Department of Mathematics
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Groups of hyperbolic isometries and their commensurability

THESIS

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Abstract

Let $\mathbb{H}^n$ be the real hyperbolic n-space, $\mathrm{Isom}(\mathbb{H}^n)$ its group of isometries and $\mathrm{Isom}^+(\mathbb{H}^n)$ the index two subgroup of orientation preserving isometries.

In this thesis we establish some new algebraic commensurability conditions for the class of hyperbolic Coxeter groups. These are discrete groups in $\mathrm{Isom}(\mathbb{H}^n)$ generated by finitely many reflections in the bounding hyperplanes of hyperbolic polyhedra all of whose dihedral angles are integral submultiples of π. At the basis is Vinberg's work associating to such a group a quadratic space (V, q), which we call the Vinberg space.

We also exploit the Clifford matrix interpretation of $\mathrm{Isom}^+(\mathbb{H}^n)$ and present our results [19], joint with S. Drewitz, about the realisability of right-angled hyperbolic polygons in any dimension.

For $n = 5$, we introduce a trace field for certain groups of complexified quaternionic 2×2 matrices. We show – in analogy to Kleinian groups in $\mathrm{PSL}(2, \mathbb{C})$ – that this trace field is an algebraic number field.

Zusammenfassung

Sei $\mathbb{H}^n$ der reelle hyperbolische Raum, $\mathrm{Isom}(\mathbb{H}^n)$ seine Isometriegruppe und $\mathrm{Isom}^+(\mathbb{H}^n)$ die Index zwei Untergruppe von orientierungserhaltenden Isometrien.

In dieser Dissertation beweisen wir einige neue algebraische Kommensurabilitätsbedingungen für die Klasse der hyperbolischen Coxeter-Gruppen. Dies sind diskrete Gruppen in $\mathrm{Isom}(\mathbb{H}^n)$, die durch endlich viele Spiegelungen an den begrenzenden Hyperebenen von hyperbolischen Polyedern, deren Diederwinkel ganzzahlige Teiler von π sind, erzeugt werden. Grundlage ist Vinbergs Arbeit, die einer solchen Gruppe einen quadratische Raum (V, q) zuordnet, den wir den Vinberg-Raum nennen.

Wir nutzen auch die Clifford-Matrix Interpretation von $\mathrm{Isom}(\mathbb{H}^n)$ und präsentieren unsere Ergebnisse [19], gemeinsam mit S. Drewitz, über die Realisierbarkeit von rechtwinkligen hyperbolischen Polygonen in jeder Dimension.

Für $n = 5$ führen wir für bestimmte Gruppen von komplexifizierten quaternionischen 2×2-Matrizen einen Spurkörper ein. Wir zeigen – analog zu Kleinschen Gruppen in $\mathrm{PSL}(2, \mathbb{C})$ – dass dieser Spurkörper ein algebraischer Zahlkörper ist.

Résumé

Soient $\mathbb{H}^n$ l'espace hyperbolique réel, $\mathrm{Isom}(\mathbb{H}^n)$ son groupe d'isométries et $\mathrm{Isom}^+(\mathbb{H}^n)$ le sous-groupe d'indice deux des isométries préservant l'orientation.

Dans cette thèse, nous établissons de nouvelles conditions algébriques pour la commensurabilité des groupes de Coxeter hyperboliques. Ce sont des groupes discrets dans $\mathrm{Isom}(\mathbb{H}^n)$ engendrés par un nombre fini de réflexions par rapport aux hyperplans bordant des polyèdres hyperboliques dont tous les angles dièdres sont des sous-multiples entiers de π. À la base de notre approche est le travail de Vinberg associant à un tel groupe un espace quadratique (V, q), que nous appelons l'espace de Vinberg.

Nous exploitons aussi l'interprétation de $\mathrm{Isom}^+(\mathbb{H}^n)$ à l'aide des matrices de Clifford et présentons nos résultats [19], partagés avec S. Drewitz, sur la réalisabilité des polygones hyperboliques à angles droits dans n'importe quelle dimension.

Pour $n = 5$, nous introduisons un corps de traces pour des matrices de format 2×2 quaternioniques complexifiées. Nous montrons – en analogie avec les groupes de Klein dans $\mathrm{PSL}(2, \mathbb{C})$ – que ce corps de traces est un corps de nombres algébriques.

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Contents

Introduction

Let $\mathbb{H}^n$ be the real hyperbolic n-space, $n \geq 2$, and denote by $\text{Isom}(\mathbb{H}^n)$ its isometry group. Hyperbolic space forms $\mathbb{H}^n/\Gamma$, where Γ is a discrete subgroup of $\text{Isom}(\mathbb{H}^n)$ are hyperbolic manifolds and orbifolds, and they can be described by polyhedral models. In general, there is a great interest to understand these objects in terms of their characteristic invariants. The most important one is their volume.

A natural step is to distinguish hyperbolic space forms up to commensurability: two space forms are commensurable if they admit a common finite-sheeted cover.

In low dimensions there is a rich theory and a wealth of results about hyperbolic space forms. For example, for $n = 3$, commensurability of Kleinian groups in $\text{PSL}(2, \mathbb{C})$ is well understood due to the work of Maclachlan and Reid. In particular, for arithmetic Kleinian groups, we dispose of a complete set of commensurability invariants, the invariant trace field and the invariant quaternion algebra.

Our contribution to the theory covers several aspects. For $n = 5$, we interpret elements of the group $\text{Isom}^+(\mathbb{H}^5)$ of direct isometries as quaternionic 2×2 matrices and pass to their complexified images. This approach allows us to introduce a trace field for a discrete group in $\text{Isom}^+(\mathbb{H}^5)$ and to prove, in analogy to the work of Maclachlan and Reid, that it is an algebraic number field.

Our main contribution deals with the commensurability of hyperbolic Coxeter groups in $\text{Isom}(\mathbb{H}^n)$, $n \geq 2$. These groups form an important class of discrete groups, and they appear often as fundamental groups of hyperbolic space forms of minimal volume. As for their commensurability, and based on the work of Vinberg, we provide new commensurability conditions in terms of the *Vinberg field* and the *Vinberg form.*

We also integrate the joint work with S. Drewitz about the realisability of right-angled hyperbolic polygons in any dimension. Our methods consist notably of the identification of direct hyperbolic isometries with Clifford matrices.

In more concrete terms, this thesis is structured as follows.

In Chapter 1 we provide the necessary theory about hyperbolic space and its isometries, geometric Coxeter groups and commensurability.

Chapter 2 contains a condensed version of the paper "On right-angled polygons in hyperbolic space" written jointly with S. Drewitz [19].

In this chapter we first describe direct isometries as Clifford matrices which allows us to give an explicit algorithm for the construction of a right-angled hyperbolic polygon (right-angled closed geodesic edge path) in arbitrary dimension. We also discuss necessary and sufficient conditions for the realisability of such polygons.

The author's contribution to the paper [19] is the algebraic and geometric arguments leading to the algorithm.

In Chapter 3 we look at elements of $\mathrm{Isom}^+(\mathbb{H}^5)$ in form of 4×4 complex block matrices with determinant one, following the work of Wilker [77]. Using this identification we define a trace field for a discrete subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^5)$. We show that this trace field is an algebraic number field if Γ is cocompact and torsion-free.

Chapter 4 is devoted to the commensurability of hyperbolic Coxeter groups. First, and motivated by the work of Vinberg [69], we associate to a hyperbolic Coxeter group a field of cycles and a regular quadratic form of signature $(n, 1)$ which we call the Vinberg field and the Vinberg form. Our main result is a necessary commensurability condition for hyperbolic Coxeter groups that holds even in the non-arithmetic case.

We also detect a weaker commensurability invariant as given by the *Vinberg ring*. We close the chapter by characterising the similarity class of a Vinberg form in terms of the Hasse invariant and the Witt invariant.

In Chapter 5 we present different ways to generate the Vinberg field $K(\Gamma)$, as an extension of $\mathbb{Q}$, of a quasi-arithmetic Coxeter group Γ. Firstly, we show that $K(\Gamma)$ can be generated by the coefficients of the characteristic polynomial of the Gram matrix of Γ. Secondly, we prove that $K(\Gamma)$ is generated by the coefficients of the characteristic polynomial of any Coxeter transformation of Γ.

We also present a result about how the extension degree of $K(\Gamma)$ has an effect on the possible finite weights of the Coxeter graph of Γ.

Attached to this work are four appendices. Appendix A provides three elementary ways to determine the coefficients of a characteristic polynomial of a complex matrix. Appendix B contains an approach to test our Conjecture 5.5.1 about the Vinberg field $K(\Gamma)$ as presented in Chapter 5.

Appendix C contains the published article “On right-angled polygons in hyperbolic space” written jointly with S. Drewitz and published in *Geometriae Dedicata, June 2019, Vol. 200, Issue 1, pp. 45–59* [19]. Finally, Appendix D consists of the Erratum “Commensurability classes of hyperbolic Coxeter groups”, due to J. Ratcliffe and S. Tschantz, which fixes a gap in the proof of Theorem 1 of [36]. The publication of the Erratum here in this work has been authorised by J. Ratcliffe and S. Tschantz.

Chapter 1

The hyperbolic space and its isometries

In this chapter we provide the definitions and notions which will be used throughout this work. We shall start by considering the vector models of the three standard geometries: hyperbolic, Euclidean and spherical. We then focus on discrete subgroups of isometries generated by reflections in these spaces, the so-called geometric Coxeter groups. Of interest for this work will be the hyperbolic space. A standard reference for this chapter is the book of Ratcliffe [59].

1.1 The three standard geometries

Let $n \geq 2$. For $\kappa \in \{-1, 0, 1\}$, equip $\mathbb{R}^{n+1}$ with the bilinear form

$$\langle \cdot, \cdot \rangle_\kappa : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$$

defined as

$$\langle x, y \rangle_\kappa = \sum_{i=1}^{n} x_i y_i + \kappa \cdot x_{n+1} y_{n+1}.$$

Let $\mathbb{X}_\kappa$ denote one of the three simply connected complete Riemannian manifolds of dimension n with constant sectional curvature κ equal to $-1, 0$ and 1, respectively. They can be modelled in the following way. The *vector space model*, or *hyperboloid model*, of the *hyperbolic n-space* $\mathbb{H}^n$ is given by the set

$$\mathcal{H}^n := \mathbb{X}_{-1} = \{x \in \mathbb{R}^{n+1} \mid ||x||_{-1}^2 = \langle x, x \rangle_{-1} = -1,\ x_{n+1} > 0\}$$

with metric

$$d_{\mathcal{H}^n}(x, y) = \operatorname{arcosh}(-\langle x, y \rangle_{-1}) \quad \forall x, y \in \mathcal{H}^n.$$

The bilinear form $\langle \cdot, \cdot \rangle_{-1}$ is called *Lorentzian product*. The space $\mathbb{R}^{n+1}$ equipped with the Lorentzian product is denoted by $\mathbb{R}^{n,1}$. From Chapter 2 onward we shall work only with the hyperbolic space. Thus we will denote the Lorentzian product only by $\langle \cdot, \cdot \rangle$. Its associated quadratic form, the *Lorentzian form*, will be denoted by $q_{-1}(x) := \langle x, x \rangle$.

The *n-dimensional Euclidean space* is given by the set

$$\mathbb{E}^n := \mathbb{X}_0 = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\},$$

with the metric given by

$$d_{\mathbb{E}^n}(x, y) = \sqrt{\langle x - y, x - y \rangle_0} \quad \forall x, y \in \mathbb{E}^n.$$

The *n-dimensional sphere* is given by the set

$$\mathbb{S}^n := \mathbb{X}_1 = \{x \in \mathbb{R}^{n+1} \mid ||x||_1^2 = \langle x, x \rangle_1 = 1\},$$

with the metric given by

$$\cos(d_{\mathbb{S}^n}(x, y)) = \langle x, y \rangle_1 \quad \forall x, y \in \mathbb{S}^n.$$

The group of isometries of $\mathbb{X}_\kappa$ is denoted by $\mathrm{Isom}(\mathbb{X}_\kappa)$. There are other models of the hyperbolic space $\mathbb{H}^n$. They are discussed in Section 1.2.

1.1.1 Hyperplanes and polyhedra

In the hyperbolic space $\mathcal{H}^n$ the orthogonal complement of a vector $e \in \mathbb{R}^{n+1}$ of Lorentzian norm 1 is

$$e^\perp := \{x \in \mathbb{R}^{n+1} \mid \langle x, e \rangle_{-1} = 0\},$$

and $H_e = e^\perp \cap \mathcal{H}^n$ is the hyperplane orthogonal to e.

In the Euclidean space $\mathbb{E}^n$, an (affine) hyperplane is given by a unit vector $e \in \mathbb{R}^n$ and a vector $t \in \mathbb{R}^n$ according to

$$H_{e,t} = \{x \in \mathbb{R}^n \mid \langle x, e \rangle_0 = 0\} + t.$$

On the sphere $\mathbb{S}^n$, analogously to the hyperbolic case, the orthogonal complement of a unit vector $e \in \mathbb{R}^{n+1}$ is defined as

$$e^\perp := \{x \in \mathbb{R}^{n+1} \mid \langle x, e \rangle_1 = 0\}.$$

The hyperplane orthogonal to e is then given by $H_e = e^\perp \cap \mathbb{S}^n$.

The relative position between two hyperplanes in $\mathbb{X}_\kappa$ is characterised as follows.

i) In the hyperbolic space $\mathcal{H}^n$ two hyperplanes H_a and H_b intersect if and only if $|\langle a, b \rangle_{-1}| < 1$. In this situation their dihedral angle is given by

$$\cos \angle(H_a, H_b) = -\langle a, b \rangle_{-1}.$$

The hyperplanes are parallel if and only if $|\langle a,b\rangle_{-1}| = 1$. In this situation their intersection angle is zero.

The hyperplanes do not intersect (they are *ultraparallel*) if and only if $|\langle a,b\rangle_{-1}| > 1$. In this situation there is a unique common perpendicular between H_a and H_b of length

$$\operatorname{arcosh}(|\langle a,b\rangle_{-1}|).$$

ii) In the Euclidean space $\mathbb{E}^n$ two hyperplanes $H_{a,t}$ and $H_{b,s}$ intersect if and only if $|\ \langle a,b\rangle_0\ | \neq 1$. In this case the angle is given by

$$\cos\angle(H_a,H_b) = -\langle a,b\rangle_0.$$

If $|\ \langle a,b\rangle_0\ | = 1$, then the two hyperplanes are parallel with intersection angle zero.

iii) On the sphere $\mathbb{S}^n$ two hyperplanes H_a and H_b always intersect. The angle between them is given by

$$\cos\angle(H_a,H_b) = -\langle a,b\rangle_1.$$

A hyperplane H_e divides $\mathbb{X}_\kappa$, for $\kappa \in \{-1,1\}$, into two half-spaces $H_e^- = \{x \in \mathbb{X}_\kappa \mid \langle x,e\rangle_\kappa \leq 0\}$ and $H_e^+ = \{x \in \mathbb{X}_\kappa \mid \langle x,e\rangle_\kappa \geq 0\}$ such that $H_e^- \cap H_e^+ = H_e$. For $\mathbb{X}_0$ the definitions are similar, namely $H_{e,t}^- = \{x \in \mathbb{E}^n \mid \langle x,e\rangle_1 \leq 0\} + t$ and $H_{e,t}^+ = \{x \in \mathbb{E}^n \mid \langle x,e\rangle_1 \geq 0\} + t$.

We can now state the definition of a polyhedron $P \subset \mathbb{X}_\kappa$; our main reference is [73, Chapter 1].

Definition 1.1.1. A (convex) *polyhedron* $P \subset \mathbb{X}_\kappa$ is the intersection with non-empty interior of finitely many half-spaces, that is, $P = \bigcap_{i=1}^N H_{e_i}^-$, $N \geq n+1$, where the unit vector e_i normal to the hyperplane H_{e_i} is pointing outwards of P.

Remark 1.1.2. If $N = n+1$, then we call P a *n-simplex*.

Definition 1.1.3. A *Coxeter polyhedron* $P \subset \mathbb{X}_\kappa$ is a polyhedron for which all the angles between the bounding hyperplanes of P are either zero or integral submultiples of π, hence of the form $\frac{\pi}{k}$ for $k \in \mathbb{N}$, $k \geq 2$.

1.2 Other models of the hyperbolic space and their isometries

In Section 1.1 we have introduced the vector space model $\mathcal{H}^n$ of the hyperbolic space. This model is very convenient in order to describe hyperplanes and polyhedra by means of their normal vectors. Moreover the group of isometries $\mathrm{Isom}(\mathcal{H}^n)$ is the Lie group of *positive Lorentzian matrices*

$$\mathrm{O}^+(n,1) = \left\{A \in \mathrm{Mat}(n+1,\mathbb{R}) \mid A^T J A = J,\ [A]_{n+1,n+1} > 0\right\}, \qquad (1.1)$$

where $J = \mathrm{diag}(1,\dots,1,-1)$ is the diagonal matrix which represents the Lorentzian form. Notice that $\mathrm{O}^+(n,1)$ is *not* an algebraic group.

Furthermore, each isometry in $\mathrm{Isom}(\mathcal{H}^n)$ is the product of finitely many reflections with respect to hyperplanes (see [7, Proposition A.2.2]).
There are further models for the hyperbolic space, each one of them has its own advantages. In particular they provide a different way to represent the isometries of the hyperbolic space. They are described in greater detail in Chapter 6 of [59].

The upper half-space model

The *upper half-space model* $\mathcal{U}^n$ is defined as the set $\mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ endowed with the metric $d_{\mathcal{U}} : (\mathbb{R}^{n-1} \times \mathbb{R}_{>0}) \times (\mathbb{R}^{n-1} \times \mathbb{R}_{>0}) \to \mathbb{R}$ given by

$$\operatorname{arcosh} d_{\mathcal{U}}(x,y) = 1 + \frac{||x-y||_0^2}{2x_n y_n}\ ,$$

for $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$. For practicality, the space $(\mathcal{U}^n, d_{\mathcal{U}})$ will be denoted by $\mathcal{U}^n$, only. This model is very convenient when dealing with orientation preserving isometries, or *direct isometries*, $\mathrm{Isom}^+(\mathcal{U}^n)$. Indeed, there is a way to represent every direct isometry of the hyperbolic space as a 2×2 matrix with entries in the Clifford group of the real Clifford algebra $\mathcal{C}_{n-2}$. This aspect will be explained in more detail and used in Section 2.1. Well-known are the cases $\mathrm{Isom}^+(\mathcal{U}^2) \cong \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{Isom}^+(\mathcal{U}^3) \cong \mathrm{PSL}(2,\mathbb{C})$.

The projective model

Consider the open unit disk

$$\mathcal{D}^n = \{x \in \mathbb{E}^n \mid ||x||_0 < 1\}.$$

We define the metric $d_{\mathcal{K}} : \mathcal{D}^n \times \mathcal{D}^n \to \mathbb{R}$ as follows

$$\operatorname{arcosh} d_{\mathcal{K}}(x,y) = \frac{1 - \langle x,y\rangle_0}{\sqrt{1-||x||_0^2}\,\sqrt{1-||y||_0^2}} \quad \text{for all} \quad x,y \in \mathcal{D}^n.$$

This gives rise to the *projective model* $\mathcal{K}^n$. It is a non-conformal model for $\mathbb{H}^n$ but it is very practical for studying the combinatorics of polyhedra and the relative positions of hyperplanes. A very important aspect is that the isometries of the hyperbolic space realised in the model $\mathcal{K}^n$ form an algebraic group. Let $\mathrm{O}(n,1) = \left\{A \in \mathrm{Mat}(n+1,\mathbb{R}) \mid A^T J A = J\right\}$ be the group of all matrices which preserve the Lorentzian form (see (1.1)). Then,

$$\mathrm{Isom}(\mathcal{K}^n) \cong \mathrm{O}(n,1)/\{\pm I\} =: \mathrm{PO}(n,1). \tag{1.2}$$

The fact that $\mathrm{Isom}(\mathcal{K}^n)$ is an algebraic group will be exploited in Section 4.3 (see Remark 4.3.3).

Remark 1.2.1. There is another important model which utilizes the open ball as its base space, the *conformal ball model* of Poincaré $\mathcal{B}^n$ ([59, §4.5]). However, we will not use this model.

1.3 Geometric Coxeter groups

Definition 1.3.1. An *abstract Coxeter group* Γ is a finitely presented group generated by the elements $s_1, \ldots, s_N \in \Gamma$ such that

$$\Gamma = \langle s_1, \ldots, s_N : (s_i s_j)^{m_{ij}} = 1 \rangle, \tag{1.3}$$

with $m_{ij} = 1$ if and only if $i = j$ and, for $i \neq j$, $m_{ij} = m_{ji} \in \{2, 3, \ldots, \infty\}$.

We set $m_{ij} = \infty$ if $s_i s_j$ is of infinite order. The set of generators $s_1, \ldots, s_N$ is denoted by S. The cardinality N of S is called the *rank* of Γ.

We are interested in Coxeter groups which admit a geometrical interpretation as reflection groups acting on $\mathbb{X}_\kappa$. Consider a hyperplane H_e in $\mathbb{X}_{-1} = \mathcal{H}^n$ with e a normal vector of Lorentzian norm 1.

Definition 1.3.2. A *reflection* with respect to the hyperplane H_e is the linear application $s_e = s_{H_e} : \mathcal{H}^n \to \mathcal{H}^n$ defined as

$$s_e(x) = x - 2\langle x, e\rangle\, e.$$

For the spherical and Euclidean cases we refer to [59, Chapter 7]. Generally, let $\Gamma < \mathrm{Isom}(\mathbb{X}_\kappa)$ be the discrete reflection group associated to a Coxeter polyhedron $P = \bigcap_{i=1}^N H_{e_i}^-$ in $\mathbb{X}_\kappa$, that is, Γ is generated by the reflections with respect to the hyperplanes bounding P, and it is denoted by $\Gamma = \langle s_{e_1}, \ldots, s_{e_N}\rangle$. Consider the reflection s_{e_i} with respect to the hyperplane H_{e_i} of P, $1 \leq i \leq N$. If two hyperplanes H_{e_i} and H_{e_j} intersect under an angle π/m_{ij}, $m_{ij} \geq 2$, then $(s_{e_i} s_{e_j})^{m_{ij}} = 1$ in Γ. If H_{e_i} and H_{e_j} are parallel or ultraparallel, $s_{e_i} s_{e_j}$ is of infinite order in Γ. Moreover $s_{e_i}^2 = 1$ for all $1 \leq i \leq N$. These relations coincide with the relations (1.3) of an abstract Coxeter group, hence Γ is an abstract Coxeter group of rank N.

Definition 1.3.3. A subgroup $\Gamma < \mathrm{Isom}(\mathbb{X}_\kappa)$ is a *(geometric) Coxeter group* if it is a discrete group generated by the reflections with respect to the hyperplanes bounding a Coxeter polyhedron P in $\mathbb{X}_\kappa$.

Let $\Gamma < \mathrm{Isom}(\mathbb{X}_\kappa)$ be a Coxeter group. If $\mathbb{X}_\kappa = \mathcal{H}^n, \mathbb{E}^n$ or $\mathbb{S}^n$, then Γ is called *hyperbolic*, *Euclidean* (*affine* or *parabolic*) or *spherical* (or *elliptic*), respectively.

In the sequel we are interested in hyperbolic Coxeter groups of *finite covolume*. For this, consider a discrete subgroup $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$. Then Γ has a fundamental domain whose closure can be assumed to be a polyhedron $P \subset \mathcal{H}^n$ (see [59, Chapter 6]).

Definition 1.3.4. A discrete subgroup $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ is said to be *cocompact* if its fundamental polyhedron $P \subset \mathcal{H}^n$ is compact. The group Γ is said to be *cofinite* if P has finite volume. For brevity, a cofinite group Γ is called a *hyperbolic lattice*.

For polyhedra in $\mathcal{H}^n$ with dihedral angles smaller than or equal to $\pi/2$, like Coxeter polyhedra, Vinberg provides a criterion for compactness and for being of finite volume [73, Proposition 4.2]. For aspects about volumes of hyperbolic polyhedra and their computation, see [3, Chapter 7].

General Assumption 1.3.5. In this thesis, unless otherwise specified, hyperbolic Coxeter groups will always be assumed to have finite covolume. While this property is explicitly stated as condition in statements such as theorems and propositions, we often omit this assumption inside the text for the sake of simplicity.

A geometric Coxeter group and its Coxeter polyhedron can be most conveniently described by means of its Coxeter graph and its Gram matrix as follows.

Definition 1.3.6. Let $\Gamma < \mathrm{Isom}(\mathbb{X}_\kappa)$ be a Coxeter group of rank N with Coxeter polyhedron $P = \bigcap_{i=1}^{N} H_{e_i}^-$, $N \geq n+1$. The *Gram matrix* associated to P and to Γ is the real symmetric matrix $G := G(P) = G(\Gamma) = (g_{ij})_{1\leq i,j\leq N}$ with coefficients

$$g_{ij} = \langle e_i, e_j \rangle_\kappa .$$

The Gram matrix G of a Coxeter group Γ is unique up to enumeration of the hyperplanes and can be characterized as follows ([3, Chapter 6]):

i) if $\Gamma < \mathrm{Isom}(\mathbb{S}^n)$, then G is positive definite with rank $n+1$;

ii) if $\Gamma < \mathrm{Isom}(\mathbb{E}^n)$, then G is positive semidefinite with rank n;

iii) if $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$, then G has signature $(n, 1)$.

Definition 1.3.7. Let Γ be a geometric Coxeter group of rank N. The *Coxeter graph* (or *Coxeter diagram*) of Γ is the graph with N vertices for which the vertex i corresponds to the hyperplane H_{e_i}. Between two vertices i and j we have:

i) an edge if the angle between H_{e_i} and H_{e_j} is π/k, $k \geq 3$. If $k \geq 4$ then the edge is labelled with k; if $k = 3$ the label is omitted. Sometimes, if $k = 4$, a double edge is used instead of labelling;

ii) an edge labelled with ∞ if H_{e_i} and H_{e_j} are parallel. Sometimes instead of labelling, a bold edge is used;

iii) a dotted edge if H_{e_i} and H_{e_j} are ultraparallel. Sometimes the dotted edge is labelled with the hyperbolic cosine of the length $d_{\mathcal{H}^n}(H_{e_i}, H_{e_j})$ of their common perpendicular.

1.3.1 Classification of geometric Coxeter groups

Spherical and Euclidean Coxeter groups have been completely classified by Coxeter in [14]. They exist in any dimension $n \geq 2$. All the irreducible groups among them are listed in Figure 1.1 and Figure 1.2.

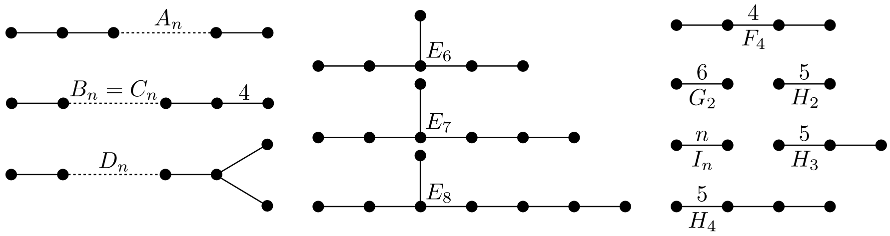


Figure 1.1: Irreducible spherical Coxeter groups of rank n.

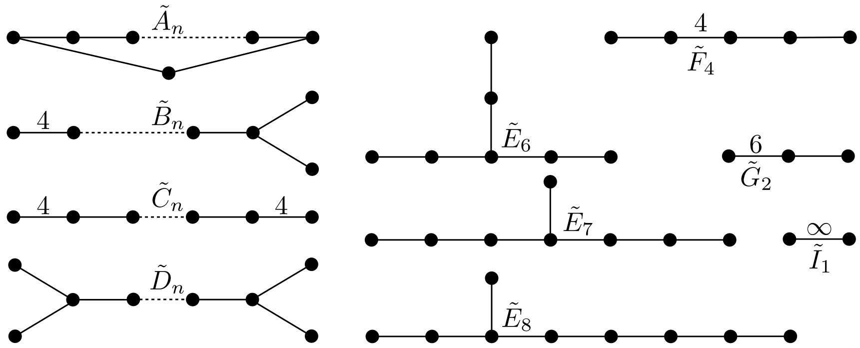


Figure 1.2: Irreducible Euclidean Coxeter groups of rank $n + 1$.

Concerning Coxeter groups acting cofinitely on $\mathcal{H}^n$, the classification is far from being achieved. They do not exist for $n \geq 996$ [58], while for $n \geq 30$ there are no cocompact hyperbolic Coxeter groups [73]. These two bounds are probably not sharp. In fact, the biggest n for which there is an example of a cofinite Coxeter group and of a cocompact Coxeter group is $n = 21$ [9] and $n = 8$ [12], respectively.

The hyperbolic Coxeter simplices ($N = n + 1$) have been classified by Lannér and Koszul. The complete list can be found in [35] where also all the volumes have been computed. For $N = n + 2$, the corresponding classification has been achieved by Kaplinskaya [37], Esselmann [22] and Tumarkin [65]. For $N = n + 3$, Esselmann [21] and Tumarkin [67] classified the compact Coxeter polyhedra while the classification of the non-compact polyhedra is not complete. They do not exist for $n > 16$ [66, Theorem 1]. Furthermore, Roberts classified all the non-compact Coxeter polyhedra with exactly one non-simple vertex [61]. For $N \geq n + 4$ much less is known. A more precise overview of the situation can be found in [29, §2.2]; see also the web page of Anna Felikson [1].

1.4 Commensurability

A central theme of this thesis is the commensurability of discrete subgroups of isometries in $\mathrm{Isom}(\mathbb{H}^n)$ with particular emphasis on Coxeter groups. We shall state here a general definition and some basic properties. The main results on this subject will appear in Chapter 4.

Definition 1.4.1. Let H be a group. Two subgroups $H_1, H_2 < H$ are *commensurable* (in the wide sense) if and only if there exists an element $h \in H$ such that $H_1 \cap h^{-1}H_2h$ has finite index in both H_1 and $h^{-1}H_2h$.

This notion is an equivalence relation. For our purpose, the group H will be the group of hyperbolic isometries $\mathrm{Isom}(\mathbb{H}^n)$ or its index 2 subgroup $\mathrm{Isom}^+(\mathbb{H}^n)$ of orientation preserving isometries. Some properties are stable under commensurability.

Let $\Gamma_1, \Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be commensurable. Then:

- Γ_1 is discrete if and only if Γ_2 is discrete,
- Γ_1 is cofinite if and only if Γ_2 is cofinite,
- the covolumes of Γ_1 and Γ_2 are rational multiples of each other,
- Γ_1 is cocompact if and only if Γ_2 is cocompact,
- Γ_1 is arithmetic if and only if Γ_2 is arithmetic (see Section 1.5).

Remark 1.4.2.

i) There is an interesting commensurability invariant for a singly cusped non-arithmetic hyperbolic orbifold V of finite volume. Write $V = \mathbb{H}^n/\Gamma$ with cusp $C \subset V$. Then the *cusp density* is defined by $\delta(C) = \mathrm{Vol}(C)/\mathrm{Vol}(V)$, and it turns out to be a commensurability invariant (see [26], for example). However, we will not make use of this analytic invariant.

ii) For non-cocompact non-arithmetic hyperbolic lattices there is a necessary and sufficient commensurability criterion involving the concept of horoball-packings ([26, Theorem 2.4]). This analytical aspect will not be treated in this work.

1.5 Arithmeticity, quasi-arithmeticity and nq-arithmeticity

We close this chapter by splitting discrete subgroups in $\mathrm{Isom}(\mathcal{H}^n)$ into three categories: *arithmetic*, *quasi-arithmetic* and *nq-arithmetic*.

Let $K \subset \mathbb{R}$ be a totally real number field and let V be a vector space of dimension $n+1$ over K endowed with a quadratic form q of signature $(n, 1)$. We denote this quadratic space by (V, q). Moreover for every non-trivial embedding $\sigma : K \hookrightarrow \mathbb{R}$ assume that the quadratic space (V, q^σ) is positive definite, where q^σ denotes the quadratic form obtained by applying σ to each coefficient of q. Let

$$\mathrm{O}(V, q) := \{U \in \mathrm{GL}(n+1, \mathbb{R}) \mid q(Ux) = q(x)\ \forall x \in V \otimes_K \mathbb{R}\}.$$

Notice that since q has the same signature and rank as the Lorentzian form q_{-1}, there exists a real invertible matrix S such that $S^{-1}\,\mathrm{O}^+(n, 1)S = \mathrm{O}^+(V, q)$.
Consider the ring of integers of K, denoted by $\mathcal{O}_K$. Let L be a $\mathcal{O}_K$-lattice[1] and denote by $\mathrm{O}(L) < \mathrm{O}^+(V, q) \cap \mathrm{GL}(n+1, K)$ the group of linear transformations with coefficients in K that preserve the lattice L. The group $\mathrm{O}(L)$ is discrete and of finite covolume [27, §2.2].

Definition 1.5.1. A discrete subgroup $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ is called *arithmetic of the simplest type* if there exist K, q and L as above such that $S^{-1}\Gamma S$ is commensurable with $\mathrm{O}(L)$ in $\mathrm{O}^+(V, q) \cap \mathrm{GL}(n+1, K)$. In this case one says that Γ *is defined over* K *with quadratic space* (V, q).

Remark 1.5.2.

i) There is a more general definition of arithmetic group (see [3, Chapter 6]). Notice that if a hyperbolic Coxeter group is arithmetic, then it is of the simplest type [69, Lemma 7]. In this thesis whenever a group in $\mathrm{Isom}(\mathcal{H}^n)$ is arithmetic it is of the simplest type. Therefore we will always refer to arithmetic groups of the simplest type as just *arithmetic* groups.

ii) There are no arithmetic Coxeter groups in $\mathrm{Isom}(\mathcal{H}^n)$ for $n \geq 30$ [74, Theorem 2.2].

[1] Let R be a ring with field of fraction K and V a vector space of dimension $n+1$ over K. An R-lattice L in V is an R-module in V of rank $n+1$ for which $\mathrm{Span}_K\{L\} = V$.

Famous examples of hyperbolic arithmetic groups are $\mathrm{PSL}(2,\mathbb{Z})$ and all the Bianchi groups $\mathrm{PSL}(2,\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

We now introduce the notion of quasi-arithmetic group, which, as the name suggests, is a weaker notion of arithmeticity. It is obtained by dropping the lattice condition.

Definition 1.5.3. A discrete subgroup $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ is called *quasi-arithmetic* if there exist K and q as above such that

$$S^{-1}\Gamma S \subset \mathrm{O}^+(V,q) \cap \mathrm{GL}(n+1,K).$$

In this case one says that Γ *is defined over K with quadratic space* (V,q).

Remark 1.5.4. If a quasi-arithmetic group in $\mathrm{Isom}(\mathcal{H}^n)$ is non-cocompact, then the associated field is $\mathbb{Q}$ [74, Chapter 6].

Clearly if a group is arithmetic, then it is quasi-arithmetic. In this thesis, when a group is quasi-arithmetic but not arithmetic, this will be pointed out explicitly.

Definition 1.5.5. A discrete subgroup $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ is called *non-quasi-arithmetic*, *nq-arithmetic* from now on, if it is neither arithmetic nor quasi-arithmetic.

Examples of such groups will appear in Chapter 4 and in Chapter 5.

Remark 1.5.6. The previous definitions apply to every discrete group in $\mathrm{Isom}(\mathcal{H}^n)$, not only Coxeter groups. Moreover, as a consequence of the construction, arithmetic groups have finite covolume while quasi-arithmetic and nq-arithmetic groups can also have infinite covolume.

Chapter 2

On right-angled polygons in hyperbolic space

In this chapter we provide a condensed version of the paper "On right-angled polygons in hyperbolic space" written jointly with Simon Drewitz and published in *Geometriae Dedicata, June 2019, Vol. 200, Issue 1, pp. 45–59* [19], which can be found entirely in Appendix C.

Throughout this chapter we consider the upper half-space model $(\mathcal{U}^n, d_{\mathcal{U}})$ for the hyperbolic space $\mathbb{H}^n$ (see Section 1.2). The main objective is to describe right-angled polygons in $\mathcal{U}^n$ by means of a set of parameters describing its sides. By right-angled hyperbolic polygon we mean an oriented closed orthogonal geodesic edge path in $\mathcal{U}^n$.

It was previously shown by Delgove and Retailleau [17] that three quaternionic parameters define a right-angled hexagon in the space $\mathcal{U}^5$. In their work, 2×2 quaternionic matrices having Dieudonné determinant 1 are used in order to describe orientation preserving isometries (or direct isometries) of $\mathcal{U}^5$. This description will be explicitly stated in Section 3.2.1. While this approach based on quaternions is very convenient in $\mathcal{U}^5$, it can not be extended to arbitrary dimensions.

A description of direct isometries in arbitrary dimensions can be achieved if we consider Clifford matrices. These are 2×2 matrices with entries in the extended Clifford group and can be used to represent the group $\mathrm{Isom}^+(\mathcal{U}^n)$ of direct isometries. In particular, with this approach, 2×2 quaternionic Clifford matrices are used to describe direct isometries of $\mathcal{U}^4$.

At first we develop the identification of direct hyperbolic isometries with Clifford matrices. We also introduce the cross ratio of Clifford vectors and its geometrical interpretation. Then we state the main result as given by Theorem 2.2.8 about right-angled polygons in $\mathcal{U}^n$. It yields an algorithmic way to construct such polygons, given a set of parameters describing the side lengths. Lastly we discuss a necessary condition on the parameters for the realisability of right-angled polygons as closed simplex edge path.

Furthermore we focus on the realisability of a right-angled pentagon in $\mathcal{U}^4$ with all the sides of equal length. For a more detailed description and for all the proofs we refer to the paper [19] and the references therein.

2.1 Clifford matrices and direct isometries

2.1.1 The real Clifford algebra

Let $m \geq 0$. The *(real) Clifford algebra* of rank m is the associative real algebra $\mathcal{C}_m$ with unit 1 and with generators $i_1, \dots, i_m$ which anticommute and whose square is -1:

$$\mathcal{C}_m = \left\langle i_1, \dots, i_m \mid i_j\, i_l = -i_l\, i_j, i_l^2 = -1 \text{ for } l \neq j \right\rangle .$$

Every element x of the algebra $\mathcal{C}_m$ can be uniquely written as $x = \sum x_I I$, where $x_I \in \mathbb{R}$ and the sum is taken over all the products $I = i_{k_1} \cdots i_{k_s}$, with $1 \leq k_1 < \cdots < k_s \leq m$ and $1 \leq s \leq m$. Here the empty product I_0 is included and identified with $i_0 := 1$. Hence $\mathcal{C}_m$ is a 2^m-dimensional real vector space. In particular we can identify $\mathcal{C}_0$ with $\mathbb{R}$, $\mathcal{C}_1$ with $\mathbb{C}$ and $\mathcal{C}_2$ with $\boldsymbol{H}$, the Hamiltonian quaternions. We can induce a Euclidean structure on $\mathcal{C}_m$ by associating to each element $x = \sum x_I I$ a norm given by $|x|^2 = \sum x_I^2$. Denote by $\Re(x)$ the coefficient x_0, called the *real part* of x, while $\Im(x) = x - \Re(x)$ is called the *non-real part* of x. If $\Re(x) = 0$ we will refer to x as a *pure element* of $\mathcal{C}_m$.

On $\mathcal{C}_m$ there are three well-known involutions. Let $x \in \mathcal{C}_m$ with $x = \sum x_I I$. Then:

i) $x^* = \sum x_I I^*$, where I^* is obtained from $I = i_{k_1} \cdots i_{k_s}$ by reversing the order of the factors, that is, $I^* = i_{k_s} \cdots i_{k_1}$;

ii) $x' = \sum x_I I'$, where I' is obtained from $I = i_{k_1} \cdots i_{k_s}$ by replacing each factor i_k with $-i_k$, that is, $I' = (-i_{k_1}) \cdots (-i_{k_s}) = (-1)^s I$;

iii) $\overline{x} = (x^*)' = (x')^*$.

The first and last involutions are anti-automorphisms, while the second one is an automorphism.

More important for our purpose are the *Clifford vectors*. These are Clifford elements of the form $x = x_0 + x_1 i_1 + \cdots + x_m i_m$. The set

$$\mathbb{V}^{m+1} = \{x_0 + x_1 i_1 + \cdots + x_m i_m \mid x_0, \dots, x_m \in \mathbb{R}\}$$

of all Clifford vectors is an $(m+1)$-dimensional real vector space and can be identified with the Euclidean space $\mathbb{E}^{m+1}$. Notice that for an element $x \in \mathbb{V}^{m+1}$ we have $x^* = x$ and hence $\overline{x} = x'$ as well as $x + \overline{x} = 2\Re(x)$ and $x\overline{x} = \overline{x}x = |x|^2$. Moreover every non-zero vector x has an inverse given by

$x^{-1} = \frac{\overline{x}}{|x|^2}$. Hence finite products of non-zero vectors are invertible and they form the so-called *Clifford group* Γ_m. Observe that we have $\Gamma_m = \mathcal{C}_m \setminus \{0\}$ only for $m \in \{0, 1, 2\}$.

2.1.2 Clifford matrices, cross ratios and hyperbolic geometry

In this section we discuss how matrices having entries in the extended Clifford group $\Gamma_{n-2} \cup \{0\}$ relate to direct isometries of the hyperbolic space $\mathcal{U}^n$, $n \geq 2$. A general reference for this is the work of Waterman [76] (all other references in this part will be omitted as they can be found in the Appendix C). After that we look at the cross ratios and exploit their geometrical applications. As in the previous section, m is a non-negative integer.

Definition 2.1.1. A *Clifford matrix* is a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \Gamma_m \cup \{0\}$ such that $ab^*, cd^*, c^*a, d^*b \in \mathbb{V}^{m+1}$ and $ad^* - bc^* \in \mathbb{R} \setminus \{0\}$. The expression $ad^* - bc^*$ is the *Ahlfors determinant* of A.

Denote the set of such matrices by $\mathrm{GL}(2, \mathcal{C}_m)$. By a result of Vahlen and Maass, the set

$$\mathrm{SL}(2, \mathcal{C}_m) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{C}_m) \mid ad^* - bc^* = 1 \right\}$$

of Clifford matrices with Ahlfors determinant 1 is a multiplicative group. It is generated by the matrices

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix},$$

where $t \in \mathbb{V}^{m+1}$ and $a \in \Gamma_m \setminus \{0\}$. Consider the projective group

$$\mathrm{PSL}(2, \mathcal{C}_m) = \mathrm{SL}(2, \mathcal{C}_m)/\{\pm I\}.$$

It is known that representatives $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of elements in $\mathrm{PSL}(2, \mathcal{C}_m)$ act bijectively on $\mathbb{V}^{m+1} \cup \{\infty\}$ by

$$T(x) = (ax + b)(cx + d)^{-1} \tag{2.1}$$

with the identification $T(-c^{-1}d) = \infty$, $T(\infty) = ac^{-1}$ if $c \neq 0$, and $T(\infty) = \infty$ otherwise.

Consider the hyperbolic space $(\mathcal{U}^n, d_{\mathcal{U}})$. The base space $\mathcal{U}^n$ can be interpreted with the help of Clifford vectors as follows:

$$\mathcal{U}^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\} \cong \mathbb{V}^{n-1} \times \mathbb{R}_{>0}.$$

The compactification $\overline{\mathcal{U}^n}$ is given by the union of $\mathcal{U}^n$ with the boundary set $\partial\mathcal{U}^n = \mathbb{V}^{n-1} \cup \{\infty\}$ of points at infinity of $\mathcal{U}^n$.

Using Poincaré extension, the action of $\mathrm{PSL}(2, \mathcal{C}_{n-2})$ given by (2.1) can be extended from $\mathbb{V}^{n-1} \cup \{\infty\}$ to the upper half-space $\mathcal{U}^n$. In this way we obtain an isomorphism between $\mathrm{PSL}(2, \mathcal{C}_{n-2})$ and the group $\mathrm{Möb}^+(n-1)$ of orientation preserving Möbius transformations of $\mathbb{V}^{n-1} \cup \{\infty\}$. Since the group $\mathrm{Isom}^+(\mathcal{U}^n)$ of orientation preserving isometries of $\mathcal{U}^n$ is isomorphic to $\mathrm{Möb}^+(n-1)$, we get the following identification:

$$\mathrm{Isom}^+ (\mathcal{U}^n) \cong \mathrm{Möb}^+ (n-1) \cong \mathrm{PSL}(2, \mathcal{C}_{n-2}). \tag{2.2}$$

Therefore any direct isometry of $\mathcal{U}^n$ can be represented by a Clifford matrix in $\mathrm{PSL}(2, \mathcal{C}_{n-2})$. Classical examples are $\mathrm{Isom}^+(\mathcal{U}^2) \cong \mathrm{PSL}(2, \mathcal{C}_0) \cong \mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{Isom}^+(\mathcal{U}^3) \cong \mathrm{PSL}(2, \mathcal{C}_1) \cong \mathrm{PSL}(2, \mathbb{C})$.

Remark 2.1.2. Möbius transformations act triply transitively on $\mathbb{V}^{n-1} \cup \{\infty\}$, meaning that for any two triplets of distinct points in $\mathbb{V}^{n-1} \cup \{\infty\}$ there exists a transformation $T \in \mathrm{Möb}(n-1)$ which maps the first triplet to the second one.

We state now the definition of a (generalised) cross ratio. It plays an important role in the algorithmic construction of a right-angled polygon as it encodes the geometrical information for the construction.

Definition 2.1.3. Let x, y, z, w be four pairwise different Clifford vectors in $\mathbb{V}^{n-1}$. Then

$$[x, y, z, w] := (x - z)(x - w)^{-1}(y - w)(y - z)^{-1} \in \Gamma_{n-2} \setminus \{0\}$$

is called the *cross ratio* of x, y, z and w.

We extend Definition (2.1.3) by continuity to $\mathbb{V}^{n-1} \cup \{\infty\}$, allowing x, y or w to be ∞, by

$$[\infty, y, z, w] = (y - w)(y - z)^{-1} \text{ for } x = \infty, \tag{2.3}$$

and similarly for $y = \infty$ and $w = \infty$. Moreover, in an analogous way, we put

$$[x, y, \infty, w] = (x - w)^{-1}(y - w).$$

The real part and the norm of the cross ratio $[x, y, z, w]$ of four vectors are invariant under the action of a $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{C}_{n-2})$. However, the cross ratio itself is not an invariant. Indeed, the following holds.

$$[T(x), T(y), T(z), T(w)] = (cz + d)^{*-1}[x, y, z, w](cz + d)^*.$$

Consider two oriented geodesics s and t in $\mathcal{U}^n$ with endpoints s^-, s^+ and t^-, t^+ all distinct in $\mathbb{V}^{n-1} \cup \{\infty\}$.

Definition 2.1.4. The *cross ratio* $\Delta(s,t)$ of the geodesics s and t is defined by

$$\Delta(s,t) := \left[s^-, s^+, t^-, t^+\right].$$

This specialised version of the cross ratio provides the following geometrical properties for geodesics in $\mathcal{U}^n$.

Lemma 2.1.5. *Let s and t be two oriented geodesics as above. If s and t intersect, then $\Delta(s,t) = \Delta(t,s)$. If s and t are disjoint, then $\Delta(s,t) = \Delta(t,s)$ if one of the endpoints is ∞ or if the cross ratios are real, otherwise the two cross ratios are conjugate.*

Proposition 2.1.6. *Two oriented hyperbolic geodesics s and t intersect if and only if their cross ratio $\Delta(s,t) \in \mathbb{R}_{<0}$. Furthermore s and t are perpendicular if and only if $\Delta(s,t) = -1$.*

Now consider three oriented geodesics r, s and t in $\mathcal{U}^n$ with pairwise different endpoints r^-, r^+, s^-, s^+ and t^-, t^+ in $\mathbb{V}^{n-1} \cup \{\infty\}$.

Definition 2.1.7. The ordered triple (r,s,t) is called a *double bridge* if s is orthogonal to r and t such that $r \neq t$.

Definition 2.1.8. The quantity

$$\Delta(r,s,t) := \left[s^+, s^-, r^+, t^+\right] \tag{2.4}$$

is called the *double bridge cross ratio* of (r,s,t).

If $\mid \Delta(r,s,t) \mid > 1$, then the intersections $r \cap s$ and $s \cap t$ do not coincide and we call the double bridge *properly oriented.*

2.2 On right-angled polygons in hyperbolic space

We now have all the tools we need to construct a right-angled polygon of p edges in $\mathcal{U}^n$. We state the precise definition of an oriented right-angled polygon. Notice that, in contrast with the usual definition, our polygons need not to be planar.

Definition 2.2.1. Let $p \geq 5$. An *(oriented) right-angled polygon with p sides* in $\mathcal{U}^n$ (or *p-gon* for short), denoted Π_p, is a p-tuple of oriented geodesics $(S_0, S_1, \ldots, S_{p-1})$ with $S_{i-1} \neq S_{i+1}$ for $i \pmod p$ and such that S_i is orthogonal to S_{i+1} for $0 \leq i \leq p-2$ and S_{p-1} is orthogonal to S_0.

Definition 2.2.2. A p-gon Π_p is said to be *non-degenerate* if, for $i \pmod p$:

i) consecutive intersections do not coincide, that is $S_{i-1} \cap S_i \neq S_i \cap S_{i+1}$;

ii) each triple (S_{i-1}, S_i, S_{i+1}) is properly oriented, that is, S_i is oriented from S_{i-1} to S_{i+1}.

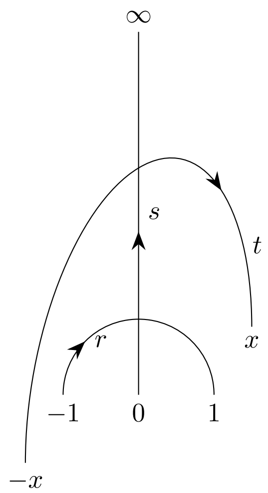


Figure 2.1: The standard double bridge configuration.

Remark 2.2.3. It is no restriction to only consider p-gons in $\mathcal{U}^{p-1}$ since the convex hull of p geodesics can at most have dimension $p-1$. Hence, we will always refer to this case.

Since the direct isometries of $\mathcal{U}^p$ act triply transitively on $\mathbb{V}^{p-1}$, we can always normalise the situation and assume that the first two geodesics of Π_p are $S_0 = (-1,1)$ and $S_1 = (0,\infty)$. This implies, due to orthogonality, that the third geodesic is of the form $S_2 = (-x,x)$. This configuration is called the *standard double bridge* (see Figure 2.1). A quick computation using (2.3) shows that the double bridge cross ratio of three geodesics in this configuration is given by

$$\Delta((-1,1),(0,\infty),(-x,x)) = [\infty,0,1,x] = x.$$

Therefore, provided that the first two geodesics in a double bridge are $(-1,1)$ and $(0,\infty)$, the double bridge cross ratio completely describes the third geodesic. Hence the main strategy for the construction of a p-gon will be to map triples of geodesics into the standard double bridge configuration and exploit the double bridge cross ratio. For this we introduce adequate mappings as follows.

Definition 2.2.4. For a set of given Clifford vectors $\{q_1,\ldots,q_{p-3}\} \subset \mathbb{V}^{p-2} \setminus \{0\}$ define, for $1 \le i \le p-3$, the isometry ϕ_i of the upper half-space $\mathcal{U}^p$ by means of the following Clifford matrix

$$\begin{pmatrix} \sqrt{-2\,q_i}^{-1} & q_i\sqrt{-2\,q_i}^{-1} \\ \sqrt{-2\,q_i}^{-1} & -q_i\sqrt{-2\,q_i}^{-1} \end{pmatrix}. \tag{2.5}$$

The inverse isometry ϕ_i^{-1} is represented by the Clifford matrix

$$\begin{pmatrix} q_i\sqrt{-2\,q_i}^{-1} & q_i\sqrt{-2\,q_i}^{-1} \\ \sqrt{-2\,q_i}^{-1} & -\sqrt{-2\,q_i}^{-1} \end{pmatrix}. \tag{2.6}$$

Remark 2.2.5.

i) For the definition of the square root of a Clifford vector see [19, Section 2.2];

ii) the isometry ϕ_i maps the two geodesics $(0,\infty)$ and $(-q_i, q_i)$ into the geodesics $(-1,1)$ and $(0,\infty)$;

iii) the concatenation $\phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1$ is denoted by Φ_i.

Definition 2.2.6. Let $(S_0, \ldots, S_{p-1})$ be a right-angled p-gon. For $i = 1$, …, $p-3$ define the *gauged (double bridge) cross ratio* $\tilde{\Delta}_i$ by the following recursive definition:

$$\begin{aligned} \tilde{\Delta}_1 &:= \Delta\left(S_0, S_1, S_2\right), \\ \tilde{\Delta}_{i+1} &:= \Delta\left(\Phi_i\left(S_i\right), \Phi_i\left(S_{i+1}\right), \Phi_i\left(S_{i+2}\right)\right), \end{aligned}$$

where the Clifford vector q_i which is needed to define the map Φ_i is calculated along the way as

$$q_i := \tilde{\Delta}_i.$$

The gauged cross ratios will be the parameters encoding the information for the construction of p-gons. For this, $p-3$ parameters are needed. Let

$$\mathcal{P}_p := \left\{(q_1, \ldots, q_{p-3}) \mid q_i \in \mathbb{V}^{p-2}, |q_i| > 1, 1 \le i \le p-3\right\}$$

be a set of $(p-3)$-tuples of non-zero Clifford vectors.

Denote by

$$\mathrm{RAP}_p := \Big\{ (S_0, \ldots, S_{p-1}) \text{ non-degenerate oriented right-angled } p\text{-gon in } \mathcal{U}^{p-1} \text{ with } S_0 = (-1,1)\,, S_1 = (0,\infty) \Big\}$$

the set of (normalised) non-degenerate oriented right-angled p-gons.

Remark 2.2.7. Notice that a $(p-3)$-tuple in $\mathcal{P}_p$, which encodes the geometrical information of the sides of a p-gon, can a priori still describe a *degenerate* p-gon.

The gauged double bridge cross ratio justifies the definition of the map

$$\tilde{\Delta} : \mathrm{RAP}_p \to \mathcal{P}_p, \qquad \Pi_p = (S_1, \ldots, S_{p-1}) \mapsto (\tilde{\Delta}_1, \ldots, \tilde{\Delta}_{p-3})$$

which to each p-gon associates its set of $p-3$ gauged double bridge cross ratios. Denote the image of this map by $\mathcal{P}_p^* := \tilde{\Delta}\,(\mathrm{RAP}_p) \subset \mathcal{P}_p$. This is the set of parameters which yield a non-degenerate Π_p. Our main result can now be stated as follows.

Theorem 2.2.8. *The map $\tilde{\Delta} : \mathrm{RAP}_p \to \mathcal{P}_p^*$ is a bijection. The inverse map is given as an explicit construction of the p-gon Π_p depending on $p-3$ parameters in $\mathcal{P}_p^*$.*

Concretely, assume we are given $p-3$ parameters $(q_1, \ldots, q_{p-3}) \in \mathcal{P}_p^*$. The construction of the Π_p goes as follows.

Start The first two geodesics are fixed as $S_0 = (-1, 1)$ and $S_1 = (0, \infty)$. Since this is the standard double bridge configuration considered above, we find $S_2 = (-q_1, q_1)$ from the condition $\Delta(S_0, S_1, S_2) = q_1$.

The geodesic $\mathbf{S_3}$ To find the endpoints of S_3, we benefit from the isometries ϕ_1 and ϕ_1^{-1} (see (2.5) and (2.6)). The isometry ϕ_1 maps $(0, \infty)$ to $(-1, 1)$, S_2 to $(0, \infty)$ and S_3 to $\phi_1(S_3)$. These three geodesics are now in the standard double bridge configuration. Since q_2 is the cross ratio of this double bridge, the third geodesic is $\phi_1(S_3) = (-q_2, q_2)$. We now apply the isometry ϕ_1^{-1} which maps $(-1, 1)$ to $(0, \infty)$, $(0, \infty)$ to S_2 and $\phi_1(S_3)$ to S_3. The geodesic S_3 is then given by $S_3 = (\phi_1^{-1}(-q_2), \phi_1^{-1}(q_2))$.

The next geodesic in the general case The further procedure expands the previous idea. First we note that the geodesic $\Phi_2(S_4)$ is given by the parameter q_3. The geodesic S_4 would then be the image of $(-q_3, q_3)$ under the isometry Φ_2^{-1} mapping $(-1, 1)$, $(0, \infty)$ and $\Phi_2(S_4)$ to S_2, S_3 and S_4, respectively.

In general, assuming we have calculated the geodesics $S_0, \ldots, S_k$ for some k with $2 \leq k \leq p-3$, we can use Φ_{k-1}^{-1} in order to obtain $S_{k+1} = \big(\Phi_{k-1}^{-1}(-q_k), \Phi_{k-1}^{-1}(q_k)\big)$.

Existence of the last geodesic After using all the parameters $q_1, \ldots, q_{p-3}$, we have determined the geodesics $S_0, \ldots, S_{p-2}$. As a consequence of Proposition 2.1.6 the last common perpendicular between S_0 and S_{p-2} exists and is unique as long as

$$\Delta\,(S_0, S_{p-2}) \notin \mathbb{R}_-.$$

This last condition is ensured since $(q_1, \ldots, q_{p-3}) \in \mathcal{P}_p^*$.

2.2.1 Realisability conditions

Given any $(p-3)$-tuple in $\mathcal{P}_p$, one can apply the construction of the previous section in order to obtain a p-gon. However, without any further assumption on this tuple, the resulting p-gon can be degenerate. A priori, we do not know when a $(p-3)$-tuple in $\mathcal{P}_p$ is actually in $\mathcal{P}_p^*$.

In the last part of this chapter we consider right-angled p-gons of the highest possible dimension. For the realisability of this type of polygons we have the following necessary criterion on the parameters.

Proposition 2.2.9. *If the parameters $q_1, \ldots, q_{p-3} \in \mathbb{V}^{p-2}$ give rise to a right-angled polygon Π_p whose intersection points are the vertices of a simplex, then the parameters together with 1 have to form a basis of the Clifford vectors according to $\langle 1, q_1, \ldots, q_{p-3} \rangle = \mathbb{V}^{p-2}$.*

Finally, we consider the realisability of a right-angled pentagon as a 4-simplex having all the edges of the same length (see Figure 2.2). Essential for this is the work of Dekster and Wilker [16]. They provide a criterion for the existence of n-simplices with vertices $p_1, \ldots, p_{n+1}$ of given side lengths and diagonal lengths $l_{ij} = d_{\mathcal{U}}(p_i, p_j)$, $1 \leq i < j \leq n+1$, in the hyperbolic space $\mathcal{U}^n$.

According to [16], a symmetric $(n+1) \times (n+1)$-matrix $L = (l_{ij})$ is *allowable* if $l_{ii} = 0$ and $l_{ij} > 0$ for $i \neq j$. The matrix L is called *realisable* in $\mathcal{U}^n$ if there are $n+1$ points $p_1, \ldots, p_{n+1}$ in $\mathcal{U}^n$ with the given distances $d_{\mathcal{U}}(p_i, p_j) = l_{ij}$.

Their realisability criterion can be quantified as follows.

Theorem 2.2.10. *Let $L = (l_{ij})$ be an allowable $(n+1) \times (n+1)$-matrix and let its entries be used to form the $(n \times n)$-matrix $S = (s_{ij})$ where*

$$s_{ij} = \cosh(l_{i,n+1}) \cosh(l_{j,n+1}) - \cosh(l_{ij}).$$

Then L is realisable if and only if the eigenvalues of S are non-negative. If L is realisable, then the dimension of such a realisation is equal to the rank of S.

As a consequence of Theorem 2.2.10, we obtain the following result for the realisability of a pentagon with all sides of equal length.

Proposition 2.2.11. *A right-angled hyperbolic pentagon $\Pi_5 = (S_0, \ldots, S_4)$ with all side lengths equal to $a > 0$ is realisable as a hyperbolic 4-simplex if and only if $\cosh(a) < \gamma$, where $\gamma = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.*

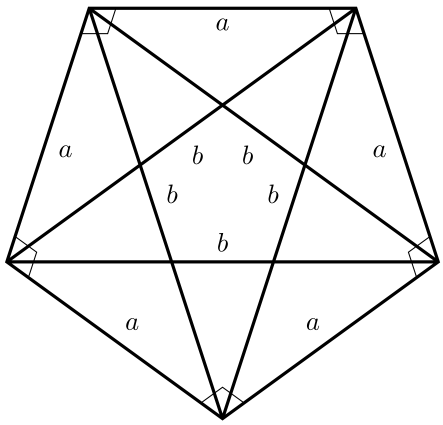


Figure 2.2: A hyperbolic pentagon with right-angled cyclic edge path of side length a.

For all the proofs and more details, we refer to our original paper [19].

Chapter 3

Trace field and commensurability of hyperbolic lattices in low dimensions

In this chapter we start to investigate the commensurability problem of hyperbolic lattices in low dimensions as introduced in Section 1.4. Throughout this chapter the model of reference for the hyperbolic n-space is the upper half-space model $\mathcal{U}^n$, $n \geq 2$.

We shall begin with a quick summary about commensurability of discrete subgroups of direct isometries acting on $\mathcal{U}^3$. As we have seen in the previous chapter, isometries in $\mathrm{Isom}^+(\mathcal{U}^3)$ can be identified with elements in $\mathrm{PSL}(2, \mathbb{C})$. In this special case the situation is well understood, and two powerful commensurability invariants are known: the *invariant trace field* and the *invariant quaternion algebra*. The main reference is the book of Maclachlan and Reid [48].

We then consider the space $\mathcal{U}^5$ and discrete groups of direct isometries acting on it. Following work of Wilker [77], we can describe elements of $\mathrm{Isom}^+(\mathcal{U}^5)$ as 2×2 quaternionic matrices with Dieudonné determinant equal to 1 (see Section 3.2.1). By complexifying these matrices, we introduce a trace field associated to a discrete subgroup Γ of $\mathrm{Isom}^+(\mathcal{U}^5)$. In this way and in analogy to the work of Maclachlan and Reid, we will show that this new trace field is an algebraic number field in the case that Γ is cocompact and torsion-free (see Section 3.3).

3.1 Commensurability of Kleinian groups

Let $\mathcal{U}^3$ denote the upper half-space model as given by $\mathbb{C} \times \mathbb{R}_+$. From Section 2.1.2 we know that the direct isometries in this model can be identified with Clifford matrices with entries in the extended Clifford group of the Clifford algebra $\mathcal{C}_1$, namely

$$\mathrm{Isom}^+(\mathcal{U}^3) \cong \mathrm{PSL}(2, \mathcal{C}_1) = \mathrm{PSL}(2, \mathbb{C}).$$

A cofinite discrete subgroup in $\mathrm{PSL}(2, \mathbb{C})$ is classically called a *Kleinian group*. Kleinian groups and their commensurability have been intensively studied by Maclachlan, Reid and others (see for example [48]). For a Kleinian group Γ define the *trace field*

$$\mathbb{Q}(\mathrm{Tr}\,\Gamma) = \mathbb{Q}(\pm \mathrm{Tr}\,\gamma \mid \gamma \in \Gamma) \tag{3.1}$$

which equals the standard trace field of a lift of Γ in $\mathrm{SL}(2, \mathbb{C})$. Obviously, $\mathbb{Q}(\mathrm{Tr}\,\Gamma)$ is a conjugacy invariant. Furthermore, it satisfies the following arithmetic property.

Theorem 3.1.1 ([48], Theorem 3.1.2)**.** *Let Γ be a Kleinian group. Then $\mathbb{Q}(\mathrm{Tr}\,\Gamma)$ is an algebraic number field.*

However $\mathbb{Q}(\mathrm{Tr}\,\Gamma)$ is *not* a commensurability invariant. Instead, associate to Γ the group

$$\Gamma^{(2)} = \langle \gamma^2 \mid \gamma \in \Gamma \rangle.$$

Following Maclachlan and Reid we introduce the following notions.

Definition 3.1.2.

i) The field $K\Gamma^{(2)} := \mathbb{Q}(\mathrm{Tr}(\gamma) \mid \gamma \in \Gamma^{(2)})$ is called the *invariant trace field*;

ii) the algebra $A\Gamma^{(2)} := \left\{\sum a_i\gamma_i \mid a_i \in K\Gamma^{(2)}, \gamma_i \in \Gamma^{(2)}\right\}$, where only finitely many a_i's are non-vanishing, is called the *invariant quaternion algebra*.

The terminologies above are justified since both $K\Gamma^{(2)}$ and $A\Gamma^{(2)}$ are commensurability invariants (see [48, Theorem 3.3.4 and Corollary 3.3.5]).

In the special case of *arithmetic* groups in $\mathrm{PSL}(2, \mathbb{C})$, the following important result holds.

Theorem 3.1.3 ([48], Theorem 8.4.1)**.** *Let Γ be an arithmetic Kleinian group. Then $\left\{K\Gamma^{(2)}, A\Gamma^{(2)}\right\}$ is a complete set of commensurability invariants.*

For a more in-depth study of these two invariants and for their practical use we refer to [48].

3.2 Analogies in higher dimensions

A nice property of $\mathcal{U}^3 = \mathbb{C} \times \mathbb{R}_+$ is the identification of direct isometries with small-sized complex matrices. Recall from Section 2.1.2 that 2×2 Clifford matrices with Ahlfors determinant 1 represent direct hyperbolic isometries according to

$$\mathrm{Isom}^+(\mathcal{U}^n) \cong \mathrm{PSL}(2, \mathcal{C}_{n-2}).$$

For $n \geq 5$, the Clifford algebra $\mathcal{C}_{n-2}$ is not commutative anymore. More specifically, the definition of a suitable trace for an element $\gamma \in \mathrm{SL}(2, \mathcal{C}_{n-2})$ which is conjugacy invariant carrying geometrical information becomes difficult. An attempt is due to Wada [75] who associates to γ a set of $n+1$ real traces. These traces are conjugacy invariant and allow one to characterise the Möbius transformation γ with respect to its fixed-point behaviour. His approach, however, does not seem adequate to single out a suitable trace field.

In the following, we look at $n = 5$ and view $\mathrm{Isom}^+(\mathcal{U}^5)$ in a different way by using quaternions and their complex interpretation.

3.2.1 Direct isometries of the hyperbolic 5-space

Let $\boldsymbol{H} = \{x_1 + x_2 i + x_3 j + x_4 k \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ denote the normed vector space of Hamiltonian quaternions and interpret $\mathcal{U}^5$ according to

$$\mathcal{U}^5 = \{x = (x_1, x_2, x_3, x_4, t) \in \mathbb{R}^5 \mid t > 0\} \cong \boldsymbol{H} \times \mathbb{R}_{>0},$$

with boundary given by $\partial\mathcal{U}^5 = \boldsymbol{H} \cup \{\infty\}$. Following Wilker [77], consider the group

$$\mathrm{SL}_\triangle(2, \boldsymbol{H}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, \boldsymbol{H}) : \mid ad - aca^{-1}b \mid = 1 \right\},$$

and the projective group

$$\mathrm{PSL}_\triangle(2, \boldsymbol{H}) = \mathrm{SL}_\triangle(2, \boldsymbol{H}) / \{\pm I\}.$$

The quantity $\mid ad - aca^{-1}b \mid$ is called the *Dieudonné determinant*. Representatives $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of elements in $\mathrm{PSL}_\triangle(2, \boldsymbol{H})$ act bijectively on $\boldsymbol{H} \cup \{\infty\}$ by

$$T(x) = (ax + b)(cx + d)^{-1} \tag{3.2}$$

with the identification $T(-c^{-1}d) = \infty$, $T(\infty) = ac^{-1}$ if $c \neq 0$, and $T(\infty) = \infty$ otherwise.

Using Poincaré extension, the action of $\mathrm{PSL}_\triangle(2, \boldsymbol{H})$ given by (3.2) can be extended to the upper half-space $\mathcal{U}^5$. In this way we obtain an isomorphism between $\mathrm{PSL}_\triangle(2, \boldsymbol{H})$ and the group $\mathrm{Möb}^+(4)$ of orientation preserving Möbius transformations of $\boldsymbol{H} \cup \{\infty\}$ (see also [77, Theorem 2]). Since the group $\mathrm{Isom}^+\left(\mathcal{U}^5\right)$ of orientation preserving isometries of $\mathcal{U}^5$ is isomorphic to $\mathrm{Möb}^+(4)$, we get the following identification:

$$\mathrm{Isom}^+\left(\mathcal{U}^5\right) \cong \mathrm{Möb}^+(4) \cong \mathrm{PSL}_{\triangle}(2,\boldsymbol{H}). \tag{3.3}$$

3.2.2 Complexification of quaternionic matrices

Consider a quaternion $q = x_1 + x_2 i + x_3 j + x_4 k$, with $i^2 = j^2 = -1$ and $ij = k$. This quaternion can be expressed in a unique way by means of two complex numbers u and v according to

$$q = (x_1 + x_2 i) + (x_3 + x_4 i)j =: u + vj \in \mathrm{span}_{\mathbb{C}}(1, j).$$

In this way q can be interpreted as the tuple $(u, v) \in \mathbb{C}^2$ so that $\partial\mathcal{U}^5 = \mathbb{C}^2 \cup \{\infty\}$. Similarly to the interpretation of a complex number by a 2×2 real matrix, we can furthermore represent q as a complex matrix via

$$q = u + vj \backsim \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}. \tag{3.4}$$

This interpretation enjoys the following properties:

i) $q^{-1} \backsim \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}^{-1}$ for $q \neq 0$;

ii) $\overline{q} = x_1 - x_2 i - x_3 j - x_4 ij \backsim \begin{pmatrix} \overline{u} & -v \\ \overline{v} & u \end{pmatrix}$;

iii) $|q|^2 = \det \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}$;

iv) $\Re(q) = x_1 = \frac{1}{2}\,\mathrm{Tr} \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}$.

Inspired by (3.4), define the map $\iota : \mathrm{SL}_{\triangle}(2,\boldsymbol{H}) \to \mathrm{SL}(4,\mathbb{C})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} a_1 & a_2 & b_1 & b_2 \\ -\overline{a_2} & \overline{a_1} & -\overline{b_2} & \overline{b_1} \\ \hline c_1 & c_2 & d_1 & d_2 \\ -\overline{c_2} & \overline{c_1} & -\overline{d_2} & \overline{d_1} \end{array}\right). \tag{3.5}$$

The map ι is an injective group homomorphism (see [4], [24]). Let $\mathcal{M}$ be the image of $\mathrm{SL}_{\triangle}(2,\boldsymbol{H})$ with respect to ι, that is, $\mathcal{M}$ is the matrix group given by

$$\mathcal{M} = \left\{ A = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ -\overline{a_2} & \overline{a_1} & -\overline{b_2} & \overline{b_1} \\ c_1 & c_2 & d_1 & d_2 \\ -\overline{c_2} & \overline{c_1} & -\overline{d_2} & \overline{d_1} \end{pmatrix} \in \mathrm{Mat}(4,\mathbb{C}) \,\middle|\, \det(A) = 1 \right\}, \tag{3.6}$$

which is a connected semisimple Lie group (see [41, Chapter I.17]). By construction, we get the group isomorphism $\iota : \mathrm{SL}_\triangle(2, \boldsymbol{H}) \to \mathcal{M}$.

Based on the correspondence (3.3), we finally obtain the following complexified picture for $\mathrm{Isom}^+(\mathcal{U}^5)$:

$$\mathrm{Isom}^+(\mathcal{U}^5) \cong \mathrm{PSL}_\triangle(2, \boldsymbol{H}) \cong P\mathcal{M} := \mathcal{M}/\{\pm I\} \tag{3.7}$$

3.3 A trace field for subgroups of $\mathrm{Isom}^+(\mathcal{U}^5)$

We start this section by defining a trace and a trace field that can be associated to every element and to every subgroup of $\mathrm{Isom}^+(\mathcal{U}^5) \cong \mathrm{PSL}_\triangle(2, \boldsymbol{H})$.

Definition 3.3.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_\triangle(2, \boldsymbol{H})$ be a quaternionic matrix of Dieudonné determinant 1. The *trace* of A is defined by

$$\mathcal{T}(A) := \Re(a + d) = \frac{1}{2}\,\mathrm{Tr}\,(\iota(A))\,.$$

Remark 3.3.2. In [56], Parker and Short define a quantity $\tau(A)$ for a matrix $A \in \mathrm{SL}_\triangle(2, \boldsymbol{H})$ which takes the role of a quaternionic trace. However, $\tau(A)$ is *not* a conjugacy invariant and hence not suitable for our purpose.

Let Γ be a discrete subgroup of $\mathrm{PSL}_\triangle(2, \boldsymbol{H})$ and consider a lift of Γ to $\mathrm{SL}_\triangle(2, \boldsymbol{H})$ which we denote by the same letter Γ if the context is clear. Notice that the traces of elements of $\mathrm{PSL}_\triangle(2, \boldsymbol{H})$ and of their lifts to $\mathrm{SL}_\triangle(2, \boldsymbol{H})$ differ only by a sign.

Definition 3.3.3. The field

$$\mathbb{Q}(\mathcal{T}(\Gamma)) := \mathbb{Q}(\mathcal{T}(\gamma) \mid \gamma \in \Gamma)$$

is called the *trace field* associated to Γ.

Obviously, the trace field $\mathbb{Q}(\mathcal{T}(\Gamma))$ is a conjugacy invariant. In the following we prove a result analogous to Theorem 3.1.1 but in a more restrictive setting.

Theorem 3.3.4. *Let $\Gamma < \mathrm{PSL}_\triangle(2, \boldsymbol{H})$ be a torsion-free cocompact discrete group. Then $\mathbb{Q}(\mathcal{T}(\Gamma))$ is an algebraic number field.*

The proof of this theorem is similar to the one of Theorem 3.1.2 in [48] but requires an adaptation to our situation. A very condensed version of this strategy can be found in the last paragraph of [8, p. 124].

Proof.
Step 1. Let $\Gamma < \mathrm{PSL}_\triangle(2, \boldsymbol{H})$ be a torsion-free cocompact discrete group.

Then, Γ is finitely presented, non-elementary[2] and irreducible (see [59, Theorem 13.5.3], [39, Kapitel 2], [40, Chapter 4], for example).

Next, consider a lift of $\Gamma < \mathrm{PSL}_{\triangle}(2, \boldsymbol{H})$ to $\mathrm{SL}_{\triangle}(2, \boldsymbol{H})$ and pass to its complexified image in $\mathcal{M}$ – denoted by Γ again – by using the group isomorphism $\iota : \mathrm{SL}_{\triangle}(2, \boldsymbol{H}) \to \mathcal{M}$ as given by (3.5).

Since Γ is finitely presented, there is a finite set of generators $A = \{A_1, \ldots, A_k\}$ with $A_i \in \mathcal{M}$ and a finite set of relations $R = \{R_1, \ldots, R_l\}$ for Γ satisfying

$$R_j(A_1, \ldots, A_k) = I \quad \forall\ 1 \leq j \leq l.$$

Since Γ is non-elementary, there are at least two loxodromic elements without common fixed-points in $\partial\mathcal{U}^5 = \mathbb{C}^2 \cup \{\infty\}$ (see [38, Corollary 3.25], for example). Pick two of them and assume that they are in A, say A_1 and A_2. Modulo conjugation, we can suppose, without loss of generality, that the fixed-points of A_1 and A_2 are 0, ∞ and 1, q, respectively, with $q = (u, v) \in \mathbb{C}^2$. Indeed, Möbius transformations act triply transitively on $\mathbb{C}^2 \cup \{\infty\}$, and the trace field $\mathbb{Q}(\mathcal{T}(\Gamma))$ is a conjugacy invariant.

Step 2. Consider the space of deformations of $\Gamma < \mathcal{M}$,

$$\mathcal{R} = \mathcal{R}(\Gamma, \mathcal{M}) = \{\rho : \Gamma \to \mathcal{M} \mid \rho \text{ is a homomorphism}\},$$

equipped with the pointwise convergence topology (see [8]). Denote by ρ_0 the identity representation in $\mathcal{R}$.

We shall construct an algebraic variety $\mathcal{V}(\Gamma)$ in $\mathcal{R}$. The first equations defining $\mathcal{V}(\Gamma)$ are given by the relations

$$R_1(A_1, \ldots, A_k) = \cdots = R_l(A_1, \ldots, A_k) = I. \tag{3.8}$$

The conditions

$$\det(A_i) = 1 \quad \text{for} \quad 1 \leq i \leq k, \tag{3.9}$$

yield further k equations.

In this way, we obtain at most $16\,l + k$ equations. To these equations we add the ones given by the fixed-points conditions according to

$$A_1(0) = 0,\ A_1(\infty) = \infty,\ A_2(1) = 1 \text{ and } A_2(q) = q. \tag{3.10}$$

In the setting (3.2), the fixed-points equations are given by the quaternionic identities

$$b_1 = 0 = c_1 \text{ and } a_2 + b_2 = c_2 + d_2,$$

which yield a total of 12 equations for the (complex) coefficients of A_1 and A_2.

[2]A hyperbolic group Γ is said to be non-elementary if it does not exist a finite Γ-orbit in $\mathbb{H}^n \cup \partial\mathbb{H}^n$.

The equations (3.8), (3.9) and (3.10) are polynomial expressions over $\mathbb{Z}$ and define an algebraic set (for each ρ) in $\mathcal{R}$ defined over $\mathbb{Q}$. Choose an irreducible subset of this algebraic set, which contains the inclusion map ρ_0, and call this algebraic variety $\mathcal{V}(\Gamma)$.

Step 3. The aim is to show that the algebraic variety $\mathcal{V}(\Gamma)$ has dimension zero. To this end, we need and cite two fundamental results. The first one is the Local Rigidity Lemma for irreducible lattices in semisimple Lie groups (see [8, Theorem 1.1]).

Theorem 3.3.5 (Local Rigidity Lemma)**.** *Let $\rho \in \mathcal{R}$ be a deformation of Γ sufficiently close to the identity ρ_0. Then ρ is an isomorphism, and $\rho(\Gamma)$ is a lattice.*

The second important ingredient is Mostow's Rigidity Theorem [52].

Theorem 3.3.6 (Mostow's Rigidity Theorem)**.** *For $n \geq 3$, let Γ_1 and Γ_2 be two isomorphic cocompact lattices in* $\mathrm{Isom}(\mathbb{H}^n)$*. Then Γ_1 and Γ_2 are conjugate in* $\mathrm{Isom}(\mathbb{H}^n)$*.*

Now, assume that $\dim(\mathcal{V}(\Gamma)) > 0$. Then, there are infinitely many distinct points in $\mathcal{V}(\Gamma)$ and thus infinitely many distinct $\rho_s \in \mathcal{R} \setminus \{\rho_0\}$ in an arbitrary neighbourhood of ρ_0. By Theorem 3.3.5, for every ρ_s, the image group $\rho_s(\Gamma)$ is a cocompact lattice isomorphic to $\rho_0(\Gamma) = \Gamma$. By Theorem 3.3.6, all these groups are conjugate to Γ in $\mathrm{Isom}(\mathbb{H}^5)$.

This is not possible. Indeed, the equations (3.10) imposed by the fixed-points conditions on A_1 and A_2 allow only four possible conjugate images of Γ. These possibilities are described by the following mappings of the points $\{0, \infty, 1, q\}$:

$$\begin{aligned}
\{0, \infty, 1, q\} &\to \{0, \infty, 1, q\}, \\
\{0, \infty, 1, q\} &\to \{0, \infty, q, 1\}, \\
\{0, \infty, 1, q\} &\to \{\infty, 0, 1, q\}, \\
\{0, \infty, 1, q\} &\to \{\infty, 0, q, 1\}.
\end{aligned}$$

Hence, $\dim(\mathcal{V}(\Gamma)) = 0$.

Step 4. By Step 3, the variety $\mathcal{V}(\Gamma)$ has dimension zero. Hence, $\mathcal{V}(\Gamma)$ consists of a single point whose coordinates are algebraic numbers (see [48, Lemma 3.1.5]). Therefore, all the coefficients of the matrices A_i , $1 \leq i \leq k$, are algebraic numbers. Since Γ is generated by $A_1, \ldots, A_k$, all the coefficients of the matrices in Γ lie in a finite extension F of $\mathbb{Q}$. Hence $\mathbb{Q}(\mathcal{T}(\Gamma)) \subset F$ is an algebraic number field. □

By Theorem 3.3.4 we know that the trace field $\mathbb{Q}(\mathcal{T}(\Gamma))$ is an algebraic number field. However it is not clear how to relate $\mathbb{Q}(\mathcal{T}(\Gamma))$ to a commensurability invariant for $\Gamma < \mathrm{Isom}^+(\mathcal{U}^5)$. In fact, in contrast to the three-dimensional case, the trace $\mathcal{T}(\gamma)$ for $\gamma \in \Gamma$ does not even contain geometrically relevant information such as the rotational effect of γ.

As a consequence, we do not continue the study of commensurability of hyperbolic lattices in higher dimensions as started above in terms of trace fields.

Instead we take another direction: we shall restrict and investigate the commensurability problem for the important class of hyperbolic Coxeter groups in $\mathrm{Isom}(\mathbb{H}^n)$, $n \geq 2$, which are characterised by a nice finite presentation. In this way, we are able to obtain new commensurability conditions which will be exposed in the next chapter.

Chapter 4

Commensurability of hyperbolic Coxeter groups

In this chapter we focus our attention on the commensurability of hyperbolic Coxeter groups. The complete classification of hyperbolic Coxeter groups into commensurability classes has been achieved for certain classes such as Coxeter simplices [36] and Coxeter pyramids [28]. This has been done using methods of algebraic and geometrical nature. However there is no general method to classify Coxeter groups up to commensurability up to date.

When considering *arithmetic* hyperbolic lattices, a necessary and sufficient criterion for commensurability is known due to Gromov and Piatetski-Shapiro [27]. Their result states that two arithmetic hyperbolic lattices are commensurable if and only if their defining fields coincide and their quadratic forms are similar over the field.

As our main contribution, we provide a necessary commensurability condition which applies to any hyperbolic Coxeter group, even the non-arithmetic ones.

In [69], Vinberg provides an arithmeticity criterion for hyperbolic Coxeter groups by associating a field and a quadratic form to them, the *Vinberg field* and *Vinberg form*. We show that they encode commensurability information. However this property applied to non-arithmetic hyperbolic Coxeter groups is a necessary but *not* sufficient commensurability condition.

In the first part of this chapter we treat arithmetic hyperbolic lattices and present the commensurability criterion due to Gromov and Piatetski-Shapiro. We then discuss the theory needed for the classification of quadratic forms into similarity classes due to Maclachlan [46].

In the second part we exploit the work of Vinberg [69] motivating us to associate a field and a quadratic form to an *arbitrary* hyperbolic Coxeter group. After that, in the third part, we present and prove our commensurability condition in Theorem 4.3.1. We then refine this result by introducing the notion of the *Vinberg ring* and showing its commensurability invariance.

Section four is devoted to the discussion of some consequences of these conditions, like the stability of quasi-arithmeticity under commensurability. In the last part, we apply the Hasse-Minkowski theorem in order to classify Vinberg forms up to similarity. All the developed theory will be illustrated with examples throughout the chapter.

4.1 Commensurability of arithmetic hyperbolic lattices

We start by considering hyperbolic lattices which are arithmetic of the simplest type (see Definition 1.5.1). Before we start discussing their commensurability, we need the following definitions.

Definition 4.1.1. Let (V_1, q_1), (V_2, q_2) be two quadratic spaces of dimension $m \geq 2$ over a field K. Then (V_1, q_1) and (V_2, q_2) are

i) *isometric* (denoted by $\cong$) if and only if there is an isomorphism $S : V_1 \to V_2$ such that

$$q_1(x) = q_2(Sx) \quad \forall x \in V_1;$$

ii) *similar* (denoted by $\backsim$) if there exist a $\lambda \in K^*$ such that (V_1, q_1) and $(V_2, \lambda q_2)$ are isometric. The scalar λ is called *similarity factor*.

Remark 4.1.2.

i) Isometry and similarity induce equivalence relations;

ii) in the sequel, we often abbreviate and speak about *isometric* (*similar*) quadratic forms instead of isometric (similar) quadratic spaces;

iii) in the literature, sometimes two isometric quadratic forms are called *equivalent*;

iv) if one represents quadratic forms with two $m \times m$ matrices Q_1 and Q_2 over K, then being isometric means that there exists an invertible matrix $S \in \mathrm{GL}(m, K)$ such that $Q_1 = S^T Q_2 S$.

Let Γ_1 and Γ_2 be two arithmetic hyperbolic lattices, that is, they are discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$, $n \geq 2$, subject to the Definition 1.5.1. Let (V_1, q_1) and (V_2, q_2) be the quadratic spaces associated to Γ_1 and Γ_2 over the fields K_1 and K_2, respectively.

Theorem 4.1.3 ([27], Theorem 2.6)**.** *Let Γ_1, $\Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be two arithmetic groups as above. Then they are commensurable if and only if $K_1 = K_2 =: K$ and their quadratic spaces are similar over K.*

Following Theorem 4.1.3, the commensurability classification of arithmetic hyperbolic lattices boils down to the similarity classification of their quadratic forms. In [46], Maclachlan provides such a criterion, describing a complete set of commensurability invariants for arithmetic hyperbolic lattices. His invariants rely strongly on two elements of the Brauer group of a quadratic form: the *Hasse invariant* and the *Witt invariant*.

In the next section we will develop the algebraic background needed for the similarity classification. Experts on the matter can skip this section and go directly to Section 4.1.2, where the criterion of Maclachlan is stated.

4.1.1 The Brauer group and the Hasse invariant

The study of similarity of quadratic forms heavily relies on quaternion algebras and elements of the Brauer group. We recall all the important definitions and properties. For a more detailed explanation we refer to [43]. Let K be a field of characteristic different from 2, and let q be a quadratic form of dimension m over K, that is, q is defined on a vector space of dimension m over K.

Definition 4.1.4. Let $a,\ b \in K^*$. A quaternion algebra $(a,b)_K$, ofted denoted by (a,b) if the context is clear, is the algebra over K generated by the elements 1, i, j, ij with the relations $i^2 = a$, $j^2 = b$ and $ij = -ji$.

A quaternion algebra is a four-dimensional, central and simple algebra[3]. Conversely, every four-dimensional, central and simple algebra is a quaternion algebra. The basic example of such an algebra is the Hamiltonian quaternion algebra $\boldsymbol{H} = (-1,-1)_{\mathbb{R}}$.

Let A be a central and simple algebra over K. Then Wedderburn's theorem states that there is a division algebra D and a natural number r such that $A \cong \mathrm{Mat}(r,D)$. The algebra D is unique up to isomorphism. One can then define an equivalence relation on the set of central simple algebras in the following way: two central simple algebras $A \cong \mathrm{Mat}(r,D)$ and $A' \cong \mathrm{Mat}(r',D')$ over K are equivalent if and only if $D \cong D'$. In particular, two central simple algebras of the same dimension are equivalent if and only if they are isomorphic.

On the set of equivalence classes of central simple algebras we define a multiplication between classes by means of $[A]\cdot[A'] := [A \otimes_K A']$. This operation has $[(1,1)_K]$ as the neutral element and the inverse of a class $[A]$ is $[A]^{-1} := [A^{op}]$ where A^{op} denotes the opposite algebra of A, defined by $a \cdot_{op} b = b \cdot a$. The set of equivalence classes endowed with this multiplication is an abelian group, the *Brauer group*, denoted by $\mathrm{Br}(K)$. In order to simplify the notation, we will write $A \cdot B$ instead of $[A]\cdot[B]$.

[3]An algebra over K is said to be central if its center is exactly K. Moreover it is said to be simple if it does not have any non-trivial two-sided ideal.

Of importance for the commensurability of arithmetic hyperbolic Coxeter groups is the set of isomorphism classes of quaternion algebras. This set gives a subgroup of $\mathrm{Br}(K)$ (see [68, Theorem 2.9]). Moreover, the following computational properties hold.

Proposition 4.1.5. *For every $a, b, c \in K^*$, we have the following isomorphisms of quaternion algebras over K:*

$$\begin{aligned}
&(a,b) \cong (b,a), \quad (a, c^2 b) \cong (a,b), \quad (a,a) \cong (a,-1),\\
&(a,1) \cong (a,-a) \cong (1,1), \quad (a, 1-a) \cong (1,1) \text{ if } a \neq 1,\\
&(a,b)\cdot(a,c) \cong (a,bc).
\end{aligned}$$

If the field K is a number field, there is a very powerful tool to decide if two quaternion algebras are isomorphic and, hence, equivalent.

Theorem 4.1.6 ([46], Theorem 4.1)**.** *Let K be a number field. Two quaternion algebras over K are isomorphic if and only if they have the same ramification set.*

Remark 4.1.7. For the definition of a ramification set $\mathrm{Ram}(A)$ for a quaternion algebra A and its theory we refer to [46]. In this thesis, the computations of ramification sets are done using the package RamifiedPlaces of Magma©.

The similarity classification of quadratic forms relies on two elements of the Brauer group, which are closely related to one another: the Hasse invariant and the Witt invariant.

Definition 4.1.8. The *Hasse invariant* $s(q)$ of a diagonal quadratic form $q = \langle a_1, \ldots, a_m \rangle$ of dimension m over K is the element of the Brauer group $\mathrm{Br}(K)$ represented by the quaternion algebra

$$s(q) = \bigotimes_{i<j} (a_i, a_j)_K\,.$$

The Hasse invariant $s(q)$ is independent of the diagonalisation chosen. It is moreover an isometry invariant (see [43, Proposition 3.18]). However it is *not* a similarity invariant. Indeed, for any quadratic form q of dimension m over K and for any $\lambda \in K^*$ we have (see also [50])

$$s(\lambda q) = \begin{cases} (\lambda, \mathrm{disc}(q)) \cdot s(q) & m \text{ even},\\ (\lambda, (-1)^{\frac{m-1}{2}}) \cdot s(q) & m \text{ odd}, \end{cases} \tag{4.1}$$

where $\mathrm{disc}(q) = (-1)^{\frac{m(m-1)}{2}} \det(q)$ denotes the *discriminant* of q (sometimes also called the *signed determinant*).

We pass now to the Witt invariant $c(q)$ of a quadratic space (V, q) over K. Associated to (V, q) is a Clifford algebra Cl and its even part Cl_0 (see for example [43, Chapter 5]).

Definition 4.1.9. Let (V, q) be a quadratic space over a field K. The *Witt invariant* $c(q)$ is the element of $\mathrm{Br}(K)$ defined as follows:

$$c(q) := \begin{cases} \big[\,\mathrm{Cl}_0(V, q)\big] & \text{if } \dim V \text{ is odd,} \\ \big[\,\mathrm{Cl}(V, q)\big] & \text{if } \dim V \text{ is even.} \end{cases}$$

We can circumvent the technicalities of the definition above since there is a close relationship between the Hasse invariant $s(q)$ and the Witt invariant $c(q)$ according to the following proposition.

Proposition 4.1.10 ([43], Chapter V, Proposition 3.20)**.** *Let (V, q) be a quadratic space over K. The Hasse invariant $s(q)$ and the Witt invariant $c(q)$ are related as follows:*

$$c(q) = \begin{cases} s(q) & \dim V \equiv 1, 2 \ (\mathit{mod}\ 8), \\ s(q) \cdot (-1, -\det q) & \dim V \equiv 3, 4 \ (\mathit{mod}\ 8), \\ s(q) \cdot (-1, -1) & \dim V \equiv 5, 6 \ (\mathit{mod}\ 8), \\ s(q) \cdot (-1, \det q) & \dim V \equiv 7, 8 \ (\mathit{mod}\ 8). \end{cases} \tag{4.2}$$

4.1.2 Maclachlan's similarity classification

As already mentioned, Maclachlan gives a complete set of invariants describing the similarity class of a quadratic form q which has *appropriate signature*, that is: q is of dimension $n + 1 \geq 3$ over a totally real field K, q is of signature $(n, 1)$, and the image of q under every non-trivial embedding $\sigma : K \hookrightarrow \mathbb{R}$ becomes positive definite[4].

Let Γ be an arithmetic group with defining field K acting on $\mathbb{H}^n$. Let (V, q) be the quadratic space of dimension $n + 1$ over K associated to Γ and put $\delta := \mathrm{disc}(q)$. Since Γ is arithmetic, q has appropriate signature. Denote by B the quaternion algebra representing the Witt invariant $c(q)$. The similarity class of (V, q) depends on the parity of n as follows.

Theorem 4.1.11 ([46], Theorem 7.2)**.** *When n is even, the similarity class of the quadratic space (V, q) of dimension $n + 1$ is in one-to-one correspondence with the isomorphism class of the quaternion algebra B.*

Theorem 4.1.12 ([46], Theorem 7.4)**.** *When n is odd, the similarity class of the quadratic space (V, q) of dimension $n+1$ is in one-to-one correspondence with the isomorphism class of the quaternion algebra $B \otimes_K K(\sqrt{\delta})$ over $K(\sqrt{\delta})$. Moreover, if δ is a square in K^*, then the similarity class is in one-to-one correspondence with the isomorphism class of B over $\mathbb{Q}$.*

Remark 4.1.13. Maclachlan's similarity classification for quadratic forms as stated above is also valid for *quasi-arithmetic* hyperbolic lattices. Indeed, their quadratic forms have appropriate signature as well (see Section 4.5).

[4]For the image of q under σ we mean applying the map σ to every coefficient of q.

4.2 Vinberg's arithmeticity criterion and Vinberg's construction

Deciding whether a hyperbolic Coxeter group is arithmetic, quasi-arithmetic or nq-arithmetic can be done by means of the well-known arithmeticity criterion of Vinberg [69, Theorem 2]. A crucial role is played by the cycles of a matrix.

Definition 4.2.1. Let $C = (c_{ij})_{1 \leq i,j \leq m}$ be a matrix. A *cycle* (or *cycle product*) is defined as

$$c_{i_1 i_2} c_{i_2 i_3} \dots c_{i_{l-1} i_l} c_{i_l i_1}$$

for any $\{i_1, i_2, \dots, i_l\} \subset \{1, 2, \dots, m\}$. A cycle is called *simple* if the indices i_j in the cycle are all distinct.

Theorem 4.2.2 (Vinberg's arithmeticity criterion)**.** *Let* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *be a Coxeter group of rank* N *and denote by* $G = (g_{ij})_{1 \leq i,j \leq N}$ *its Gram matrix. Let* $\widetilde{K}$ *be the field generated by the entries of* G *and let* $K(\Gamma)$ *be the field generated by all the possible cycles of* G*. Then* Γ *is quasi-arithmetic if and only if:*

i) $\widetilde{K}$ *is totally real;*

ii) for any embedding $\sigma : \widetilde{K} \hookrightarrow \mathbb{R}$ *which is not the identity on* $K(\Gamma)$*, the matrix* G^σ *is positive semidefinite.*

Moreover, a quasi-arithmetic group Γ *is arithmetic if and only if*

iii) the cycles of $2\,G$ *are algebraic integers in* $K(\Gamma)$*.*

In both cases, Γ *is defined over* $K(\Gamma)$*.*

Remark 4.2.3.

i) As usual G^σ is the matrix obtained by applying the embedding σ the every coefficient of G;

ii) by Remark 1 of [69], a non-cocompact hyperbolic Coxeter group is quasi-arithmetic if and only if it is defined over $\mathbb{Q}$. This implies that a non-cocompact Coxeter group is arithmetic if and only if the cycles of $2\,G$ are rational integers;

iii) if the Coxeter graph of a quasi-arithmetic hyperbolic Coxeter group does not have any dotted edges, then the group is arithmetic ([69, Remark 3]).

By Theorem 4.1.3, we can detect the commensurability of arithmetic hyperbolic Coxeter groups with the help of their defining fields and quadratic forms. The criterion of Vinberg explicitly describes these fields in terms of the cycles of their Gram matrices. Furthermore, the construction of the quadratic form is given in the proof of Theorem 4.2.2 and goes as follows (see also [48, §10.4], [47] and [23]).

Let Γ be a hyperbolic Coxeter group of rank N and let $e_1, \dots, e_N \in \mathbb{R}^{n,1}$ be the outer normal unit vectors of its Coxeter polyhedron. Let $G = (g_{ij})_{1 \le i,j \le N}$ be the Gram matrix of Γ. For any $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, N\}$ consider the cyclic product of $2\,G$

$$b_{i_1 i_2 \dots i_k} := 2^k g_{i_1 i_2} g_{i_2 i_3} \dots g_{i_{k-1} i_k} g_{i_k i_1}. \tag{4.3}$$

Define the field $K(\Gamma) := \mathbb{Q}(\{b_{i_1 i_2 \dots i_k}\})$ of all cycles of $2\,G$. It is obvious that $K(\Gamma)$ is generated by the simple cycles.

Remark 4.2.4.

i) A non-zero cycle in $K(\Gamma)$ corresponds to a closed path on the Coxeter graph of Γ;

ii) the power of 2 in the definition (4.3) has no influence on the computation of the cycle field $K(\Gamma)$. It is left in the definition anyway for coherence with Vinberg's arithmeticity criterion part *iii*). In Section 4.3.4, we will use the cycles (4.3) to define a certain ring.

Next, for $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, N\}$, define the vectors

$$v_1 := 2e_1 \text{ and } v_{i_1 i_2 \dots i_k} := 2^k g_{1 i_1} g_{i_1 i_2} \dots g_{i_{k-1} i_k} e_{i_k}. \tag{4.4}$$

Consider the $K(\Gamma)$-vector space V spanned by the vectors $\{v_{i_1 i_2 \dots i_k}\}$ according to (4.4). By [23, Lemma 1], V is of dimension $n+1$. Moreover, V is left invariant by Γ. Indeed, let s_{e_j} be a generating reflection of Γ. We get

$$\begin{aligned} s_{e_j}(v_{i_1 i_2 \dots i_k}) &= v_{i_1 i_2 \dots i_k} - 2\langle v_{i_1 i_2 \dots i_k}, e_j \rangle e_j \\ &= v_{i_1 i_2 \dots i_k} - 2^{k+1} g_{1 i_1} g_{i_1 i_2} \dots g_{i_{k-1} i_k} \langle e_{i_k}, e_j \rangle e_j \\ &= v_{i_1 i_2 \dots i_k} - v_{i_1 i_2 \dots i_k j}. \end{aligned} \tag{4.5}$$

Let us compute the Lorentzian product of two spanning vectors of V:

$$\begin{aligned} \langle v_{i_1 \dots i_k}, v_{j_1 j_2 \dots j_l} \rangle &= 2^k g_{1 i_1} g_{i_1 i_2} \dots g_{i_{k-1} i_k} \cdot 2^l g_{1 j_1} g_{j_1 j_2} \dots g_{j_{l-1} j_l} \cdot g_{i_k j_l} \\ &= \frac{1}{2}(2^{k+l+1} g_{1 i_1} g_{i_1 i_2} \dots g_{i_{k-1} i_k} g_{i_k j_l} g_{j_l j_{l-1}} \dots g_{j_1 1}) \\ &= \frac{1}{2} \cdot b_{1 i_1 \dots i_k j_l \dots j_1} \in K(\Gamma). \end{aligned} \tag{4.6}$$

Since $2\,G$ is of signature $(n,1)$, the restriction of the Lorentzian product on V yields a quadratic form $q = q_V$ of signature $(n,1)$ on V.

By combining the equations (4.5) and (4.6) we also get

$$\langle s_{e_j}(v_{i_1 i_2 \dots i_k}), s_{e_j}(v_{j_1 j_2 \dots j_l})\rangle = \langle v_{i_1 i_2 \dots i_k}, v_{j_1 j_2 \dots j_l}\rangle. \tag{4.7}$$

Indeed, by linearity of the Lorentzian product and by commuting factors appropriately $\langle s_{e_j}(v_{i_1 i_2 \dots i_k}), s_{e_j}(v_{j_1 j_2 \dots j_l})\rangle$ is equal to

$$\frac{1}{2} \cdot b_{1 i_1 \dots i_k j_l \dots j_1} - b_{1 i_1 \dots i_k j j_l \dots j_1} + \frac{1}{2} \cdot b_{1 i_1 \dots i_k j j j_l \dots j_1}.$$

Since $g_{jj} = 1$, it follows that $\frac{1}{2} \cdot b_{1 i_1 \dots i_k j j j_l \dots j_1} = b_{1 i_1 \dots i_k j j_l \dots j_1}$, and we get the equality (4.7).

Let us make the following important observation: by the construction of the $K(\Gamma)$-vector space V in terms of the vectors (4.4) and the form q_V we obtain a natural embedding $\Gamma \hookrightarrow \mathrm{O}(V,q)$. This construction is independent of the arithmetic nature of Γ.

Therefore *any* hyperbolic Coxeter group has an associated field and quadratic form which justifies the following definition.

Definition 4.2.5. Let Γ be a hyperbolic Coxeter group. Then

i) the field $K(\Gamma) = \mathbb{Q}(\{b_{i_1 i_2 \dots i_l}\})$ is called the *Vinberg field* of Γ;

ii) the quadratic form $q = q_V$ is called the *Vinberg form* of Γ;

iii) the quadratic space (V, q) is called the *Vinberg space* of Γ.

Example 4.2.6. Let us illustrate the Vinberg construction with an example. Consider the nq-arithmetic Coxeter pyramid group Γ acting cofinitely on $\mathbb{H}^4$ and defined by the graph given in Figure 4.1.

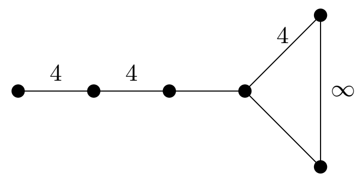


Figure 4.1: The Coxeter group Γ in $\mathrm{Isom}(\mathbb{H}^4)$.

Its Gram matrix is

$$G = \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 & 0 & 0 & 0 \\ -\sqrt{2}/2 & 1 & -\sqrt{2}/2 & 0 & 0 & 0 \\ 0 & -\sqrt{2}/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & -\sqrt{2}/2 & -1/2 \\ 0 & 0 & 0 & -\sqrt{2}/2 & 1 & -1 \\ 0 & 0 & 0 & -1/2 & -1 & 1 \end{pmatrix}.$$

The Coxeter pyramid has the following outer normal unit vectors:

$$
\begin{aligned}
e_1 &= (1, 0, 0, 0, 0),\\
e_2 &= \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\right),\\
e_3 &= (0, -1, 1, 0, 1),\\
e_4 &= \left(0, 0, -3/4, \frac{-1}{\sqrt{2}}, \frac{-1}{4}\right),\\
e_5 &= (0, 0, 0, 1, 0),\\
e_6 &= \left(0, 0, 1+\sqrt{2}, -1, 1+\sqrt{2}\right).
\end{aligned}
$$

The group Γ has Vinberg field $K(\Gamma) = \mathbb{Q}\left(\sqrt{2}\right)$. By the construction of Vinberg, let V be the $\mathbb{Q}(\sqrt{2})$-vector space of dimension 5 spanned by the vectors as given by (4.4). The following vectors form a basis of V:

$$
\begin{aligned}
v_1 &:= 2e_1 = (2, 0, 0, 0, 0),\\
v_2 &:= 4g_{11}g_{12}e_2 = (2, -2, 0, 0, 0),\\
v_3 &:= 8g_{11}g_{12}g_{23}e_3 = (0, -4, 4, 0, -4),\\
v_4 &:= 16g_{11}g_{12}g_{23}g_{34}e_4 = \left(0, 0, 3, 2\sqrt{2}, 1\right),\\
v_5 &:= 32g_{11}g_{12}g_{23}g_{34}g_{45}e_5 = \left(0, 0, 0, 4\sqrt{2}, 0\right).
\end{aligned}
$$

The matrix representing the Vinberg form q equals

$$
Q = \langle v_i, v_j \rangle_{1 \le i,j \le 5} = \begin{pmatrix} 4 & 4 & 0 & 0 & 0\\ 4 & 8 & 8 & 0 & 0\\ 0 & 8 & 16 & 8 & 0\\ 0 & 0 & 8 & 16 & 16\\ 0 & 0 & 0 & 16 & 32 \end{pmatrix}.
$$

4.2.1 The Vinberg field and the Vinberg form of a Kleinian group

In this part we see how the Vinberg field and the Vinberg form of a Coxeter group in $\mathrm{Isom}^+(\mathbb{H}^3)$ are linked with commensurability invariants of Kleinian groups.

By Section 3.1, we know that $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$. In this context, we have the two commensurability invariants as given by the invariant trace field $K\Gamma^{(2)}$ and the invariant quaternion algebra $A\Gamma^{(2)}$. Recall that these invariants form a complete set of commensurability invariants if Γ is arithmetic. Furthermore, they are connected to the Vinberg field and the Vinberg form as follows.

Theorem 4.2.7 ([47], Theorem 3.1)**.** *Let Γ be a Kleinian group with Vinberg field $K(\Gamma)$ and diagonalised Vinberg form $q = \mathrm{diag}(q_1, q_2, q_3, q_4)$ with discriminant δ. Then*

i) $K\Gamma^{(2)} = K(\Gamma)(\sqrt{\delta})$;

ii) $A\Gamma^{(2)} \cong (-q_1\,q_2, -q_1\,q_3)_{K\Gamma^{(2)}}$.

This theorem suggests that the Vinberg field and the Vinberg form should contain commensurability information for discrete groups in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geq 2$, whether they are arithmetic or non-arithmetic. A corresponding result is given by Theorem 4.3.1.

4.2.2 The field of definition of a hyperbolic Coxeter group

Let us provide first some elements about rings of definition which are needed later in this chapter. Fundamental is the paper of Vinberg [71].

Definition 4.2.8. Let H be a Lie group. The *adjoint trace field* $\mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,H)$ of H is defined as the field generated by the traces of the adjoint representation of the elements of H, namely

$$\mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,H) = \mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}(h) \mid h \in H).$$

Let U be a finite dimensional vector space over a field F, and let $R \subset F$ be an integrally closed Noetherian ring. Denote by Δ a family of linear transformations of U.

Definition 4.2.9. The ring R is said to be a *ring of definition* for Δ if U contains an R-lattice which is invariant under Δ.

If a principal ideal domain R is a ring of definition for Δ, then we can find a basis of U such that every element of Δ can be written as a matrix having entries in R.

Remark 4.2.10. When R is a field we call R a *field of definition.* In this case any R-lattice is a vector space. Recall that fields are always integrally closed Noetherian principal ideal domains.

Let us specialise the context and consider a Coxeter group $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$. As we have seen in Section 4.2, the space $\mathbb{R}^{n,1}$ contains the $K(\Gamma)$-module V which is invariant under Γ. That is, the Vinberg field $K(\Gamma)$ is a field of definition for Γ. This is not a surprise as $K(\Gamma)$ is defined by means of the cycles of the Gram matrix of Γ.

Remark 4.2.11. The Vinberg field $K(\Gamma)$ is actually the *smallest* field of definition associated to Γ. This is a consequence of the following result.

Lemma 4.2.12 ([71], Lemma 11 and Lemma 12)**.** *Let Γ be a hyperbolic Coxeter group with Gram matrix G and let F be a field of characteristic 0. An integrally closed Noetherian ring $R \subset F$ is a ring of definition for Γ if and only if R contains all the simple cycles of $2\,G$.*

4.3 New commensurability conditions for hyperbolic Coxeter groups

4.3.1 Stating the theorem

By Theorem 4.1.3, we know that if two arithmetic hyperbolic lattices are commensurable, then their defining fields are equal and the associated quadratic spaces are similar over this field. This holds in particular for arithmetic hyperbolic Coxeter groups. In the sequel, we extend this result to *arbitrary* hyperbolic Coxeter groups by exploring their Vinberg fields and Vinberg forms. This will give us new necessary commensurability conditions as follows.

Theorem 4.3.1. *Let Γ_1 and Γ_2 be two commensurable cofinite hyperbolic Coxeter groups acting on $\mathbb{H}^n$, $n \geq 2$. Then their Vinberg fields coincide and the two associated Vinberg forms are similar over this field.*

4.3.2 Proof of the theorem

We begin the proof by showing that two commensurable Coxeter groups in in $\mathrm{Isom}(\mathbb{H}^n)$ have the same Vinberg field. This proof relies on Theorem 5 of the paper [71] of Vinberg. We recapitulate here a more specific version suitable to our context.

Theorem 4.3.2. *Let Γ be a cofinite hyperbolic Coxeter group with Vinberg space (V, q) and Gram matrix G. Let R be an integrally closed Noetherian ring. Then the following are equivalent:*

i) R is a ring of definition of Γ;

ii) R is a ring of definition of $\mathrm{Ad}\,\Gamma$;

iii) R contains all the simple cyclic products of $2\,G$.

Remark 4.3.3. It is important to notice that in [71] Vinberg considers Zariski dense groups generated by reflections of a quadratic space defined over an *algebraically closed* field. This hypothesis does not apply directly to our situation since the isometry group $\mathrm{PO}(n, 1)$ of Klein's projective model $\mathcal{K}^n$ is defined over the reals.

Our version of the theorem can be retrieved from the original one as follows. Pass to the complexified space $\mathbb{R}^{n+1} \otimes_{\mathbb{R}} \mathbb{C}$ endowed with the standard (real) Lorentzian form q_{-1}. Let $\mathrm{O}_{\mathbb{C}}(n,1)$ be the group of complex $(n+1) \times (n+1)$ matrices which preserve q_{-1}, and form the projective group $\mathrm{PO}_{\mathbb{C}}(n,1) = \mathrm{O}_{\mathbb{C}}(n,1)/\{\pm I\}$.

Recall that a cofinite hyperbolic Coxeter group is Zariski dense (over $\mathbb{R}$) in $\mathrm{PO}(n,1)$ (see [40, Chapter 4]). This property remains valid in the complexified context of $\mathrm{PO}_{\mathbb{C}}(n,1)$ over $\mathbb{C}$. We can now apply the original Theorem 5 of [71] which implies Theorem 4.3.2.

By means of Theorem 4.3.2, we can prove the following result.

Proposition 4.3.4. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a cofinite Coxeter group, $n \geq 2$. Then the associated Vinberg field and the adjoint trace field coincide, that is*

$$K(\Gamma) = \mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma). \tag{4.8}$$

Proof. By Remark 4.2.11, the Vinberg field $K(\Gamma)$ is the smallest field of definition of Γ. By point *i)* of Theorem 4.3.2, $K(\Gamma)$ is a field of definition of $\mathrm{Ad}\,\Gamma$ as well and by point *iii)* $K(\Gamma)$ is contained in every field of definition of $\mathrm{Ad}\,\Gamma$. On the other hand, $\mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma)$ is the smallest field of definition of $\mathrm{Ad}\,\Gamma$ ([71, Corrollary of Theorem 1]). Hence, the equality (4.8) follows.

□

Corollary 4.3.5. *Let Γ_1, $\Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be two cofinite Coxeter groups, $n \geq 2$. If Γ_1 and Γ_2 are commensurable, then their associated Vinberg fields coincide, that is,*

$$K(\Gamma_1) = K(\Gamma_2).$$

Proof. By Proposition 4.3.4 we know that $K(\Gamma_1) = \mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma_1)$ and $K(\Gamma_2) = \mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma_2)$. Since the adjoint trace field of a cofinite group in $\mathrm{Isom}(\mathbb{H}^n)$ is a commensurability invariant (see [18, Proposition 12.2.1]), the claim follows.

□

Remark 4.3.6.

i) By Corollary 1 of Theorem 4 of [71], if a hyperbolic Coxeter group Γ is generated by elements γ with traces $\mathrm{Tr}(\gamma) \in \mathbb{Q} \setminus \{0\}$, then the smallest field of definition of Γ is $\mathbb{Q}(\mathrm{Tr}\,\Gamma)$. Since reflections in $\mathrm{O}^+(n,1)$ have traces equal to $n-1$, we also get the equality

$$K(\Gamma) = \mathbb{Q}(\mathrm{Tr}\,\Gamma). \tag{4.9}$$

ii) By the Local Rigidity Theorem [55, Chapter 1], the adjoint trace field $\mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma)$ of a Coxeter group in $\mathrm{Isom}(\mathbb{H}^n)$ is a number field for $n \geq 4$.

Therefore, by Proposition 4.3.4, the Vinberg field $K(\Gamma)$ is a number field. Moreover, $K(\Gamma)$ is a number field for $n = 3$ as well. This is a consequence of the connection between $K(\Gamma)$ and the invariant trace field $K\Gamma^{(2)}$ (see Theorem 4.2.7) and the fact that $K\Gamma^{(2)}$ is a number field.

iii) The Vinberg field $K(\Gamma)$ is totally real for any quasi-arithmetic hyperbolic Coxeter group. Some tests support the conjecture that this remains true for nq-arithmetic hyperbolic Coxeter groups. This question was also raised by V. Emery.

For the proof of our theorem we now have to show that two commensurable hyperbolic Coxeter groups have similar Vinberg forms. The proof will follow the same strategy as indicated by Gromov and Piateski-Shapiro in Theorem 2.6 of [27] for arithmetic groups, and which has been elaborated by Johnson, Kellerhals, Ratcliffe and Tschantz in Theorem 1 of [36] for the special case of hyperbolic Coxeter simplex groups.

Consider two commensurable hyperbolic Coxeter groups Γ_1 and Γ_2 represented in $\mathrm{O}^+(n,1)$ and denote by K their Vinberg field. There is a matrix $X \in \mathrm{O}^+(n,1)$ and there are two subgroups $H_1 < \Gamma_1$ and $H_2 < \Gamma_2$, each of finite index, such that $H_1 = X^{-1}H_2X$. We may assume that H_1 and H_2 are contained in $\mathrm{SO}^+(n,1)$, the index two subgroup of $\mathrm{O}^+(n,1)$ of determinant one matrices. Let (V_1, q_1) be the Vinberg space over K associated to Γ_1 and equipped with a basis $\{v_1, \dots, v_{n+1}\}$ according to (4.4) such that all the elements of Γ_1 are matrices over K (since K is a field of definition of Γ). It is clear that the forms q_1 and q_{-1} are equivalent over $\mathbb{R}$. The same reasoning applies to the Vinberg space (V_2, q_2). Denote by Q_1 and Q_2 the matrix representations of the Vinberg forms q_1 and q_2 in the relative bases. Let us represent the isometries between the Vinberg forms and the Lorentzian form q_{-1} with the real matrices T_1 and T_2. We get that the matrix

$$S := T_2^{-1}XT_1 \tag{4.10}$$

represents an isometry between q_1 and q_2, since $Q_1 = S^TQ_2S$. Furthermore define the groups $H_1' := T_1^{-1}H_1T_1$ and $H_2' := T_2^{-1}H_2T_2$.

Consider the isomorphism between the orthogonal groups $\mathrm{O}(q_1)$ and $\mathrm{O}(q_2)$ given by

$$\phi : A \to SAS^{-1}. \tag{4.11}$$

Lemma 4.3.7. *The map ϕ restricts to a K-linear map on* $\mathrm{Mat}(n+1, K)$.

Proof. Let $i \in \{1,2\}$. The isometry between q_i and q_{-1} gives a group isomorphism between $\mathrm{O}(q_i)$ and $\mathrm{O}(q_{-1})$. This isomorphism maps $\mathrm{O}^+(q_i)$ onto $\mathrm{O}^+(n,1)$, where $\mathrm{O}^+(q_i)$ is the group of q_i-orthogonal maps which leave each sheet of the hyperboloid $\mathcal{H}_i^{n+1} = \{x \in \mathbb{R}^{n+1} \mid q_i(x) = -1\}$ invariant. Analogously, $\mathrm{SO}^+(q_i)$ is mapped onto $\mathrm{SO}^+(n,1)$ and hence $H_i' \subset \mathrm{SO}^+(q_i)$. Since $\mathrm{SO}^+(n,1)$ is a non-compact connected simple Lie group, the same can be said for $\mathrm{SO}^+(q_i)$. Since H_i' has finite covolume, by the Borel density theorem [10] we get that $\mathrm{Span}_{\mathbb{R}}(H_i') = \mathrm{Span}_{\mathbb{R}}(\mathrm{SO}^+(q_i))$ in $\mathrm{Mat}(n+1,\mathbb{R})$. Now, the action of $\mathrm{SO}^+(n,1)$ on $\mathbb{C}^{n+1}$ is irreducible[5]. Therefore the action of $\mathrm{SO}^+(q_i)$ is irreducible as well. By Burnside's theorem [13] (see also [42]) we get that $\mathrm{Span}_{\mathbb{R}}(\mathrm{SO}^+(q_i)) = \mathrm{Mat}(n+1,\mathbb{R})$, which implies that $\mathrm{Span}_{\mathbb{R}}(H_i') = \mathrm{Mat}(n+1,\mathbb{R})$.

Recall that $H_i' \subset \mathrm{Mat}(n+1,K)$, since K is a field of definition for H_i. Note that for each $\alpha \in K$ and $C \in \mathrm{Mat}(n+1,K)$ we have $\phi(\alpha C) = \alpha\phi(C)$. By the same arguments as before, we have that $\mathrm{Span}_K(H_i') = \mathrm{Mat}(n+1,K)$. Moreover, by (4.10),

$$\phi(H_1') = \phi(T_1^{-1}H_1T_1) = T_2^{-1}XH_1X^{-1}T_2 = T_2^{-1}H_2T_2 = H_2'.$$

We deduce that $\phi(\mathrm{Span}_K(H_1')) = \mathrm{Span}_K(H_2')$. Therefore ϕ restricts to a K-linear map on $\mathrm{Mat}(n+1,K)$.

□

Based on Lemma 4.3.7 we are finally ready to prove the last step as given by the following proposition. Its proof is a direct adaptation of the corresponding step in the proof of [36, Theorem 1].

Proposition 4.3.8. *Let Γ_1, Γ_2 be two commensurable Coxeter groups in $\mathrm{Isom}(\mathbb{H}^n)$, $n \geq 2$, with Vinberg field $K(\Gamma_1) = K(\Gamma_2) =: K$. Then the two Vinberg forms q_1 and q_2 are similar over K. Moreover, the similarity factor is positive.*

Proof. Let $1 \leq i,j \leq n+1$ and define the matrix $I_{ij} \in \mathrm{Mat}(n+1,K)$ with coefficient $[I]_{ij} = 1$ and all the other coefficients equal to zero. Consider the isomorphism ϕ according to (4.11) and define $M_{ij} := \phi(I_{ij}) = SI_{ij}S^{-1}$. By Lemma 4.3.7, $M_{ij} \in \mathrm{Mat}(n+1,K)$. The matrix $SI_{ij} =: S_{ij}$ has the j-th column which is equal to the i-th column of S and all the other coefficients are equal to zero. Observe that $[M_{ij}]_{kl} = [S]_{ki}[S^{-1}]_{jl}$ for all k,l,i,j. Since the matrix S^{-1} is invertible, we can always find a pair $\{j,l\}$ such that $[S^{-1}]_{jl} \neq 0$. Let us denote the inverse of the coefficient $[S^{-1}]_{jl}$ by λ. In this way, every coefficient of S can be written as λ multiplied with an entry of a matrix of the form $M_{ij} \in \mathrm{Mat}(n+1,K)$. Therefore there exists a matrix $M \in \mathrm{Mat}(n+1,K)$ such that $S = \lambda M$. Since $Q_1 = S^TQ_2S$ holds, we have $Q_1 = \lambda^2 M^TQ_2M$, with Q_1, Q_2 and M all in $\mathrm{Mat}(n+1,K)$. Thus λ^2 is a positive element belonging to K so that q_1 is isometric to $\lambda^2 q_2$. □

[5]For the detailed proof of this statement clarifying a gap in the original proof of Theorem 1 in [36], see "Erratum: Commensurability classes of hyperbolic Coxeter groups" by J. Ratcliffe and S. Tschantz given in Appendix D.

4.3.3 About the converse of the theorem

By combining Theorem 4.1.3 with Theorem 4.3.1 we have a necessary and sufficient commensurability criterion for hyperbolic Coxeter groups which are *arithmetic*. There is the question whether the converse of Theorem 4.3.1 remains valid in the non-arithmetic case. This is *not* the case, as shown by the following example.

Example 4.3.9. Consider the two non-cocompact quasi-arithmetic (but *not* arithmetic) Coxeter cube groups Γ_1 and Γ_2 acting on $\mathbb{H}^3$ and defined by the graphs in Figure 4.2 (see [34]). Observe that they both have $\mathbb{Q}$ as Vinberg field. Since Γ_1 and Γ_2 are quasi-arithmetic, we can apply Maclachlan's criterion to decide whether the Vinberg spaces (V_1, q_1) and (V_2, q_2) are similar (see Remark 4.1.13).

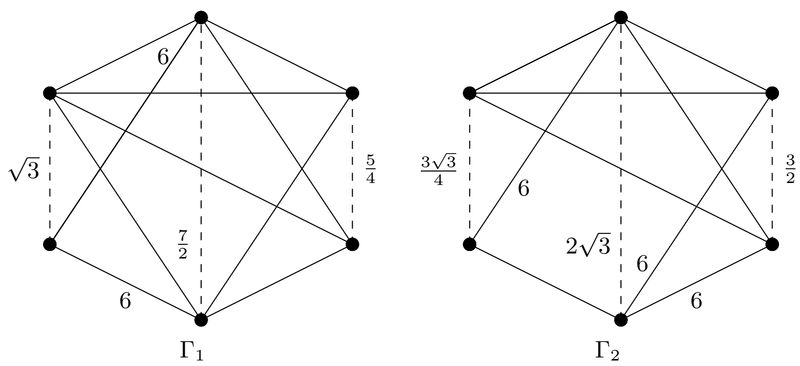


Figure 4.2: Quasi-arithmetic Coxeter cube groups Γ_1 and Γ_2 acting on $\mathbb{H}^3$.

With Vinberg's construction, we can compute the matrices representing q_1 and q_2 and diagonalise them over $\mathbb{Q}$. We obtain

$$q_1 = \operatorname{diag}\left(4, 3, \frac{-225}{4}, \frac{225}{4}\right) \text{ and } q_2 = \operatorname{diag}(4, 3, 15, -15).$$

The quadratic forms q_1 and q_2 have both $(-1, 3)$ as Hasse invariant and therefore they have identical Witt invariant represented by the quaternion algebra $B = (1, 1)$ (see Section 4.1.1). Hence, the ramification set of $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-1})$ over $\mathbb{Q}(\sqrt{-1})$ is identical for both groups. This implies that the Vinberg spaces are similar. However, as shown in [78] by means of a geometric argument, Γ_1 and Γ_2 are *not* commensurable.

4.3.4 The Vinberg ring

In this section we are looking for additional commensurability invariants for arbitrary hyperbolic Coxeter groups.

As a motivation consider the following situation. Let Γ_1 and Γ_2 be two hyperbolic Coxeter groups. Assume that Γ_1 is arithmetic and Γ_2 is quasi-arithmetic but not arithmetic. Hence they can not be commensurable. Denote by G_1 and G_2 their Gram matrices, and assume that they both have the same Vinberg field K. Let $\mathcal{O}$ be the ring of integers of the number field K (see Remark 4.3.6, part *ii*)). Due to Vinberg's arithmeticity criterion, the cycles of $2\,G_1$ are in $\mathcal{O}$ while the cycles of $2\,G_2$ are not all in $\mathcal{O}$. Based on this observation let us introduce the following ring.

Definition 4.3.10. Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 2$, be a cofinite Coxeter group with Gram matrix G. Consider all cycles $b_{i_1 i_2 \ldots i_k} = 2^k g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_{k-1} i_k} g_{i_k i_1}$ of $2\,G$. The ring

$$R(\Gamma) := \mathcal{O}(\{b_{i_1 i_2 \ldots i_k}\})$$

is called the *Vinberg ring* of Γ.

We show that the Vinberg ring is a ring of definition for certain groups and hence a commensurability invariant.

Proposition 4.3.11. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 2$, be a cofinite Coxeter group with Gram matrix G. Assume that its Vinberg field K is a number field. Then the Vinberg ring $R(\Gamma)$ is a commensurability invariant.*

Proof. The main idea for the proof is to use some results of Davis about overrings[6] ([15]) in the same way as used by Mila in [51, Section 2.1]. By hypothesis the Vinberg field K is a number field. Thus there exists a minimal ring of definition R for Γ ([71, Corollary to Theorem 1]) which equals the integral closure of $\mathbb{Z}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$ in K. Clearly R is integrally closed and therefore contains the ring of integers $\mathcal{O}$ of K. Thus R is the integral closure of $\mathcal{O}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma] =: R'$ in K. The ring R' is an overring of $\mathcal{O}$. Since $\mathcal{O}$ is a Noetherian integral Dedekind domain, its overring R' is integrally closed ([15, Theorem 1]). Therefore $R = R'$ which means that $\mathcal{O}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$ is the smallest ring of definition for Γ. Moreover, the Vinberg ring $R(\Gamma)$ is also an overring of $\mathcal{O}$, so that it is integrally closed as well. It is furthermore Noetherian since it is a subring of the number field K (see [25, Theorem]). Hence $R(\Gamma)$ is a ring of definition for Γ. By Theorem 4.3.2, $R(\Gamma) \subset R'$. Now, R' is the smallest ring of definition so that $R(\Gamma) = \mathcal{O}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$. By Theorem 3 of [71], rings of definition are commensurability invariants. Thus $R(\Gamma)$ is a commensurability invariant. □

Remark 4.3.12. Notice that the Vinberg ring as commensurability invariant is superfluous when considering arithmetic groups. Indeed, let Γ_1 and Γ_2 be two arithmetic hyperbolic Coxeter groups with Vinberg fields K_1, K_2

[6] An overring of an integral domain is a subring of the quotient field containing that given ring. In our case, the integral domain is the ring of integers $\mathcal{O}$ of the Vinberg field K, which has the Vinberg field as its quotient field.

and Gram matrices G_1 and G_2, respectively. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ denote the rings of integers of K_1 and K_2. By arithmeticity, every cycle of $2\,G_i$ is in $\mathcal{O}_i$, and hence $R(\Gamma_i) = \mathcal{O}_i$, $i \in \{1,2\}$. Now, if $K_1 \neq K_2$, the groups Γ_1 and Γ_2 are incommensurable. However, if $K_1 = K_2$, then $R(\Gamma_1) = R(\Gamma_2)$. Hence, in the latter case, we do not gain any additional commensurability information.

4.4 Some applications

We now investigate to which extent the Vinberg form of a hyperbolic Coxeter group is a commensurability invariant. We shall see that it encodes information such as appropriate signature and cocompactness, which both detect incommensurability.

Proposition 4.4.1. *Let Γ_1, $\Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be two commensurable cofinite Coxeter groups with identical Vinberg field K and with Vinberg forms q_1 and q_2. Then q_1 has appropriate signature if and only if q_2 has appropriate signature.*

Proof. It is enough to show one implication. Assume that q_1 has appropriate signature. Let Q_1 and Q_2 be matrix representations of q_1 and q_2, respectively. By assumption we know that Q_1^σ is positive definite for every non-trivial embedding $\sigma : K \hookrightarrow \mathbb{R}$. Since Γ_1 and Γ_2 are commensurable we know that there is a matrix $S \in \mathrm{Mat}(n+1, K)$ such that $Q_1 = \lambda S^T Q_2 S$ for some $\lambda \in K$, $\lambda > 0$ (see Proposition 4.3.8). Let $M := \sqrt{\lambda} S$. It follows that

$$Q_1^\sigma = (M^T Q_2 M)^\sigma = (M^\sigma)^T Q_2^\sigma M^\sigma.$$

By Silvester's law of inertia, two congruent matrices have the same signature. Hence Q_2^σ is positive definite for all non-trivial embeddings σ as well. It follows that Γ_2 has appropriate signature. □

The fact that the appropriate signature property for a quadratic form is stable under commensurability implies the following, probably folklore result.

Corollary 4.4.2. *Let Γ_1, $\Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be two commensurable cofinite Coxeter groups. Then Γ_1 is quasi-arithmetic if and only if Γ_2 is quasi-arithmetic.*

Another property which is preserved by similarity of the Vinberg form is the *isotropy property.*

Definition 4.4.3. Let (V, q) be a quadratic space. The form q is said to be *isotropic* if there is a $v \in V \setminus \{0\}$ such that $q(v) = 0$.

It is known that an arithmetic hyperbolic lattice is non-cocompact if its quadratic form is isotropic (see [27, Section 2.3], for example). This result translates into the context of hyperbolic Coxeter groups as follows. Let q_1 and q_2 be the Vinberg forms of two commensurable hyperbolic Coxeter groups defined over K.

Lemma 4.4.4. *The form q_1 is isotropic if and only if q_2 is isotropic.*

Proof. Let Q_1 and Q_2 be matrix representations of q_1 and q_2, respectively. By commensurability, $Q_1 = \lambda S^T Q_2 S$ for some invertible matrix S and a positive $\lambda \in K$, as above. Let $M = \sqrt{\lambda} S$ so that $Q_1 = M^T Q_2 M$. Since q_1 is isotropic, there exists a non-zero $v \in V$ such that $Q_1 v = 0$. Hence $(M^T Q_2 M)v = 0$. Let $v' = Mv$. Then $v' \neq 0$ since M is invertible and $v \neq 0$. If $Q_2 v' \neq 0$, then $M^T Q_2 v' \neq 0$, contradicting $Q_1 v = 0$. Therefore $Q_2 v' = 0$, and the result follows. $\square$

Incommensurability test using the Vinberg field

The Vinberg field is a very powerful commensurability invariant since it is fairly easy to compute even by hand, depending on the complexity of the graphs, of course. Consider the two non-cocompact nq-arithmetic Coxeter pyramid groups Γ_1 and Γ_2 acting on $\mathbb{H}^4$ as shown in Figure 4.3.

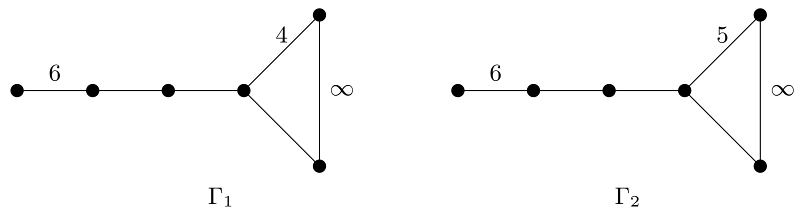


Figure 4.3: Two Coxeter pyramid groups Γ_1 and Γ_2 in $\mathrm{Isom}(\mathbb{H}^4)$.

The Gram matrices $G(\Gamma_1) =: G$ and $G(\Gamma_2) =: G'$ are

$$G = \begin{pmatrix} 1 & -\sqrt{3}/2 & 0 & 0 & 0 & 0 \\ -\sqrt{3}/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & -\sqrt{2}/2 & -1/2 \\ 0 & 0 & 0 & -\sqrt{2}/2 & 1 & -1 \\ 0 & 0 & 0 & -1/2 & -1 & 1 \end{pmatrix},$$

$$G' = \begin{pmatrix} 1 & -\sqrt{3}/2 & 0 & 0 & 0 & 0 \\ -\sqrt{3}/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & -(1+\sqrt{5})/4 & -1/2 \\ 0 & 0 & 0 & -(1+\sqrt{5})/4 & 1 & -1 \\ 0 & 0 & 0 & -1/2 & -1 & 1 \end{pmatrix}.$$

For G we have the cycle $b_{456} = 2^3 g_{45} g_{56} g_{64} = -2\sqrt{2}$ while all the other simple cycles of G lie in $\mathbb{Q}$. For G' we consider the cycle $b'_{45} = 2^2 g'_{45} g'_{54} = \frac{1}{2}\left(3+\sqrt{5}\right)$. All other simple cycles of G' are either in $\mathbb{Q}$ or $\mathbb{Q}\left(\sqrt{5}\right)$. Therefore the Vinberg fields are $K(\Gamma_1) = \mathbb{Q}\left(\sqrt{2}\right)$ and $K(\Gamma_2) = \mathbb{Q}\left(\sqrt{5}\right)$. Thus the two groups Γ_1 and Γ_2 are incommensurable.

Incommensurability test using the Vinberg ring

Consider the two non-cocompact quasi-arithmetic (but not arithmetic) Coxeter cube groups Γ_1 and Γ_2 in $\mathrm{Isom}(\mathbb{H}^3)$ defined in Figure 4.4. Similarly to the groups defined in Figure 4.2, Γ_1 and Γ_2 have $\mathbb{Q}$ as Vinberg field and similar quadratic forms. Their Vinberg rings are given by $R(\Gamma_1) = \mathbb{Z}[1/3]$ and $R(\Gamma_2) = \mathbb{Z}[1/2]$, respectively. By Proposition 4.3.11 the groups Γ_1 and Γ_2 are therefore incommensurable.

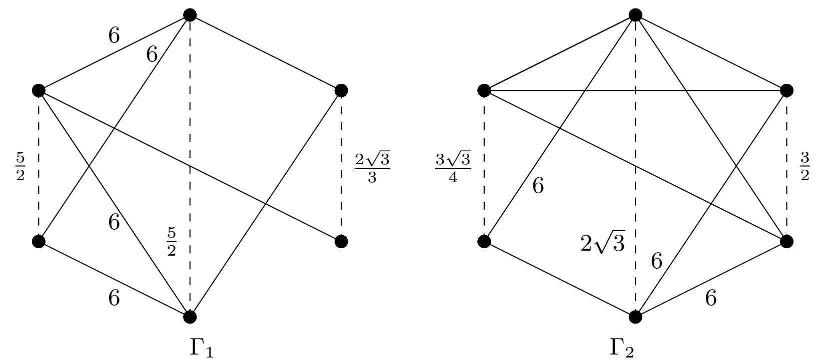


Figure 4.4: Two Coxeter cube groups Γ_1 and Γ_2 in $\mathrm{Isom}(\mathbb{H}^3)$.

Caution 4.4.5. Two hyperbolic Coxeter groups having the same Vinberg field, the same Vinberg ring and similar Vinberg forms do *not* have to be commensurable. As an example, consider the two groups as given by Figure 4.2. They have the same Vinberg field, similar quadratic forms and identical Vinberg ring $\mathbb{Z}[1/2]$. However they are incommensurable!

4.5 Similarity classification of the Vinberg forms

Consider two Coxeter groups in $\mathrm{Isom}(\mathbb{H}^n)$ having the same Vinberg field and the same Vinberg ring. Our aim is to classify their Vinberg forms up to similarity. This classification should be compatible with the one of Maclachlan in the case of quasi-arithmetic hyperbolic Coxeter groups (see Section 4.1.2). As we shall see, the results here depend on the parity of the dimension n. For n even, a similarity criterion can be stated. For n odd, we can provide a necessary condition for similarity, only. We start by recalling the Hasse-Minkowski Theorem in terms of the Hasse invariant see ([43], [5]).

Theorem 4.5.1. *Let K be a number field and let q_1 and q_2 be two quadratic forms over K. For a $\lambda \in K^*$, q_1 and λq_2 are isometric if and only if the following properties are satisfied:*

i) $\dim(q_1) = \dim(\lambda q_2)$*;*

ii) $\det(q_1) \equiv \det(\lambda q_2)$ *in* $K^* \mod (K^*)^2$*;*

iii) $s(q_1) = s(\lambda q_2)$*;*

iv) $\operatorname{sgn}(\sigma(q_1)) = \operatorname{sgn}(\sigma(\lambda q_2))$ *for all real embeddings* $\sigma : K \hookrightarrow \mathbb{R}$.

The even dimensional case

For n even, let $\Gamma_1, \Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be Coxeter groups with the same Vinberg field K, and denote by q_1 and q_2 the associated Vinberg forms over K. Recall that $\dim(q_1) = \dim(q_2) = n + 1 =: m$, i.e. the dimension of both quadratic forms is odd. Then, condition *ii*) of the Hasse-Minkowski Theorem 4.5.1 implies that $\det(q_1) \equiv \lambda \det(q_2)$ in $K^* \mod (K^*)^2$. This means that λ can only be the value which balances the two determinants, that is, $\lambda = \frac{\det(q_1)}{\det(q_2)} \in K^*/(K^*)^2$ (see also the proof of [50, Proposition 5.4]). Using (4.1), we get the following simplification for the Hasse invariant $s(\lambda q_2)$:

$$s(\lambda q_2) = \begin{cases} s(q_2), & m \equiv 1 \mod 4; \\ (\lambda, -1) \cdot s(q_2), & m \equiv 3 \mod 4. \end{cases}$$

Hence, for $\lambda = \frac{\det(q_1)}{\det(q_2)} \in K^*/(K^*)^2$, we obtain the complete set of similarity invariants for Vinberg forms as shown in Table 4.1.

n	Similarity criterion
$n \equiv 0 \mod 4$	$s(q_1) = s(q_2)$ $\operatorname{sgn}(\sigma(q_1)) = \operatorname{sgn}(\sigma(\lambda q_2))$
$n \equiv 2 \mod 4$	$s(q_1) = (\lambda, -1) \cdot s(q_2)$ $\operatorname{sgn}(\sigma(q_1)) = \operatorname{sgn}(\sigma(\lambda q_2))$

Table 4.1: Similarity criterion for Vinberg forms of hyperbolic Coxeter groups.

By specialising to quasi-arithmetic groups, whose forms q_1 and q_2 have appropriate signature, it is easy to check that the equation $\operatorname{sgn}(\sigma(q_1)) =$

n	Similarity criterion
$n \equiv 0 \mod 4$	$s(q_1) = s(q_2)$
$n \equiv 2 \mod 4$	$s(q_1) = (\lambda, -1) \cdot s(q_2)$

Table 4.2: Similarity criterion for Vinberg forms of quasi-arithmetic hyperbolic Coxeter groups.

$\mathrm{sgn}(\sigma(\lambda q_2))$ is always satisfied. Therefore, for quasi-arithmetic hyperbolic Coxeter groups, the criterion can be reduced according to Table 4.2.

Table 4.2 is compatible with the classification of Maclachlan. Indeed, since the dimensions and the determinants of q_1 and λq_2 are the same, having the same Hasse invariant implies having the same Witt invariant by (4.2), i.e. $c(q_1) = c(\lambda q_2)$. Since the quadratic forms are of odd dimension, one even gets $c(q_1) = c(q_2)$.

The odd dimensional case

For n odd, let $\Gamma_1,\ \Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be Coxeter groups. If they are quasi-arithmetic, we refer to the similarity classification provided by Maclachlan. Otherwise, the similarity problem for their even-dimensional Vinberg forms q_1 and q_2 is more involved. We present here a partial result, only.

Applying condition *ii)* of the Hasse-Minkowski Theorem 4.5.1 we get $\det(q_1) \equiv \det(\lambda q_2)$ in $K^* \mod (K^*)^2$ which reduces to $\det(q_1) \equiv \det(q_2) \mod (K^*)^2$. In contrast to the previous case, we can not extract any information about λ. This fact can be stated in the following lemma, sometimes referred to as the *ratio-test*.

Lemma 4.5.2. *Let $\Gamma_1,\ \Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$, n odd, be two commensurable cofinite Coxeter groups with Vinberg field K and Vinberg forms q_1 and q_2, respectively. Then,* $\det(q_1) \equiv \det(q_2) \in K^* \mod (K^*)^2$.

Example 4.5.3. Consider the two cocompact quasi-arithmetic (but not arithmetic) Coxeter groups Γ_1, Γ_2 in $\mathrm{Isom}(\mathbb{H}^4)$ given in Figure 4.5. The groups Γ_1 and Γ_2 are so-called crystallographic Napier cycles (see [33]).

The weights l_i and l'_i of the dotted edges in the Coxeter graphs are

$$l_1 = \sqrt{\frac{1}{11}\left(10 + 3\sqrt{5}\right)}, \qquad l'_1 = \sqrt{\frac{2}{11}\left(7 + \sqrt{5}\right)},$$

$$l_2 = \frac{1}{2}\sqrt{\left(5 + \sqrt{5}\right)}, \qquad l'_2 = \sqrt{\frac{2}{19}\left(9 + \sqrt{5}\right)},$$

$$l_3 = \sqrt{\frac{1}{11}\left(16 + 7\sqrt{3}\right)}, \qquad l'_3 = \sqrt{\frac{1}{209}\left(233 + 104\sqrt{5}\right)}.$$

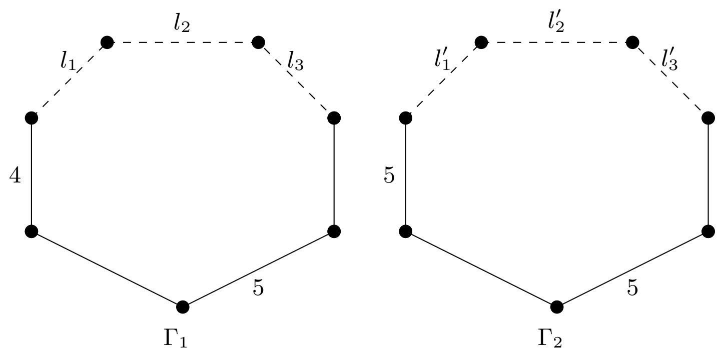


Figure 4.5: The Coxeter groups Γ_1 and Γ_2 in $\mathrm{Isom}(\mathbb{H}^4)$.

The groups Γ_1 and Γ_2 have both $K = \mathbb{Q}\left(\sqrt{5}\right)$ as their Vinberg field. The diagonalised associated Vinberg forms over K are

$$\begin{aligned} q_1 &= \mathrm{diag}\left(4, 4, 4, -2-2\sqrt{5}, 20+8\sqrt{5}\right), \\ q_2 &= \mathrm{diag}\left(4, \frac{5}{2}+\frac{1}{2}\sqrt{5}, 2+\frac{2}{5}\sqrt{5}, \frac{-37}{2}-\frac{17}{2}\sqrt{5}, \frac{312}{19}+\frac{136}{19}\sqrt{5}\right). \end{aligned}$$

Using the theory as presented in Section 4.1.1, we compute their Hasse invariants as follows.

$$\begin{aligned} c(\Gamma_1) &= \left(-2-2\sqrt{5}, 5+2\sqrt{5}\right), \\ c(\Gamma_2) &= \left(10+2\sqrt{5}, -1\right) \cdot \left(-74-34\sqrt{5}, 1482+646\sqrt{5}\right). \end{aligned}$$

The ramification set $\mathrm{Ram}(\Gamma_1)$ contains two prime ideals, one generated by 2, and the other generated by 5 and $-1+2\sqrt{5}$. The ramification set $\mathrm{Ram}(\Gamma_2)$ is empty. Since $\mathrm{Ram}(\Gamma_1) \neq \mathrm{Ram}(\Gamma_2)$, the two quaternion algebras representing $c(\Gamma_1)$ and $c(\Gamma_2)$ are not isomorphic. Hence the Vinberg forms q_1 and q_2 are not similar, and the groups Γ_1 and Γ_2 are incommensurable.

Chapter 5

New generators for the Vinberg field

In this last chapter we discuss various views on the Vinberg field of a hyperbolic Coxeter group. We start with a summary of known results about possible Vinberg fields associated to arithmetic hyperbolic Coxeter groups. We also present a result about possible Vinberg fields associated to quasi-arithmetic Coxeter groups, imposing restrictions on the Coxeter graph.

We then provide a new result for a finite volume hyperbolic Coxeter n-polyhedron P defining a Coxeter group with Vinberg field of degree d. This result gives the range of the admissible dihedral angles of P in terms of d.

In the last two parts we describe new generators for the Vinberg field of a *quasi-arithmetic* hyperbolic Coxeter group. The first set of generators is given by the coefficients of the characteristic polynomial of the associated Gram matrix while the second set consists of the coefficients of the characteristic polynomial of any Coxeter transformation of the group.

We conclude by formulating a conjecture generalising the above results for nq-arithmetic hyperbolic Coxeter groups.

5.1 The Vinberg field of a quasi-arithmetic hyperbolic Coxeter group

In this part we present some results about possible Vinberg fields associated to quasi-arithmetic Coxeter groups in $\mathrm{Isom}(\mathbb{H}^n)$, $n \geq 2$. Recall from part *ii)* of Remark 4.2.3 that a non-cocompact quasi-arithmetic Coxeter group has Vinberg field $\mathbb{Q}$. Moreover recall from Remark 1.5.2 that arithmetic Coxeter groups exist only for $n < 30$.

We start by considering compact hyperbolic Coxeter n-simplices, which were classified by Lannér and exist only for $n \leq 4$.

Definition 5.1.1. A *Lannér graph* is the Coxeter graph of a compact hyperbolic Coxeter simplex. The complete list of Lannér graphs is given in Table 5.1[7].

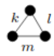
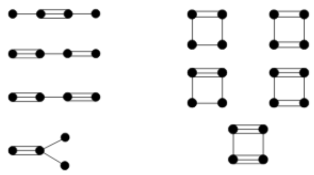
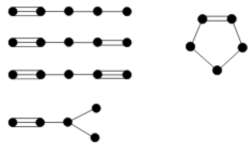
order	diagrams
2	
3	 $(2 \leq k, l, m < \infty,$ $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1)$
4	
5	

Table 5.1: Complete list of Lannér graphs.

Essential for the following is the fact [67, Corollary 2.1] that the Coxeter graph of a cocompact hyperbolic Coxeter group contains a subgraph which is a Lannér graph (called a *Lannér subgraph*).

Consider a cocompact quasi-arithmetic hyperbolic Coxeter group. Its Vinberg form satisfies the appropriate signature condition. This property is crucial for the proof of the following result of Vinberg.

Theorem 5.1.2 ([72], Proposition 17)**.** *For a cocompact quasi-arithmetic hyperbolic Coxeter group, the Vinberg field is generated by the determinant of any Lannér subgraph*[8] *of the Coxeter graph.*

Vinberg exploited the above result for larger dimensions as follows.

Proposition 5.1.3 ([72], Theorem 2 and Theorem 3)**.** *Let* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *be a cocompact arithmetic Coxeter group. For* $n \geq 14$ *the only possible Vinberg fields for* Γ *are*

$$\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}(\sqrt{2}, \sqrt{5}), \mathbb{Q}(\cos 2\pi/m)$$

with $m = 7, 9, 11, 15, 16, 20$*. More specifically, for* $n \geq 22$ *the possible Vinberg fields are*

$$\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\cos 2\pi/7).$$

[7]A double edge between vertices corresponds to the weight 4 while a triple edge corresponds to the weight 5. As usual, a single edge corresponds to the weight 3.

[8]The determinant of a Coxeter graph is the determinant of the corresponding Gram matrix.

We exploit Theorem 5.1.2 by assuming that the Coxeter graph contains a Lannér subgraph of order at least three and present the following result.

Proposition 5.1.4. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a cocompact quasi-arithmetic Coxeter group such that its Coxeter graph contains a Lannér subgraph of order at least three. The possible Vinberg fields for Γ are*

$$\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}(\sqrt{2}, \sqrt{5}), \mathbb{Q}(\cos 2\pi/m)$$

for $m = 7, 9, 11, 15, 16, 20$.

Proof. By Theorem 5.1.2, the Vinberg field of Γ is the extension of $\mathbb{Q}$ by the determinant of any Lannér subgraph. Since Γ is quasi-arithmetic, any such Lannér subgraph of order at least three describes an *arithmetic* Lannér group. Indeed, the subgroups corresponding to these Lannér subgraphs must have the appropriate signature property. Moreover, Lannér graphs of order at least three do not have dotted edges. These two conditions imply that the Lannér subgraphs describe arithmetic groups (see Remark 4.2.3, *iii*)).

The arithmetic Lannér groups of order three have been determined by Takeuchi [64]: there are finitely many examples. The Lannér groups of orders four and five are all arithmetic with one exception (see [36], for example). Hence, we have to compute finitely many determinants, and the result follows. □

Remark 5.1.5. When the Coxeter graph of Γ contains only Lannér subgraphs of order two, other fields can arise. In his PhD thesis [21], Esselmann gives examples of such cocompact arithmetic hyperbolic Coxeter groups with Vinberg fields $\mathbb{Q}(\sqrt{13})$, $\mathbb{Q}(\sqrt{17})$ and $\mathbb{Q}(\sqrt{21})$.

We close this part by summarising known results about the extension degree $[K : \mathbb{Q}] =: d$ of the Vinberg field K over $\mathbb{Q}$ of an arithmetic Coxeter group. A lot of work has been done in this direction, especially by V. Nikulin, in order to find an effective upper bound for d. For details about the related Vinberg fields, see [53]. The Table 5.2 provides a rough survey.

n	$d \leq$	Reference
2	11	Maclachlan [45]
3	35	Belolipetsky [6]
$4, 5$	35	Nikulin [54]
≥ 6	25	Nikulin [54]

Table 5.2: Possible extension degree d of the Vinberg field of an arithmetic hyperbolic Coxeter group in $\mathrm{Isom}(\mathbb{H}^n)$.

5.2 The Vinberg field and the dihedral angles of a hyperbolic Coxeter group

Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$, $n > 2$, be a cofinite Coxeter group with Coxeter polyhedron P and Coxeter graph Σ. Suppose that two hyperplanes H_{i_1} and H_{i_2} bounding P intersect under the angle of $\frac{\pi}{m}$, $m \geq 2$. This implies that Σ contains the subgraph

$$\bullet\overset{m}{\text{—}}\bullet \tag{5.1}$$

which consists only of two nodes if $m = 2$.

Let K be the Vinberg field of Γ. By Remark 4.3.6, *ii*), K is an algebraic number field of degree $d := [K : \mathbb{Q}] \geq 1$.

Proposition 5.2.1. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$, $n > 2$, be a cofinite Coxeter group with Coxeter polyhedron P and Vinberg field K of degree d. Then, for any dihedral angle $\frac{\pi}{m}$ of P one has*

$$\phi(m) \leq 2\,d,$$

where $\phi(m)$ is the Euler's totient function.

Proof. Let $a = a_m := \cos\left(\frac{2\pi}{m}\right)$ and $\xi := e^{2\pi i/m}$. For Euler's totient function $\phi(m)$, which is the number of positive integers, relatively coprime to m, between 1 and m, both included, it is well-known that

$$[\mathbb{Q}(\xi) : \mathbb{Q}] = \phi(m). \tag{5.2}$$

We first compute the degree $[\mathbb{Q}(a_m) : \mathbb{Q}]$, by using a standard method. For $m = 2$ this degree is 1. Let $m > 2$. Since $\frac{\xi+\xi^{-1}}{2} = a_m$, the field $\mathbb{Q}(\xi)$ contains $\mathbb{Q}(a_m)$. Moreover,

$$\xi^2 - 2a\,\xi + 1 = 0.$$

Hence ξ is a zero of $P(t) = t^2 - 2at + 1 \in \mathbb{Q}(a)[t]$ together with its complex conjugate $\overline{\xi}$. Since $m > 2$, ξ and $\overline{\xi}$ are non-real, and consequently they are not in $\mathbb{Q}(a)$. This implies that $P(t)$ is irreducible over $\mathbb{Q}(a)$, and we have $[\mathbb{Q}(\xi) : \mathbb{Q}(a)] = \deg P(t) = 2$. By the tower property, we obtain

$$[\mathbb{Q}(\xi) : \mathbb{Q}(a)] \cdot [\mathbb{Q}(a) : \mathbb{Q}] = [\mathbb{Q}(\xi) : \mathbb{Q}]. \tag{5.3}$$

By (5.2) and (5.3), for $m > 2$, we get

$$[\mathbb{Q}(a) : \mathbb{Q}] = \phi(m)/2. \tag{5.4}$$

Next, consider the Vinberg field K of Γ. To every dihedral angle $\frac{\pi}{m}$ of P corresponds a subgraph of the Coxeter graph as in (5.1) and the cycle $b_{i_1 i_2} = 4\cos^2\left(\frac{\pi}{m}\right) \in K$ (see (4.3)). This forces $[\mathbb{Q}(4\cos^2\left(\frac{\pi}{m}\right)) : \mathbb{Q}] \leq d$.

By the angle doubling property of the cosine function we have

$$\mathbb{Q}\left(4\cos^2\left(\frac{\pi}{m}\right)\right) = \mathbb{Q}\left(\cos\left(\frac{2\pi}{m}\right)\right).$$

Therefore, by (5.4), the weight m must satisfy the inequality

$$[\mathbb{Q}\left(\cos\left(\frac{2\pi}{m}\right)\right) : \mathbb{Q}] = \phi(m)/2 \leq d.$$

$\square$

Example 5.2.2. Let Γ be an *arithmetic* Coxeter group in $\mathrm{Isom}(\mathbb{H}^n)$, $n \geq 14$. By Proposition 5.1.3, the degree $d = [K : \mathbb{Q}]$ is smaller than or equal to five. For $d = 5$, Proposition 5.2.1 yields $\phi(m) \leq 10$, for any dihedral angle $\frac{\pi}{m}$. In general, for x not equal to 2 or 6, Euler's function $\phi(x)$ satisfies

$$\phi(x) \geq \sqrt{x}.$$

Thus, for $m \leq 100$, we have to check when $\phi(m) \leq 10$. As a result, for $n \geq 14$, all the possible values for m are

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 22, 24, 30.$$

5.3 The Gram field of a hyperbolic Coxeter group

For $n \geq 2$, consider a Coxeter group Γ in $\mathrm{Isom}(\mathbb{H}^n)$ of rank N. Let G be its Gram matrix of signature $(n, 1)$ with characteristic polynomial

$$\chi_G(t) = a_0 + a_1 t + \cdots + a_N t^N, \quad a_N = 1.$$

The matrix G is uniquely defined by Γ up to simultaneous permutation of its lines and columns which would yield a similar matrix G' with identical characteristic polynomial. Let us remark that, by [70, Proposition 16], G' can be related to G by means of a positive diagonal matrix D such that

$$G' = D^{-1} G D.$$

Notice that $a_{N-1} = (-1)^{N-1}\,\mathrm{Tr}(G) = (-1)^{N-1} N$. Moreover, each coefficient a_r of χ_G, $r < N$, can be expressed as the sum of all the principal minors of size $N - r$ (see equation (A.1) of Appendix A). In particular, a_r vanishes for all $r < N - (n+1)$.

Denote by $\widetilde{K}$ the field generated by all the entries of G as in Theorem 4.2.2. Clearly all the coefficients $a_0, \ldots, a_N$ of χ_G are in $\widetilde{K}$. We know that $\widetilde{K}$ contains the Vinberg field K. The following two examples show that the coefficients of χ_G are not only elements but even *generate* the Vinberg field K.

Consider the Coxeter group Γ_1 in $\mathrm{Isom}(\mathbb{H}^4)$ defined by the Coxeter graph given in Figure 5.1 (see [61]). The group Γ_1 is arithmetic and non-cocompact. We have that $\widetilde{K}_1 = \mathbb{Q}(\sqrt{2})$ and $K_1 = \mathbb{Q}$. Furthermore, the characteristic polynomial of the Gram matrix $G(\Gamma_1)$ turns out to be

$$t^8 - 8t^7 + \frac{33}{2}t^6 + 19t^5 - \frac{1711}{16}t^4 + \frac{493}{4}t^3 - \frac{171}{4}t^2.$$

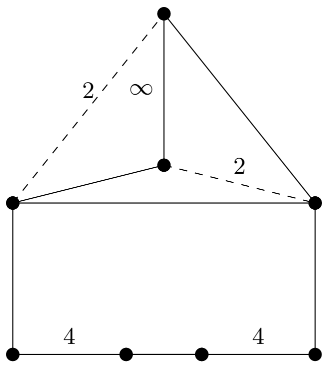


Figure 5.1: The Coxeter group Γ_1 in $\mathrm{Isom}(\mathbb{H}^4)$.

Consider the Coxeter group Γ_2 in $\mathrm{Isom}(\mathbb{H}^4)$ defined by the Coxeter graph given in Figure 5.2 (see [33]). The group Γ_2 is arithmetic and cocompact. We have that $\widetilde{K}_2 = \mathbb{Q}\left(\cos\left(\frac{\pi}{8}\right), \sqrt{2}\right) = \mathbb{Q}\left(\sqrt{2+\sqrt{2}}\right)$ and $K_2 = \mathbb{Q}(\sqrt{2})$.

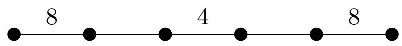


Figure 5.2: The Coxeter group Γ_2 in $\mathrm{Isom}(\mathbb{H}^4)$.

The characteristic polynomial of the Gram matrix $G(\Gamma_2)$ is

$$\begin{aligned} t^6 - 6t^5 &+ \frac{1}{64}\left(832 - 32\sqrt{2}\right)t^4 + \frac{1}{64}\left(-768 + 128\sqrt{2}\right)t^3 \\ &+ \frac{1}{64}\left(268 - 152\sqrt{2}\right)t^2 + \frac{1}{64}\left(-24 + 48\sqrt{2}\right)t \end{aligned}$$

In summary, the coefficients of $\chi_{G(\Gamma_1)}$ and $\chi_{G(\Gamma_2)}$ generate the Vinberg fields K_1 and K_2, respectively.

These examples motivate the following definition and are at the basis of the next result.

Definition 5.3.1. Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a Coxeter group of rank N. Let G be its Gram matrix with characteristic polynomial $\chi_G(t) = a_0 + a_1 t +$

$\cdots + a_N t^N, \quad a_N = 1$. The *Gram field* $K(G)$ is the field generated by the coefficients of $\chi_G(t)$ over $\mathbb{Q}$, namely

$$K(G) = \mathbb{Q}(a_j \mid 0 \leq j \leq N).$$

Proposition 5.3.2. *Let Γ be a cofinite quasi-arithmetic hyperbolic Coxeter group with Vinberg field K. Then*

$$K = K(G).$$

Proof. We prove first the inclusion $K \supseteq K(G)$. For this, let Σ be the Coxeter graph of Γ and denote by γ a cyclic subgraph of length ≥ 2 of Σ. Denote by $p(\gamma)$ the product of all the entries of G corresponding to the weights of γ. In particular, for the cyclic subgraph γ of length 2 given by $\bullet\overset{m}{\text{—}}\bullet$, we get $p(\gamma) = \cos^2\left(\frac{\pi}{m}\right)$.

In this way the determinant $\det(G)$ of the Gram matrix G can be expressed according to ([72, Proposition 11])

$$\det(G) = \sum (-1)^s p(\gamma_1) \cdots p(\gamma_s), \tag{5.5}$$

where $\{\gamma_1, \ldots, \gamma_s\}$ ranges over all unordered collections of pairwise disjoint cyclic subgraphs of Σ, including the empty one. The same result applies to every principal submatrix of G.

By Appendix A, equation (A.1), the coefficients of χ_G can be expressed as the sum of principal minors of G which together with (5.5) yields $K \supseteq K(G)$.

Assume that $K \supsetneq K(G)$. Then there exists a non-trivial embedding $\sigma : K \hookrightarrow \mathbb{R}$ which is the identity on $K(G)$. Let G^σ be the matrix obtained by applying σ to every coefficient of G. If $\chi_G = \sum_{i=0}^N a_i x^i$ is the characteristic polynomial of G, then $\chi_{G^\sigma} = \sum_{i=0}^N \sigma(a_i) x^i$ (since σ is a field homomorphism). Since σ fixes the coefficients of χ_G, we get

$$\chi_G = \chi_{G^\sigma}.$$

In particular, G^σ has signature $(n, 1)$ and is not positive semidefinite. This is a contradiction to part *ii*) of Theorem 4.2.2 and the claim follows. $\square$

5.4 The Coxeter field of a hyperbolic Coxeter group

Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 2$, be a cofinite Coxeter group with natural set of generators $\{s_1, \ldots, s_N\}$. Our aim is to study a Coxeter transformation $C = s_1 \cdots s_N$ of Γ defined up to the ordering of the factors.

With the real coefficients $a_0, \ldots, a_{n+1}$ of the characteristic polynomial $\chi_C(t)$ we define a new field, the *Coxeter field* $K(C) = \mathbb{Q}(a_j \mid 1 \leq j \leq$

$n+1$), and prove that it coincides with the Vinberg field $K(\Gamma)$ if Γ is quasi-arithmetic.

The proof is based on the work of Howlett [31] and the theory of M-matrices which we are going to review briefly.

5.4.1 Abstract Coxeter groups and M-matrices

Let $W = (W, S)$ be a Coxeter system with generating set $S = \{s_1, \ldots, s_N\}$ satisfying the relations (1.3). By Tits' theory, it is known that W can be represented as a subgroup $\rho(W)$ of $\mathrm{GL}(V)$ for a real vector space V of dimension N equipped with a suitable symmetric bilinear form B (see [32], for example).

Recall that W is finite (affine) if B is positive definite (positive semidefinite). We denote by $\mathrm{rad}(V) = \{v \in V \mid B(v, v') = 0 \quad \forall v' \in V\}$ the *radical* of B which will play a role later on.

A *Coxeter element* $c \in W$ is the product of the N generators in S arranged in any order. The representative $C_T \in \mathrm{GL}(V)$ of c is called a *Coxeter transformation* of W.

For a Coxeter element $c = s_1 \cdots s_N$, the matrix of C_T with respect to a basis $\{v_1, \ldots, v_N\}$ of V, denoted again by C_T, can be written according to (see [31], for example)

$$C_T = -U^{-1}U^T, \tag{5.6}$$

where $U \in \mathrm{GL}(N, \mathbb{R})$ is given by

$$U = \begin{pmatrix} 1 & & & & \\ & 1 & & * & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix}, \tag{5.7}$$

with $[U]_{st} = 2B(v_s, v_t)$ for $t > s$. Notice that

$$U + U^T = 2B. \tag{5.8}$$

By means of the theory of M-matrices, Howlett ([31, Theorem 4.1], see also [2]) characterised abstract Coxeter groups in terms of a Coxeter transformation C_T and its eigenvalues. For example, W is finite if and only if C_T is of finite order with eigenvalues on the unit circle.

More concretely, an *M-matrix* is a real matrix with non-positive off-diagonal entries all of whose principal minors are positive. For example, the matrix U given by (5.7) is an M-matrix.

The proof of Howlett's Theorem 4.1 in [31] is based on the following results.

Lemma 5.4.1 ([31], Lemma 3.1). *Let U be a real matrix such that $U+U^T$ is positive definite. Then U is invertible and $-U^{-1}U^T$ is diagonalisable over $\mathbb{C}$ with all of its eigenvalues having modulus one.*

Lemma 5.4.2 ([31], Lemma 3.2 and Corollary 3.3). *Let U be an M-matrix such that $U+U^T$ is not positive definite. Then $-U^{-1}U^T$ has a real eigenvalue $\lambda \geq 1$. If $U+U^T$ is not positive semidefinite, then $\lambda > 1$. If $U+U^T$ is positive semidefinite, all the eigenvalues of $-U^{-1}U^T$ have modulus one and $-U^{-1}U^T$ is not diagonalisable.*

Later we will also need another lemma, which is implicitly stated in Howlett's proof of Lemma 5.4.2.

Lemma 5.4.3. *Let U be an invertible real matrix such that $U+U^T$ is positive semidefinite. Then the eigenvalues of $-U^{-1}U^T$ have all modulus one.*

Proof. For $\epsilon > 0$ define the matrix $U^\epsilon := U + \epsilon I$. Since $U+U^T$ is positive semidefinite, $U^\epsilon + (U^\epsilon)^T$ is positive definite. By Lemma 5.4.1, all the eigenvalues of $-(U^\epsilon)^{-1}(U^\epsilon)^T$ have modulus one. The entries of U^ϵ depend continuously on ϵ. The same can be said for $-(U^\epsilon)^{-1}(U^\epsilon)^T$ and the coefficients of its characteristic polynomial. Hence the eigenvalues of $-(U^\epsilon)^{-1}(U^\epsilon)^T$ and their modulus depend continuously on ϵ, and the claim follows. □

5.4.2 Coxeter transformations of a hyperbolic Coxeter group

Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ be a hyperbolic Coxeter group with generating reflections $s_1, \ldots, s_N$. In this way Γ represents a geometric realisation of an abstract Coxeter group. Let $P \in \mathcal{H}^n$ be its Coxeter polyhedron with outer unit normal vectors $e_1, \ldots, e_N$ and associated Gram matrix $G \in \mathrm{Mat}(N, \mathbb{R})$.

Let $C \in \Gamma$ be a Coxeter transformation of Γ. Our goal is to construct a new field $K(C)$ associated to C which we can identify later with the Vinberg field $K(\Gamma)$. Our motivation comes from [60, Theorem 1.8, (iv)], due to Reiner, Ripoll and Stump, relating Coxeter transformations of a finite complex reflection group to its field of definition[9] (see also Malle in [49, Section 7A]).

Inspired by this, we state the following definition.

Definition 5.4.4. Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ be a Coxeter group. Let $C \in \Gamma$ be a Coxeter transformation with characteristic polynomial $\chi_C(t) = a_0 + a_1 t + \cdots + a_{n+1}t^{n+1}$, $a_{n+1} = 1$. The *Coxeter field* $K(C)$ is the field generated by the coefficients of $\chi_C(t)$ over $\mathbb{Q}$, namely

$$K(C) = \mathbb{Q}(a_j \mid 0 \leq j \leq n+1).$$

[9] In [60], the field of definition of a Coxeter group W is the field generated by the traces of all elements in $\rho(W)$.

Consider the characteristic polynomial $\chi_C(t) = a_0 + a_1 t + \cdots + a_{n+1} t^{n+1}$ as above. It is not difficult to see that $\chi_C(t)$ is palindromic ($a_j = a_{n+1-j}$) if $N = n + 1 + 2k$ and it is pseudo-palindromic ($a_j = -a_{n+1-j}$) if $N = n + 1 + (2k + 1)$, for some $k \geq 0$.

Furthermore, $N - (n+1)$ is the dimension of the radical $\operatorname{rad}\left(\mathbb{R}^N\right)$ for the Tits representation space $\left(\mathbb{R}^N, G\right)$. Clearly, every element in Γ viewed in $\mathrm{GL}\left(\mathbb{R}^N\right)$ acts as the identity on $\operatorname{rad}\left(\mathbb{R}^N\right)$. Hence the same is true for every Coxeter transformation $C_T \in \mathrm{GL}\left(\mathbb{R}^N\right)$ of Γ. Since $\dim\left(\mathbb{R}^N / \operatorname{rad}\left(\mathbb{R}^N\right)\right) = n + 1$, the characteristic polynomials χ_C and χ_{C_T} are related by

$$(t-1)^{(N-(n+1))} \chi_C(t) = \chi_{C_T}(t). \tag{5.9}$$

With this preparation we are ready to prove the following result.

Proposition 5.4.5. *Let Γ be a cofinite quasi-arithmetic hyperbolic Coxeter group with Vinberg field K, and let C be any Coxeter transformation of Γ. Then*

$$K = K(C).$$

Proof. We first show that $K \supseteq K(C)$. Since the Vinberg field K is a field of definition (see Remark 4.2.11), the Coxeter transformation C can be written as a matrix with coefficients in K by means of a suitable basis. Since a basis change leaves the characteristic polynomial invariant, we have that $K \supseteq K(C)$.

Assume that $K \supsetneq K(C)$. Then there exists a non-trivial embedding $\sigma : K \hookrightarrow \mathbb{R}$ that is the identity on $K(C)$. Let C_T be the Coxeter transformation acting on $\left(\mathbb{R}^N, G\right)$ which corresponds to C in the sense of Tits. By (5.6), $C_T = -U^{-1}U^T$ where U is an M-matrix. By Lemma 5.4.2, C_T has a real eigenvalue $\lambda > 1$. Moreover, as a consequence of (5.9), λ is an eigenvalue of C as well.

Apply σ to the coefficients of the Gram matrix G of Γ, and denote the resulting matrix by G^σ. Notice that U^σ is invertible but in general not an M-matrix anymore (its off-diagonal entries may become positive). Define $C^\sigma := -(U^\sigma)^{-1}(U^\sigma)^T$. By (5.8), we have the equation $U^\sigma + (U^\sigma)^T = 2\, G^\sigma$ and thus, by part *ii*) of Theorem 4.2.2, $U^\sigma + (U^\sigma)^T$ is positive semidefinite. Moreover, since σ is a field homomorphism, the characteristic polynomial of C^σ is obtained by applying σ to the coefficients of the characteristic polynomial of C_T. Since σ is the identity on $K(C)$ leaving the characteristic polynomial χ_C invariant, the identity (5.9) yields

$$\chi_{C_T} = \chi_{C^\sigma}.$$

Hence the two polynomials have the same eigenvalues. This is impossible, since C_T has an eigenvalue $\lambda > 1$ and since, by Lemma 5.4.3, all the eigenvalues of C^σ have modulus one. $\square$

5.5 Conclusion

By Proposition 5.3.2 and Proposition 5.4.5, the Vinberg field, the Gram field and the Coxeter field of a quasi-arithmetic hyperbolic Coxeter group Γ coincide. Essential for their proofs is that the matrix G^{σ} becomes positive semidefinite for *every* non-trivial embedding.

A natural question is whether this phenomenon holds also for nq-arithmetic hyperbolic Coxeter groups. Based on various tests, we formulate the following conjecture.

Conjecture 5.5.1. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a cofinite Coxeter group. Then the Vinberg field, the Gram field and the Coxeter field of Γ coincide, that is,*

$$K(\Gamma) = K(G) = K(C).$$

For the proof of the Conjecture new ideas have to be developed since we do not dispose of the appropriate signature property anymore. In Appendix B, we illustrate the testing of the above Conjecture by means of graph-theoretical tools on the level of the Coxeter graphs.

Appendix A

On the coefficients of the characteristic polynomial of a matrix

Let G be an $N \times N$ complex matrix, $N \geq 2$. Denote its characteristic polynomial by

$$\chi_G(t) = a_0 + a_1 t + \cdots + a_N t^N, \quad a_N = 1.$$

In this appendix we present three ways to express the coefficients of χ_G.

In the following, let m be an integer with $1 \leq m \leq N$.

The coefficients in terms of principal minors

Let $S_m(G)$ denote the sum of all the principal minors of G of size m, of which there are $\binom{N}{m}$. Then (see [30, p. 53], for example)

$$a_{N-m} = S_m(G). \tag{A.1}$$

The coefficients in terms of traces

Denote by $TR_m(G)$ the trace of the m-th power matrix G^m of the matrix G. Moreover consider a partition of m as partial sums of sizes $s_1, \ldots, s_j$ according to

$$m = (r_1 + r_1 + \cdots + r_1) + (r_2 + \cdots + r_2) + \cdots + (r_j + \cdots + r_j) = \sum_{i=1}^{j} r_i s_i,$$

with integers $r_1, \ldots, r_j$ satisfying $r_1 > r_2 > \cdots > r_j > 0$.

Using Newton's formulae for symmetric functions, one can show that (see [44], for example)

$$a_{N-m} = -\sum \prod_{i=1}^{j} \frac{(-TR_{r_i}(G))^{s_i}}{r_i^{s_i} \cdot s_i!}, \tag{A.2}$$

where the sum is taken over all the partitions of m.

The coefficients in terms of eigenvalues

Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of G. Define the function $Eig_m(G)$ as

$$Eig_m(G) = \sum_{1 \leq k_1 < \cdots < k_m \leq N} \prod_{i=1}^{m} \lambda_{k_i}.$$

Then (see [30, p. 54] and [11], for example)

$$a_{N-m} = Eig_m(G). \tag{A.3}$$

Appendix B

An approach to test the Conjecture 5.5.1

In this appendix we show a way how to test the Conjecture 5.5.1 formulated as follows: let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a cofinite Coxeter group of rank N. Then the Vinberg field, the Gram field and the Coxeter field of Γ coincide, that is,

$$K(\Gamma) = K(G) = K(C).$$

The test $K(\Gamma) = K(G)$ is easily computable given the Gram matrix G of Γ. Therefore we only check the equality $K(\Gamma) = K(C)$ for *every* Coxeter transformation C of Γ.

The difficulty of this is that Γ has at most $N!$ Coxeter elements. However, since the characteristic polynomials are conjugacy invariants, regrouping Coxeter elements into conjugacy classes allows to compute $K(C)$ for a representative of each class, only.

In the following we present a graph theoretical tool to regroup Coxeter elements into conjugacy classes in general which will be exploited later in our particular context.

B.1 Some graph terminology

Let Σ be a graph and consider two vertices u and v in its vertex set. A *path* ω is a sequence of vertices $u = v_{i_1}, v_{i_2}, \ldots, v_{i_r} = v$ such that v_l is adjacent to v_{l+1} for every $i_1 \leq l \leq i_r - 1$. The *length* of a path is the number of edges of the path. The *inverse path* $-\omega$ is obtained by reversing the sequence, going from v to u. If a path is defined by a cycle of the graph, that is $u = v$, we call it a *cyclic path*.

Assume we have an orientation on Σ. When the edge between two vertices v_i and v_j is oriented from v_i to v_j we write $v_i \to v_j$. Let v_i and v_{i+1} be two adjacent vertices of a path ω. The edge between v_i and v_{i+1} is *forward* if $v_i \to v_{i+1}$. Otherwise the edge is *backward*. A *directed path* is a sequence of vertices $u = v_{i_1}, v_{i_2}, \ldots, v_{i_r} = v$ such that $v_l \to v_{l+1}$ holds for every $i_1 \leq l \leq i_r - 1$. If $u = v$, we call ω a *directed cycle*. An oriented graph is called *acyclic* if it does not contain any directed cycle.

A vertex v of Σ is called a *source* if for every adjacent vertex u we have $v \to u$. Furthermore, v is said to be a *sink* if $u \to v$. We can define an operation on an oriented graph which changes an acyclic orientation into another acyclic orientation as follows.

Definition B.1.1. Let v be a source of an oriented graph. A *source-to-sink operation* on v, or *ss-operation*, means reversing all the orientations around v. With this operation v is turned into a sink (see Figure B.1).

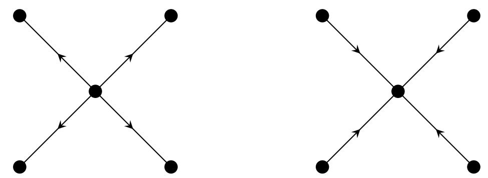


Figure B.1: Source-to-sink operation.

The ss-operation yields an equivalence relation on the set of all orientations on a graph ([63, p. 4]): two orientations O_1 and O_2 are ss-equivalent if and only if O_2 can be obtained from O_1 using only ss-operations.

B.2 Conjugacy classes of Coxeter elements

Let Γ be an abstract Coxeter group of rank N with Coxeter graph Σ. If Σ is a tree, then the equality $K(\Gamma) = K(C)$ for hyperbolic Coxeter groups can be efficiently tested due to the following proposition.

Proposition B.2.1 ([32], Proposition 3.16)**.** *Let Γ be a hyperbolic Coxeter group. If the Coxeter graph of Γ is a tree, then all the Coxeter elements are conjugate.*

Suppose that the Coxeter graph Σ of Γ is not a tree. The partition of Coxeter elements into conjugacy classes becomes more involved. In the following we shall discuss this problem.

Put an enumeration on the vertices $v_1, \ldots, v_N$ of Σ. Notice that to each enumeration corresponds a Coxeter element obtained by multiplying the corresponding generators following the order of the enumeration. There is the following theorem which links enumerations and acyclic orientations of Σ.

Theorem B.2.2 ([62], Theorem 1.5)**.** *Let Σ be a Coxeter graph. Then there is a one-to-one correspondence between the set of acyclic orientations* *on Σ and the enumerations of its vertices $v_1, \dots, v_N$ up to shuffling, that is, switching the enumeration index of two vertices if there is no edge between them.*

Remark B.2.3. Notice that if two vertices of the Coxeter graph Σ have no edge between them, then the two corresponding generators of the associated Coxeter group commute with each other. Therefore two enumerations obtained by shuffling one into the other correspond to the same Coxeter element.

Let O_1 be an acyclic orientation on the Coxeter graph Σ with corresponding enumeration $v_1, \dots, v_N$. Then there is a sink which we denote by v. It is not difficult to check that, by shuffling enumerations, one can assume that $v_N = v$. Apply the ss-operation on v and get the new acyclic orientation O_2. The enumeration (up to shuffling) corresponding to O_2 is then $v, v_1, \dots, v_{N-1}$.

Therefore the Coxeter element $c_1 = s_{v_1} \cdots s_{v_{N-1}} s_v$ of Γ associated to the first enumeration is conjugate to the Coxeter element $c_2 = s_v s_{v_1} \cdots s_{v_{N-1}}$ given by the second enumeration, namely

$$c_1 = s_v^{-1} c_2 s_v.$$

Thus two Coxeter elements corresponding to two ss-equivalent acyclic orientations are in the same conjugacy class.

Hence, the number of conjugacy classes of Coxeter elements is bounded from above by the number of ss-classes of acyclic orientations of the Coxeter graph Σ. Even more is true: Eriksson and Eriksson [20] have proved that the number of conjugacy classes of Coxeter elements is equal to the number of ss-classes.

For example, if the Coxeter graph Σ contains only one cycle, then the number of conjugacy classes of Coxeter elements is $\nu - 1$, where ν is the number of vertices of the cycle (see also [63]).

In order to state the next result we need to introduce the following notion for an arbitrary graph.

Definition B.2.4. Let O be an orientation assigned to a graph. The *flow difference* of a cycle is the integer d_O obtained by subtracting the number of backward edges from the number of forward edges.

With the above definition we can state the following theorem.

Theorem B.2.5 ([57], Theorem 1)**.** *Consider a graph with an acyclic orientation O_1. Another orientation O_2 can be obtained from O_1 by ss-operations if and only if $d_{O_1} = d_{O_2}$ for every cycle of the graph.*

In the following simple example, we apply the above knowledge to pass from a Coxeter element to its conjugacy class and then test more easily the equality $K(\Gamma) = K(C)$ of the Conjecture 5.5.1.

Example B.2.6. Consider the nq-arithmetic hyperbolic Coxeter simplex group Γ with Coxeter graph Σ given in Figure B.2.

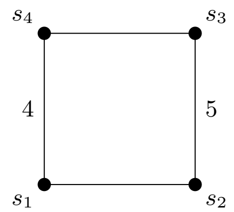


Figure B.2: The Coxeter group Γ in $\mathrm{Isom}(\mathbb{H}^3)$.

The graph Σ is a cycle with $\nu = 4$ vertices. Hence there are three conjugacy classes of Coxeter elements.

Let the bottom left vertex on the graph be the start and end vertex for a walk on the cycle going counter-clockwise. Consider the three acyclic orientations a, b and c of Σ with flow difference 2, 0 and -2, respectively (see Figure B.3).

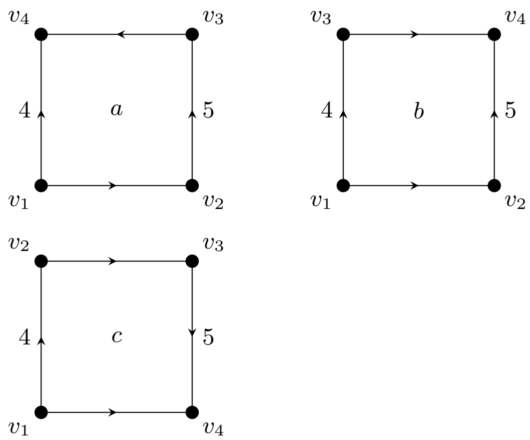


Figure B.3: The three acyclic orientations a, b and c of Σ.

Enumerating the vertices of the graph Σ as shown in Figure B.3, we compute the three Coxeter elements c_a, c_b, c_c each representing a conjugacy class: $c_a = s_1 s_2 s_3 s_4$, $c_b = s_1 s_2 s_4 s_3$ and $c_c = s_1 s_4 s_3 s_2$.

The three corresponding Coxeter transformations C_a, C_b, $C_c \in \mathrm{GL}(\mathbb{R}^4)$ can be computed as described in the proof of Theorem 5.3 in [29]. Consequently we obtain the characteristic polynomials

$$
\begin{aligned}
\chi_{C_a}(t) &= t^4 - \beta t^3 - t^2 - \beta t + 1, \\
\chi_{C_b}(t) &= t^4 - \frac{1}{2}\left(3+\sqrt{5}\right)t^3 - \left(1+\sqrt{2}+\sqrt{10}\right)t^2 - \frac{1}{2}\left(3+\sqrt{5}\right)t + 1, \\
\chi_{C_c}(t) &= t^4 - \beta t^3 - t^2 - \beta t + 1,
\end{aligned}
$$

where $\beta = \frac{1}{2}\left(3+\sqrt{2}+\sqrt{5}+\sqrt{10}\right)$.

Therefore $K(C_a) = K(C_b) = K(C_c) = \mathbb{Q}\left(\sqrt{2}, \sqrt{5}\right) = K(\Gamma)$ as conjectured.

Appendix C

On right-angled polygons in hyperbolic space

Attached to this appendix is the paper "On right-angled polygons in hyperbolic space" written jointly with Simon Drewitz and published in *Geometriae Dedicata, June 2019, Vol. 200, Issue 1, pp. 45–59* [19]. DOI: https://doi.org/10.1007/s10711-018-0357-y. A condensed version of this paper is presented in Chapter 2.

Appendix D

Erratum to the paper "Commensurability classes of hyperbolic Coxeter groups"

Attached to this appendix is the Erratum, due to J. Ratcliffe and S. Tschantz, fixing a gap in the proof of Theorem 1 of [36]. We used some ideas of this proof to demonstrate Proposition 4.3.7. The publication of the Erratum here in this work has been authorised by J. Ratcliffe and S. Tschantz.

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ORIGINAL PAPER

On right-angled polygons in hyperbolic space

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Abstract We study oriented right-angled polygons in hyperbolic spaces of arbitrary dimensions, that is, finite sequences $(S_0, S_1, \ldots, S_{p-1})$ of oriented geodesics in the hyperbolic space $\boldsymbol{H}^{n+2}$ such that consecutive sides are orthogonal. It was previously shown by Delgove and Retailleau (Ann Fac Sci Toulouse Math 23(5):1049–1061, 2014. https://doi.org/10.5802/afst.1435) that three quaternionic parameters define a right-angled hexagon in the 5-dimensional hyperbolic space. We generalise this method to right-angled polygons with an arbitrary number of sides $p \geq 5$ in a hyperbolic space of arbitrary dimension.

Keywords Hyperbolic space · Clifford matrix · Cross ratio · Right-angled polygon · Golden ratio

Mathematics Subject Classification (2000) 51M10 · 15A66 · 51M20

1 Introduction

For $n \geq 0$, let $\boldsymbol{H}^{n+2}$ denote the real hyperbolic $(n+2)$-space. The boundary of this space can be described with Clifford vectors. These are special elements of the Clifford algebra $\mathcal{C}_n$, which is the unitary associative algebra generated by n elements $i_1, \ldots, i_n$ such that $i_j i_l = -i_l i_j$, $i_l^2 = -1$ for $l \neq j$. The group of orientation preserving isometries $\mathrm{Isom}^+(\boldsymbol{H}^{n+2})$ of the hyperbolic space $\boldsymbol{H}^{n+2}$ can be expressed with Clifford matrices. These are 2×2 matrices

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with coefficients in the multiplicative group generated by Clifford vectors and with Ahlfors determinant 1.

In this context, we describe hyperbolic right-angled polygons for which we mean right-angled closed edge paths in $n+2$ dimensions. We show how to construct a hyperbolic right-angled polygon Π_p of p sides, $p > 4$, by prescribing a parameter set consisting of $p-3$ Clifford vectors in $\partial \boldsymbol{H}^{n+2}$. Such a construction is achieved in an arbitrary dimension. No connection between the dimension of the space and the number of sides of the polygon is required.

Similar objects have already been studied in dimension 2 and 3 by Thurston [8] and by Fenchel [9], who studied right-angled hexagons. Costa and Martínez [5] studied right-angled polygons with an arbitrary number of sides in the hyperbolic plane. More recently Delgove and Retailleau [7] classified right-angled hexagons in $\boldsymbol{H}^5$. In their work, 2×2 quaternionic matrices having Dieudonné determinant 1 are used in order to describe the direct isometries of $\boldsymbol{H}^5$. While this approach based on quaternions is very convenient, it can not be extended to arbitrary dimensions. By using Clifford matrices instead, we are able to generalise the construction to any dimension. Particularly, 2×2 quaternionic Clifford matrices are used to describe direct isometries of $\boldsymbol{H}^4$.

In the first section we develop more precisely the connection between hyperbolic space and the Clifford algebra. Then we discuss the role of the cross ratio for Clifford vectors and its geometrical interpretation. Our main result, the algorithmic construction of Π_p, is presented in the second section. In the last part we treat the case when the convex hull of the p vertices of the polygon Π_p give rise to a hyperbolic $(p-1)$-simplex. A necessary condition for its realisation is stated. As a conclusion we discuss in more details a special case in 4 dimensions, supposing that all the edges of the edge path have equal length. By exploiting the work of Dekster and Wilker [6] we explicitly state a necessary and sufficient condition for realisability depending on such a side length. Surprisingly, it turns out that the side length must be related to the golden ratio $\gamma = \frac{1+\sqrt{5}}{2}$.

2 The real Clifford algebra and hyperbolic space

In this section we present the notion of Clifford algebra and its relation to isometries of hyperbolic space. For a more complete description we refer to the works of Ahlfors [1,2], Vahlen [13] and Waterman [14] (see also [11, Section 7]).

2.1 The real Clifford algebra $\mathcal{C}_n$

Consider the real Clifford algebra $\mathcal{C}_n$ generated by $i_1, \ldots, i_n$, that is

$$\mathcal{C}_n = \left\langle i_1, \ldots, i_n \mid i_j\, i_l = -i_l\, i_j,\, i_l^2 = -1 \text{ for } l \neq j \right\rangle,$$

which is a unitary associative real algebra. Every element x of the algebra $\mathcal{C}_n$ can be uniquely written as $x = \sum x_I I$, where $x_I \in \mathbb{R}$ and the sum is taken over all the products $I = i_{k_1} \cdots i_{k_m}$, with $1 \le k_1 < \cdots < k_m \le n$ and $1 \le m \le n$. Here the empty product I_0 is included and identified with $i_0 := 1$. Hence $\mathcal{C}_n$ is a 2^n-dimensional real vector space. In particular we can identify $\mathcal{C}_0$ with $\mathbb{R}$, $\mathcal{C}_1$ with $\mathbb{C}$ and $\mathcal{C}_2$ with $\mathbb{H}$, the Hamiltonian quaternions. To each element $x = \sum x_I I$ we associate a norm as given by $|x|^2 = \sum x_I^2$, inducing a Euclidean structure on $\mathcal{C}_n$. Denote with $\Re(x)$ the coefficient x_0, called the *real part* of x, while $\Im(x) = x - \Re(x)$ is called the *non-real part* of x. If $\Re(x) = 0$ we will refer to x as a *pure element* of $\mathcal{C}_n$.

On $\mathcal{C}_n$ there are three well-known involutions. Let $x \in \mathcal{C}_n$, $x = \sum x_I I$. Then:

(i) $x^* = \sum x_I I^*$, where I^* is obtained from $I = i_{k_1} \cdots i_{k_m}$ by reversing the order of the factors, that is $I^* = i_{k_m} \cdots i_{k_1}$;
(ii) $x' = \sum x_I I'$, where I' is obtained from $I = i_{k_1} \cdots i_{k_m}$ by replacing each factor i_k with $-i_k$, that is $I' = (-i_{k_1}) \cdots (-i_{k_m}) = (-1)^m I$;
(iii) $\overline{x} = (x^*)' = (x')^*$.

The involutions (i) and (iii) are anti-automorphisms, while the involution (ii) is an automorphism.

Of particular interest are Clifford elements of the form $x = x_0 + x_1 i_1 + \cdots + x_n i_n$, called *Clifford vectors*. The set

$$\mathbb{V}^{n+1} = \{x_0 + x_1 i_1 + \cdots + x_n i_n \mid x_0, \ldots, x_n \in \mathbb{R}\}$$

of all Clifford vectors is an $(n+1)$-dimensional real vector space, naturally isomorphic to the Euclidean space $\mathbb{R}^{n+1}$. Notice that for an element $x \in \mathbb{V}^{n+1}$ we have $x^* = x$ and hence $\overline{x} = x'$ as well as $x + \overline{x} = 2\Re(x)$ and $x\overline{x} = \overline{x}x = |x|^2$. Moreover every non-zero vector x has an inverse given by $x^{-1} = \frac{\overline{x}}{|x|^2}$. Hence finite products of non-zero vectors are invertible and they form the so-called *Clifford group* Γ_n. Observe that we have $\Gamma_n = \mathcal{C}_n \backslash \{0\}$ only for $n \in \{0, 1, 2\}$.

2.2 Square root of a Clifford vector

Next we introduce the notion of the square root of a Clifford vector. It will be a generalisation of the square root of quaternions (see [10] for example) in the following way:

Proposition 1 *Let $y \in \mathbb{V}^{n+1} \backslash \{0\}$ be a Clifford vector. If $y \notin \mathbb{R}_{<0}$, then there exist exactly two elements $x_1, x_2 \in \mathbb{V}^{n+1}$ such that $x_1^2 = x_2^2 = y$; x_1 and x_2 are both called a* square root *of y. If $y \in \mathbb{R}_{<0}$, we have the three following situations depending on n:*

- *If $n = 0$, then there is no element $x \in \mathbb{V}^1$ such that $x^2 = y$,*
- *If $n = 1$, then there are exactly two elements $x_1, x_2 \in \mathbb{V}^2$ such that $x_1^2 = x_2^2 = y$,*
- *If $n \geq 2$, then there are uncountably many square roots of y.*

Proof Suppose that $x^2 = y$, with $x, y \in \mathbb{V}^{n+1} \backslash \{0\}$. Then $\overline{x}^2 = \overline{y}$ and $|x|^2 = |y|$. We have the following two equations:

$$x(\overline{x} + x) = x\overline{x} + x^2 = |y| + y, \tag{1}$$
$$(x + \overline{x})^2 = x^2 + 2x\overline{x} + \overline{x}^2 = y + 2|y| + \overline{y} = 2(\Re(y) + |y|). \tag{2}$$

Observe that the term $2(\Re(y) + |y|) \geq 0$.

Now let $y \notin \mathbb{R}_{<0}$, then we have $\Re(y) + |y| > 0$, and the element

$$x := \frac{|y| + y}{\sqrt{2(\Re(y) + |y|)}} \in \mathbb{V}^{n+1} \tag{3}$$

satisfies $x^2 = y$. Indeed,

$$x^2 = \frac{|y|^2 + 2|y|y + y^2}{2\Re(y) + 2|y|} = \frac{(\overline{y} + 2|y| + y)y}{2\Re(y) + 2|y|} = y.$$

Notice that in the special case if $y \in \mathbb{R}_{>0}$, the identity (3) yields $x = \pm\sqrt{y}$ as desired. For $y \notin \mathbb{R}$ the square roots of y have to lie in the plane spanned by 1 and y which is isomorphic to $\mathbb{C}$, ensuring the non-existence of more than two roots. By abuse of notation the square root x of y is denoted by $\sqrt{y} := x$.

Let $y \in \mathbb{R}_{<0}$. For $n = 0$ or 1 the assertion is trivial. Let $n \geq 2$. We can write $y = -z^2$ for some $z \in \mathbb{R}_{>0}$. In this case consider $x := z \cdot u$ where u is a pure Clifford vector with norm 1. In general for any pure Clifford vector we have

$$0 = (u + \overline{u})u = u\overline{u} + u^2 = |u|^2 + u^2,$$

which implies $u^2 = -|u|^2$. Hence $x^2 = z^2u^2 = -z^2|u|^2 = -z^2$. $\square$

Remark 1 Notice that Proposition 1 remains true for $y \in \Gamma_2 = \mathbb{H}\backslash\{0\}$ since $y + \overline{y} = 2\Re(y)$ still holds. However, it does not hold for a general element of $\mathcal{C}_n$ or even Γ_n, $n \geq 3$. Indeed, for an arbitrary $y \in \Gamma_n$ one has $y + \overline{y} \neq 2\Re(y)$. For example let $y = i_1i_2i_3 \in \Gamma_n, n \geq 3$. Then $y + \overline{y} = 2i_1i_2i_3$. Hence Eq. (2) does not hold.

Remark 2 For the square root $\sqrt{y}$ of a Clifford vector $y \in \mathbb{V}^{n+1}\backslash\mathbb{R}_{\leq 0}$ we have:

- For all positive $\mu \in \mathbb{R}_{>0}$, $\sqrt{\mu y} = \sqrt{\mu}\sqrt{y}$,
- For the inverse $\sqrt{y^{-1}} = \sqrt{y}^{-1} = \frac{1}{|y|}\sqrt{\overline{y}}$.
- The square root of $-y$ can be found by a rotation of 90°: $\sqrt{-y} = i\sqrt{y}$ for some pure Clifford vector i with $i^2 = -1$. This also holds for negative $y \in \mathbb{R}_{<0}$.

2.3 Clifford matrices and hyperbolic isometries

We now take a look at matrices having entries in the extended Clifford group $\Gamma_n \cup \{0\}$. These matrices will be used to explicitly represent direct isometries of the hyperbolic space $\boldsymbol{H}^{n+2}$ (see for example [14] and [11, Section 7]).

A *Clifford matrix* is a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with

$$a, b, c, d \in \Gamma_n \cup \{0\}, \quad ab^*, cd^*, c^*a, d^*b \in \mathbb{V}^{n+1}, \quad ad^* - bc^* \in \mathbb{R}\backslash\{0\},$$

where $ad^* - bc^*$ is the *Ahlfors determinant* of A. Denote the set of such matrices by $\mathrm{GL}(2, \mathcal{C}_n)$. By a result of Vahlen and Maass [2, p. 221] the set

$$\mathrm{SL}(2, \mathcal{C}_n) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{C}_n) \mid ad^* - bc^* = 1 \right\} \tag{4}$$

of Clifford matrices with Ahlfors determinant 1 is a multiplicative group.

Each element $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{C}_n)$ has the inverse matrix $T^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$.

Furthermore $\mathrm{SL}(2, \mathcal{C}_n)$ is generated by the matrices

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix},$$

where $t \in \mathbb{V}^{n+1}$ and $a \in \Gamma_n$ (see for example [11, Section 7]).

The group $\mathrm{SL}(2, \mathcal{C}_n)$ plays an important role in our investigation since it is closely related to the group of orientation preserving isometries of the hyperbolic $(n+2)$-space realised in the upper half-space according to

$$\begin{aligned} \boldsymbol{H}^{n+2} &= \left\{ x = (x_0, x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+2} \mid x_{n+1} > 0 \right\} \\ &\cong \mathbb{V}^{n+1} \times \mathbb{R}_{>0}. \end{aligned}$$

The compactification $\overline{\boldsymbol{H}^{n+2}}$ is given by the union of $\boldsymbol{H}^{n+2}$ with the boundary set $\partial\boldsymbol{H}^{n+2} = \mathbb{V}^{n+1} \cup \{\infty\}$ of points at infinity of $\boldsymbol{H}^{n+2}$.

Consider the projective group

$$\mathrm{PSL}(2, \mathcal{C}_n) = \mathrm{SL}(2, \mathcal{C}_n)/\{\pm I\}.$$

It is known that this group acts bijectively on $\mathbb{V}^{n+1} \cup \{\infty\}$ by

$$T(x) = (ax + b)(cx + d)^{-1} \tag{5}$$

with $T(-c^{-1}d) = \infty$, $T(\infty) = ac^{-1}$ if $c \neq 0$, and $T(\infty) = \infty$ otherwise. By Poincaré extension, the action (5) can be extended to the upper half-space $\boldsymbol{H}^{n+2}$. In this way we obtain an isomorphism between $\mathrm{PSL}(2, \mathcal{C}_n)$ and the group $\mathrm{Möb}^+(n + 1)$ of orientation preserving Möbius transformations of $\mathbb{V}^{n+1} \cup \{\infty\}$ (see [4,14]). Since the group $\mathrm{Isom}^+\left(\boldsymbol{H}^{n+2}\right)$ of orientation preserving isometries of $\boldsymbol{H}^{n+2}$ is isomorphic to $\mathrm{Möb}^+(n+1)$, we get the following identification:

$$\mathrm{Isom}^+\left(\boldsymbol{H}^{n+2}\right) \cong \mathrm{Möb}^+\,(n + 1) \cong \mathrm{PSL}(2, \mathcal{C}_n). \tag{6}$$

Therefore any direct isometry of $\boldsymbol{H}^{n+2}$ can be represented by a Clifford matrix in $\mathrm{PSL}(2, \mathcal{C}_n)$.

Finally, we remark that Möbius transformations act triply transitively on $\mathbb{V}^{n+1} \cup \{\infty\}$ (see [15, Section 6], for example). That is, given two triplets $\{x_1, x_2, x_3\}$ and $\{x_1', x_2', x_3'\}$ of distinct points in the boundary, there always exists a transformation $T \in \mathrm{Möb}(n + 1)$ with $T(x_i) = x_i'$. For $n = 0$ this map is unique and for $n = 1$ it is unique if one demands that it preserves the orientation. In higher dimensions this map is not unique anymore.

2.4 The cross ratio

As in the classical case, we shall use the cross ratio to study configurations of points in $\mathbb{V}^{n+1} \cup \{\infty\}$.

Definition 1 Let x, y, z, w be four pairwise different Clifford vectors in $\mathbb{V}^{n+1}$. Then

$$[x, y, z, w] := (x - z)(x - w)^{-1}(y - w)(y - z)^{-1} \in \Gamma_n\backslash\{0\} \tag{7}$$

is called the *cross ratio* of x, y, z and w.

We extend the definition (7) by continuity to $\mathbb{V}^{n+1} \cup \{\infty\}$, allowing x, y or w to be ∞, by

$$[\infty, y, z, w] = (y - w)(y - z)^{-1} \quad \text{for } x = \infty, \tag{8}$$

and similarly for $y = \infty$ and $w = \infty$. Moreover in an analogous way we put

$$[x, y, \infty, w] = (x - w)^{-1}(y - w).$$

The cross ratio satisfies the following transformation behaviour (see [4, Lemma 6.2]):

$$[T(x), T(y), T(z), T(w)] = (cz + d)^{*-1}[x, y, z, w](cz + d)^*, \tag{9}$$

for all $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{C}_n)$.

Hence, the real part and the norm of the cross ratio $[x, y, z, w]$ of four vectors are invariant under the action of T. However, the cross ratio itself is not an invariant.

We specialise the cross ratio in the following way: consider two oriented geodesics s, t in $\boldsymbol{H}^{n+2}$ whose endpoints s^-, s^+ and t^-, t^+ are four distinct points in $\mathbb{V}^{n+1} \cup \{\infty\}$.

Definition 2 The *cross ratio* $\Delta(s,t)$ *of* s *and* t is defined by

$$\Delta(s,t) := \left[s^-, s^+, t^-, t^+\right]. \tag{10}$$

Lemma 1 *Let* s *and* t *be two geodesics as above. If* s *and* t *intersect then* $\Delta(s,t) = \Delta(t,s)$. *If* s *and* t *are disjoint, then* $\Delta(s,t) = \Delta(t,s)$ *if one of the endpoints is* ∞ *or if the cross ratios are real, otherwise the two cross ratios are conjugate.*

Proof Assuming one of the endpoints to be infinity, let $s = (x, \infty)$ with $x \in \mathbb{V}^{n+1}$. We can apply a translation $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ such that s is mapped to $(0, \infty)$. By (9), any translation leaves the cross ratio unchanged. Using (8) it is easy to see that $\Delta(s,t) = \Delta(t,s)$.

Let now s and t be two arbitrary geodesics with no endpoint at infinity. We know that we can always find an isometry $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{C}_n)$ mapping the two endpoints of one of the geodesics to 0 and ∞. Using (9) and what we have just discussed above we get

$$\begin{aligned}(ct^- + d)^{*-1}[s^-, s^+, t^-, t^+](ct^- + d)^* &= \left[T(s^-), T(s^+), T(t^-), T(t^+)\right] \\ &= \left[T(t^-), T(t^+), T(s^-), T(s^+)\right] \\ &= (cs^- + d)^{*-1}[t^-, t^+, s^-, s^+](cs^- + d)^*.\end{aligned}$$

Hence the two cross ratios $\Delta(s,t)$ and $\Delta(t,s)$ are conjugate. This implies that if the cross ratios are real, then the equality $\Delta(s,t) = \Delta(t,s)$ holds. In particular, if two geodesics intersect, then $\Delta(s,t) = \Delta(t,s)$ by Proposition 2 below. □

Now consider three geodesics r, s and t in $\boldsymbol{H}^{n+2}$ with pairwise different endpoints r^-, r^+, s^-, s^+ and t^-, t^+ in $\mathbb{V}^{n-1} \cup \{\infty\}$.

Definition 3 The quantity

$$\Delta(r,s,t) := \left[s^+, s^-, r^+, t^+\right] \tag{11}$$

is called the *double bridge cross ratio* of (r,s,t).

Definition 4 The ordered triple (r,s,t) is called a *double bridge* if s is orthogonal to r and t such that $r \neq t$. If $|\Delta(r,s,t)| > 1$, then the intersections $r \cap s$ and $s \cap t$ do not coincide and we call the double bridge *properly oriented*.

Consider a properly oriented double bridge (r,s,t). The norm of $\Delta(r,s,t)$ encodes the hyperbolic length of the geodesic segment $[r,t]$ on s between r and t. Indeed, assume w.l.o.g. that the endpoints of s in the double bridge (r,s,t) are $s^- = 0$ and $s^+ = \infty$ (see Fig. 1). The hyperbolic distance δ of two points $p, q \in s$ in $\boldsymbol{H}^{n+2}$ with $p_{n+2} > q_{n+2}$ is equal to (see [3, p. 131])

$$\delta = \log\left(\frac{p_{n+2}}{q_{n+2}}\right).$$

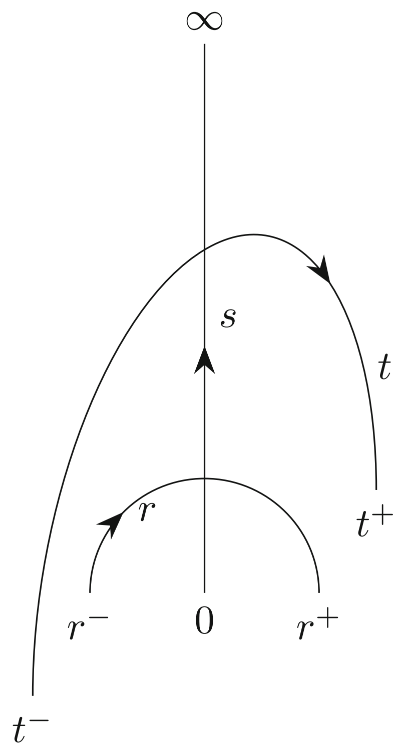
On the other hand, by (7) we get

$$|\Delta(r,s,t)| = |[\infty, 0, r^+, t^+]| = \frac{|t^+|}{|r^+|}.$$

If we take $p = s \cap t$ and $q = r \cap s$, we conclude that $\delta = \log(|\Delta(r,s,t)|)$.

The following results will be of importance:

Fig. 1 Double bridge



Proposition 2 *Two hyperbolic geodesics s and t intersect if and only if their cross ratio $\Delta(s,t) \in \mathbb{R}_{<0}$. Furthermore s and t are perpendicular if and only if $\Delta(s,t) = -1$.*

Proof Since hyperbolic isometries act triply transitively, there is an isometry represented by $A \in \mathrm{SL}(2, \mathcal{C}_n)$ mapping s and t into $(0,\infty)$ and $(1,x)$, $x \in \mathbb{V}^{n+1}$. Then, by (7) and (8), the cross ratio of $A(s)$ and $A(t)$ equals $\Delta(A(s), A(t)) = [0, \infty, 1, x] = x^{-1}$, and the assertions follow for $A(s)$ and $A(t)$. Moreover, by (9), a real cross ratio stays invariant under isometry. □

Proposition 3 *Let $s = (0,\infty)$ and $t = (1, y)$ with $y \neq 0, \infty$ be two disjoint geodesics in $\boldsymbol{H}^{n+2}$. Then the common perpendicular l is $(-\sqrt{y}, \sqrt{y})$. This perpendicular is unique up to orientation.*

Proof Let $l = (z, w)$ denote the common perpendicular between s and t. By Proposition 2 and by (8), we get

$$\Delta(s,l) = [0, \infty, z, w] = -1. \tag{12}$$

and

$$\Delta(t,l) = [1, y, z, w] = -1. \tag{13}$$

Equation (12) yields $z = -w$. The Eq. (13) states that

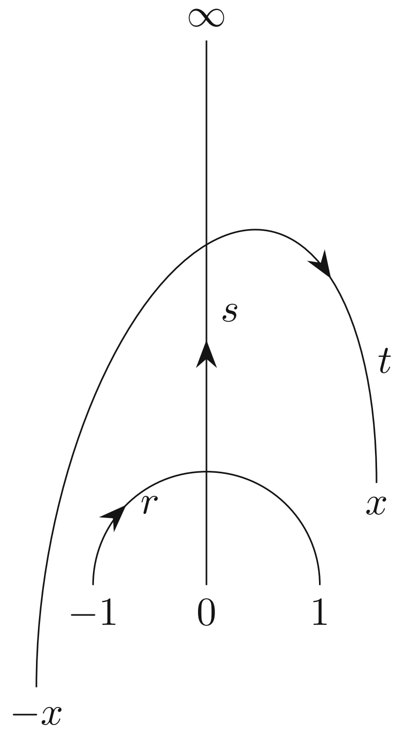
$$(1-z)(1+z)^{-1} = -(y-z)(y+z)^{-1}. \tag{14}$$

It is easy to see that $(1-z)(1+z)^{-1} = (1+z)^{-1}(1-z)$, so that

$$(1-z)(y+z) = -(1+z)(y-z).$$

By expanding the above equation we obtain $y = z^2$. Notice that by construction, since s and t are disjoint, we have $y \notin \mathbb{R}_{<0}$. Hence, by applying Proposition 1, the result follows for $l = (\pm\sqrt{y}, \mp\sqrt{y})$. □

Fig. 2 Standard configuration double bridge



3 The main theorem

3.1 Preliminaries

Our aim is to construct oriented right-angled polygons in hyperbolic space from a minimal number of prescribed parameters.

Definition 5 An oriented right-angled polygon with p sides in $\boldsymbol{H}^{n+2}$ (or p-gon for short), $n \geq 0$, is a p-tuple of oriented geodesics $(S_0, S_1, \ldots, S_{p-1})$ with $S_{i-1} \neq S_{i+1}$ for $i \pmod p$ and such that S_i is orthogonal to S_{i+1} for $0 \leq i \leq p-2$ and S_{p-1} is orthogonal to S_0.

We usually denote it by Π_p.

We call such a p-gon Π_p *non-degenerate* if consecutive intersections do not coincide (that is $S_{i-1} \cap S_i \neq S_i \cap S_{i+1}$ for $i \pmod p$) and the double bridges (S_{i-1}, S_i, S_{i+1}), $i \pmod p$, are properly oriented.

We can take $p \geq 5$ since the simplest case of a right-angled polygon is the pentagon. There cannot be a hyperbolic rectangle since the common perpendicular of two geodesics S_0 and S_2 is unique. Hence if there was a hyperbolic rectangle (S_0, S_1, S_2, S_3), two geodesics would have to be identical.

Note that it is no restriction to only consider p-gons in $\boldsymbol{H}^{p-1}$ since the convex hull of p geodesics can at most have dimension $p-1$. Hence, we will always refer to this case.

Recall that the one-point compactified vector space $\mathbb{V}^{p-2} \cup \{\infty\}$ forms the boundary of hyperbolic $(p-1)$-space

$$\boldsymbol{H}^{p-1} = \left\{(x, y) \in \mathbb{V}^{p-2} \times \mathbb{R}_{>0}\right\}.$$

Consider the standard configuration double bridge (r, s, t) similar to Sect. 2.4 with $r = (-1, 1)$, $s = (0, \infty)$ and $t = (-x, x)$ for $x \in \mathbb{V}^{p-2} \backslash \{-1, 0, 1\}$ (see Fig. 2).

A small computation shows that the double bridge cross ratio is given by

$$\Delta((-1, 1), (0, \infty), (-x, x)) = [\infty, 0, 1, x] = x. \tag{15}$$

If conversely the first two geodesics of this double bridge and a desired double bridge cross ratio q are given, one can construct the third geodesic as $(-q, q)$. In the general case this is not easy since the Clifford vectors do not commute. In view of (15) we shall start with the configuration given by the geodesics $(-1, 1)$, $(0, \infty)$ and $(-x, x)$. If the double bridges are supposed to be properly oriented, this poses the immediate restriction $|x| > 1$.

To construct more geodesics we will have to apply certain isometries to achieve this configuration from a general double bridge. These isometries depend on the double bridge cross ratios in the right-angled polygon Π_p they are part of.

Definition 6 For a set of given Clifford vectors $\{q_1, \ldots, q_{p-3}\} \subset \mathbb{V}^{p-2} \setminus \{0\}$ define the isometries ϕ_i of upper half-space by the following Möbius transformations:

$$\phi_i : x \mapsto \sqrt{-2\,q_i}^{-1} (x + q_i)(x - q_i)^{-1} \sqrt{-2\,q_i}, \quad 1 \le i \le p-3. \tag{16}$$

If $q_i \in \mathbb{R}_{>0}$, choose $\sqrt{-2\,q_i} := \sqrt{2\,q_i}\, i_1$.

Let Φ_i be the concatenation $\Phi_i := \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1$.

Note that the isometries ϕ_i carry the two geodesics $(0, \infty)$ and $(-q_i, q_i)$ into the geodesics $(-1, 1)$ and $(0, \infty)$ of a double bridge in the aforementioned setting. However, these isometries are not uniquely defined by this property. We will always apply these ϕ_i if we need an isometry which maps given geodesics to specific other geodesics in a polygon Π_p.

The Clifford matrix corresponding to ϕ_i is

$$\begin{pmatrix} \sqrt{-2\,q_i}^{-1} & q_i \sqrt{-2\,q_i}^{-1} \\ \sqrt{-2\,q_i}^{-1} & -q_i \sqrt{-2\,q_i}^{-1} \end{pmatrix}. \tag{17}$$

The inverse $\phi_i^{-1}(x) = \sqrt{-q_i}\,(1 + x)(1 - x)^{-1} \sqrt{-q_i}$ is represented by the matrix

$$\begin{pmatrix} q_i \sqrt{-2\,q_i}^{-1} & q_i \sqrt{-2\,q_i}^{-1} \\ \sqrt{-2\,q_i}^{-1} & -\sqrt{-2\,q_i}^{-1} \end{pmatrix}. \tag{18}$$

Repeatedly applying these isometries to geodesics in a Π_p enables us to standardise the cross ratio of a double bridge in a p-gon and eliminate the problem of the cross ratio not being invariant under isometries (Fig. 3).

3.2 The theorem

Definition 7 Let $(S_0, \ldots, S_{p-1})$ be a right-angled p-gon. Define the *gauged double bridge cross ratios* $\tilde{\Delta}_i$ for $i = 1, \ldots, p-3$ by the following recursive definition:

$$\tilde{\Delta}_1 := \Delta(S_0, S_1, S_2), \tag{19}$$

$$\tilde{\Delta}_{i+1} := \Delta(\Phi_i(S_i), \Phi_i(S_{i+1}), \Phi_i(S_{i+2})). \tag{20}$$

The Clifford vectors q_i which are needed to define the maps Φ_i are calculated along the way as

$$q_i = \tilde{\Delta}_i. \tag{21}$$

These gauged double bridge cross ratios will be the parameters describing the non-degenerate right-angled p-gons in $\boldsymbol{H}^{p-1}$ in the Theorem 1 below. Hence consider the set

$$\mathcal{P}_p := \{(q_1, \ldots, q_{p-3}) \mid q_i \in \mathbb{V}^{p-2}, |q_i| > 1, 1 \le i \le p-3\} \tag{22}$$

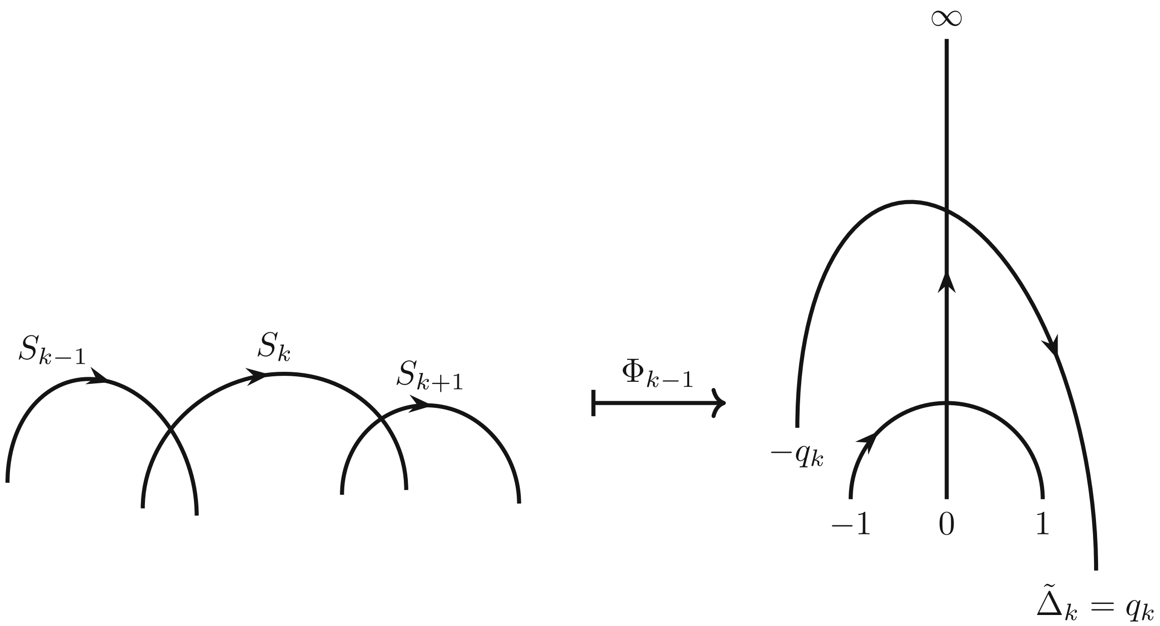


Fig. 3 Gauging by an isometry

of $(p-3)$-tuples of non-zero Clifford vectors. Denote by

$$\mathrm{RAP}_p := \Big\{ \left(S_0, \ldots, S_{p-1}\right) \text{ non-degenerate right-angled polygon in } \boldsymbol{H}^{p-1} \\ \text{with } S_0 = (-1, 1)\,, S_1 = (0, \infty) \Big\} \tag{23}$$

the set of non-degenerate right-angled polygons with p sides. The calculation of the gauged double bridge cross ratios gives a map $\tilde{\Delta} : \mathrm{RAP}_p \to \mathcal{P}_p$. Denote the image of this map by $\mathcal{P}_p^* := \tilde{\Delta}\left(\mathrm{RAP}_p\right) \subset \mathcal{P}_p$. This is the set of parameters which will yield a non-degenerate Π_p in the construction below.

Theorem 1 *The map $\tilde{\Delta} : \mathrm{RAP}_p \to \mathcal{P}_p^*$ is a bijection. The inverse map can be given as an explicit construction of a right-angled p-gon Π_p from a tuple of $p-3$ parameters in $\mathcal{P}_p^*$.*

3.3 Proof of Theorem 1

Bijectivity It is sufficient to prove the injectivity of $\tilde{\Delta}$ since it is surjective by definition. Note that in the standard configuration double bridge of Fig. 2, there is a one-to-one correspondence of Clifford vectors x and geodesics $t = (-x, x)$ as given by Eq. (15). Now assume there are two p-gons $\Pi_p = (S_0, \ldots, S_{p-1})$, $\Pi_p' = (S_0', \ldots, S_{p-1}') \in \mathrm{RAP}_p$ such that $\tilde{\Delta}\left(\Pi_p\right) = \tilde{\Delta}\left(\Pi_p'\right) = \left(q_1, \ldots, q_{p-3}\right)$. By definition $S_0 = S_0'$ and $S_1 = S_1'$. By the above correspondence we also have $S_2 = S_2'$. Furthermore the maps $\phi_1, \ldots, \phi_{p-3}$ are the same for both Π_p and Π_p' since these maps are defined by $q_1, \ldots, q_{p-3}$ as given in Eq. (16). Therefore the map Φ_i yields the same one-to-one correspondence between geodesics and Clifford vectors in both p-gons.

Construction of the polygon Π_p The inverse map $\tilde{\Delta}^{-1}$ is given by the construction of a Π_p from $p-3$ parameters $q_1, \ldots, q_{p-3} \in \mathbb{V}^{p-2}$.

Assume we are given $p-3$ parameters $(q_1, \ldots, q_{p-3}) \in \mathcal{P}_p^*$.

Start The first two geodesics are fixed as $S_0 = (-1, 1)$ and $S_1 = (0, \infty)$. Since this is the standard configuration double bridge considered above, we find $S_2 = (-q_1, q_1)$ if we demand $\Delta(S_0, S_1, S_2) = q_1$.

The geodesic S_3 To find the endpoints of S_3, we benefit from the isometry ϕ_1^{-1} above which maps $(-1, 1)$ to $(0, \infty)$ and $(0, \infty)$ to S_2. If q_2 was the cross ratio of a double bridge involving $(-1, 1)$ and $(0, \infty)$, the third geodesic would be $(-q_2, q_2)$. Since S_3 is part of the double bridge starting with $(0, \infty)$ and S_2, S_3 can be found by applying ϕ_1^{-1} to $(-q_2, q_2)$, that is $S_3 = (\phi_1^{-1}(-q_2), \phi_1^{-1}(q_2))$.

The next geodesic in the general case The further procedure expands the previous idea. First we note that the next geodesic is given by the parameter q_3. The geodesic S_4 would then be the image of $(-q_3, q_3)$ under the isometry Φ_2^{-1} mapping $(-1, 1)$ and $(0, \infty)$ to S_2 and S_3, respectively.

In general, assuming we have calculated the geodesics $S_0, \ldots, S_k$ for some k with $2 \leq k \leq p-3$, we can use Φ_{k-1}^{-1} in order to obtain $S_{k+1} = (\Phi_{k-1}^{-1}(-q_k), \Phi_{k-1}^{-1}(q_k))$.

Existence of the last geodesic After using all the parameters $q_1, \ldots, q_{p-3}$, we have determined the geodesics $S_0, \ldots, S_{p-2}$. As a consequence of Proposition 2 the last common perpendicular between S_0 and S_{p-2} exists and is unique as long as

$$\Delta\left(S_0, S_{p-2}\right) \notin \mathbb{R}_-. \tag{24}$$

This is ensured by the set $\mathcal{P}^* \subset \mathcal{P}$. □

Remark 3 Since the Clifford vectors do not commute, one cannot directly compute the common perpendicular S_{p-1} using the equations

$$\Delta\left(S_{p-1}, S_0\right) = -1, \quad \Delta\left(S_{p-1}, S_{p-2}\right) = -1. \tag{25}$$

However, one can use an isometry to obtain a nice configuration where the terms in the equations above commute. Writing $S_{p-2} = (a, b)$, consider the isometry

$$\psi : x \mapsto \alpha^{-1}\,(1+x)(1-x)^{-1}\,\alpha^{-1} \tag{26}$$

where $\alpha := \sqrt{-(1+a)(1-a)^{-1}}$. This isometry maps S_0 to $(0, \infty)$ and S_{p-2} to $(1, c)$ where $c := \alpha^{-1}\,(1+b)(1-b)^{-1}\,\alpha^{-1}$.

Hence, by Proposition 3

$$S_{p-1} = \left(\psi^{-1}\left(-\sqrt{c}\right), \psi^{-1}\left(\sqrt{c}\right)\right) \tag{27}$$

modulo orientation where ψ^{-1} is given by

$$\psi^{-1}(x) = (\alpha\, x\, \alpha - 1)\,(\alpha\, x\, \alpha + 1)^{-1}. \tag{28}$$

Remark 4 A major drawback is that we cannot explicitly describe $\mathcal{P}_p^*$. One can take a set of parameters $\left(q_1, \ldots, q_{p-3}\right) \in \mathcal{P}_p$, apply the above construction and afterwards check whether the created object actually is a non-degenerate right-angled p-gon.

If all the parameters q_i have norm $|q_i| > 1$ the proper orientation of the geodesics $S_1, \ldots, S_{p-3}$ is automatically guaranteed. So one needs to check the orientation of S_0, S_{p-2} and S_{p-1}. This can be done by calculating the norm of the double bridge cross ratios with the respective geodesic as the central one. Since the norm of the cross ratio is invariant under isometry we do not have to use the gauged double bridge cross ratios at this point. If the orientation of S_{p-1} is wrong, one can just invert it. If the orientation of S_{p-2} is wrong, one needs to replace the parameter q_{p-3} by $-q_{p-3}$ and the construction yields the same Π_p just with the inverted orientation of S_{p-2}. If the orientation of S_0 is wrong, one can replace the parameter q_1 by $-q_1$. This introduces a factor i to the left and to the right of the map ϕ_1^{-1}, where i is a root of -1 in the plane spanned by 1 and q_1; respectively $i = i_1$ if q_1 is real. Such a map is a rotation of 180° in the plane spanned by 1 and i.

After some exemplary calculations, we conjecture that for $p = 5$ the set

$$\{(q_1, q_2) \in \mathcal{P}_5 \mid \Re(q_1) \neq 0, q_1 \not\angle q_2\} \tag{29}$$

yields non-degenerate right-angled 5-gons up to orientation.

4 Right-angled polygons with full span

One natural question which arises when studying right-angled polygons is the question of the dimension of the resulting object. In this section we consider right-angled p-gons which have the highest possible dimension. This is the case if the p intersection points are the vertices of a $(p-1)$-simplex. Thus the parameters will be taken from a $(p-2)$-dimensional Clifford vector space $\mathbb{V}^{p-2} \subset \mathcal{C}_{p-3}$.

4.1 A necessary condition for the realisation of $(p-1)$-simplices

If we want some set of parameters to yield a simplex, we need to pass to a new dimension with every new geodesic in the construction. This basic idea results in the following theorem:

Theorem 2 *If the parameters $q_1, \dots, q_{p-3} \in \mathcal{C}_{p-3}$ give rise to a right-angled polygon Π_p whose intersection points are the vertices of a simplex, then the parameters together with 1 have to form a basis of the Clifford vectors according to $\langle 1, q_1, \dots, q_{p-3}\rangle = \mathbb{V}^{p-2}$.*

This theorem is a consequence of the following lemma:

Lemma 2 *Let $(S_0, S_1, \dots, S_k)$, $k \geq 2$ be a finite sequence of geodesics in upper-half space $\boldsymbol{H}^{p-1}$ such that $S_0 = (-1, 1)$, $S_1 = (0, \infty)$ and $S_{i-1} \perp S_i$ for $i = 1, \dots, k$. Furthermore denote by $q_i := \tilde{\Delta}(S_{i-1}, S_i, S_{i+1})$ the gauged double bridge cross ratios of the respective double bridges for $i = 1, \dots, k-1$ and write $S_i = (S_i^-, S_i^+)$ for all geodesics.*

Then the linear subspace of $\mathbb{V}^{p-2}$ spanned by the endpoints of the geodesics is the same as the subspace spanned by $\{1, q_0, q_1, \dots, q_{k-1}\}$. In symbols this means

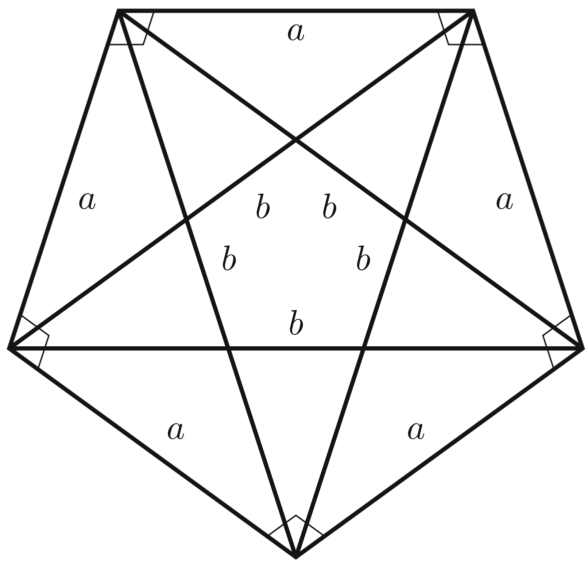
$$\left\langle S_0^{\pm}, S_2^{\pm}, S_3^{\pm}, \dots, S_k^{\pm}\right\rangle = \langle 1, q_1, q_2, \dots, q_{k-1}\rangle\,. \tag{30}$$

The geodesic S_1 is left out since $\infty \notin \mathbb{V}^{p-2}$.

Proof We prove this by induction over k. For $k = 2$ the lemma is plain, since $S_2 = (-q_1, q_1)$.

Hence, we have to prove $\left\langle 1, q_1, q_2, \dots, q_{k-1}, S_{k+1}^{\pm}\right\rangle = \langle 1, q_1, q_2, \dots, q_k\rangle$. We know that S_{k+1} is given as the image of $(-q_k, q_k)$ under the isometry Φ_{k-1}^{-1}. This isometry is given as a concatenation of the maps $\phi_i^{-1} : x \mapsto \sqrt{-q_i}\,(1+x)(1-x)^{-1}\sqrt{-q_i}$, $1 \leq i \leq k-1$. If $q_i \notin \mathbb{R}$, ϕ_i^{-1} restricts to an isometry on $\boldsymbol{H}^3$ where the boundary is given as $\partial \boldsymbol{H}^3 = \langle 1, q_i\rangle \cup \{\infty\}$. Likewise, ϕ_i restricts to an isometry on $\boldsymbol{H}^4$ where the boundary is given as $\partial \boldsymbol{H}^4 = \langle 1, q_i, q_k\rangle \cup \{\infty\}$. The case $q_i \in \mathbb{R}$ follows in the same manner, by yielding isometries leaving corresponding subspaces $\boldsymbol{H}^2$ and $\boldsymbol{H}^3$ invariant. Thus follows the statement. □

Fig. 4 Hyperbolic pentagon with right-angled cyclic edge path



Notice that the theorem above does not give a sufficient condition. If the parameters q_i are pairwise orthogonal to each other and pure Clifford vectors then the geodesics S_0 and S_{p-2} will contribute sides of length 0.

4.2 Hyperbolic 4-simplices with an orthogonal cyclic edge path

In the end, it would be nice to have an a priori condition on the parameters of at least some family of pentagons. Dekster and Wilker [6] proved a criterion for the existence of n-simplices with vertices $p_1, \dots, p_{n+1}$ with given side and diagonal lengths $l_{ij} = \mathrm{d}\left(p_i, p_j\right)$, $1 \le i < j \le n+1$ in a Euclidean, spherical or hyperbolic space $X \in \left\{\boldsymbol{E}^n, \boldsymbol{S}^n, \boldsymbol{H}^n\right\}$. They call a symmetric $(n+1) \times (n+1)$-matrix $L = (l_{ij})$ *allowable* if $l_{ii} = 0$ and $l_{ij} > 0$ for $i \neq j$. The matrix L is called *realisable* in the space X if there are $n+1$ points $p_1, \dots, p_{n+1}$ in X with the given distances $\mathrm{d}\left(p_i, p_j\right) = l_{ij}$. They gave a criterion for realisability in each of the three above cases. We are especially interested in the hyperbolic case.

Theorem 3 [6, Theorem 1 (hyperbolic case)] *Let $L = (l_{ij})$ be an allowable $(n+1) \times (n+1)$-matrix and let its entries be used to form the $(n \times n)$-matrix $S = (s_{ij})$ where*

$$s_{ij} = \cosh l_{i,n+1} \cosh l_{j,n+1} - \cosh l_{ij}.$$

Then L is realisable if and only if the eigenvalues of S are non-negative. If L is realisable then the dimension of each realisation is equal to the rank of S.

Now we can easily treat the case of a hyperbolic pentagon having a cyclic edge path along which all sides have the same length (Fig. 4). With [6] we can get a criterion on the side lengths and due to symmetry it might be possible to find the corresponding orientations of the sides.

Lemma 3 *A right-angled hyperbolic pentagon $\Pi_5 = (S_0, \dots, S_4)$ with all side lengths equal to $a > 0$ is realisable as a 4-simplex if and only if $\cosh(a) < \gamma$, where $\gamma = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.*

Proof By using hyperbolic trigonometry (see for example [12, Section 3.5]) we obtain the relation $\cosh(b) = \cosh^2(a)$. We can now construct the two matrices L and S as in [6, Theorem 1]. We get

$$L = \begin{pmatrix} 0 & a & b & b & a \\ a & 0 & a & b & b \\ b & a & 0 & a & b \\ b & b & a & 0 & a \\ a & b & b & a & 0 \end{pmatrix}.$$

Let us define $x := \cosh(a)$. We then have

$$S = \begin{pmatrix} x^2-1 & x^3-x & x^3-x^2 & 0 \\ x^3-x & x^4-1 & x^4-x & x^3-x^2 \\ x^3-x^2 & x^4-x & x^4-1 & x^3-x \\ 0 & x^3-x^2 & x^3-x & x^2-1 \end{pmatrix}.$$

By Dekster's and Wilker's Theorem, the matrix L is realisable as a 4-simplex if and only if all the eigenvalues of S are positive. This is true if and only if S is positive definite. By Sylvester's criterion, it is enough to check that all the top left minors of S have positive determinant:

$$\begin{aligned} \det{}_1 &= x^2 - 1, \\ \det{}_2 &= x^4 - 2x^2 + 1 = (x^2-1)^2 = (x+1)^2(x-1)^2, \\ \det{}_3 &= -x^8 + 2x^7 + x^6 - 2x^5 - 2x^4 + 3x^2 - 1, \\ \det{}_4 = \det(S) &= 2x^{10} - 10x^9 + 15x^8 - 15x^6 + 2x^5 + 10x^4 - 5x^2 + 1. \end{aligned}$$

Notice that $x > 1$ since a must be greater than 0. Hence $\det_1$ and $\det_2$ are always greater than 0. Furthermore, $\det_3$ is positive whenever $-1 < x < \frac{1-\sqrt{5}}{2}$ or $1 < x < \frac{1+\sqrt{5}}{2}$, hence only the latter has to be considered. The determinant of S is positive everywhere except in $\frac{1-\sqrt{5}}{2}$, 1, $\frac{1+\sqrt{5}}{2}$, where it vanishes. Combining everything we obtain that S is positive definite whenever $1 < x < \gamma$, giving us the desired result. $\square$

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ERRATUM: COMMENSURABILITY CLASSES OF HYPERBOLIC COXETER GROUPS

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ABSTRACT. In this note we fill a gap in our previously published paper [2].

1. INTRODUCTION

In our paper [2], there is a gap in the proof of Theorem 1. The gap is caused by a miss application of Burnside's theorem [1]. Burnside's theorem requires an algebraically closed field, and so we need to work over $\mathbb{C}$ rather than $\mathbb{R}$. The gap is filled by the following lemma and a few comments after the proof.

The positive special Lorentz group $\mathrm{SO}^+(n,1)$ is the connected component of the identity of the Lorentz group $\mathrm{O}(n,1)$. In our paper $\mathrm{SO}^+(n,1)$ is denoted by $\mathrm{PSO}(n,1)$, and in the following we will use the notation of our paper [2]. We will assume $n \geq 2$.

Lemma 1.1. *The group* $\mathrm{PSO}(n,1)$ *acts irreducibly on* $\mathbb{C}^{n+1}$*, that is, there is no proper complex vector subspace of* $\mathbb{C}^{n+1}$ *that is invariant under the action of* $\mathrm{PSO}(n,1)$ *by left matrix multiplication.*

Proof. On the contrary, assume that W is a proper complex vector subspace of $\mathbb{C}^{n+1}$ that is invariant under the action of $\mathrm{PSO}(n,1)$ by left matrix multiplication. As stated in the proof of Theorem 1, there is no proper real vector subspace of $\mathbb{R}^{n+1}$ that is invariant under the action of $\mathrm{PSO}(n,1)$ by left matrix multiplication. This follows from the fact that $\mathrm{PSO}(n,1)$ acts transitively on the sets of k-dimensional time-like, space-like, and light-like vector subspaces of $\mathbb{R}^{n,1}$ for each $k = 1, \ldots, n$.

Let $W_\mathbb{R} = W \cap \mathbb{R}^{n+1}$. Then $\mathbf{i}\, W_\mathbb{R} \subset W$. Now $W_\mathbb{R}$ is invariant under the action of $\mathrm{PSO}(n,1)$. Hence $W_\mathbb{R} = \{0\}$ or $W_\mathbb{R} = \mathbb{R}^{n+1}$, but the latter case implies that $W = \mathbb{C}^{n+1}$, which is not the case. Therefore $W_\mathbb{R} = \{0\}$.

Let $w \in W$. Then we can write $w = u + v\mathbf{i}$, with $u, v \in \mathbb{R}^{n+1}$. Notice that if $u = 0$, then $v = 0$, since if $u = 0$, then $v = -\mathbf{i}w \in W_\mathbb{R} = \{0\}$.

Let U be the set of $u \in \mathbb{R}^{n+1}$ such that $u + v\mathbf{i} \in W$ for some $v \in \mathbb{R}^{n+1}$. Then U is a subspace of $\mathbb{R}^{n+1}$ which is invariant under the action of $\mathrm{PSO}(n,1)$. Therefore $U = \{0\}$ or $U = \mathbb{R}^{n+1}$, but the former case implies that $W = \{0\}$ which is not the case. Therefore $U = \mathbb{R}^{n+1}$.

Suppose $w = u + v\mathbf{i} \in W$ with $u, v \in \mathbb{R}^{n+1}$. We claim that v is uniquely determined by u. Suppose $u + v_j\mathbf{i} \in W$ with $v_j \in \mathbb{R}^{n+1}$ for $j = 1, 2$. The $(v_1 - v_2)\mathbf{i} \in W$, and so $(v_1 - v_2) = 0$. We write $w = u + u'\mathbf{i}$. The map $\kappa : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $\kappa(u) = u'$ is obviously a linear transformation over $\mathbb{R}$.

Now $-\mathbf{i}(u + u'\mathbf{i}) = u' - u\mathbf{i}$, and so $u'' = -u$. Therefore κ is a complex structure on $\mathbb{R}^{n+1}$. Let C be the matrix for κ with respect to the standard basis of $\mathbb{R}^{n+1}$.

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Then $C^2 = -I_{n+1}$. Hence $(\det C)^2 = (-1)^{n+1}$. Therefore n must be odd, and so n is at least 3. Moreover $C^2 = -I_{n+1}$ implies that the eigenvalues of C are $\pm\mathbf{i}$.

Let $u \in \mathbb{R}^{n+1}$, and let $A \in \mathrm{PSO}(n,1)$. Then $A(u + u'\mathbf{i}) = Au + (Au')\mathbf{i}$. Hence $(Au)' = Au'$. Therefore $CA = AC$. Thus C commutes with every element of $\mathrm{PSO}(n,1)$.

Now let A be the block diagonal $(n+1)\times(n+1)$ matrix whose first block is $-I_2$ and whose second block is I_{n-1}. Then $A \in \mathrm{PSO}(n,1)$. Now $CA = AC$ implies that $C = (c_{ij})$ is a block diagonal matrix whose first block is 2×2 and whose second block is $(n-1)\times(n-1)$. In particular $c_{ij} = 0$ for $i = 3,\ldots,n+1$ and $j = 1,2$.

Let B be the block diagonal $(n+1)\times(n+1)$ matrix whose first block is I_1, whose second block is $-I_2$, and whose third block is I_{n-2}. Then $A \in \mathrm{PSO}(n,1)$. Now $CA = AC$ implies that $c_{12} = 0 = c_{21}$. Therefore c_{11} and c_{22} are real eigenvalues of C, which is a contradiction, since the eigenvalues of C are $\pm\mathbf{i}$. □

The gap in the proof of Theorem 1 of [2] is bridged by the following argument. The above lemma implies that $\mathrm{PSO}(F_i)$ acts irreducibly on $\mathbb{C}^{n+1}$ for $i = 1,2$. Therefore by Burnside's theorem [1] (Theorem IX.3 [3]), we have that $\mathrm{Span}_{\mathbb{C}}(\mathrm{PSO}(F_i)) = \mathrm{M}(n+1,\mathbb{C})$ for $i = 1,2$. As the elements of $\mathrm{PSO}(F_i)$ are real matrices, we have that $\mathrm{Span}_{\mathbb{R}}(\mathrm{PSO}(F_i)) = \mathrm{M}(n+1,\mathbb{R})$.

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