A NOTE ON NONNEGATIVE CURVATURE AND (RATIONAL) HOMOTOPY TYPE

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1. Introduction

The study of Riemannian manifolds with given curvature bounds and their topology is a central theme of global differential geometry, and much attention has here been paid to the interplay between curvature and homotopy type. In this regard there is also the following question, which has been raised by Félix, Oprea, and Tanré (see Question 6.17 in their book [FOT08]):

Question 1 ([FOT08], p. 249) *If two closed manifolds have the same (rational) homotopy type and one manifold has a metric of non-negative sectional curvature, does the other?*

In turns out that the answer to the above question is negative for many classes of manifolds, even if one restricts to positive sectional curvature and the simply connected case. The main purpose of the present note - further details will appear elsewhere in a broader context - consists in providing some of these examples and in proposing a substitute for Question 1.

Concerning the first point, examples of closed simply connected manifolds M with positive sectional curvature that have homotopy equivalent cousins N which do not admit a metric of nonnegative scalar curvature (and hence nonnegative sectional curvature) can be obtained in the following way:

1. Let $M = S^n$ be the round sphere in dimension $n = 8k+1$ or $8k+2$ and let N be a homotopy n-sphere with non-vanishing α-invariant. Then N does not admit a metric of positive scalar curvature by the work of Hitchin [Hi74], and the well-known deformation properties of scalar curvature imply that any metric of nonnegative sectional curvature on N must be flat. However, since homotopy spheres do not carry flat metrics, N does not admit a metric of nonnegative sectional curvature.

Actually, the homotopy sphere N does not even admit a metric of nonnegative scalar curvature: As before, any such metric g must be scalar-flat. The non-vanishing of the α-invariant now implies that N carries a nontrivial parallel spinor and that (N, g) must have special holonomy (compare

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[Hi74]). However, since homotopy spheres do always have generic holonomy, N doesn't carry a metric of nonnegative scalar curvature.

Essentially the same reasoning can be applied to any spin manifold M of positive sectional curvature in dimension $8k+1$ or $8k+2$ to construct a manifold which is homotopy equivalent to M but which does not admit a metric of nonnegative sectional curvature.

2. Let $M = \mathbb{H}P^m$ be the quaternionic projective space of dimension $n = 4m \geq 8$ equipped with its natural metric of positive sectional curvature. Since the universal expressions for the $\hat{A}$-genus and the signature in terms of Pontrjagin numbers are linearly independent, we may, using surgery theory, choose a manifold N with non-vanishing $\hat{A}$-genus in the homotopy type of M. Thus M has positive sectional curvature whereas N does not admit any metric of positive scalar curvature. Moreover, holonomy considerations as above show that N does not even admit a metric of nonnegative scalar curvature.

In search of suitable substitutes for Question 1, given the above examples, one might first restrict Question 1 to manifolds which already admit metrics of nonnegative or even positive scalar curvature, e.g.: *If two closed manifolds of positive scalar curvature have the same (rational) homotopy type and one manifold has a metric of non-negative sectional curvature, does the other?*

However, there is strong evidence that the answer to this question might be negative as well. Indeed, if, for example, Stolz' conjecture about the vanishing of the Witten genus of string manifolds with positive Ricci curvature is true, one can construct counterexamples.

On the other hand, one may also restrict Question 1 to manifolds which a priori admit metrics with positive Ricci curvature:

Question 1′ *If two closed manifolds of positive Ricci curvature have the same (rational) homotopy type and one manifold has a metric of non-negative sectional curvature, does the other?*

This question is at present completely open and, moreover, any kind of answer to it would be interesting. In particular, a negative answer will show that there exist manifolds of positive Ricci curvature which do not admit metrics with nonnegative sectional curvature, but whose homology satisfies the bounds given by Gromov's Betti number theorem for nonnegatively curved manifolds.

References

[FOT08] Y. Félix, J. Oprea, and D.Tanré. *Algebraic Models in Geometry.* Oxford Graduate Texts in Mathematics, vol. 17, Oxford University Press, New York, 2008.

[Hi74] N. Hitchin. *Harmonic spinors*, Advances in Math. 14 (1974), 1–55.

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