A POLYHEDRAL APPROACH TO THE ARITHMETIC AND GEOMETRY OF HYPERBOLIC LINK COMPLEMENTS

RUTH KELLERHALS

Abstract. Motivated by the works of Meyer, Millichap and Trapp [18], Neumann and Reid [20, Section 5] as well as Thurston [23, Chapter 6], we provide an elementary polyhedral approach to study and deduce results about the arithmeticity and commensurability of an infinite family of hyperbolic link complements M_n for $n \geq 3$. The manifold M_n is the complement of $\mathbb{S}^3$ by the $(2n)$-link chain $\mathcal{D}_{2n}$ and has $2n$ cusps.

The hyperbolic structure of M_n stems from an ideal right-angled polyhedron that can be cut into four copies of an ideal right-angled n-gonal antiprism. Each of these polyhedra gives rise to a hyperbolic Coxeter orbifold that is commensurable to a hyperbolic orbifold with a single cusp. Vinberg's arithmeticity criterion and certain cusp density and volume computations allow us to reproduce some of the main results in [20] and [18] about M_n in a comparatively elementary and direct way. This approach works in several other cases of link complements as well.

As a by-product of this polyhedral viewpoint, we give a rigorous proof of Thurston's volume formula for M_n and deduce that, for $n \geq 6$, the volume of M_n is strictly bigger than the volume of the $(2n-1)$-cyclic cover over one component of the Whitehead link. This property, without proof, was indicated to Agol by Ventzke and hinted more concretely by Masai; see [1, 13].

Keywords. Hyperbolic chain link complement, Coxeter orbifold, antiprism, non-arithmeticity, commensurability, cusp density, volume.

1. Introduction

For an integer $n \geq 3$, consider the manifold M_n given by the complement of $\mathbb{S}^3$ by a $(2n)$-link chain $\mathcal{D}_{2n}$ exemplary illustrated in Figure 1. The link complement M_n is a multiply cusped hyperbolic manifold which comes with a decomposition into four isometric copies of an ideal right-angled n-gonal antiprism A_n as described by Thurston [23, Section 6.8].

The manifold M_n is a minimally twisted $(2n)$-chain link complement in the terminology of Agol [1], and M_n can also be interpreted as the complement of an untwisted fully augmented pretzel link $\mathcal{P}_n$, in short an untwisted pretzel FAL; see [18]. It is known that any hyperbolic link $L \subset \mathbb{S}^3$ can be obtained via Dehn surgery on a hyperbolic FAL, which explains the interest

Date: June 12, 2023.

2010 *Mathematics Subject Classification.* 57M25, 51M25, 51F15.

in the family of links $\mathcal{D}_{2n}$ (see [9], and [18] for further references). Another important feature of the manifolds M_n is that they are virtually fibered with a fundamental group that is LERF. These results are due to Chesebro, DeBlois and Wilton [3, Corollary 1.2 and Corollary 1.4].

Regarding the volume of non-compact hyperbolic 3-manifolds, Agol [1] conjectures that for $3 \leq m \leq 10$, the minimally twisted m-chain link complement has smallest volume among all orientable hyperbolic manifolds with exactly m cusps. This conjecture was proven for $m = 4$ by K. Yoshida [26].

A natural step in the study of this and other infinite families of hyperbolic manifolds is to regroup them according to arithmeticity and commensurability. For example, Chesebro and DeBlois [4] constructed a certain infinite family of hyperbolic link complements, and infinite subfamilies of them obtained by mutation. Their respective incommensurability is detected by the Bloch invariant and the cusp parameters as described in [4], and by the set of maximal disjoint horoballs associated to the cusps as completed by H. Yoshida [27].

In the recent work of Meyer, Millichap and Trapp [18], based in parts on the work of [21], a satisfactory answer to this circle of questions has been delivered for the manifolds M_n (and their half-twist partners) by using various methods involving the study of short geodesics and (hidden) symmetries of M_n.

In this work, we exploit the beautiful polyhedral structure as given by the n-gonal antiprism A_n and related Coxeter polyhedra underlying $M_n = \mathbb{S}^3 \setminus \mathcal{D}_{2n}$. Indeed, each polyhedron A_n can be further dissected into isometric copies of another non-compact Coxeter polyhedron R_n whose Coxeter orbifold is commensurable to a *1-cusped* hyperbolic orbifold in an obvious way. Based on this observation, we are able to provide an *alternative* and comparatively *elementary approach* to decide about the arithmeticity and commensurability of the manifolds M_n for all $n \geq 3$. As a by-product, we provide rigorous proofs for Thurston's volume formula for M_n and the fact that the volume of $M_n\,, n \geq 6\,,$ is strictly bigger than the volume of the $(2n-1)$-cyclic cover over one component of the Whitehead link. The latter fact has been stated without proof first by Ventzke and then by Masai in a more concrete way; see [1, 13].

In this context, recall that two hyperbolic orbifolds $O_1 = \mathbb{H}^3/G_1$ and $O_2 = \mathbb{H}^3/G_2$ are *commensurable* if they have a common finite sheeted cover. Equivalently, their fundamental groups, and hence, $G_1, G_2 \subset \operatorname{Isom}\mathbb{H}^3$ are commensurable in the wide sense, that is, there exists an element $\gamma \in \operatorname{Isom}\mathbb{H}^3$ such that $G_1 \cap \gamma G_2 \gamma^{-1}$ has finite index in both G_1 and $\gamma G_2 \gamma^{-1}$. The commensurability property for groups in $\operatorname{Isom}\mathbb{H}^3$ is an equivalence relation preserving characteristics such as discreteness, finite covolume and arithmeticity. As for the latter property, a fundamental result of Margulis (see [17, Theorem 10.3.5], for example) states that a hyperbolic lattice given by a discrete group $G \subset \operatorname{Isom}\mathbb{H}^3$ of finite covolume is non-arithmetic if and only if its commensurator

$$\mathrm{Comm}(G) = \{\, \gamma \in \mathrm{Isom}\,\mathbb{H}^n \mid G \text{ and } \gamma G \gamma^{-1} \text{ are commensurable} \,\} \tag{1.1}$$

is a hyperbolic lattice, and containing G as a subgroup of finite index.

For an algorithmic approach to find the commensurator of a cusped non-arithmetic hyperbolic manifold and to decide about the commensurability of cusped non-arithmetic manifolds, see [10].

Here, we provide new and simple proofs of the following main results in [18] and [20, Sections 5-8].

Theorem A. *Let $n \geq 3$. The manifold $M_n = \mathbb{S}^3 \setminus \mathcal{D}_{2n}$ is arithmetic if and only if $n = 3, 4$.*

Theorem B. *For $m, n \geq 3$ with $m \neq n$, the manifold M_n is incommensurable to M_m.*

Furthermore, we give a detailed proof of Thurston's volume formula for $\mathrm{vol}(M_n)$ as stated in [23, Example 6.8.7] and the complete reasoning in the spirit of Masai's remark [13, Remark 1.1]. More specifically, we will rigorously prove the following result.

Theorem C.

(1) *For $n \geq 3$, the volume of the manifold M_n is given by*
$$\mathrm{vol}(M_n) = 8n\,\{\text{Л}(\frac{\pi}{4} + \frac{\pi}{2n}) + \text{Л}(\frac{\pi}{4} - \frac{\pi}{2n})\}\,,$$
where $\text{Л}(\omega)$ is the Lobachevsky function.

(2) *Let $\widehat{W}_n$ be the $(2n-1)$-cyclic cover over one component of the Whitehead link of volume $\mathrm{vol}(\widehat{W}_n) = 8\,(2n-1)\,\text{Л}(\frac{\pi}{4})$. Then, for $n \geq 6$, $\mathrm{vol}(M_n) > \mathrm{vol}(\widehat{W}_n)$.*

For the proofs of Theorem A and Theorem B, we use the commensurability of the fundamental group $\pi_1(M_n)$ to the reflection group Γ_n associated to the antiprism A_n and to the reflection group Λ_n associated to the Coxeter polyhedron R_n. The polyhedron R_n is combinatorially a triangular prism with only two ideal vertices, and whose Coxeter graph is depicted in Figure 4. For the arithmeticity check, we use Vinberg's criterion in the non-compact case by looking at the cycles of twice the Gram matrix of R_n. The 6×6 Gram matrix of R_n is a symmetric matrix whose two non-zero coefficients above the diagonal are equal to $\cos\frac{\pi}{n}$ and its inverse. In this way, Theorem A is an immediate consequence of Vinberg's criterion; see Section 2.1.

The proof of Theorem B is more involved, and this part represents the main achievement of the paper. First, we observe that it is sufficient to consider the case $n \geq 5$ and non-arithmetic manifolds M_n, only. In fact, it is not difficult to see that Γ_3 is commensurable to the Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$ while the group Γ_4 is commensurable to the Bianchi group $\mathrm{PSL}(2, \mathcal{O}_2)$ whose coefficients belong to the ring of integers $\mathcal{O}_2$ of the number field $\mathbb{Q}(\sqrt{-2})$.

Hence, M_3 and M_4 are incommensurable to each other and to each M_n for $n \geq 5$.

In order to prove the claim of Theorem B for $n \geq 5$, we develop a new approach and study the (maximal) cusp density of a certain 1-cusped non-arithmetic hyperbolic 3-orbifold O_n commensurable to M_n and use the result [21, Proposition 1] that the cusp density of O_n is a commensurability invariant. The orbifold O_n arises as follows. The polyhedron R_n can be dissected by its obvious symmetry plane, also apparent in the Coxeter graph $\Sigma(R_n)$, into two copies of a (non-Coxeter) polyhedron $Q_n = Q(\frac{\pi}{n})$ with a single ideal vertex. The polyhedron Q_n belongs to a 1-parameter family of polyhedra $Q(\alpha)\,, \alpha \in (0, \frac{\pi}{2})\,$, whose volumes can be determined, yielding also the volume of M_n with a (detailed) proof of Thurston's volume formula [23, Example 6.8.7]; see Section 3.1, Proposition 2 and the Corollary of Section 3.2. For the other ingredient of the cusp density of O_n, we determine the volume of the maximal polyhedral cusp $C(\alpha)$ embedded in $Q(\alpha)$ using basic hyperbolic trigonometry, only. In this way, we obtain a closed formula for the polyhedral cusp density $\delta(\alpha) = \operatorname{vol} C(\alpha)/\operatorname{vol} Q(\alpha)$ of $Q(\alpha)$ and hence for the cusp density of the orbifold O_n; see Theorem 2. In Section 3.3, we show by means of Schläfli's volume differential formula that the cusp density of O_n is strictly monotone with respect to n which finishes the proof of Theorem B. At the end, in Section 3.4, we again use Schläfli's differential formula for the 1-parameter family $R(\frac{\pi}{x})\,, x \in [6, \infty)\,$, in order to show that the function $h(x) = x \operatorname{vol}(R(\frac{\pi}{x})) - (2x-1)\,Л(\frac{\pi}{4})$ is strictly monotonically increasing with $h(6) > 0$. This result together with the Corollary finishes the proof of Theorem C.

Acknowledgements. The author would like to thank the referee for the comments. She is indebted to Hidetoshi Masai for helpful explanations in connection with Theorem C. She is grateful to William Schaller for the support with the drawings.

2. Polyhedral models for hyperbolic $(2n)$-chain link complements

Let $n \geq 3$, and denote by $\mathcal{D}_{2n}$ a $(2n)$-link chain as depicted in Figure 1.

Following [23, Section 6.8] and [22], if each component of $\mathcal{D}_{2n}$ is spanned by a disk in a natural way, the complement of the complex is an open solid torus. This torus with its cell decomposition yields the manifold $M_n = \mathbb{S}^3 \setminus \mathcal{D}_{2n}$ by deleting its vertices and by identifying appropriately faces of the tiling of the torus boundary by rectangles.

The manifold M_n is homeomorphic to the complement $\mathbb{S}^3 \setminus \mathcal{P}_n$ with respect to a pretzel FAL $\mathcal{P}_n$ with n knot circles and n untwisted crossing circles. Indeed, the links $\mathcal{D}_{2n}$ and $\mathcal{P}_n$ have equivalent diagrams.

For a description of $\mathcal{P}_n$ and its half-twist partners, see for instance [18] and Figures 1 and 11 therein.

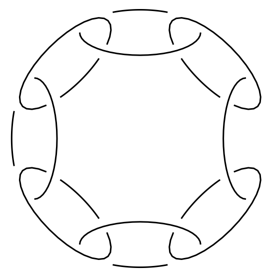


FIGURE 1. The diagram of a 8-link chain $\mathcal{D}_8$

The manifold M_n carries a hyperbolic structure induced by an ideal right-angled polyhedron $P = P_n$ that arises by suitably glueing four copies of an ideal right-angled n-gonal antiprism A_n (or *drum* in the terminology of Thurston [23, Section 6.8]). The polyhedron A_n has $2n+2$ facets, $4n$ edges and $2n$ vertices on the ideal boundary $\partial\mathbb{H}^3$. More precisely, the antiprism A_n consists of two disjoint copies π and π' of an ideal regular n-gon that are connected by an alternating band of $2n$ triangles. In particular, the two polygons π, π' are twisted by an angle of $\frac{\pi}{n}$, and their centres c, c' are the endpoints of the common perpendicular of π and π'.

In [18], the polyhedral decomposition of a pretzel FAL complement $\mathbb{S}^3 \setminus \mathcal{P}_n$ into two isometric copies $P_\pm$ of the ideal right-angled polyhedron P is used to prove the above results, one of them arising by reflection of one antiprism A_n in the facet plane carrying π, say, and glueing both copies together. In fact, the resulting decomposition of the polyhedron P according to $P_- \cup P_+$ allows one to study a checkerboard coloring of the facets of each of the polyhedra $P_\pm$ having different implications for M_n (for further information, see [16, Appendix], [9] and [3]).

As a consequence, the volume M_n is given by $\mathrm{vol}(M_n) = 4\,\mathrm{vol}(A_n) = 2\,\mathrm{vol}(P_+)$, and the (non-)arithmeticity of (the fundamental group of) M_n and $\mathbb{S}^3 \setminus \mathcal{P}_n$ follows from the corresponding property of the groups generated by the reflections in the facet planes of A_n and in the facet planes of P_+, respectively.

In the sequel, we focus on the antiprism representation for M_n in order to decide about their arithmeticity and commensurability for all distinct $n \geq 3$.

2.1. **The antiprism A_n, hyperbolic Coxeter groups and the arithmeticity of M_n.** The antiprism A_n is an ideal right-angled polyhedron and hence a *Coxeter polyhedron*, that is, all of its dihedral angles are submultiples of π (see Figure 2 for $n = 8$)

In this way, the reflections in the facet planes of A_n generate a discrete group $\Gamma_n \subset \mathrm{Isom}\,\mathbb{H}^3$ which is the hyperbolic realisation of a Coxeter group by Tits' representation. By construction, the fundamental group of M_n is a subgroup of index 4 in Γ_n so that the (non-)arithmeticity of $\pi_1(M_n)$ follows directly from the corresponding one of Γ_n.

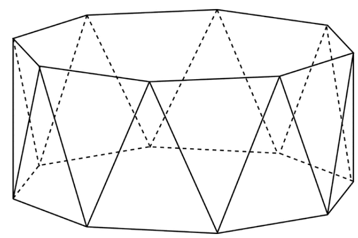


FIGURE 2. The antiprism A_8

In view of the symmetry properties of A_n, we decompose the polyhedron A_n further into $2n$ isometric pieces, each again a non-compact but non-ideal Coxeter polyhedron of simple combinatorial type. The decomposition goes as follows. Pick the basis polygon π which is an ideal regular n-gon. Denote by c its centre and decompose π barycentrically into $2n$ right-angled triangles, each with angle $\frac{\pi}{n}$ at c and one ideal vertex v. Denote by $\Delta = [c,m,v]$ one of these triangles where m is the midpoint of an edge of π containing v. Perform the identical decomposition for the opposite polygon π' of A_n and choose the triangle $\Delta' = [c',m',v']$ such that v and v' form an edge and so that m,v and v' lie in the same triangle of A_n. The convex hull of the six vertices of Δ and Δ' is a non-compact Coxeter polyhedron $R_n = R(\frac{\pi}{n})$ all of whose dihedral angles are equal to $\frac{\pi}{2}$ apart form the angle $\frac{\pi}{n}$ at the ridge segment $[c,c']$. Only the vertices v and v' of R_n are ideal (and non-simple). In the schematic picture as given in Figure 3, they are marked bold.

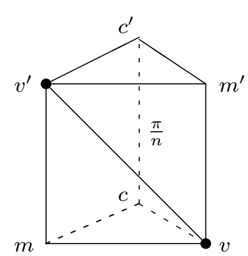


FIGURE 3. The Coxeter polyhedron $R_n \subset \mathbb{H}^3$

Our next aim is to treat the arithmeticity problem for M_n for $n \geq 3$. To this end, we study the Gram matrix of R_n in order to apply Vinberg's arithmeticity criterion stated below.

Consider an arbitrary (convex) polyhedron $P \subset \mathbb{H}^3$ with $k \geq 4$ bounding planes $H_1, \ldots, H_k$. The Gram matrix $G(P) = (g_{ij})$ is a real symmetric $k \times k$ matrix of signature $(3,1)$ with $g_{ii} = 1$ whose coefficients g_{ij} for $i \neq j$ encode the intersection behavior of the facets $H_i \cap P$, $1 \leq i \leq k$, of P as follows.

$$-g_{ij} = \begin{cases} \cos\alpha_{ij} & \text{if } \measuredangle(H_i,H_j)=\alpha_{ij}\,, \\ 1 & \text{if } H_i,H_j \text{ are parallel}\,, \\ \cosh l_{ij} & \text{if } d_{\mathbb{H}}(H_i,H_j)=l_{ij}>0\,. \end{cases} \tag{2.1}$$

Example 1. *Consider the* 1*-parameter family* $R(\alpha)$, $\alpha\in[0,\frac{\pi}{2})$, *of polyhedra of the same combinatorial type as and comprising* $R_n=R(\frac{\pi}{n})$ *for each* $n\geq 3$. *The Gram matrix of* $R(\alpha)$ *is given by*

$$G(R(\alpha)) = \begin{pmatrix} 1 & -\cos\alpha & 0 & 0 & 0 & -1 \\ -\cos\alpha & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -\cosh l_\alpha & 0 \\ 0 & 0 & 0 & -\cosh l_\alpha & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

where the length l_α *of the edge* $[c,c']$ *can be computed by exploiting the fact that* $\det(G(R(\alpha)))=0$. *It follows that*

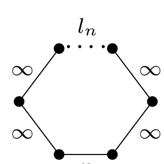
$$\cosh l_\alpha = \frac{1}{\cos\alpha}\;. \tag{2.2}$$

A convenient and at times more efficient description of the polyhedron R_n is given by its Coxeter graph, or more generally, by the Vinberg graph when considering the comprehensive family $R(\alpha)$ as given in Example 1.

For a Coxeter polyhedron $P\subset\mathbb{H}^3$ and its group $\Gamma\subset\operatorname{Isom}\mathbb{H}^3$ generated by the reflections $s_1,\ldots,s_k$ in the facet planes $H_1,\ldots,H_k$ of P, the *Coxeter graph* $\Sigma(\Gamma)=\Sigma(P)$ of Γ (and of P) is constructed as follows. Each node i of $\Sigma(\Gamma)$ corresponds to a generator s_i (and to the plane H_i). Two nodes i,j are not joined by an edge if H_i and H_j are orthogonal. They are joined by a simple edge if the corresponding planes intersect under the angle $\frac{\pi}{3}$. The edge carries the weight $m_{ij}\geq 4$, ∞, or is replaced by a dotted edge (often with weight l_{ij}), if the hyperplanes H_i,H_j intersect under the angle $\frac{\pi}{m_{ij}}$, are parallel, or at the positive hyperbolic distance l_{ij}, respectively.

The *Vinberg graph* of an arbitrary polyhedron with dihedral angles α_{ij} between intersecting planes H_i,H_j is formed similarly to the case of the Coxeter graph, however, by replacing the edge weight $m_{ij}>3$ by the real value α_{ij}.

Example 2. *The Coxeter graph of* R_n, $n\geq 3$, *is given by Figure 4.*



where $\quad\cosh l_n = \frac{1}{\cos\frac{\pi}{n}}$.

FIGURE 4. The Coxeter graph of $R_n\subset\mathbb{H}^3$

For hyperbolic Coxeter groups of finite covolume, Vinberg [24, Theorem 2], [25, pp. 226-227], proved an efficient criterion for arithmeticity. In the case of a Coxeter group $\Gamma \subset \operatorname{Isom}\mathbb{H}^3$ with a *non-compact* Coxeter polyhedron $P \subset \mathbb{H}^3$ of finite volume, Vinberg's criterion can be stated in terms of the Gram matrix $G(P)$ as follows. Consider twice the Gram matrix of P and write $2G(P) =: (h_{ij})$. Form the coefficient cycles (of length l) of the form

$$h_{i_1 i_2 \dots i_l} := h_{i_1 i_2} h_{i_2 i_3} \cdot \ldots \cdot h_{i_{l-1} i_l} h_{i_l i_1} \,, \tag{2.3}$$

with distinct indices i_j in $2G(P)$. The field $K(\Gamma) := \mathbb{Q}(\{h_{i_1 i_2 \dots i_l}\})$ generated by all cycles of $2G(P)$ is called the *Vinberg field* of Γ. It is the smallest field of definition for Γ, and it is an algebraic number field coinciding with the adjoint trace field of Γ. As a consequence, the Vinberg field $K(\Gamma)$ is a commensurability invariant for Γ. For more details, see [7, Section 4] and [8, Section 3].

Vinberg's criterion. *The Coxeter group $\Gamma \subset \operatorname{Isom}\mathbb{H}^3$ as given above (and its associated Coxeter orbifold $\mathbb{H}^3/\Gamma$) is arithmetic with field of definition $\mathbb{Q}$ if and only if all the cycles of $2G(P)$ are rational integers.*

Example 3. *The triangular antiprism A_3 is an ideal regular octahedron with Schläfli symbol $\{3,4\}$. It can be barycentrically decomposed into 48 copies of a Coxeter tetrahedron with Coxeter graph* $\bullet\!\!-\!\!\!-\!\!\bullet\overset{4}{-\!\!\!-}\bullet\overset{4}{-\!\!\!-}\bullet$ *and Gram matrix*

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}.$$

Vinberg's criterion implies the well known fact that the associated Coxeter group, denoted by $[3,4,4]$, is arithmetic (see [12], for example). Since the Coxeter group Γ_3 related to A_3 is of finite index in the group $[3,4,4]$, Γ_3 is arithmetic as well.

Example 4. *The square antiprism A_4 is an ideal right-angled polyhedron that is decomposable into 8 Coxeter polyhedra R_4 with Coxeter graph given by Figure 4. Since $\cosh l_4 = \sqrt{2}$ by (2.2), it is easy to see that all coefficient cycles of $2G(R_4)$ are in $\mathbb{Z}$. By Vinberg's criterion, the Coxeter group generated by the reflections in the facets of R_4 and the group Γ_4 commensurable to it are arithmetic.*

Remark 1. *By Example 3 and Example 4, the groups Γ_3 and Γ_4 are arithmetic groups. Since both groups are commensurable to non-cocompact arithmetic Kleinian groups (as discrete subgroups of $\mathrm{PSL}(2;\mathbb{C})$), they are commensurable to certain Bianchi groups by* [17, Theorem 8.2.3]. *Recall that a Bianchi group is of the from $\mathrm{PSL}(2;\mathcal{O}_d)$ where $\mathcal{O}_d$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ for some positive square-free integer d.*

More precisely, the group $[3,4,4]$ *is commensurable to the Picard group* $\mathrm{PSL}(2;\mathcal{O}_1)$*, while the group* Γ_4 *is commensurable to the group* $\mathrm{PSL}(2;\mathcal{O}_2)$*; see* [23, Example 6.8.7] *and* [17, Section 9]*.*

As a consequence, the groups Γ_3 *and* Γ_4 *are incommensurable.*

We are now able to provide a new proof about the arithmeticity of the manifolds $M_n\,, n\geq 3\,,$ in a comparatively elementary way (see Theorem A in the Introduction). For another and more involved approach using short geodesics and invariant trace field calculations, see [18, Sections 3–5].

Theorem 1. *The manifold* $M_n=\mathbb{S}^3\setminus\mathcal{D}_{2n}$ *is arithmetic if and only if* $n=3,4$*.*

Proof. In view of Example 3 and Example 4, we have to show that the manifold M_n is *non-arithmetic* for $n\geq 5$. Since arithmeticity is preserved with respect to commensurability, it suffices to prove non-arithmeticity for a group commensurable to $\pi_1(M_n)$, and we do so by considering the Coxeter group Γ_n associated to the right-angled antiprism A_n. Furthermore, Γ_n is a subgroup of index n in, and hence, commensurable to the Coxeter group Λ_n generated by the reflections in the facet planes of R_n. The Coxeter graph of Λ_n (and R_n) is given by Figure 4, where

$$4\cosh^2 l_n=\frac{4}{\cos^2\frac{\pi}{n}}$$

according to (2.2). We show that Λ_n is non-arithmetic for $n\geq 5$ by using Vinberg's criterion, that is, by showing that *not all* cycles of the matrix $2\,G(R_n)$ are rational integers. It is easy to see that the non-trivial cycles in $2\,G(R_n)$ are of the form $4, 4\cos^2\frac{\pi}{n}$ and $4\cosh^2 l_n$. Since $\cos^2\frac{\pi}{n}$ is rational only for $n=1,2,3,4$ and 6, with $4\cos^2\frac{\pi}{6}=3\,,$ we deduce that for all $n\geq 5$, the cycle $4\cosh^2 l_n$ as specified above is *not* in $\mathbb{Z}$. $\square$

Remark 2. *The proof of Theorem 1 shows that the field of definition of the fundamental group of* $M_n\,, n\geq 5\,,$ *is given by the Vinberg field* $K(\Gamma_n)$ *which is equal to*

$$K(\pi_1(M_n))=K(\Gamma_n)=\mathbb{Q}(\cos^2\frac{\pi}{n})=\mathbb{Q}(\cos\frac{2\pi}{n})=:K_n\;. \tag{2.4}$$

The extension degree $[K_n:\mathbb{Q}]$ *of* K_n *is given by* $\varphi(n)/2$ *where* $\varphi(n)$ *denotes the Euler totient function which counts the positive integers smaller than or equal to* n *that are relatively prime to* n*. Since the function* $\varphi(n)$ *is not injective, we cannot deduce from* (2.4) *that the manifolds* M_m *and* M_n *are incommensurable for all distinct* $m,n\geq 5$ *since the Vinberg fields* K_m *and* K_n *(as commensurability invariants) may coincide. In Section 3, we shall prove that* M_m *and* M_n *are incommensurable for all distinct* $m,n\geq 3$ *by means of cusp density computations.*

Remark 3. *Consider the polyhedral description of the untwisted pretzel FAL complements* $\mathbb{S}^3\setminus\mathcal{P}_n$ *as described above. It follows from Theorem 1 together* *with* [18, Proposition 3.2] *that* $\mathbb{S}^3 \setminus \mathcal{P}_n$ *(and its half-twist partners) are non-arithmetic for* $n \geq 5$*.*

3. Commensurability of the manifolds M_n

Our next goal is to decide about the commensurability of the manifolds M_n, $n \geq 3$. We can restrict the investigation to the case $n \geq 5$. In fact, the arithmetic fundamental groups of M_3 and M_4 are incommensurable, and by the non-arithmeticity of M_n for $n \geq 5$, it follows that M_3 and M_4 are incommensurable to M_n as well.

Let $n \geq 5$, and consider the hyperbolic Coxeter groups Γ_n and Λ_n, both commensurable to $\pi_1(M_n)$. The groups Γ_n and Λ_n are generated by the reflections in the facet planes of the ideal right-angled antiprism A_n and the non-compact polyhedron $R_n = R(\frac{\pi}{n})$ with two ideal vertices, respectively.

The Coxeter graph $\Sigma(\Lambda_n)$ of the group Λ_n and its Coxeter polyhedron R_n are depicted in Figure 4. Obviously, the graph $\Sigma(\Lambda_n)$ has a vertical symmetry implying that the polyhedron R_n has a symmetry plane decomposing R_n into two isometric copies of a polyhedron $Q_n = Q(\frac{\pi}{n})$, each having exactly one ideal vertex (for a general description, see Section 3.1). Denote by $\tau \in \operatorname{Isom}\mathbb{H}^3$ the half-turn which identifies the two copies of Q_n. Then, the group extension $\Lambda_n^* := \Lambda_n \star \langle\tau\rangle$ is a discrete group, containing Λ_n with index two, and having Q_n as a fundamental polyhedron. Since Q_n has only one ideal vertex, the orbifold $\mathbb{H}^3/\Lambda_n^*$ is a *1-cusped* hyperbolic orbifold with a finite cover given by M_n.

In this context, consider an arbitrary non-compact orbifold $O = \mathbb{H}^3/\Gamma$ of finite volume where $\Gamma \subset \operatorname{Isom}\mathbb{H}^3$ is a discrete group with non-compact fundamental polyhedron $P \subset \mathbb{H}^3$, say. A *cusp* $C \subset O$ is a subset of O that lifts to a set of horoballs with disjoint interiors in $\mathbb{H}^3$. The cusp C corresponds to an ideal vertex $v \in P$ whose stabiliser $\Gamma_v < \Gamma$ is non-trivial. In this way, we can write $C = B_v/\Gamma_v$ where $B_v \subset \mathbb{H}^3$ is a precisely invariant horoball internally tangent to $\partial\mathbb{H}^3$ at v. The group Γ_v is a crystallographic group acting cocompactly by Euclidean isometries on the horosphere $H_v = \partial B_v$.

Assume that O is *1-cusped*, that is, O has precisely one cusp C, and that C is *maximal* in O, that is, C is tangent to itself at one or more points. The *cusp density* $\delta(O) = \delta(C)$ is defined by the volume quotient

$$\delta(O) = \delta(C) = \frac{\operatorname{vol}(C)}{\operatorname{vol}(O)} \,. \tag{3.1}$$

In the sequel, we usually work in the upper half space model

$$\mathcal{U}^3 = \left(\mathbb{E}^3_+ \,,\, ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}\right) \tag{3.2}$$

for $\mathbb{H}^3$ and use the transitivity properties of the isometry group of $\mathcal{U}^3$. In particular, we can suppose that the (maximal, single) cusp C of O is of the form $C = B_\infty/\Gamma_\infty$ with boundary horosphere $H_\infty = \partial B_\infty$ at height $\rho > 0$

from the ground space $\{x_3 = 0\}$. More specifically, for $\rho = 1$, and by (3.2), the hyperbolic length of a horocyclic segment on H_∞ coincides with the standard Euclidean length, and the induced area of a polygon on H_∞ is given by its Euclidean area.

In general, the numerator $\mathrm{vol}(C)$ of (3.1) can be computed by means of the induced area of a fundamental polygon $P_\infty \subset H_\infty$ for the action of Γ_∞ according to the classical formula

$$\mathrm{vol}(C) = \frac{\mathrm{vol}(P_\infty)}{2}\,. \tag{3.3}$$

Now, we can cite the following result in the context of commensurability of non-arithmetic hyperbolic orbifolds which will be of importance for what follows.

Proposition 1 ([21, Proposition 1], [10, Section 2])**.** *The cusp density is a commensurability invariant for 1-cusped non-arithmetic hyperbolic orbifolds.*

In the sequel, we shall study the cusp density of $\mathbb{H}^3/\Lambda_n^*$ from a polyhedral point of view. Our strategy is to determine the volume of the polyhedral half $Q(\alpha) \subset \mathbb{H}^3$ of $R(\alpha)$ and the volume of a *maximal embedded polyhedral cusp neighborhood* $C(\alpha)$ of the ideal vertex of $Q(\alpha)$ in terms of the angle parameter α; see Section 3.1 and Section 3.2. As a result, we shall get an explicit formula for the *polyhedral cusp density* of $Q(\frac{\pi}{n})$ and for the cusp density of $\mathbb{H}^3/\Lambda_n^*$ (see Theorem 2). Then, we prove strict monotonicity of the density function for $Q(\alpha)$ which, by Proposition 1, implies that the manifolds M_m and M_n are pairwise incommensurable for distinct m, n.

3.1. **The building block $Q(\alpha)$.** Consider the polyhedron $R(\alpha) \subset \mathbb{H}^3$ whose Gram matrix is given in Example 1 and whose Vinberg graph arises from the Coxeter graph for R_n depicted in Figure 4. The polyhedron $R(\alpha)$ is related to an *orthoscheme* $\widehat{R}(\alpha)$ defined by the Vinberg graph $\bullet\overset{\infty}{\text{—}}\bullet\overset{\alpha}{\text{—}}\bullet\overset{\infty}{\text{—}}\bullet$ having two ultra-ideal vertices p_0, p_3 (characterised by hyperbolic triangles with identical Vinberg graph $\bullet\overset{\alpha}{\text{—}}\bullet\overset{\infty}{\text{—}}\bullet$), both cut off by their corresponding polar planes H_0, H_3. In this way, $R(\alpha)$ is a doubly truncated orthoscheme. Furthermore, the truncating planes H_0, H_3 have a common perpendicular of length l_α and touch the facets opposite to p_0, p_3 at the ideal vertices $p_1 := v$ and $p_2 := v'$ of $R(\alpha)$. Observe that the vertices $p_0, \ldots, p_3$ form an orthogonal edge path in $\widehat{R}(\alpha)$. Denote by $F_i\,,\, 0 \leq i \leq 3\,$, the facets opposite to p_i in $\widehat{R}(\alpha)$. By means of the bisector H of H_0, H_3 we divide $R(\alpha)$ into two copies of a new polyhedron with a single vertex, the *building block* $Q(\alpha)$. We consider $Q(\alpha)$ as being the part of $R(\alpha)$ with vertex p_1. The Vinberg graph of $Q(\alpha)$ is depicted in Figure 5. The plane H yields a face F of $Q(\alpha)$ that is at distance $\frac{l_\alpha}{2}$ from the plane H_0, and F intersects the face F_3 at the dihedral angle β_α, the face F_0 at the complement $\pi - \beta_\alpha$ and all other ones orthogonally.

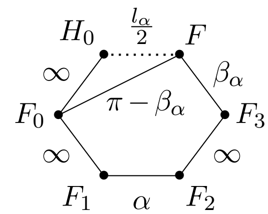


FIGURE 5. The Vinberg graph of the building block $Q(\alpha) \subset \mathbb{H}^3$

Here, the angle β_α can easily be computed by using (2.2) as follows.

$$\cos\beta_\alpha = \frac{1}{\sqrt{2\,(1+\cosh l_\alpha)}} = \sqrt{\frac{\cos\alpha}{2\,(1+\cos\alpha)}}\,. \tag{3.4}$$

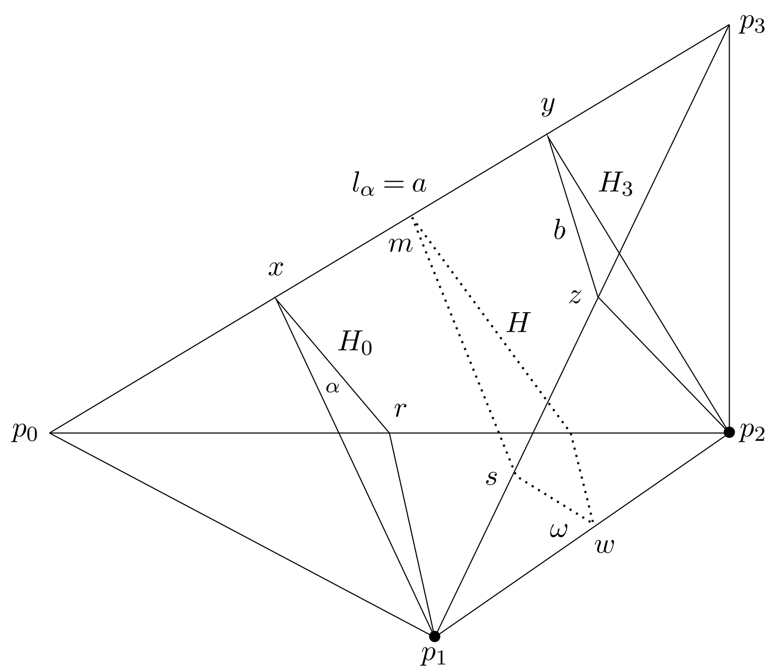


FIGURE 6. Synthetic view of the building block $Q(\alpha) \subset \mathbb{H}^3$

3.2. **The cusp density of $Q(\alpha)$.** Our aim is to derive an explicit formula for the polyhedral cusp density of the building block $Q(\alpha) \subset \mathbb{H}^3$, $\alpha \in (0, \frac{\pi}{3}]$, defined by the volume quotient $\delta(\alpha)$ of a maximal embedded polyhedral cusp neighbood $C(\alpha)$ of the ideal vertex of $Q(\alpha)$ by the volume of $Q(\alpha)$. More precisely, the maximal embedded cusp neighborhood $C(\alpha)$ of the ideal vertex p_1 of $Q(\alpha)$ is the horoball cone with apex p_1 whose horospherical boundary is tangent to the closest of the two facets of $Q(\alpha)$ not incident to

p_1, see Figure 6. Again, we normalise the setting so that in the upper half space $\mathcal{U}^3$, the apex p_1 is identified with ∞ and the horospherical boundary of $C(\alpha)$ is a horizontal plane at positive distance from $\{x_3 = 0\}$.
In the special case $\alpha = \frac{\pi}{n}$, the set $C(\frac{\pi}{n})$ covers the maximal cusp in $\mathbb{H}^3/\Lambda_n^*$.

We start with the denominator $\mathrm{vol}(Q(\alpha)) = \frac{1}{2}\,\mathrm{vol}(R(\alpha))$ of $\delta(\alpha)$. As mentioned in Section 3.1, $R(\alpha)$ is a doubly truncated orthoscheme whose truncating polar hyperplanes H_0, H_3 associated to the vertices p_0, p_3 are at distance l_α but touch their opposite facets F_0, F_3 at the ideal vertices p_1, p_2, respectively. In particular, by [14, Theorem II], we dispose of an explicit volume formula in terms of the Lobachevsky function $Л(\omega)$ and the additional angle parameter $\theta = \theta(\alpha) \in [0, \frac{\pi}{2})$ given by

$$\tan\theta = \cos\alpha \tag{3.5}$$

as follows.

$$\mathrm{vol}(R(\alpha)) = \frac{1}{4}\left\{Л(\frac{\pi}{2}+\alpha-\theta) + Л(\frac{\pi}{2}-\alpha-\theta) + 4\,Л(\theta) + 2\,Л(\frac{\pi}{2}-\theta)\right\}. \tag{3.6}$$

The Lobachevsky function is given by $Л(\omega) = -\int_0^\omega \log|2\ \sin t|\,\mathrm{d}t$, and $Л(\omega)$ is odd, π-periodic and satisfies the distribution relation

$$\frac{1}{k}\,Л(kx) = \sum_{r=0}^{k-1} Л\left(x + \frac{r\pi}{k}\right)\ ,\ k \in \mathbb{N}\,. \tag{3.7}$$

As an example, Catalan's constant $2\,Л(\frac{\pi}{4}) \approx 0.91596$ and the (maximum) value $Л(\frac{\pi}{6})$ can be expressed according to

$$\frac{4}{3}\,Л(\frac{\pi}{4}) = Л(\frac{\pi}{12}) + Л(\frac{5\pi}{12}) \quad , \quad Л(\frac{\pi}{6}) = \frac{3}{2}Л(\frac{\pi}{3}) \approx 0.50747\,. \tag{3.8}$$

For computations, the series representation

$$Л(\omega) = \omega\left(1 - \log|2\omega| + \sum \frac{B_n\,(2\,\omega)^{2n}}{2n\,(2n+1)!}\right) \tag{3.9}$$

with Bernoulli coefficients $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}, \ldots$ converges rapidly for $|\omega| \le \pi$ (see [19, Appendix]).

The formula (3.6) is based on Schläfli's differential formula for the volume $v = v(\alpha) := \mathrm{vol}(R(\alpha))$ in terms of the angle α, and the differential can be expressed according to

$$\frac{d}{d\alpha}\,v = -\frac{1}{2}\,l_\alpha = -\frac{1}{2}\,\mathrm{arcosh}\left(\frac{1}{\cos\alpha}\right)\,. \tag{3.10}$$

As a consequence, the volume $v(\alpha)$ is a strictly decreasing function with respect to α.

The volume expression (3.6) for $R(\alpha)$ can be simplified *without* the use of the additional parameter θ as follows.

Proposition 2. *For* $\alpha \in [0, \frac{\pi}{2})$,

$$\mathrm{vol}(R(\alpha)) = Л(\frac{\pi}{4}+\frac{\alpha}{2}) + Л(\frac{\pi}{4}-\frac{\alpha}{2})\,. \tag{3.11}$$

Proof. Associate to $R(\alpha)$ the polyhedral subset $\widehat{S}(\alpha)$ of the (infinite volume) orthoscheme $\widehat{R}(\alpha)$ with vertices $p_0,\dots,p_3$ by omitting the polar plane H_3 associated to the ultra-ideal vertex p_3. In particular, by cutting $\widehat{S}(\alpha)$ by means of the plane H_3, we get back $R(\alpha)$. Let $r=[p_0,p_2]\cap H_0$ denote the vertex in the facet plane H_0 on the geodesic defined by p_0 and p_2 in $\widehat{R}(\alpha)$. Then, the polyhedron $\widehat{S}(\alpha)$ can be cut into two (infinite volume) orthoschemes $\widehat{R}_1$ and $\widehat{R}_2$ by means of the plane defined by the vertices p_1, r, and p_3 (see also [5, Theorem 2]).

Suppose for a moment that the vertices p_2 and p_3 of $\widehat{S}(\alpha)$ are *ordinary* points of $\mathbb{H}^3$. Then, $\widehat{S}(\alpha)$ is a finite volume pyramid with ideal apex p_1 over a quadrilateral face, depending on further angular parameters. Now, for an arbitrary hyperbolic pyramid $P\subset\mathbb{H}^3$ with ideal apex q over an n-gon π with vertices $a_1,\dots,a_n$, Vinberg [24, pp. 129–130] obtained a closed formula in terms of the dihedral angles (up to minor sign errors). In the particular case of a pyramid $P=P(\alpha_1,\dots,\alpha_4)$ whose apex q at infinity is the intersection of 4 edges with right (interior) dihedral angles (an example is $\widehat{S}(\alpha)$), Vinberg's formula can be stated as follows in terms of the dihedral angles $\alpha_1,\dots,\alpha_4$ at the edges of the quadrilateral π (circularly enumerated with indices modulo 4); see also [11, (2.12)].

$$
\begin{aligned}
2\,\mathrm{vol}(P)=\sum_{k=1}^{4}\,\{&\text{Л}((\frac{\pi}{2}+\alpha_k+\alpha_{k+1})/2)+\text{Л}((\frac{\pi}{2}+\alpha_k-\alpha_{k+1})/2)\\
&+\text{Л}((\frac{\pi}{2}-\alpha_k+\alpha_{k+1})/2)+\text{Л}((\frac{\pi}{2}-\alpha_k-\alpha_{k+1})/2)\}\;. \qquad (3.12)
\end{aligned}
$$

Furthermore, and in a similar way as above, $\widehat{S}(\alpha)$ is cut into the two ordinary (finite volume) orthoschemes $\widehat{R}_1$ and $\widehat{R}_2$ so that their volumes add up to the one of $\widehat{S}(\alpha)$.

Next, suppose that the vertex p_3 of $\widehat{S}(\alpha)$ is ultraideal and cut off by its polar plane H_3 so that $d_{\mathbb{H}}(H_3,F_3)\geq 0$. By [14, Theorem II], the analytical expressions of $\mathrm{vol}(\widehat{R}_i)\,,\,i=1,2\,,$ and hence of their sum $\mathrm{vol}(\widehat{S}(\alpha))$ in terms of the dihedral angles and Lobachevsky's function remain unchanged under this truncation process. Moreover, in the limiting case $d_{\mathbb{H}}(H_3,F_3)=0$ where p_2 becomes an ideal vertex of $\widehat{R}(\alpha)$, the polyhedra $\widehat{S}(\alpha)$ and $R(\alpha)$ coincide.

As a consequence, the volume of $R(\alpha)$ equals the volume of the polarly truncated square pyramid $\widehat{S}(\alpha)$ with angles $\alpha_1=\alpha$, $\alpha_2=0$ and $\alpha_3=\alpha_4=\frac{\pi}{2}$. By (3.12) and (3.7), it follows that

$$
\mathrm{vol}(R(\alpha))=\text{Л}(\frac{\pi}{4}+\frac{\alpha}{2})+\text{Л}(\frac{\pi}{4}-\frac{\alpha}{2})\;.
$$

□

By Proposition 2, a formula for the volume of the $(2n)$-link chain complement $\mathbb{S}^3\setminus\mathcal{D}_{2n}$ can now be deduced, and it agrees with Thurston's formula

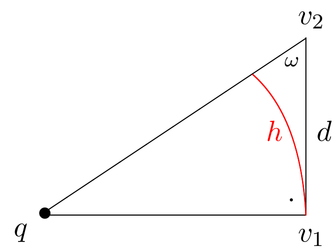


FIGURE 7. A horoarc in the right-angled triangle $T = [q, v_1, v_2]$ with ideal vertex q

presented without proof in [23, Example 6.8.7]. The following result corresponds to part (1) of Theorem C (the part (2) will be proved in Section 3.4).

Corollary. *The volume of the $(2n)$-link chain complement $\mathbb{S}^3 \setminus \mathcal{D}_{2n}$ equals*

$$\mathrm{vol}(M_n) = 4\,\mathrm{vol}(A_n) = 8n\,\mathrm{vol}(R(\frac{\pi}{n})) = 8n\,\{Л(\frac{\pi}{4}+\frac{\pi}{2n}) + Л(\frac{\pi}{4}-\frac{\pi}{2n})\}\ ,\ n \geq 3\ .$$

Remark 4. *The comparison of formula (3.6) with the one (3.11) of Proposition 2 yields a functional equation for the inscrutable Lobachevsky function $Л(\omega)$ in a geometric way. For $\alpha, \theta \in [0, \frac{\pi}{2})$ connected by $\tan\theta = \cos\alpha$,*

$$4\,\{Л(\frac{\pi}{4}+\frac{\alpha}{2}) + Л(\frac{\pi}{4}-\frac{\alpha}{2})\} =$$
$$Л(\frac{\pi}{2}+\alpha-\theta) + Л(\frac{\pi}{2}-\alpha-\theta) + 4\,Л(\theta) + 2\,Л(\frac{\pi}{2}-\theta)\ .$$

Next, we determine the numerator of the cusp density function $\delta(\alpha)$. By viewing the maximal cusp $C(\alpha) \subset Q(\alpha)$ in $\mathcal{U}^3$ and identifying its centre given by the 4-valent ideal vertex p_1 with ∞, $C(\alpha)$ is a cone over a right-angled quadrilateral with induced edge lengths, h, k, say, along the horocycles according to (3.2). For the volume, one has

$$\mathrm{vol}(C(\alpha)) = \frac{hk}{2}\ . \tag{3.13}$$

The following classical results about horocycle geometry will be useful in order to determine h, k and hence $\mathrm{vol}(C(\alpha))$. Consider first a hyperbolic triangle T with one ideal vertex q, a right angle at the vertex v_1 and the angle ω at the vertex v_2. Let $d = d_{\mathbb{H}}(v_1, v_2)$, and consider the horoarc segment in T, based at q and touching v_1, of hyperbolic length h. The situation is depicted in Figure 7 in a synthetic way.

Lemma 1 ([6, Section 4])**.** *Denote by h the hyperbolic length along the horoarc based at the ideal vertex q and touching the vertex v_1 in the right-angled triangle $T = [q, v_1, v_2]$. Let ω be the angle of T at v_2 and $d = d_{\mathbb{H}}(v_1, v_2)$ according to Figure 7. Then,*

$$h = \cos\omega = \tanh d\ .$$

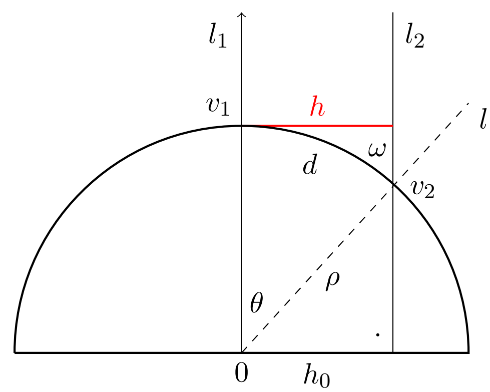


FIGURE 8. A horoarc in the right-angled triangle $T = [\infty, v_1, v_2]$ with ideal vertex ∞

We provide a short proof Lemma 1.

Proof. Consider the triangle T in the upper half plane $\mathcal{U}^2 \subset \mathbb{E}^2_+$ so that its ideal vertex q is identified with ∞ and the vertex $v_1 = (0, \rho) \in \mathcal{U}^2$ lies at height $\rho > 0$ on the geodesic line l_1 passing through 0 and ∞. Then, the vertex v_2 lies on the half-circle centred at 0 and of radius ρ, and its hyperbolic distance d to v_1 is given by the formula (see [2, (7.20.3)])

$$\tanh d = \sin\theta\,, \tag{3.14}$$

where θ is the angle formed by the line l_1 and the euclidean line l defined by the points 0 and v_2; see Figure 8. By construction, $\theta = \frac{\pi}{2} - \omega$. Let l_2 be the vertical line through v_2. For the induced length h along the (red colored) horoarc on height ρ from $\{x_2 = 0\}$ delimited by l_1 and l_2, (3.2) yields

$$h = \frac{h_0}{\rho}\,, \tag{3.15}$$

where h_0 denotes the Euclidean distance between l_1 and l_2. On the other hand side, $\cos\omega = \sin\theta = \frac{h_0}{\rho}$ which yields $h = \cos\omega = \tanh d$ as desired. □

Next, consider a Lambert quadrilateral $L = L(a,b) \subset \mathbb{H}^2$ with one ideal vertex q and opposite edges of lengths a and b, respectively. Furthermore, L has three right angles at the ordinary vertices x, y and z, and the lengths a and b of the edges $[x,y]$ and $[y,z]$ are related by the well known formula $\sinh a \cdot \sinh b = 1$. Put a horocycle σ based at q in such a way that it starts at the vertex z and has non-empty intersection with L. Denote by s the intersection point of σ with the geodesic defined by q and x. The point s can lie outside of L. Let h be the hyperbolic length of the horoarc $\sigma_s \subset \sigma$ delimited by s and z. The situation is depicted in Figure 9.

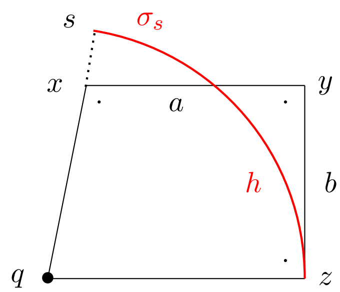


FIGURE 9. The Lambert quadrilateral L and a horoarc σ_s of length h

Lemma 2. *Let $a = d_{\mathbb{H}}(x,y)$ be the edge length of the Lambert quadrilateral $L = [q,x,y,z]$ with ideal vertex q according to Figure 9. Denote by h the hyperbolic length of the horoarc σ_s based at q and delimited by the points s and z related to L. Then,*

$$h = \cosh a\,.$$

Furthermore, the intersection point s lies outside of the edge $[q,x]$ of L if and only if $h > \sqrt{2}$.

Remark 5. *The condition $h > \sqrt{2}$ in Lemma 2 is equivalent to the property that $\sinh a > 1$ and hence to $\sinh b < 1$ for the edge length $b = d_{\mathbb{H}}(y,z)$ in L. As a consequence, the horoarc σ'_q of hyperbolic length h' based at q and starting at x towards L has its intersection point s' on the edge $[q,z]$ of L and satisfies $h' = \cosh b$.*

Proof. We provide a proof which is very similar to the one of Lemma 1. View the Lambert quadrilateral L in the upper half plane $\mathcal{U}^2$ in such a way that its ideal vertex q coincides with ∞ and that the vertex z is at height r from the boundary $\{x_2 = 0\}$; see Figure 10.

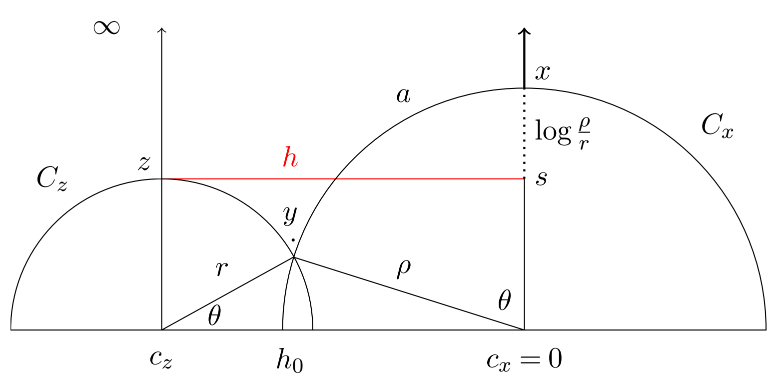


FIGURE 10. The Lambert quadrilateral L in the upper half plane

The edges $[x,y]$ and $[y,z]$ lie on semicircles C_x and C_z, both perpendicular to the boundary $\{x_2 = 0\}$ and intersecting orthogonally each other at y.

Consider the semicircle C_x with its centre c_x at 0 and with radius ρ. Let θ be the angle between the radii $[c_x, x]$ and $[c_x, y]$. By the same correspondence as given by (3.14), we have here that

$$\tanh a = \sin\theta \qquad (\text{and similarly, } \tanh b = \cos\theta\,)\,. \tag{3.16}$$

The hyperbolic length h of the horoarc is given in terms of the Euclidean distance $h_0 = d_0(c_x, c_z)$ and the radius r according to $h = \frac{h_0}{r}$. Furthermore, we easily see that

$$h_0^2 = r^2 + \rho^2\,, \tag{3.17}$$

$$\tan\theta = \frac{\rho}{r}\,. \tag{3.18}$$

Putting (3.16)–(3.18) together, we obtain

$$h^2 = \frac{h_0^2}{r^2} = \frac{r^2\,(1+\tan^2\theta)}{r^2} = \frac{1}{\cos^2\theta} = \cosh^2 a$$

as claimed.

In order to finish the proof, we need to show that the hyperbolic distance $d_{\mathbb{H}}(x,s)$ is positive if and only if $h > \sqrt{2}$. In terms of the Euclidean radii ρ of C_x and r of C_z, this condition is equivalent to the property $\log\frac{\rho}{r} > 0$. By (3.18), $\rho > r$ holds if and only if $\theta > \frac{\pi}{4}$, that is, by (3.16), that $\tanh a > \frac{1}{\sqrt{2}}$. Since

$$\frac{1}{\cosh^2 a} = 1 - \tanh^2 a\,,$$

we deduce that $\cosh a = h > \sqrt{2}$. □

Finally, consider a horocyclic sector bounded by a horoarc of length h based at the ideal point q and a concentric horoarc of length k with $0 < k < h$ (lengths with respect to the induced metric) at hyperbolic distance d; see Figure 11. By means of (3.15) in the upper half plane setting, it is easy to

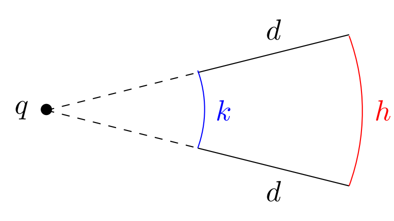


FIGURE 11. Two concentric horoarcs based at q and at hyperbolic distance d

derive the following result.

Lemma 3 ([6, Section 5], [23, Section 3.7]). *Denote by d the hyperbolic distance of two concentric horoarcs based at the ideal point q and of induced* *hyperbolic lengths h and k with $0<k<h$ along the respective horocycles according to Figure 11. Then,*

$$\frac{h}{k}=e^{d}\,.$$

Now, we are in the position to prove the following result.

Proposition 3. *For $\alpha\in(0,\frac{\pi}{3}]$, the cusp volume $\operatorname{vol}(C(\alpha))$ is given by*

$$\operatorname{vol}(C(\alpha))=\frac{1}{2\,(2+\cos\alpha)}\,.$$

Proof. Consider the building block $Q(\alpha)$ as part of the doubly truncated orthoscheme $R(\alpha)$. By (3.13), we have to quantify the lengths h,k of the base quadrilateral on the horosphere boundary of the cone $C(\alpha)$ with apex p_1. Let us introduce some notations according to Figure 6. The triangle $[p_0,p_1,p_3]$ gives rise to the Lambert quadrilateral $L=L(a,b)$ in $R(\alpha)$ with vertices p_1,x,y and z and edge lengths $d_{\mathbb{H}}(x,y)=l_\alpha=:a$ and $d_{\mathbb{H}}(y,z)=:b$.

The ordinary vertices $r=[p_0,p_2]\cap H_0$ and $s=[p_1,p_3]\cap H$ of $Q(\alpha)$ belong both to the right-angled triangular facets with ideal vertex p_1 of $Q(\alpha)$. Let $m=[p_0,p_3]\cap H$ and $w=[p_1,p_2]\cap H$ be vertices of $Q(\alpha)$ defined by the facet plane H. By construction, the triangle $[p_1,r,x]$ has angle α at x giving rise to the horoarc based at p_1, starting at r and of length $h_\alpha=\cos\alpha$ according to Lemma 1. In a similar way, the triangle $[p_1,s,w]$ has angle ω at w and contains the horoarc h_ω based at p_1, starting at s and of length $h_\omega=\cos\omega$. The angle $\omega\in(0,\frac{\pi}{2})$ depends on $\beta=\beta_\alpha$ and hence on α as follows. Consider the spherical vertex triangle associated to w which, by construction, is right-angled with angles β and $\pi-\beta$ and edge length ω opposite to its vertex with angle β. We easily deduce that

$$\cos\omega=\cot\beta\,,$$

and hence, $h_\omega=:h_\beta=\cot\beta$.

We start by determining an edge length h of the horospherical quadrilateral bounding $C(\alpha)$. We claim that $h_\alpha>h_\beta$ for $\alpha\in(0,\frac{\pi}{3}]$ so that $h=h_\beta$. Implementing the expressions for h_α and h_β, and using the identity (3.4) between α and β, we see that $h_\alpha>h_\beta$ is equivalent to the following inequality.

$$\cos^2\alpha\,\tan^2\beta=\cos^2\alpha\left(1+\frac{2}{\cos\alpha}\right)>1\,.$$

Since $\alpha\in(0,\frac{\pi}{3}]$, the term above is indeed bigger than 1.

Next, we determine the other edge length k of the base quadrilateral of $C(\alpha)$ which is the length of the horoarc κ based at p_1 and starting at s towards the facet plane H_0. For this, consider the Lambert quadrilateral $L=L(a,b)=[p_1,x,y,z]\subset R(\alpha)$. The edge length $a=l_\alpha$ is given by (2.2) while the length b of the edge $[y,z]$ can be deduced by hyperbolic trigonometry for the right-angled triangle $[y,z,p_2]$ according to

$$\cosh b=\frac{1}{\sin\alpha}\,.$$

Consider the horocycle σ based at p_1 and starting at z towards the geodesic $[p_1,x]$ in L. Denote by t the intersection point of σ with the geodesic defined by p_1 and x. By Lemma 2, the length K of the horoarc $\sigma_t \subset \sigma$ delimited by t and z is given by

$$K = \cosh l_\alpha = \frac{1}{\cos\alpha}$$

Observe that σ_t is concentric with κ, and the hyperbolic distance between σ_t and κ is given by $d = d_{\mathbb{H}}(s,z)$. The distance $d_{\mathbb{H}}(s,z)$ can easily be computed by looking at the (compact) Lambert quadrilateral $[s,m,y,z]$ with two neighboring edges of known lengths b and $\frac{l_\alpha}{2}$ (see [2, Theorem 7.17.1]). It follows that

$$\tanh d = \cosh b \cdot \tanh\frac{l_\alpha}{2} = \frac{1}{\sin\alpha}\cdot\frac{\sin\alpha}{1+\cos\alpha} = \frac{1}{1+\cos\alpha}\,.$$

By Lemma 3 and Euler's identity for e^d, we deduce that

$$\begin{aligned}\frac{K}{k} &= e^d = \cosh d + \sinh d = \cosh d\,(1+\tanh d)\\ &= \frac{1+\cos\alpha}{\sqrt{\cos\alpha\,(2+\cos\alpha)}}\left(1+\frac{1}{1+\cos\alpha}\right) = \frac{2+\cos\alpha}{\sqrt{\cos\alpha\,(2+\cos\alpha)}}\,.\end{aligned}$$

As a consequence of the above identities, we get

$$\begin{aligned}\mathrm{vol}(C(\alpha)) &= \frac{hk}{2} = \frac{1}{2}h_\beta\,K\,e^{-d}\\ &= \frac{1}{2}\sqrt{\frac{\cos\alpha}{2+\cos\alpha}}\cdot\frac{1}{\cos\alpha}\cdot\frac{\sqrt{\cos\alpha\,(2+\cos\alpha)}}{2+\cos\alpha} = \frac{1}{2\,(2+\cos\alpha)}\end{aligned}$$

as claimed. □

Putting the results of Proposition 2 and Proposition 3 together, we obtain the following explicit formula for the polyhedral cusp density of the building block $Q(\alpha)$.

Theorem 2. *For $\alpha \in (0,\frac{\pi}{3}]$, the polyhedral cusp density $\delta(\alpha)$ of $Q(\alpha)$ is given by*

$$\delta(\alpha) = \frac{\mathrm{vol}(C(\alpha))}{\mathrm{vol}(Q(\alpha))} = \frac{1}{2\,(2+\cos\alpha)\left\{Л(\frac{\pi}{4}+\frac{\alpha}{2}) + Л(\frac{\pi}{4}-\frac{\alpha}{2})\right\}}\,.$$

By means of Theorem 2 and the explicit representation of the density function $\delta(\alpha)$, we can prove the following important property of $\delta(\alpha)$.

Theorem 3. *For $\alpha \in (0,\frac{\pi}{3}]$, the polyhedral cusp density $\delta(\alpha)$ of $Q(\alpha)$ is a strictly increasing function.*

Proof. By Theorem 2 and Proposition 3, the numerator $\mathrm{vol}(C(\alpha))$ of the quotient $\delta(\alpha)$ is given by

$$\mathrm{vol}(C(\alpha)) = \frac{1}{2\,(2+\cos\alpha)}\,,$$

which is a strictly increasing function with respect to α. For the denominator $\mathrm{vol}(Q(\alpha))$ of $\delta(\alpha)$, we conclude by means of Schläfli's differential formula

$$\frac{d}{d\alpha}\,\mathrm{vol}(Q(\alpha)) = -\frac{1}{4}\,l_\alpha = -\frac{1}{4}\,\mathrm{arcosh}\left(\frac{1}{\cos\alpha}\right) < 0\,,$$

as given by (3.10), that $\mathrm{vol}(Q(\alpha))$ is strictly decreasing. Hence, the function $\delta(\alpha)$ is strictly increasing.

□

In the next section, we use Theorem 3 to prove the incommensurability of the manifolds M_n given by the complements of $\mathbb{S}^3$ by the $(2n)$-link chains $\mathcal{D}_{2n}$ by specialising to the case $\alpha = \frac{\pi}{n}$ and $Q_n = Q(\frac{\pi}{n})$ for all $n \geq 3$.

3.3. **Cusp density and commensurability of the manifolds M_n.** In this section, we apply our previous results to give a new and more elementary combinatorial-geometric proof of the following result due to Meyer, Millichap and Trapp [18, Section 6] (see Theorem B in the Introduction).

Theorem 4. *For $n \geq 3$, let $M_n = \mathbb{S}^3 \setminus \mathcal{D}_{2n}$ denote the complement of $\mathbb{S}^3$ by the $(2n)$-link chain $\mathcal{D}_{2n}$. Then, M_n is incommensurable to M_m for all distinct $m, n \geq 3$.*

Proof. As already pointed out at the beginning of Section 3 (see also Remark 1), it is sufficient to consider the non-arithmetic case, that is, to compare manifolds M_n and M_m up to commensurability for distinct $m, n \geq 5$.

Recall that the fundamental group $\pi_1(M_n)$ of M_n is commensurable to the group Λ_n generated by the reflections in the facet planes of the doubly truncated Coxeter orthoscheme $R_n = R(\frac{\pi}{n})$ with two ideal vertices. The polyhedron R_n decomposes into two isometric copies of the building block $Q_n = Q(\frac{\pi}{n})$ with one ideal vertex which are identified by the half-turn τ. In particular, $\pi_1(M_n)$ is commensurable to the group extension $\Lambda_n^* = \Lambda_n \star \langle\tau\rangle$ of Λ_n.

For $n \geq 5$, and alike $\pi_1(M_n)$, the group Λ_n^* is non-arithmetic by Theorem 1. Since the orbifold $\mathbb{H}^3/\Lambda_n^*$ is 1-cusped, its cusp density $\delta_n := \delta(\mathbb{H}^3/\Lambda_n^*)$ is a commensurability invariant by Proposition 1.

Finally, the cusp density function δ_n is strictly monotone with respect to n by Theorem 3. Therefore, $\delta_m \neq \delta_n$ for distinct $m, n \geq 5$ implying that the manifolds M_m and M_n finitely covering the orbifolds $\mathbb{H}^3/\Lambda_m^*$ and $\mathbb{H}^3/\Lambda_n^*$ are incommensurable.

□

Remark 6. *Our approach to prove arithmeticity and incommensurability of hyperbolic link complements by means of their polyhedral building blocks works also for other infinite families of manifolds. However, for a convenient arithmeticity check, it is useful to detect building blocks that are Coxeter polyhedra of simple combinatorial type. In the case of the link complements M_n, we have a choice of two types of Coxeter polyhedra, ideal right-angled* *antiprisms with* $2n+2$ *facets and – more conveniently – doubly truncated Coxeter orthoschemes with* 6 *facets. Their associated reflections groups give rise to Coxeter orbifolds or reflection orbifolds in the terminology of* [18, Section 3] *and* [3, Section 7]*, for example. In this context, note that Chesebro, DeBlois and Wilton* [3, Section 7.2] *describe an infinite family of FAL manifolds that are* not *commensurable to any Coxeter orbifold.*

Remark 7. *As already mentioned, Meyer, Millichap and Trap* [18, Section 6] *provide a different proof of Theorem 4. In fact, they study the symmetry group of* $M_n\,,\, n\geq 5\,,$ *which is isomorphic to the quotient group* $N(\Pi_n)/\Pi_n$ *of the normaliser* $N(\Pi_n)$ *of* $\Pi_n:=\pi_1(M_n)$ *in* $\operatorname{Isom}\mathbb{H}^3$ *by* Π_n*.* Hidden symmetries *of* M_n *correspond to non-trivial elements in* $C(\Pi_n)/N(\Pi_n)$ *where* $C(\Pi_n)$ *is the commensurator of the group* Π_n*. Then, the authors classify the symmetries and hidden symmetries of* M_n*. This study allows them to show that the manifolds* $M_n\,,\, n\geq 5\,,$ *admit* no *hidden symmetry, and to deduce that the manifolds* M_m *and* M_n *are incommensurable for distinct* m,n*. See* [18, Theorem 6.1, Corollary 6.3 and Corollary 6.4]*.*

3.4. **Comparing the volumes of M_n and $\widehat{W}_n$.** In this paragraph, we complete the proof of Theorem C and verify its part (2) which states that for $n\geq 6$, the volume of M_n is strictly bigger than the volume of the $(2n-1)$-cyclic cover over one component of the Whitehead link $\widehat{W}_n$. This property, without proof, was indicated to Agol by Ventzke and hinted more concretely by Masai; see [1, 13].

By the Corollary of Section 3.2, the volume of M_n is given by $8n\,\mathrm{vol}(R(\frac{\pi}{n}))$ and in terms of the Lobachevsky function. The volume of $\widehat{W}_n$ equals $(2n-1)\,\mathrm{vol}(O_{reg}^{\infty})$ where O_{reg}^{∞} is an ideal regular octahedron of dihedral angle $\frac{\pi}{4}$ which can be dissected into 6 copies of the Coxeter polyhedron $R(\frac{\pi}{3})$; see [15, Part (d), pp. 326-328]. Hence, by (3.8) and (3.11), the volume of the manifold $\widehat{W}_n$ equals $6\,(2n-1)\,\{Л(\frac{\pi}{12})+Л(\frac{5\pi}{12})\}=8\,(2n-1)\,Л(\frac{\pi}{4})$.

For the real parameter $x\in[6,\infty)$, define the help function

$$h(x)=x\,\mathrm{vol}(R(\frac{\pi}{x}))-(2x-1)\,Л(\frac{\pi}{4})\,. \tag{3.19}$$

It follows that $8\,h(n)=\mathrm{vol}(M_n)-\mathrm{vol}(\widehat{W}_n)$. Furthermore, one can show that $h(6)>0$. In fact, (3.8) and (3.11) give

$$h(6)=6\,\mathrm{vol}(R(\frac{\pi}{6}))-11\,Л(\frac{\pi}{4})=10\,Л(\frac{\pi}{6})-11\,Л(\frac{\pi}{4})\,,$$

and by means of the series representation (3.9) in the form

$$Л(\omega)=\omega\left(1-\log|2\omega|+\frac{\omega^2}{18}+\frac{\omega^4}{900}+\dots\right)\,,$$

one deduces that $h(6)=10\,Л(\frac{\pi}{6})-11\,Л(\frac{\pi}{4})\approx 0.036$.

Hence, in order to prove part (2) of Theorem C, it suffices to show that $h(x)$ is strictly monotonically increasing. To this end, we use Schläfli's differential formula presented in Section 3.2 and express the volume of $R(\frac{\pi}{x})$ in integral form

$$\mathrm{vol}(R(\frac{\pi}{x})) = -\frac{1}{2}\int\limits_0^{\frac{\pi}{x}} l_\alpha \, d\alpha + 2\,\text{Л}(\frac{\pi}{4}) \,,$$

where the integrand l_α is the edge length associated to the dihedral angle α of the family $R(\alpha)$; see (3.10). As for the second term at the right hand side, we used (3.11) of Proposition 2 for the identification $\mathrm{vol}(R(0)) = 2\,\text{Л}(\frac{\pi}{4})$.

By (3.19), it follows that

$$h(x) = -\frac{x}{2}\int\limits_0^{\frac{\pi}{x}} l_\alpha \, d\alpha + \text{Л}(\frac{\pi}{4}) \,,$$

and by taking derivatives,

$$h'(x) = -\frac{1}{2}\int\limits_0^{\frac{\pi}{x}} l_\alpha \, d\alpha + \frac{\pi}{2x}\, l_{\frac{\pi}{x}} \,.$$

Consider the edge length l_α given by $\cosh l_\alpha = 1/\cos\alpha$ according to (2.2). Obviously, l_α is strictly monotonically increasing. This fact implies that $h'(x) > 0$. Hence, $h(x)$ is strictly monotonically increasing with $h(6) > 0$, and this conclusion yields part (2) of Theorem C.

References

[1] I. Agol, *The minimal volume orientable hyperbolic 2-cusped 3-manifolds*, Proc. Amer. Math. Soc. 138 (2010), 3723–3732.

[2] A. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York, 1983.

[3] E. Chesebro, J. DeBlois, H. Wilton, *Some virtually special hyperbolic 3-manifold groups*, Comment. Math. Helv. 87 (2012), 727–787.

[4] E. Chesebro, J. DeBlois, *Algebraic invariants, mutation, and commensurability of link complements*, Pacific J. Math. 267 (2014), 341–398.

[5] H. Coxeter, *Trisecting an orthoscheme*, Comput. Math. Appl. 17 (1989), 59–71.

[6] H. Coxeter, *Arrangements of equal spheres in non-Euclidean spaces*, Acta Math. Acad. Sci. Hungar. 5 (1954), 263–274.

[7] E. Dotti, *Groups of hyperbolic isometries and their commensurability*, Ph.D. thesis no. 2213, University of Fribourg, 2020.

[8] E. Dotti, *On the commensurability of hyperbolic Coxeter groups*, arXiv:2101.10024v1, 25 January 2021.

[9] D. Futer, J. Purcell, *Links with no exceptional surgeries*, Comment. Math. Helv. **82** (2007), 629–664.

[10] O. Goodman, D. Heard, C. Hodgson, *Commensurators of cusped hyperbolic manifolds*, Experiment. Math. 17 (2008), 283–306.

[11] R. Guglielmetti, M. Jacquemet, R. Kellerhals, *Commensurability of hyperbolic Coxeter groups: theory and computation*, RIMS Kôkyûroku Bessatsu B66 (2017), 57–113.

[12] N. Johnson, R. Kellerhals, J. Ratcliffe, S. Tschantz, *Commensurability classes of hyperbolic Coxeter groups*, Linear Algebra Appl. 345 (2002), 119–147.

[13] J. Kaiser, J. Purcell, C. Rollins, *Volumes of chain links*, J. Knot Theory Ramifications 21 (2012), 1250115, 17pp.

[14] R. Kellerhals, *On the volume of hyperbolic polyhedra*, Math. Ann. 285 (1989), 541–569.

[15] R. Kellerhals, *The dilogarithm and volumes of hyperbolic polytopes*, Structural properties of polylogarithms, 301–336, Math. Surveys Monogr., 37, Amer. Math. Soc., Providence, RI, 1991.

[16] M. Lackenby, *The volume of hyperbolic alternating link complements*, Proc. London Math. Soc. **88** (2004), 204–224, With an appendix by Ian Agol and Dylan Thurston.

[17] C. Maclachlan, A. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, vol.219, Springer-Verlag, New York, 2003.

[18] J. Meyer, C. Millichap, R. Trapp, *Arithmeticity and hidden symmetries of fully augmented pretzel link complements*, New York J. Math. 26 (2020), 149–183.

[19] J. Milnor, *Hyperbolic geometry: the first 150 years*, Bull. Amer. Math. Soc. (N.S.) 6 (1982), 9–24.

[20] W. Neumann, A. Reid, *Arithmetic of hyperbolic manifolds*, in: Topology '90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, 273–310, 1992.

[21] W. Neumann, A. Reid, *Notes on Adams' small volume orbifolds*, in: Topology '90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, 311–314, 1992.

[22] J. Purcell, *An introduction to fully augmented links*, Interactions between hyperbolic geometry, quantum topology and number theory, 205–220, Contemp. Math., 541, Amer. Math. Soc., Providence, RI, 2011.

[23] W. Thurston, *The geometry and topology of three-manifolds*, electronic version, 1.1 - March 2002, `http://www.msri.org/publications/books/gt3m/`.

[24] È. Vinberg, *Discrete groups generated by reflections in Lobačevskiĭ spaces*, Mat. Sb. (N.S.) **72** (114) (1967), 471–488; correction, ibid. **73** (115) (1967), 303.

[25] È. Vinberg, O. Shvartsman, *Discrete groups of motions of spaces of constant curvature*, Geometry, II, Encyclopaedia Math. Sci., vol. 29, Springer, Berlin, 1993, pp. 139–248.

[26] K. Yoshida, *The minimal volume orientable hyperbolic 3-manifold with 4 cusps*, Pacific J. Math. 266 (2013), 457–476.

[27] H. Yoshida, *Commensurability of link complements*, Osaka J. Math. 54 (2017), 635–645.

Department of Mathematics, University of Fribourg, CH-1700 Fribourg, Switzerland

Email address: `ruth.kellerhals@unifr.ch`