# The $\hat{A}$-genus of $S^1$-manifolds with finite second homotopy group

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**Abstract**

We construct simply connected smooth manifolds $M$ of dimension $4k \geq 8$ with the following properties: the second homotopy group $\pi_2(M)$ is finite, $M$ admits a smooth action by the circle $S^1$ and the $\hat{A}$-genus $\hat{A}(M)$ is non-zero. *To cite this article: Manuel Amann, Anand Dessai, C. R. Acad. Sci. Paris, Ser. I ... (...).*

**Résumé**

**Le $\hat{A}$-genre de $S^1$-variétés avec le deuxième groupe homotopie fini.** Nous construisons des variétés $M$ simplement connexes de dimension $4k \geq 8$ avec les propriétés suivantes : le deuxième groupe d'homotopie $\pi_2(M)$ est fini, $M$ admet une action lisse du cercle $S^1$ et le $\hat{A}$-genre $\hat{A}(M)$ est non nulle. *Pour citer cet article : Manuel Amann, Anand Dessai, C. R. Acad. Sci. Paris, Ser. I ... (...).*

## 1. Introduction

In this note we prove that the existence of a smooth non-trivial circle action on a simply connected manifold with finite second homotopy group does not force the $\hat{A}$-genus to vanish. Our result is related to a paper of Haydeé and Rafael Herrera [6] on 12-dimensional positive quaternionic Kähler manifolds. To explain this we begin with a short incomplete survey of the classification problem for positive quaternionic Kähler manifolds (QK-manifolds) with special focus on the 12-dimensional case. We refer to the survey article [9] of Salamon for more information on QK-manifolds and references.

The only known examples of positive QK-manifolds are the symmetric examples studied by Wolf. LeBrun and Salamon showed that up to homothety there are only finitely many positive QK-manifolds in any fixed dimension and they conjectured that any positive QK-manifold is symmetric.

One knows that any positive QK-manifold $M$ is simply connected and that the second homotopy group $\pi_2(M)$ is trivial, isomorphic to $\mathbb{Z}$ or finite with 2-torsion. In the first two cases $M$ is homothetic to the quaternionic projective space $\mathbb{H}P^n = Sp(n+1)/(Sp(n) \times Sp(1))$ or the complex Grassmannian $Gr_2(\mathbb{C}^{n+2}) = U(n+2)/(U(n) \times U(2))$, respectively. There are symmetric examples, e.g. the Grassmannian $Gr_4(\mathbb{R}^{n+4}) = SO(n+4)/(SO(n) \times SO(4))$, which realize the third case. The question remains whether there exist non-symmetric positive QK-manifolds with finite second homotopy group.

The LeBrun-Salamon conjecture has been proved by Hitchin, Poon-Salamon and LeBrun-Salamon in dimension $\leq 8$. Haydeé and Rafael Herrera [6] showed that any 12-dimensional positive QK-manifold $M$ is symmetric if the $\hat{A}$-genus of $M$ vanishes. If $M$ is a spin manifold this condition is always fulfilled by a classical result of Lichnerowicz since a positive QK-manifold has positive scalar curvature. One also knows from the work of Borel and Hirzebruch [2, Th. 23.3] that $\hat{A}(M)$ vanishes on the symmetric examples which are $\pi_2$-finite (i.e. which have finite second homotopy group).

Atiyah and Hirzebruch [1] showed that the $\hat{A}$-genus vanishes on spin manifolds with smooth effective $S^1$-action. In [6] Haydeé and Rafael Herrera offered a proof for the vanishing of the $\hat{A}$-genus on *any* $\pi_2$-finite manifold with smooth effective $S^1$-action. Since one knows from the work of Salamon that the dimension of the isometry group of a 12-dimensional positive QK-manifold is at least 5 this would lead to a proof of the LeBrun-Salamon conjecture in dimension 12.

The argument in [6] essentially consists of three parts. In the first part Haydeé and Rafael Herrera argue that any smooth $S^1$-action on a $\pi_2$-finite manifold is of even or odd type (this condition means that the sum of rotation numbers at the $S^1$-fixed points is always even or always odd). Then they argue that the proof of Bott-Taubes [3] for the rigidity of the elliptic genus may be adapted to non-spin manifolds if the $S^1$-action is of even or odd type. Finally they use an argument of Hirzebruch-Slodowy [8] to derive the vanishing of the $\hat{A}$-genus from the rigidity of the elliptic genus.

Unfortunately, the first part of their argument cannot be correct. In fact, as was noticed by the first named author of this note, there are $S^1$-actions on the Grassmannian $Gr_4(\mathbb{R}^{n+4})$ for any odd $n \geq 3$ which are neither even nor odd. For example, the 12-dimensional Grassmannian $Gr_4(\mathbb{R}^7)$ admits an $S^1$-action such that the fixed point components of the corresponding involution are of dimension 4 and 6 (the components are diffeomorphic to $S^4$ and $Gr_2(\mathbb{R}^5) = SO(5)/(SO(3) \times SO(2))$ and both contain $S^1$-fixed points). However, for odd $n \geq 3$, $Gr_4(\mathbb{R}^{n+4})$ is a non-spin positive QK-manifold with finite second homotopy group. The error in [6] can be traced back to an application of a result of Bredon [4, Th. V, p. 527] on the representations at different fixed points which requires that $\pi_2(M)$ *and* $\pi_4(M)$ are finite (see the paragraph after Th. 4 in [6] and the forthcoming erratum [7]).

This prompts the question whether one can prove the vanishing of the $\hat{A}$-genus on $\pi_2$-finite manifolds with smooth effective $S^1$-action by other means. The purpose of this note is to answer this question in the negative. More precisely, we show the following

**Theorem 1.1** *For any $k > 1$ there exists a smooth simply connected $4k$-dimensional $\pi_2$-finite manifold $M$ with smooth effective $S^1$-action and $\hat{A}(M) \neq 0$.*

The examples in the theorem may be chosen to have $\pi_2 = \mathbb{Z}/2\mathbb{Z}$. In dimension 4 the $\hat{A}$-genus does vanish on a simply connected $\pi_2$-finite manifold since it is a multiple of the signature.

It remains a challenging task to determine whether the $\hat{A}$-genus vanishes on $\pi_2$-finite positive QK-manifolds as predicted by the LeBrun-Salamon conjecture.

## 2. Proof

Theorem 1.1 is a consequence of the following

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**Lemma 2.1 (Surgery lemma)** *Let $G$ be a compact Lie group and let $M$ be a smooth simply connected $G$-manifold. Suppose the fixed point manifold $M^G$ contains a submanifold $N$ of dimension $\geq 5$ such that the inclusion map $N \hookrightarrow M$ is 2-connected. Then $M$ is $G$-equivariantly bordant to a simply connected $G$-manifold $M'$ with $\pi_2(M') \subset \mathbb{Z}/2\mathbb{Z}$.*

**Proof:** We adapt the classical elementary surgery theory (see [5, Chapter IV]) to the equivariant setting. Let $f : M \to BSO$ be a classifying map for the stable normal bundle of $M$. We fix a finite set of generators for the kernel of $f_* : \pi_2(M) \to \pi_2(BSO) \cong \mathbb{Z}/2\mathbb{Z}$. Since the inclusion map $N \hookrightarrow M$ is 2-connected and $\dim N \geq 5$ we may represent these generators by disjointly embedded 2-spheres in $N$. By construction the normal bundle in $M$ of each such 2-sphere is trivial as a non-equivariant bundle and equivariantly diffeomorphic to a $G$-equivariant vector bundle over the trivial $G$-space $S^2$. For each embedded 2-sphere we identify the normal bundle $G$-equivariantly with a tubular neighborhood of the sphere and perform $G$-equivariant surgery for all of these 2-spheres. The result of the surgery is a simply connected $G$-manifold $M'$ with $\pi_2(M') \subset \mathbb{Z}/2\mathbb{Z}$ which is $G$-equivariantly bordant to $M$ (if $M$ is a spin manifold then $M'$ is actually 2-connected).

**Proof of Theorem 1.1:** We begin with some linear effective $S^1$-action on the complex projective space $\mathbb{C}P^{2k}$ such that the fixed point manifold $M^{S^1}$ contains a component $N$ diffeomorphic to $\mathbb{C}P^l$ for some $l \geq 3$. Since $N \hookrightarrow \mathbb{C}P^{2k}$ is 2-connected, the manifold $\mathbb{C}P^{2k}$ is $S^1$-equivariantly bordant to a simply connected $S^1$-manifold $M'$ with $\pi_2(M')$ finite by the surgery lemma (in fact, $\pi_2(M') \cong \mathbb{Z}/2\mathbb{Z}$ since $\mathbb{C}P^{2k}$ is not a spin manifold). It is well-known that the $\hat{A}$-genus does not vanish on $\mathbb{C}P^{2k}$. Since $M'$ is bordant to $\mathbb{C}P^{2k}$ we get $\hat{A}(M') = \hat{A}(\mathbb{C}P^{2k}) \neq 0$.

It is straightforward to produce examples with much larger symmetry using the construction above. We leave the details to the reader.

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