manuscripta math.

Edoardo Dotti

On the commensurability of hyperbolic Coxeter groups

Received: 22 April 2022 / Accepted: 1 December 2022

Abstract. In this paper we study the commensurability of hyperbolic Coxeter groups of finite covolume, providing three necessary conditions for commensurability. Moreover we provide two new sets of generators for the field of definition in case of quasi-arithmetic hyperbolic Coxeter groups. This work is a concise version of chapters 4 and 5 of the author's Ph.D. thesis Dotti (Groups of hyperbolic isometries and their commensurability. PhD thesis, Department of mathematics, University of Fribourg 2020).

1. Introduction

For $n \geq 2$, let $\mathbb{H}^n$ be the real hyperbolic space of dimension n and denote by $\mathrm{Isom}(\mathbb{H}^n)$ its isometry group. Consider a space form $\mathbb{H}^n/\Gamma$, where Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. Two such space forms are commensurable if they admit a common finite-sheeted cover. We are interested in distinguishing hyperbolic space forms up to commensurability.

The situation is well understood in dimensions two and three. For $n = 3$ the group $\mathrm{Isom}^+(\mathbb{H}^3)$ of orientation preserving isometries can be identified with the group $\mathrm{PSL}(2, \mathbb{C})$. Due to the work of Maclachlan and Reid [24] there are two powerful commensurability invariants for Kleinian groups in $\mathrm{PSL}(2, \mathbb{C})$, the *invariant trace field* and *invariant quaternion algebra*, which form a complete set of invariants for arithmetic Kleinian groups.

In higher dimensions the situation needs more investigation. When dealing with arithmetic (of the simplest type) hyperbolic lattices, Gromov and Piatetski-Shapiro [11] provide a complete commensurability criterion. Consider an arithmetic lattice with its associated totally real field and quadratic form. Then, their commensurability criterion states that two such lattices are commensurable if and only if the two associated fields coincide and the two forms are similar over their field.

However, in the non-arithmetic context, no general commensurability criterion is known up to date.

In this paper we study the problem of commensurability of hyperbolic Coxeter groups of finite covolume. These are discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ generated by

E. Dotti (✉): Department of Mathematics, University of Fribourg, 1700 Fribourg, Switzerland
e-mail: edotti.edo@gmail.com

Mathematics Subject Classification: 20F55 · 22E40 · 11R21

https://doi.org/10.1007/s00229-022-01451-6
Published online: 21 January 2023

finitely many reflections in the bounding hyperplanes of Coxeter polyhedra, which are polyhedra whose angles are integral submultiples of π. We always suppose that the volume of the Coxeter polyhedra is finite. Following the work of Vinberg in [30], we shall associate a field of cycles and a quadratic form to every hyperbolic Coxeter group: the *Vinberg field* and the *Vinberg form*. Inspired by the result of Gromov and Piatetski-Shapiro, we provide the following necessary conditions for commensurability: if Γ_1 and Γ_2 are two commensurable cofinite hyperbolic Coxeter groups acting on $\mathbb{H}^n$, then their Vinberg fields coincide and the two associated Vinberg forms are similar over this field.

We are then able to refine the previous statement by associating to a Coxeter group as above a ring, the *Vinberg ring*, and show that this ring is also a commensurability invariant.

The paper concludes with two new sets of generators for the Vinberg field of a quasi-arithmetic Coxeter group. Specifically, we shall see that the Vinberg field of such a Coxeter group is generated by the coefficients of the characteristic polynomial of its Gram matrix on one side and by the coefficients of the characteristic polynomial of any Coxeter transformation on the other side.

The paper is structured as follows. In Sect. 2 we present all the basic theory needed for the rest of the paper such as hyperbolic Coxeter groups and commensurability. In Sect. 3 we present the theory on field of definition and we prove the commensurability statement above, the commensurability property of the Vinberg ring and we discuss the similarity classification of Vinberg forms. In the last section we study the Vinberg field and provide new sets of generators as mentioned above. This work is a concise version of chapters 4 and 5 of the author's Ph.D. thesis [8], and the proofs in this work are a direct adaptation from those in [8].

2. Preliminaries

2.1. Hyperbolic space, Coxeter polyhedra and Coxeter groups

Let $n \geq 2$ and denote by $\mathbb{H}^n$ the real hyperbolic space of dimension n. We interpret it with the *hyperboloid model*. Denote by $\mathbb{R}^{n,1}$ the space $\mathbb{R}^{n+1}$ equipped with the *Lorentzian product* defined as

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

The associated quadratic form, the *Lorentzian form*, will be denoted by $q_{-1}(x) := \langle x, x\rangle$. The hyperboloid model for $\mathbb{H}^n$ is then given by the set

$$\mathcal{H}^n := \{x \in \mathbb{R}^{n+1} \mid ||x||^2 = \langle x, x\rangle = -1,\ x_{n+1} > 0\}$$

with metric $d_{\mathcal{H}^n}(x, y) = \operatorname{arcosh}(-\langle x, y\rangle)$ for all $x, y \in \mathcal{H}^n$.

The group of isometries $\mathrm{Isom}(\mathcal{H}^n)$ is the Lie group of *positive Lorentzian matrices*

$$\mathrm{O}^+(n, 1) = \left\{A \in \mathrm{Mat}(n+1, \mathbb{R}) \mid A^T J A = J,\ [A]_{n+1,n+1} > 0\right\},$$

where $J = \mathrm{diag}(1, \ldots, 1, -1)$ is the diagonal matrix which represents the Lorentzian form.

Remark 2.1. The group $\mathrm{O}^+(n, 1)$ is *not* an algebraic group. However, one can project $\mathcal{H}^n$ to the open unit ball and consider its *projective model* $\mathcal{K}^n$. A very important aspect of $\mathcal{K}^n$ is that its isometries form an algebraic group. Consider the group of all matrices which preserve the Lorentzian form $\left\{A \in \mathrm{Mat}(n+1, \mathbb{R}) \mid A^T J A = J\right\} = \mathrm{O}(n, 1)$. Then,

$$\mathrm{Isom}(\mathcal{K}^n) \cong \mathrm{O}(n, 1)/\{\pm I\} =: \mathrm{PO}(n, 1). \tag{1}$$

The fact that $\mathrm{Isom}(\mathcal{K}^n)$ is an algebraic group will be exploited in Sect. 3.2 (see Remark 3.4).

Let $H_e = e^\perp$ be a hyperbolic hyperplane given as the orthogonal complement of a vector $e \in \mathbb{R}^{n+1}$ of Lorentzian norm 1 and consider the half-space $H_e^- = \{x \in \mathcal{H}^n \mid \langle x, e\rangle \leq 0\}$.

A (convex) *polyhedron* $P \subset \mathcal{H}^n$ is the intersection with non-empty interior of finitely many half-spaces, that is,

$$P = \bigcap_{i=1}^{N} H_{e_i}^-,$$

$N \geq n+1$, where the unit vector e_i normal to the hyperplane H_{e_i} is pointing outwards of P. If $N = n+1$, then P is called an *n-simplex*.

Particularly, a *Coxeter polyhedron* is a polyhedron all of whose angles between its bounding hyperplanes are either zero or sub-multiples of π, hence of the form $\frac{\pi}{k}$ for $k \in \mathbb{N}, k \geq 2$.

Let $\Gamma = \langle s_{e_1}, \ldots, s_{e_N}\rangle < \mathrm{Isom}(\mathcal{H}^n)$ be the discrete group generated by the reflections in the hyperplanes bounding a Coxeter polyhedron P. Then Γ is a geometric representation of an abstract Coxeter group in $\mathrm{O}^+(n, 1)$. The group Γ is called a *hyperbolic Coxeter group*. The number of its generating reflections N is called the rank of Γ.

In the sequel, hyperbolic Coxeter groups will always be assumed to be *cofinite*, that is, the associated Coxeter polyhedron P has finite volume. A hyperbolic Coxeter group Γ is said to be *cocompact* if P is compact.

The *Gram matrix* associated to P and to Γ is the real symmetric matrix $G := G(P) = G(\Gamma) = (g_{ij})_{1\leq i,j\leq N}$ with coefficients $g_{ij} = \langle e_i, e_j\rangle$. The Gram matrix G is unique up to enumeration of the hyperplanes and has signature $(n, 1)$ (see [2, Chapter 6]).

The *Coxeter graph* of Γ is the graph with N vertices for which the vertex i corresponds to the hyperplane H_{e_i}. Between two vertices i and j we have:

(i) an edge if the angle between H_{e_i} and H_{e_j} is π/k, $k \geq 3$. If $k \geq 4$ then the edge is labelled with k; if $k = 3$ the label is omitted;

(ii) an edge labelled with ∞ if H_{e_i} and H_{e_j} are parallel;

(iii) a dotted edge if H_{e_i} and H_{e_j} are ultraparallel. The dotted edge is labelled with the hyperbolic cosine of the length $l = d_{\mathcal{H}^n}(H_{e_i}, H_{e_j})$ of their common perpendicular.

2.2. *Commensurability and arithmeticity*

Let H be a group. Two subgroups $H_1, H_2 \subset H$ are *commensurable* (in the wide sense) if and only if there exists an element $h \in H$ such that $H_1 \cap h^{-1}H_2h$ has finite index in both H_1 and $h^{-1}H_2h$.

This notion defines an equivalence relation. In our context, the group H will be $\mathrm{Isom}(\mathcal{H}^n)$. Stable under commensurability are some properties of subgroups of $\mathrm{Isom}(\mathcal{H}^n)$ such as discreteness, cofiniteness, cocompactness and arithmeticity. This latter notion can be further refined by splitting Coxeter groups in $\mathrm{Isom}(\mathcal{H}^n)$ into four categories: *arithmetic*, *quasi-arithmetic*, *properly quasi-arithmetic* and *nq-arithmetic*.

For this, the following criterion due to Vinberg turns out to be very practical (see [30, Theorem 2]).

Theorem 2.2. (Vinberg's arithmeticity criterion) *Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ be a Coxeter group of rank N and denote by $G = (g_{ij})_{1\le i,j\le N}$ its Gram matrix. Let $\widetilde{K}$ be the field generated by the entries of G, and let $K(\Gamma)$ be the field generated by all the possible cycles of G*[1]*. Then Γ is quasi-arithmetic if and only if:*

(i) *$\widetilde{K}$ is totally real;*
(ii) *for any embedding $\sigma : \widetilde{K} \hookrightarrow \mathbb{R}$ which is not the identity on $K(\Gamma)$, the matrix G^σ, obtained by applying σ to all the coefficients of G, is positive semidefinite.*
Moreover, a quasi-arithmetic group Γ is arithmetic if and only if
(iii) *the cycles of $2\,G$ are algebraic integers in $K(\Gamma)$.*

Thus, if the Coxeter group Γ above satisfies conditions $i)$, $ii)$ and $iii)$, we call Γ an *arithmetic* Coxeter group. Such a group has always finite covolume.

If Γ above satisfies conditions $i)$ and $ii)$, but not necessarily $iii)$, then we call Γ a *quasi-arithmetic* Coxeter group. If only $i)$ and $ii)$ are satisfied while $iii)$ fails then Γ is called a *properly quasi-arithmetic* Coxeter group.

Finally, Γ is called *non-quasi-arithmetic*, *nq-arithmetic* from now on, if it is not quasi-arithmetic.

Remark 2.3. There is a more general type of arithmeticity for groups of hyperbolic isometries (see [2, Chapter 6]). However, if a hyperbolic Coxeter group is arithmetic, then it is of the simplest type [30, Lemma 7]. Since we will be working only with hyperbolic Coxeter groups, we will always refer to arithmetic groups of the simplest type as just *arithmetic* groups.

3. Commensurability of hyperbolic Coxeter groups

In this section we prove the commensurability statements stated in the Introduction and we show the commensurability property of the Vinberg ring. An important role will be played by the theory of fields of definition, which will be recalled in this section.

[1] A *cycle* (or *cyclic product*) of G is defined as $g_{i_1i_2}g_{i_2i_3}\cdots g_{i_{l-1}i_l}g_{i_li_1}$ for any $\{i_1, i_2, \ldots, i_l\} \subset \{1, 2, \ldots, m\}$. A cycle is called *simple* if the indices i_j in the cycle are all distinct.

3.1. The Vinberg construction

We now associate a quadratic space to a hyperbolic Coxeter group following a construction due to Vinberg in [30].

Let Γ be a hyperbolic Coxeter group of rank N and let $e_1, \ldots, e_N \in \mathbb{R}^{n,1}$ be the outer normal unit vectors of its Coxeter polyhedron. Let $G = (g_{ij})_{1 \le i,j \le N}$ be the Gram matrix of Γ. For any $\{i_1, i_2, \ldots, i_l\} \subset \{1, 2, \ldots, N\}$ consider the cyclic product of $2\,G$

$$b_{i_1 i_2 \ldots i_l} := 2^l g_{i_1 i_2} g_{i_2 i_3} \ldots g_{i_{l-1} i_l} g_{i_l i_1}. \tag{2}$$

Define the field $K(\Gamma) := \mathbb{Q}(\{b_{i_1 i_2 \ldots i_l}\})$ of all cycles of $2\,G$. It is obvious that $K(\Gamma)$ is generated by the simple cycles.

Next, for $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, N\}$, define the vectors

$$v_1 := 2e_1 \text{ and } v_{i_1 i_2 \ldots i_k} := 2^k g_{1 i_1} g_{i_1 i_2} \ldots g_{i_{k-1} i_k} e_{i_k}, \tag{3}$$

and consider the $K(\Gamma)$-vector space V spanned by the vectors $\{v_{i_1 i_2 \ldots i_k}\}$ according to (3). By [9, Lemma 1], V is of dimension $n + 1$.

Moreover, as shown in [23] for example, V is left invariant by Γ. Indeed $s_{e_j}(v_{i_1 i_2 \ldots i_k}) = v_{i_1 i_2 \ldots i_k} - v_{i_1 i_2 \ldots i_k j}$, and $\langle v_{i_1 i_2 \ldots i_k}, v_{j_1 j_2 \ldots j_l} \rangle \in K(\Gamma)$. By combining these equations, a quick computation shows that

$$\langle s_{e_j}(v_{i_1 i_2 \ldots i_k}), s_{e_j}(v_{j_1 j_2 \ldots j_l}) \rangle = \langle v_{i_1 i_2 \ldots i_k}, v_{j_1 j_2 \ldots j_l} \rangle.$$

Since $2\,G$ is of signature $(n, 1)$, the restriction of the Lorentzian product on V yields a quadratic form $q = q_V$ of signature $(n, 1)$ on V. By the construction of the $K(\Gamma)$-vector space V in terms of the vectors (3) and the form q_V, we obtain a natural embedding $\Gamma \hookrightarrow \mathrm{O}(V, q)$.

Observe that this construction is independent of the arithmetic nature of Γ. Therefore *any* hyperbolic Coxeter group has an associated field and quadratic form which justifies the following definition.

Definition 3.1. Let Γ be a hyperbolic Coxeter group. Then

(i) the field $K(\Gamma) = \mathbb{Q}(\{b_{i_1 i_2 \ldots i_l}\})$ is called the *Vinberg field* of Γ;
(ii) the quadratic form $q = q_V$ is called the *Vinberg form* of Γ;
(iii) the quadratic space (V, q) is called the *Vinberg space* of Γ.

The next objective is to show that the Vinberg field and the similarity class of the Vinberg form are two commensurability invariants. Before that, we need more terminology.

Let (V_1, q_1), (V_2, q_2) be two quadratic spaces of dimension $m \ge 2$ over a field K. Then (V_1, q_1) and (V_2, q_2) are *isometric* (denoted by $\cong$) if and only if there is an isomorphism $S : V_1 \to V_2$ such that $q_1(x) = q_2(Sx)$ for all $x \in V_1$. They are *similar* (denoted by $\backsim$) if there exist a $\lambda \in K^*$ such that (V_1, q_1) and $(V_2, \lambda q_2)$ are isometric. The scalar λ is called *similarity factor*.

Isometry and similarity induce equivalence relations. In the sequel, we often abbreviate and speak about *isometric* (*similar*) quadratic forms instead of isometric (similar) quadratic spaces. Furthermore, if one represents two quadratic forms by two $m \times m$ matrices Q_1 and Q_2 over K, then being isometric means that there exists an invertible matrix $S \in \mathrm{GL}(m, K)$ such that $Q_1 = S^T Q_2 S$.

3.2. Theory on fields of definition

We present here some aspects about fields of definition, which will play an essential role for the upcoming proofs. All the theory presented here is taken from Vinberg's paper [31].

Let U be a finite dimensional vector space over a field F, and let $R \subset F$ be an integrally closed Noetherian ring. Denote by Δ a family of linear transformations of U. The ring R is said to be a *ring of definition* for Δ if U contains an R-lattice which is invariant under Δ. When R is a field we call R a *field of definition*.

If a principal ideal domain R is a ring of definition for Δ, then we can find a basis of U such that every element of Δ can be written as a matrix having entries in R.

Let us specialise the context and consider a Coxeter group $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$. As we have seen in Sect. 3.1, the space $\mathbb{R}^{n,1}$ contains the $K(\Gamma)$-module V which is invariant under Γ. That is, the Vinberg field $K(\Gamma)$ is a field of definition for Γ. The next lemma implies that the Vinberg field $K(\Gamma)$ is actually the *smallest* field of definition associated to Γ.

Lemma 3.2. ([31], Lemma 11 and Lemma 12) *Let Γ be a hyperbolic Coxeter group with Gram matrix G and let F be a field of characteristic 0. An integrally closed Noetherian ring $R \subset F$ is a ring of definition for Γ if and only if R contains all the simple cycles of $2\,G$.*

Lastly, for the following proofs we need the result [31, Theorem 5] of Vinberg. We recapitulate here a more specific version suitable to our context.

Theorem 3.3. *Let Γ be a cofinite hyperbolic Coxeter group with Vinberg space (V, q) and Gram matrix G. Let R be an integrally closed Noetherian ring. Then the following is equivalent:*

(i) R is a ring of definition of Γ,
(ii) R is a ring of definition of $\mathrm{Ad}\,\Gamma$,
(iii) R contains all the simple cyclic products of $2\,G$.

Remark 3.4. It is important to notice that in [31] Vinberg considers Zariski dense groups generated by reflections of a quadratic space defined over an *algebraically closed* field. This hypothesis does not apply directly to our situation since the isometry group $\mathrm{PO}(n, 1)$ of Klein's projective model $\mathcal{K}^n$ is defined over the reals.

Our version of the theorem can be retrieved from the original one as follows. Pass to the complexified space $\mathbb{R}^{n+1} \otimes_{\mathbb{R}} \mathbb{C}$ endowed with the standard (real) Lorentzian form q_{-1}. Let $\mathrm{O}_{\mathbb{C}}(n, 1)$ be the group of complex $(n+1) \times (n+1)$ matrices which preserve q_{-1}, and form the projective group $\mathrm{PO}_{\mathbb{C}}(n, 1) = \mathrm{O}_{\mathbb{C}}(n, 1)/\{\pm I\}$.

Recall that a cofinite hyperbolic Coxeter group is Zariski dense (over $\mathbb{R}$) in $\mathrm{PO}(n, 1)$ (see [19, Chapter 4]). This property remains valid in the complexified context of $\mathrm{PO}_{\mathbb{C}}(n, 1)$ over $\mathbb{C}$. We can now apply the original Theorem 5 of [31] which implies Theorem 3.3.

3.3. Commensurability conditions for hyperbolic Coxeter groups

We are now able to prove the commensurability statements stated in the Introduction.

Theorem 3.5. *Let Γ_1 and Γ_2 be two commensurable cofinite hyperbolic Coxeter groups acting on $\mathcal{H}^n$, $n \geq 2$. Then their Vinberg fields coincide and the two associated Vinberg forms are similar over this field.*

We start the proof by showing that two commensurable Coxeter groups have the same Vinberg field. For this, denote by $\mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma) = \mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}(\gamma) \mid \gamma \in \Gamma)$ the *adjoint trace field* of Γ.

Proposition 3.6. *Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ be a cofinite Coxeter group, $n \geq 2$. Then the associated Vinberg field and the adjoint trace field coincide, that is*

$$K(\Gamma) = \mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma). \tag{4}$$

Proof. Lemma 3.2 implies that the Vinberg field $K(\Gamma)$ is the smallest field of definition of Γ. By point $i)$ of Theorem 3.3, $K(\Gamma)$ is a field of definition of $\mathrm{Ad}\,\Gamma$ as well and by point $iii)$ $K(\Gamma)$ is contained in every field of definition of $\mathrm{Ad}\,\Gamma$. By [31, Corollary of Theorem 1], $\mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma)$ is the smallest field of definition of $\mathrm{Ad}\,\Gamma$. Hence, the equality (4) follows. □

Corollary 3.7. *Let $\Gamma_1, \Gamma_2 < \mathrm{Isom}(\mathcal{H}^n)$ be two cofinite Coxeter groups, $n \geq 2$. If Γ_1 and Γ_2 are commensurable, then their associated Vinberg fields coincide, that is,*

$$K(\Gamma_1) = K(\Gamma_2).$$

Proof. By Proposition 3.6 we know that $K(\Gamma_1) = \mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma_1)$ and $K(\Gamma_2) = \mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma_2)$. The adjoint trace field of a hyperbolic lattice is a commensurability invariant (see [7, Proposition 12.2.1]). Therefore the claim follows. □

Remark 3.8. By the Local Rigidity Theorem [28, Chapter 1], the adjoint trace field $\mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma)$ of a Coxeter group in $\mathrm{Isom}(\mathcal{H}^n)$ is a number field for $n \geq 4$. Therefore, by Proposition 3.6, the Vinberg field $K(\Gamma)$ is a number field. Moreover, $K(\Gamma)$ is a number field for $n = 3$ as well. This is a consequence of the connection between $K(\Gamma)$ and the invariant trace field $K\Gamma^{(2)}$ ([23, Theorem 3.1]) and the fact that $K\Gamma^{(2)}$ is a number field ([24, Theorem 3.1.2]).

Example 3.9. Consider the two non-cocompact nq-arithmetic Coxeter pyramid groups Γ_1 and Γ_2 acting on $\mathcal{H}^4$ as shown in Fig. 1.

For Γ_1 we have the cycle $-2\sqrt{2}$ obtained by following the triangular tail path, while all the other simple cycles lie in $\mathbb{Q}$ or $\mathbb{Q}\left(\sqrt{2}\right)$. For Γ_2, the triangular tail path gives the cycle $\frac{1}{2}\left(3+\sqrt{5}\right)$. All other simple cycles are either in $\mathbb{Q}$ or $\mathbb{Q}\left(\sqrt{5}\right)$. Therefore the Vinberg fields are $K(\Gamma_1) = \mathbb{Q}\left(\sqrt{2}\right)$ and $K(\Gamma_2) = \mathbb{Q}\left(\sqrt{5}\right)$. Thus Γ_1 and Γ_2 are incommensurable.

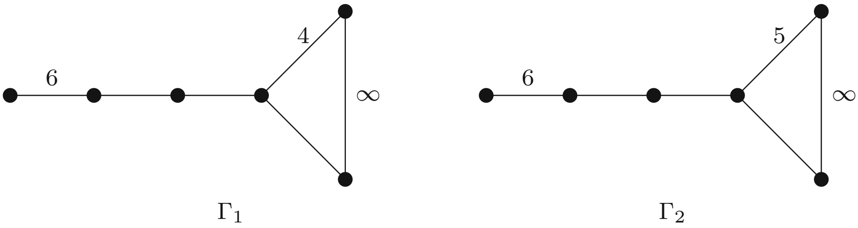

Fig. 1. Two Coxeter pyramid groups Γ_1 and Γ_2 in $\mathrm{Isom}(\mathcal{H}^4)$

Let us return to the proof of Theorem 3.5 and show that two commensurable hyperbolic Coxeter groups have similar Vinberg forms. The proof will follow the same strategy as indicated by Gromov and Piateski-Shapiro in Theorem 2.6 of [11] for arithmetic groups, and which has been elaborated by Johnson, Kellerhals, Ratcliffe and Tschantz in Theorem 1 of [18] for the special case of hyperbolic Coxeter simplex groups. We will apply the same algebraic terminology as used in [18].

Consider two commensurable hyperbolic Coxeter groups Γ_1 and Γ_2 represented in $\mathrm{O}^+(n, 1)$ and denote their Vinberg field by K. There is a matrix $X \in \mathrm{O}^+(n, 1)$ and there are two subgroups $H_1 < \Gamma_1$ and $H_2 < \Gamma_2$, each of finite index, such that $H_1 = X^{-1} H_2 X$. One can assume that H_1 and H_2 are contained in $\mathrm{SO}^+(n, 1)$, the index two subgroup of $\mathrm{O}^+(n, 1)$ of determinant one matrices. Let (V_1, q_1) be the Vinberg space over K associated to Γ_1 and equipped with the basis $\{v_1, \dots, v_{n+1}\}$ according to (3). With respect to this basis, all elements of Γ_1 are matrices over K, since K is a field of definition of Γ. Clearly the forms q_1 and the Lorentzian form q_{-1} are equivalent over $\mathbb{R}$. The same reasoning applies to the Vinberg space (V_2, q_2). Let Q_1 and Q_2 be the matrix representations of the Vinberg forms q_1 and q_2 in the relative bases. Let the real matrices T_1 and T_2 be the representations of the isometries between the Vinberg forms and the Lorentzian form q_{-1}. Then the matrix

$$S := T_2^{-1} X T_1 \tag{5}$$

represents an isometry between q_1 and q_2, since $Q_1 = S^T Q_2 S$. Moreover define the two groups $H_1' := T_1^{-1} H_1 T_1$ and $H_2' := T_2^{-1} H_2 T_2$.

Consider the isomorphism between the orthogonal groups $\mathrm{O}(q_1)$ and $\mathrm{O}(q_2)$ given by

$$\phi : A \to S A S^{-1}. \tag{6}$$

Lemma 3.10. *The map ϕ restricts to a K-linear map on* $\mathrm{Mat}(n + 1, K)$.

Proof. Let $i \in \{1, 2\}$. Denote by $\mathrm{O}^+(q_i)$ the group of q_i-orthogonal maps which leave each sheet of the hyperboloid $\mathcal{H}_i^{n+1} = \{x \in \mathbb{R}^{n+1} \mid q_i(x) = -1\}$ invariant. The isometry between q_i and the Lorentzian form q_{-1} gives a group isomorphism between $\mathrm{O}(q_i)$ and $\mathrm{O}(q_{-1})$. This isomorphism maps $\mathrm{O}^+(q_i)$ onto $\mathrm{O}^+(n, 1)$. Analogously, $\mathrm{SO}^+(q_i)$ is mapped onto $\mathrm{SO}^+(n, 1)$ and hence $H_i' \subset \mathrm{SO}^+(q_i)$. Now, $\mathrm{SO}^+(n, 1)$ is a non-compact connected simple Lie group and thus the same can be said for $\mathrm{SO}^+(q_i)$. Since H_i' has finite covolume, by the Borel density theorem [4] we get that $\mathrm{Span}_{\mathbb{R}}(H_i') = \mathrm{Span}_{\mathbb{R}}(\mathrm{SO}^+(q_i))$ in $\mathrm{Mat}(n+1, \mathbb{R})$. Furthermore, the action of $\mathrm{SO}^+(n, 1)$ on $\mathbb{C}^{n+1}$ is irreducible[2], and hence that the action of $\mathrm{SO}^+(q_i)$ is irreducible as well. By Burnside's theorem [5] (see also [20]) we get the equality $\mathrm{Span}_{\mathbb{R}}(\mathrm{SO}^+(q_i)) = \mathrm{Mat}(n+1, \mathbb{R})$, which implies that $\mathrm{Span}_{\mathbb{R}}(H_i') = \mathrm{Mat}(n+1, \mathbb{R})$.

Notice that for each $\alpha \in K$ and $C \in \mathrm{Mat}(n+1, K)$ we have $\phi(\alpha C) = \alpha\phi(C)$. Recall that K is a field of definition for H_i, thus $H_i' \subset \mathrm{Mat}(n+1, K)$. By the same arguments as before, we have that $\mathrm{Span}_K(H_i') = \mathrm{Mat}(n+1, K)$. Moreover, by (5),

$$\phi(H_1') = \phi(T_1^{-1} H_1 T_1) = T_2^{-1} X H_1 X^{-1} T_2 = T_2^{-1} H_2 T_2 = H_2'.$$

We deduce that $\phi(\mathrm{Span}_K(H_1')) = \mathrm{Span}_K(H_2')$. Therefore ϕ restricts to a K-linear map on $\mathrm{Mat}(n+1, K)$. □

Based on Lemma 3.10 we are finally ready to prove the last step as given by the following proposition. Its proof is a direct adaptation of the corresponding step in the proof of [18, Theorem 1].

Proposition 3.11. *Let Γ_1, Γ_2 be two commensurable Coxeter groups in* $\mathrm{Isom}(\mathcal{H}^n)$*, $n \geq 2$, with Vinberg field $K(\Gamma_1) = K(\Gamma_2) =: K$. Then the two Vinberg forms q_1 and q_2 are similar over K. Moreover, the similarity factor is positive.*

Proof. Let $1 \leq i, j \leq n+1$. Define the matrix $I_{ij} \in \mathrm{Mat}(n+1, K)$ with coefficient $[I]_{ij} = 1$ and all the other coefficients equal to zero. Consider the isomorphism ϕ according to (6). Define $M_{ij} := \phi(I_{ij}) = S I_{ij} S^{-1}$, which is in $\mathrm{Mat}(n+1, K)$ by Lemma 3.10. The matrix $S I_{ij} =: S_{ij}$ has the j-th column which is equal to the i-th column of S and all the other coefficients are equal to zero. Observe that $[M_{ij}]_{kl} = [S]_{ki}[S^{-1}]_{jl}$ for all k, l, i, j. The matrix S^{-1} is invertible, thus we can always find a pair $\{j, l\}$ such that $[S^{-1}]_{jl} \neq 0$. Let λ denote the inverse of the coefficient $[S^{-1}]_{jl}$. In doing so, every coefficient of S can be written as λ multiplied with an entry of a matrix of the form $M_{ij} \in \mathrm{Mat}(n+1, K)$. Hence there exists a matrix $M \in \mathrm{Mat}(n+1, K)$ such that $S = \lambda M$. Recall that $Q_1 = S^T Q_2 S$ holds, therefore $Q_1 = \lambda^2 M^T Q_2 M$, with Q_1, Q_2 and M all in $\mathrm{Mat}(n+1, K)$. Finally λ^2 is a positive element belonging to K so that q_1 is isometric to $\lambda^2 q_2$, and the claim follows. □

3.4. Similarity classification of the Vinberg forms

The study of similarity of quadratic forms heavily relies on isomorphisms of quaternion algebras and others elements of the Brauer group. For a more detailed explanation on this topic we refer to [12]. Let K denotes a field of characteristic different from 2, and let q be a quadratic form of dimension m over K, that is, q is defined on a vector space of dimension m over K.

[2] See the Erratum to the paper "Commensurability classes of hyperbolic Coxeter groups", due to J. Ratcliffe and S. Tschantz, presented in [8, Appendix D].

For two elements $a, b \in K^*$, we denote by (a, b) the quaternion algebra over K generated by the elements $1, i, j, ij$ with the relations $i^2 = a$, $j^2 = b$ and $ij = -ji$.

Two quaternion algebras are said to be equivalent if and only if they are isomorphic. Equivalence classes of quaternion algebras form a group, which is a subgroup of the Brauer group $\mathrm{Br}(K)$. For some computational rules about the multiplication between quaternion algebras, we refer to [21, Proposition 3.20]. If the field K is a number field, then two quaternion algebras over K are isomorphic if and only if they have the same ramification set (see [22, Theorem 4.1]).

For the definition of a ramification set $\mathrm{Ram}(A)$ for a quaternion algebra A and its theory we refer to [22]. In this paper, the computations of ramification sets are done using the package RamifiedPlaces of Magma©.

The similarity classification of quadratic forms relies on two elements of the Brauer group, which are closely related to one another. The first one is the *Hasse invariant* $s(q)$ of a diagonal quadratic form $q = \langle a_1, \dots, a_m \rangle$. This is the element of the Brauer group $\mathrm{Br}(K)$ represented by the quaternion algebra

$$s(q) = \bigotimes_{i<j} (a_i, a_j)_K \; .$$

The Hasse invariant $s(q)$ is independent of the diagonalisation chosen. It is moreover an isometry invariant (see [21, Proposition 3.18]). However it is *not* a similarity invariant (see [26, Lemma 4.3]).

The second element of the Brauer group we are interested in is the *Witt invariant* $c(q)$ of a quadratic space (V, q) over K. It is obtained from the Hasse invariant according to [21, Chapter V, Proposition 3].

Let Γ be a quasi-arithmetic Coxeter group with Vinberg field K acting on $\mathcal{H}^n$. Let (V, q) be the Vinberg space of dimension $n + 1$ over K associated to Γ and put $\delta := (-1)^{\frac{n(n+1)}{2}} \det(q)$, the *discriminant* of q. Denote by B the quaternion algebra representing the Witt invariant $c(q)$. The similarity class of (V, q) depends on the parity of n as follows.

Theorem 3.12. ([22], Theorem 7.2) *When n is even, the similarity class of the Vinberg space (V, q) of dimension $n + 1$ is in one-to-one correspondence with the isomorphism class of the quaternion algebra B.*

Theorem 3.13. ([22], Theorem 7.4) *When n is odd, the similarity class of the Vinberg space (V, q) of dimension $n + 1$ is in one-to-one correspondence with the isomorphism class of the quaternion algebra $B \otimes_K K(\sqrt{\delta})$ over $K(\sqrt{\delta})$. Moreover, if δ is a square in K^*, then the similarity class is in one-to-one correspondence with the isomorphism class of B over $\mathbb{Q}$.*

Consider now a nq-arithmetic group Γ acting on $\mathcal{H}^n$. For n even, a similarity criterion can be stated. For n odd, we provide a necessary condition for similarity, only. We start by recalling the Hasse-Minkowski Theorem in terms of the Hasse invariant see ([3,21]).

Table 1. Similarity criterion for Vinberg forms of hyperbolic Coxeter groups

n	Similarity criterion
$n \equiv 0 \mod 4$	$s(q_1) = s(q_2)$ $\mathrm{sgn}(\sigma(q_1)) = \mathrm{sgn}(\sigma(\lambda q_2))$
$n \equiv 2 \mod 4$	$s(q_1) = (\lambda, -1) \cdot s(q_2)$ $\mathrm{sgn}(\sigma(q_1)) = \mathrm{sgn}(\sigma(\lambda q_2))$

Theorem 3.14. *Let K be a number field and let q_1 and q_2 be two quadratic forms of dimension m over K. For a $\lambda \in K^*$, q_1 and λq_2 are isometric if and only if the following properties are satisfied:*

(i) $\dim(q_1) = \dim(\lambda q_2)$*,*
(ii) $\det(q_1) \equiv \det(\lambda q_2)$ *in* $K^* \mod (K^*)^2$*,*
(iii) $s(q_1) = s(\lambda q_2)$*,*
(iv) $\mathrm{sgn}(\sigma(q_1)) = \mathrm{sgn}(\sigma(\lambda q_2))$ *for all real embeddings* $\sigma : K \hookrightarrow \mathbb{R}$*.*

For n even, let Γ_1 and Γ_2 be two hyperbolic Coxeter groups with the same Vinberg field K, and denote by q_1 and q_2 the associated Vinberg forms over K. Recall that $\dim(q_1) = \dim(q_2) = n + 1 =: m$, i.e. the dimension of both quadratic forms is odd. Then, condition ii) of the Hasse-Minkowski Theorem 3.14 implies that Γ_1 and Γ_2 can be isometric only if $\det(q_1) \equiv \lambda \det(q_2)$ in $K^* \mod (K^*)^2$. This means that λ can only be the value which balances the two determinants, that is, $\lambda = \frac{\det(q_1)}{\det(q_2)} \in K^*/(K^*)^2$ (see also the proof of [26, Proposition 5.4]). Using [26, Lemma 4.3], we get $s(\lambda q_2) = (\lambda, -1) \cdot s(q_2)$ for $\lambda = \frac{\det(q_1)}{\det(q_2)} \in K^*/(K^*)^2$, and we obtain the complete set of similarity invariants for Vinberg forms as shown in Table 1.

Notice that, for quasi-arithmetic groups, this similarity classification is compatible with the one provided by Maclachlan in the even dimensional case. Indeed, for these groups, as a consequence of property ii) of Theorem 2.2, the equality between signatures is always satisfied. Moreover, a computation on the Hasse invariants leads to the equality between Witt invariants. Notice moreover that for $n = 2$, one has first to make sure that the Vinberg field is a number field in order to use the previous classification.

For n odd, let Γ_1 and Γ_2 be two hyperbolic Coxeter groups. If they are quasi-arithmetic, we refer to the similarity classification provided by Maclachlan (see Theorem 3.13). Otherwise, the similarity problem for their even-dimensional Vinberg forms q_1 and q_2 is more involved. We present here a partial result, only.

Applying condition ii) of the Hasse-Minkowski Theorem 3.14 we get that Γ_1 and Γ_2 can be isometric only if $\det(q_1) \equiv \det(\lambda q_2)$ in $K^* \mod (K^*)^2$ which reduces to $\det(q_1) \equiv \det(q_2) \mod (K^*)^2$. In contrast to the previous case, we can not extract any information about λ. This fact can be stated in the following lemma, sometimes referred to as the *ratio-test*.

Lemma 3.15. *Let $\Gamma_1, \Gamma_2 < \mathrm{Isom}(\mathcal{H}^n)$, n odd, be two commensurable Coxeter groups with Vinberg field K and Vinberg forms q_1 and q_2, respectively. Then,* $\det(q_1) \equiv \det(q_2) \in K^* \mod (K^*)^2$.

Example 3.16. As an incommensurability example using the Vinberg form, consider the two cocompact Coxeter groups Γ_1, Γ_2 in $\mathrm{Isom}(\mathcal{H}^4)$ given in Fig. 2. The groups Γ_1 and Γ_2 are so-called crystallographic Napier cycles (see [16]). Observe that both groups are properly quasi-arithmetic.

The weights l_i and l_i' of the dotted edges in the Coxeter graphs can be computed and are

$$l_1 = \sqrt{\frac{1}{11}\left(10+3\sqrt{5}\right)}, \qquad l_1' = \sqrt{\frac{2}{11}\left(7+\sqrt{5}\right)},$$
$$l_2 = \frac{1}{2}\sqrt{\left(5+\sqrt{5}\right)}, \qquad l_2' = \sqrt{\frac{2}{19}\left(9+\sqrt{5}\right)},$$
$$l_3 = \sqrt{\frac{1}{11}\left(16+7\sqrt{3}\right)}, \qquad l_3' = \sqrt{\frac{1}{209}\left(233+104\sqrt{5}\right)}.$$

The groups Γ_1 and Γ_2 have both $K = \mathbb{Q}\left(\sqrt{5}\right)$ as their Vinberg field. The diagonalised associated Vinberg forms over K are

$$q_1 = \mathrm{diag}\left(4, 4, 4, -2-2\sqrt{5}, 20+8\sqrt{5}\right),$$
$$q_2 = \mathrm{diag}\left(4, \frac{5}{2}+\frac{1}{2}\sqrt{5}, 2+\frac{2}{5}\sqrt{5}, \frac{-37}{2}-\frac{17}{2}\sqrt{5}, \frac{312}{19}+\frac{136}{19}\sqrt{5}\right).$$

These forms have the following Hasse invariants:

$$c(\Gamma_1) = \left(-2-2\sqrt{5}, 5+2\sqrt{5}\right),$$
$$c(\Gamma_2) = \left(10+2\sqrt{5}, -1\right)\cdot\left(-74-34\sqrt{5}, 1482+646\sqrt{5}\right).$$

The ramification set $\mathrm{Ram}(\Gamma_1)$ contains two prime ideals, one generated by 2, and the other generated by 5 and $-1+2\sqrt{5}$. The ramification set $\mathrm{Ram}(\Gamma_2)$ is empty. Since $\mathrm{Ram}(\Gamma_1) \neq \mathrm{Ram}(\Gamma_2)$, the two quaternion algebras representing $c(\Gamma_1)$ and $c(\Gamma_2)$ are not isomorphic. Hence the Vinberg forms q_1 and q_2 are not similar, and the groups Γ_1 and Γ_2 are incommensurable.

3.5. *The Vinberg ring*

In this section we are looking for additional commensurability invariants for an arbitrary hyperbolic Coxeter group $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$. Let us assume until the end of the chapter that the Vinberg field of Γ is a number field. For $n \geq 3$ this is always true (see Remark 3.8). Denote by $\mathcal{O}$ the ring of integers of the Vinberg field.

Definition 3.17. Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$, $n \geq 2$, be a cofinite Coxeter group with Gram matrix G. Consider all the cycles $b_{i_1 i_2 \dots i_l} = 2^l g_{i_1 i_2} g_{i_2 i_3} \dots g_{i_{l-1} i_l} g_{i_l i_1}$ of $2G$. The ring

$$R(\Gamma) := \mathcal{O}(\{b_{i_1 i_2 \dots i_l}\})$$

is called the *Vinberg ring* of Γ.

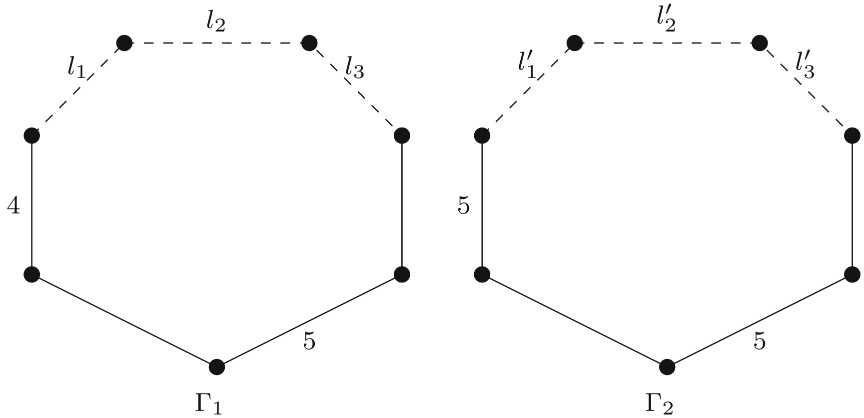

Fig. 2. The Coxeter groups Γ_1 and Γ_2 in $\mathrm{Isom}(\mathcal{H}^4)$

We show that the Vinberg ring is a ring of definition for certain groups and hence a commensurability invariant. Notice that the Vinberg ring as commensurability invariant is superfluous when considering arithmetic groups.

Proposition 3.18. *Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$, $n \geq 2$, be a cofinite Coxeter group with Gram matrix G. Assume that its Vinberg field K is a number field. Then the Vinberg ring $R(\Gamma)$ is a commensurability invariant.*

Proof. For this proof we use some results of Davis about overrings[3] ([6]) in the same way as used by Mila in [27, Section 2.1]. Since by hypothesis the Vinberg field K is a number field, there exists a minimal ring of definition R for Γ ([31, Corollary to Theorem 1]) which equals the integral closure of $\mathbb{Z}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$ in K. Clearly R is integrally closed and therefore contains the ring of integers $\mathcal{O}$ of K. Thus R is the integral closure of $\mathcal{O}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma] =: R'$ in K. The ring R' is an overring of $\mathcal{O}$. Since $\mathcal{O}$ is a Noetherian integral Dedekind domain, by [6, Theorem 1] its overring R' is integrally closed. This implies $R = R'$ which means that $\mathcal{O}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$ is the smallest ring of definition for Γ. Moreover, the Vinberg ring $R(\Gamma)$ is also an overring of $\mathcal{O}$ in such a way that it is integrally closed as well, and it is furthermore Noetherian since it is a subring of the number field K (see [10, Theorem]). Hence $R(\Gamma)$ is a ring of definition for Γ. By Theorem 3.3, $R(\Gamma) \subset R'$. Now, R' is the smallest ring of definition so that $R(\Gamma) = \mathcal{O}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$. By Theorem 3 of [31], rings of definition are commensurability invariants. Thus $R(\Gamma)$ is a commensurability invariant. □

Example 3.19. As an example, consider the two non-cocompact properly quasi-arithmetic Coxeter cube groups Γ_1 and Γ_2 in $\mathrm{Isom}(\mathcal{H}^3)$ defined in Fig. 3 (see

[3] An overring of an integral domain is a subring of the quotient field containing that given ring. In our case, the integral domain is the ring of integers $\mathcal{O}$ of the Vinberg field K, which has the Vinberg field as its quotient field.

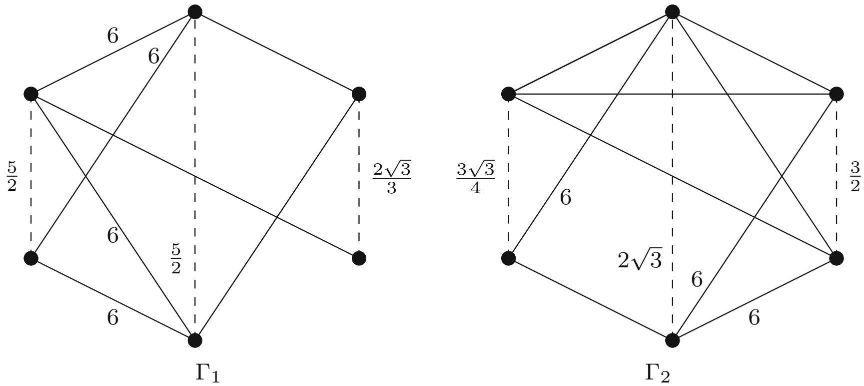

Fig. 3. Two Coxeter cube groups Γ_1 and Γ_2 in $\mathrm{Isom}(\mathcal{H}^3)$

[17]). They both have $\mathbb{Q}$ as Vinberg field and similar quadratic forms. Their Vinberg rings are given by $R(\Gamma_1) = \mathbb{Z}[1/3]$ and $R(\Gamma_2) = \mathbb{Z}[1/2]$, respectively. By Proposition 3.18 the groups Γ_1 and Γ_2 are therefore incommensurable.

Caution 3.20. Two hyperbolic Coxeter groups having the same Vinberg field, the same Vinberg ring and similar Vinberg forms do *not* have to be commensurable. For example, consider the two non-cocompact properly quasi-arithmetic Coxeter cube groups Γ_2 as above and Γ_3 in $\mathrm{Isom}(\mathcal{H}^3)$ defined by the graph in Fig. 4. Observe that for both groups, the Vinberg field is $\mathbb{Q}$ and the Vinberg ring is $\mathbb{Z}[1/2]$. Since Γ_2 and Γ_3 are quasi-arithmetic, we can apply Maclachlan's criterion to decide whether the Vinberg spaces (V_2, q_2) and (V_3, q_3) are similar (see Theorem 3.13).

With Vinberg's construction, we can compute the matrices representing q_1 and q_2 and diagonalise them over $\mathbb{Q}$. We obtain

$$q_2 = \mathrm{diag}(4, 3, 15, -15) \text{ and } q_3 = \mathrm{diag}\left(4, 3, \frac{-225}{4}, \frac{225}{4}\right).$$

The quadratic forms q_2 and q_3 have both $(-1, 3)$ as Hasse invariant and therefore they have identical Witt invariant represented by the quaternion algebra $B = (1, 1)$. Hence, the ramification set of $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-1})$ over $\mathbb{Q}(\sqrt{-1})$ is identical for both groups. This implies that the Vinberg spaces are similar. However, as shown in [33] by means of a geometric argument, Γ_2 and Γ_3 are *not* commensurable.

4. New generators for the Vinberg field

In this last section we provide two new sets of generators for the Vinberg field of a *quasi-arithmetic* hyperbolic Coxeter group. The first set is given by the coefficients of the characteristic polynomial of the Gram matrix, while the second is given by the coefficients of the characteristic polynomial of any Coxeter transformation of the Coxeter group. These sets can make the computation of the Vinberg field much more efficient in case of a group with a complicated Coxeter graph. Hopefully, these results can be extended to nq-arithmetic groups as well.

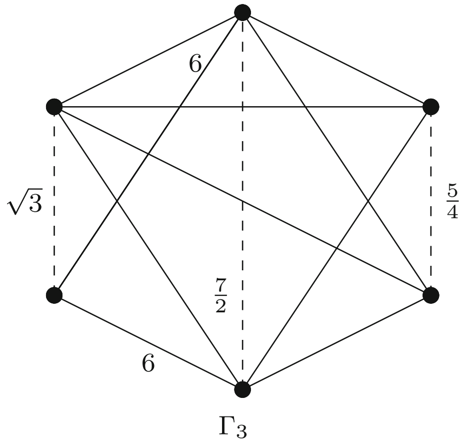

Fig. 4. Quasi-arithmetic Coxeter cube group Γ_3 acting on $\mathcal{H}^3$

4.1. The gram field of a hyperbolic Coxeter group

For $n \geq 2$, consider a Coxeter group Γ in $\mathrm{Isom}(\mathcal{H}^n)$ of rank N. Let G be its Gram matrix of signature $(n, 1)$ with characteristic polynomial $\chi_G(t) = a_0 + a_1 t + \cdots + a_N t^N$, $a_N = 1$. The matrix G is uniquely defined by Γ up to simultaneous permutation of its lines and columns which would yield a similar matrix G' with identical characteristic polynomial.

Notice that $a_{N-1} = (-1)^{N-1}\,\mathrm{Tr}(G) = (-1)^{N-1} N$. Moreover, each coefficient a_r of χ_G, $r < N$, can be expressed as the sum of all the principal minors of size $N-r$ (see [13, p. 53], for example). In particular, a_r vanishes for all $r < N-(n+1)$.

Definition 4.1. Let Γ be a hyperbolic Coxeter group of rank N. Let G be its Gram matrix with characteristic polynomial $\chi_G(t)$. The *Gram field* $K(G)$ is the field generated by the coefficients of $\chi_G(t)$ over $\mathbb{Q}$, namely

$$K(G) = \mathbb{Q}(a_j \mid 0 \leq j \leq N).$$

Proposition 4.2. *Let Γ be a cofinite quasi-arithmetic hyperbolic Coxeter group with Vinberg field K. Then*

$$K = K(G).$$

Proof. We prove first the inclusion $K \supseteq K(G)$. By [32, Proposition 11], the determinant of the Gram matrix G is given by a sum of cyclic products. The same result applies to every principal submatrix of G. Since the coefficients of χ_G can be expressed as the sum of principal minors of G (see [13, p. 53], for example), we get $K \supseteq K(G)$.

Assume that $K \supsetneq K(G)$. Then there exists a non-trivial embedding $\sigma : K \hookrightarrow \mathbb{R}$ which is the identity on $K(G)$. Let G^σ be the matrix obtained by applying σ to every coefficient of G and let $\chi_G = \sum_{i=0}^{N} a_i x^i$ be the characteristic polynomial of G.

Since σ is a field homomorphism, then $\chi_{G^\sigma} = \sum_{i=0}^{N} \sigma(a_i) x^i$. The embedding σ fixes the coefficients of χ_G, thus $\chi_G = \chi_{G^\sigma}$. In particular, G^σ has signature $(n, 1)$ and is not positive semidefinite. This is a contradiction to part $ii)$ of Theorem 2.2 and the claim follows. $\square$

4.2. The Coxeter field of a hyperbolic Coxeter group

Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$, $n \geq 2$, be a cofinite Coxeter group with natural set of generators $\{s_1, \ldots, s_N\}$. Consider a Coxeter transformation $C = s_1 \cdots s_N$ of Γ defined up to the ordering of the factors. With the real coefficients of the characteristic polynomial $\chi_C(t)$ we define a new field, the *Coxeter field*, and prove that it coincides with the Vinberg field $K(\Gamma)$ if Γ is quasi-arithmetic.

The proof is based on the work of Howlett [14] and the theory of M-matrices which we are going to review briefly.

Let $W = (W, S)$ be a Coxeter system with generating set $S = \{s_1, \ldots, s_N\}$ satisfying the relations of a Coxeter group. By Tits' theory, it is known that W can be represented as a subgroup of $\mathrm{GL}(V)$ for a real vector space V of dimension N equipped with a suitable symmetric bilinear form B (see [15], for example). Denote by $\mathrm{rad}(V) = \{v \in V \mid B(v, v') = 0 \quad \forall v' \in V\}$ the *radical* of B which will play a role later on. A *Coxeter element* $c \in W$ is the product of the N generators in S arranged in any order. The representative $C_T \in \mathrm{GL}(V)$ of c is called a *Coxeter transformation* of W.

For a Coxeter element $c = s_1 \cdots s_N$, the matrix of C_T with respect to a basis $\{v_1, \ldots, v_N\}$ of V, denoted again by C_T, can be written according to (see [14], for example)

$$C_T = -U^{-1}U^T, \tag{7}$$

where $U \in \mathrm{GL}(N, \mathbb{R})$ is the upper triangular matrix given by $[U]_{st} = 2B(v_s, v_t)$ for $t > s$, $[U]_{st} = 1$ on the main diagonal and $[U]_{st} = 0$ for $t < s$. Notice that

$$U + U^T = 2B. \tag{8}$$

By means of the theory of M-matrices, Howlett ([14, Theorem 4.1], see also [1]) characterised abstract Coxeter groups in terms of a Coxeter transformation C_T and its eigenvalues. More concretely, an *M-matrix* is a real matrix with non-positive off-diagonal entries all of whose principal minors are positive. For example, the matrix U described above is an M-matrix.

The proof of Howlett's Theorem 4.1 in [14] is based on the following results.

Lemma 4.3. ([14], Lemma 3.1) *Let U be a real matrix such that $U + U^T$ is positive definite. Then U is invertible and $-U^{-1}U^T$ is diagonalisable over $\mathbb{C}$ with all of its eigenvalues having modulus one.*

Lemma 4.4. ([14], Lemma 3.2 and Corollary 3.3) *Let U be an M-matrix such that $U+U^T$ is not positive definite. Then $-U^{-1}U^T$ has a real eigenvalue $\lambda \geq 1$. If $U+U^T$ is not positive semidefinite, then $\lambda > 1$. If $U+U^T$ is positive semidefinite, all the eigenvalues of $-U^{-1}U^T$ have modulus one and $-U^{-1}U^T$ is not diagonalisable.*

Later we will also need the following lemma, which is stated in Howlett's proof of Lemma 4.4.

Lemma 4.5. *Let U be an invertible real matrix such that $U+U^T$ is positive semidefinite. Then the eigenvalues of $-U^{-1}U^T$ have all modulus one.*

Proof. For $\epsilon > 0$ define the matrix $U^\epsilon := U + \epsilon I$. Since $U+U^T$ is positive semidefinite, $U^\epsilon + (U^\epsilon)^T$ is positive definite. By Lemma 4.3, all the eigenvalues of $-(U^\epsilon)^{-1}(U^\epsilon)^T$ have modulus one. The entries of U^ϵ depend continuously on ϵ. The same can be said for $-(U^\epsilon)^{-1}(U^\epsilon)^T$ and the coefficients of its characteristic polynomial. Hence the eigenvalues of $-(U^\epsilon)^{-1}(U^\epsilon)^T$ and their modulus depend continuously on ϵ, and the claim follows. □

Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)$ be a hyperbolic Coxeter group with generating reflections $s_1, \ldots, s_N$. In this way Γ represents a geometric realisation of an abstract Coxeter group. Let $P \in \mathcal{H}^n$ be its Coxeter polyhedron with outer unit normal vectors $e_1, \ldots, e_N$ and associated Gram matrix $G \in \mathrm{Mat}(N, \mathbb{R})$.

Let $C \in \Gamma$ be a Coxeter transformation of Γ. Our goal is to construct a new field $K(C)$ associated to C which we can identify later with the Vinberg field $K(\Gamma)$. Our motivation comes from [29, Theorem 1.8, (iv)], due to Reiner, Ripoll and Stump, relating Coxeter transformations of a finite complex reflection group to its field of definition[4] (see also Malle in [25, Section 7A]).

Inspired by this, we state the following definition.

Definition 4.6. Let Γ be a hyperbolic Coxeter group. Let $C \in \Gamma$ be a Coxeter transformation with characteristic polynomial $\chi_C(t) = a_0 + a_1 t + \cdots + a_{n+1}t^{n+1}$, $a_{n+1} = 1$. The *Coxeter field* $K(C)$ is the field generated by the coefficients of $\chi_C(t)$ over $\mathbb{Q}$, namely

$$K(C) = \mathbb{Q}(a_j \mid 0 \leq j \leq n+1).$$

It is not difficult to see that $\chi_C(t)$ is palindromic ($a_j = a_{n+1-j}$) if $N = n+1+2k$ and it is pseudo-palindromic ($a_j = -a_{n+1-j}$) if $N = n+1+(2k+1)$, for some $k \geq 0$.

Furthermore, $N-(n+1)$ is the dimension of the radical $\mathrm{rad}\left(\mathbb{R}^N\right)$ for the Tits representation space $\left(\mathbb{R}^N, G\right)$. Clearly, every element in Γ viewed in $\mathrm{GL}\left(\mathbb{R}^N\right)$ acts as the identity on $\mathrm{rad}\left(\mathbb{R}^N\right)$. Hence the same is true for every Coxeter transformation $C_T \in \mathrm{GL}\left(\mathbb{R}^N\right)$ of Γ. Since $\dim\left(\mathbb{R}^N / \mathrm{rad}\left(\mathbb{R}^N\right)\right) = n+1$, the characteristic polynomials χ_C and χ_{C_T} are related by

$$(t-1)^{(N-(n+1))}\chi_C(t) = \chi_{C_T}(t). \tag{9}$$

[4] In [29], the field of definition of a Coxeter group W is the field generated by all the traces of the matrices representing the elements of W.

In particular the field generated by the coefficients of χ_C and the field generated by the coefficients of χ_{C_T} coincide. With this preparation we are ready to prove the following result.

Proposition 4.7. *Let Γ be a cofinite quasi-arithmetic hyperbolic Coxeter group with Vinberg field K, and let C be any Coxeter transformation of Γ. Then*

$$K = K(C).$$

Proof. We first show that $K \supseteq K(C)$. The Vinberg field K is a field of definition (see Sect. 3.1). Thus, by means of a suitable basis, the Coxeter transformation C can be written as a matrix with coefficients in K. Since the characteristic polynomial is invariant under a basis change, we have that $K \supseteq K(C)$.

Assume that $K \supsetneq K(C)$. Then there exists a non-trivial embedding $\sigma : K \hookrightarrow \mathbb{R}$ that is the identity on $K(C)$. Consider the Coxeter transformation C_T acting on $\left(\mathbb{R}^N, G\right)$ which corresponds to C in the sense of Tits. By (7), we can express $C_T = -U^{-1}U^T$ where U is an M-matrix. As a consequence of Lemma 4.4, C_T has a real eigenvalue $\lambda > 1$.

Consider the matrices G^σ and U^σ, obtained by applying σ to the coefficients of the Gram matrix G of Γ and U. The matrix U^σ is invertible but in general not an M-matrix anymore since its off-diagonal entries may become positive. Define $C^\sigma := -(U^\sigma)^{-1}(U^\sigma)^T$. By (8), we have the equation $U^\sigma + (U^\sigma)^T = 2\,G^\sigma$. By part $ii)$ of Theorem 2.2, $U^\sigma + (U^\sigma)^T$ is therefore positive semidefinite. The embedding σ is a field homomorphism, thus the characteristic polynomial of C^σ is obtained by applying σ to the coefficients of the characteristic polynomial of C_T.

Since σ is the identity on $K(C)$, it leaves the characteristic polynomial χ_C invariant. The identity (9) then yields $\chi_{C_T} = \chi_{C^\sigma}$. This is contradiction, since C_T has an eigenvalue $\lambda > 1$ and since, by Lemma 4.5, all the eigenvalues of C^σ have modulus one. □

Acknowledgements The author would like to thank Prof. Dr. Ruth Kellerhals for the support, valuable discussions and helpful suggestions. This work was supported by the Swiss National Science Foundation, projects 200020_156104 and 200021_172583.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] A'Campo, N.: Sur les valeurs propres de la transformation de Coxeter. Invent. Math. **33**(1), 61–67 (1976)

[2] Alekseevskij, D., Vinberg, È., Solodovnikov, A.: Geometry of spaces of constant curvature. In: Geometry II- Spaces of constant curvature, pp. 1–138. Springer (1993)

[3] Bayer, E.: Lecture notes - Formes quadratiques sur des corps. Notes written by R. Guglielmetti, École polytechnique Fédérale de Lausanne (2011)

[4] Borel, A.: Density properties for certain subgroups of semi-simple groups without compact components. Ann. Math. **72**(1), 179–188 (1960)

[5] Burnside, W.: On the condition of reducibility of any group of linear substituions. Proc. Lond. Math. Soc. **2**(1), 430–434 (1905)

[6] Davis, E.: Overrings of commutative rings. II. Integrally closed overrings. Trans. Am. Math. Soc. **110**(2), 196–212 (1964)

[7] Deligne, P., Mostow, G.: Monodromy of hypergeometric functions and non-lattice integral monodromy. Publications Mathématiques de l'IHÉS **63**(1), 5–89 (1986)

[8] Dotti, E.: Groups of hyperbolic isometries and their commensurability. PhD thesis, Department of mathematics, University of Fribourg (2020)

[9] Everitt, B., Maclachlan, C.: Constructing hyperbolic manifolds. Computational and Geometric Aspects of Modern Algebra. Lond. Maths. Soc. Lect. Notes **275**, 78–86 (2000)

[10] Gilmer, R.: Integral domains with Noetherian subrings. Commentarii Mathematici Helvetici **45**(1), 129–134 (1970)

[11] Gromov, M., Piatetski-Shapiro, I.: Non-arithmetic groups in Lobachevsky spaces. Publications Mathématiques de l'Institut des Hautes Études Scientifiques **66**(1), 93–103 (1987)

[12] Guglielmetti, R., Jacquemet, M., Kellerhals, R.: Commensurability of hyperbolic Coxeter groups: theory and computation. RIMS Kôkyûroku Bessatsu **B66**, 057–113 (2017)

[13] Horn, R., Johnson, C.: Matrix Analysis. Cambridge University Press, Cambridge (2013)

[14] Howlett, R.: Coxeter groups and M-matrices. Bull. Lond. Math. Soc. **14**(2), 137–141 (1982)

[15] Humphreys, J.: Reflection groups and Coxeter groups, vol. 29. Cambridge University Press, Cambridge (1992)

[16] ImHof, H.: Napier cycles and hyperbolic Coxeter groups. Bull. Soc. Math. Belg. Sér. A **42**(3), 523–545 (1990)

[17] Jacquemet, M.: On hyperbolic Coxeter n-cubes. Eur. J. Comb. **59**, 192–203 (2017)

[18] Johnson, N., Kellerhals, R., Ratcliffe, J., Tschantz, S.: Commensurability classes of hyperbolic Coxeter groups. Linear Algebra Appl. **345**(1–3), 119–147 (2002)

[19] Kim, I.: Rigidity on symmetric spaces. Topology **43**(2), 393–405 (2004)

[20] Lam, T.: A theorem of Burnside on matrix rings. Am. Math. Mon. **105**(7), 651–653 (1998)

[21] Lam, T.: Introduction to Quadratic Forms Over Fields, vol. 67. American Mathematical Soc, New York (2005)

[22] Maclachlan, C.: Commensurability classes of discrete arithmetic hyperbolic groups. Groups Geom. Dyn. **5**(4), 767–785 (2011)

[23] Maclachlan, C., Reid, A.: Invariant trace-fields and quaternion algebras of polyhedral groups. J. Lond. Math. Soc. **58**(3), 709–722 (1998)

[24] Maclachlan, C., Reid, A.: The Arithmetic of Hyperbolic 3-Manifolds, vol. 219. Springer Science & Business Media, Berlin (2013)

[25] Malle, G.: On the rationality and fake degrees of characters of cyclotomic algebras. J. Math. Sci. Univ. Tokyo **6**(4), 647–678 (1999)

[26] Meyer, J.: Totally geodesic spectra of arithmetic hyperbolic spaces. Trans. Am. Math. Soc. **369**(11), 7549–7588 (2017)

[27] Mila, O.: Nonarithmetic hyperbolic manifolds and trace rings. Algeb. Geom. Topol. **18**(7), 4359–4373 (2018)
[28] Onishchik, A., Vinberg, È.: Lie groups and Lie algebras II: I. Discrete subgroups of Lie groups. In Encyclopaedia of Mathematical Sciences, vol. 21. Springer, Berlin (2000)
[29] Reiner, V., Ripoll, V., Stump, C.: On non-conjugate Coxeter elements in well-generated reflection groups. Math. Z. **285**(3–4), 1041–1062 (2017)
[30] Vinberg, È.: Discrete groups generated by reflections in Lobačevskiĭ spaces. Sbornik Math. **1**(3), 429–444 (1967)
[31] Vinberg, È.: Rings of definition of dense subgroups of semisimple linear groups. Math. USSR-Izvestiya **5**(1), 45 (1971)
[32] Vinberg, È.: The absence of crystallographic groups of reflections in Lobačevskiĭ spaces of large dimensions. Trans. Moscow Math. Soc. **47**, 75–112 (1985)
[33] Yoshida, H.: Commensurability of ideal hyperbolic Coxeter 3-cubic groups. Preprint (2017)