Canad. J. Math. Vol. 73 (2), 2021 pp. 465–492
http://dx.doi.org/10.4153/S0008414X20000036

Ideal Uniform Polyhedra in $\mathbb{H}^n$ and Covolumes of Higher Dimensional Modular Groups

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In memoriam Norman Johnson

Abstract. Higher dimensional analogues of the modular group $PSL(2,\mathbb{Z})$ are closely related to hyperbolic reflection groups and Coxeter polyhedra with big symmetry groups. In this context, we develop a theory and dissection properties of ideal hyperbolic k-rectified regular polyhedra, which is of independent interest. As an application, we can identify the covolumes of the quaternionic modular groups with certain explicit rational multiples of the Riemann zeta value $\zeta(3)$.

1 Introduction

First and eminent prototypes of arithmetic groups are the modular group $PSL(2,\mathbb{Z})$, the Eisenstein modular group $PSL(2,\mathbb{Z}[\omega])$, where $\omega = \frac{1}{2}(-1+i\sqrt{3})$ is a primitive third root of unity, and the Hamilton modular group $PSL(2,\mathbb{Z}[i,j])$. They act by orientation preserving isometries on real hyperbolic n-space for $n = 2, 3$, and 4, respectively, and they are isomorphic to finite index subgroups of discrete hyperbolic reflections groups (see [11, 13, 21]). In this way, their arithmetic, combinatorial, and geometric structure can be characterised by means of finite volume hyperbolic Coxeter polyhedra of particularly nice shape. Best known is the geometry of $PSL(2,\mathbb{Z})$, which is related to the Coxeter group with Coxeter symbol $[3,\infty]$ and an ideal hyperbolic triangle of angle $\frac{\pi}{3}$ and area π. The group $PSL(2,\mathbb{Z}[\omega]) \subset PSL(2,\mathbb{C})$ is a Bianchi group and isomorphic to a subgroup of index 4 in the hyperbolic Coxeter simplex group with Coxeter symbol $[3,3,6]$. In this way, the volume of a fundamental domain, or the *covolume* of $PSL(2,\mathbb{Z}[\omega])$, can be expressed by means of Humbert's volume formula for imaginary quadratic number fields [6] as well as by means of Lobachevsky's volume formula (see [27, part I, Chapter 7] and [22], for example) according to

$$\mathrm{covol}_3(PSL(2,\mathbb{Z}[\omega])) = \frac{3^{3/2}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-3})}(2) = \frac{1}{2}Л_2\Big(\frac{\pi}{3}\Big).$$

Here, $\zeta_k(s)$ denotes the Dedekind zeta function of the algebraic number field k, and $Л_2(x)$ is Lobachevsky's function (see Section 2.4).

Received by the editors June 14, 2019; revised December 30, 2019.
Published online on Cambridge Core January 27, 2020.
AMS subject classification: **51M10**, **20F55**, **51M25**, 11F06, 20G20, 11M06.
Keywords: Ideal hyperbolic polyhedron, k-rectification, Napier cycle, Coxeter group, quaternionic modular group, hyperbolic volume, zeta value.

The aim of this work is twofold. First, we study the geometry of the higher dimensional modular and pseudo-modular groups of 2×2 matrices whose coefficients form a *basic system of algebraic integers* in an associative normed real division algebra. In this way, they act by fractional linear transformations on the boundary $\mathbb{R}^{n-1} \cup \{\infty\}$ and—by Poincaré extension—as hyperbolic isometries in the upper half space U^n. Since the octonion multiplication is no longer associative, such groups can be realised by means of quaternionic matrices with unit Dieudonné determinant Δ and by certain Clifford matrices for $n \leq 5$ only. Of particular interest is the quaternionic modular group $PS_\Delta L(2, \mathbb{H}\mathrm{yb})$ with coefficients in the ring of *hybrid integers* $\mathbb{H}\mathrm{yb} = \mathbb{Z}[\omega, j]$. This group is closely related to the Coxeter pyramid group with Coxeter symbol $[6,3,3,3,3,6]$ whose explicit covolume computation, however, is very difficult.

In this context, we discovered a beautiful, highly symmetric, hyperbolic polyhedron, the *ideal birectified* 6*-cell* $r_2 S_{reg}$, which can be interpreted as a 5-dimensional analogue of the ideal regular tetrahedron S_{reg} (see Example 2.3). Based on this, and as our second and main achievement, we introduce and develop the theory of *ideal hyperbolic k-rectified regular polyhedra* $r_k P$ in hyperbolic n-space viewed in the projective model of Klein-Beltrami. Such a polyhedron is uniform; that is, its symmetry group acts transitively on the set of its vertices. For the important families of regular simplices $P = S_{reg}$ and regular orthoplexes (or cross-polytopes) $P = O_{reg}$, we derive explicit dissection relations by means of certain truncated characteristic simplices $\widehat{U}_i(\alpha_k^n)$ and $\widehat{V}_i(\alpha_k^n)$ characterised by a dihedral angle α_k^n depending on the dimension n and on the rectification degree k (for notation and the construction, see Section 3.2). Denote by $\delta_{ik} \in \{0,1\}$ the Kronecker-Delta function defined for elements i, k in an index set I. Then one of our main results can be stated as follows (see Theorem 3.6).

Theorem 1.1 *Let $n \geq 3$ and $1 \leq k \leq n-2$ be integers. For a regular polyhedron $P \subset \mathbb{E}^n$ with Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$, the ideal k-rectified regular n-polyhedron $\widehat{P} = r_k P \subset \mathbb{H}^n$ admits the following dissections.*

(i) *If P is a simplex S_{reg} with $p_1 = \cdots = p_{n-1} = 3$, then $r_k S_{reg}$ admits for $0 \leq i \leq n$ a dissection into $i!(n+1-i)!$ of $(2-\delta_{0i})$-truncated orthoschemes isometric to $\widehat{U}_i(\alpha_k^n)$, and each of these splits into $\binom{n+1}{i}$ simply-truncated orthoschemes isometric to $\widehat{U}_0(\alpha_k^n)$, where $\alpha_k^n = \arccos\sqrt{\frac{n-k}{2(n-k-1)}}$.*

(ii) *If P is an orthoplex O_{reg} with $p_1 = \cdots = p_{n-2} = 3$ and $p_{n-1} = 4$, then $r_k O_{reg}$ admits for $0 \leq i \leq n-1$ a dissection into $2^n i!(n-i)!$ of $(2-\delta_{0i})$-truncated orthoschemes isometric to $\widehat{V}_i(\alpha_k^n)$, and each of these splits into $\binom{n}{i}$ simply-truncated orthoschemes isometric to $\widehat{V}_0(\alpha_k^n)$, where $\alpha_k^n = \arccos\frac{1}{\sqrt{n-k-1}}$.*

Our proof is based on Debrunner's Theorem [4] and the theory of Napier cycles as introduced by Im Hof [9].

As a consequence, the ideal birectified 6-cell $r_2 S_{reg}$ admits a dissection into 6! isometric copies of the Coxeter prism $[6,3,3,3,3,6]$, which in turn is part of a crystallographic Napier cycle (see Section 2.3). This relationship allows us to determine the volume of $r_2 S_{reg}$ and the covolume of $PS_\Delta L(2, \mathbb{H}\mathrm{yb})$ as follows (see Theorem 4.1).

Theorem 1.2 *For the hybrid modular group $PS_\Delta L(2, \mathbb{H}\mathrm{yb}) = PS_\Delta L(2, \mathbb{Z}[\omega, j])$,*

$$\mathrm{covol}_5(PS_\Delta L(2, \mathbb{H}\mathrm{yb})) = \frac{13}{180}\zeta(3).$$

As a curious by-product, we obtain the following and seemingly new expression for $\zeta(3)$ by combining the two different volume representations for the Coxeter polyhedron $[6,3,3,3,3,6]$ (see Section 2, (2.11)):

$$\zeta(3) = \frac{360}{13}\left[\frac{\pi}{4}Л_2\left(\frac{\pi}{3}\right) + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left\{Л_2\left(\frac{\pi}{6} + \omega(t)\right) + Л_2\left(\frac{\pi}{6} - \omega(t)\right)\right\} dt\right], \quad \text{where}$$

$$\cos\omega(t) = \frac{\sin t}{\sqrt{4\sin^2 t - 1}}.$$

This work is organised as follows. In Section 2, we provide the basic concepts about hyperbolic polyhedra and Coxeter orthoschemes. We discuss Napier cycles and present some distinguished examples. At the end of the section, we supply the volume identities that will play a crucial role. In Section 3, we present Debrunner's classical dissection result for regular simplices and orthoplexes in a standard geometric space. Then we develop the theory of ideal hyperbolic k-rectified regular polyhedra and provide an interpretation by (polarly) truncated polyhedra. Our key dissection result as given by Theorem 1.1 can then be established. In the last part, in Section 4, we exploit the relation between certain quaternionic (pseudo)-modular groups and arithmetic hyperbolic Coxeter groups as described by Johnson [11]. By combining various of our results and applying them to the ideal birectified 6-cell, we will be able to establish our second main result as stated in Theorem 1.2. The work ends with the Remark 4.2 about the incommensurability of the modular 5-orbifolds given by $\mathbb{H}^5/PS_\Delta L(2, \mathbb{H}\mathrm{am})$ and $\mathbb{H}^5/PS_\Delta L(2, \mathbb{H}\mathrm{yb})$. In fact, there is no (orientable) hyperbolic 5-manifold that is a finite cover of both orbifolds.

2 Napier Cycles and Volumes in Hyperbolic 5-space

In this section, we present the necessary background about hyperbolic polyhedra and their description as fundamental polyhedra for discrete hyperbolic reflection groups. Of particular interest are regular polyhedra and their characteristic simplices. The truncation by (polar) hyperplanes will lead us to the notion of Napier cycles. Finally, some related volume formulas will be presented that will form a key ingredient in the proofs of our main results.

2.1 Hyperbolic Polyhedra and Coxeter Orthoschemes

Denote by $\mathbb{X}^n$ either the Euclidean space $\mathbb{E}^n$, the sphere $\mathbb{S}^n$, or the hyperbolic space $\mathbb{H}^n$, together with its isometry group $\mathrm{Isom}(\mathbb{X}^n)$. In the sequel, we will focus on the hyperbolic case assuming that the corresponding classical concepts in the euclidean-affine and spherical cases are well known.

The hyperbolic space $\mathbb{H}^n$ can be viewed in different ways, for example, by means of the Poincaré models in the upper half space U^n and in the unit ball B^n of $\mathbb{E}^n$, as well as the projective unit ball model K^n of Klein-Beltrami, closely related to the vector space model $\mathcal{H}^n$ in the space $\mathbb{E}^{n,1}$ of Lorentz–Minkowski.

For the description of polyhedral objects, the vector space model $\mathcal{H}^n$ (and its projective counterpart K^n) of hyperbolic space is most convenient (see [26, Chapter I], for example). More concretely, let $\mathcal{H}^n$ be defined in the quadratic space $\mathbb{E}^{n,1} = (\mathbb{R}^{n+1}, \langle x, y\rangle_{n,1} = \sum_{i=1}^n x_i y_i - x_{n+1}y_{n+1})$ of signature $(n,1)$, that is,

$$\mathcal{H}^n = \{x \in \mathbb{E}^{n,1} \mid \langle x, x\rangle_{n,1} = -1, x_{n+1} > 0\},$$

with distance function $d_{\mathcal{H}}(x, y) = \operatorname{arcosh}(-\langle x, y\rangle_{n,1})$. A non-zero vector $x \in \mathbb{E}^{n,1}$ is *time-like, light-like*, or *space-like*, if its square Lorentzian norm is negative, zero, or positive, respectively. The boundary $\partial\mathcal{H}^n$ can be identified with the set of light-like vectors $x \in \mathbb{E}^{n,1} \cap \mathbb{S}^n$ on the unit sphere such that $x_{n+1} > 0$. Furthermore, for an integer $1 \le k \le n-1$, a hyperbolic k-plane in $\mathcal{H}^n$ is the non-empty intersection of a $(k+1)$-dimensional subspace of $\mathbb{R}^{n+1}$ with $\mathcal{H}^n$. Of importance will be the fact that the Lorentz-orthogonal complement of a hyperbolic hyperplane is generated by a space-like vector. Let us add that the group $\mathrm{Isom}(\mathcal{H}^n)$ coincides with the group $PO(n,1)$ of positive Lorentzian matrices. For more details, see [23, Chapter 3] and [26].

Now, any discrete subgroup of $\mathrm{Isom}(\mathcal{H}^n)$ has a convex fundamental domain in $\mathcal{H}^n$ whose closure can be assumed to be polyhedral. Of particular interest will be discrete groups generated by finitely many reflections in hyperplanes of $\mathcal{H}^n$. For their description, we rely on the combinatorics and geometry of their fundamental polyhedra.

To this end, represent a hyperplane $H \subset \mathcal{H}^n$ by a unit normal vector, that is, by a space-like vector $e \in \mathbb{E}^{n,1}$ of Lorentzian norm 1 such that

$$H = \{x \in \mathcal{H}^n \mid \langle x, e\rangle_{n,1} = 0\},$$

and which bounds two closed half-spaces, for example,

$$H^- = \{x \in \mathcal{H}^n \mid \langle x, e_i\rangle_{n,1} \le 0\}.$$

A (convex, closed, and indecomposable) polyhedron $P \subset \mathcal{H}^n$ is of the form

$$P = \bigcap_{i \in I} H_i^-,$$

where the index set I is always supposed to be finite. A polyhedron P is compact (or of finite volume) if P is the convex hull of finitely many points of $\mathcal{H}^n$, called the *vertices* of P (or of $\mathcal{H}^n \cup \partial\mathcal{H}^n$). The polyhedron P is *ideal* if all vertices belong to $\partial\mathcal{H}^n$. In the sequel, we will consider acute-angled polyhedra, that is, polyhedra with (interior) dihedral angles $\measuredangle(H_i, H_j) \le \pi/2$ for $i, j \in I$. Consider the Gram matrix $G(P)$ of P formed by the products $\langle e_i, e_j\rangle_{n,1}$, $i, j \in I$. It is known that an indecomposable real symmetric $N \times N$ matrix A with diagonal elements $[A]_{ii} = 1$ and non-diagonal elements $[A]_{ij} \le 0$, $i \ne j$, is the Gram matrix $G(P)$ of an acute-angled polyhedron $P \subset \mathcal{H}^n$ (uniquely determined up an isometry) if and only if the signature of A equals $(n,1)$. Similar results for acute-angled polyhedra in $\mathbb{E}^n$ and $\mathbb{S}^n$ are well known (see [27, Part I, Chapter 6]). Furthermore, many of the combinatorial, metrical, and arithmetic properties of P can be read off from $G(P)$. In particular, for $i \ne j$, the coefficients $\langle e_i, e_j\rangle_{n,1}$ characterise the mutual position of the hyperplanes H_i, H_j as follows:

$$-\langle e_i, e_j\rangle_{n,1} = \begin{cases} \cos\alpha_{ij} & \text{if } H_i, H_j \text{ intersect at the angle } \alpha_{ij} \text{ in } \mathcal{H}^n, \\ 1 & \text{if } H_i, H_j \text{ meet at } \partial\mathcal{H}^n, \\ \cosh l_{ij} & \text{if } H_i, H_j \text{ are at distance } l_{ij} \text{ in } \mathcal{H}^n. \end{cases} \tag{2.1}$$

For more details, see Vinberg's seminal work [26], [27, Part I, Chapter 6].

In case of many orthogonal bounding hyperplanes and small $|I|$, it is convenient to represent a given polyhedron $P \subset \mathcal{H}^n$ in terms of a (weighted) graph $\Sigma = \Sigma(P)$ of order $|I|$. With each bounding hyperplane H with normal vector $e \in \mathbb{E}^{n,1}$ directed outwards with respect to P, we associate a node v in Σ. Two different nodes v_i, v_j are connected by an edge with a weight c_{ij} if the hyperplanes H_i, H_j are not orthogonal in $\mathcal{H}^n$. The weight c_{ij} is given by $\langle e_i, e_j\rangle_{n,1}$. However, if $-1 < c_{ij} < 0$, then $c_{ij} = -\cos\alpha_{ij}$, and we replace c_{ij} by α_{ij}. An edge with weight -1 will be decorated by the symbol ∞. Edges with weights $|c_{ij}| > 1$ are replaced by a dashed edge, and the weights are omitted in most cases.

More specifically, if P is a *Coxeter polyhedron* having by definition dihedral angles of the form $\alpha_{ij} = \pi/m_{ij}$ for integers $m_{ij} \geq 2$, only, the corresponding weights traditionally carry only the label $m_{ij} > 3$. Hence, simple edges indicate an intersection angle equal to $\pi/3$, and nodes not connected by an edge symbolise orthogonal hyperplanes. In order to depict a Coxeter graph in an abbreviated way, we often use the *Coxeter symbol*. In particular, $[p_1, \ldots, p_k]$ or $[q_1, \ldots, q_l, \infty]$ with integer labels $p_i, q_j \geq 3$ are associated with linear Coxeter graphs with $k+1$ or $l+2$ edges marked by the respective weights. The Coxeter symbol $[(p,q)^{[r]}]$ describes a group with cyclic Coxeter graph consisting of $r \geq 2$ consecutive Coxeter graphs $[p,q]$ (see [12, Appendix], for example). In the sequel, we often represent a Coxeter polyhedron by quoting its Coxeter symbol.

Recall that the reflections with respect to the bounding hyperplanes of a Coxeter polyhedron $P \subset \mathcal{H}^n$ generate a discrete group $\Gamma_P \subset \mathrm{Isom}(\mathcal{H}^n)$ that is called a *hyperbolic Coxeter group*. The group Γ_P is denoted by the Coxeter graph and Coxeter symbol of P, and we do not distinguish between Coxeter polyhedron and reflection group. Notice that in contrast to the hyperbolic ones, the irreducible spherical (finite) and euclidean (or affine) Coxeter groups are completely classified. For a list, see [8, Chapter 2] or [27, Chapter 5].

A polyhedron $R \subset \mathbb{X}^n$ with linear graph $\Sigma_n = \Sigma_n(\alpha_1, \ldots, \alpha_n)$ of order $n+1 = |I| \geq 2$ given by Figure 1:

Σ_n :

Figure 1: The graph of an n-orthoscheme $R \subset \mathbb{X}^n$.

is a geometric n-simplex whose bounding hyperplanes are indexed by $H_i, 0 \leq i \leq n$, in such a way that $H_i \perp H_j$ for $|i-j| > 1$. The polyhedron R is parametrised by the dihedral angles $\alpha_i = \measuredangle(H_{i-1}, H_i), 1 \leq i \leq n$. These simplices are called *orthoschemes* and were introduced by L. Schläfli [25] in the spherical case. They play an important role when studying polyhedra with high degree of symmetry and provide the *characteristic simplices* in the barycentric decomposition of regular polyhedra in $\mathbb{X}^n$ (see Section 3.1, [3, Section 7.9], and [27, Part II, Chapter 5]).

Consider the special case of a graph $\Sigma_n(\alpha,\beta)$ as given in Figure 2, where the parameters $\alpha,\beta \in [0,\pi/2)$ are such that the graph $\Sigma_n(\alpha,\beta)$ relates to a polyhedron $R(\alpha,\beta) \subset \mathbb{X}^n$ (possibly of infinite volume).

$\Sigma_n(\alpha,\beta)$:

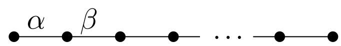


Figure 2: Graphs related to certain characteristic n-orthoschemes.

- The graph $\Sigma_n(\alpha,\frac{\pi}{3})$ describes the characteristic (or barycentric) simplex of a finite volume regular n-simplex $S_{reg}(2\alpha) \subset \mathbb{X}^n$ with dihedral angle 2α, which is spherical, euclidean, or hyperbolic if $-1 < \cos(2\alpha) < \frac{1}{n}$, $\cos(2\alpha) = \frac{1}{n}$, or $\frac{1}{n} < \cos(2\alpha) \le \frac{1}{n-1}$, respectively (see [17], for example). Indeed, by barycentric decomposition from its in-center, $S_{reg}(2\alpha)$ can be decomposed into $(n+1)!$ isometric copies of $R(\alpha,\frac{\pi}{3})$. Observe that finite volume regular simplices tesselating hyperbolic space $\mathcal{H}^n$ exist only for $n = 2, 3$, and 4 (see [27, Part II, Chapter 5, Section 3]). A particular role is played by the non-compact, finite volume orthoscheme $R(\frac{\pi}{6},\frac{\pi}{3})$ with Coxeter symbol $[6,3,3]$, which is the characteristic simplex of an ideal regular hyperbolic tetrahedron or *4-cell* $S_{reg}(\frac{\pi}{3}) \subset \mathcal{H}^3$.
- The graph $\Sigma_n(\alpha,\frac{\pi}{4})$ arises with respect to the barycentric decomposition into $2^n n!$ isometric copies of the characteristic simplex of an n-dimensional regular cross-polytope or *n-orthoplex* $O_{reg}(2\alpha)$ (in the terminology of J. Conway) with dihedral angle 2α, which is hyperbolic of finite volume if $\frac{1}{\sqrt{n}} < \cos\alpha \le \frac{1}{\sqrt{n-1}}$. In particular, the Coxeter orthoscheme $[4,4,3]$ is related to an ideal regular octahedron $O_{reg}(\frac{\pi}{2}) \subset \mathcal{H}^3$. The graph $\Sigma_n(\alpha,\frac{\pi}{4})$ (read from right to left in Figure 2) describes the dual polyhedron of $O_{reg}(2\alpha)$, that is, a regular n-cube $T_{reg}(2\alpha) \subset \mathbb{X}^n$ with dihedral angle 2α. Of course, the polyhedron $T_{reg}(2\alpha)$ exists with finite volume in $\mathcal{H}^n$ under the identical angle constraint.

2.2 Napier Cycles

An n-orthoscheme $R \subset \mathcal{H}^n$ as given by the graph in Figure 1 is characterised by $n+1$ outer (unit) normal vectors $e_0,\dots,e_n \in \mathbb{E}^{n,1}$, forming a basis of $\mathbb{E}^{n,1}$ and satisfying $-1 < \langle e_{i-1}, e_i\rangle_{n,1} < 0$ and $\langle e_i, e_j\rangle_{n,1} = 0$ for $i \neq j$. In this respect, an orthoscheme is a part of and generates a so-called Napier cycle as introduced by Im Hof [9].

Definition 2.1 A *Napier cycle* in $\mathbb{E}^{n,1}$ is a set $\mathcal{N} = \{e_i \in \mathbb{E}^{n,1} \mid i \in \mathbb{Z}/(n+3)\}$ of $n+3$ vectors subject to the conditions

$$(2.2)\qquad \begin{aligned} c_i &:= \langle e_{i-1}, e_i\rangle_{n,1} < 0 && \text{for all } i,\\ \langle e_i, e_j\rangle_{n,1} &= 0 && \text{for } j \neq i-1, i, i+1. \end{aligned}$$

Any $n+1$ consecutive vectors in a Napier cycle $\mathcal{N}$ form a basis of $\mathbb{E}^{n,1}$. Furthermore, the deletion of two non-adjacent vectors from $\mathcal{N}$ defines two Lorentz-orthogonal subspaces of $\mathbb{E}^{n,1}$. Either these subspaces are both light-like, or one of them is space-like, while the other one is time-like (or negative). A Napier cycle $\mathcal{N}$ can be of three different types. Either all vectors of $\mathcal{N}$ are space-like, or exactly one vector is not space-like, or exactly two vectors are not space-like. For $n \geq 4$, the vectors in $\mathcal{N}$ admit a numbering and can be normalised in such a way that, for $0 \leq i \leq n$, $\langle e_i, e_i \rangle_{n,1} = 1$ with $-1 < c_i = \langle e_{i-1}, e_i \rangle_{n,1} < 0$. Consider such a normalised Napier cycle $\mathcal{N}$ with given negative numbers $c_1, \dots, c_n$. The set $\mathcal{N}$ is of *type* d if it contains precisely $n+d$ vectors of positive Lorentzian norm, equal to 1, say, whose non-vanishing Lorentzian products c_i admit the interpretation according to (2.1). Moreover, the respective additional products $c_{n+1} = \langle e_n, e_{n+1} \rangle_{n,1}$, $c_{n+2} = \langle e_{n+1}, e_{n+2} \rangle_{n,1}$ and $c_0 = \langle e_{n+2}, e_0 \rangle_{n,1}$ can be easily computed by means of a suitable Gram determinant calculation in terms of the values $c_1, \dots, c_n$. In fact, let $\delta(c_1, \dots, c_n)$ be the determinant of the Gram matrix

$$G(e_0, \dots, e_n) = \begin{pmatrix} 1 & c_1 & & & \\ c_1 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1 & c_n \\ & & & c_n & 1 \end{pmatrix}$$

of the vectors $e_0, \dots, e_n$, which satisfies nice recursion formulas. In particular, one easily verifies that

$$\delta(c_1, \dots, c_n) = \frac{n+1}{2^n}, \quad \delta\Big(-\frac{1}{\sqrt{2}}, c_2, \dots, c_n\Big) = \frac{1}{2^{n-1}} \quad \text{if all} \quad c_i = -\frac{1}{2}. \tag{2.3}$$

In the same spirit, by [9, Proposition 1.6] (or [25, Section 27]), one can show that

$$\begin{aligned} c_0^2 &= \frac{\delta(c_1, \dots, c_n)}{\delta(c_2, \dots, c_n)}, \quad c_{n+1}^2 = \frac{\delta(c_1, \dots, c_n)}{\delta(c_1, \dots, c_{n-1})}, \\ 1 - c_{n+2}^2 &= \frac{\delta(c_1, \dots, c_n)\delta(c_2, \dots, c_{n-1})}{\delta(c_1, \dots, c_{n-1})\delta(c_2, \dots, c_n)}. \end{aligned} \tag{2.4}$$

Now, since the Lorentzian orthogonal complement of each unit space-like vector e in $\mathcal{N}$ defines a hyperplane H bounding two half-spaces in $\mathcal{H}^n$, it is not difficult to understand the polyhedral configuration provided by a Napier cycle of type d. In fact, type 1 Napier cycles correspond to orthoschemes in $\mathcal{H}^n$. Type 2 Napier cycles (with e_{n+2} not space-like, say) are infinite volume orthoschemes bounded by $H_0, \dots, H_n$ and truncated by H_{n+1} to yield finite volume, *simply-truncated* orthoschemes in $\mathcal{H}^n$. In a similar way, Napier cycles of type 3 are finite volume, *doubly-truncated* orthoschemes bounded by hyperplanes $H_0, \dots, H_{n+2}$ in $\mathcal{H}^n$. They arise from infinite volume orthoschemes bounded by $n+1$ hyperbolic hyperplanes by truncation by means of the two remaining ones. For details, see [9].

2.3 Crystallographic Napier Cycles

Consider a Napier cycle of type d in $\mathbb{E}^{n,1}$ and assume that it yields a $(d-1)$-truncated Coxeter orthoscheme in $\mathcal{H}^n$. This means that for each weight c_i with $-1 < c_i < 0$, one has that $c_i = -\cos\frac{\pi}{m_i}$ for some integer $m_i \geq 3$. In [9], Im Hof completely classified these particular Coxeter polyhedra, referring to them as *crystallographic* Napier cycles, and showed that they exist in $\mathcal{H}^n$ for $n \leq 9$, only. In the sequel, the following examples will be of particular interest.

Example 2.2 The Coxeter polyhedron $[6,3,3,3,\infty]$ in $\mathcal{H}^4$ with graph given by Figure 3 gives rise to a crystallographic Napier cycle of type 2 (here with e_6 not space-like) and describes a simply-truncated Coxeter orthoscheme. Underlying is the infinite volume orthoscheme $R_0 = [6,3,3,3]$ bounded by the hyperplanes $H_0,\dots,H_4$ according to the graph in Figure 3. Denote by p_k the vertices of R_0 opposite to the bounding hyperplane H_k for $0 \leq k \leq 5$. By construction, they form an orthogonal edge path $p_0p_1,\dots,p_4p_5$ in $\mathbb{E}^{4,1}$. The polyhedron R_0 can be interpreted as the characteristic simplex (see Section 3.1) of an infinite volume regular hyperbolic 4-simplex $S_{reg}(\frac{\pi}{3})$ with in-center equal to p_0 and all of whose vertices $v_0,\dots,v_4$ are given by space-like vectors. In particular, one vertex of $S_{reg}(\frac{\pi}{3})$ corresponds to p_4 whose neighborhood in R_0 is a cone over the hyperbolic Coxeter tetrahedron $[6,3,3]$. The truncating hyperplane H_5 intersects H_4 at the point p_3 on the boundary $\partial\mathcal{H}^4$, indicated by a circle in Figure 3. In this way, the polyhedron $[6,3,3,3,\infty]$ is a Coxeter pyramid with apex at infinity p_3 over a product of two (euclidean) Coxeter simplices with symbols $[6,3]$ and $[\infty]$ (see [7], for example). In particular, all edges $\overline{v_iv_j}$ of $S_{reg}(\frac{\pi}{3})$ are bisected at an ideal point, denoted by q_{ij}, on $\partial\mathcal{H}^4$. The convex hull of q_{ij}, $0 \leq i < j \leq 4$, gives rise to an ideal polyhedron of finite volume in $\partial\mathcal{H}^4$, with dihedral angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$, called the *ideal rectified 5-cell*, denoted by r_1S_{reg} (see Section 3.2).

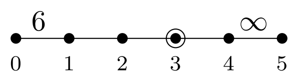


Figure 3: The 4-dimensional finite volume analogue of $[6,3,3]$.

Example 2.3 Consider the Coxeter polyhedron $[6,3,3,3,3,6]$ in $\mathcal{H}^5$ given by Figure 4. It is a simply-truncated Coxeter orthoscheme which belongs to a crystallographic Napier cycle of type 2. The Coxeter orthoscheme $R_0 = [6,3,3,3,3]$ bounded by the hyperplanes $H_0,\dots,H_5$ and with vertices $p_0,\dots,p_5$ opposite to them is of infinite volume. The vertices p_4, p_5 of R_0 are given by space-like vectors described by principal submatrices of $G(R_0)$ of signature $(4,1)$, while the vertex p_3 is given by a light-like vector. The neighborhood of p_3 is a cone over the product of two (euclidean) Coxeter triangles with symbol $[6,3]$ as indicated by a circle in Figure 4. The polyhedron R_0 is the characteristic simplex arising in the barycentric decomposition of an infinite volume hyperbolic regular 5-simplex $S_{reg}(\frac{\pi}{3})$ with in-center p_0. All vertices of $S_{reg}(\frac{\pi}{3})$ are given by space-like vectors $v_0,\dots,v_5$ such that they span space-like planes $\mathrm{span}(v_i,v_j)$ in $\mathbb{E}^{5,1}$. By applying Vinberg's theory about the face complex of acute-angled hyperbolic polyhedra, the graph in Figure 4 indicates that the barycenter of each 2-face (the vertex p_3, for example) of $S_{reg}(\frac{\pi}{3})$ is a point on the boundary $\partial\mathcal{H}^5$. Taking the convex hull of all these ideal barycenters yields a finite volume

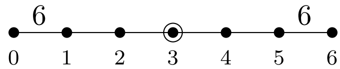


Figure 4: The 5-dimensional finite volume analogue of $[6,3,3]$.

hyperbolic polyhedron, of dihedral angles equal to $\frac{\pi}{3}$ and $\frac{\pi}{2}$, which is called the *ideal birectified 6-cell* and denoted by $r_2 S_{reg} \subset \mathcal{H}^5$ (see Section 3.2).

Example 2.4 Similarly to Example 2.2, the Coxeter pyramid $[4,4,3,3,\infty]$ in $\mathcal{H}^4$ associated with an infinite volume regular 4-orthoplex $O_{reg}(\frac{\pi}{2})$ describes a simply-truncated orthoscheme that decomposes the *ideal rectified 4-orthoplex* $r_1 O_{reg} \subset \mathcal{H}^4$ with all dihedral angles equal to $\frac{\pi}{2}$ into 384 isometric copies (see Section 3.2).

Similarly to Example 2.3, the Coxeter polyhedron $[4,4,3,3,3,6]$ in $\mathcal{H}^5$ associated with an infinite volume regular 5-orthoplex $O_{reg}(\frac{\pi}{2})$ describes a simply-truncated orthoscheme which decomposes the *ideal birectified 5-orthoplex* $r_2 O_{reg} \subset \mathcal{H}^5$ with dihedral angles $\frac{\pi}{2}$ and $\frac{\pi}{3}$ into 3,840 isometric copies (see Section 2.4, Remark 2.7, and Section 3.2).

Example 2.5 The Coxeter polyhedron $[3,4,3^5,6]$ in $\mathcal{H}^7$ is a simply-truncated orthoscheme that decomposes barycentrically the *ideal birectified 7-orthoplex* $r_2 O_{reg} \subset \mathcal{H}^7$ with dihedral angles $\frac{2\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ into 645,120 isometric copies (see Section 3.2 with Corollary 3.7).

2.4 Some Volume Identities

Consider a normalised Napier cycle $\mathcal{N} = \{e_i \in \mathbb{E}^{n,1} \mid i \in \mathbb{Z}/(n+3)\}$ of type d, $1 \le d \le 3$, in $\mathbb{E}^{n,1}$ such that $-1 < c_i = \langle e_{i-1}, e_i \rangle_{n,1} < 0$ for $1 \le i \le n$. As described in Section 2.2, $\mathcal{N}$ yields a k-truncated orthoscheme R_k in $\mathcal{H}^n$ for $k = d-1$. We are interested in finding explicit volume expressions for crystallographic Napier cycles containing the building blocks of tesselating k-rectified regular polyhedra in $\mathcal{H}^n$ (see Section 3.2). By [14, 15], we dispose of closed volume formulae available for dimension $n = 3$ and for even dimensions.

- For $n = 3$, the volume of any k-truncated orthoscheme can be expressed in terms of its dihedral angles and by means of the Lobachevsky function $Л_2(x)$, which is defined as follows (see [14]):

$$Л_2(x) = \frac{1}{2}\sum_{r=1}^{\infty}\frac{\sin(2rx)}{r^2} = -\int_0^x \log|2\sin t|dt, \quad x \in \mathbb{R}.$$

For example, by the classical Lobachevsky formula [14, (2)], the volume of a non-compact (0-truncated) orthoscheme $R \subset \mathbb{H}^3$ with linear graph $\Sigma_3(\frac{\pi}{2}-\alpha,\alpha,\beta)$ (see Figure 1) is given by the expression

$$\mathrm{vol}_3(R) = \frac{1}{2}Л_2(\alpha) + \frac{1}{4}\Big\{Л_2\Big(\frac{\pi}{2}-\alpha+\beta\Big) - Л_2\Big(\frac{\pi}{2}+\alpha+\beta\Big)\Big\}.$$

In particular, the volume of the Coxeter orthoscheme $[6,3,3]$ equals $\frac{1}{8}Л_2(\frac{\pi}{3})$ so that $\mathrm{vol}_3(S_{reg}(\frac{\pi}{3})) = 3\,Л_2(\frac{\pi}{3})$.

- For $n = 2m$ even, the volume of a k-truncated hyperbolic n-orthoscheme R_k with graph $\Sigma_k = \Sigma(R_k)$ can be expressed in terms of certain real-valued modified volume functions F_n and f_l defined on (the set of measurable subsets of) $\mathbb{H}^n$ and $\mathbb{S}^l$, respectively, in a more elegant way. More concretely, let $f_0 = F_0 = 1$, and define for $l = 1, \dots, n-1$, $n \geq 2$, the volume functions

$$f_l := \nu_l \ \mathrm{vol}_l, \quad F_n := i^n \nu_n \, \mathrm{vol}_n \quad \text{with} \quad i^2 = -1, \quad \nu_l = \frac{2^{l+1}}{\mathrm{vol}_l(\mathbb{S}^l)}.$$

Then the so-called Reduction Formula as presented in [15, Section 3] allows one to express the volume of a k-truncated n-orthoscheme R_k with graph Σ_k in terms of its dihedral angles as follows:

$$\text{(2.5)} \qquad F_{2m}(\Sigma_k) = \sum_{r=0}^{m} \frac{(-1)^r}{r+1} \binom{2r}{r} \sum_{\sigma} f_{2m-(2r+1)}(\sigma), \quad \sum_{\sigma} f_{-1} := 1.$$

Here, σ runs through all *spherical* subgraphs of order $2(m-r)$ of Σ_k all of whose connected components are of *even* order. Observe that, for $m > r$, each such component describes a spherical orthoscheme of odd dimension $< 2(m-r)$. By means of formula (2.5) for $n = 2m$ and by Schläfli's results about the order of a finite Coxeter group providing the values f_l for $l \leq m-1$ (see [25, No. 23, p. 268 ff]), the volumes of all k-truncated Coxeter n-orthoschemes were determined in [15, Appendix] (up to some minor calculation errors). In particular, for $n = 4$, the simply-truncated Coxeter orthoscheme $[6, 3, 3, 3, \infty]$ is of volume $\pi^2/540$, while for $n = 6$, the simply-truncated Coxeter orthoscheme $[3, 4, 3, 3, 3, 3, \infty]$ is of volume $\pi^3/259,200$.

In the sequel, we are particularly interested in the volume computation for those polyhedra in $\mathbb{H}^5$ that are related to orbit spaces by certain quaternionic modular groups. In general, it is very difficult to find a closed volume formula for a family of polyhedra of fixed combinatorial-metrical type in $\mathcal{H}^5$. However, there are some partial but very useful results for k-truncated 5-orthoschemes. For $k = 0$, consider a 5-orthoscheme

$$\text{(2.6)} \qquad R(\alpha, \beta, \gamma) = \bigcap_{0 \leq i \leq 5} H_i^- \quad \text{such that} \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

which is defined by the graph in Figure 5.

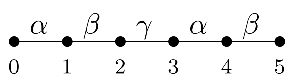


Figure 5: The orthoscheme $R(\alpha, \beta, \gamma) \subset \mathcal{H}^5$ with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

The angle condition in (2.6) ensures that the vertices p_0 and p_5 of $R(\alpha, \beta, \gamma)$ opposite to the bounding hyperplanes H_0 and H_5 are ideal points. In [16, Theorem, p. 659], we obtained the following volume formula:

$$\text{(2.7)} \qquad \mathrm{vol}_5\big(R(\alpha, \beta, \gamma)\big) = \frac{1}{4}\Big\{\text{Л}_3(\alpha) + \text{Л}_3(\beta) - \frac{1}{2}\text{Л}_3\Big(\frac{\pi}{2} - \gamma\Big)\Big\} - \frac{1}{16}\Big\{\text{Л}_3\Big(\frac{\pi}{2} + \alpha + \beta\Big) + \text{Л}_3\Big(\frac{\pi}{2} - \alpha + \beta\Big)\Big\} + \frac{3}{64}\zeta(3).$$

Here, the Lobachevsky function of order three $Л_3(\omega)$, $\omega \in \mathbb{R}$, is related to the classical trilogarithm function $\mathrm{Li}_3(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^3}$, $|z| \le 1$, satisfying $\mathrm{Li}_3(1) = \zeta(3)$, as follows:

$$Л_3(x) = \frac{1}{4}\Re\left(\mathrm{Li}_3(e^{2ix})\right) = \frac{1}{4}\sum_{r=1}^{\infty}\frac{\cos(2rx)}{r^3}, \quad x \in \mathbb{R}. \tag{2.8}$$

The function $Л_3(x)$ is even, π-periodic, and fulfils the distribution law

$$\frac{1}{m^2}Л_3(mx) = \sum_{r=0}^{m-1} Л_3\left(x + \frac{r\pi}{m}\right), \quad \text{where} \quad x \in \mathbb{R} \quad \text{and} \quad m \in \mathbb{Z}_{>0}, \tag{2.9}$$

which allows one to identify some special values of $Л_3(x)$ with certain rational multiples of $\zeta(3)$ (see [16, Section 2.3]).

As a consequence of the identities (2.7), (2.8), and (2.9), the Coxeter orthoscheme $R(\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3})$ in $\mathcal{H}^5$ has volume

$$\mathrm{vol}_5([3,4,3,3,4]) = \frac{7}{4,608}\zeta(3). \tag{2.10}$$

However, the volume computation for the simply-truncated Coxeter orthoscheme $[6,3,3,3,3,6]$ in $\mathcal{H}^5$ discussed in Example 2.3 cannot be performed in an exact manner by exploiting results in the spirit of (2.7). Nevertheless, based on Schläfli's volume differential formula (see [18, Section 2.1], for example), its volume can be represented by a single integral as follows. Denote by $R(\omega) \subset \mathcal{H}^3$ a non-compact orthoscheme with dihedral angles $\omega, \frac{\pi}{3}, \frac{\pi}{6}$ (see Figure 1). As a function of ω, its volume is given by the classical Lobachevsky formula [14, (2)] according to

$$\mathrm{vol}_3(R(\omega)) = \frac{1}{2}Л_2\left(\frac{\pi}{3}\right) + \frac{1}{4}\left\{Л_2\left(\frac{\pi}{6} + \omega\right) + Л_2\left(\frac{\pi}{6} - \omega\right)\right\}.$$

With these preparations, the volume of $[6,3,3,3,3,6]$ can be written as follows:

$$\begin{aligned} \mathrm{vol}_5([6,3,3,3,3,6]) &= \frac{1}{4}\left[\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \mathrm{vol}_3(R(\omega(t)))dt - \frac{1}{48}Л_2\left(\frac{\pi}{3}\right)\right] \\ &= \frac{1}{16}\left[\frac{\pi}{4}Л_2\left(\frac{\pi}{3}\right) + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left\{Л_2\left(\frac{\pi}{6} + \omega(t)\right) + Л_2\left(\frac{\pi}{6} - \omega(t)\right)\right\}dt\right] \\ &\simeq 0.0027129757, \quad \text{where } \cos\omega(t) = \frac{\sin t}{\sqrt{4\sin^2 t - 1}}. \end{aligned} \tag{2.11}$$

Remark 2.6 By a structural result of Prasad (see [5, Proposition 2.1 (1)], for example), the volume of the non-compact arithmetic orbifold defined over $\mathbb{Q}$ as given by the orbit space of $[6,3,3,3,3,6]$ is a rational multiple of $\zeta(3)$. Now, numerical evidence suggests that the value (2.11) for the volume of $[6,3,3,3,3,6]$ is equal to $\frac{13}{5{,}760}\zeta(3)$. By means of a combinatorial-metrical argument, we will prove rigorously the conjectural identity $\mathrm{vol}_5([6,3,3,3,3,6]) \stackrel{?}{=} \frac{13}{5{,}760}\zeta(3)$ (see Theorem 4.1).

Remark 2.7 In a similar way, the volume of the simply-truncated Coxeter orthoscheme $[6,3,3,3,4,4]$ in $\mathcal{H}^5$ can be identified as follows:

$$\mathrm{vol}_5\left([6,3,3,3,4,4]\right) = \frac{1}{16}\left[\frac{3\pi}{8}Л_2\left(\frac{\pi}{3}\right) + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left\{Л_2\left(\frac{\pi}{6}+\omega(t)\right) + Л_2\left(\frac{\pi}{6}-\omega(t)\right)\right\}dt\right].$$

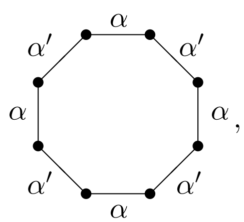
In [24], Ratcliffe and Tschantz determined—among other things—the covolume of the group of units of the quadratic form $f_3^n(x) = x_1^2 + \cdots + x_n^2 - 3x_{n+1}^2$, which is known to contain a (maximal) reflection subgroup Γ of finite index for $n \le 13$. In the case of $n = 5$, the Coxeter group $\Gamma \subset \mathrm{Isom}(\mathcal{H}^5)$ is given by $[6,3,3,3,4,4]$. Their volume computation yields the precise expression

$$\mathrm{vol}_5([6,3,3,3,4,4]) = \frac{\sqrt{3}}{320}L(3,12) \simeq 0.0053587488,$$

where $L(s,D) = \sum_{r=1}^{\infty}\left(\frac{D}{r}\right)\frac{1}{r^s}$ denotes the Dirichlet L-series with Kronecker symbol $\left(\frac{D}{r}\right)$ (see [24, Table 1] and compare with [5, Proposition 2.1 (2)]). Notice that the two arithmetic hyperbolic Coxeter groups $[6,3,3,3,4,4]$ and $[6,3,3,3,3,6]$ have non-compact fundamental polyhedra of identical combinatorial type being pyramids over a product of two simplices. If the two groups were commensurable, the volumes would be necessarily $\mathbb{Q}$-proportional. However, in [7], we proved that the groups are incommensurable.

We finish the volume considerations by quoting and applying the following result for doubly-truncated 5-orthoschemes as proved in [18, (24)].

$\Omega(\alpha)$:



, $\quad \alpha' := \frac{\pi}{2} - \alpha$

Figure 6: A doubly-truncated 5-orthoscheme with cyclic graph $\Omega(\alpha)$.

Proposition 2.8 *Consider a doubly-truncated orthoscheme in $\mathcal{H}^5$ with cyclic graph $\Omega(\alpha)$ of order* 8 *as given in Figure* 6. *Then its volume equals*

$$\mathrm{vol}_5(\Omega(\alpha)) = \frac{1}{32}\zeta(3) - \frac{1}{2}\left\{Л_3(\alpha) + Л_3\left(\frac{\pi}{2}-\alpha\right)\right\}.$$

Example 2.9 By the above proposition together with the identity (2.9), we obtain that

$$\mathrm{vol}_5\left([(3,6)^{[4]}]\right) = \frac{13}{288}\zeta(3) = 20\cdot\frac{13}{5,760}. \tag{2.12}$$

By comparing (2.12) with the result of Remark 2.6, we deduce the following conjectural volume identity between $\Omega(\frac{\pi}{3}) = [(3,6)^{[4]}]$ and $[6,3,3,3,3,6]$:

$$\mathrm{vol}_5([(3,6)^{[4]}]) \stackrel{?}{=} \binom{6}{3} \cdot \mathrm{vol}_5\left([6,3,3,3,3,6]\right). \tag{2.13}$$

This volume identity will be a direct consequence of a dissection result for an ideal birectified regular simplex in $\mathbb{H}^5$ (see Section 3.5, Theorem 3.6(i) for $n = 5$ and $k = 2$).

3 Rectifying Hyperbolic Regular Polyhedra

In this section, we present Debrunner's dissection result for regular simplices and orthoplexes by means of certain orthoschemes. Then we develop the theory of ideal hyperbolic k-rectified regular polyhedra in the projective model of hyperbolic n-space and provide an interpretation by polarly truncated polyhedra. In this way, we can describe the two families of ideal k-rectified regular simplices and orthoplexes by means of Napier cycles and prove one of our main results as given by Theorem 3.6.

3.1 Regular Polyhedra in $\mathbb{X}^n$ and Debrunner's Result

Consider a polyhedron $P \subset \mathbb{X}^n$ and its *flags* of the form $\mathcal{F} = \{F_0, \dots, F_{n-1}\}$, $F_{-1} := \varnothing$, consisting of k-dimensional faces F_k of P such that $F_{k-1} \subset F_k$ for $k = 0, \dots, n-1$. The polyhedron P is *regular* (and denoted by P_{reg} at times) if its symmetry group $Sym(P)$ acts (simply) transitively on its flags (see [27, Part II, Chapter 5, Section 3], for example). It follows that each face of P is itself a regular polyhedron and that the symmetry group of P has a unique fixed point, the *(bary- or in-)center of* P, denoted by b_n. The point b_n is the center of the diverse in- and circumspheres attached to P. Fix a flag $\mathcal{F}$ of P and consider the centers b_k of its k-faces ($0 \le k \le n-1$). In particular, the point b_0 coincides with a vertex $v \in P$. Each sequence $b_0, \dots, b_{i-1}, b_{i+1}, \dots, b_n$ defines an (affine) hyperplane $H_i, 0 \le i \le n-1$, which bounds the half-space H_i^- in $\mathbb{X}^n$ containing the point b_i. Consider the polyhedral cone $C = \cap_{i=0}^{n-1} H_i^-$ with apex b_n in $\mathbb{X}^n$ whose edges pass through $b_0, \dots, b_{n-1}$. It provides a fundamental domain for $Sym(P)$, which is generated by the n reflections in the hyperplanes $H_0, \dots, H_{n-1}$. It is not difficult to see that $R := C \cap P$ is an n-orthoscheme, called the *characteristic simplex* of P. The regular polyhedron P is of dihedral angle 2α if the hyperplane H_n opposite to b_n in the boundary of R (and of P) and the hyperplane H_{n-1} form the angle $\measuredangle(H_{n-1}, H_n) = \alpha$. For $k = 1, \dots, n-1$, let p_k denote the number of k-dimensional faces of P containing the face F_{k-2} and being contained in the face F_{k+1} where $F_n := P$. Then the *Schläfli symbol* of P is the ordered set $\{p_1, \dots, p_{n-1}\}$. Reading a given Schläfli symbol in reversed order yields the Schläfli symbol of the regular polyhedron dual to P. Observe that the cone C and the reflection group generating $Sym(P)$ can be represented by the Coxeter graph Σ_C as given in Figure 7. Furthermore, the graph Σ_C relates to a finite Coxeter group (of type $A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4$, or $I_2(m)$; see [8, part I, Section 2]).

The *vertex figure* P_v *at* v of an arbitrary polyhedron P is the intersection of P with a sphere centered at v of sufficiently small radius in $\mathbb{X}^n$ (intersecting only the edges passing through v). Up to a normalisation, each vertex figure of an (ordinary) vertex v in $\mathbb{X}^n$ is a spherical $(n-1)$-polyhedron. In the hyperbolic case, a vertex q of

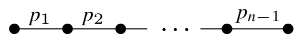


Figure 7: The Coxeter graph Σ_C associated with the cone $C \subset \mathbb{X}^n$.

a finite volume polyhedron P may be on the boundary at infinity $\partial\mathcal{H}^n$, in which case its vertex figure—as intersection of P with a suitable horosphere centered at q—turns into a Euclidean $(n-1)$-polyhedron. It is easy to see that each vertex figure P_v of a regular polyhedron P is itself a regular polyhedron and admits a barycentric decomposition into $(n-1)$-orthoschemes isometric to $R \cap P_v$. In this way, the Schläfli symbol $\{p_1, \dots, p_{n-1}\}$ can be interpreted as the union of the Schläfli symbol $\{p_1, \dots, p_{n-2}\}$ of a facet with the Schläfli symbol $\{p_2, \dots, p_{n-1}\}$ of a vertex figure of a regular polyhedron P. The graph Σ_C is the Coxeter graph associated with the spherical vertex figure $R \cap P_{b_n}$. Finally and most importantly, a regular polyhedron $P \subset \mathbb{X}^n$ of dihedral angle 2α can be entirely described by means of its characteristic simplex R with graph $\Sigma_n(\frac{\pi}{p_1}, \dots, \frac{\pi}{p_{n-1}}, \alpha)$ according to Figure 1 (see [3, Sections 7.5–7.9]).

Examples are the Schläfli symbols $\{3, \dots, 3\}$ and $\{4, 3, \dots, 3\}$, which describe (self-dual) regular simplices and regular hypercubes in $\mathbb{X}^n$, while the Schläfli symbol $\{3, \dots, 3, 4\}$ describes a regular orthoplex (or cross-polytope). Notice that these are the only regular polyhedra existing in every dimension. For these types of regular polyhedra, realised on the sphere $\mathbb{S}^n$, Schläfli [25, Sections 29 and 31] obtained volume identities for certain related orthoscheme families. More precisely, these are the orthoscheme family $U_i(\alpha), 0 \le i \le n+1$, associated with a regular n-simplex $S_{reg}(2\alpha) \subset \mathbb{X}^n$ and the orthoscheme family $V_i(\alpha), 0 \le i \le n$, associated with a regular orthoplex $O_{reg}(2\alpha) \subset \mathbb{X}^n$ as given by the graphs in Figures 8 and 9. For their realisation conditions, see Section 2.1 and [4, Remark (7.5), Remark (7.9)]. Furthermore, the orthoschemes $U_0(\alpha) = R(\alpha, \frac{\pi}{3})$ and $V_0(\alpha) = R(\alpha, \frac{\pi}{4})$ are isometric to the orthoschemes $U_{n+1}(\alpha)$ and $V_n(\alpha)$, respectively, the latter being the corresponding characteristic simplices (see Section 2.1 and Figure 2).

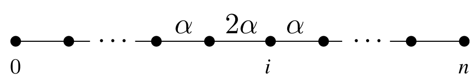


Figure 8: Orthoschemes $U_i(\alpha), 0 \le i \le n+1$, tiling $S_{reg}(2\alpha) \subset \mathbb{X}^n$.

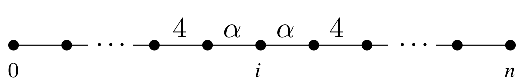


Figure 9: Orthoschemes $V_i(\alpha), 0 \le i \le n$, tiling $O_{reg}(2\alpha) \subset \mathbb{X}^n$.

In [4, Theorems (7.4) and (7.8)], Debrunner provided an alternative proof for Schläfli's volume identities, in the cases of $S_{reg}(2\alpha)$ and $O_{reg}(2\alpha)$ in $\mathbb{X}^n$, by using only a dissection argument. More precisely, he deduced the following result (see also the proof of Theorem 3.6 below).

Theorem 3.1 (Debrunner)

(i) *A regular simplex* $S_{reg}(2\alpha) \subset \mathbb{X}^n$ *of dihedral angle* 2α *admits for* $0 \le i \le n$ *a dissection into* $i!(n+1-i)!$ *orthoschemes isometric to* $U_i(\alpha)$*, and each of these splits into* $\binom{n+1}{i}$ *orthoschemes isometric to* $U_0(\alpha)$.

(ii) *A regular orthoplex* $O_{reg}(2\alpha) \subset \mathbb{X}^n$ *of dihedral angle* 2α *admits for* $0 \le i \le n-1$ *a dissection into* $2^n i!(n-i)!$ *orthoschemes isometric to* $V_i(\alpha)$*, and each of these splits into* $\binom{n}{i}$ *orthoschemes isometric to* $V_0(\alpha)$.

In the next section, we will extend these results to their ideal truncated or k-*rectified* hyperbolic counterparts.

3.2 Rectification of Regular Polyhedra

Let $P \subset \mathbb{E}^n$ be a regular Euclidean polyhedron with f_k faces F_k of dimension k for $0 \le k \le n$. Consider a flag $F_0, \dots, F_{n-1}, F_n$ of faces F_k with centers b_k as above. Then the orbit $M_k := Sym(P)b_k$ consists of the centers (or midpoints) of all k-dimensional faces of P and has cardinality equal to f_k.

For $0 \le k \le n-1$, the *k-rectified regular polyhedron* $r_kP \subset \mathbb{E}^n$ *of* P is the (Euclidean) convex hull of the f_k points in M_k. The polyhedron r_kP arises from P by shrinking all the k-dimensional faces of P to their centers. Hence, $r_0P = P$, while r_1P is the result of the truncation from P of each vertex cone, denoted by $\mathrm{cone}(v, P_v)$, $v \in M_0$, by the affine hyperplane E_v determined by the centers of the edges of P ending at v. The facets of r_1P consist of the polyhedra $E_v \cap P$ associated with the vertices $v \in M_0$, and the truncated facets of P. As a reference, see [3, Chapter VIII].

Observe that the polyhedron $r_{n-1}P$ coincides with the dual of the regular polyhedron P. Consequently, we will consider k-rectified regular polyhedra in $\mathbb{E}^n$ for $1 \le k \le n-2$, only.

A k-rectified regular polyhedron $r_kP \subset \mathbb{E}^n$ gives rise to an ideal hyperbolic n-polyhedron (of finite volume) in the following way. Interpret hyperbolic n-space $\mathbb{H}^n$ in the projective unit ball model K^n of Klein–Beltrami. Next, consider the insphere S of $r_kP \subset \mathbb{E}^n$ centered at b_n, which touches all points $m_1, \dots, m_{f_k}$ of M_k. By normalising appropriately, the sphere S can be identified with the unit boundary sphere $\mathbb{S}^{n-1}$ of K^n. Since $k \ge 1$, the vertices of P are *ultra-ideal* points, lying outside of S with respect to K^n, and $P \cap \mathbb{H}^n$ is a convex region bounded by the hyperbolic hyperplanes $Sym(P)F_{n-1}$, which is of infinite volume.

Definition 3.2 Let $1 \le k \le n-2$ be an integer. The ideal hyperbolic k-rectified regular polyhedron $\widehat{P} \subset \mathbb{H}^n$ associated with P is the (hyperbolic) convex hull of the ideal points $m_1, \dots, m_{f_k} \in \mathbb{S}^{n-1}$ of $\mathbb{H}^n$. We identify $\widehat{P}$ with r_kP and write $\widehat{P} = r_kP$ accordingly.

An ideal hyperbolic k-rectified regular polyhedron $\widehat{P} \subset \mathbb{H}^n$ is of finite volume and determined uniquely up to isometry by the Schläfli symbol of the underlying regular polyhedron P and the degree of rectification k. Of particular interest will be the ideal hyperbolic k-rectified regular simplex r_kS_{reg} and the ideal hyperbolic k-rectified regular orthoplex r_kO_{reg} in $\mathbb{H}^n$ (for $k = 2$ and $n = 5, 7$, see Examples 2.3 and 2.5).

We will provide a complete description of their facial structure and dihedral angles (see Section 3.4, Remark 3.5).

3.3 Rectification and Polar Truncation

Our aim is to extend Debrunner's Theorem 3.1 to the families of ideal k-rectified regular simplices r_kS_{reg} and orthoplexes r_kO_{reg} in $\mathbb{H}^n$, forming the single categories of such polyhedra (up to duality) existing in all dimensions. To this end, we exploit the properties of the Klein–Beltrami model K^n in real projective space P^n in order to adjust Debrunner's proof appropriately. In fact, this approach will allow us to interpret an ideal rectified regular polyhedron as a regular polyhedron, which is suitably (polarly) truncated.

First, there is the well known relationship between points $X = [x] \in P^n$, represented by non-zero vectors $x \in \mathbb{E}^{n,1}$, and hyperplanes $\pi_X = \{[y] \in P^n \mid \langle x, y\rangle_{n,1} = 0\}$ relative to the quadric $Q_{n,1} = \{[x] \in P^n \mid \langle x, x\rangle_{n,1} = 0\}$, which yields a bijection between the set of all points or *poles* $X =: \mathrm{pol}(\pi_X)$ and the set of all hyperplanes or *polar hyperplanes* $\pi_X =: \mathrm{pol}(X)$ of P^n. This duality principle for P^n relative to $Q_{n,1}$ is characterised by the following important properties (see [14, Section 1], for example).

Properties 3.3 (i) The polar hyperplane π_X of $X = [x] \in P^n$ respectively intersects, touches, or avoids the quadric $Q_{n,1}$ if and only if the vector x is space-like, light-like, or time-like.

(ii) If two lines g, h in P^2 intersect at $I = g \cap h$, then $\mathrm{pol}(I)$ is the line determined by $\mathrm{pol}(g)$ and $\mathrm{pol}(h)$.

(iii) If a hyperplane π_1 in P^n contains the pole $\mathrm{pol}(\pi_2)$ of the hyperplane π_2, then $\pi_1 \perp \pi_2$ holds.

Consider an ideal hyperbolic 1-rectified regular polyhedron $\widehat{P} = r_1P \subset K^n$ with underlying regular polyhedron P having the barycenter b_n. For each vertex $v \in P$, interpreted as a unit space-like vector in $\mathbb{E}^{n,1}$, the polar hyperplane $\pi_{[x]}$ consists of all points $[y]$ with $y \in E_v$ where E_v is the hyperplane determined by the (ideal) centers m_i of the edges $vv_i, 1 \le i \le N$, of P ending at v ($N \ge n$). Indeed, E_v is the hyperbolic vector subspace of $\mathbb{E}^{n,1}$ generated by the (non-zero) light-like vectors $v + v_i$ representing the centers m_i, up to normalisation ($1 \le i \le N$). For a vector $y = \sum_i \lambda_i(v + v_i) \in E_v$, one gets $\langle y, v\rangle_{n,1} = \sum_i \lambda_i\{1+\langle v_i, v\rangle_{n,1}\} = \sum_i \lambda_i\{1-\|v_i\|_{n,1}\,\|v\|_{n,1}\} = 0$ since v, v_i lie on opposite sides of the light-like line generated by m_i (see [23, Theorem 3.2.9, Theorem 3.2.10], for example).

As a consequence (see Property (iii) above), the hyperbolic hyperplane E_v intersects orthogonally all those hyperplanes, bounding P or bounding a characteristic simplex $R = C \cap P$ for a flag $\mathcal{F} = \{F_0, \ldots, F_{n-1}\}$ of P, which contain the vertex $v = F_0 \in P$ (see Section 3.1). Write $R = \cap_{i=0}^n H_i^-$, where H_0 is the hyperplane opposite to $v = b_0$, as usually. Since $\widehat{P} = r_1P$ is 1-rectified, the hyperplanes H_0 and E_v are (hyperbolic) parallel (intersecting at $b_1 \in \partial K^n$) with $\measuredangle(H_0, E_v) = 0$, while $\measuredangle(H_i, E_v) = \frac{\pi}{2}$ for $1 \le i \le n$. For later purpose, it is convenient to write $H_{-1} := E_v$.

Next, consider all vertices $v =: v_1, \ldots, v_{f_0}$ of P. For $1 \le i \le f_0$, define (affine) rays ρ_i through b_n and v_i, parametrised by $t \ge 0$ such that $b_n = \rho_i(0)$ and $v_i = \rho_i(t_1)$

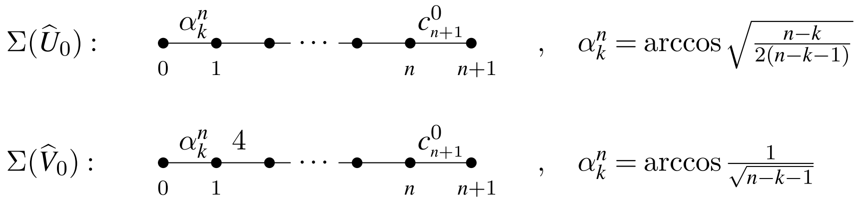


Figure 10: The graphs of $\widehat{U}_0(\alpha_k^n)$ and $\widehat{V}_0(\alpha_k^n)$.

for some $t_1 > 0$. Moreover, intersecting hyperplanes $E_i(t_1) := E_{v_i} = v_i^{\perp}$ meet at the points of $M_1 = Sym(P)b_1$, being the centers of the edges $v_i v_j$ for $i \neq j$. Recall that the (hyperbolic) convex hull of the points of $M_1 = Sym(P)b_1$ equals $r_1 P = \widehat{P} =: \widehat{P}(t_1)$. In this context, for $k > 1$, the ideal hyperbolic k-rectified regular polyhedron $r_k P = \widehat{P}$ can be interpreted as $\widehat{P}(t_k)$ for some (unique) $t_k > t_{k-1}$. Indeed, by Property (ii) (describing a way to construct polar hyperplanes), for $t \nearrow t_k$, the points $\rho_i(t)$ go uniformly away from the polyhedron P, while the polar hyperplanes represented by $E_i(t)$ tend inwards of P until they meet points of $M_k = Sym(P)b_k$ at time $t = t_k$. Observe that $E_1(t_k)$ intersects the hyperplane H_0 bounding R under a certain non-zero dihedral angle, while $E_1(t_1) = E_v$ and H_0 are (hyperbolic) parallel. The dihedral angle will be made explicit in the cases $r_k S_{reg}$ and $r_k O_{reg}$ (see (3.1)).

3.4 Napier Cycles Associated with $r_k S_{reg}$ and $r_k O_{reg}$

Let $n \geq 3$ and $1 \leq k \leq n-2$. The above considerations restricted to the special cases $r_k S_{reg}$ and $r_k O_{reg}$ motivate us to look at the simply-truncated orthoschemes $\widehat{U}_0 = \widehat{U}_0(\alpha_k^n)$, where $\alpha_k^n = \arccos\sqrt{\frac{n-k}{2(n-k-1)}}$, and $\widehat{V}_0 = \widehat{V}_0(\alpha_k^n)$, where $\alpha_k^n = \arccos\frac{1}{\sqrt{n-k-1}}$ in $\mathbb{H}^n$, whose graphs are given by Figure 10 and carry the additional weights $c_{n+1}^0(\widehat{U}_0)$ and $c_{n+1}^0(\widehat{V}_0)$ (denoted by c_{n+1}^0, for short). The polyhedra $\widehat{U}_0$ and $\widehat{V}_0$ are bounded by $n+2$ hyperbolic hyperplanes H_i whose intersection behavior is indicated in Figure 10, by associating with H_i the node labeled by i, $0 \leq i \leq n+1$. The underlying orthoschemes U_0 and V_0, bounded by the hyperplanes $H_0, \dots, H_n$, respectively, are such that the vertex opposite to the hyperplane H_{n-k} is an ideal point. Furthermore, the hyperplane H_{n+1} corresponds to the polar hyperplane (denoted earlier by H_{-1}), and it lies opposite to the ultra-ideal vertex opposite to H_n. The vertex figure of U_0 of the vertex opposite to H_0 is of type $A_{n-1} = [3, \dots, 3]$, while the vertex figure of V_0 of the vertex opposite to H_0 is of type $B_{n-1} = [4, 3, \dots, 3]$. These facts allow us to deduce the explicit formulas for α_k^n. Indeed, for $\widehat{U}_0$, for example, since the vertex opposite of H_{n-k} is an ideal point, having a Euclidean vertex neighborhood, the leading principal minor of order $n-k$ of the Gram determinant associated with $H_0, \dots, H_{n+1}$ must vanish. For example, in the case of $\widehat{U}_0$, and based on (2.3), the requirement $\delta(-\cos\alpha_k^n, -\frac{1}{2}, \dots, -\frac{1}{2}) = 0$ yields the expression for α_k^n as mentioned above (see also the proof of Lemma 3.4).

Lemma 3.4 *Let $n \geq 3$ and $1 \leq k \leq n-2$. Then, $c_{n+1}^0(\widehat{U}_0) = c_{n+1}^0(\widehat{V}_0) = -\sqrt{\frac{k+1}{2k}}$.*

Proof The polyhedra $\widehat{U}_0$ and $\widehat{V}_0$ are bounded by $n+2$ hyperplanes $H_0, \ldots, H_{n+1}$ in $\mathcal{H}^n$, characterised by (space-like) normal vectors $e_0, \ldots, e_{n+1} \in \mathbb{E}^{n,1}$, which are linearly dependent in $\mathbb{R}^{n+1}$. Hence, their Gram matrices $G_U = G(\widehat{U}_0)$ and $G_V = G(\widehat{V}_0)$ have vanishing determinant. For $3 \le m \le n$, consider the principal submatrix G_U^m of G_U formed by the vectors $e_0, \ldots, e_m$. By some well known recursion formulas for the determinant of such matrices (see [3, Sections 7.74–7.76], for example), we easily deduce that

$$\det G_U^m = \det A_m - \cos^2 \alpha_n^k \det A_{m-1} = \frac{1}{2^m}\Big(1 - \frac{m}{n-k-1}\Big),$$

since $\det A_l = (l+1)/2^l$. Therefore, we obtain

$$\big(c_{n+1}^0(\widehat{U}_0)\big)^2 = \frac{\det G_U^n}{\det G_U^{n-1}} = \frac{k+1}{2k} \le 1,$$

which implies the assertion. In a similar vein, one identifies the determinant of the submatrix G_V^m of G_V formed by the vectors $e_0, \ldots, e_m$ with

$$\det G_V^m = \det B_m - \cos^2 \alpha_n^k \det A_{m-1} = \frac{1}{2^{m-1}}\Big(1 - \frac{m}{n-k-1}\Big),$$

since $\det B_l = 1/2^{l-1}$. Hence, we deduce that

$$\big(c_{n+1}^0(\widehat{V}_0)\big)^2 = \frac{\det G_V^n}{\det G_V^{n-1}} = \frac{k+1}{2k} \le 1,$$

which finishes the proof. ∎

Remark 3.5 Let P be a regular simplex S_{reg} or a regular orthoplex O_{reg} in Euclidean n-space. The ideal hyperbolic k-rectified regular polyhedron $\widehat{P} = r_k P \subset \mathbb{H}^n$ has facets (faces of codimension 1) of two sorts: (truncated) facets belonging to P and polar facets, that is, facets contained in the polar hyperplanes π_v of the ultra-ideal vertices v of P ($k \ge 1$). Furthermore, $r_k P$ has dihedral angles $2\alpha_k^n$ formed by intersecting facets of P, dihedral angles $\frac{\pi}{2}$ between each polar facet in a π_v and the facets of P containing the pole v, as well as dihedral angles $2\gamma_k^n$ attached to the intersection of a polar facet in π_v with a facet of P *not* containing v. By Lemma 3.4, the angle $2\gamma_k^n \in]0, \frac{\pi}{2}]$ can be identified as follows:

$$\gamma_k^n = \begin{cases} 0 & \text{for } k = 1, \\ \arccos\sqrt{\frac{k+1}{2k}} & \text{for } k > 1. \end{cases} \tag{3.1}$$

Each of the polyhedra $\widehat{U}_0$ and $\widehat{V}_0$ is part of a Napier cycle of type 2 (see Section 2.2). Furthermore, for $i \ge 1$, we can form doubly-truncated orthoschemes $\widehat{U}_i = \widehat{U}_i(\alpha_k^n)$ and $\widehat{V}_i = \widehat{V}_i(\alpha_k^n)$ in $\mathcal{H}^n$ with graphs and additional weights as indicated in Figure 11 and in Figure 12, respectively, each defining a Napier cycle of type 3.

The additional weights $c_h^i(\widehat{U}_i)$ and $c_h^i(\widehat{V}_i)$, where $h = n+1, n+2, 0$, as indicated in Figures 11 and 12 are determined by formula (2.4) upon passing from angular weights ω such as $\frac{\pi}{3}$, $\frac{\pi}{4}$ or α_k^n to $c = -\cos\omega$ (see (2.2)).

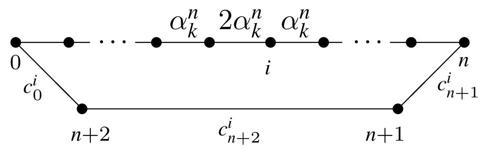


Figure 11: The doubly-truncated orthoschemes $\widehat{U}_i(\alpha_k^n), 1 \le i \le n$.

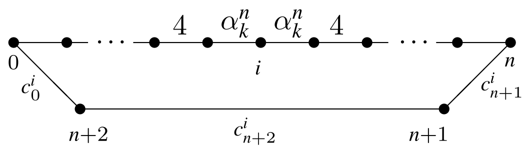


Figure 12: The doubly-truncated orthoschemes $\widehat{V}_i(\alpha_k^n), 1 \le i \le n-1$.

3.5 Dissecting Ideal Rectified Regular Simplices and Orthoplexes

With the above preparations, we can formulate and prove our first main result as stated in Theorem 1.1. Denote by $\delta_{ik} \in \{0,1\}$ the Kronecker-Delta function defined for elements i, k in an index set I.

Theorem 3.6 *Let $n \ge 3$ and $1 \le k \le n-2$ be integers. For a regular polyhedron $P \subset \mathbb{E}^n$ with Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$, the ideal k-rectified regular n-polyhedron $\widehat{P} = r_k P \subset \mathbb{H}^n$ admits the following dissections.*

(i) *If P is a simplex S_{reg} with $p_1 = \cdots = p_{n-1} = 3$, then $r_k S_{reg}$ admits for $0 \le i \le n$ a dissection into $i!(n+1-i)!$ of $(2-\delta_{0i})$-truncated orthoschemes isometric to $\widehat{U}_i(\alpha_k^n)$, and each of these splits into $\binom{n+1}{i}$ simply-truncated orthoschemes isometric to $\widehat{U}_0(\alpha_k^n)$, where $\alpha_k^n = \arccos\sqrt{\frac{n-k}{2(n-k-1)}}$.*

(ii) *If P is an orthoplex O_{reg} with $p_1 = \cdots = p_{n-2} = 3$ and $p_{n-1} = 4$, then $r_k O_{reg}$ admits for $0 \le i \le n-1$ a dissection into $2^n i!(n-i)!$ of $(2-\delta_{0i})$-truncated orthoschemes isometric to $\widehat{V}_i(\alpha_k^n)$, and each of these splits into $\binom{n}{i}$ simply-truncated orthoschemes isometric to $\widehat{V}_0(\alpha_k^n)$, where $\alpha_k^n = \arccos\frac{1}{\sqrt{n-k-1}}$.*

Proof We follow roughly the strategy of Debrunner's proof of Theorem 3.1 and recapitulate the most important ingredients.

Ad (i): Suppose that $P =: \langle v_0, \ldots, v_n \rangle$ is a Euclidean regular simplex with center b_n and with vertices $v_0, \ldots, v_n$ so that $\widehat{P} = r_k S_{reg} \subset \mathbb{H}^n$. We interpret $\mathbb{H}^n$ in the projective model K^n. Since $k \ge 1$, each vertex $v_i, 0 \le i \le n$, of P is outside of the quadric $Q_{n,1}$ and therefore pole of its polar hyperplane represented by the hyperbolic hyperplane E_i, say. In order to describe the dissections of P, we adopt Debrunner's elegant notation as follows. Let $[i,k] := \{i, i+1, \ldots, k-1, k\}$ for integers $0 \le i \le k \le n$. For a set $I \subset [0,n]$, denote by F_I the face of P with vertices $v_i, i \in I$, and let $F_\varnothing := \varnothing$ and $F_{[0,n]} := P$. Each F_I is a regular simplex with (bary-)center $b(F_I) =: B_I$. In particular, for $0 \le i \le n$, one has $B_{\{i\}} = B_{[i,i]} = v_i$, as well as $B_{[0,n]} = b_n$. The Euclidean

n-simplex $U_0 := \langle B_{[0,n]} \dots B_{[n,n]} \rangle$ is a characteristic simplex of P and fundamental domain for the action of $Sym(P) \cong S_{n+1}$. Therefore, P dissects into $(n+1)!$ isometric copies of U_0. Furthermore, U_0 is an orthoscheme whose vertex figure at $B_{[0,n]} = b_n$ is a spherical $(n-1)$-orthoscheme of type A_{n-1}. Among the vertices of U_0 not in K^n, we see that $B_{[l,n]}, \dots, B_{[n,n]}$ are ultra-ideal points precisely for $l = n-k+1, \dots, n$, while the vertex $B_{[n-k,n]}$ is an ideal point on ∂K^n.

Next, consider the Euclidean n-simplices

$$U_i := \langle B_{[0,0]} \dots B_{[0,i-1]} B_{[i,n]} \dots B_{[n,n]} \rangle, \quad 0 \le i \le n. \tag{3.2}$$

Each simplex U_i arises from the following construction. For $0 \le i \le n$, consider the partition of $[0,n]$ by $I = [0, i-1]$ and $J = [i,n]$. Dissect both, the regular simplex $(i-1)$-face F_I into its $i!$ characteristic simplices $\sigma(R_I) =: R_I^\sigma$ $(\sigma \in S_i)$, and the regular simplex $(n+1-i)$-face F_J into its $(n+1-i)!$ characteristic simplices $\tau(R_J) =: R_I^\tau$ $(\tau \in S_{n+1-i})$. Since $P = S_{reg}$ is the join $F_I \circ F_J$, P splits into $i!(n+1-i)!$ simplices $R_I^\sigma \circ R_I^\tau$, which are permuted by the elements of S_{n+1} that stabilise F_I (and F_J). One of these simplices is U_i for a suitable ordering of the vertices. By Theorem 3.1, we know that each U_i is an orthoscheme admitting a dissection into $\binom{n+1}{i}$ copies of U_0.

Let us pass to the k-rectified polyhedron $\widehat{P}$ associated with $P = S_{reg}$. The hyperbolic polar hyperplanes $E_0, \dots, E_n$ associated with the ultra-ideal vertices $v_0, \dots, v_n$ induce a (simple or double) truncation of U_i $(0 \le i \le n)$ as given by (3.2) and its isometric copies in the decomposition of P as described above, making a bridge to the polyhedra $\widehat{U}_i = \widehat{U}_i(\alpha_k^n)$. Indeed, we will show that

$$\widehat{U}_i = \widehat{P} \cap U_i \quad \text{for} \quad 0 \le i \le n,$$

which will finish the proof of (i). We consider the following two cases.

(a) Let $i = 0$, and consider the characteristic simplex $U_0 = \langle B_{[0,n]} \dots B_{[n,n]} \rangle$ with vertex $B_{[n,n]} = v_n$ of P that is simply-truncated by E_0. The hyperplane E_0 meets the ideal vertex $B_{[n-k,n]}$ of U_0 at infinity, and the vertex $B_{[0,n]}$, being the in-center of $P = S_{reg}$, is of type A_{n-1}. For $0 \le l \le n$, denote by H_{n-l} the hyperplane bounding U_0, which is opposite to $B_{[l,n]}$, and write $H_{n+1} := E_0$. By Property (iii), H_{n+1} is orthogonal to the hyperplanes $H_0, \dots, H_{n-1}$, while $\measuredangle(H_j, H_{j+1}) = \frac{\pi}{3}$ for $1 \le j \le n-1$. Since $B_{[n-k,n]}$ is ideal, we get $\measuredangle(H_0, H_1) = \alpha_k^n$ as above. Furthermore, by Lemma 3.4 and its proof, $\measuredangle(H_n, H_{n+1}) = \gamma_k^n$ according to (3.1). Hence, the truncated orthoscheme $U_0 \cap \widehat{P}$ coincides with the polyhedron $\widehat{U}_0(\alpha_k^n)$ given by Figure 10.

Notice that the dihedral angle 2α formed by two facets of $P = S_{reg}$ is identical to the dihedral angle formed by the corresponding pair of (truncated) facets of $\widehat{P}$, and this angle is therefore equal to $2\alpha_k^n$. Hence, the orthoscheme U_0 can be described by the graph $\Sigma(U_0) = \Sigma_n(\alpha_k^n, \frac{\pi}{3})$ according to Figure 2. Furthermore, the orthoschemes $U_i(\alpha)$ arising in the dissection of $P = S_{reg}$ according to Debrunner's Theorem 3.1(i) are given by $U_i(\alpha_k^n)$ for $0 \le i \le n$ (see Figure 8).

(b) Let $i > 0$. It remains to show that the polyhedra $U_i \cap \widehat{P}$, where $U_i = U_i(\alpha_k^n)$, coincide with the polyhedra $\widehat{U}_i(\alpha_k^n)$ as given by Figure 11. To this end, observe that each of the orthoschemes U_i, $i > 0$, shares the vertices $B_{[0,0]} = v_0$ and $B_{[n,n]} = v_n$ with $P = S_{reg}$ (see (3.2)). Both vertices of the U_i are truncated by the hyperbolic polar hyperplanes $H_{n+1} = E_0$ and $H_{n+2} := E_n$ of P. Fix such a U_i, and denote by $e_l, 0 \leq l \leq n+2$, the unit normal space-like vector with $H_l = e_l^{\perp}$ directed outwards of U_i. It follows that $e_0, \ldots, e_{n+2}$ (indices modulo $n+3$) form a Napier cycle $\mathcal{N}$ of type 3 (see Section 2.2). For $1 \leq l \leq n$, the weights $c_l = \langle e_{l-1}, e_l \rangle_{n,1}$ are given by $-\cos \omega_l$ where $\omega_l \in \{\alpha_k^n, 2\alpha_k^n, \frac{\pi}{3}\}$ denote the angular weights of the orthoscheme $U_i = U_i(\alpha_k^n)$ according to Figure 8. The remaining weights c_{n+1}, c_{n+2}, c_0 of the cycle $\mathcal{N}$ are given by the formulas (2.4). This proves that the hyperplanes $H_0, \ldots, H_{n+2}$ bounding $U_i \cap \widehat{P}$ are described by the cyclic graph of Figure 11, and that, finally, $U_i \cap \widehat{P}$ coincides with $\widehat{U}_i(\alpha_k^n)$.

Ad (ii): The proof is similar to (i). Let us describe the dissection procedure of an ideal hyperbolic k-rectified regular orthoplex $r_k O_{reg}$ and the respective appearance of the polyhedra $\widehat{V}_i(\alpha_k^n)$ as claimed. Consider a Euclidean regular n-orthoplex O_{reg}, and denote by v_i, v_{-i} the n pairs of vertices of O_{reg} such that the segments $\langle v_i v_{-i} \rangle, 1 \leq i \leq n$, meet orthogonally in their common midpoint (and in-center) z. The n symmetry hyperplanes generated by all vertices except one pair induce a dissection of the polyhedron O_{reg} into 2^n simplices, all isometric to $S := \langle z v_1 \ldots v_n \rangle$. The facet $\langle v_1 \ldots v_n \rangle$ of O_{reg} (and of S) is a regular $(n-1)$-simplex whose faces $\langle v_1 \ldots v_i \rangle$ and $\langle v_{i+1} \ldots v_n \rangle$ can be dissected barycentrically into $i!$ and $(n-i)!$ characteristic orthoschemes R and R', respectively. Form the join $V_i := R \circ \langle z \rangle \circ R'$ for $1 \leq i \leq n$, and denote by B_I the barycenter of the face F_I with vertices $v_i, i \in I$, of O_{reg}, as above. In this picture, for $i = 0$, the polyhedron O_{reg} is cut into its $2^n n!$ characteristic orthoschemes all isometric to $V_0 = \langle z B_{[1,n]} \ldots B_{[n,n]} \rangle \subset S$, and for $1 \leq i \leq n$, the simplex $V_i = \langle B_{[1,n]} \ldots B_{[1,i]} z B_{[i+1,n]} \ldots B_{[n,n]} \rangle$ is one of the $2^n i!(n-i)!$ pairwise isometric simplices dissecting O_{reg}. Again, it can be shown that each such simplex is an orthoscheme. For details, see [4, pp. 150–151]. As in the proof of (i), one easily verifies that the truncated orthoschemes $\widehat{V}_i(\alpha_k^n) = r_k O_{reg} \cap V_i$ provide the decomposition of $r_k O_{reg}$ as claimed. ■

As a first application of Theorem 3.6, let us consider the ideal birectified regular orthoplex $r_2 O_{reg}$ in hyperbolic 7-space. We can prove the following result.

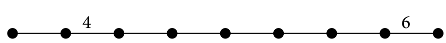
Corollary 3.7 *The volume of the ideal birectified regular orthoplex $r_2 O_{reg} \subset \mathbb{H}^7$ is given by*

$$\mathrm{vol}_7(r_2 O_{reg}) = \frac{153}{16} \sqrt{3} L(4, -3) \simeq 27.3241,$$

where $L(s, D) = \sum_{r \geq 1} \left(\frac{D}{r}\right) r^{-s}$ is the Dirichlet L-series defined by the Kronecker symbol $\left(\frac{D}{r}\right)$.

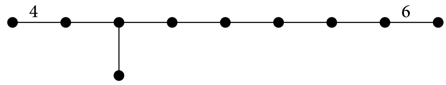
Proof By Remark 3.5, the ideal birectified regular orthoplex $r_2 O_{reg}$ has dihedral angles $2\alpha_2^7 = \frac{2\pi}{3}, 2\gamma_2^7 = \frac{\pi}{3}$ and $\frac{\pi}{2}$, and part (ii) of the above theorem implies that $r_2 O_{reg}$ can be cut into $2^7 7! = 645{,}120$ polyhedra isometric to the (truncated) characteristic simplex $\widehat{V}_0 = [3, 4, 3^5, 6]$ with Coxeter graph given as follows.

$\Sigma(\widehat{V}_0)$:



In [7, Section 2.2], we showed that the (arithmetic) hyperbolic Coxeter group Γ_0 with graph $\Sigma(\widehat{V}_0)$ is commensurable to the Coxeter group Γ_1 with graph

$\Sigma(\Gamma_1)$:



by identifying Γ_1 as a subgroup of index 3 in the group Γ_0. The group Γ_1 itself is the (maximal) reflection subgroup of the group of units of the quadratic form f_3^7 whose covolume has been determined by Ratcliffe and Tschantz (see Remark 2.7). More precisely, according to [24, Table 1], one has that

$$\mathrm{covol}_5(\Gamma_1) = \frac{51\sqrt{3}}{2^{15}\cdot 5\cdot 7} L(4,-3) \simeq 7.240232999\cdot 10^{-5}.$$

As a consequence, $\mathrm{vol}_7(r_2 O_{reg}) = \frac{2^7 7!}{3}\cdot\frac{51\sqrt{3}}{2^{15}\cdot 5\cdot 7}L(4,-3) = \frac{153}{16}\sqrt{3}L(4,-3)$ as asserted. ■

4 Quaternionic (pseudo-)modular Groups and their Covolumes

In the sequel, we interpret the orientation-preserving isometries of hyperbolic 4- and 5-spaces by means of certain quaternionic 2×2 matrices. In this way, and following Johnson [11], we can relate the quaternionic (pseudo)-modular groups to certain arithmetic hyperbolic Coxeter groups. By combining our previous results and applying them to the ideal birectified 6-cell, we will be able to prove our second main result about the covolume of the hybrid modular group as given by Theorem 4.1.

4.1 Quaternionic 2×2 Matrices and Hyperbolic Isometries

Consider hyperbolic n-space in the upper half space $U^n = \mathbb{E}^{n-1}\times\mathbb{R}_+$. In this model, the group $\mathrm{Isom}^+(U^n)$ of orientation preserving or *direct* hyperbolic isometries is isomorphic to the group $M^+(U^n)$ of direct Möbius transformations of $\mathbb{E}^n$, which leave U^n invariant. By Poincaré extension, the latter group is isomorphic to the group of direct Möbius transformations of the extended ground space $\widehat{\mathbb{E}}^{n-1} = \mathbb{E}^{n-1}\cup\{\infty\}$. As in the classical case of $\mathrm{Isom}^+(U^2)\cong PSL(2,\mathbb{R})$, the group $\mathrm{Isom}^+(U^{n+1})$, $n\geq 1$, can be identified with a projective group of 2×2 matrices according to Vahlen, Maass, and Ahlfors (see [1, 2]). To this end, interpret the real vector space $\mathbb{R}^n$ as the set of *Clifford vectors* $x = x_0 + x_1 i_1 + \cdots + x_{n-1}i_{n-1}\in C_n$ of the *Clifford algebra* C_n, which is the associative real algebra generated by $n-1$ elements $i_1,\ldots,i_{n-1}$ subject to the relations $i_k i_l = -i_l i_k$ $(k\neq l)$ and $i_k^2 = -1$. A typical element $a\in C_n$ can written in the form $a = \sum_I a_I I, a_I\in\mathbb{R}$, where I runs through all products $i_{k_1}\cdots i_{k_r}$ with $0\leq k_1<\cdots<k_r<n$, where we include the empty product in the form $i_{k_0} = i_0 := 1$. We call $S(a) := a_0$ the *scalar part* of a. Accordingly, C_n is a real vector space of dimension 2^{n-1} that can be equipped with a Euclidean norm defined by $|a|^2 = \sum_I a_I^2$. In particular, $C_1 = \mathbb{R}$, $C_2 = \mathbb{C}$ and $C_3 = \mathbb{H}$.

On C_n, there are three important involutions. The mapping $a\mapsto a^*$ is defined by sending each $I = i_{k_1}\cdots i_{k_r}$ to $I^* := i_{k_r}\cdots i_{k_1}$, while $a\mapsto a'$ is given by replacing each factor i_k by $-i_k$. The conjugation $a\mapsto\overline{a}$ is the composition $\overline{a} := a'^*$. Obviously, Clifford vectors $x\in C_n$ satisfy $x = x^*$, and a non-zero Clifford vector x is invertible with inverse $x^{-1} = x'/|x|^2$. The non-zero Clifford vectors form a multiplicative group, which is termed the *Clifford group* G_n. A *Clifford matrix* is an element of the set

$$SL(2, C_n) := \left\{ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in G_n \cup \{0\}; ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^n; ad^* - bc^* = 1 \right\}.$$

The quantity $ad^* - bc^*$ is called the *pseudo-determinant* of T and is such that the set $SL(2, C_n)$ becomes a group under matrix multiplication (see [1, p. 221]). Its associated projective group

$$PSL(2, C_n) = SL(2, C_n)/\{\lambda I_2 | \lambda \in \mathbb{R}^*\}$$

acts bijectively on $\widehat{\mathbb{E}}^n$ by fractional linear transformations $T(x) = (ax+b)(cx+d)^{-1}$, $T(0) = bd^{-1}$, $T(\infty) = ac^{-1}$, and this action appropriately extended to $\widehat{\mathbb{E}}^{n+1}$ preserves U^{n+1}. As a consequence, the group $PSL(2, C_n)$ is isomorphic to $\mathrm{Isom}^+(U^{n+1})$.

In particular, in the quaternionic case, we have that $PSL(2, \mathbb{H}) \cong \mathrm{Isom}^+(U^4)$ (for some geometric properties of its discrete subgroups, see [19]). There is another matrix group over the quaternions closely related to hyperbolic isometries. Following Wilker [28], consider a 2×2 matrix M with coefficients in the quaternion algebra $\mathbb{H}$ given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{H}.$$

Since quaternions can be interpreted by means of complex 2×2 matrices, M can be identified with a block matrix $\mathcal{M} \in \mathrm{Mat}(4, \mathbb{C})$ whose (ordinary) determinant is a non-negative real number that can be written according to

$$\det \mathcal{M} = |ad|^2 + |bc|^2 - 2S(a\bar{c}d\bar{b}) = |ad - aca^{-1}b|^2 \quad \text{for} \quad a \neq 0.$$

Based on this, the *Dieudonné determinant* of M is defined by

$$\Delta = \Delta(M) :=_+ \sqrt{|ad|^2 + |bc|^2 - 2S(a\bar{c}d\bar{b})} = |ad - aca^{-1}b|,$$

and Δ satisfies all required properties of a determinant function. Moreover, the set

$$S_\Delta L(2, \mathbb{H}) = \left\{ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, \mathbb{H}) | \Delta(T) = 1 \right\},$$

is a group whose elements act on $\widehat{\mathbb{H}}$ by fractional linear transformations. Finally, it can be shown that the projective analogue

$$PS_\Delta L(2, \mathbb{H}) = S_\Delta L(2, \mathbb{H})/\{\pm I_2\}$$

is isomorphic to the group $\mathrm{Isom}^+(U^5)$ (for some geometric properties of its discrete subgroups, see [20]).

4.2 Basic Systems of Quaternionic Integers

Consider the normed real associative algebra $\mathbb{H}$ of quaternions $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$ where $i = i_1$, $j = i_2$, and $k = ij$, as usual. The norm $N(q)$ and the trace $T(q)$ of q are given by

$$N(q) = q_0^2 + q_1^2 + q_2^2 + q_3^2, \quad T(q) = 2S(q) = 2q_0,$$

which appear naturally in the quadratic equation

$$q^2 - T(q)q + N(q) = 0. \tag{4.1}$$

Quaternions of norm 1 are called *units* and form a group that is isomorphic to the special unitary group $SU(2)$. According to Johnson–Weiss [13], [11, Chapter 15], a *basic system* of elements in $\mathbb{H}$ is a set $\mathcal{S}$ such that

(a) each element in $\mathcal{S}$ is a (quadratic) algebraic integer;
(b) $\mathcal{S}$ is a subring of $\mathbb{H}$; the elements of norm 1 in $\mathcal{S}$ form a subgroup of the group of unit quaternions;
(c) in the real vector space $\mathbb{H}$, the elements of $\mathcal{S}$ are the points of a four-dimensional lattice spanned by the units.

By means of (4.1), condition (a) holds for a quaternion q if $S(q) = q_0 \in \mathbb{Z} \cup \frac{1}{2}\mathbb{Z}$ and $N(q) \in \mathbb{Z}$. By [13, Theorem 4.1], there are precisely three basic systems of integral quaternions which can be described—briefly—as follows.

The first basic system is the ring $\mathbb{H}\mathrm{am} = \mathbb{Z}[i, j]$ whose four units $1, i, j$, and $k = ij$ span the lattice of *Hamilton integers*. The ring $\mathbb{H}\mathrm{am}$ can be regarded as a quaternionic analogue of the ring of Gaussian integers $\mathbb{Z}[i]$.

The second basic system is the ring $\mathbb{H}\mathrm{ur}= \mathbb{Z}[u, v]$ of *Hurwitz integers* where the quaternions u, v are defined by $u = \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k$ and $v = \frac{1}{2} + \frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k$. One verifies that each Hurwitz integer is an integral combination of $1, u, v, w$ where u, v, w satisfy the relations

$$u - u^2 = v - v^2 = w - w^2 = uvw = 1.$$

The ring $\mathbb{H}\mathrm{ur}$ has 24 units consisting of the 8 Hamilton units $\pm 1, \pm i, \pm j, \pm k$ together with the 16 units of the form $\pm\frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k$. The ring $\mathbb{H}\mathrm{ur}$ contains $\mathbb{H}\mathrm{am}$ as a subring and can be viewed as a quaternionic analogue of the ring $\mathbb{Z}[\omega]$, $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$, of Eisenstein integers since $(uv)^{-1} = \omega$.

The third basic system is the ring $\mathbb{H}\mathrm{yb} = \mathbb{Z}[\omega, j]$ of *hybrid integers* where $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$ and j satisfy the relations

$$\omega + \omega^2 = j^2 = (\omega j)^2 = (\omega^2 j)^2 = -1.$$

There are 12 hybrid units, given by $1, \omega, \omega^2, j, \omega j, \omega^2 j$ and their negatives.

4.3 Covolumes of Some Quaternionic (Pseudo-)modular Groups

Consider the group $PSL(2, \mathbb{H}) \cong \mathrm{Isom}^+(U^4)$ represented by quaternionic Clifford matrices and restrict the coefficient ring from $\mathbb{H}$ to one of the basic systems $\mathcal{S}$ of quadratic integers in $\mathbb{H}$ as described in Section 4.2. Each of the three groups $PSL(2, \mathcal{S})$ is a particularly nice arithmetic discrete group of direct isometries acting on hyperbolic 4-space with finite covolume. In [13] (see also [11, Section 15.2]), Johnson and Weiss studied various properties of $PSL(2, \mathcal{S})$, which they call a quaternionic *pseudo-modular group* (and denote it there by $PS^*L_2(\mathcal{S})$ referring to the underlying unit pseudo-determinant of Ahlfors). They show that each group $PSL(2, \mathcal{S})$ can be identified with a finite index subgroup of a certain hyperbolic Coxeter group, and the identification is made explicit in terms of suitable generators.

In this way, the Hamilton pseudo-modular group $PSL(2, \mathbb{H}\mathrm{am}) = PSL(2, \mathbb{Z}[i, j])$ turns out to be isomorphic to an index 12 subgroup of the hyperbolic Coxeter simplex group $[3, 4, 3, 4]$ (see [13, p. 173]). By the Reduction Formula (2.5) of Section 2.4, the volume of the Coxeter orthoscheme $[3, 4, 3, 4]$ can be computed to be $\pi^2/864$ so that the following result holds:

$$\mathrm{covol}_4(PSL(2, \mathbb{H}\mathrm{am})) = \frac{\pi^2}{72}.$$

We refer to [21] for further and more algebraic, arithmetic, and geometric details about the group $PSL(2, \mathbb{Z}[i, j])$.

The Hurwitz pseudo-modular group $PSL(2, \mathbb{H}\mathrm{ur}) = PSL(2, \mathbb{Z}[u, v])$ can be identified with a semidirect product of $PSL(2, \mathbb{H}\mathrm{am})$ with a cyclic group of order 3 generated by an element transforming Hamilton integers into Hurwitz integers (see [21, p. 751] and [13, p. 174]). As a consequence, one obtains that

$$\mathrm{covol}_4(PSL(2, \mathbb{H}\mathrm{ur})) = \frac{\pi^2}{24}.$$

The hybrid pseudo-modular group $PSL(2, \mathbb{H}\mathrm{yb}) = PSL(2, \mathbb{Z}[\omega, j])$ is isomorphic to a certain index 4 subgroup of the Coxeter pyramid group $[6, 3, 3, 3, \infty]$ of covolume $\pi^2/540$ (see Section 2.4), which in turn relates to the symmetry group of the ideal rectified 5-cell $r_1 S_{reg}(\frac{\pi}{3})$ (see Example 2.2). It follows that

$$\mathrm{covol}_4(PSL(2, \mathbb{H}\mathrm{yb})) = \frac{\pi^2}{135}.$$

Let us pass to the case of higher *modular groups* in $PS_\Delta L(2, \mathbb{H}) \cong \mathrm{Isom}^+(U^5)$. In particular, the modular groups $PS_\Delta L(2, \mathbb{H}\mathrm{am})$ and $PS_\Delta L(2, \mathbb{H}\mathrm{ur})$ are arithmetic discrete groups of finite covolume that are intimately related to one another. In fact, by work of Johnson and Weiss [13, Section 7] (see also [11, Section 15.3]), it is known that the Hamilton modular group $PS_\Delta L(2, \mathbb{H}\mathrm{am})$ is isomorphic to a certain subgroup of index 12 in the Coxeter simplex group $[3, 4, 3, 3, 4]$. Furthermore, the Hurwitz modular group $PS_\Delta L(2, \mathbb{H}\mathrm{ur})$ is closely related to the Coxeter simplex group $[3, 4, 3, 3, 3]$ and contains the group $PS_\Delta L(2, \mathbb{H}\mathrm{am})$ as a subgroup of index 30 (see [13, Section 9] and [11, Section 15.4]). By means of our volume expression (2.10) for $[3, 4, 3, 3, 4]$, we are able to derive the following results:

$$\mathrm{covol}_5\left(PS_\Delta L(2, \mathbb{H}\mathrm{am})\right) = \frac{7}{384}\zeta(3), \tag{4.2}$$
$$\mathrm{covol}_5\left(PS_\Delta L(2, \mathbb{H}\mathrm{ur})\right) = \frac{7}{11,520}\zeta(3).$$

4.4 The Ideal Birectified 6-cell and the Covolume of the Hybrid Modular Group $PS_\Delta L(2, \mathbb{H}\mathrm{yb})$

Much less has been known about the hybrid modular group $PS_\Delta L(2, \mathbb{H}\mathrm{yb}) \subset \mathrm{Isom}^+(U^5)$ with coefficient ring $\mathbb{Z}[\omega, j]$, $\omega = \frac{1}{2}(-1 + i\sqrt{3})$. Recently, in [10] and [11, Section 15.5], Johnson analysed the group $S_\Delta L(2, \mathbb{H}\mathrm{yb})$ in detail and determined the generators as follows:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}.$$

Note that $A^2 = -I$ and that $M^3 = N^4 = I$. This analysis enabled him to relate the group $PS_\Delta L(2, \mathbb{H}\mathrm{yb})$ to the hyperbolic Coxeter prism group $[6,3,3,3,3,6]$. Based on this work, we are able to prove the following result (see Theorem 1.2).

Theorem 4.1 *For the hybrid modular group $PS_\Delta L(2, \mathbb{H}\mathrm{yb}) = PS_\Delta L(2, \mathbb{Z}[\omega, j])$,*

$$\mathrm{covol}_5(PS_\Delta L(2, \mathbb{H}\mathrm{yb})) = \frac{13}{180}\zeta(3).$$

Proof By [11, Section 15.5], the group $PS_\Delta L(2, \mathbb{H}\mathrm{yb}) = PS_\Delta L(2, \mathbb{Z}[\omega, j])$ is isomorphic to the semidirect product of a certain commutator subgroup of index 8 in $[6,3,3,3,3,6]$ and the cyclic group of order 4 generated by the element $N \in S_\Delta L(2, \mathbb{Z}[\omega, j])$ above. This implies that $\mathrm{covol}_5(PS_\Delta L(2, \mathbb{H}\mathrm{yb})) = 32 \cdot \mathrm{covol}_5([6,3,3,3,3,6])$. It remains to show that $\mathrm{covol}_5([6,3,3,3,3,6]) = \frac{13}{5{,}760}\zeta(3)$ as already announced in Remark 2.6.

The Coxeter polyhedron $[6,3,3,3,3,6]$ is associated with the ideal birectified regular 6-cell $r_2 S_{reg} \subset \mathbb{H}^5$ of dihedral angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$. In fact, it is the truncated characteristic orthoscheme $\widehat{U}_0$ of the regular 6-cell of dihedral angle $\frac{\pi}{3}$ with ultra-ideal vertices all of whose triangles are replaced by an ideal point (see Section 3.2). Consider the truncated orthoschemes $\widehat{U}_0$ and $\widehat{U}_i$ with graphs and weights given according to Figures 10 and 11 (see also Lemma 3.4 and formula (2.4)). In fact, the distinguished weights for $\widehat{U}_0$ are $\alpha_2^5 = \frac{\pi}{6} = \gamma_2^5$ in view of (3.1). By Theorem 3.6(i), the polyhedron $r_2 S_{reg}$ admits a dissection into 6! simply-truncated orthoschemes $\widehat{U}_0$ isometric to $[6,3,3,3,3,6]$ and, by taking $i = 3$, a dissection into $(3!)^2$ doubly-truncated orthoschemes $\widehat{U}_3$. Each of the polyhedra $\widehat{U}_3$ can be dissected into $\binom{6}{3}$ copies of $\widehat{U}_0$. These dissection relations provide the volume identity

$$\mathrm{vol}_5([6,3,3,3,3,6]) = \mathrm{vol}_5(\widehat{U}_0) = \frac{1}{20}\,\mathrm{vol}_5(\widehat{U}_3).$$

Finally, it remains to identify the polyhedron $\widehat{U}_3$, as given by the graph in Figure 11 for $i = 3$, with the Coxeter polyhedron $[(3,6)^{[4]}]$ of the cyclic graph $\Omega(\frac{\pi}{3})$ whose volume is given by $\frac{13}{288}\zeta(3)$ according to Proposition 2.8 and (2.13). This can be done in two ways.

The cyclic graph of $\widehat{U}_3$ has symbol $[(3,6,3,6,3,c_5^3,c_6^3,c_0^3)]$ where the squares of the ingredients $c_l^3, l = 5,6,0$, are computable by formula (2.4). Without computation, the values c_5^3, c_6^3, c_0^3 can be determined directly by using the Napier cycle property (see Section 2.2) that the deletion of two non-adjacent nodes among $0,\dots,7$, representing vectors of the Napier cycle $\mathcal{N}$, defines two Lorentz-orthogonal subspaces of $\mathbb{E}^{5,1}$. Hence, the deletion of the nodes 2, 6 and 3, 7, respectively, yields $c_0^3 = c_5^3 = -\cos\frac{\pi}{6}$, while the deletion of the pair 0, 4 shows that $c_5^3 = -\cos\frac{\pi}{3}$. In fact, all corresponding subgraphs (after deletion) are products of Euclidean type $[3,6] \times [3,6]$. As a consequence, the polyhedron $\widehat{U}_3$ is isometric to $[(3,6)^{[4]}]$, and the assertion follows. ∎

Remark 4.2 In [7], it was shown that the (arithmetic) Coxeter groups with Coxeter symbols $[3, 4, 3, 3, 4]$ and $[6, 3, 3, 3, 3, 6]$ are incommensurable. Although the quotient of their covolumes is a rational number by (4.2) and Theorem 4.1, there is no (orientable) hyperbolic 5-manifold covering both, the modular Hamilton (or Hurwitz) orbifold $\mathbb{H}^5/PS_\Delta L(2, \mathbb{H}\,\mathrm{am})$ and the modular hybrid orbifold $\mathbb{H}^5/PS_\Delta L(2, \mathbb{H}\,\mathrm{yb})$.

Acknowledgments The author would like to thank Matthieu Jacquemet for helpful discussions. The author was partially supported by Schweizerischer Nationalfonds 200021–172583.

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