Torus actions on homotopy complex projective spaces

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May 22, 2003

Abstract

We prove that an effective action of a torus T on a homotopy $\mathbb{C}P^m$ is linear if $m < 4\cdot\mathrm{rk}(T)-1$. Examples show that the bound is optimal. Combining this with a theorem of Hattori we conclude that the total Pontrjagin class of such a manifold is given by the usual formula $(1+x^2)^{m+1}$.

1 Introduction

In this paper we study torus actions of large rank on homotopy complex projective spaces. Let T be a torus and let M be a homotopy $\mathbb{C}P^m$ with smooth T-action. We fix a generator $x \in H^2(M;\mathbb{Z})$ and denote by γ a T-equivariant complex line bundle over M with first Chern class equal to x.

By restricting the tangent bundle TM and the line bundle γ to T-fixed points one obtains a set of T-representations. The action is called linear if for some linear T-action on $\mathbb{C}^{m+1}$ the induced action on the canonical line bundle over $\mathbb{C}P^m$ gives the same representations (see Section 2 for a precise definition).

In [6, 8] Petrie constructed examples of S^1-actions on homotopy $\mathbb{C}P^{4r-1}$'s which are not linear. These "exotic actions" extend to effective actions by a torus of rank r. The linear actions and Petrie's exotic actions are the only known actions on homotopy complex projective spaces. In this paper we show

Theorem 1.1. *Let M be a homotopy $\mathbb{C}P^m$ with smooth effective action by a torus T of rank r. If $m < 4r-1$ then the T-action is linear.*

Petrie conjectured that the total Pontrjagin class of a homotopy $\mathbb{C}P^m$ is standard (i.e. of the form $(1+x^2)^{m+1}$) if the manifold admits a smooth effective S^1-action. He proved his conjecture for S^1-actions which extend to smooth effective actions by a torus of rank m [7]. Hattori has shown that the conjecture is true for linear S^1-action [2]. Combining his result with Theorem 1.1 gives

*The second named author is an Alfred P. Sloan Research Fellow and was partly supported by an NSF grant.

Corollary 1.2. *Let M be a homotopy $\mathbb{C}P^m$ which admits a smooth effective action by a torus T of rank r. If $m < 4r - 1$ then the total Pontrjagin class of M is standard.* ■

It follows from simply-connected surgery theory that for fixed $m \geq 3$ the set of diffeomorphism classes of homotopy $\mathbb{C}P^m$'s is infinite and partitioned into finite subsets by their total Pontrjagin class. Hence, Corollary 1.2 implies

Corollary 1.3. *For $m < 4r - 1$, $m \neq 2$, the class of homotopy $\mathbb{C}P^m$'s which admit a smooth effective action by a torus T of rank r contains only finitely many diffeomorphism types.* ■

Remark 1.4. a) It is known that a compact manifold M which is homotopically equivalent to $\mathbb{C}P^m$ has a standard Pontrjagin class if and only if M is tangentially homotopically equivalent to M, i.e., there is a homotopy equivalence $h\colon M \to \mathbb{C}P^m$ such that the pull back bundle $h^*T\mathbb{C}P^m$ is stabily isomorphic to the tangent bundle TM. This in turn is equivalent to saying that for some $k > 0$ the manifolds $M \times \mathbb{R}^k$ and $\mathbb{C}P^m \times \mathbb{R}^k$ are diffeomorphic.

b) The paper was partly motivated by [11], where it is shown that a simply connected positively curved n–dimensional ($n \neq 7$) Riemannian manifold (M^n, g) that supports an isometric effective action of a r-dimensional torus with $r \geq \frac{n}{4} + 1$ is homeomorphic to $\mathbb{S}^n$ or $\mathbb{H}P^{n/4}$ or homotopically equivalent to $\mathbb{C}P^{n/2}$. By Corollary 1.2 the conclusion can be improved to tangentially homotopically equivalent.

The idea for this paper was developed during a stay of the authors at the University of Pennsylvania. The first named author would like to thank Wolfgang Ziller and the University of Pennsylvania for hospitality. The second named author would like to thank Igor Belegradek for pointing out Remark 1.4 a).

2 Basic properties

In this section we recall basic properties of torus actions on integral cohomology $\mathbb{C}P^m$'s. These are used in the next section to prove a slightly more general version of Theorem 1.1. As a general reference we recommend [6, 1, 4].

Recall that a smooth closed manifold M is an integral cohomology $\mathbb{C}P^m$ if $H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/(x^{m+1})$, where x has degree 2. Any homotopy $\mathbb{C}P^m$ is an integral cohomology $\mathbb{C}P^m$. The converse is true for simply-connected manifolds.

Assume a torus T acts smoothly on M. Let $\gamma \to M$ be a complex line bundle over M with $c_1(\gamma) = x$ (the "Hopf bundle"). Hattori and Yoshida have shown that the T-action lifts to γ and any two lifts differ by a complex one-dimensional T-representation [3]. We fix a lift.

Let X be a connected component of the fixed point manifold M^T and let $pt \in X$. By restricting γ to pt one obtains a complex one-dimensional T-representation χ_X, the *Hopf representation* at X. At the fixed point pt the tangent bundle splits as a direct sum of the tangent space of X and the *normal representation* N_X which, by definition, is the normal bundle of X restricted to pt. Since X is a trivial T-space the isomorphism class of the real representation N_X and the isomorphism class of the complex representation χ_X are independent of the choice of the point pt in X. Note also that for two connected components $X, Y \subset M^T$ the isomorphism class of the complex representation $\chi_Y \cdot \chi_X^{-1}$ is independent of the lift by [3].

It will be convenient to use the following notation: Let W_1, W_2 be two T-representations (real or complex) and $\widetilde{T} \subset T$.

- $W_1 \cong_{\mathbb{R}} W_2$ if W_1 and W_2 are isomorphic as real T-representations.
- $W_1 \cong_{\mathbb{C}} W_2$ if W_1 and W_2 are complex representations which are isomorphic as complex T-representations.
- $W_1 \cong_{(\widetilde{T},\mathbb{R})} W_2$ (resp. $W_1 \cong_{(\widetilde{T},\mathbb{C})} W_2$) if W_1 and W_2 are isomorphic as real (resp. complex) $\widetilde{T}$-representations.

The localization theorem for cohomology or K-theory leads to strong relations between M and M^T (cf. [6]; [1], Ch. VII; [4], Ch. VI).

Proposition 2.1. *Let M be an integral cohomology $\mathbb{C}P^m$ with smooth T-action. Then the following holds:*

1. *The restriction of $x \in H^2(M;\mathbb{Z})$ to a connected component $X \subset M^T$ generates $H^*(X;\mathbb{Z})$. In particular, X is an integral cohomology complex projective space.*
2. $\dim_{\mathbb{C}} M + 1 = \sum_{X \subset M^T} (\dim_{\mathbb{C}} X + 1)$.
3. *Two connected components $X, Y \subset M^T$ are equal if and only if $\chi_X \cong_{\mathbb{C}} \chi_Y$.*
4. *The normal representation N_X, $X \subset M^T$, admits a T-equivariant complex structure such that*

$$\det N_X \cong_{\mathbb{C}} \det \left(\bigoplus_{Y \subset M^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1} \right).$$

■

Here $\dim_{\mathbb{C}}$ denotes half of the dimension. If M is the complex projective space $\mathbb{C}P^m$ and the action is induced by a linear T-action on $\mathbb{C}^{m+1}$ then the normal representation N_X at $X \subset M^T$ is isomorphic to the direct sum of the representations $(\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}$, where the sum runs over the connected components $Y \subset M^T$ different from X. For an integral cohomology $\mathbb{C}P^m$ we make the

Definition 2.2. *The T-action is linear if for every $X \subset M^T$*

$$N_X \cong_{\mathbb{R}} \bigoplus_{Y \subset M^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{1}$$

Remarks 2.3. 1. *If (1) holds then the tangential representations $T_{pt}M$, $pt \in X \subset M^T$, and the Hopf representations χ_X are isomorphic to the ones for a T-action on the canonical line bundle over $\mathbb{C}P^m$ induced by a linear action on $\mathbb{C}^{m+1}$.*

2. *A smooth S^1-action on an integral cohomology projective space M is linear if M^{S^1} has less than 4 connected components [9, 10, 12].*

3. *A smooth torus action on an integral cohomology projective space M is linear if $\dim_{\mathbb{C}} M < 3$ (to see this apply the last remark and Proposition 2.1, Part 2, to a suitable S^1-subgroup).*

Next we extend the notion of linearity to certain normal subspaces. Let $\widetilde{T}$ be a subtorus of T, $\widetilde{V} := (N_X)^{\widetilde{T}}$ and $\widetilde{F}$ the connected component of $M^{\widetilde{T}}$ which contains X.

Definition 2.4. *The T-action is linear on $\widetilde{V}$ if*

$$\widetilde{V} \cong_{\mathbb{R}} \bigoplus_{Y \subset \widetilde{F}^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{2}$$

Note that T acts linearly on M if and only if T acts linearly on N_X for every connected component $X \subset M^T$.

We shall be interested in the case where $\widetilde{T}$ is the identity component of the kernel of an irreducible T-subrepresentation $\widetilde{R} \subset N_X$ (i.e. $\widetilde{T}$ is the maximal subtorus of T acting trivially on $\widetilde{R}$). Let $T_1, \ldots, T_k$ denote the different subtori arising in this way and let $V_j := (N_X)^{T_j}$. Note that N_X is the direct sum of the V_j and that two irreducible representations R, $\widetilde{R}$ belong to the same V_j if and only if their kernels have the same identity components. The next lemma shows that linearity can be detected locally.

Lemma 2.5. *The T-action on M is linear if and only if the T-action is linear on $V_j \subset N_X$ for all X and all V_j.*

Proof: Assume the T-action on M is linear. By restricting to trivial T_j-representations in (1) one obtains

$$V_j \cong_{\mathbb{R}} \bigoplus_{Y \subset M^T, Y \neq X, \chi_Y \cong_{(T_j, \mathbb{C})} \chi_X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}.$$

Let F_j denote the connected component of M^{T_j} which contains X. By Proposition 2.1 $\chi_Y \cong_{(T_j, \mathbb{C})} \chi_X$ if and only if $Y \subset F_j$. Hence, T acts linearly on V_j.

Next assume that for all X and all $V_j \subset N_X$ the T-action is linear on V_j, i.e.

$$V_j \cong_{\mathbb{R}} \bigoplus_{Y \subset F_j^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{3}$$

To show that the T-action on M is linear it suffices to show that T acts linearly on N_X. Consider a connected component $Y \subset M^T$ with $X \neq Y$. By Proposition 2.1 the representation $\chi_Y \cdot \chi_X^{-1}$ is nontrivial and hence the identity component of the kernel is a codimension one subtorus $\widetilde{T}$. Again by Proposition 2.1 X and Y are contained in the same component $\widetilde{F}$ of $M^{\widetilde{T}}$. This proves $\widetilde{T} = T_j$ and $\widetilde{F} = F_j$ for j suitable. Conversely if $Y \subset F_j$, then T_j is necessarily given by the identity component of the kernel of $\chi_Y \cdot \chi_X^{-1}$. In summary we can say that Y belongs to precisely one F_j. Also $N_X = \bigoplus_j V_j$. By summing up (3) it follows that T acts linearly on N_X. ■

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1 for integral cohomology $\mathbb{C}P^m$'s by induction on the rank of the action. In the induction step we will use the fact that a T-action is linear if some S^1-subgroup acts with low codimension.

Proposition 3.1. *Let M be an integral cohomology $\mathbb{C}P^m$ with smooth effective T-action. The T-action is linear if one of the following holds:*

1. $\operatorname{codim}_{\mathbb{C}} M^{S^1} < 3$ *for some* $S^1 \subset T$.
2. $\operatorname{codim}_{\mathbb{C}} M^{S^1} = 3$ *for some* $S^1 \subset T$ *and* $\dim_{\mathbb{C}} M \neq 3$.

Proof: For the S^1-subgroup itself linearity follows from work of Masuda, Tsukada-Washiyama, Wang, Yoshida and others: If M^{S^1} has at most 3 connected components then the S^1-action is linear [9, 10, 12]. By Proposition 2.1 this is the case if $\operatorname{codim}_{\mathbb{C}} M^{S^1} < 3$. If M^{S^1} has more than 3 connected components and if $\operatorname{codim}_{\mathbb{C}} M^{S^1} = 3$ then the number of connected components is 4 by Proposition 2.1. Masuda has shown that an S^1-action with 4 fixed point components is linear if the components don't have the same dimension (cf. [5], Lemma 5.4). Since $\dim_{\mathbb{C}} M \neq 3$ the fixed point component of complex codimension 3 has positive dimension. The other components are isolated fixed points by Proposition 2.1. Hence, T acts linearly by [5]. This completes the proof in the case that the rank of T is one.

So assume the rank of T is ≥ 2. Let $S^1 \subset T$ be as in the proposition and let $M_0^{S^1} \subset M^{S^1}$ be a component of minimal codimension. By the above S^1 acts linearly on M. To show linearity for T it suffices to show that the T-action is linear on $V_j \subset N_X$ for all $X \subset M^T$ and all V_j by Lemma 2.5.

We claim that T acts linearly on V_j if S^1 acts non-trivially on V_j or if $X \not\subset M_0^{S^1}$. Assume first that S^1 acts non-trivially on V_j. In particular, T is generated by T_j and S^1. Since S^1 acts linearly on M the S^1-action is linear on V_j by Lemma 2.5. Since T_j acts trivially on F_j (notation as in the proof of Lemma 2.5) it follows that T acts linearly on V_j. Next assume $V_j \subset N_X$ and $X \not\subset M_0^{S^1}$. By the previous case we may assume that S^1 acts trivially on V_j and hence by Proposition 2.1 $\dim_{\mathbb{C}} F_j \leq 2$. The claim now follows from Remark 2.3.

Next consider the representations $V_l \subset N_X$, where X is a component of $M_0^{S^1} \cap M^T$. By the above claim we may assume that V_l is tangential to the fixed point component $M_0^{S^1}$. Fix a connected component X_0 of M^T which is not contained in $M_0^{S^1}$, and fix the T-action on the Hopf bundle γ for which χ_{X_0} is a trivial T-representation.

Let $T_j \subset T$ be the identity component of the kernel of χ_X and let $V_j := N_X^{T_j}$. Since F_j contains X (by definition) and X_0 (apply Proposition 2.1, Part 3, to T_j) S^1 acts non-trivially on F_j. Let ν_{F_j} denote the normal bundle of $F_j \subset M$. By construction $V_l \subset \nu_{F_j}$. To understand the T-action on $V_l \subset N_X^{S^1} = (\nu_{F_j\,|pt})^{S^1}$ we will compare $\nu_{F_j\,|pt}$ with $\nu_{F_j\,|q_0}$, $q_0 \in X_0$, and use the established linearity at X_0.

Note that $\nu_{F_j\,|pt} \cong_{(T_j,\mathbb{R})} \nu_{F_j\,|q_0} \cong_{\mathbb{R}} N_{X_0} \ominus N_{X_0}^{T_j}$. Since T acts linearly on N_{X_0}

$$N_{X_0} \cong_{\mathbb{R}} \bigoplus_{Y\subset M^T, Y\neq X_0} (\dim_{\mathbb{C}} Y+1)\cdot\chi_Y\cdot\chi_{X_0}^{-1} \cong_{(T_j,\mathbb{C})} \bigoplus_{Y\subset M^T, Y\neq X_0} (\dim_{\mathbb{C}} Y+1)\cdot\chi_Y\cdot\chi_X^{-1}.$$

By Proposition 2.1, Part 3, $\chi_Y \cdot \chi_X^{-1}$ is a trivial T_j-representation if and only if $Y \subset F_j$. Hence,

$$\nu_{F_j\,|pt} \cong_{(T_j,\mathbb{R})} N_{X_0} \ominus N_{X_0}^{T_j} \cong_{(T_j,\mathbb{R})} \bigoplus_{Y\subset M^T, Y\not\subset F_j} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \qquad (4)$$

Recall that T acts linearly on all subrepresentations V_i of $\nu_{F_j\,|pt} \ominus (\nu_{F_j\,|pt})^{S^1}$. Hence,

$$\nu_{F_j\,|pt} \ominus (\nu_{F_j\,|pt})^{S^1} \cong_{\mathbb{R}} \bigoplus_{Y\subset F_i^T,\ i\neq j,\ Y\neq X,\ (V_i)^{S^1}\neq V_i} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \qquad (5)$$

Note that $\{Y \subset F_i^T \mid Y \neq X,\ V_i \subset (\nu_{F_j\,|pt})^{S^1}\} \subset \{Y \subset M^T \mid Y \not\subset F_j\}$ is the complement of the index set of the direct sum in (5). Thus (4) and (5) imply

$$(\nu_{F_j\,|pt})^{S^1} \cong_{(T_j,\mathbb{R})} \bigoplus_{Y\subset F_i^T, Y\neq X,\ V_i\subset(\nu_{F_j\,|pt})^{S^1}} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}.$$

Since $\chi_Y \cdot \chi_X^{-1}$ is a trivial S^1-representation for $Y \subset F_i^T$, $V_i \subset (\nu_{F_j\,|pt})^{S^1}$, the isomorphism extends to an isomorphism of T-representations

$$(\nu_{F_j\,|pt})^{S^1} \cong_{\mathbb{R}} \bigoplus_{Y\subset F_i^T, Y\neq X,\ V_i\subset(\nu_{F_j\,|pt})^{S^1}} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \qquad (6)$$

By restricting to the trivial T_l-subrepresentations (for fixed $l \neq j$) on both sides of (6) it follows that T acts linearly on any $V_l \subset (\nu_{F_j}{}_{|pt})^{S^1} = N_X^{S^1}$. This completes the proof of the proposition. ■

Theorem 3.2. *Let M be an integral cohomology $\mathbb{C}P^m$ with effective smooth action by a torus T of rank r. If $m < 4r - 1$ then the T-action is linear.*

Proof: We prove the statement for almost effective actions (i.e. actions with finite kernel) by induction on the rank of the action. If $r = 1$ then $\dim_{\mathbb{C}} M \leq 2$ and the T-action is linear as pointed out before. So assume $r \geq 2$.

Let X be a connected component of M^T. By Lemma 2.5 it suffices to show that T acts linearly on every $V_i \subset N_X$.

For fixed $V_i \subset N_X$ let V_{max} be a maximal element (with respect to inclusion) of the set of representations

$$\{V \subset N_X \mid V_i \subset V,\ V^{S^1} = V \text{ for some subgroup } S^1 \subset T\}$$

and let S^1 denote the subtorus of T which acts trivially on V_{max}. Since V_{max} is maximal a complementary subtorus $\widetilde{T} \subset T$ of rank $r-1$ acts almost effectively on V_{max}.

Let $M_0^{S^1}$ denote the connected component of M^{S^1} which contains X. By Proposition 2.1 $M_0^{S^1}$ is an integral cohomology complex projective space. Note that $\widetilde{T}$ acts almost effectively on $M_0^{S^1}$.

If $\dim_{\mathbb{C}} M_0^{S^1} < 4r - 5$ then $\widetilde{T}$ acts linearly on $M_0^{S^1}$ by the induction hypothesis. In this case $\widetilde{T}$ acts linearly on V_i which implies the same for the T-action on V_i. If $\mathrm{codim}_{\mathbb{C}} M_0^{S^1} < 3$ then T acts linearly on M by Proposition 3.1. The remaining case ($\dim_{\mathbb{C}} M = 4r - 2 \neq 3$ and $\mathrm{codim}_{\mathbb{C}} M_0^{S^1} = 3$) also follows from Proposition 3.1. ■

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