ON THE TOPOLOGY OF SCALAR-FLAT MANIFOLDS

ANAND DESSAI

Abstract

Let M be a simply connected closed manifold of dimension greater than 4 which does not admit a metric with positive scalar curvature. We give necessary conditions for M to admit a scalar-flat metric. These conditions involve the first Pontrjagin class and the cohomology ring of M. As a consequence, any simply connected scalar-flat manifold of dimension greater than 4 with vanishing first Pontrjagin class admits a metric with positive scalar curvature. We also describe some relations between scalar-flat metrics, almost complex structures and the free loop space.

1. *Introduction*

In this paper we give restrictions on the possible topological type of simply connected closed scalar-flat manifolds. A Riemannian manifold is called *scalar-flat* if its scalar curvature vanishes identically. One way to obtain such manifolds (in dimension greater than 2) is to start with a metric with positive scalar curvature and decrease the scalar curvature by changing the metric. By the results of Kazdan and Warner [**11**], one can deform the metric globally by a conformal change to obtain a scalar-flat metric. Recently, Lohkamp [**13**] has shown that scalar decreasing deformations can even be carried out locally to achieve changes of the scalar curvature arbitrarily close to a prescribed decrease.

However, not every scalar-flat metric arises from a metric with positive scalar curvature. The positive energy theorem and the Lichnerowicz formula for Spin-manifolds imply that it is, in general, impossible to increase scalar curvature locally or globally. In the discussion that follows we shall call a scalar-flat manifold *strongly scalar-flat* if it does not admit a metric with positive scalar curvature. By a result of Bourguignon, such manifolds are Ricci-flat. Otherwise, one could deform the metric by using the Ricci tensor to obtain a new metric which would be pointwise conformal to one with positive scalar curvature [**11**].

Although strongly scalar-flat manifolds seem to be quite special, very little is known about their topological type. In contrast, the surgery theorems of Gromov and Lawson and of Schoen and Yau (see [**4**, **15**]) and Stolz' solution of the Gromov–Lawson conjecture [**16**] give a complete topological classification for simply connected manifolds of dimension greater than 4 which admit a metric with positive scalar curvature.

In Theorem 2.1 we give partial information on the topological type of strongly scalar-flat manifolds. Our result shows that many manifolds are not strongly scalar-flat. In particular, it implies the following theorem.

Received 26 March 1999; revised 22 November 1999.

2000 *Mathematics Subject Classification* primary 53C25, secondary 53C27, 53C29, 55P35.

THEOREM 1.1. *Let M be a simply connected scalar-flat manifold of dimension greater than 4. If the first real Pontrjagin class of M vanishes, then M is not strongly scalar-flat; that is, M admits a metric with positive scalar curvature.*

As a consequence of Stolz' theorem [**16**], a simply connected strongly scalar-flat manifold of dimension greater than 4 admits a non-vanishing harmonic spinor. Futaki proved in [**3**] that this implies that the manifold decomposes as a Riemannian product of irreducible strongly scalar-flat manifolds which are Ricci-flat Kähler or have holonomy Spin(7) (see also [**7**]). Theorem 2.1 follows from the properties of these factors. Futaki used this decomposition to give an upper bound for the absolute value of the $\hat{A}$-genus of strongly scalar-flat manifolds.

The paper is structured in the following way. In Section 2 we state our main result on the topology of strongly scalar-flat manifolds (see Theorem 2.1), and give a partial answer to the question of whether such manifolds are almost complex (see Proposition 2.2). In Section 3 we prove these statements, and observe that irreducible Ricci-flat manifolds with vanishing first Pontrjagin class have generic holonomy. In Section 4 we rephrase some of the results in terms of the free loop space of the manifold.

2. *Topology of strongly scalar-flat manifolds*

In this section we state some necessary conditions for a manifold to be strongly scalar-flat. We also give a partial answer to the question of whether strongly scalar-flat manifolds are almost complex. In the discussion that follows, all manifolds are smooth, connected, oriented and closed.

THEOREM 2.1. *Let M be a simply connected manifold of dimension greater than 4. If M is strongly scalar-flat, then M carries a closed 2-form ω and a closed 4-form Ω such that*

$$\langle \mathbf{e}^{[\omega]} \cdot \mathbf{e}^{[\Omega]} \cdot p_1(M), \mu_M \rangle \neq 0.$$

Here $[\omega]$ and $[\Omega]$ denote the induced real cohomology classes, μ_M is the fundamental cycle of M and $\langle , \rangle$ denotes the Kronecker pairing. Theorem 1.1 follows directly from Theorem 2.1.

We deduce Theorem 2.1 in the next section from the properties of Ricci-flat Kähler manifolds and manifolds with holonomy Spin(7). The first compact examples of Spin(7)-holonomy manifolds were constructed by Joyce [**9**]. Very little is known about their topological type. In the examples of [**9**, **10**], the signature is divisible by 8. In particular, their Euler characteristic is even. If this were true for all Spin(7)-holonomy manifolds, then any strongly scalar-flat manifold would be almost complex. A more precise statement is given in the following proposition.

PROPOSITION 2.2. *Let M be a simply connected strongly scalar-flat manifold of dimension greater than 4. Assume for some scalar-flat metric on M that any Spin(7)-holonomy factor of M has even Euler characteristic. Then M admits an almost complex structure with vanishing first Chern class. If* $\dim M = 4k$, *then*

$$\operatorname{sign}(M) \equiv (-1)^k \cdot \chi(M) \bmod 4.$$

We close this section with a few remarks.

REMARKS 2.3. (1) The method used to prove Theorem 2.1 also shows that the odd Stiefel–Whitney classes of a simply connected strongly scalar-flat manifold vanish.

(2) In contrast to the case of positive scalar curvature, strongly scalar-flat structures are not preserved under surgery. In fact, performing surgery (of any codimension) turns a strongly scalar-flat manifold with infinite fundamental group into a manifold for which the scalar curvature of any metric is negative somewhere. The situation for finite fundamental groups is similar.

3. *Holonomy of strongly scalar-flat manifolds*

In this section we prove Theorem 2.1, Proposition 2.2 and the following observation concerning the question of whether there exist irreducible Ricci-flat manifolds with generic holonomy (see [**1**, p. 19]).

OBSERVATION 3.1. *A simply connected irreducible Ricci-flat manifold with vanishing first real Pontrjagin class has generic holonomy.*

Theorem 2.1 relies on the following result of Futaki, which he deduces from the surgery theorem (see [**4**, **15**] and Stolz' solution of the Gromov–Lawson conjecture (see [**16**]) using the de Rham splitting theorem and Berger's classification of holonomy groups (see [**1**, Chapter 10]).

THEOREM 3.2. ([**3**, Theorem 1]). *Let M be a simply connected strongly scalar-flat manifold of dimension greater than 4. Then M is a Spin-manifold and splits as a Cartesian product of irreducible manifolds, where each factor admits a Ricci-flat Kähler metric or a Riemannian metric with holonomy* Spin (7).

In the next proposition we collect those topological properties of Ricci-flat Kähler manifolds and Spin(7)-holonomy manifolds which are used in the proof of Theorem 2.1. Note that a holonomy reduction leads to a corresponding topological reduction of the frame bundle. We always fix the orientation of the manifold induced by this reduction.

PROPOSITION 3.3. *Let (M, g) be a simply connected irreducible Riemannian manifold of dimension $n > 0$. Let H be the holonomy group of (M, g).*

(1) *If $H \subset \mathrm{SU}(n/2)$ (that is, (M, g) is Ricci-flat Kähler) and ω denotes the Kähler form, then*
$$\langle e^{[\omega]} \cdot p_1(M), \mu_M \rangle < 0 \quad \textit{and} \quad \langle e^{[\omega]}, \mu_M \rangle > 0.$$

(2) *If $H = \mathrm{Spin}(7)$ (implying that n=8), then M admits a closed 4-form Ω such that*
$$\langle e^{[\Omega]} \cdot p_1(M), \mu_M \rangle < 0 \quad \textit{and} \quad \langle e^{[\Omega]}, \mu_M \rangle > 0.$$

Proof. Let $H \subset \mathrm{SU}(m)$ where $m = n/2$. We note that $\langle e^{[\omega]}, \mu_M \rangle > 0$ since the m-fold wedge of the Kähler form ω is (up to a positive constant) the volume form. Also, $c_1(M) = 0$ implies $p_1(M) = -2 \cdot c_2(M)$. Thus $\langle e^{[\omega]} \cdot p_1(M), \mu_M \rangle$ is equal to

$\langle [\omega]^{m-2} \cdot c_2(M), \mu_M \rangle$ times a negative constant. We now use the following formula of Apte (see [**1**, Formula (2.80a)]):

$$\langle [\omega]^{m-2} \cdot c_2(M), \mu_M \rangle = C \cdot \int_M |R|^2 \mu_g.$$

Here, C is a positive constant and R denotes the curvature tensor viewed as a symmetric endomorphism on 2-forms. Note that (M, g), as a simply connected manifold of positive dimension, cannot be flat. Hence

$$\langle e^{[\omega]} \cdot p_1(M), \mu_M \rangle < 0.$$

Next, let $\dim M = 8$ and $H = \mathrm{Spin}(7) \hookrightarrow \mathrm{SO}(8)$. We adapt the argument for G_2-manifolds given in [**8**, p. 333]. First, we recall some standard facts about Spin(7)-holonomy manifolds (see [**2**, **9**]). Associated to the holonomy reduction is a parallel closed self-dual 4-form $\tilde{\Omega}$. (For an explicit local description see, for example, [**9**, p. 510].) Its stabilizer at each point is isomorphic to Spin(7). Up to a positive constant, $\tilde{\Omega} \wedge \tilde{\Omega}$ is the volume form of M.

The space of k-forms $\Gamma(\Lambda^k(M))$ splits orthogonally into components parametrized by the irreducible representations of the Spin(7)-module $\Lambda^k(\mathbb{R}^8)$. In particular, $\Gamma(\Lambda^2(M))$ decomposes as $\Gamma(\Lambda^2_7) \oplus \Gamma(\Lambda^2_{21})$, where $\Gamma(\Lambda^2_7)$ corresponds to the Spin(7)-representation $\Lambda^1(\mathbb{R}^7)$, and $\Gamma(\Lambda^2_{21})$ corresponds to the adjoint representation of Spin(7). Local computations (see [**2**, p. 546]) show that $\int_M \tilde{\Omega} \wedge \xi \wedge \xi = -\int_M |\xi|^2 \mu_g$ for any $\xi \in \Gamma(\Lambda^2_{21})$. So, for $\Omega := -\tilde{\Omega}$, we obtain

$$\int_M \Omega \wedge \Omega > 0 \quad \text{and} \quad \int_M \Omega \wedge \xi \wedge \xi = \int_M |\xi|^2 \mu_g.$$

By the Ambrose–Singer theorem (see [**1**, p. 291]), the components R_{ij} of the curvature tensor R are in $\Gamma(\Lambda^2_{21})$. Since the first Pontrjagin form $p_1(M, g)$ is given by a negative multiple of the trace $\mathrm{tr}(R \wedge R) = \sum R_{ij} \wedge R_{ij}$, we conclude that

$$\int_M \Omega \wedge p_1(M, g) = C \cdot \int_M |R|^2 \mu_g,$$

where $C < 0$ is a constant. Again, the integral is non-zero since (M, g) is not flat. Hence, $\langle e^{[\Omega]} \cdot p_1(M), \mu_M \rangle = \int_M \Omega \wedge p_1(M, g) < 0$.

Proof of Theorem 2.1. Let $M = K_1 \times \cdots \times K_r \times J_1 \times \cdots \times J_s$ be the splitting of the strongly scalar-flat manifold M into irreducible factors, where the holonomy of K_i is contained in the special unitary group (that is, K_i is Ricci-flat Kähler) and the holonomy of J_j is Spin(7) (see Theorem 3.2). On each factor we choose the orientation induced by the holonomy reduction. This gives an orientation of M. Let ω_i denote the Kähler form of K_i, and let Ω_j denote the 4-form given in Proposition 3.3. We put $\omega := \sum_i \omega_i, \Omega := \sum_j \Omega_j, A_i := \langle e^{[\omega_i]}, \mu_{K_i} \rangle, B_j := \langle e^{[\Omega_j]}, \mu_{J_j} \rangle$, $A := \prod_{i=1}^r A_i$ and $B := \prod_{j=1}^s B_j$. Next, we note that $\langle e^{[\omega]} \cdot e^{[\Omega]} \cdot p_1(M), \mu_M \rangle$ may be written as a sum, where each summand has one of the following forms:

$$\langle e^{[\omega_i]} \cdot p_1(K_i), \mu_{K_i} \rangle \cdot (A \cdot B)/A_i \quad \text{or} \quad \langle e^{[\Omega_j]} \cdot p_1(J_j), \mu_{J_j} \rangle \cdot (A \cdot B)/B_j.$$

By Proposition 3.3, each summand is negative. This concludes the proof.

Proof of Observation 3.1. Assume that M is an irreducible Ricci-flat manifold with reduced holonomy. We apply Berger's classification (see [**1**, Chapter 10]), and conclude that the holonomy of M is G_2, Spin(7) or contained in the special unitary group. In all these cases, the first real Pontrjagin class of M is non-zero, giving the desired contradiction (see [**8**] and Proposition 3.3 above).

For the proof of Proposition 2.2, we use the following lemma.

LEMMA 3.4. *Let N be an 8-dimensional Spin-manifold which admits a topological* Spin(7)*-reduction. Then N admits an almost complex structure with vanishing first Chern class if and only if the Euler characteristic $\chi(N)$ is even.*

Proof. In [**5**], Heaps showed, using results of Massey, that an oriented 8-dimensional manifold X admits a stable almost complex structure if and only if $w_2(X)$ is integral and $w_8(X) \in Sq^2(H^6(X;\mathbb{Z}))$. Note that the Steenrod operation $Sq^2 : H^6(N;\mathbb{Z}/2\mathbb{Z}) \to H^8(N;\mathbb{Z}/2\mathbb{Z})$ vanishes identically, since it is given by multiplication with $w_2(N) = 0$. Since $\chi(N)$ reduces to $w_8(N)$ modulo 2, we conclude that N admits a stable almost complex structure if and only if $\chi(N)$ is even. This gives one implication. Next, we assume that $\chi(N)$ is even, and we let ξ denote a stable almost complex structure. We may assume that $c_1(\xi) = 0$. (We replace ξ by $\xi + L - \bar{L}$, where L is a complex line bundle with $2 \cdot c_1(L) = -c_1(\xi)$.) It is well known that ξ induces an almost complex structure if and only if $c_4(\xi)$ is equal to the Euler class $e(N)$. We now use the fact that N admits a topological Spin(7)-reduction. In cohomological terms, the existence of such a reduction is equivalent to $8 \cdot e(N) = 4 \cdot p_2(N) - p_1(N)^2$; see, for example, [**12**, p. 349]. Expressing the Pontrjagin classes of N in terms of the Chern classes of ξ, we obtain $c_4(\xi) = e(N)$. Thus ξ induces an almost complex structure on N with vanishing first Chern class.

Proof of Proposition 2.2. By Theorem 3.2, the strongly scalar-flat manifold M splits as a Cartesian product of manifolds with holonomy contained in the special unitary group, and manifolds with Spin(7)-holonomy. The former are complex manifolds with vanishing first Chern class. The latter support almost complex structures with vanishing first Chern class, by Lemma 3.4 and the assumption. For the last statement, we recall from [**6**, p. 777] that any $4k$-dimensional almost complex manifold M satisfies the congruence $\mathrm{sign}(M) \equiv (-1)^k \cdot \chi(M) \bmod 4$.

4. *Relations to the free loop space*

Some of the previous results indicate that the first Pontrjagin class of a manifold M takes a special role in questions concerning strongly scalar-flat metrics or Ricci-flat metrics with generic holonomy. On the other hand, the vanishing of $p_1(M)$ is closely related to the existence of a Spin-structure on the free loop space $\mathscr{L}M$ of M.

Motivated by a paper of Stolz [**17**], we rephrase Theorem 1.1 and Observation 3.1 in terms of $\mathscr{L}M$ at the end of this section. In [**17**], Stolz gives a heuristic treatment of a 'Weitzenböck formula' for the free loop space of a Spin-manifold M with $\frac{p_1}{2}(M) = 0$, and conjectures that the Witten genus of M vanishes if M admits a metric with positive Ricci curvature. The conjecture implies the existence of simply connected Riemannian manifolds with positive scalar curvature which do not admit metrics with positive Ricci curvature.

Let M be a closed oriented connected n-dimensional Riemannian manifold with free loop space $\mathscr{L}M = C^\infty(S^1, M)$, where we identify S^1 with $\mathbb{R}/\mathbb{Z}$. Following [**17**], we define the scalar curvature of $\mathscr{L}M$ to be the map $\mathrm{scal}_{\mathscr{L}M} : \mathscr{L}M \to \mathbb{R}$, which assigns to a loop $\gamma \in \mathscr{L}M$ the average of the Ricci curvature in the direction of γ; that is,

$$\mathrm{scal}_{\mathscr{L}M}(\gamma) := \int_0^1 \mathrm{Ric}(\gamma'(z))\, dz.$$

Hence $\mathrm{scal}_{\mathscr{L}M}$ is positive on non-constant loops if M has positive Ricci curvature. Also, if M is Ricci-flat, then $\mathscr{L}M$ is scalar-flat; that is, $\mathrm{scal}_{\mathscr{L}M}$ vanishes identically if M is Ricci-flat. Conversely, if $\mathrm{Ric}(v) \neq 0$ for some unit tangent vector v, then it is easy to find a loop γ with $\gamma'(0) = v$ such that $\mathrm{scal}_{\mathscr{L}M}(\gamma) \neq 0$. Hence, if $\mathscr{L}M$ is scalar-flat, then M must be Ricci-flat.

Now let M be a simply connected manifold which carries a Spin-structure given by a $\mathrm{Spin}(n)$-principal bundle $P \to M$. The free loop space $\mathscr{L}M$ of M is called Spin if the $\mathscr{L}\,\mathrm{Spin}(n)$-principal bundle $\mathscr{L}P \to \mathscr{L}M$ admits a reduction to the basic central extension of $\mathscr{L}\,\mathrm{Spin}(n)$ by S^1. (For $n \neq 4$ this is the universal extension.) As shown in [**14**], the free loop space is Spin if $\frac{p_1}{2}(M)$ vanishes. Here, $\frac{p_1}{2}$ denotes the generator of $H^4(B\,\mathrm{Spin}(n); \mathbb{Z})$, which is half of the universal first Pontrjagin class. The converse holds at least if M is two-connected.

Next, assume that $\mathscr{L}M$ is Spin. Guided by the finite-dimensional situation, one would expect $\mathscr{L}M$ to carry a 'Dirac operator' acting on sections of the spinor bundle associated to the Spin-structure of $\mathscr{L}M$. In [**18**], Witten formally applied the Lefschetz fixed-point formula to this hypothetical Dirac operator, and derived a bordism invariant of the underlying manifold M. This invariant, known as the *Witten genus*, expresses the equivariant index of the 'Dirac operator' localized at the constant loops $M \subset \mathscr{L}M$.

In the finite-dimensional setting, the α-invariant (that is, the index of the ordinary Dirac operator) is an obstruction to positive scalar curvature metrics on a Spin-manifold, by the Weitzenböck formula (see [**7**]). Analogously, one would expect an obstruction (namely the Witten genus) to positive scalar curvature on the free loop space if $\mathscr{L}M$ is Spin. The first part of the following corollary shows that if one replaces positive scalar curvature by scalar-flatness, then the α-invariant is an obstruction.

Corollary 4.1. *Let M be a Riemannian manifold for which its free loop space $\mathscr{L}M$ is simply connected and Spin.*

(1) *Assume that M has dimension greater than 4. If $\mathscr{L}M$ is scalar-flat, then $\alpha(M) = 0$; that is, M admits a metric with positive scalar curvature.*
(2) *Assume that M is irreducible. If $\mathscr{L}M$ is scalar-flat, then M is Ricci-flat with generic holonomy group.*

Proof. Note that the homotopy groups of $\mathscr{L}M$ may be computed from the homotopy groups of M using the path fibration $\Omega M \to \Gamma M \to M$ and the fibration $\Omega M \to \mathscr{L}M \to M$. In particular, M is two-connected since $\mathscr{L}M$ is simply connected. As mentioned before, the existence of a Spin-structure for $\mathscr{L}M$ implies that $p_1(M)$ vanishes. Now the statements follow directly from Theorem 1.1 and Observation 3.1.

Acknowledgements. I thank D. Joyce, J. Lohkamp and U. Semmelmann for helpful discussions. I also thank the referee for valuable suggestions, and for pointing out a mistake in a previous version. Part of this work was completed during a visit at the SFB 478 of the University of Münster. I am grateful to the SFB for its hospitality.

NOTE ADDED IN PROOF. Some of the results of this paper have counterparts for non-trivial fundamental groups. For example, Theorem 1.1 is still true if $\pi_1(M)$ is a finite group for which the Gromov–Lawson–Rosenberg conjecture holds. (In this connection, see also the recent paper of Botvinnik and McInnes, 'On rigidly scalar-flat manifolds', math.DG/9911023, for relations between holonomy groups and the Gromov–Lawson–Rosenberg conjecture.) The class of simply connected strongly scalar-flat manifolds of dimension $n \geqslant 5$ which satisfy an upper diameter bound and a lower curvature bound contains only finitely many diffeomorphism classes. This follows from Cheeger's finiteness theorem or the results of Cheeger, Fukaya and Gromov on collapsed manifolds, described in 'Nilpotent structures and invariant metrics of collapsed manifolds', *J. Amer. Math. Soc.* 5 (1992) 327–372. Recently, Joyce has constructed Spin(7)-holonomy manifolds with odd Euler characteristic. Hence, in Proposition 2.2, the condition on the Euler characteristic is necessary.

References

1. A. L. BESSE, *Einstein manifolds* (Springer, Berlin, 1987).
2. R. L. BRYANT, 'Metrics with exceptional holonomy', *Ann. of Math.* 126 (1987) 525–576.
3. A. FUTAKI, 'Scalar-flat closed manifolds not admitting positive scalar curvature metrics', *Invent. Math.* 112 (1993) 23–29.
4. M. GROMOV and H. B. LAWSON, 'The classification of simply-connected manifolds of positive scalar curvature', *Ann. of Math.* 111 (1980) 209–230.
5. T. HEAPS, 'Almost complex structures on eight- and ten-dimensional manifolds', *Topology* 9 (1970) 111–119.
6. F. HIRZEBRUCH, *Gesammelte Abhandlungen* (Springer, Berlin, 1987).
7. N. HITCHIN, 'Harmonic spinors', *Adv. Math.* 14 (1974) 1–55.
8. D. JOYCE, 'Compact Riemannian manifolds with holonomy G_2, I, II', *J. Differential Geom.* 43 (1996) 291–375.
9. D. JOYCE, 'Compact 8-manifolds with holonomy Spin(7)', *Invent. Math.* 123 (1996) 507–552.
10. D. JOYCE, 'A new construction of compact 8-manifolds with holonomy Spin(7)', math.DG/9910002 (1999).
11. J. KAZDAN and F. WARNER, 'Prescribing curvature', *Proc. Sympos. Pure Math.* 27 (1975) 309–319.
12. H. B. LAWSON and M. L. MICHELSON, *Spin geometry* (Princeton University Press, 1989).
13. J. LOHKAMP, 'Scalar curvature and hammocks', *Math. Ann.* 313 (1999) 385–407.
14. D. A. MCLAUGHLIN, 'Orientation and string structures on loop spaces', *Pacific J. Math.* 155 (1992) 143–156.
15. R. SCHOEN and S.-T. YAU, 'On the structure of manifolds with positive scalar curvature', *Manuscripta Math.* 28 (1979) 159–183.
16. S. STOLZ, 'Simply connected manifolds with positive scalar curvature', *Bull. Amer. Math. Soc.* 23 (1990) 427–432.
17. S. STOLZ, 'A conjecture concerning positive Ricci curvature and the Witten genus', *Math. Ann.* 304 (1996) 785–800.
18. E. WITTEN, 'The index of the Dirac operator in loop space', *Elliptic curves and modular forms in algebraic topology*, Lecture Notes in Math. 1326 (ed. P. S. Landweber, Springer, New York, 1988) 161–181.

Department of Mathematics
University of Augsburg
D-86135 Augsburg
Germany

e-mail:
dessai@math.Uni-Augsburg.DE