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SOME REMARKS ON ALMOST AND STABLE ALMOST COMPLEX MANIFOLDS

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Abstract. In the first part we give necessary and sufficient conditions for the existence of a stable almost complex structure on a 10-manifold M with $H_1(M;\mathbb{Z})=0$ and no 2-torsion in $H_i(M;\mathbb{Z})$ for $i=2,3$. Using the Classification Theorem of Donaldson we give a reformulation of the conditions for a 4-manifold to be almost complex in terms of Betti numbers and the dimension of the $\pm$-eigenspaces of the intersection form. In the second part we give general conditions for an almost complex manifold to admit infinitely many almost complex structures and apply these to symplectic manifolds, to homogeneous spaces and to complete intersections.

1. Introduction

In this paper we study the problem of classifying the (stable) almost complex structures on a smooth closed oriented manifold M. A **(stable) almost complex structure** on M is a complex vector bundle F together with an orientation preserving isomorphism between the underlying (stable) real vector bundle of F and the (stable) tangent bundle of M. In a similar way one defines a (stable) complex structure for any oriented real vector bundle.

We will restrict to the question of existence and enumeration of (almost) complex structures on M. Since the answer only depends on the diffeomorphy type of M we try to describe the answer in terms of invariants such as characteristic classes, the cohomology ring and the Steenrod algebra of M.

In the first part we study the necessary and sufficient conditions (in the above sense) for M to be almost or stably almost complex. These are only known in dimension ≤ 8 (cf. [Wu52], [Eh50], [Ma61], [Th67] and [He70]). For a 10-manifold M they are known if $H_1(M;\mathbb{Z}/2\mathbb{Z})=0$ and $w_4(M)=0$ (cf. [Th67] and [He70]).

In Theorem 1.2 we give necessary and sufficient conditions for a 10-manifold M to be stably almost complex if $H_1(M;\mathbb{Z})=0$ and $H_i(M;\mathbb{Z})$, $i=2,3$, has no 2-torsion. No

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assumption on $w_4(M)$ is made. The theorem simplifies if M has "nice" cohomology, such as the one of complete intersections (cf. Corollary 1.3).

In Theorem 1.4 we give, using the Classification Theorem of Donaldson, a reformulation of the classical conditions for a 4-manifold M to be almost complex in terms of the Betti numbers and the dimension of the $\pm$-eigenspaces of the intersection form.

In the second part we study the question whether an almost complex $2n$-dimensional manifold admits infinitely many almost complex structures. In Theorem 2.2 and Theorem 2.3 we give a simple answer for a large class of manifolds. We apply these theorems to symplectic manifolds (cf. Corollary 2.4), homogeneous spaces (cf. Corollary 2.5) and complete intersections (cf. Example 2.6).

1. Part I

1.1. Stable almost complex structures on 10-manifolds

Let M be a 10-manifold with $H_1(M;\mathbb{Z}) = 0$ and assume $H_i(M;\mathbb{Z})$ contains no 2-torsion for $i = 2, 3$. Let $c \in H^2(M;\mathbb{Z})$ be a fixed class congruent to $w_2(M)$ mod 2.

Definition 1.1. Let $\mathcal{D}(M)$ be the linear subspace of $H^2(M;\mathbb{Z})$ consisting of elements $x \in H^2(M;\mathbb{Z})$, s.t.

$$x^2 + x \cdot c = 2z \tag{1.1}$$

for some (uniquely determined) $z \in H^4(M;\mathbb{Z})$ depending on x. $\mathcal{D}(M)$ does not depend on the choice of c.

Let $p : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and $q : \mathbb{Z} \hookrightarrow \mathbb{Q}$ denote the canonical projection and injection. For any $x \in \mathcal{D}(M)$ we can make the following choices (cf. Lemma 1.6.1 and Lemma 1.8):

- Choose $v_x \in H^6(M;\mathbb{Z})$, s.t. $p_*(v_x) = Sq^2(p_*(z))$.
- Choose a complex vector bundle F_x over M trivial over the 3-skeleton of M, s.t.

$$e^{\frac{c}{2}} \cdot ch(\widetilde{L_x} - \widetilde{F_x}) \equiv x + (\frac{x^3}{6} - \frac{xc^2}{8} - \frac{v_x}{2}) \bmod H^{\geq 8}(M;\mathbb{Q}).$$

Here L_x denotes the line bundle with first Chern class equal to x and $\widetilde{\ }$ denotes the reduced vector bundle. For a complex vector bundle F over M let $\hat{A}_c(M;F)$ be the twisted $Spin^c$-index of M. In cohomology $\hat{A}_c(M;F) = (\hat{\mathcal{A}}(M) \cdot e^{\frac{c}{2}} \cdot ch(F))[M]$, where $\hat{\mathcal{A}}$ denotes the multiplicative sequence of the $\hat{A}$-genus and $[M]$ denotes evaluation on the fundamental cycle of M. Let $G_x := \widetilde{L_x} - \widetilde{F_x}$. We are now ready to state

Theorem 1.2. *Let M be a 10-manifold with $H_1(M;\mathbb{Z}) = 0$ and assume $H_i(M;\mathbb{Z})$ contains no 2-torsion for $i = 2, 3$. Let c be a fixed integral class congruent to $w_2(M)$ mod 2. Then M admits a stable complex structure if and only if*

$$\hat{A}_c(M; G_x \otimes (\widetilde{\tau(M)} - \widetilde{L_c}) \otimes_{\mathbb{R}} \mathbb{C}) \equiv 0 \bmod 2 \tag{1.2}$$

holds for every $x \in \mathcal{D}(M)$.

The congruence (1.2) does not depend on the choice of v_x. In fact, a straight-forward calculation, using $w_4(M) \equiv \frac{c^2 - p_1(M)}{2} \bmod 2$, shows that (1.2) has the explicit form

$$\frac{4x^3p_1(M) + xp_1(M)^2 - 4xp_2(M) - 2c^2xp_1(M) - 4c^2x^3 + c^4x}{24} \equiv$$

$$\equiv Sq^2(p_*(z))w_4(M) \bmod 2.$$

We remark that one only needs to check (1.2) for a basis of $\mathcal{D}(M)$. Theorem 1.2 simplifies if M has "nice" cohomology, such as the one of complete intersections:

Corollary 1.3. *Let M be as in Theorem 1.2. Assume in addition that $H^2(M;\mathbb{Z})$ is generated by h and $h^2 \not\equiv 0 \bmod 2$. Then M admits a stable almost complex structure if and only if*

$$w_2(M) \cdot w_4(M)^2 = 0.$$

Thus the existence of a stable almost complex structure only depends on the homotopy type of M.

1.2. Almost complex structures on 4-manifolds

For an oriented 4-manifold M a classical Theorem of Wu (cf. [Wu52]) asserts that M is almost complex if and only if there is an integral class w with mod 2 reduction equal to $w_2(M)$, s.t.

$$(w^2)[M] = 3sign(M) + 2e(M). \tag{1.3}$$

Combining the Classification Theorem of Donaldson (the intersection form of a 4-manifold, which is definite, is standard, cf. [Do87]) with the classification of indefinite unimodular bilinear forms we give a reformulation in terms of Betti numbers and the dimension $b_\pm$ of the $\pm$-eigenspaces of the intersection form.

Theorem 1.4. *Let M be an oriented 4-manifold with Euler number $e(M)$, intersection form S and signature $sign(M) = b_+ - b_-$.*

1. *S indefinite: M is almost complex $\iff b_1 \not\equiv b_+ \bmod (2)$.*
2. *S positive definite: M is almost complex $\iff b_1 \not\equiv b_2 \bmod (2)$ and $b_1 - b_2 \leq 1$.*
3. *S negative definite: M is almost complex $\iff b_1 \not\equiv 0 \bmod (2)$ and if $b_2 \leq 2$ in addition $4(b_1 - 1) + b_2$ is equal to the sum of b_2 integer squares.*

This result is probably well-known but we only know a reference in the case that the intersection form is even (cf. [Ma88], Remark B2).

1.3. Proof of Theorem 1.2 and Corollary 1.3

In this section let M be always an oriented closed 10-manifold with $H_1(M;\mathbb{Z}) = 0$ and no 2-torsion in $H_i(M;\mathbb{Z})$ for $i = 2, 3$.

A real vector bundle E over a finite CW-complex X admits a stable complex structure if and only if the classifying map $f : X \to BO$ of the stable vector bundle lifts in the fibration $BU \to BO$ induced by the inclusion $U \hookrightarrow O$. Let

$$\mathcal{O}_k(h) \in H^k(X; \pi_{k-1}(O/U))$$

denote the obstruction to extending a given stable complex structure $h : X^{(k-1)} \to BU$ of $E|_{X^{(k-1)}}$ to the k-skeleton. For $k \leq 10$ the only obstructions which do not vanish in general are $\mathcal{O}_3(h) \in H^3(X;\mathbb{Z})$, $\mathcal{O}_7(h) \in H^7(X;\mathbb{Z})$, $\mathcal{O}_8(h) \in H^8(X;\mathbb{Z}/2\mathbb{Z})$ and $\mathcal{O}_9(h) \in H^9(X;\mathbb{Z}/2\mathbb{Z})$. These obstructions were studied by Massey:

Lemma 1.5. (cf. [Ma61]) *The first obstructions satisfy*

1. $\mathcal{O}_3(h) = \delta^*(w_2(E))$, $\mathcal{O}_7(h) = \delta^*(w_6(E))$ *and*
2. $\mathcal{O}_8(h)$ *has indeterminacy* $Sq^2 \circ p_*(H^6(X;\mathbb{Z}))$, *i.e. by changing* h *the obstruction* $\mathcal{O}_8(h)$ *takes all values of a certain coset* $\mathcal{O}_8 \in H^8(M;\mathbb{Z}/2\mathbb{Z})/Sq^2 \circ p_*(H^6(X;\mathbb{Z}))$.

Here δ denotes the Bockstein operator of the long exact cohomology sequence for

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \to 0.$$

The proof of Theorem 1.2 consists of several lemmata. The general idea is to describe $\mathcal{O}_8$ in terms of K-theory and the Chern character and use the integrality theorem for $Spin^c$-manifolds to derive the mod 2 condition given in Theorem 1.2.

Lemma 1.6. *M has the following properties:*

1. $p_* : H^i(M;\mathbb{Z}) \to H^i(M;\mathbb{Z}/2\mathbb{Z})$ *is surjective for* $i \neq 4, 5$ *and* $p_* \circ q_*^{-1}$ *is well-defined on* $q_*(H^8(M;\mathbb{Z}))$.
2. *The annihilator of* $Sq^2 \circ p_*(H^6(M;\mathbb{Z}))$ *with respect to the cup-product is equal to* $p_*(\mathcal{D}(M))$.
3. *Any stable complex vector bundle over the 7-skeleton of* M *extends to one over* M.
4. *A stable real vector bundle* E *over* M *admits a stable complex structure over the 7-skeleton and any stable complex structure* F *of* E *over the 8-skeleton can be extended to one over* M.

Proof. Ad 1: Since $H_i(M;\mathbb{Z})$ has no 2-torsion for $i = 0, 1, 2, 3, 10$ the same is true for $H^i(M;\mathbb{Z})$, $i \neq 5, 6$ (universal coefficient theorem and Poincaré Duality). Applying the long exact Bockstein sequence gives the first statement. The kernel of

$q_* : H^8(M;\mathbb{Z}) \to H^8(M;\mathbb{Q})$ is odd torsion which maps to zero under p_*. This proves the second statement.

Ad 2: Let $y \in H^2(M;\mathbb{Z}/2\mathbb{Z})$. Applying the Cartan formula and the definition of the Stiefel-Whitney classes in terms of Steenrod operations gives the following first equivalence

$$\begin{aligned} & y \cdot Sq^2 \circ p_*(z) = 0 \text{ for all } z \in H^6(M;\mathbb{Z}) \\ \iff & (w_2(M)\cdot y + y^2)\cdot p_*(z) = 0 \text{ for all } z \in H^6(M;\mathbb{Z}) \\ \iff & w_2(M)\cdot y + y^2 = 0 \text{ by Lemma 1.6.1} \end{aligned}$$

Since $p_* : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}/2\mathbb{Z})$ is surjective the statement follows from the definition of $\mathcal{D}(M)$.

Ad 3: Observe that the only obstruction for an extension of a map $M^{(7)} \to BU$ to M lives in $H^9(M;\mathbb{Z})$ and vanishes by Poincaré Duality.

Ad 4: The first statement follows from the assumptions about M and Lemma 1.5.1. The only obstruction to extending F lives in $H^9(M;\mathbb{Z}/2\mathbb{Z})$ which vanishes if $H_1(M;\mathbb{Z})$ is zero. This proves the second statement. □

Let ph denote the Pontrjagin character which is graded by $ph_k = ch_k \circ (\otimes_{\mathbb{R}}\mathbb{C})$. Let $r : K(M) \to KO(M)$ denote the realification map.

Lemma 1.7. *Let $E : M \to BSO$ be a stable oriented vector bundle over M. Then E admits a stable complex structure if and only if*

$$p_* \circ q_*^{-1}(ph_4(r(F) - E)) \in Sq^2 \circ p_*(H^6(M;\mathbb{Z}))$$

for some stable complex structure $F \in K(M)$ of E over the 7-skeleton.

Proof. One direction is trivial. So assume $p_* \circ q_*^{-1}(ph_4(r(F) - E))$ is an element of $Sq^2 \circ p_*(H^6(M;\mathbb{Z}))$. It suffices to show that $r(F) - E$ admits a stable complex structure. So without loss of generality we may assume E is trivial over the 7-skeleton and $p_* \circ q_*^{-1}(ph_4(E)) \in Sq^2 \circ p_*(H^6(M;\mathbb{Z}))$. According to [Ad61] there is an element $\xi \in K(M)$, trivial over the 5-skeleton, s.t. $2ch_4(\xi) = ph_4(E)$. Since $\pi_i(BO) = 0$ for $i = 6, 7$ the bundle $r(\xi)$ is trivial over the 7-skeleton. Again it suffices to show that $E' := E - r(\xi)$ admits a stable complex structure.

By construction E' is trivial over $M^{(7)}$ and $ph_4(E') = 0$. Since $H^8(M;\mathbb{Z})$ contains no 2-torsion an argument in the Atiyah-Hirzebruch spectral sequence (cf. [De91] for details) shows that E' admits a stable complex structure over the 8-skeleton and by Lemma 1.6.4 also over M.

Here we give a more transparent argument suggested by the referee. By the integrality of the Chern character (cf. [AtHi61], [Ad61]) the Pontrjagin character ph_4 for the universal bundle over the 7 connected cover $BO\langle 8\rangle$ of BO factorizes over $\mathbb{Z}$:

$$ph_4 : BO\langle 8\rangle \xrightarrow{ph_{4,0}} K(\mathbb{Z}, 8) \xrightarrow{q_*} K(\mathbb{Q}, 8).$$

Let F denote the fibre of the induced map $BO\langle 8\rangle \xrightarrow{ph'_{4,0}} K(\mathbb{Z}_{(2)}, 8)$, where $\mathbb{Z}_{(2)}$ denotes the localization of $\mathbb{Z}$ at the prime 2. Since E' is trivial over $M^{(7)}$ we may fix a lift

$f : M \to BO\langle 8\rangle$ of the classifying map for E'. Since $H^8(M;\mathbb{Z})$ has no 2-torsion and $ph_4(E') = 0$ the composition $ph'_{4,0} \circ f$ is nullhomotopic. Thus f factors via

$$M \to F \to BO\langle 8\rangle.$$

Now $H^8(F;\mathbb{Z}/2\mathbb{Z}) = 0$ (e.g. F is 6-connected, $\pi_7(F)$ is odd torsion and $\pi_8(F) = 0$). Thus the obstruction $\mathcal{O}_8$ for the universal bundle over F vanishes, so that $\mathcal{O}_8(E')$ vanishes, too.

□

The formulation of Theorem 1.2 is based on certain vector bundles $G_x = \widetilde{L_x} - \widetilde{F_x}$, which we describe in the following lemma.

Lemma 1.8. *Let c be an integral class congruent to $w_2(M)$ mod 2. For $x \in \mathcal{D}(M)$ there is a vector bundle $F_x \in K(M)$ trivial over the 3-skeleton, s.t.*

$$e^{\frac{c}{2}} \cdot ch(\widetilde{L_x} - \widetilde{F_x}) \equiv x + (\frac{x^3}{6} - \frac{xc^2}{8} - \frac{v_x}{2}) \bmod H^{\geq 8}(M;\mathbb{Q}).$$

Proof. Let $x \in \mathcal{D}(M)$. By the definition of $\mathcal{D}(M)$ there is a class $z \in H^4(M;\mathbb{Z})$, s.t. $x^2 + x\cdot c = 2z$. Then there exists a stable complex vector bundle $F_x \in K(M)$ trivial over $M^{(3)}$ and a cohomology class $v_x \in H^6(M;\mathbb{Z})$, s.t. $ch_2(F_x) = q_*(z)$, $q_*(v_x) = 2ch_3(F_x)$ and $p_*(v_x) = Sq^2(p_*(z))$. Moreover any cohomology class $v_x \in H^6(M;\mathbb{Z})$ satisfying $p_*(v_x) = Sq^2(p_*(z))$ can occur in this way (cf. [Ad61]). Let L_x be the complex line bundle with $c_1(L_x) = x$. Then it is straight-forward to check that

$$e^{\frac{c}{2}} \cdot ch(\widetilde{L_x} - \widetilde{F_x}) \equiv x + (\frac{x^3}{6} - \frac{x \cdot c^2}{8} - \frac{v_x}{2}) \bmod H^{\geq 8}(M;\mathbb{Q}).$$

□

We now come to the proof of Theorem 1.2. It is convenient to state the theorem in a more general form. Theorem 1.2 follows from the next one by choosing $E = \tau(M)$ and $d = c$.

Theorem 1.9. *Let M be a 10-manifold with $H_1(M;\mathbb{Z}) = 0$ and assume $H_i(M;\mathbb{Z})$ contains no 2-torsion for $i = 2, 3$. Let E be a stable real vector bundle over M. Choose integral classes c (resp. d) congruent to $w_2(M)$ mod 2 (resp. $w_2(E)$ mod 2). Then E admits a stable complex structure if and only if*

$$\hat{A}_c(M; G_x \otimes (\widetilde{E} - \widetilde{L_d}) \otimes_{\mathbb{R}} \mathbb{C}) \equiv 0 \bmod 2$$

holds for every $x \in \mathcal{D}(M)$.

Proof. We first assume E is spin and choose $d = 0$. The vector bundle E admits a stable complex structure F over the 7-skeleton (cf. Lemma 1.6.4). Without loss of generality we may assume $c_1(F) = 0$. By Lemma 1.6.3 the complex vector bundle F over $M^{(7)}$ may be extended to one over M (also denoted by F). Let

$$\mathcal{O}(F) := ph_4(E - r(F)) = ph(\widetilde{E} - \widetilde{r(F)}).$$

We want to show that

$$p_* \circ q_*^{-1}(\mathcal{O}(F)) \in Sq^2(p_*(H^6(M;\mathbb{Z}))),\text{ i.e. } (\mathcal{D}(M)\cdot\mathcal{O}(F))[M] \equiv 0 \bmod 2.$$

By Lemma 1.8 for every $x \in \mathcal{D}(M)$ there exists $G_x = \widetilde{L_x} - \widetilde{F_x}$, s.t.

$$e^{\frac{c}{2}} \cdot ch(G_x) = x + (\frac{x^3}{6} - \frac{xc^2}{8} - \frac{v_x}{2}) + \text{ higher terms.}$$

Thus

$$(x \cdot \mathcal{O}(F))[M] = (e^{\frac{c}{2}} \cdot ch(G_x) \cdot ph(\widetilde{E} - \widetilde{r(F)}) \cdot \hat{\mathcal{A}}(M))[M] =$$

$$(e^{\frac{c}{2}} \cdot ch(G_x) \cdot ph(\widetilde{E}) \cdot \hat{\mathcal{A}}(M))[M] - (e^{\frac{c}{2}} \cdot ch(G_x) \cdot ch(\widetilde{F} + \widetilde{\overline{F}}) \cdot \hat{\mathcal{A}}(M))[M]$$

for every $x \in \mathcal{D}(M)$. The class $e^{\frac{c}{2}} \cdot ch(G_x)$ has no terms of degree 0 and 4. Also $c_1(F) = 0$. This implies

$$(e^{\frac{c}{2}} \cdot ch(G_x) \cdot ch(F) \cdot \hat{\mathcal{A}}(M))[M] = (e^{\frac{c}{2}} \cdot ch(G_x) \cdot ch(\overline{F}) \cdot \hat{\mathcal{A}}(M))[M].$$

From the integrality theorem for $Spin^c$-manifolds (cf. [AtHi59]) follows

$$(x \cdot \mathcal{O}(F))[M] \equiv (e^{\frac{c}{2}} \cdot ch(G_x) \cdot ph(\widetilde{E}) \cdot \hat{\mathcal{A}}(M))[M] \bmod 2. \tag{1.4}$$

If E admits a stable complex structure we may choose F, s.t. $\mathcal{O}(F) = 0$. This proves one direction of the theorem for E spin.

If $(e^{\frac{c}{2}} \cdot ch(G_x) \cdot ph(\widetilde{E}) \cdot \hat{\mathcal{A}}(M))[M] \equiv 0 \bmod 2$ for every $x \in \mathcal{D}(M)$ the equation (1.4) implies

$$(\mathcal{D}(M) \cdot \mathcal{O}(F))[M] \equiv 0 \bmod 2.$$

Thus $p_* \circ q_*^{-1}(\mathcal{O}(F)) \in Sq^2 \circ p_*(H^6(M;\mathbb{Z}))$. By Lemma 1.7 E admits a stable complex structure over M. This shows the other direction for E spin.

Now assume E is not spin. Since E is stably almost complex if and only if the spin vector bundle $E - L_d$ is stably almost complex we can apply the above to $E - L_d$. This proves the general case. □

Proof. (Corollary 1.3) If M is spin the map $Sq^2 \circ p_* : H^6(M;\mathbb{Z}) \to H^8(M;\mathbb{Z}/2\mathbb{Z})$ is surjective by Lemma 1.6.2. Thus the obstruction $\mathcal{O}_8$ vanishes. By Lemma 1.6.4 and Lemma 1.7 M is stably almost complex.

If M is not spin let $c = h$. Then $\mathcal{D}(M) = \mathbb{Z}\langle h\rangle$. By Theorem 1.2 M is stably almost complex if and only if congruence (1.2) holds for $x = -h$. Since $x^2 + xc = 0$ we may choose $F_x = 0$ and $G_x = \widetilde{L_{-h}}$. Then (1.2) is equivalent to

$$\hat{A}_h(M; (1 - L_h^{-1}) \otimes \widetilde{\tau(M)} \otimes_{\mathbb{R}} \mathbb{C}) \equiv \hat{A}_h(M; (1 - L_h^{-1}) \otimes (\widetilde{L_h} + \widetilde{L_{-h}})) \bmod 2.$$

The right hand side of the last congruence is always even. This follows from the integrality theorem for $Spin^c$-manifolds and

$$\hat{A}_h(M; (1 - L_h^{-1}) \otimes \widetilde{L_h}) = \hat{A}_h(M; (1 - L_h^{-1}) \otimes \widetilde{L_{-h}}).$$

Thus M is stably almost complex if and only if

$$\hat{A}_h(M; (1 - L_h^{-1}) \otimes \widetilde{\tau(M)} \otimes_{\mathbb{R}} \mathbb{C}) \equiv 0 \bmod 2.$$

Let $N \overset{i}{\hookrightarrow} M$ be Poincaré dual to h. Then N is spin and the last congruence is equivalent to

$$\hat{A}(N; \tau(N) \otimes_{\mathbb{R}} \mathbb{C}) \equiv 0 \bmod 2. \tag{1.5}$$

For any oriented 8-manifold X

$$sign(X) = -\hat{A}(X, \tau(X) \otimes_{\mathbb{R}} \mathbb{C}) + 24\hat{A}(X).$$

A conceptual way to find this and other identities between twisted $\hat{A}$'s and twisted signatures is given in [Hi92]. Thus equation (1.5) is equivalent to $sign(N) \equiv 0 \bmod 2$, and hence is equivalent to $w_4(N)^2 = 0$ (since N is spin the reduction of the equation $45sign(N) = 7p_1(N)^2 - p_2(N)$ modulo 2 yields $sign(N) \equiv w_4(N)^2 \bmod 2$). Since $(w_2(M) \cdot w_4(M)^2)[M] = (w_4(N)^2)[N]$ the condition $w_4(N)^2 = 0$ can be rephrased as

$$w_2(M) \cdot w_4(M)^2 = 0.$$

This proves the Corollary. □

1.4. Proof of Theorem 1.4

Proof. From the classification of indefinite unimodular bilinear forms (cf. [MiHu73]) and the work of Donaldson (the intersection form of an oriented 4-manifold, which is definite, is standard, cf. [Do87]) follows that the intersection form

$$S : H^2(M; \mathbb{Z})/H^2_{torsion}(M; \mathbb{Z}) \to \mathbb{Z}$$

is isomorphic to either

$$\bigoplus_{b_+}(+1) \oplus \bigoplus_{b_-}(-1) \text{ if } S \text{ is odd or}$$

$$\bigoplus_{n} H \oplus m \bigoplus_{m} E_8, n > 0, m \in \mathbb{Z}, \text{ if } S \text{ is even.}$$

Here H denotes the standard hyperbolic form and E_8 the irreducible 8-dimensional positive definite even form. Let $w \in H^2(M; \mathbb{Z})/H^2_{torsion}(M; \mathbb{Z})$ be congruent to 0 mod 2 (S even) or congruent to the standard basis of $\bigoplus_{b_+}(+1) \oplus \bigoplus_{b_-}(-1)$ mod 2 (S odd). Then w lifts to a class in $H^2(M; \mathbb{Z})$ with $\mathbb{Z}/2\mathbb{Z}$ reduction equal to $w_2(M)$ (cf. [HiHo58], page 169).

By definition $(w^2)[M] \equiv sign(M) \bmod 8$. A short calculation shows

$$2sign(M) + 2e(M) \equiv 0 \bmod 8 \iff b_1 \not\equiv b_+ \bmod 2.$$

Hence

$$(w^2)[M] \equiv 3sign(M) + 2e(M) \bmod 8 \iff b_1 \not\equiv b_+ \bmod 2.$$

By the Theorem of Wu (cf. equation (1.3)) $b_1 \not\equiv b_+ \bmod 2$ is a necessary condition for M to be almost complex.

Ad 1: If S is indefinite $(w^2)[M]$ has indeterminacy $8\mathbb{Z}$. This proves 1.

Ad 2: Assume S to be positive definite. If M is almost complex, the Classification Theorem of Donaldson, the signature theorem and $p_*(c_1(M)) = w_2(M)$ imply $3sign(M) + 2e(M) \geq b_2$. This inequality is equivalent to $b_1 - b_2 \leq 1$. This gives one direction. For the other direction we need to show that $3sign(M)+2e(M)$ is the sum of b_2 odd squares (write w as a linear combination w.r.t. the standard basis). This follows for $b_2 < 3$ by examination and for $b_2 \geq 3$ by the Gauss lemma (compare [Se70], p.79).

Ad 3: If S is negative definite the argument is similar to the one given in 2. and is left to the reader. □

2. Part II

2.1. Enumeration of almost complex structures

Let M be a $2n$-dimensional manifold with almost complex structure τ. We study the enumeration problem, i.e. the question, whether M admits infinitely many almost complex structures. Here we identify two (stable) almost complex structures if they are isomorphic as (stable) complex vector bundles. The stable almost complex structures of M are given by the coset $\tau + \ker(r)$, where $r : K(M) \to KO(M)$ denotes the realification map. The connection to almost complex structures is given in the following lemma (cf. [Th67], [Pu88] or [De91]).

Lemma 2.1. *Let i be the inclusion $U(n) \hookrightarrow U$.*

1. *Then i induces a bijection $(Bi)_* : [M, BU(n)] \to [M, BU]$, e.g. a stable almost complex structure on M induces a complex vector bundle of rank n, which is unique up to isomorphism.*
2. *A stable almost complex structure F on M induces an almost complex structure if and only if $c_n(F)$ is equal to the Euler class of M.*

Thus the almost complex structures on M are given by $\tau + F$, where $F \in \ker(r)$ satisfies $c_n(\tau + F) = c_n(\tau)$, i.e. satisfies

$$c_n(F) + c_{n-1}(F) \cdot c_1(\tau) + \cdots + c_1(F) \cdot c_{n-1}(\tau) = 0. \tag{2.1}$$

This shows that the set of almost complex structures only depends on $c(\tau)$ and the homotopy type of M. The next two theorems give a complete answer to the enumeration problem for a large class of manifolds.

Theorem 2.2. *Let M be a $2n$-dimensional manifold with almost complex structure τ. Let b_i denote the i^{th}-Betti number of M and define $\mathcal{B} := \{2i \in 4\mathbb{Z}+2 \quad | \quad b_{2i} \geq 1\}$.*

1. *M has only finitely many almost complex structures if $\sum_{2i\in\mathcal{B}} b_{2i} \leq 1$.*

2. *M has infinitely many almost complex structures if* $\mathcal{B}$ *contains* $2i$, $2j$, *s.t.* $2i \neq 2j$ *and* $2i + 2j \neq 2n$ *or contains* $2l \neq n$, *s.t.* $b_{2l} \geq 2$.

3. *Let* $n = 2k$, k *odd and* $\mathcal{B} = \{2k\}$. *M has infinitely many almost complex structures if and only if the intersection form is indefinite and if* $(c_k(\tau)^2)[M] \neq 0$ *in addition* $b_{2k} \geq 3$ *holds.*

We remark that Theorem 2.2.2 remains true if we identify almost complex structures whose Chern classes are in the same orbit under the action of $Aut(H^*(M;\mathbb{Z}))$ (cf. [De91]).

The next theorem deals with 8-manifolds not covered by Theorem 2.2. Let M be an 8-manifold with $b_2(M) = 1$ and almost complex structure τ. Let h be a generator of $H^2(M;\mathbb{Z})/H^2_{torsion}(M;\mathbb{Z})$ and let d, c_1, γ and c_3 be integers defined by $(h^4)[M] = d$, $c_1(\tau) \equiv c_1 \cdot h$ modulo torsion, $(c_2(\tau) \cdot h^2)[M] = \gamma$ and $(c_3(\tau) \cdot h)[M] = c_3$.

Theorem 2.3. *Let* M, τ, d, c_1, γ *and* c_3 *be as above. Then M admits infinitely many almost complex structures if and only if*

$$c_1(3dc_1^3 - 12\gamma c_1 + 24c_3) = 0. \tag{2.2}$$

The next Corollary follows directly from Theorem 2.2.2 and generalizes the case $M = \mathbb{C}P^n$, for which the enumeration problem has been solved by Heaps (cf. [Th67]) and Puschnigg (cf. [Pu88] or [Hi87], page 776).

Corollary 2.4. *Let M be a* $2n$*-dimensional almost complex manifold. Assume there exists* $x \in H^2(M;\mathbb{Q})$, *s.t.* $x^n \neq 0$ *(e.g. M is a symplectic manifold). Then M admits infinitely many almost complex structures if* $n \neq 1, 2, 4$.

In the remaining cases not covered by Theorem 2.2 and Theorem 2.3 the enumeration problem is more complicated (cf. [De91]).

Next we apply the above to certain homogeneous spaces. Let G be a compact connected simple Lie group and U a maximal closed connected subgroup of maximal rank. A complete list of such inclusions is known. Let U be the connected centralizer of an element of order 3 or 5 or of an 1-torus. Then the manifold $M = G/U$ is homogeneous almost complex (cf. [BoHi58], p. 500, p. 521 and the literature cited there). By applying Theorem 2.2 and Theorem 2.3 to the list one may easily deduce the following

Corollary 2.5. *Let* $M = G/U$ *be as above. Then M admits only finitely many almost complex structures if and only if M is isomorphic to*

$$S^2, S^6, \mathbb{C}P^2, \mathbb{C}P^4, \frac{SO(6)}{SO(4) \times SO(2)} \text{ or } \frac{U(4)}{U(2) \times U(2)}.$$

2.2. Proof of Theorem 2.2 and Theorem 2.3

In the following we will often use the Newton formula

$$k \cdot c_k = c_{k-1} \cdot ch_1 \cdot 1! - c_{k-2} \cdot ch_2 \cdot 2! \pm \ldots + (-1)^{k-1} ch_k \cdot k!.$$

Proof. (Theorem 2.2) Atiyah and Hirzebruch proved $K(M)$ is finitely generated and the Chern character induces an isomorphism $K(M) \otimes \mathbb{Q} \cong H^*(M;\mathbb{Q})$ (cf. [AtHi61]). Thus $\ker(ch)$ is finite. It follows that M admits infinitely many almost complex structures if and only if M admits infinitely many almost complex structures with pairwise distinct Chern character. In the following we will study the enumeration problem by looking at the Chern character of the almost complex structures.

Ad 1: For any $F \in \ker(r)$ the Chern character lives in $H^{4*+2}(M;\mathbb{Q})$. If $\mathcal{B}$ is empty or $\mathcal{B} = \{2n\}$ condition (2.1) implies $F \in \ker(ch)$. The other case $\mathcal{B} = \{n\}$ and $b_n = 1$ follows from the third statement.

Ad 2: Choose $a \in \mathbb{N}$, s.t. $aH^*(M;\mathbb{Z})$ is torsion-free and for any $x \in aH^{4*+2}(M;\mathbb{Z})$ there exists $F_x \in \ker(r)$ with $ch(F_x) = x$. Let $\{x_i\}$ be a basis of homogeneous elements for the free $\mathbb{Z}$-module $aH^{4*+2}(M;\mathbb{Z})$. Since $c(F_{a_i \cdot x_i}) = e^{(l-1)! \cdot a_i \cdot x_i}$ for $a_i \in \mathbb{Z}$ and x_i of degree $2l$, the left hand side of (2.1) restricted to the $\mathbb{Z}$-span of the F_{x_i}'s may be considered as a polynomial in a_i with coefficients in $\mathbb{Q}$. Now the second statement follows from a straight-forward computation. For example assume $\mathcal{B} = \{2i, 2n\}$, $i < n$, and let x_i (resp. x_n) be a generator of $aH^{2i}(M;\mathbb{Z})$ (resp. $aH^{2n}(M;\mathbb{Z})$). Then (2.1) takes for $F = F_{a_i \cdot x_i} + F_{a_n \cdot x_n}$ the form $a_n = R(a_i)$, where R is a polynomial with rational coefficients and constant term zero. This equation has infinitely many solutions $(a_i, a_n) \in \mathbb{Z}^2$. The other cases are similar; details are left to the reader.

Ad 3: Let $L := H^n(M;\mathbb{Z})/H^n_{torsion}(M;\mathbb{Z})$ and let Φ be the quadratic form

$$\Phi : L \to \mathbb{Z}, \quad x \mapsto (x^2)[M].$$

Then equation (2.1) is equivalent to

$$\Phi((k-1)! \cdot ch_k(F) + c_k(\tau)) = \Phi(c_k(\tau)). \tag{2.3}$$

If S is definite the last equation has only finitely many solutions. In fact the equation $\Phi(x) = \mathit{constant}$ has only finitely many solutions restricted to a lattice of $H^n(M;\mathbb{Q})$, which contains all possible elements $(k-1)!ch_k(F) + c_k(\tau)$ for $F \in \ker(r)$. If S is indefinite and $\Phi(c_k(\tau)) = 0$ equation (2.3) has infinitely many solutions (choose F_x, where x is a multiple of $c_k(\tau)$ or a multiple of an indefinite element).

Next assume S is indefinite and $\Phi(c_k(\tau)) \neq 0$. If $b_n = 2$ the intersection form S is isomorphic to H or $(+1) \oplus (-1)$, where H denotes the standard hyperbolic form. In any case only finitely many elements are mapped under Φ to $\Phi(c_k(\tau))$.

We are left with the case $b_n \geq 3$. It suffices to prove that the set

$$\mathcal{C} := \{x \in \lambda \cdot L \quad | \quad \Phi(x + c_k(\tau)) = \Phi(c_k(\tau))\}$$

contains infinitely many elements, where $\lambda = (k-1)! \cdot a$. Choose decompositions $L' \oplus H$ and $c_{L'} + c_H$ for L and $c_k(\tau)$, where H denotes the standard hyperbolic form with basis v_1, v_2. If $\Phi(c_H) = 0$, any multiple of λc_H or λv_1 will be in $\mathcal{C}$ and we are done.

So assume $\Phi(c_H) \neq 0$, i.e. $c_H = \alpha v_1 + \beta v_2$ with $\alpha \cdot \beta \neq 0$. Now choose $d_{L'} \in L'$ congruent to $c_{L'}$ mod $2\beta \cdot \lambda$ and $d_H := (\alpha - m \cdot \lambda)v_1 + \beta \cdot v_2$, where m is defined by $2m \cdot \beta \cdot \lambda = \Phi(d_{L'}) - \Phi(c_{L'})$. Then $d := d_{L'} + d_H$ is congruent to $c_k(\tau)$ mod λ and $\Phi(d) = \Phi(c_k(\tau))$. This completes the proof. □

Proof. (Theorem 2.3) Let $g \in H^6(M;\mathbb{Z})/H^6_{torsion}(M;\mathbb{Z})$ be the class dual to h, i.e. $(h \cdot g)[M] = 1$. For $F \in \ker(r)$ let $(A(F), B(F))$ be the pair of rational numbers defined by $ch(F) = A(F) \cdot h + B(F) \cdot g$. Then equation (2.1) takes the form

$$\beta \cdot Q(\alpha) = P(\alpha) \tag{2.4}$$

for $(\alpha, \beta) = (A(F), B(F))$, where

$$Q(\alpha) = 2(c_1 + \alpha) \text{ and } P(\alpha) = -\alpha \frac{d(\alpha^3 + 4\alpha^2 \cdot c_1) + 12\alpha \cdot \gamma + 24c_3}{24}.$$

Since $24P(-c_1)$ is equal to the left hand side of equation (2.2), we want to show that M admits infinitely many almost complex structures if and only if $P(-c_1) = 0$.

Let $P(-c_1) = 0$. By the argument from the beginning of the proof of Theorem 2.2 there is a natural number a, s.t. any pair $(\alpha, \beta) \in a\mathbb{Z}^2$ can be realized as $(A(F), B(F))$ for some $F \in \ker(r)$. If $c_1 = 0$ any pair $(0, \beta) \in a\mathbb{Z}^2$ is a solution of (2.4). If $c_1 \neq 0$ the quotient $\frac{P(\alpha)}{Q(\alpha)}$ is a polynomial in α with constant term equal to zero. In this case we may choose a suitable non-zero integer, s.t. for any multiple k of this integer $(ka, \frac{P(ka)}{Q(ka)})$ is in $a\mathbb{Z}^2$ and is a solution of (2.4). Thus M admits infinitely many almost complex structures if $P(-c_1) = 0$.

Let $P(-c_1) \neq 0$. If we divide equation (2.4) by $Q(\alpha)$ and multiply by a suitable non-zero integer (2.4) takes the form $C \cdot (3\beta) = \hat{P}(\alpha) + \frac{R}{Q(\alpha)}$, where C and R are integers, $R \neq 0$ and $\hat{P}(\alpha)$ is a polynomial with integer coefficients. If $(\alpha, \beta) = (A(F), B(F))$ for some $F \in \ker(r)$ then α and 3β have to be integers. Thus we are looking for $\alpha \in \mathbb{Z}$, s.t. $\hat{P}(\alpha) + \frac{R}{Q(\alpha)} \in \mathbb{Z}$. This is only possible for finitely many choices. Thus in this case there are only finitely many almost complex structures. This proves the theorem. □

We remark that it is possible to determine the complete set of all almost complex structures in this situation (cf. [De91]). Applications are given in Section 2.3..

2.3. Applications

In this section we apply Theorem 2.3 to certain complete intersections. Let M^d be a hypersurface of complex dimension 4. Then the left hand side of (2.2) is a polynomial in d. A calculation shows that the only integer solutions are $d = 2$ and $d = 6$. From Theorem 2.3 follows that M^d admits infinitely many almost complex structures if and only if $d = 2$ or $d = 6$. We remark that a complete intersection $M^{d_1,\dots,d_r}$ of complex dimension 4 admits only finitely many almost complex structures if $M^{d_1,\dots,d_r}$ is not spin or $d_i \gg r$ for all i (cf. [De91]).

Let h be the standard generator of $H^2(M^d;\mathbb{Z})$ and let $g \in H^6(M^d;\mathbb{Z})$ be the class dual to h, i.e. $(h \cdot g)[M] = 1$. For $d \leq 4$ the almost complex structures are given in the following list (the standard complex structure has Todd genus equal to 1).

Example 2.6.

$M^{(1)}$:

$$\begin{array}{llll}
c(E) & = 1 + h - 2h^2 + 2g + 5g \cdot h & Td(E) & = 0 \\
c(E) & = 1 + 5h + 10h^2 + 10g + 5g \cdot h & Td(E) & = 1 \\
c(E) & = 1 + 25h + 310h^2 + 1922g + 5g \cdot h & Td(E) & = 1001
\end{array}$$

$M^{(3)}$:

$$\begin{array}{llll}
c(E) & = 1 + h + 2h^2 - 30g + 27g \cdot h & Td(E) & = 0 \\
c(E) & = 1 + 3h + 6h^2 + 6g + 27g \cdot h & Td(E) & = 1 \\
c(E) & = 1 + 9h + 42h^2 + 290g + 27g \cdot h & Td(E) & = 55 \\
c(E) & = 1 + 29h + 422h^2 + 9210g + 27g \cdot h & Td(E) & = 5565 \\
c(E) & = 1 + 87h + 3786h^2 + 247134g + 27g \cdot h & Td(E) & = 447931 \\
c(E) & = 1 + 261h + 34062h^2 + 6667930g + 27g \cdot h & Td(E) & = 36256870
\end{array}$$

$M^{(4)}$:

$$\begin{array}{llll}
c(E) & = 1 + 2h + 7h^2 - 32g + 188g \cdot h & Td(E) & = 1 \\
c(E) & = 1 + 4h + 13h^2 + 44g + 188g \cdot h & Td(E) & = 6 \\
c(E) & = 1 + 8h + 37h^2 + 322g + 188g \cdot h & Td(E) & = 56 \\
c(E) & = 1 + 14h + 103h^2 + 1504g + 188g \cdot h & Td(E) & = 441 \\
c(E) & = 1 + 28h + 397h^2 + 11252g + 188g \cdot h & Td(E) & = 6566 \\
c(E) & = 1 + 56h + 1573h^2 + 88366g + 188g \cdot h & Td(E) & = 103096
\end{array}$$

$M^{(2)} : i \in \mathbb{Z}$

$$\begin{array}{ll}
c(E_i) & = 1 - h^2 + 2i \cdot g + 6g \cdot h \\
Td(E_i) & = 0 \\
c(\hat{E}_i) & = 1 + (4 + 2i)h + (7 + 8i + 2i^2)h^2 + (12 + 22i + 12i^2 + 2i^3)g + 6g \cdot h \\
Td(\hat{E}_i) & = \frac{12+28i+23i^2+8i^3+i^4}{12}
\end{array}$$

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