$Spin^{\mathbf{c}}$-MANIFOLDS WITH $Pin(2)$-ACTION

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Abstract

We study $Spin^{\mathbf{c}}$-manifolds with $Pin(2)$-action. The main tool is a vanishing theorem for certain indices of twisted $Spin^{\mathbf{c}}$-Dirac operators. This theorem is used to show that the Witten genus vanishes on such manifolds provided the first Chern class and the first Pontrjagin class are torsion. We apply the vanishing theorem to cohomology complex projective spaces and give partial evidence for a conjecture of Petrie. For example we prove that the total Pontrjagin class of a cohomology $\mathbb{C}P^{2n}$ with S^3-action has standard form if the first Pontrjagin class has standard form. We also determine the intersection form of certain 4-manifolds with $Pin(2)$-action.

1 Introduction

An important way to study smooth group actions on manifolds is based on the Lefschetz fixed point formula (cf. [AtSeII68], [AtSiIII68]). The classical example is the famous theorem of Atiyah and Hirzebruch which asserts that the index of the Dirac operator on a $Spin$-manifold M vanishes if the group S^1 acts non-trivially on M (cf. [AtHi70]). Consequently this index, the $\hat{A}$-genus, obstructs non-trivial actions by compact connected Lie groups on $Spin$-manifolds. Hattori extended this result to $Spin^{\mathbf{c}}$-manifolds in [Ha78] (cf. also [MaSc73], [Kr76]) and gave various applications for S^1-equivariant stable almost complex manifolds including cohomology complex projective spaces, complex hypersurfaces and 4-manifolds (cf. [Ha78] for details and other applications).

In [Wi86] Witten considered the index of "classical operators" on the free loop space $\mathcal{L}M$ of a manifold M. Although a mathematically precise definition of such operators has yet to be given Witten computed what their index should be by formally applying the Lefschetz fixed point formula to the natural S^1-action on $\mathcal{L}M$. In this way Witten derived invariants of the underlying manifold. It turned out that the "signature" of $\mathcal{L}M$ gives the elliptic genus (of level 2) which had been studied before by Ochanine, Landweber, Stong and others (cf. [Oc86], [Oc87], [LaSt88]). The invariant which corresponds to the index of the "Dirac operator" on $\mathcal{L}M$ is known as the Witten genus, the "Dolbeault operator" leads to the family of elliptic genera of higher level. Witten conjectured that the elliptic genus is rigid on $Spin$-manifolds with S^1-action. Soon afterwards Taubes and subsequently Bott and Taubes proved this conjecture (cf. [Ta89], [BoTa89]). The rigidity of the elliptic genera of higher level was proven by Hirzebruch in [Hi88] (cf. als [HiBeJu92], Appendix III). In [De96] we extended these rigidity results to $Spin^{\mathbf{c}}$-manifolds (cf. also [De98]). We also showed a vanishing theorem for certain indices of $Spin^{\mathbf{c}}$-Dirac operators.

The main aim of this paper is to illustrate how this result may be used to study $Spin^{\mathbf{c}}$-manifolds with $Pin(2)$-action. In the next section we explain the vanishing theorem. We introduce a series $\varphi^{\mathbf{c}}(M;V,W)$ of indices of twisted $Spin^{\mathbf{c}}$-Dirac operators depending on a $Spin^{\mathbf{c}}$-manifold M and a pair of vector bundles (V,W) over M. Given an S^1-action on M, V and W Theorem 2.2 states that the series of equivariant indices $\varphi^{\mathbf{c}}(M;V,W)_{S^1}$ vanishes identically provided certain conditions on the first Chern class and equivariant first Pontrjagin class are satisfied.

In order to apply Theorem 2.2 to $Spin^{\mathbf{c}}$-manifolds with S^1-action we need to know that the action lifts to the $Spin^{\mathbf{c}}$-structure and to the vector bundles V and W. Also the condition on the equivariant first Pontrjagin class needs to be satisfied. It turns out that for S^1-actions which extend to nice $Pin(2)$-actions (see Definition 3.4) these conditions may be fulfilled in many cases. This is the topic of Section 3.

In the last section we give applications of Theorem 2.2. We show that the Witten genus vanishes on a $Spin^{\mathbf{c}}$-manifold with nice $Pin(2)$-action if the first Chern class and first Pontrjagin class are torsion (see Theorem 4.1). A conjecture of Petrie states that the total Pontrjagin class of a homotopy $\mathbb{C}P^m$ with S^1-action has standard form. We give partial evidence for this conjecture (see Theorem 4.2). In particular we show that the conjecture is true for S^3-actions if the first Pontrjagin class has standard form and m is even (see Corollary 4.3). We also consider 4-manifolds with nice $Pin(2)$-action. Using Theorem 2.2 we determine their intersection form in certain cases (see Theorem 4.8, Theorem 4.9).

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2 A Vanishing Theorem for $Spin^{c}$-indices

In this section we state a vanishing theorem (see Theorem 2.2) for certain equivariant indices of twisted $Spin^{\mathbf{c}}$-Dirac operators (in the following also called equivariant twisted $Spin^{\mathbf{c}}$-indices). This theorem which is proven in [De96] (cf. also [De98]) will be applied to $Spin^{c}$-manifolds with $Pin(2)$-action in Section 4.

Let M be a closed smooth connected manifold and let G be a compact Lie group (not necessarily connected) which acts smoothly on M. For a G-equivariant elliptic differential operator D on M the equivariant index $ind_G(D)$ is defined as the (formal) difference of kernel and cokernel of D:

$$ind_G(D) = ker(D) - coker(D).$$

Since D is elliptic both spaces are finite dimensional G-representations and $ind_G(D)$ is an element of the representation ring $R(G)$. If $ind_G(D)$ is trivial, i.e. any $g \in G$ acts as the identity on $ind_G(D)$, we call the operator and also its index **rigid**. If G is connected $ind_G(D)$ is trivial if and only if the restriction to any S^1-subgroup is rigid.

It is well-known that the S^1-equivariant index is rigid for the following operators: the signature operator for oriented manifolds, the Dirac operator for *Spin*-manifolds and the Dolbeault operator for complex manifolds.

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In contrast the equivariant index of a $Spin^{\mathbf{c}}$-Dirac operator is in general not rigid. As an example we consider $\mathbb{C}P^2$ with the S^1-action induced by $\lambda(x_0, x_1, x_2) := (x_0, \lambda \cdot x_1, \lambda^2 \cdot x_2)$, $\lambda \in S^1$. Let P be a $Spin^{\mathbf{c}}$-structure for which the associated complex line bundle over $\mathbb{C}P^2$ is isomorphic to γ^5, where γ denotes the Hopf bundle. One can show that the S^1-action lifts to P. It turns out that for any lift the index of the $Spin^{\mathbf{c}}$-Dirac operator has the form $\lambda^d(1 + \lambda + \lambda^2)$, in particular the index is never rigid.

In the remaining part of this section we recall a vanishing theorem of [De96]. Let M be a $2m$-dimensional S^1-equivariant $Spin^{\mathbf{c}}$-manifold and let $\partial_{\mathbf{c}}$ denote the $Spin^{\mathbf{c}}$-Dirac operator. Let V be an S^1-equivariant s-dimensional complex vector bundle V over M and W an S^1-equivariant $2t$-dimensional $Spin$-vector bundle over M.

From these data we build the q-power series $R \in K_{S^1}(M)[[q]]$ of virtual S^1-equivariant vector bundles defined by

$$R := \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \Lambda_{-1}(V^*) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(\widetilde{V} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \triangle(\widetilde{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C}).$$

Here q is a formal variable, $\widetilde{E}$ denotes the reduced vector bundle $E - \dim(E)$, $\triangle(W)$ is the full complex spinor bundle associated to the $Spin$-vector bundle W and $\Lambda_t := \sum \Lambda^i \cdot t^i$ (resp. $S_t := \sum S^i \cdot t^i$) denotes the exterior (resp. symmetric) power operation. The tensor product is, if not indicated otherwise, taken over the complex numbers. We extend the index function ind_G to power series.

Definition 2.1. *Let $\varphi^{\mathbf{c}}(M; V, W)_{S^1}$ be the S^1-equivariant index of the $Spin^{\mathbf{c}}$-Dirac operator twisted with R, i.e.*

$$\varphi^{\mathbf{c}}(M; V, W)_{S^1} := ind_{S^1}(\partial_{\mathbf{c}} \otimes R) \in R(S^1)[[q]].$$

In the non-equivariant situation we write $\varphi^{\mathbf{c}}(M; V, W)$.

Note that $\varphi^{\mathbf{c}}(M; V, W)_{S^1}$ evaluated on the identity element of S^1 is equal to $\varphi^{\mathbf{c}}(M; V, W)$. The series of equivariant $Spin^{\mathbf{c}}$-indices $\varphi^{\mathbf{c}}(M; V, W)_{S^1}$ vanishes provided certain conditions on the first Chern classes and first equivariant Pontrjagin classes are satisfied. In order to state these conditions we need to introduce some notation. Let N be a manifold with G-action, G a compact Lie group. For a G-equivariant virtual vector bundle E over N and a characteristic class $u(E)$ the corresponding equivariant characteristic class will be denoted by $u(E)_G$. A G-equivariant $Spin^{\mathbf{c}}$-structure of N induces a G-equivariant complex line bundle $L_{\mathbf{c}}$ over N (see the beginning of the next section). Its equivariant first Chern class $c_1(L_{\mathbf{c}})_G$ will also be denoted by $c_1(N)_G$.

Finally, let π denote the projection of the fixed point manifold M^{S^1} to a point pt and let x be a fixed generator of $H^2(BS^1; \mathbb{Z})$.

Theorem 2.2. ([**De96**], Th. 3.6, [**De98**], Th. 3.2) *Assume that the equivariant class $p_1(V + W - TM)_{S^1}$ restricted to M^{S^1} is equal to $\pi^*(\mathcal{I} \cdot x^2)$ modulo torsion for some integer $\mathcal{I}$ and assume that $c_1(M)$ and $c_1(V)$ are equal modulo torsion. If $\mathcal{I} < 0$ then $\varphi^{\mathbf{c}}(M; V, W)_{S^1}$ vanishes identically.* ∎

The $Spin^{\mathbf{c}}$-indices $\varphi^{\mathbf{c}}(M; V, W)$ may be calculated in terms of cohomology (cf. [AtSiIII68]). If one restricts to the coefficient of q^0 in the cohomological description of $\varphi^{\mathbf{c}}(M; V, W)$ in the above theorem one obtains the

Corollary 2.3. *Assume the conditions given in the beginning of Theorem 2.2. If $\mathcal{I} < 0$ then*

$$\left\langle \prod_{i=1}^{m} \left(\frac{x_i}{e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}} \right) \cdot \prod_{j=1}^{s} \left(e^{\frac{v_j}{2}} - e^{-\frac{v_j}{2}} \right) \cdot \prod_{k=1}^{t} \left(e^{\frac{w_k}{2}} + e^{-\frac{w_k}{2}} \right), [M] \right\rangle = 0.$$

■

Here $\pm x_i$ (resp. v_j and $\pm w_k$) denote the formal roots of M (resp. V and W), $[M]$ denotes the fundamental cycle of M and $\langle \quad , \quad \rangle$ denotes the Kronecker pairing.

3 Nice $Pin(2)$-actions

In this section we give conditions under which S^1- and $Pin(2)$-actions lift to $Spin^{\mathbf{c}}$-structures or complex line bundles. For the facts on spectral sequences used below we refer to [Mc85]. Let G be a compact Lie group (not necessarily connected) which acts smoothly on the $2m$-dimensional $Spin^{\mathbf{c}}$-manifold M and let S^1 denote a fixed subgroup of G. Let V (resp. W) be a complex (resp. $Spin$-) vector bundle over M. In order to apply Theorem 2.2 we need to know that the S^1-action lifts to the $Spin^{\mathbf{c}}$-structure and the vector bundles V and W. We will use results of Hattori, Yoshida and Petrie given below. For a G-space X let $X_G := EG \times_G X$ denote the Borel construction, where EG is a classifying space for G.

Theorem 3.1. (**[HaYo76]**, Cor. 1.2) *Let X be a smooth manifold with smooth G-action and let L be a complex line bundle over X. Then the G action lifts to L if and only if $c_1(L)$ is in the image of the forget homomorphism $H^2(X_G;\mathbb{Z}) \to H^2(X;\mathbb{Z})$.* ■

We recall some basic facts about $Spin^{\mathbf{c}}$-structures (for details we refer to [AtBoSh64]). Let P denote a given $Spin^{\mathbf{c}}$-structure on M. The $Spin^{\mathbf{c}}$-principal bundle P induces two complex line bundles. The first one is a complex line bundle $L_{\mathbf{c}}$ over M defined by the $U(1)$-principal bundle $P/Spin(2m) \to P/Spin^{\mathbf{c}}(2m) \cong M$ using the standard embedding of $Spin(2m)$ into $Spin^{\mathbf{c}}(2m)$. The class $c_1(L_{\mathbf{c}})$ will be called the first Chern class of M and is also denoted by $c_1(P)$ or $c_1(M)$.

The group $U(1)$ acts on P via the embedding $U(1) \hookrightarrow Spin^{\mathbf{c}}(2m)$. The quotient $P/U(1)$ may be identified with the $SO(2m)$-principal bundle Q of orthonormal frames (for the metric induced by P). The projection $\xi : P \to Q$ is a $U(1)$-principal bundle and defines the second complex line bundle which we also denote by ξ. It is well-known that the pull-back of $L_{\mathbf{c}}$ to Q is isomorphic to ξ^2. Note that the S^1-action on M lifts to Q via differentials.

Theorem 3.2. (**[Pe72]**, Th. 6.2) *If the S^1-action lifts to the $U(1)$-principal bundle $\xi : P \to Q$ then for a modified lift the $Spin^{\mathbf{c}}$-structure $P \to M$ is S^1-equivariant.* ■

Combining the preceding two theorems with a spectral sequence argument one obtains the

Proposition 3.3. *If the first Betti number $b_1(M)$ vanishes or $c_1(M)$ is a torsion element then the S^1-action lifts to the $Spin^c$-structure P.*

Proof: A proof of the first statement is given in [Ha78]. For the convenience of the reader we prove both statements. By Theorem 3.1 the S^1-action on the principal bundle of orthonormal frames Q lifts to the $U(1)$-principal bundle $\xi : P \to Q$ if and only if the first Chern class of ξ is in the image of $H^2(Q_{S^1};\mathbb{Z}) \to H^2(Q;\mathbb{Z})$. To show this we inspect the Leray-Serre spectral sequence $\{E_r^{p,q}\}$ for $Q_{S^1} \to BS^1$ in integral cohomology. Note that $H^*(BS^1;\mathbb{Z})$ is a polynomial ring in one generator of degree 2. Thus all differentials in the spectral sequence restricted to the subgroup of bi-degree $(0,2)$ are trivial except maybe

$$d_2 : E_2^{0,2} \to E_2^{2,1} \cong H^2(BS^1; H^1(Q;\mathbb{Z})).$$

If $b_1(M)$ vanishes then $b_1(Q)$ vanishes, too (use for example the Leray-Serre spectral sequence for $Q \to M$). In this case $H^2(BS^1; H^1(Q;\mathbb{Z})) = 0$ and $d_2 : E_2^{0,2} \to E_2^{2,1}$ is the zero map. If $c_1(M)$ is a torsion class then the first Chern class of ξ is also torsion since the pull-back of L_c to Q is isomorphic to ξ^2. Since $E_2^{2,1} \cong H^2(BS^1; H^1(Q;\mathbb{Z}))$ is always torsion free, the image of $c_1(\xi)$ under d_2 is zero.

Thus, in any case the class $c_1(\xi)$ lives forever, i.e. all differentials vanish on $c_1(\xi)$. This implies that $c_1(\xi)$ is in the image of $H^2(Q_{S^1};\mathbb{Z}) \to H^2(Q;\mathbb{Z})$. By Theorem 3.1 and Theorem 3.2 the S^1-action on Q admits a lift to P for which the $Spin^c$-structure $P \to M$ is S^1-equivariant. This completes the proof. ■

In the next section we give applications of Theorem 2.2 for certain $Pin(2)$-actions which we introduce next. Recall that $Pin(2)$ is isomorphic to the normalizer of a torus in S^3. The group $Pin(2)$ is a non-trivial 2-fold cover of the orthogonal group $O(2)$ and may be presented by the closure of

$$\langle \lambda, g \mid g\lambda g^{-1} = \lambda^{-1},\, g^2 = -1 \rangle,$$

where λ is a topological generator of S^1.

Definition 3.4. *A $Pin(2)$-action on M is called* **nice** *if and only if the action is almost effective (i.e. the kernel of the action is finite) and the induced action on $H^*(M;\mathbb{Z})$ is trivial.*

We remark that a non-trivial semi-simple group action $G \times M \to M$ always induces a nice $Pin(2)$-action. If $\{\pm r_i\}$ denotes the set of roots of G and if H_i is the subgroup of G which corresponds to $\pm r_i$ then H_i is isomorphic to $SO(3)$ or S^3. Since the action is non-trivial at least one subgroup H_i acts non-trivially on M. After passing, if necessary, to the two-fold cover it follows that the group S^3 acts on M with finite kernel. Since S^3 is connected its action on the integral cohomology ring is trivial. Hence, the induced action of $Pin(2)$, the normalizer of S^1 in S^3, is nice. Also an almost effective $O(2)$-action which is trivial on the integral cohomology ring gives rise to a nice $Pin(2)$-action induced by the covering map $Pin(2) \to O(2)$. In the next lemma we collect some cohomological data of $BPin(2)$ which may be derived easily from the Leray-Serre spectral sequence for $\mathbb{R}P^2 \to BPin(2) \to BS^3$.

Lemma 3.5. *The first few integral cohomology groups of $BPin(2)$ are*

$$H^0(BPin(2);\mathbb{Z}) \cong \mathbb{Z},\ H^1(BPin(2);\mathbb{Z}) = 0,\ H^2(BPin(2);\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},\ H^3(BPin(2);\mathbb{Z}) = 0.$$

The rational cohomology of $BPin(2)$ is concentrated in degree $4\mathbb{Z}$. ■

Next we give conditions under which $Pin(2)$-actions lift to complex line bundles.

Proposition 3.6. *Let M be a manifold with nice $Pin(2)$-action and let L be a complex line bundle over M. If $b_1(M)$ vanishes or L is spin then the $Pin(2)$-action lifts to L.*

Proof: For both statements we will use the Leray-Serre spectral sequence $\{E_r^{p,q}\}$ in integral cohomology for the fibre bundle $M \overset{i}{\hookrightarrow} M_{Pin(2)} \xrightarrow{\pi} BPin(2)$. By Theorem 3.1 the $Pin(2)$-action lifts to L if $c_1(L)$ is in the image of $i^* : H^2(M_{Pin(2)};\mathbb{Z}) \to H^2(M;\mathbb{Z})$. Since the action is nice $Pin(2)$ acts trivially on the integral cohomology ring of M. Thus the E_2-term is given by $E_2^{p,q} \cong H^p(BPin(2); H^q(M;\mathbb{Z}))$. Note that $H^3(BPin(2);\mathbb{Z}) = 0$ (see Lemma 3.5). If $b_1(M) = 0$ all differentials of the spectral sequence restricted to the subgroup of bi-degree $(0,2)$ are zero. In particular they vanish on $c_1(L)$. If L is spin the same holds true since $c_1(L)$ is divisible by 2 and

$$d_2 : E_2^{0,2} \to E_2^{2,1} \cong H^2(BPin(2); H^1(M;\mathbb{Z})) \cong (\mathbb{Z}/2\mathbb{Z})^{b_1(M)}$$

takes values in a $\mathbb{Z}/2\mathbb{Z}$-module. Thus $c_1(L)$ is in the image of $i^* : H^2(M_{Pin(2)};\mathbb{Z}) \to H^2(M;\mathbb{Z})$ and the $Pin(2)$-action lifts to L by Theorem 3.1. ■

Proposition 3.7. *Let M be a manifold with nice $Pin(2)$-action. Consider the S^1-action induced by $S^1 \hookrightarrow Pin(2)$. Let π denote the projection $M_{S^1} \to BS^1$ and let x be a generator of $H^2(BS^1;\mathbb{Z})$. If the first Pontrjagin class $p_1(M)$ is torsion then the S^1-action is fixed point free or $p_1(M)_{S^1}$ is equal to $-\pi^*(\mathcal{I}\cdot x^2)$ modulo torsion, where $\mathcal{I}$ is a negative integer.*

Proof: Assume the action has a fixed point $pt \in M^{S^1}$. Consider the Leray-Serre spectral sequence $\{E_r^{p,q}\}$ in rational cohomology. Note that $H^*(BPin(2);\mathbb{Q})$ is concentrated in degree $4\mathbb{Z}$ (see Lemma 3.5). Since the action is nice $Pin(2)$ acts trivially on the rational cohomology of M. Hence

$$E_2^{p,q} \cong H^p(BPin(2);\mathbb{Q}) \otimes H^q(M;\mathbb{Q})$$

vanishes if $p \not\equiv 0 \bmod 4$. We claim that

$$H^4(BPin(2);\mathbb{Q}) \xrightarrow{\pi^*} H^4(M_{Pin(2)};\mathbb{Q}) \xrightarrow{i^*} H^4(M;\mathbb{Q}) \tag{1}$$

is exact: Since $\pi \circ i$ maps the fibre to a point one direction is trivial. So assume $i^*(y) = 0$. Since i^* factorizes as

$$H^4(M_{Pin(2)};\mathbb{Q}) \twoheadrightarrow E_\infty^{0,4} \subset E_2^{0,4} = H^4(M;\mathbb{Q})$$

and $E_\infty^{0,4} = H^4(M_{Pin(2)};\mathbb{Q})/E_\infty^{4,0}$ the element y is in $E_\infty^{4,0}$. Since π^* factorizes as

$$H^4(BPin(2);\mathbb{Q}) = E_4^{4,0} \twoheadrightarrow E_\infty^{4,0} \subset H^4(M_{Pin(2)},\mathbb{Q})$$

we conclude that y is in the image of π^*.

Next let k_* denote the homomorphism in cohomology induced by $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Since $k_*(p_1(M))$ vanishes by assumption and (1) is exact $k_*(p_1(M)_{Pin(2)})$ is in the image of π^*. By the naturality of (1) with respect to the inclusion $S^1 \hookrightarrow Pin(2)$ we conclude that

$$k_*(p_1(M)_{S^1}) = -\pi^*(\mathcal{I}\cdot k_*(x)^2).$$

A priori, $\mathcal{I}$ is a rational number. However, the restriction of the integral class $p_1(M)_{S^1}$ to the fixed point pt gives $\mathcal{I} \in \mathbb{Z}$ and $\mathcal{I} < 0$. In fact, $-\mathcal{I}$ is equal to the sum of squares of rotation numbers of the S^1-action at the fixed point pt. Since the $Pin(2)$-action is almost effective the induced S^1-action is non-trivial and this sum is positive. ∎

We close this section with a simple application of the classical Lefschetz fixed point formula for the Euler characteristic.

Lemma 3.8. *Assume $Pin(2)$ acts on M without fixed point. Then the Euler characteristic of M is even.*

Proof: Let $g \in Pin(2)$ be an element with $g\lambda g^{-1} = \lambda^{-1}$ for any $\lambda \in S^1$. Since the $Pin(2)$-action has no fixed point g acts freely on M^{S^1}. By the Lefschetz fixed point formula for the Euler characteristic $e(M) = e(M^{S^1})$. Now $M^{S^1} \to M^{S^1}/\langle g\rangle$ is a two-fold covering. Since the Euler characteristic is multiplicative in coverings we conclude

$$e(M) = e(M^{S^1}) = 2 \cdot e(M^{S^1}/\langle g\rangle) \equiv 0 \bmod 2.$$

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4 Applications

In this section we apply Theorem 2.2 and its Corollary 2.3 to $Spin^{\mathbf{c}}$-manifolds with nice $Pin(2)$-action.

4.1 We prove the vanishing of $\varphi^{\mathbf{c}}(M;0,0)$ if $c_1(M)$ and $p_1(M)$ are torsion elements. This leads to a vanishing theorem for the Witten genus. First we recall the classical situation.

Using the Lefschetz fixed point formula Atiyah and Hirzebruch proved in [AtHi70] that the $\hat{A}$-genus of a $Spin$-manifold M with non-trivial S^1-action vanishes. The $\hat{A}$-genus of M is the index of the Dirac operator and Atiyah-Hirzebruch in fact showed that the equivariant index of the Dirac operator vanishes identically.[2] This result has the following generalization. If M is an S^1-equivariant $Spin^{\mathbf{c}}$-manifold with first Chern class torsion the same argument applies to show that the equivariant $Spin^{\mathbf{c}}$-index vanishes identically. Note that the non-equivariant $Spin^{\mathbf{c}}$-index coincides with the $\hat{A}$-genus of M if $c_1(M)$ is torsion and coincides with the index of the Dirac operator if the $Spin^{\mathbf{c}}$-structure is induced from a $Spin$-structure on M.

Next we consider the Witten genus φ_W (the index of the hypothetical Dirac operator on the free loop space) which is defined by the even power series (cf. [Wi86], p. 165)

$$\frac{x}{e^{x/2} - e^{-x/2}} \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})}.$$

If M is a $Spin$-manifold and ∂ the Dirac operator then $\varphi_W(M) = ind(\partial \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM}))$.

The q-power series $\varphi^{\mathbf{c}}(M;0,0)$ of twisted $Spin^{\mathbf{c}}$-indices is related to the Witten genus in a similar way as the $Spin^{\mathbf{c}}$-index is related to the $\hat{A}$-genus. If $c_1(M)$ is a torsion element $\varphi^{\mathbf{c}}(M;0,0)$ coincides with the cohomological definition of the Witten genus. If M is spin then $\varphi^{\mathbf{c}}(M;0,0)$ coincides with the index-theoretical definition of the Witten genus.

[2] We remark that the 2-fold action always lifts to the $Spin$-structure.

Theorem 4.1. *Let M be a $Spin^c$-manifold for which the first Chern class $c_1(M)$ and the first Pontrjagin class $p_1(M)$ are torsion elements. Assume M admits a nice $Pin(2)$-action. Then the S^1-action induced by $S^1 \hookrightarrow Pin(2)$ lifts to the $Spin^c$-structure and for any such lift the q-power series of equivariant twisted $Spin^c$-indices $\varphi^{\mathbf{c}}(M; 0, 0)_{S^1}$ vanishes identically. In particular, the Witten genus $\varphi_W(M)$ vanishes.*

Proof: By Proposition 3.3 the S^1-action lifts to the $Spin^c$-structure. If M^{S^1} is empty the Lefschetz fixed point formula implies that all S^1-equivariant indices on M vanish identically (cf. [AtSeII68]). In particular, the theorem follows in this case. So assume the action has a fixed point $pt \in M^{S^1}$. By Proposition 3.7 $p_1(M)_{S^1}$ is equal to $-\pi^*(\mathcal{I} \cdot x^2)$ modulo torsion for some negative integer $\mathcal{I}$. Thus, we are in the position to apply Theorem 2.2 for $V = W = 0$ to get the vanishing of $\varphi^{\mathbf{c}}(M; 0, 0)_{S^1}$. Since $\varphi_W(M) = \varphi^{\mathbf{c}}(M; 0, 0)$ it follows that the Witten genus $\varphi_W(M)$ vanishes, too. This completes the proof. ■

As a consequence the S^1-equivariant Witten genus vanishes on a $BO\langle 8\rangle$-manifold M with nice $Pin(2)$-action. There is a "converse" to this: Let $\Omega_*^{\langle 8\rangle}$ denote the bordism ring of $BO\langle 8\rangle$-manifolds. In [De96], Proposition 4.12, we proved that the kernel of the rational Witten genus restricted to $\Omega_*^{\langle 8\rangle} \otimes \mathbb{Q}$ is generated by $BO\langle 8\rangle$-manifolds with nice $Pin(2)$-action, in fact with non-trivial S^3-action. Note that the kernel of the rational $\hat{A}$-genus is generated by *Spin*-manifolds with non-trivial S^1-action (cf. [AtHi70]).

Theorem 4.1 generalizes previous results. The vanishing of the Witten genus was proven by the author for $BO\langle 8\rangle$-manifolds with non-trivial S^3-action in [De94] and independently by Höhn in unpublished work. As Stolz pointed out in [St96] the vanishing of the Witten genus leads to some evidence for the following conjecture of Stolz and Höhn: If M is a Riemannian $BO\langle 8\rangle$-manifold with positive Ricci curvature then the Witten genus of M vanishes. For details we refer to [St96].

4.2 The next application of Theorem 2.2 deals with manifolds having the same integral cohomology ring as $\mathbb{C}P^m$. Such manifolds will be called cohomology $\mathbb{C}P^m$'s. Obviously any homotopy $\mathbb{C}P^m$ is a cohomology $\mathbb{C}P^m$. The converse holds in the simply connected case. The motivation is a conjecture of Petrie (cf. [Pe72], Strong conjecture, p. 105) which we state in the following equivalent form:

If M is a homotopy $\mathbb{C}P^m$ with non-trivial smooth S^1-action then the total Pontrjagin class has standard form, i.e. $p(M) = (1 + x^2)^{m+1}$, where x is a generator of $H^2(M; \mathbb{Z})$.

The conjecture has an affirmative answer if $m < 5$ or if the number of connected components of the fixed point manifold M^{S^1} is less than 5 (cf. [Wa75], [De76], [Yo76], [TsWa79], [Ma81], [Ja85]; for related results cf. [Pe72], [Pe73], [Ha78], [Ma88], [DoMa90]). Hattori proved the conjecture in the case that M is stably almost complex, the S^1-action preserves the stable almost complex structure and the first Chern class has the standard form, i.e. $c_1(M) = (m + 1)x$ (cf. [Ha78], Prop. 4.7). The next theorem which relies on Corollary 2.3 gives a partial answer to the conjecture of Petrie.

Theorem 4.2. *Let M be a cohomology $\mathbb{C}P^m$ with nice $Pin(2)$-action. If m is odd assume that the $Pin(2)$-action has a fixed point. Let x be a generator of $H^2(M; \mathbb{Z})$ and let b be the integer defined by $p_1(M) = b \cdot x^2$. Then $b \leq m + 1$ and*

$$b = m + 1 \Longrightarrow p(M) = (1 + x^2)^{m+1}.$$

Before we give the proof we point out some consequences. In each complex dimension $m \geq 5$ there are infinitely many differentiable manifolds M_i of the same homotopy type of $\mathbb{C}P^m$ such that all these manifolds have standard first Pontrjagin class but non-standard total Pontrjagin class. I.e. for each i the class $p_1(M_i)$ is equal to $(m+1)\cdot x^2$ and the class $p(M_i)$ is not equal to $(1+x^2)^{m+1}$, where x denotes a generator of $H^2(M_i;\mathbb{Z})$ (cf. [Hs66] and [Li89]). Theorem 4.2 implies that for any $m \geq 5$ these manifolds do not admit a $Pin(2)$-action with the properties stated in the theorem. We single out the following special case.

Corollary 4.3. *Let M be a cohomology $\mathbb{C}P^m$, m even, with standard first Pontrjagin class. If $p(M) \neq (1+x)^{m+1}$ then M does not support a non-trivial S^3-action.* ■

We remark that for $m \equiv 1 \bmod 4$ the α-invariant also obstructs S^3-actions on M: The α-invariant is a KO-theoretical generalization of the $\hat{A}$-genus and takes values in $\mathbb{Z}/2\mathbb{Z}$ for *Spin*-manifolds of dimension congruent 2 mod 8. Given a non-trivial S^3-action Lawson and Yau (cf. [LaYa74]) constructed a metric with positive scalar curvature on M. By a result of Hitchin (cf. [Hi74]) the α-invariant vanishes on *Spin*-manifolds with such metrics.

The proof of Theorem 4.2 will use the following theorem which we derive from Theorem 2.2 and from results of Section 3. Let V and W be sums of complex line bundles over a $2m$-dimensional $Spin^c$-manifold M with $b_1(M) = 0$. Assume W is spin. Let v_i (resp. w_k) denote the first Chern class of the i-th complex line bundle occurring as a summand of V (resp. of the k-th complex line bundle occurring as a summand of W).

Theorem 4.4. *Assume M admits a nice $Pin(2)$-action with fixed point. Assume $c_1(V)$ is equal to the first Chern class of M modulo torsion and $p_1(V+W-TM)$ is a torsion element. Then the S^1-action induced by $S^1 \hookrightarrow Pin(2)$ can be lifted to the $Spin^c$-structure and V and W in such a way that $\varphi^{\mathbf{c}}(M;V,W)_{S^1} = 0$.*

Proof: Since $b_1(M) = 0$ the induced S^1-action lifts to the $Spin^{\mathbf{c}}$-structure by Proposition 3.3. Let $L \to M$ denote one of the complex line bundles occurring as a summand of V or W. By Proposition 3.6 the $Pin(2)$-action lifts to L. We fix a lift of the $Pin(2)$-action to each complex line bundle occurring as a summand in V or W.

Next we consider the equivariant class $p_1(V+W-TM)_{Pin(2)}$. By assumption the restriction of this class to M is zero after passing to rational cohomology. The argument in the proof of Proposition 3.7 applies to show that

$$p_1(V+W-TM)_{S^1} \equiv \pi^*(\mathcal{I}\cdot x^2)$$

modulo torsion, where $\mathcal{I}$ is an integer. At the fixed point $pt \in M^{Pin(2)}$ the tangent bundle reduces to a non-trivial real S^1-representation and may be (non-canonically) identified with a complex representation with character $\sum \lambda^{m_i}$. At pt a line bundle L_i of V (resp. W) reduces to a complex one dimensional S^1-representation with character λ^{a_i} (resp. λ^{b_i}). Thus the restriction of $p_1(V+W-TM)_{S^1}$ to pt is equal to

$$\mathcal{I}\cdot x^2 = (\sum a_i^2 + \sum b_i^2 - \sum m_i^2)\cdot x^2,$$

where x is a generator of $H^2(BS^1;\mathbb{Z})$. Since the restriction of a complex one-dimensional $Pin(2)$-representation to S^1 is a trivial representation $a_i = b_i = 0$. Thus $\mathcal{I}$ is a negative number and Theorem 2.2 gives the vanishing of $\varphi^{\mathbf{c}}(M;V,W)_{S^1}$. ■

Proof of Theorem 4.2: It follows from the ring structure of $H^*(M;\mathbb{Z}/2\mathbb{Z})$ and the Wu formulas that $w_2(M) \equiv (m+1)\cdot x \bmod 2$. The relation $p_1(M) \equiv w_2(M)^2 \bmod 2$ implies $b \equiv m+1 \bmod 2$. Since $H^3(M;\mathbb{Z})$ is zero M admits a $Spin^{\mathbf{c}}$-structure and any class $a\cdot x$, $a \equiv m+1 \bmod 2$, can be realized as the first Chern class of a $Spin^{\mathbf{c}}$-structure of M.

Note that the $Pin(2)$-action has a fixed point. If m is odd this is part of the assumptions. If m is even this follows from Lemma 3.8 since the Euler characteristic of M is odd. Let $pt \in M^{Pin(2)}$.

We will now show that the assumption $b > m+1$ leads to a contradiction. Let $b \geq m+3$. Choose a $Spin^c$-structure on M with $c_1(M) = (m+1)x$. Since $b_1(M) = 0$ the induced S^1-action on M lifts to any $Spin^c$-structure (see Proposition 3.3). Let L_{nx} denote the complex line bundle with first Chern class equal to nx. We consider the bundles $V = L_{2x} + (m-1)\cdot L_x$ and $W = (b-m-3)\cdot L_x$. Then $c_1(V) = c_1(M)$, the vector bundle W is spin and $p_1(V+W-TM) = 0$. By Theorem 4.4 the S^1-action lifts to the $Spin^c$-structure and V and W in such a way that $\varphi^{\mathbf{c}}(M;V,W)_{S^1} = 0$. By Corollary 2.3 we have

$$\left\langle \prod_{i=1}^{m}\left(\frac{x_i}{e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}}\right)\cdot (e^{\frac{x}{2}} - e^{-\frac{x}{2}})^m \cdot (e^{\frac{x}{2}} + e^{-\frac{x}{2}})^{b-m-2}, [M]\right\rangle = 0,$$

where $\pm x_1, \ldots, \pm x_m$ denote the formal Pontrjagin roots of M. So $\langle x^m, [M]\rangle = 0$ which gives a contradiction. Hence, $b \leq m+1$.

Next we want to show that $b = m+1$ implies $p(M) = (1+x^2)^{m+1}$. This is trivial for $m=1$ and follows for $m=2$ from the signature theorem (cf. [Hi56]). So assume $m \geq 3$ and $b = m+1$. Let

$$\hat{\mathcal{A}} = 1 + \hat{\mathcal{A}}_1 + \hat{\mathcal{A}}_2 + \ldots = 1 + \frac{-p_1}{24} + \frac{7p_1^2 - 4p_2}{5760} + \ldots$$

denote the multiplicative series for $x/(e^{x/2} - e^{-x/2})$, i.e. $\hat{\mathcal{A}}$ is equal to $\prod_{i=1}^{l}(x_i/(e^{x_i/2} - e^{-x_i/2}))$ after replacing p_j by the j-th elementary symmetric function in $x_1^2, \ldots, x_l^2$. Let $\hat{\mathcal{A}}(M)$ be the series which one obtains by substituting the j-th Pontrjagin class of M for p_j. We use Theorem 4.4 for $V_k := L_{2x} + (m-3-2k)\cdot L_x$ and $W_k := 2k \cdot L_x$, where $k = 0, 1, \ldots, [\frac{m-3}{2}]$. For each k we choose a $Spin^{\mathbf{c}}$-structure with first Chern class equal to $c_1(V_k)$. Then Theorem 4.4 and Corollary 2.3 give

$$\left\langle \hat{\mathcal{A}}(M)\cdot (e^x - e^{-x})\cdot (e^{\frac{x}{2}} - e^{-\frac{x}{2}})^{m-3-2k} \cdot (e^{\frac{x}{2}} + e^{-\frac{x}{2}})^{2k}, [M]\right\rangle = 0. \tag{2}$$

For $k=0$ the corresponding identity determines $\hat{\mathcal{A}}_1(M)\cdot x^{m-2}$, for $k=1$ the corresponding identity determines $\hat{\mathcal{A}}_2(M)\cdot x^{m-4}$ and so on. Note that the multiplication map

$$H^{2m-2k}(M;\mathbb{Z}) \to H^{2m}(M;\mathbb{Z}), \quad y \mapsto y \cdot x^k,$$

is injective. So these identities together determine $\hat{\mathcal{A}}_j(M)$ for $j = 1, \ldots, [\frac{m-1}{2}]$. The coefficient of p_j in $\hat{\mathcal{A}}_j$ is a non-zero rational number (cf. [Hi56], §1). Since $H^*(M;\mathbb{Z})$ has no torsion we conclude that $p_j(M)$ is uniquely determined by the equations (2) for $j = 1, \ldots, [\frac{m-1}{2}]$.

We will now show that the total Pontrjagin class of M has the standard form. Consider first the case m odd. Then $p_j(M) = 0$, $j > \frac{m-1}{2}$, for dimensional reasons. So $p(M)$ is already uniquely determined by the equations (2). Since $\mathbb{C}P^m$ admits a non-trivial $Pin(2)$-action with fixed point the same identities also hold true for $\mathbb{C}P^m$. This implies $p(M) = (1+x^2)^{m+1}$.

Next assume m is even. Then all Pontrjagin classes of M except $p_{\frac{m}{2}}(M)$ are already determined. The top Pontrjagin class $p_{\frac{m}{2}}(M)$ can be calculated from the other Pontrjagin classes and the signature formula. Here we use the fact that the coefficient of p_j in the j-th term of the multiplicative series of the signature genus is non-zero (cf. [Hi56], §1). So again the total Pontjagin class of M is determined. Since all these identities also hold true for $\mathbb{C}P^m$ we get $p(M) = (1 + x^2)^{m+1}$. ■

The last proof can be easily modified to study more general situations, for example homotopy complete intersections which support a nice $Pin(2)$-action with fixed point. The next proposition shows that the first Pontrjagin class of such manifolds cannot be very large. The proof which is essentially contained in the proof of Theorem 4.2 is left to the reader.

Proposition 4.5. *Let M be a $2m$-dimensional $Spin^{\mathbf{c}}$-manifold with $b_1(M) = 0$, $H^2(M;\mathbb{Z}) = \mathbb{Z}\langle x\rangle$ and $x^m \neq 0$. Assume M admits a nice $Pin(2)$-action with fixed point. If $p_1(M) = b \cdot x^2$ then $b \leq m + 1$.* ■

The proposition may be applied to homotopy complex hypersurfaces. In this connection we mention the following result of Hattori.

Proposition 4.6. ([**Ha78**], Prop 3.15) *Let M be a non-singular hypersurface of degree d in $\mathbb{C}P^{m+1}$, $m > 1$. Let τ denote the stable almost complex structure induced from the complex structure of M. If M admits a non-trivial S^1-action which lifts to τ then $d \leq m + 3$.* ■

4.3 In this section we consider 4-manifolds with nice $Pin(2)$-action. We use Theorem 2.2 to determine their intersection form in certain cases.

Let us first take a look at the problem of classifying (non-trivial) S^1- and S^3-actions on closed manifolds in low dimensions. S^1-actions on 3-manifolds were completely classified by Orlik and Raymond in [OrRa67] (for S^3-actions cf. [As76]). The classification of 4-dimensional manifolds with S^3-action was given in [As76] and [MePa86]. Such manifolds fall into the classes: S^4, $\pm\mathbb{C}P^2$, homogeneous S^3-bundles over S^1, S^2-bundles over S^2 and quotients of S^2-bundles over S^2 by involutions. It turns out that the classification of 4-manifolds with S^1-action is much harder. Fintushel (cf. [Fi77], [Fi78]) reduced it to the (difficult) classification of "legally-weighted" 3-manifolds. In the simply connected case he and also Yoshida (cf. [Yo78]) gave a classification up to homotopy 4-spheres: Any simply connected 4-manifold with S^1-action is diffeomorphic to $\Sigma\sharp k(\mathbb{C}P^2)\sharp m(-\mathbb{C}P^2)\sharp n(S^2 \times S^2)$, where Σ is a homotopy 4-sphere. All these results are proved using techniques from the theory of transformation groups.

If one restricts to oriented 4-manifolds another tool is available: Since any such manifold admits a $Spin^{\mathbf{c}}$-structure one may try to use the Lefschetz fixed point formula (cf. [AtSeII68], [AtSiIII68]) for the $Spin^{\mathbf{c}}$-Dirac operator to characterize G-actions on oriented 4-manifolds. In [Ha78] Hattori proved the vanishing of the equivariant index of the $Spin^{\mathbf{c}}$-Dirac operator in certain cases and used this result to derive the following

Proposition 4.7. ([**Ha78**], Prop. 3.14) *Let M be an oriented 4-manifold with $b_1(M) = 0$ and let τ be an S^1-equivariant almost complex structure on M. If $n \in \mathbb{N}$ divides $c_1(\tau)$ then $n \leq 3$. If $n = 3$ then the Euler characteristic of M is equal to three times the signature of M, i.e. $e(M) = 3sign(M)$.* ■

The second part of this theorem may be rephrased in terms of the intersection form S of M: If $c_1(\tau)$ is divisible by 3 then S is isomorphic to $(2q+1)(+\mathbb{1}) \oplus q(-\mathbb{1})$, i.e. S is isomorphic to the intersection form of $(2q+1)(\mathbb{C}P^2)\sharp q(-\mathbb{C}P^2)$, for some non-negative integer q. Next we illustrate how Theorem 2.2 may be applied to 4-manifold with nice $Pin(2)$-action.

Proposition 4.8. *Let M be an oriented 4-manifold with nice $Pin(2)$-action. Let S denote the intersection form. Assume the $Pin(2)$-action has a fixed point.*

1. *Then S is either of odd type or trivial.*
2. *If M admits a $Pin(2)$-equivariant $Spin^{\mathbf{c}}$-structure then S is definite.*

Proof: In the course of the proof we will use the fact that the induced S^1-action lifts to any $Spin^{\mathbf{c}}$-structure: Let Q be the principal bundle of orthonormal frames. We lift the $Pin(2)$-action to Q via differentials. Let P be a $Spin^{\mathbf{c}}$-structure and $\xi : P \to Q$ the associated $U(1)$-principal bundle. We will show that the first Chern class of ξ is in the image of $H^2(Q_{S^1};\mathbb{Z}) \to H^2(Q;\mathbb{Z})$. To this end we compare the Leray-Serre spectral sequences $\{E_r^{p,q}\}$ and $\{\hat{E}_r^{p,q}\}$ for $Q_{S^1} \to BS^1$ and $Q_{Pin(2)} \to BPin(2)$, respectively. Note that in both spectral sequences all differentials restricted to the subgroup of bi-degree $(0,2)$ are trivial except maybe the second one. Since the action has a fixed point $Pin(2)$ acts trivially on $H^1(Q;\mathbb{Z})$. Thus these differentials are given by

$$H^2(Q;\mathbb{Z}) \cong E_2^{0,2} \xrightarrow{d_2} E_2^{2,1} \cong H^2(BS^1; H^1(Q;\mathbb{Z})) \cong \mathbb{Z}^{b_1(M)} \text{ and}$$

$$H^2(Q;\mathbb{Z}) \cong \hat{E}_2^{0,2} \xrightarrow{\hat{d}_2} \hat{E}_2^{2,1} \cong H^2(BPin(2); H^1(Q;\mathbb{Z})) \cong (\mathbb{Z}/2\mathbb{Z})^{b_1(M)}.$$

Since d_2 factorizes over $\hat{d}_2$ and the homomorphism induced by $S^1 \hookrightarrow Pin(2)$ we conclude that d_2 is zero on $E_2^{0,2}$. Thus $H^2(Q_{S^1};\mathbb{Z}) \to H^2(Q;\mathbb{Z})$ is surjective. By Theorem 3.1 and Theorem 3.2 the S^1-action lifts to P.

Ad 1: Assume the intersection form S is even. We want to show that S is trivial, i.e. $b_2(M) = 0$. Since S is even $w_2(M)$ is the mod 2 reduction of an integral torsion class (cf. [HiHo58]). We choose a $Spin^{\mathbf{c}}$-structure P_0 on M with $c_1(P_0)$ a torsion class and lift the S^1-action to P_0. As remarked earlier the argument in [AtHi70] shows that the index of the $Spin^{\mathbf{c}}$-Dirac operator $\partial_{\mathbf{c}}$ for P_0 vanishes:

$$ind(\partial_{\mathbf{c}}) = \langle e^{c_1(P_0)/2} \cdot (1 - p_1(M)/24), [M]\rangle = 0.$$

Since $c_1(P_0)$ is a torsion class $p_1(M)$ vanishes and by the signature theorem (cf. [Hi56]) the signature $sign(M) = \langle p_1(M)/3, [M]\rangle$ vanishes, too.

If $b_2(M) \neq 0$ the intersection form must be indefinite and we may choose classes $x, y \in H^2(M;\mathbb{Z})$ with $x^2 = y^2 = 0$ and $xy \neq 0$. Let $V = L_{2x} + L_{2y}$, where L_z denotes the complex line bundle with first Chern class z. By Proposition 3.6 the $Pin(2)$-action lifts to each line bundle of V. Next choose a $Spin^{\mathbf{c}}$-structure P with $c_1(P) \equiv c_1(V)$ modulo torsion and lift the S^1-action to P. Note that $p_1(V) = 0$. We are in the position to apply Corollary 2.3 for V and P as above and $W = 0$. It follows that $xy = 0$ contradicting the choice of x and y. Thus $b_2(M) = 0$ and S is trivial.

Ad 2: Assume S is indefinite. By the first part S is odd and hence of the form $p(+\mathbb{1}) \oplus q(-\mathbb{1})$ with basis $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$. Let L_c denote the complex line bundle over M induced by the given $Pin(2)$-equivariant $Spin^{\mathbf{c}}$-structure. Then $c_1(L_c)$ reduces to $w_2(M)$ modulo 2. Since L_c is $Pin(2)$-equivariant the differential $H^2(M;\mathbb{Z}) \to H^2(BPin(2); H^1(M;\mathbb{Z})) \cong$

$(\mathbb{Z}/2\mathbb{Z})^{b_1(M)}$ vanishes on $c_1(L_c)$ and any other integral lift of $w_2(M)$. Let $u \equiv \sum x_i + \sum y_j$ modulo torsion with mod 2 reduction equal to $w_2(M)$ and lift the $Pin(2)$-action to L_u. By Proposition 3.6 the $Pin(2)$-action also lifts to L_{2x_1} and L_{2y_1}. Let $V = L_u + L_{2x_1}$ and let $W = L_u + L_u + L_{2y_1}$. Finally we choose an S^1-equivariant $Spin^{\mathbf{c}}$-structure on M with first Chern class equal to $c_1(V)$.

By the signature theorem (cf. [Hi56]) $\langle p_1(M), [M]\rangle = 3(p-q)$. Since $p_1(V+W) = p_1(M)$ we may apply Corollary 2.3 for V, W and P as above to derive the contradiction $x_1^2 = 0$. Thus S is definite. ■

One may use Donaldson's deep classification theorem [Do87] (any negative definite intersection form of an oriented closed 4-manifold is of the form $(-\mathbb{1}) \oplus \ldots \oplus (-\mathbb{1})$) to improve the last result. Details are left to the reader.

Proposition 4.9. *Let M be an oriented 4-manifold with nice $Pin(2)$-action. Assume the $Pin(2)$-action has a fixed point and M admits a $Pin(2)$-equivariant $Spin^{\mathbf{c}}$-structure. Then $b_2(M) \leq 1$, i.e. the intersection form is trivial or of the form $(\pm\mathbb{1})$.* ■

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