(\frac{1}{d})$-ball B_x is internally tangent to the border of D (see Figure 18). In this situation, and since the radius of B_x equals $\frac{1}{2d^2}$, we deduce the inequality

$$d \geq \frac{1}{d^2} + \frac{1}{d} \, ,$$

whose solution $d \geq 1.324718$ yields the cusp volume bound

$$\mathrm{vol}(C) \geq 0.189971 > v_* \, .$$

Order 2. Suppose that a full-sized horoball $B = B_1$ is centred at the singular point $b_1 = a_2$ in D, and let B_2, B_3 be full-sized horoballs centred at b_2, b_3 in D, both at distance d from b_1. Then, the edge length of the cusp triangle D equals $2d$ while the distance from b_1 to the centre of D is given by $r = \frac{d}{\sqrt{3}}$ (representing the inradius of D). In particular, we have $\mathrm{vol}(C) = \frac{\sqrt{3}d^2}{12}$, and there are disks of no tangency centred at the vertices of D. Hence, we cannot assume that $w \geq 1$.

However, the case that Γ_∞ is orientation-preserving can be excluded since the corresponding cusp volume for $d > 1$ immediately yields the estimate

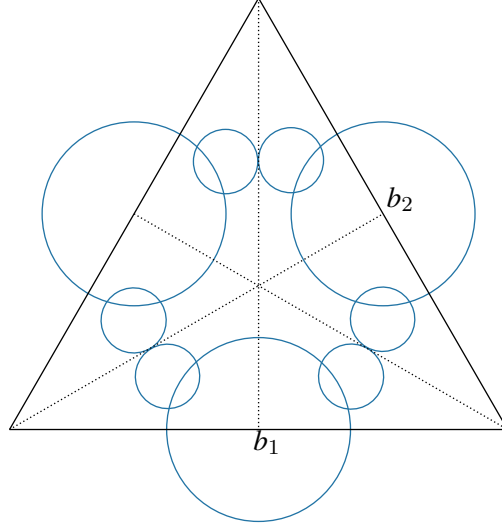$$\mathrm{vol}(C) = \frac{\sqrt{3}d^2}{6} > \frac{\sqrt{3}}{6} > v_* \, .$$

Figure 19: Full-sized horoballs centred at singular points of order 2

Assume that D has a mirror symmetry. We analyse the different tangency possibilities of the $(\frac{1}{d})$-balls with respect to one or several full-sized horoballs and among themselves. To this end, the following fact is very useful.

Fact. The centres of the four $(\frac{1}{d})$-balls touching one full-sized horoball form a rectangle whose diagonals intersect at the angle $\frac{\pi}{3}$.

• If a single $(\frac{1}{d})$-ball touches three full-sized horoballs in D, then it is centred at a_3. It follows that $r = \frac{1}{d}$, and hence, $d = \sqrt[4]{3}$ so that $\mathrm{vol}(C) = \frac{1}{4} > v_*$.

• If a single $(\frac{1}{d})$-ball touches a pair of neighboring full-sized horoballs but does not touch a third full-sized one, then the smallest cusp volume arises if the centres of the $(\frac{1}{d})$-ball and of the full-sized horoballs which it touches are aligned. This follows easily from the above *fact*. We deduce that $d = \sqrt{2}$ and that $\mathrm{vol}(C) = \frac{\sqrt{3}}{6} > v_*$.

• Suppose that each $(\frac{1}{d})$-ball touches a unique full-sized horoball. As in the case when a full-sized horoball B is centred at a singular point a_6 of order 6 in D with distance w from a_6 to the centre x of a $(\frac{1}{d})$-ball not touching it (see Figure 5), we obtain the identical equation (3–7) for w when the full-sized horoball B is centered at a singular point a_2 in D with the condition $0 \le \theta \le \frac{\pi}{3}$ (see Figure 19).

There are again several cases to consider. The case that three $(\frac{1}{d})$-balls are mutually tangent in D can be excluded by Lemma 4. If two $(\frac{1}{d})$-balls are tangent and do not touch the third one in D, then the smallest volume arises if their centres and the centres of the associated full-sized horoballs form an isosceles trapezoid; see Figure 19. In this case, by Ptolemy's theorem, the square of the diagonal of the trapezoid equals

$$w^2 = \frac{1}{d} + \frac{1}{d^2} \ .$$

By means of equation (3–7) with $0 \le \theta \le \frac{\pi}{3}$, we deduce that $d^3 - d - 1 \ge 0$ and $d \ge 1.32471$. Hence, $\mathrm{vol}(C) > \frac{1}{4} > v_*$.
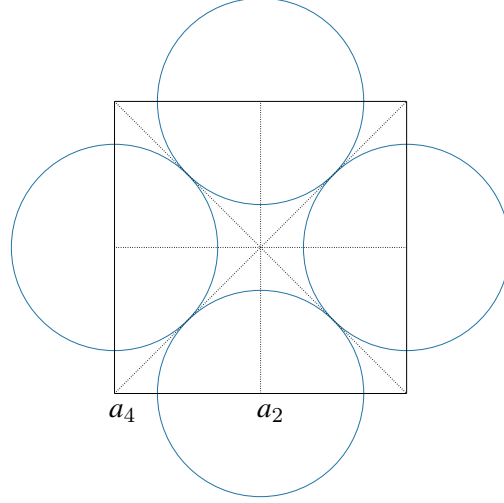
Figure 20: A full-sized horoball centred at a_2

Suppose that the $(\frac{1}{d})$-balls do not touch each other. In particular, the minimum distance μ of their centres is bigger than $\frac{1}{d^2}$. Since D has mirror symmetry, we get a (possibly degenerate) isosceles trapezoid formed by the centres of two closest $(\frac{1}{d})$-balls in D and the centres of the two full-sized horoballs which they touch. By Ptolemy's theorem, we deduce that

$$(3\text{--}17) \qquad\qquad w^2 = \mu d + \frac{1}{d^2} > \frac{1}{d} + \frac{1}{d^2} \,,$$

and hence $\mathrm{vol}(C) > \frac{1}{4} > v_*$ in a similar way as above.

As a summary of all the investigations in Section 3.4.1, we obtain the following result.

Proposition 8 *Let V be a non-arithmetic hyperbolic 3-orbifold with a single cusp $C = B_\infty/\Gamma_\infty$ which is rigid of type $\{2,3,6\}$. Suppose that Γ_∞ gives rise to only one equivalence class of full-sized horoballs. Then,*

$$\mathrm{vol}(V) \geq \mathrm{vol}(V_*) \,,$$

and equality holds if and only if the orbifold V is isometric to $V_* = \mathbb{H}^3/[5,3,6]$.

3.4.2 The case $\{2,4,4\}$

(i) As in Section 3.4.1, (i), we start by assuming that there are at least two full-sized horoballs, which are centred at equivalent singular points (see also (3–4)) and which touch one another in the (square) cusp diagram D. That is, the minimal distance d of the centres of full-sized horoballs equals 1 (see also [3, Section 5, p. 11]).

Order 4. Suppose that there is a full-sized horoball centred at one of the two singular points a_4 of order 4 in Δ. Then, $d_0(a_2, a_4) = \frac{1}{2}$, and the associated cusp volume $\mathrm{vol}(C)$ equals $\frac{1}{16}$. Similarly to the case of Section 3.4.1, one can check that there is a unique orbifold V related to this cusp configuration. In fact, it is the quotient of

$\mathbb{H}^3$ by the *arithmetic* Coxeter group $[3, 4, 4]$ (see also [3, p. 13]). The quotient space $V = \mathbb{H}^3/[3, 4, 4]$ has volume $\frac{\omega_3}{48} = \frac{1}{6}\, \text{JI}(\frac{\pi}{4}) \approx 0.0763304$ (see Section 2.3). By [3, Theorem 5.2 and Theorem 6.1], we know that the space $\mathbb{H}^3/[3, 4, 4]$ has minimal volume among *all* hyperbolic 3-orbifolds with at least one cusp of type $\{2, 4, 4\}$.

Order 2.   If there is a full-sized horoball centred at $a_2 \in \Delta$, then we get $d_0(a_2, a_4) = \frac{1}{\sqrt{2}}$, and the cusp volume equals $\frac{1}{8}$ (see [3, Figure 6(b), p. 12] and Figure 20). Again, there is a unique orbifold corresponding to this configuration. Its fundamental group is given by the Coxeter group $[4^{1,1}, 3]$ and therefore commensurable to the arithmetic Coxeter group $[3, 4, 4]$ (see [18, p. 130]).

(ii)   Suppose now that the full-sized horoballs do *not* touch one another implying that the minimal distance d between the centres a_s (for $s = 2$ or $s = 4$) of full-sized horoballs satisfies $d > 1$.

Notice that if the stabiliser Γ_∞ is orientation-preserving, then the cusp volume equals

$$(3\text{--}18) \qquad\qquad\qquad \text{vol}(C) = \frac{d^2}{2\,s} > \frac{1}{2s}\ .$$

In particular, for $s = 2$, we deduce that the cusp volume yields $\text{vol}(C) > \frac{1}{4} > v_*$. Moreover, the case $s = 4$ can also be excluded as follows. If there is a disk of no tangency, then $d \geq \sqrt{2}$ and $\text{vol}(C) > v_*$. If there is *no* disk of no tangency, then by Lemma 6 we have that the distance w of the centre of a $(\frac{1}{d})$-ball touching B to the centre of a neighboring full-sized horoball satisfies $w \geq 1$. The equation $(3\text{--}7)$ for w remains valid under the adapted constraint $0 \leq \theta \leq \frac{\pi}{4}$ in the square diagram D. The inequality $w \geq 1$ now implies that

$$d^4 - (1 + \sqrt{2})\, d^2 + 1 \geq 0\ ,$$

and hence $\text{vol}(C) = \frac{d^2}{8} > \frac{1.88}{8} > 0.2 > v_*$.

Consequence. We can assume that the cusp diagram D has a mirror symmetry.

As in the case 3.4.1, (ii), we study the $(\frac{1}{d})$-balls and their tangency behavior with respect to the full-sized horoballs centred at a singular point a_s. Observe that for each full-sized horoball B, there are four $(\frac{1}{d})$-balls touching B. The angles formed by their centres measured from the center B is a multiple of $\frac{\pi}{4}$. This leads to the following first cases; see Figure 21.

(a)   Suppose that one $(\frac{1}{d})$-ball B_x is touching four full-sized horoballs so that the center x coincides with a singular point of order $s = 4$. By symmetry, we can suppose that x is the center of the cusp diagram D.

•   Assume first that the centres of the full-sized horoballs lie at the singular points of order 2 in D, that is, D has circumradius d. Since the distance between the center x and the centre of each of the full-sized horoballs equals $\frac{1}{d}$, we deduce that $d = \sqrt[4]{2}$. Hence, for the cusp volume, we get $\text{vol}(C) = \frac{d^2}{8} = \frac{\sqrt{2}}{8} \approx 0.176777$ and therefore $\text{vol}(C) > v_*$ (see $(3\text{--}18)$).

•   Assume next that the centres of the full-sized horoballs lie at the vertices (of order 4) of D. Hence, D has circumradius $\frac{1}{d}$. We deduce that $d = \sqrt[4]{2}$ as well. However, for the cusp volume, we obtain $\text{vol}(C) = \frac{\sqrt{2}}{16}$. As shown by [3, p. 13] in the oriented
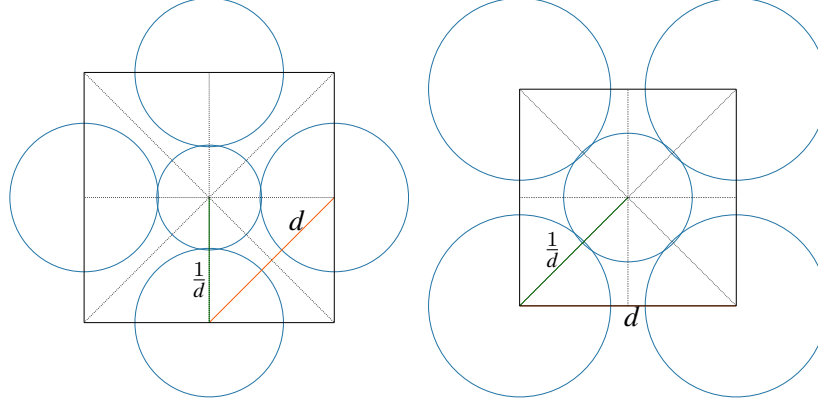
Figure 21: $(\frac{1}{d})$-balls touching four full-sized balls

case, there is a unique orbifold corresponding to this configuration. It is the arithmetic quotient of an ideal Coxeter tetrahedron with dihedral angles $\frac{\pi}{4}$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$ by its orientation-preserving symmetry group. Hence, in the non-oriented case, a hyperbolic orbifold with cusp volume $\frac{\sqrt{2}}{16}$ would have a fundamental group commensurable to the arithmetic Coxeter group $[4^{[4]}]$ and therefore can be excluded from our consideration.

(b)   Suppose that one $(\frac{1}{d})$-ball B_x is touching exactly two full-sized horoballs.

•   If these full-sized ones are centred at singular points of order 2 in D, then the circumradius of D equals d. Since the center x is aligned with the centres of the full-sized horoballs which B_x touches, we deduce that $d = \sqrt{2}$ and $\mathrm{vol}(C) = \frac{d^2}{8} = \frac{1}{4} > v_*$.

•   If the two full-sized horoballs are centred at vertices (of order 4) of D, then again $d = \sqrt{2}$. It follows that $\mathrm{vol}(C) = \frac{1}{8}$. Observe that there is a disk of no tangency based at the centre of D and touching the full-sized horoballs. Given this, the only possible such configuration, and which leads to a singly cusped orbifold, arises by halving the Coxeter orthoscheme $[4,4,4]$ with vertices q, p_1, p_2, p_3 by means of the plane P passing through the edge $p_2 p_3$ and orthogonal to the (doubly infinite) edge $q p_4$ (see Figure 1). However, a simple computation shows that the horoball sector $B_\infty \cap [4,4,4]$ associated to q, has non-empty intersection with P in contradiction to the maximality of the cusp. Hence, this configuration can not be realised.

(c) Suppose that a $(\frac{1}{d})$-ball is touching only one full-sized horoball B.
Observe that, and similarly to the distance w, the distance v of the centre of B to the centre of its neighboring $(\frac{1}{d})$-balls as given by (3–6) remains valid under the constraint $0 \le \theta \le \frac{\pi}{4}$ (the distance u between the centres of two $(\frac{1}{d})$-balls needs a slight modification, though).

•   Assume that B is centred at a singular point of order 2 in D. As in the case of the cusp of type $\{2,3,6\}$, and by symmetry, we get a (possibly degenerate) isosceles trapezoid formed by the centres of two neighboring $(\frac{1}{d})$-balls at distance $\mu \ge \frac{1}{d^2}$ and by the centres of the full-sized horoballs which they touch. We obtain the identity (3–17) which, combined with the equation (3–7) for w^2 and the condition $0 \le \theta \le \frac{\pi}{4}$, yields $d^3 - \sqrt{2}d - 1 \ge 0$ and the estimate $d \ge 1.450405 > \sqrt{2}$. As a consequence, the volume of the cusp C satisfies $\mathrm{vol}(C) = \frac{d^2}{8} > \frac{1}{4} > v_*$.

• More delicate is the case when B is centred at a singular point of order 4. By symmetry, we can suppose that the centre of B coincides with a vertex of D. For the cusp volume, we obtain $\mathrm{vol}(C) = \frac{d^2}{16}$.

As in [4, Section 5], we consider the mutual positions of the $(\frac{1}{d})$-balls associated to the full-sized horoballs. The case that four $(\frac{1}{d})$-balls are closest to one another, that is, each one is tangent to two other ones and they all are symmetrically arranged around the centre of D, can be excluded by applying verbatim Adams' corresponding argument.

Suppose next that two $(\frac{1}{d})$-balls are tangent, and no other $(\frac{1}{d})$-ball is tangent to them. Since D has a mirror symmetry, this can only happen if they are tangent at a singular point a_2 of order 2 with centres being aligned with the centres of the corresponding full-sized horoballs. We deduce that $d = \frac{2}{d} + \frac{1}{d^2}$ and hence $d = 2\cos\frac{\pi}{5}$. It follows that

$$\frac{\mathrm{vol}(C)}{d_3(\infty)} > 0.19 > v_* \,.$$

It remains to investigate the case when the $(\frac{1}{d})$-balls do not touch. As in the analogous case of a cusp of type $\{2,3,6\}$, we consider the $(\frac{1}{w})$-balls with their position relative to the $(\frac{1}{d})$-balls. Assume first that four $(\frac{1}{w})$-balls coincide and, hence, have center equal to the centre of the square D. Then, the circumradius of D equals $\frac{1}{w}$ so that $w = \frac{\sqrt{2}}{d}$. By means of (3–7), one can compute that $d = \sqrt[4]{5}$ and $\cos\theta = \frac{2}{\sqrt{5}}$; see Figure 31. Hence, the cusp diagram D has no mirror symmetry, and the corresponding cusp volume yields $\mathrm{vol}(C) = \frac{\sqrt{5}}{8} > \frac{1}{4} > v_*$. By performing analogous computations as in the case of a type $\{2,3,6\}$-cusp, we can show furthermore that there is a unique corresponding oriented orbifold that is arithmetic. For more details, see Appendix B.

The case that two of the $(\frac{1}{w})$-balls coincide in the interior of D can be excluded by symmetry.

Hence, it remains to investigate the situation when all $(\frac{1}{w})$-balls are distinct. In order to finish this case, it is sufficient to check the following two extremal possibilities for a $(\frac{1}{w})$-ball touching two $(\frac{1}{d})$-balls associated to two different full-sized balls.

• Suppose first that a $(\frac{1}{w})$-ball touches two $(\frac{1}{d})$-balls in such a way that their centres are aligned on an edge of D. Then, $\frac{1}{w} = \frac{d}{2}$ which by means of (3–8) and $\theta = 0$ implies that $d^4 - 2d^2 - 3 = 0$ with solution $d = \sqrt{3}$. Hence, $\mathrm{vol}(C) = \frac{3}{16} > v_*$.

• Suppose next that a $(\frac{1}{w})$-ball touches two $(\frac{1}{d})$-balls whose centres lie on the diagonals of D. Notice that there are eight $(\frac{1}{w})$-balls around each full-sized horoball. Then, one can show in a similar way as in the case of a cusp of type $\{2,3,6\}$ that $d = \sqrt{1+\sqrt{2}}$. It follows that $\frac{\mathrm{vol}(C)}{d_3(\infty)} = \frac{1+\sqrt{2}}{16\,d_3(\infty)} > 0.176 > v_*$.

Furthermore, we can describe the associated 1-cusped orbifold in an analogous way as in the case of a cusp of type $\{2,3,6\}$ and the explanations referring to Figure 13. The orbifold has a non-arithmetic fundamental group and is built upon a polyhedron arising from halving the Coxeter tetrahedron with cyclic Coxeter symbol $[(4^3,3)]$ and Coxeter graph with internal symmetry as given by
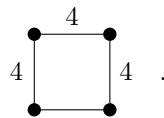
As a result, and by [17, p. 348], the volume of the orbifold is about 0.27814 and hence too large.

Summarising the investigations in Section 3.4.2, we can state the following result.

Proposition 9 *Let V be a non-arithmetic hyperbolic 3-orbifold with a single cusp $C = B_\infty/\Gamma_\infty$ which is rigid of type $\{2,4,4\}$. Suppose that Γ_∞ gives rise to only one equivalence class of full-sized horoballs. Then,*

$$\mathrm{vol}(V) > \mathrm{vol}(V_*)\,.$$

3.5 More than one equivalence class of full-sized horoballs

Assume that the crystallographic group Γ_∞ gives rise to more than one equivalence class of full-sized horoballs. We show that the corresponding orbifold has volume strictly bigger than $v_* = \mathrm{vol}(\mathbb{H}^3/[5,3,6])$ (if it exists) by treating the cases of a cusp C of type $\{2,3,6\}$ or of type $\{2,4,4\}$ separately. For simplicity, we assume that the group Γ is *orientation-preserving* by passing to its orientation-preserving subgroup of index 2 if necessary.

3.5.1 The case $\{2,3,6\}$

As in the case of one equivalence class, inequivalent horoballs have to be centred in the singular points a_s, $s \in \{2,3,6\}$ of the cusp diagram D because otherwise already the cusp volume becomes too big (see Section 3.4 and (3–4)).

Suppose first that there are three equivalence classes of full-sized horoballs with respect to the action of Γ_∞. The smallest volume then arises if $d_0(a_2, a_3) = 1$. It follows that the (oriented) cusp volume satisfies $\mathrm{vol}(C) = \frac{\sqrt{3}}{2} > 2v_*$, and that the orbifold volume is too big.

Consider the case of exactly two equivalence classes of full-sized horoballs. Continue to denote by τ the shortest translation length in Γ_∞ and by d the shortest distance between two equivalent full-sized horoballs. In the case where the full-sized horoballs are centred in singular points of order 2 and 3, we get $\tau \geq 2$ which leads to a cusp volume of at least $\mathrm{vol}(C) \geq \frac{\sqrt{3}}{2} > 2v_*$.

Using the fact that the full-sized horoballs are centred in singular points a_s, we can deduce the following slightly generalised version of Adams' Lemma 5 in the case of a (oriented) cusp C of type $\{2,3,6\}$.

Lemma 8 *Suppose that the full-sized horoballs of a $\{2,3,6\}$-cusp are centred in singular points a_s in the cusp diagram D. Then, there is a rotation of order 2 in Γ whose axis is tangent to a full-sized horoball and the horoball B_∞.*

Proof Take a full-sized horoball centred at a singular point $a_s = (u, 1)$ where $u \in \mathbb{R}^2$ belongs to the boundary ∂U^3. Since the group Γ acts transitively on the set of horoballs covering C, there is an isometry $\gamma \in \Gamma$ mapping u to ∞. Since the axis l_s formed by u and ∞ is the rotational axis of an isometry in Γ_∞, the image geodesic $\gamma(l_s)$ is also the axis of a rotation of the same order. Thus, by composing γ with a suitable isometry in Γ_∞, we can assume that γ maps ∞ back to u. It follows that the element γ has to be a rotation of order 2 in Γ whose axis is tangent to the full-sized horoball and B_∞. $\square$

There are two cases to consider for a possible placement of the centres of the full-sized horoballs.

Centres at a_6 and a_3. Assume that the full-sized horoballs are centred in singular points a_s of order 6 and 3, respectively. Denote by $e = \frac{\tau}{\sqrt{3}}$ the distance between a_6 and a_3 in the cusp diagram D.

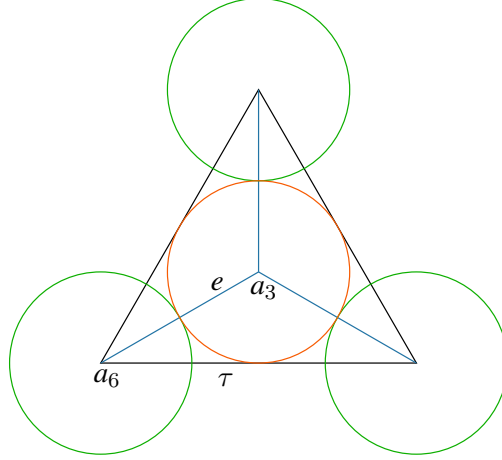In the minimal case, $e = 1$; see Figure 22. By Lemma 8, we have the existence of
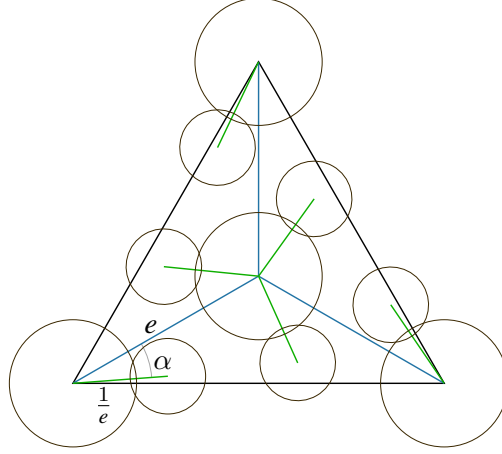

Figure 22: Inequivalent full-sized horoballs touch one another

rotations of order 2 which exchange the full-sized horoballs centred at $a_6 = (u_6, 1)$ and at $a_3 = (u_3, 1)$ with the horoball B_∞. It is not difficult to see that the rotation which sends ∞ to u_3 and fixes u_6, together with the rotations in the singular axes of order 2, give a face identification of the ideal regular tetrahedron S_{reg}^∞ with ideal vertices defined by the 3 vertices of the regular triangle D and ∞. One can also see that the above rotations are generated by reflections in the sides of the characteristic orthoscheme $[3, 3, 6]$ associated to S_{reg}^∞. That implies that these rotations generate an arithmetic subgroup of finite index in Γ so that Γ itself is arithmetic. The rotations around the singular axis of order 3 and the rotation of order 2 mapping u_6 to ∞ generate 6 isometries of S_{reg}^∞. As a by-product, we can calculate the volume by means of (2–5) and obtain

$$\text{vol}\left(\mathbb{H}^3/\Gamma\right) = \frac{1}{6}\,\text{vol}\left(S_{\text{reg}}^\infty\right) = \frac{1}{2}\,\text{JI}\left(\frac{\pi}{3}\right).$$

In the case $e > 1$, there is a similar notion to a $(\frac{1}{d})$-ball in the case of one equivalence class. More precisely, the rotations of Lemma 8 map full-sized horoballs at distance e from a full-sized horoball to horoballs of diameter $\frac{1}{e^2}$ at distance $\frac{1}{e}$ from the full-sized image of B_∞. Accordingly, we call these balls $(\frac{1}{e})$-*balls*. The general situation is depicted in Figure 23.

The minimal distance e is obtained by decreasing e to the point where the $(\frac{1}{e})$-balls touch simultaneously both inequivalent full-sized horoballs; see Figure 24. For a given angle $\alpha \in [0, \frac{\pi}{6}]$ as defined in Figure 23, one can calculate the minimal possible e by using the isosceles triangle with base e, opposite angle α and sides $\frac{1}{e}$. This implies

Figure 23: Inequivalent full-sized horoballs at distance $e > 1$

that $\cos\alpha = \frac{e^2}{2}$ and that

$$(3\text{--}19) \qquad e = \sqrt{2\cos\alpha} \geq \sqrt{2\cos\frac{\pi}{6}} = \sqrt[4]{3}.$$
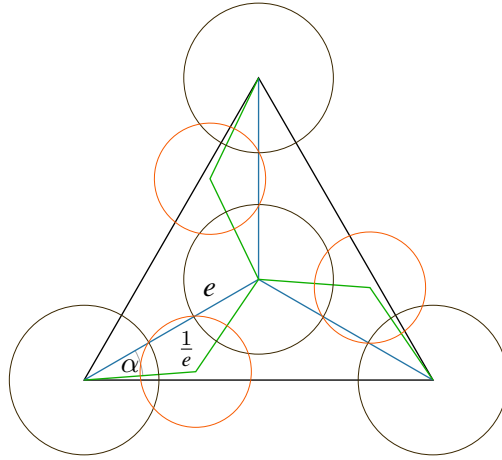
Figure 24: A $(\frac{1}{e})$-ball touching two inequivalent full-sized horoballs

The lower bound on e yields a lower bound for the (oriented) cusp volume. It follows that

$$\mathrm{vol}(C) = \frac{\tau e}{8} = \frac{\sqrt{3}e^2}{8} \geq \frac{3}{8} > 2v_* \,,$$

which is enough to exclude this case this from our considerations.

Centres at a_6 and a_2. Assume that the full-sized horoballs are centred in singular points a_6 and a_2, of order 6 and 2, respectively. Denote by $e \geq 1$ the shortest distance between a_6 and a_2 in the diagram D. The next result shows that $e > 1$.

Proposition 10 *Suppose that a 1-cusped oriented hyperbolic 3-orbifold has a cusp of type* $\{2,3,6\}$ *with precisely two equivalence classes of full-sized horoballs centred in the singular points* a_6 *and* a_2 *of the horoball diagram* D. *Then,* $d_0(a_6,a_2) > 1$, *and the full-sized horoballs cannot touch each other.*

Proof On the contrary, assume there is such an orbifold with touching full-sized horoballs giving rise to a horoball configuration as in Figure 25. By Lemma 8, there is a rotation of order 2 exchanging the horoball B_1 centred in the singular point $b_1 = (u_1, 1)$ of order 6 and the horoball B_∞ at infinity. This rotation sends the touching full-sized horoballs around B_1 onto each other. After possibly using a rotation around the axis $l_1 = (u_1, \infty)$, we can assume that there is such a rotation r exchanging B_1 and B_∞ while fixing B_2 and B_2' centred in $b_2 = (u_2, 1)$ and $b_2' = (u_2', 1)$, respectively. This implies that the image geodesic (u_1, u_2) of the order 2 axis $l_2 = (u_2, \infty)$ is also a rotation axis of order 2.

Using Lemma 8 again, there has to be a rotation r' around a common tangent of B_2 and B_∞. Denote by s the rotation of order 2 around the axis l_2. Then sr' is also a rotation through a common tangent of B_2 and B_∞. The axis of sr' is perpendicular to the rotation axis of r'. Both rotations r' and sr' have to map full-sized horoballs touching B_2 to full-sized horoballs touching B_2. Hence one of these rotations has to fix two of the touching full-sized horoballs. (If r' does not fix two full-sized horoballs touching B_2, then it has to map two neighbouring full-sized horoballs touching B_2 onto each other. The rotational axis of r' then has to pass between those horoballs. Since the rotational axis of sr' is orthogonal, it has to pass through the centres of two full-sized horoballs touching B_2.) Due to symmetry it is enough to consider the cases where B_1 or B_3 is fixed. Assuming that r' fixes B_1, then the axis (u_1, u_2) has to have order 6 because of its image (u_1, ∞). However, we already noted that it has order 2. Assuming that r' fixes B_3 centred at $b_3 = (u_3, 1)$, then the axis (u_2, u_3) has to have order 2 – the same as its image (u_3, ∞). The geodesic (u_2, u_4) related to b_2 and $b_4 = (u_4, 1)$, has to have order 6 as its image (u_1, ∞). However, they have to have the same order as they get mapped to each other by rotations around the singular point a_3 representing the centre of the cusp diagram D. $\qquad\square$

Proposition 10 implies that the distance between two inequivalent full-sized horoballs satisfies $e > 1$. Again, consider the associated $(\frac{1}{e})$-balls. The minimal distance e is achieved if a $(\frac{1}{e})$-ball lies on the angle bisector in D and touches three full-sized horoballs as in Figure 26. This gives the lower bound $e \geq \sqrt[4]{3}$. As a consequence, the (oriented) cusp volume yields the estimate

$$\mathrm{vol}(C) = \frac{e^2}{2\sqrt{3}} \geq \frac{1}{2} > 2v_* \,,$$

so that the corresponding orbifold volume becomes too big.

3.5.2 The case $\{2,4,4\}$

Assume that there is a $\{2,4,4\}$-cusp with at least two equivalence classes of full-sized horoballs. As in Section 3.5.1, we can exclude the case of three equivalence classes,
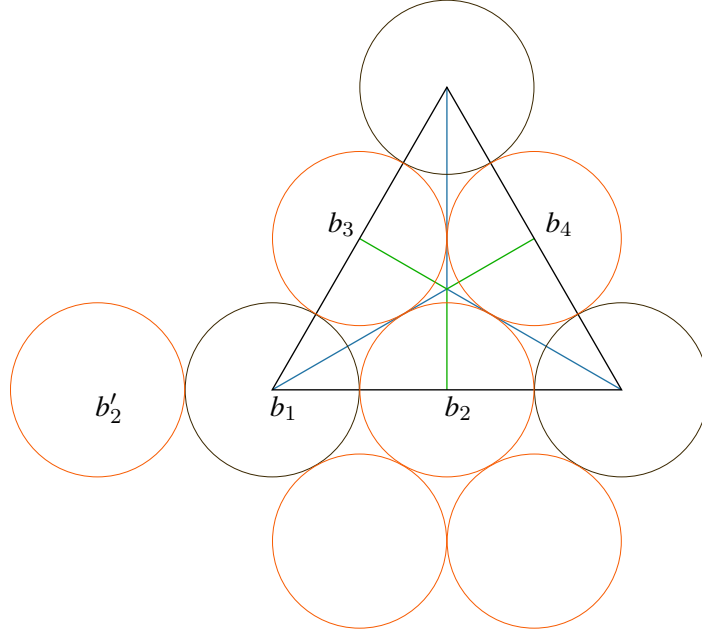
Figure 25: Two equivalence classes of full-sized horoballs centred in a_6 and a_2

since the smallest volume would arise if all the full-sized horoballs were centred in the singular points yielding a (oriented) cusp volume $\mathrm{vol}(C) = \frac{1}{2} > 2v_*$.

Furthermore, given two equivalence classes of full-sized horoballs, these horoballs have to be centred in the singular points because otherwise the cusp volume would be too big. If there are full-sized horoballs in 4- and 2-fold singular points, the shortest translation length satisfies $\tau \geq 2$ which gives a cusp volume $\mathrm{vol}(C) = \frac{\tau^2}{8} \geq \frac{1}{2} > 2v_*$. Thus we only have to consider the case where the full-sized horoballs are centred in the 4-fold singularities.

However, for a cusp of type $\{2, 4, 4\}$, there is no equivalent of Lemma 8 since there are non-conjugate rotations of the same order in Γ_∞. This is the reason why there are two cases to distinguish if one assumes that the full-sized horoballs, representing different equivalence classes, touch each other. We analyse this situation first.

• Suppose that $e = 1$. There is an (orientation preserving) isometry $r \in \Gamma$ mapping the full-sized horoball B_1 centred at $b_1 = (u_1, 1)$ to B_∞. After possibly using an isometry in Γ_∞, we can assume that r maps B_∞ to B_1 or to B_2 where B_2 is an inequivalent full-sized horoball centred in $b_2 = (u_2, 1)$ in D, respectively; see Figure 27.

Case 1. If r sends ∞ to u_1, then r is a rotation of order 2 and has to map the 4 touching full-sized horoballs onto each other. There are 4 such rotations, and any one of them helps us. In fact, the order 2 rotations around the midpoints a_2 of the sides of the square diagram D with side lengths τ and their conjugates by r give a side pairing of the ideal regular octahedron O_{reg}^∞ formed by the ideal points defined by the 4 vertices and the in-centre of the square diagram D and ∞. Since this side pairing generates a finite index subgroup of Γ as well as a finite index subgroup of the arithmetic reflection group $[4, 4, 4]$, they are all commensurable. In particular, Γ has to be arithmetic.
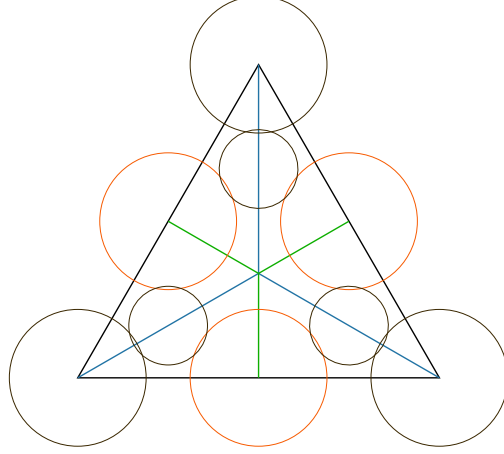
Figure 26: A $(\frac{1}{e})$-ball centred on the angle bisector in D

Case 2.   If r sends ∞ to u_2, we can assume that $r(u_2) = u_1$. That implies that r is a rotation of order 3, which permutes ∞, u_1 and u_2. It also permutes the centres b_1', b_2' of neighbouring full-sized horoballs and a $(\frac{1}{e})$-ball touching all of them as depicted in Figure 27. Denote by s the rotation of order 2 around the vertical axis defined by the centre of this $(\frac{1}{e})$-ball in $\mathbb{H}^3$. Then, rsr^{-1} is a rotation of order 2 exchanging B_∞ and the $(\frac{1}{e})$-ball. It follows that the two rotations of order 4 around the axes associated to b_1 and b_1', together with their conjugates by rsr^{-1}, give a side-pairing of the ideal regular octahedron O_{reg}^∞. With the same reasoning as above, Γ turns out to be arithmetic.

•   Suppose that $e > 1$. We can use the ideas from the proofs of Lemma 8 and Adams' Lemma 5 as follows. A full-sized horoball B_u centred at $u \in \partial \mathbb{H}^3 \setminus \{\infty\}$ can be mapped to B_∞, and B_∞ gets mapped to a horoball B_v centred at v. By hypothesis, there is a full-sized horoball at distance e from B_u which gets mapped to a horoball touching B_v and of diameter $\frac{1}{e^2}$. This is the equivalent of the $(\frac{1}{e})$-ball as above.

The minimal possible distance e is obtained if the $(\frac{1}{e})$-ball touches inequivalent full-sized horoballs as illustrated in Figure 28. We can perform the same calculation which led to equation (3–19) by using the isosceles triangle with angle α satisfying $\alpha \in [0, \frac{\pi}{4}]$. We obtain the lower bound according to

$$e = \sqrt{2\cos\alpha} \ge \sqrt{2\cos\frac{\pi}{4}} = \sqrt[4]{2}\,.$$

Hence, the (oriented) cusp volume yields the estimate

$$\mathrm{vol}(C) = \frac{e^2}{4} \ge \frac{\sqrt{2}}{4} > 0.35 > 2v_*.$$

As a summary of the investigations in Sections 3.5.1 and 3.5.2, we can formulate the following result.

Proposition 11 *A non-arithmetic cusped hyperbolic* 3*-orbifold* $\mathbb{H}^3/\Gamma$ *of minimal volume with cusp of the form* $C = B_\infty/\Gamma_\infty$ *has the property that* Γ_∞ *permutes all full-sized horoballs.*
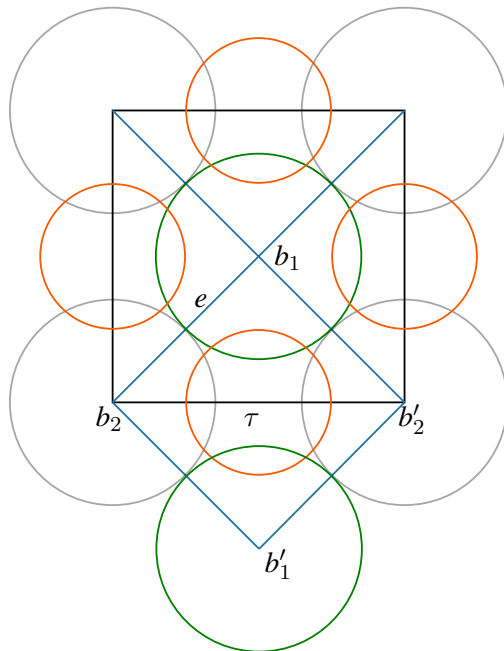
Figure 27: Two equivalence classes of full-sized horoballs in the diagram of a cusp of type $\{2,4,4\}$

In combination with Proposition 1, Proposition 2, Proposition 8 and Proposition 9, Proposition 11 is the final step in the proof of the Theorem as stated in Section 1 and in Section 3.

4   Final remarks and a conjecture

The theory and investigations exploited in this work are valid for cusped hyperbolic n-orbifolds of dimensions $n \geq 2$. In fact, for fixed dimension n with $2 \leq n \leq 9$, the cusped hyperbolic n-orbifold of minimal volume is known and intimately related to an arithmetic Coxeter simplex group (see [16] and [14,15]). In comparison to this, it is much more difficult to identify the *non-arithmetic* cusped hyperbolic n-orbifolds of minimal volume for $n \geq 4$ by providing a presentation of their fundamental groups. In view of the Gromov–Piatetski-Shapiro construction, observe that non-arithmetic non-cocompact hyperbolic Coxeter simplex groups do not exist anymore for $n \geq 4$.

However, for dimension $n = 2$, one can derive in a fairly easy way the following result mentioned in Section 1 (for details, see [9, Chapter V]).

Proposition 12 *Among all non-arithmetic cusped hyperbolic 2-orbifolds, the 1-cusped quotient space of $\mathbb{H}^2$ by the triangle Coxeter group $[5, \infty]$ has minimal area. As such the orbifold is unique, and its area is given by $\frac{3\pi}{10}$.*

By Selberg's Lemma, any hyperbolic orbifold has a finite sheeted (smooth) cover manifold. In the following, we are interested in cusped hyperbolic 3-manifolds of small
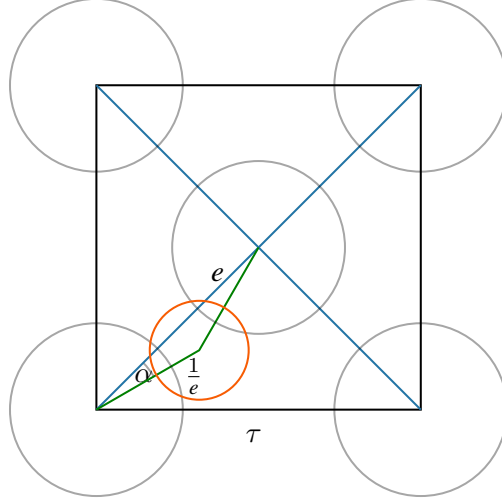
Figure 28: Two inequivalent full-sized horoballs at distance e with $(\frac{1}{e})$-ball in the diagram of a cusp of type $\{2,4,4\}$

volume. These manifolds form a large family of spaces. For a list of smallest ones, see [12] and [22, Section 13.6]. Here, we focus on the subclass of *non-arithmetic Coxeter manifolds*, that is, manifolds whose fundamental groups are commensurable with non-arithmetic hyperbolic Coxeter groups. Consider the 1-cusped orbifold $V_* = \mathbb{H}^3/[5,3,6]$ of smallest volume among all non-arithmetic cusped hyperbolic 3-orbifolds. The minimal index of a torsion-free subgroup of $[5,3,6]$ is given by the l.c.m. of the orders of finite subgroups of $[5,3,6]$ and equals 120. In [10], it is shown that there are 10 such subgroups which are non-conjugate in $[5,3,6]$, yielding non-isometric non-arithmetic hyperbolic Coxeter 3-manifolds, all orientable with one or two cusps, covering the orbifold V_*. Each of these manifolds is of volume $120 \cdot \mathrm{vol}([5,3,6]) \approx 20.580199$.

Compare V_* with the non-arithmetic 2-cusped orbifold $V_\circ$ of volume

$$\frac{5}{8}\,\mathrm{JI}(\frac{\pi}{3}) + \frac{1}{3}\,\mathrm{JI}(\frac{\pi}{4}) \approx 0.364107$$

whose fundamental group is given by the tetrahedral Coxeter group $[(3^3,6)]$ (see Example 2.2 and (2–8)). There is a torsion-free subgroup Λ of $[(3^3,6)]$ of minimal possible index equal to 24 so that the resulting Coxeter manifold $M = \mathbb{H}^3/\Lambda$ is of volume ≈ 8.738570. Moreover, the manifold M is non-orientable and multiply-cusped. Since the groups $[5,3,6]$ and $[(3^3,6)]$ are incommensurable (see Remark 1), the manifold M does not have a common covering manifold with V_*. As a consequence, one gets the following result stated in Section 1 (for details, see [9, Chapter IV]).

Proposition 13 *The fundamental group of a non-arithmetic cusped hyperbolic Coxeter 3-manifold M_* of minimal volume is incommensurable to the Coxeter group $[5,3,6]$; the volume of M_* is smaller than or equal to $24 \cdot \mathrm{vol}([(3^3,6)]) \approx 8.738570$.*

Let us add that there are precisely four conjugacy classes of torsion-free subgroups of index 24 in $[(3^3,6)]$. The four related non-isometric quotient manifolds $M_1,\ldots,M_4$ are non-orientable and have 2, 2, 3 and 4 cusps. Inspired by these results, we propose the following conjecture.
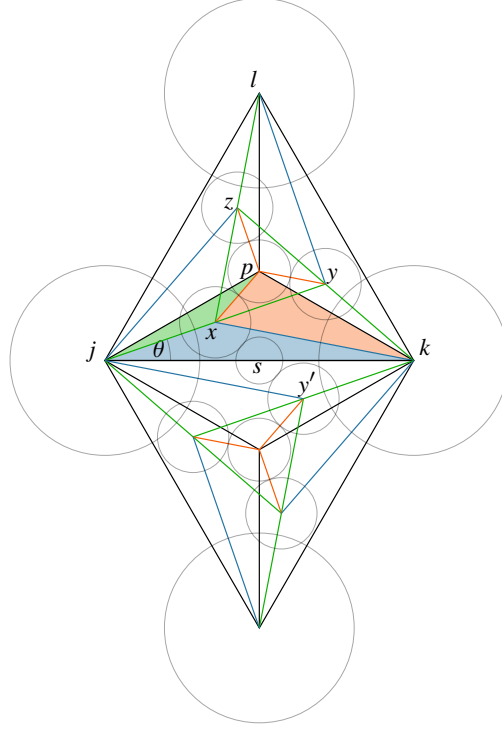
Figure 29: A single $(\frac{1}{w})$-ball in a $\{2,3,6\}$-cusp triangle D

Conjecture *The four manifolds $M_1, \ldots, M_4$ are of minimal volume among all non-arithmetic cusped hyperbolic Coxeter 3-manifolds.*

For more details and the proof of Proposition 13, we refer to [9, Chapter IV].

Appendix

A   The case $\{2,3,6\}$

Consider a discrete group $\Gamma \subset \mathrm{Isom}^+ \mathbb{H}^3$ of orientation preserving isometries whose quotient space O has exactly one cusp of the form $C = B_\infty / \Gamma_\infty$ giving rise to full-sized horoballs of diameter 1 in the upper half space U^3.

Suppose that C is of type $\{2,3,6\}$ and that the group Γ_∞ induces only one equivalence class of full-sized horoballs, all centred at singular points of order 6 in the cusp diagram D.

In the sequel, we consider the special case when the full-sized horoballs are at distance $d > 1$ from one another and when the associated $(\frac{1}{d})$-balls do not touch one another. In particular, these $(\frac{1}{d})$-balls give rise to $(\frac{1}{w})$-balls. Suppose that the $(\frac{1}{w})$-balls of the three $(\frac{1}{d})$-balls in D coincide with center at the singular point $p = a_3$ of order 3 in D; see Figure 29. As explained in Section 3.4.1, one obtains $w = \frac{\sqrt{3}}{d}$, $\cos\theta = \frac{5}{2\sqrt{7}}$ and
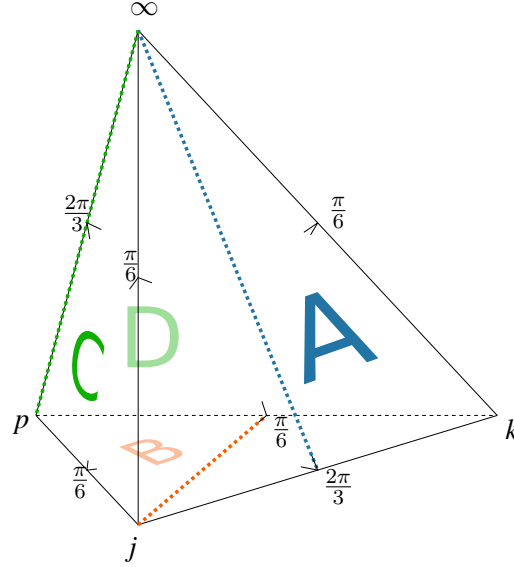
Figure 30: Side pairings of *P* for a subgroup of Γ

$d = \sqrt[4]{7}$. In particular, $\mathrm{vol}(C) = \frac{\sqrt{21}}{24}$.

In the following, we correct Adams' determination of a fundamental polyhedron for Γ and the volume computation for *O* as given in [3, p. 10]. Furthermore, we show that *O* is arithmetic (see Remark 4 in Section 3.4.1).

Consider the full-sized horoballs B_j, B_k and B_l, as well as the $(\frac{1}{d})$-balls centred at *x*, *y* and *z* as depicted in Figure 29. Identify all horoball centres at height $t = 1$ in *D* with the corresponding points in the plane $\{t = 0\}$. Recall that the centres *j*, *x*, and *y* are located on one line. Denote by *r* the order 2 rotation which interchanges the horoballs B_∞ and B_j as well as B_k and B_x.

The $(\frac{1}{w})$-ball B_s is actually the image of B_y under this rotation *r* and centred in a_2. It is fixed by a rotation $s \in \Gamma_\infty$ of order 2 around a singular point a_2.
The image of the axis connecting B_∞ and B_p under the rotation *r* is the geodesic connecting B_j and B_y. Hence the conjugate rotation $s' := rsr$ swaps B_x and B_∞ while fixing B_j and B_y.

It is easy enough to find a fundamental polyhedron for a subgroup of Γ described by the cusp diagram. Consider the ideal tetrahedron with vertices ∞, *j*, *k*, and *p*. Each of its ideal vertex triangles has angles $\frac{2\pi}{3}$, $\frac{\pi}{6}$ and $\frac{\pi}{6}$. Denote by ρ the clockwise rotation of order 3 around the singular axis connecting the centres of B_∞ and B_p. The symmetries *s*, s', and ρ give face identifications which are visualised in Figure 30. The faces *A* and *B* are identified with themselves by the appropriate rotation around the given axes. The face *C* gets mapped to *D* by ρ. This gives a *subproper side pairing* according to [26, Section 13.4] implying that *P* is a fundamental polyhedron for a subgroup of finite index in Γ. Denote this subgroup by Θ.

From this we can deduce the arithmeticity of Γ as follows. A representation of Θ in

$\mathrm{PSL}_2(\mathbb{C})$ is given by

$$s \mapsto \begin{pmatrix} i & -i \\ 0 & -i \end{pmatrix}, \quad \rho \mapsto \begin{pmatrix} -\omega & \omega^2 \\ 0 & \omega^2 \end{pmatrix}, \quad s' \mapsto \begin{pmatrix} i & 0 \\ i\,d^2\,\mathrm{e}^{-i\beta} & -i \end{pmatrix} = \begin{pmatrix} i & 0 \\ -\frac{\sqrt{3}}{2}+i\frac{5}{2} & -i \end{pmatrix},$$

where $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a primitive sixth root of unity. Note that this representation is based on the normalisation that the shortest translation length is one. Full-sized horoballs have then diameter equal to $1/\sqrt[4]{7}$. The trace field is $k\Gamma = \mathbb{Q}(\omega)$. This can be seen by utilising [22, Theorem 3.5.9] and by taking ρ^{-1}, $\rho^{-1}s$, and $s\,rsr$ as generators for the subgroup Θ. By means of [22, Theorem 8.3.2], it follows that Γ is arithmetic, and the result [22, Theorem 8.4.1] in conjunction with [22, Theorem 3.3.8] allows us to deduce that Γ is even commensurable to the $[3,3,6]$.

In order to obtain Γ, one needs to add r as a generator to Θ. Replacing the generator ρ by $s\rho$ (a rotation around the axis connecting ∞ and j) makes it easy to show that $\Gamma = \Theta \rtimes \langle r \rangle$ is a semidirect product. The subgroup Θ has index 2 in Γ, so half of P is a fundamental polyhedron for Γ. This allows us to calculate the volume of the fundamental polyhedron P and of the orbifold O as follows (see Section 2.3).

$$\mathrm{vol}(O) = \mathrm{vol}(P) = \frac{1}{2}\left\{ \text{Ћ}\left(\frac{2\pi}{3}\right) + 2\,\text{Ћ}\left(\frac{\pi}{6}\right) \right\} = \text{Ћ}\left(\frac{\pi}{3}\right) \approx 0.338314\,.$$

Remark 5 *Based on Adams' original explanation [3, p. 10], there is a dissection of the polyhedron P and its image under rsr into three ideal tetrahedra $T(\infty,j,k,x)$, $T(\infty,j,p,x)$, and $T(\infty,k,p,x)$. They are represented by the coloured triangles in Figure 29. The blue and green tetrahedra have angles $\theta, \frac{\pi}{6} - \theta$ and $\frac{5\pi}{6}$, where $\cos\theta = \frac{5}{2\sqrt{7}}$ as above. The red tetrahedron has angles θ, $\frac{\pi}{3} - \theta$, and $\frac{2\pi}{3}$. This is useful in order to obtain the following new identity*[1] *for the Lobachevsky function evaluated at $\frac{\pi}{3}$.*

$$\text{Ћ}\left(\frac{\pi}{3}\right) = \frac{1}{4}\left(2\left(\text{Ћ}(\theta) + \text{Ћ}\left(\frac{\pi}{6} - \theta\right) + \text{Ћ}\left(\frac{5\pi}{6}\right)\right) + \text{Ћ}(\theta) + \text{Ћ}\left(\frac{2\pi}{3} - \theta\right) + \text{Ћ}\left(\frac{\pi}{6}\right) \right)$$

$$= \frac{3}{4}\,\text{Ћ}(\theta) + \frac{1}{2}\,\text{Ћ}\left(\frac{\pi}{6} - \theta\right) + \frac{1}{4}\,\text{Ћ}\left(\frac{2\pi}{3} - \theta\right) - \frac{1}{4}\,\text{Ћ}\left(\frac{\pi}{6}\right) \quad \text{with} \quad \cos\theta = \frac{5}{2\sqrt{7}}\,.$$

B   The case $\{2,4,4\}$

As in Appendix A, consider a discrete group $\Gamma \subset \mathrm{Isom}^+\mathbb{H}^3$ of orientation preserving isometries whose quotient space O has exactly one cusp of the form $C = B_\infty/\Gamma_\infty$ giving rise to full-sized horoballs of diameter 1 in the upper half space U^3.

Suppose that C is of type $\{2,4,4\}$ and that the group Γ_∞ has only one equivalence class of full-sized horoballs, all centred at singular points of order 4 in the cusp diagram D, and at distance $d > 1$ from one another. Assume that the associated $(\frac{1}{d})$-balls do not touch one another so that each $(\frac{1}{d})$-ball yields $(\frac{1}{w})$-balls.

Suppose that the four $(\frac{1}{w})$-balls associated to the four $(\frac{1}{d})$-balls in D coincide and, hence, have centre equal to the centre of the square D. Furthermore, the circumradius of D coincides with $\frac{1}{w}$ so that $w = \frac{\sqrt{2}}{d}$.

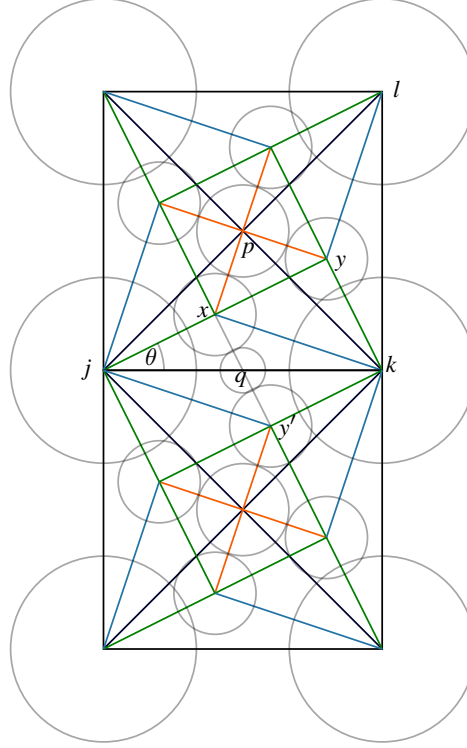[1] To the best of the authors knowledge, this identity is new.

Figure 31: The cusp diagram D for the cusp of type $\{2,4,4\}$ with a single $(\frac{1}{w})$-ball at its centre

In a completely analogous manner as in Appendix A, and by means of (3–7), one can show that $d = \sqrt[4]{5}$ and $\cos\theta = \frac{2}{\sqrt{5}}$. Since the cusp diagram D has no mirror symmetry, the cusp volume yields the large value $\mathrm{vol}(C) = \frac{\sqrt{5}}{8} > v_*$; see Figure 31.

Furthermore, by performing similar computations as in Appendix A, one can deduce that this configuration gives rise to a unique oriented orbifold $O = \mathbb{H}^3/\Gamma$ that is arithmetic and in fact commensurable to the Coxeter group $[3,4,4]$.

Remark 6 *By a suitable decomposition of a fundamental polyhedron for Γ as above, a similar identity for the value $\text{JI}\left(\frac{\pi}{4}\right)$ of the Lobachevsky function in terms of the angle $0 < \theta < \frac{\pi}{4}$ satisfying $\cos\theta = \frac{2}{\sqrt{5}}$ is obtained. This value also gives the volume of the orbifold O as follows.*

$$
\begin{aligned}
\text{JI}\left(\frac{\pi}{4}\right) &= \frac{1}{4}\left\{2\left(\text{JI}(\theta) + \text{JI}\left(\frac{\pi}{4}-\theta\right) + \text{JI}\left(\frac{3\pi}{4}\right)\right) + \text{JI}(\theta) + \text{JI}\left(\frac{\pi}{2}-\theta\right) + \text{JI}\left(\frac{\pi}{2}\right)\right\} \\
&= \frac{3}{4}\,\text{JI}(\theta) + \frac{1}{2}\,\text{JI}\left(\frac{\pi}{4}-\theta\right) + \frac{1}{4}\,\text{JI}\left(\frac{\pi}{2}-\theta\right) - \frac{1}{2}\,\text{JI}\left(\frac{\pi}{4}\right) \\
&\approx 0.457983 \quad \text{with} \quad \cos\theta = \frac{2}{\sqrt{5}}\,.
\end{aligned}
$$

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