

# Obstructions to positive curvature and symmetry

Anand Dessai

## Abstract

We show that the indices of certain twisted Dirac operators vanish on a *Spin*-manifold  $M$  of positive sectional curvature if the symmetry rank of  $M$  is  $\geq 2$  or if the symmetry rank is one and  $M$  is two connected. We also give examples of simply connected manifolds of positive Ricci curvature which do not admit a metric of positive sectional curvature and positive symmetry rank.<sup>1</sup>

## 1 Introduction

An important application of index theory in Riemannian geometry is in the study of manifolds of positive scalar curvature. Soon after Atiyah and Singer proved the index theorem, Lichnerowicz used a Bochner type formula to show that the index of the Dirac operator vanishes on closed *Spin*-manifolds of positive scalar curvature.

Whereas the relation between index theory and positive scalar curvature (for high dimensional simply connected manifolds) is well understood [20, 45, 50, 38] possible relations to stronger curvature conditions such as positive Ricci or positive sectional curvature remain obscure (see however the fascinating conjecture in [51]).

In this paper we give obstructions to metrics of positive sectional curvature (positive curvature for short) with symmetry. We show that the indices of certain twisted Dirac operators vanish on a positively curved closed *Spin*-manifold  $M$  provided that the symmetry rank (i.e. the rank of the isometry group of  $M$ ) is at least two and the dimension of  $M$  is sufficiently large. These indices occur as coefficients in an expansion of the elliptic genus. A similar result holds if  $M$  is 2-connected and the symmetry rank is  $\geq 1$ .

The elliptic genus  $\varphi$  is a ring homomorphism from the oriented bordism ring to the ring of modular forms for

$$\Gamma_0(2) := \{A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{2}\}$$

(in particular,  $\varphi$  vanishes in all dimensions not divisible by 4). On the complex projective spaces  $\mathbb{C}P^{2k}$  it is given by

$$\sum_{k \geq 0} \varphi(\mathbb{C}P^{2k}) t^{2k} = (1 - 2\delta \cdot t^2 + \epsilon \cdot t^4)^{-1/2},$$

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<sup>1</sup>This paper replaces the preprint “On the elliptic genus of positively curved manifolds with symmetry” (April 2001) which appeared at the arXiv (<http://arxiv.org/abs/math.DG/0104256>).

where  $\delta$  and  $\epsilon$  are modular forms of weight 2 and 4, respectively.

The normalized elliptic genus  $\Phi(M) := \varphi(M)/\epsilon^{k/2}$  of an oriented  $4k$ -dimensional manifold  $M$  expands in one of the cusps of  $\Gamma_0(2)$  as a series of twisted signatures. Following Witten [57] this series is best thought of as the index of a hypothetical signature operator on the free loop space of  $M$ . In the other cusp of  $\Gamma_0(2)$  the elliptic genus expands as a series  $\Phi_0(M)$  of characteristic numbers. If  $M$  is *Spin* the coefficients of this expansion are indices of twisted Dirac operators

$$\begin{aligned} \Phi_0(M) &= q^{-k/2} \cdot \hat{A}(M, \bigotimes_{n=2m+1>0} \Lambda_{-q^n} TM \otimes \bigotimes_{n=2m>0} S_{q^n} TM) \\ &= q^{-k/2} \cdot (\hat{A}(M) - \hat{A}(M, TM) \cdot q + \hat{A}(M, \Lambda^2 TM + TM) \cdot q^2 + \dots). \end{aligned}$$

Here  $\hat{A}(M, E)$  denotes the index of the Dirac operator twisted with the complexification  $E_{\mathbb{C}}$  of a real vector bundle  $E$  over  $M$ .

For a *Spin*-manifold  $M$  the first coefficient of the series  $\Phi_0(M)$ , the  $\hat{A}$ -genus, vanishes if  $M$  admits a metric of positive scalar curvature [35] or if  $M$  admits a non-trivial smooth  $S^1$ -action [4]. Our main result asserts that additional coefficients vanish if  $M$  admits a metric of positive curvature with symmetry rank  $\geq 2$ .

**Theorem 1.1.** *Let  $M$  be a closed connected *Spin*-manifold of dimension  $> 12r - 4$ . If  $M$  admits a metric of positive curvature and symmetry rank  $\geq 2r$  then the indices of twisted Dirac operators occurring as the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  vanish.*

We remark that all simply connected manifolds known to carry a metric of positive curvature have a lot of symmetry. Besides the biquotients found by Eschenburg [14, 15] and Bazaikin [5], all other examples admit a homogeneous metric of positive curvature. The latter were classified by Berger [7], Aloff, Wallach [55, 2] and Bérard Bergery [6] (for recent progress on cohomogeneity one manifolds see [23, 24, 46, 40, 41, 54]).

In the case that the symmetry rank of  $M$  is at least two Theorem 1.1 states:

*Let  $M$  be a closed connected *Spin*-manifold of dimension  $> 8$ . If  $M$  admits a metric of positive curvature and symmetry rank  $\geq 2$  then  $\hat{A}(M)$  and  $\hat{A}(M, TM)$  vanish.*

Note that the index  $\hat{A}(M, TM)$  does not vanish for the quaternionic plane. Since the symmetry rank of  $\mathbb{H}P^2$  (with its standard metric) is three the lower bound on the dimension of  $M$  is necessary.

We believe that the vanishing of  $\hat{A}(M, TM)$  also holds under weaker symmetry assumptions. For 2-connected manifolds we show

**Theorem 1.2.** *Let  $M$  be a closed 2-connected manifold of dimension  $> 8$ . If  $M$  admits a metric of positive curvature with effective isometric  $S^1$ -action then  $\hat{A}(M)$  and  $\hat{A}(M, TM)$  vanish.*

The proofs of Theorem 1.1 and Theorem 1.2 are rather indirect. For both statements we study the fixed point manifold of isometric cyclic subactions. The Bott-Taubes-Witten rigidity theorem [57, 53, 8] for elliptic genera implies that

the codimension of the fixed point manifold is bounded from above by a constant which depends on the pole order of the expansion  $\Phi_0(M)$  [30, 12]. Further restrictions arise from the curvature assumption. A component of the fixed point manifold is a totally geodesic submanifold of the positively curved manifold  $M$ . By an old result of Frankel [17] totally geodesic submanifolds of sufficiently large dimension must intersect. This property imposes additional restrictions on the fixed point manifold. The consequences of the rigidity theorem and Frankel's result indicated above are the main ingredients in the proofs of Theorem 1.1 and Theorem 1.2 which also rely on the work of Grove and Searle on isometric  $S^1$ -actions of codimension two [21] and recent work of Wilking on the connectivity of the inclusion of totally geodesic submanifolds [57].

We don't know how to prove Theorem 1.1 and Theorem 1.2 by more direct methods such as the Bochner formula for twisted Dirac operators. Already for the proof of the vanishing of  $\hat{A}(M, TM)$  we need to use the entire elliptic genus. Note that in view of the above discussion for  $\mathbb{H}P^2$  a Bochner type argument for the vanishing of  $\hat{A}(M, TM)$  would not apply in dimension eight!

Manifolds of positive curvature (no assumptions on the symmetry) are classified in dimension  $< 4$  [25]. In dimension  $\geq 4$  the only known obstructions to positive curvature are given by restrictions for the fundamental group (cf. [37, 52, 19], see also [43]), Gromov's Betti number theorem [19] and the Lichnerowicz-Hitchin vanishing theorem [35, 31] for the  $\alpha$ -invariant of *Spin*-manifolds.

Further progress concerning obstructions and classification has been obtained for positively curved manifolds with a lot of symmetry, e.g. manifolds with large isometry dimension, large (discrete) symmetry rank or small cohomogeneity [32, 21, 56, 47, 22, 44, 48, 58, 42, 16, 13].

All these results require that the dimension of the manifold is bounded from above by a constant depending on the symmetry. In contrast Theorem 1.1 and Theorem 1.2 only require a *lower* bound on the dimension of the manifold.

Theorem 1.1 allows to distinguish positive curvature from weaker curvature properties under assumptions on the symmetry rank. For example, consider the product of  $\mathbb{H}P^2$  and a Ricci-flat  $K_3$ -surface. The Riemannian manifold  $M = \mathbb{H}P^2 \times K_3$  (equipped with the product metric) has symmetry rank three and positive scalar curvature as well as non-negative Ricci curvature. The index  $\hat{A}(M, TM)$  does not vanish. Hence, if one restricts to metrics with symmetry rank  $\geq 2$  it follows from Theorem 1.1 that  $M$  admits a metric of positive scalar curvature but no metric of positive curvature.

This kind of reasoning can be pushed further to yield examples of simply connected manifolds of positive Ricci curvature which do not admit a metric of positive curvature if one restrict to metrics with a prescribed lower bound on the symmetry rank (see Section 6 for precise statements). Using different arguments (based on [12, 56]) it is possible to distinguish positive Ricci from positive curvature under rather mild assumptions on the symmetry. We shall call an  $S^1$ -action on a Riemannian manifold *finite-order-isometric* of order  $o$  if the cyclic subgroup of order  $o$  acts effectively and isometrically.

**Theorem 1.3.** *For every  $d \in \mathbb{N}$  and every  $o \geq 2$  there exists a simply connected closed manifold  $M$  of dimension greater than  $d$  such that:*

1.  *$M$  admits a metric of positive Ricci curvature with finite-order-isometric  $S^1$ -action of order  $o$ .*
2.  *$M$  does not admit a metric of positive curvature with finite-order-isometric  $S^1$ -action of order  $o$ .*

We note that the examples (given in Section 6) admit metrics of positive Ricci curvature and symmetry rank  $\geq 3$ . In particular, one obtains simply connected manifolds of positive Ricci curvature and positive symmetry rank which do not admit a metric of positive curvature with positive symmetry rank.

The paper is structured in the following way. In the next section we review basic properties of positive curvature used in the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3. In Section 3 we recall the rigidity theorem for elliptic genera and discuss applications to cyclic actions. The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 4 and Section 5. In Section 4 we also discuss related results for positive  $k$ th Ricci curvature, finite isometric actions and integral cohomology  $\mathbb{H}P^k$ 's. In the final section we show Theorem 1.3.

## 2 Geodesic submanifolds

In this section we review basic properties of positively curved manifolds used in the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3. A main ingredient is an old result of Frankel on the intersection property for totally geodesic submanifolds.

**Theorem 2.1** ([17]). *Let  $N_1$  and  $N_2$  be totally geodesic submanifolds of a positively curved connected manifold  $M$ . If  $\dim N_1 + \dim N_2 \geq \dim M$  then  $N_1$  and  $N_2$  have non-empty intersection. ■*

The proof uses a Synge type argument for the parallel transport along a geodesic from  $N_1$  to  $N_2$  which minimizes the distance. Whereas it is difficult to find totally geodesic submanifolds for generic metrics they do occur naturally as fixed point components in the presence of symmetry. Theorem 2.1 clearly imposes restrictions on the fixed point manifold of isometric actions. The following consequence is immediate.

**Corollary 2.2.** *Let  $\sigma$  be an isometry of a positively curved connected manifold  $M$  and let  $F$  be a connected component of the fixed point manifold  $M^\sigma$  of minimal codimension. Then the dimension of every other component is less than the codimension of  $F$ . ■*

In [18] Frankel applied Theorem 2.1 to show that the inclusion  $N \hookrightarrow M$  of a totally geodesic submanifold of codimension  $k$  is 1-connected provided  $k$  is not larger than half of the dimension of the positively curved manifold  $M$ . Recently, Wilking generalized this result significantly using Morse theory.

**Theorem 2.3 ([56]).** *Let  $M$  be an  $n$ -dimensional connected Riemannian manifold and  $N$  a connected totally geodesic submanifold of codimension  $k$ . If  $M$  is positively curved then the inclusion  $N \hookrightarrow M$  is  $(n - 2k + 1)$ -connected. ■*

The theorem imposes severe restrictions on the topology. For example, if  $M$  is simply connected and the codimension of  $N$  is two then either  $M$  is homeomorphic to an odd dimensional sphere or all odd Betti numbers of  $M$  and  $N$  vanish [56]. The case where  $N$  is fixed under an isometric  $S^1$ -action was studied before by Grove and Searle.

**Theorem 2.4 ([21]).** *Let  $M$  be a simply connected positively curved manifold with isometric  $S^1$ -action. If  $\text{codim } M^{S^1} = 2$  then  $M$  is diffeomorphic to a sphere or a complex projective space. ■*

### 3 Rigidity and cyclic actions

In this section we recall the rigidity theorem for elliptic genera and discuss applications to cyclic actions. For more information on elliptic genera we refer to [34, 29].

A genus is a ring homomorphism from the oriented bordism ring  $\Omega_*^{SO}$  to a  $\mathbb{Q}$ -algebra  $R$  [27]. The genus is called elliptic (of level 2) if its logarithm  $g(u)$  is given by a formal elliptic integral

$$g(u) = \int_0^u \frac{dt}{\sqrt{1 - 2 \cdot \delta \cdot t^2 + \epsilon \cdot t^4}},$$

where  $\delta, \epsilon \in R$  [39]. Classical examples of elliptic genera are the signature ( $\delta = \epsilon = 1$ ) and the  $\hat{A}$ -genus ( $\delta = -\frac{1}{8}, \epsilon = 0$ ).

The ring of modular forms  $M_*(\Gamma_0(2))$  is a polynomial ring with generators  $\delta$  and  $\epsilon$  of weight 2 and 4, respectively [29]. The corresponding elliptic genus

$$\varphi : \Omega_*^{SO} \rightarrow M_*(\Gamma_0(2))$$

is universal since  $\delta$  and  $\epsilon$  are algebraically independent.

As in [57, 30] we shall consider for a  $4k$ -dimensional oriented manifold  $M$  the normalized elliptic genus  $\Phi(M) = \varphi(M)/\epsilon^{k/2}$  which is a modular function of weight 0 (with  $\mathbb{Z}/2\mathbb{Z}$ -character). In one of the cusps (the signature cusp)  $\Phi(M)$  has an expansion which is equal to the following series of twisted signatures

$$\text{sign}(q, \mathcal{L}M) := \text{sign}(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM) \in \mathbb{Z}[[q]].$$

Here  $\text{sign}(M, E)$  denotes the index of the signature operator twisted with the complexified vector bundle  $E_{\mathbb{C}}$ . The series  $\text{sign}(q, \mathcal{L}M)$  describes the “signature” of the free loop space  $\mathcal{L}M$  localized at the manifold  $M$  of constant loops [57].

In the other cusp (the  $\hat{A}$ -cusp)  $\Phi(M)$  expands as a series of characteristic numbers<sup>2</sup>

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<sup>2</sup>For a suitable change of cusps.

$$\begin{aligned} \Phi_0(M) &:= q^{-k/2} \cdot \hat{A}(M, \bigotimes_{n=2m+1>0} \Lambda_{-q^n} TM \otimes \bigotimes_{n=2m>0} S_{q^n} TM) \\ &= q^{-k/2} \cdot (\hat{A}(M) - \hat{A}(M, TM) \cdot q + \hat{A}(M, \Lambda^2 TM + TM) \cdot q^2 + \dots). \end{aligned}$$

Here  $\hat{A}(M, E) := \langle \hat{\mathcal{A}}(M) \cdot ch(E_{\mathbb{C}}), [M] \rangle$ , where  $\hat{\mathcal{A}}(M)$  denotes the multiplicative sequence for the  $\hat{A}$ -genus,  $E_{\mathbb{C}}$  is the complexification of the vector bundle  $E$ ,  $[M]$  denotes the fundamental cycle and  $\langle \ , \ \rangle$  is the Kronecker pairing. If  $M$  is *Spin*  $\hat{A}(M, E)$  is equal to the index of the Dirac operator twisted with  $E_{\mathbb{C}}$  by the Atiyah-Singer index theorem [3]. In this case  $\Phi_0(M)$  has an interpretation as a series of indices of twisted Dirac operators (twisted Dirac-indices for short).

The main feature of the elliptic genus is its rigidity under actions of compact connected Lie groups. The rigidity was explained by Witten in [57] using heuristic arguments from quantum field theory and proved rigorously by Taubes and Bott-Taubes [53, 8] (cf. also [28, 36]).

Assume the *Spin*-manifold  $M$  carries an action by a compact Lie group  $G$  preserving the *Spin*-structure (note that any smooth  $G$ -action lifts to the *Spin*-structure after passing to a two-fold covering action, if necessary). Then the indices occurring in the expansions of  $\Phi(M)$  refine to virtual  $G$ -representations which we identify with their characters. If  $G$  is connected the elliptic genus is rigid, i.e. the characters do not depend on  $G$ .

**Theorem 3.1 ([53, 8]).** *Let  $M$  be a  $G$ -equivariant *Spin*-manifold. If  $G$  is connected then each twisted signature (resp. each twisted Dirac-index) occurring as coefficient in the expansion of  $\Phi(M)$  in the signature cusp (resp. in the  $\hat{A}$ -cusp) is constant as a character of  $G$ . ■*

The rigidity of  $\Phi(M)$  also holds for certain non-*Spin* manifolds such as *Spin*<sup>c</sup>-manifolds with first Chern class a torsion class [10] or orientable manifolds with finite second homotopy group [26].

In the remaining part of this section we discuss consequences of Theorem 3.1 for cyclic actions which are used in the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Assume  $S^1$  acts on the *Spin*-manifold  $M$  (not necessarily preserving the *Spin*-structure). Let  $\sigma \in S^1$  be the element of order 2. In [30] Hirzebruch and Slodowy showed that the expansion of the elliptic genus in the signature cusp can be expressed in terms of the transversal self-intersection  $M^\sigma \circ M^\sigma$  of the fixed point manifold  $M^\sigma$

$$sign(q, \mathcal{L}M) = sign(q, \mathcal{L}(M^\sigma \circ M^\sigma)). \quad (1)$$

The formula is equivalent to  $\Phi_0(M) = \Phi_0(M^\sigma \circ M^\sigma)$  which implies the following generalization of the Atiyah-Hirzebruch vanishing theorem [4] for the  $\hat{A}$ -genus.

**Theorem 3.2 ([30]).** *Let  $M$  be a *Spin*-manifold with  $S^1$ -action and let  $\sigma$  be the element of order two in  $S^1$ . If  $\text{codim } M^\sigma > 4r$  then the first  $(r + 1)$  coefficients of  $\Phi_0(M)$  vanish. ■*

Here  $\text{codim } M^\sigma$  denotes the minimal codimension of the connected components of  $M^\sigma$  in  $M$ .

Recall that the  $S^1$ -action is called even if it lifts to the *Spin*-structure (otherwise the action is called odd). In the even case the codimension of all fixed point components of  $M^\sigma$  is divisible by 4 whereas in the odd case the codimensions are always  $\equiv 2 \pmod{4}$  (cf. [4], Lemma 2.4). Note that for an odd action the series  $q^{\dim M/8} \cdot \Phi_0(M)$  is an element in  $\mathbb{C}[[q]]$  whereas  $q^{\dim M/8} \cdot \Phi_0(M^\sigma \circ M^\sigma) \in q^{1/2} \cdot \mathbb{C}[[q]]$ . Thus formula (1) implies

**Corollary 3.3.** *Let  $M$  be a *Spin*-manifold with  $S^1$ -action. If the action is odd then  $\Phi(M)$  vanishes identically. ■*

In the remaining part of this section we recall a generalization of Theorem 3.2 to cyclic actions of arbitrary order [12]. Let  $M$  be a *Spin*-manifold with  $S^1$ -action and let  $\sigma \in S^1$  be of order  $o \geq 2$ . At a connected component  $Y$  of the fixed point manifold  $M^{S^1}$  the tangent bundle  $TM$  splits equivariantly as the direct sum of  $TY$  and the normal bundle  $\nu$ . The latter splits (non-canonically) as a direct sum  $\nu = \bigoplus_{k \neq 0} \nu_k$  corresponding to the irreducible real 2-dimensional  $S^1$ -representations  $e^{i \cdot \theta} \mapsto \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}$ ,  $k \neq 0$ . We fix such a decomposition of  $\nu$ . For each  $k \neq 0$  choose  $\alpha_k \in \{\pm 1\}$  such that  $\alpha_k k \equiv \tilde{k} \pmod{o}$ ,  $\tilde{k} \in \{0, \dots, [\frac{o}{2}]\}$ . On each vector bundle  $\nu_k$  introduce a complex structure such that  $\lambda \in S^1$  acts on  $\nu_k$  by scalar multiplication with  $\lambda^{\alpha_k k}$ . Finally define

$$m_o(Y) := \left( \sum_k \dim_{\mathbb{C}} \nu_k \cdot \tilde{k} \right) / o \quad \text{and} \quad m_o := \min_Y m_o(Y),$$

where  $Y$  runs over the connected components of  $M^{S^1}$ .

**Theorem 3.4 ([12]).** *Let  $M$  be a *Spin*-manifold with  $S^1$ -action. If  $m_o > r$  then the first  $(r + 1)$  coefficients of  $\Phi_0(M)$  vanish. ■*

To prove this result one analyzes the expansion of the equivariant elliptic genus in the  $\hat{A}$ -cusp using the Lefschetz fixed point formula [3] and Theorem 3.1 (see [12] for details). Since  $\text{codim } M^\sigma \leq 2o \cdot m_o$  Theorem 3.4 implies the following corollary which generalizes Theorem 3.2 to cyclic actions of arbitrary finite order.

**Corollary 3.5.** *Let  $M$  be a *Spin*-manifold with  $S^1$ -action and let  $\sigma \in S^1$  be of order  $o \geq 2$ . If  $\text{codim } M^\sigma > 2o \cdot r$  then the first  $(r + 1)$  coefficients of  $\Phi_0(M)$  vanish. ■*

## 4 Positive curvature and elliptic genera

In this section we prove the vanishing results for positively curved manifolds stated in the introduction. Recall from Section 2 that positive curvature restricts the dimension of fixed point components of isometric actions. This property is an essential ingredient in the proofs of Theorem 1.1 and Theorem 1.2.

**Definition 4.1.** Let  $M$  be a closed connected manifold with smooth action by a torus  $T$ . We say  $M$  has *restricted fixed point dimension* for the prime  $p$  if for all cyclic subgroups  $H \subset H' \subset T$  of order a power of  $p$  and every connected component  $X \subset M^H$  the dimension of two different connected components  $F_1$  and  $F_2$  of  $X^{H'}$  is restricted by  $\dim F_1 + \dim F_2 < \dim X$ .

By Theorem 2.1 a positively curved manifold with isometric  $T$ -action has restricted fixed point dimension for any prime  $p$ . Other examples are given by  $T$ -manifolds with the same integral cohomology ring as a projective space (see [9], Chapter VII).

A main step in the study of the elliptic genus for these manifolds is the following lemma which is a consequence of Theorem 3.2 and Theorem 3.4 (see the next section for the proof).

**Lemma 4.2 (Main lemma).** *Let  $T$  be a torus of rank  $2r > 0$  and let  $M$  be a Spin-manifold with effective  $T$ -action. If  $M$  has restricted fixed point dimension for the prime 2 and  $\dim M > 12r - 4$  then at least one of the following possibilities holds:*

1. *The first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  vanish.*
2. *For some subgroup  $S^1 \subset T$  with involution  $\sigma \in S^1$  the fixed point manifolds  $M^\sigma$  and  $M^{S^1}$  are orientable and connected of codimension 4 and 6, respectively.*

With this information at hand we now prove Theorem 1.1 stated in the introduction.

**Theorem 4.3.** *Let  $M$  be a closed connected Spin-manifold of dimension  $> 12r - 4$ . If  $M$  admits a metric of positive curvature and symmetry rank  $\geq 2r$  then the indices of twisted Dirac operators occurring as the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  vanish.*

**Proof:** For  $r = 0$  the theorem follows from [35]. So assume  $r > 0$ . Let  $T$  denote a torus of rank  $2r$  acting isometrically and effectively on  $M$ . Assume the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  do not vanish. In particular,  $M$  is of even dimension and simply connected [52].

The  $T$ -action has the properties given in the second part of Lemma 4.2, i.e.  $X := M^\sigma$  and  $X^{S^1}$  are orientable connected submanifolds of codimension 4 and 6, respectively. Note that  $X$ , being a totally geodesic submanifold of  $M$ , inherits positive curvature from  $M$ . Since the submanifold  $X^{S^1}$  of  $X$  has codimension two it follows from Theorem 2.4 that  $X$  is diffeomorphic to a sphere or a complex projective space. It is well-known that a non-trivial  $S^1$ -action on a complex projective space has more than one connected fixed point component (cf. [9], Chapter VII, Theorem 5.1). Thus  $X$  is diffeomorphic to a sphere of codimension 4. Since  $\dim M > 8$  the Euler class  $e(\nu_X) \in H^4(X; \mathbb{Z})$  of the normal bundle  $\nu_X$  of  $X \subset M$  is trivial (we fix compatible orientations for  $X$  and  $\nu_X$ ).

This implies that the expansion of the elliptic genus in the signature cusp vanishes as we will explain next (see [12] for details). Recall that the coefficients of this expansion are twisted signatures  $sign(q, \mathcal{L}M) = \sum_{l \geq 0} sign(M, E_l) \cdot q^l$ , where  $E_l$  is a virtual complex vector bundle associated to the tangent bundle  $TM$ . The  $S^1$ -action on  $TM$  induces an action on each  $E_l$ . Let  $sign_{S^1}(M, E_l)$  denote the  $S^1$ -equivariant twisted signature and  $sign_{S^1}(q, \mathcal{L}M)$  the equivariant expansion of the elliptic genus. Recall from Theorem 3.1 that  $sign(q, \mathcal{L}M) = sign_{S^1}(q, \mathcal{L}M)(\lambda)$  for any  $\lambda \in S^1$ . By the Lefschetz fixed point formula [3]



$sign_{S^1}(M, E_l)(\sigma)$  is equal to

$$\left\langle \prod_i \left( x_i \cdot \frac{1 + e^{-x_i}}{1 - e^{-x_i}} \right) \cdot \prod_j \left( y_j \cdot \frac{1 + e^{-y_j}}{1 - e^{-y_j}} \right)^{-1} \cdot ch(E_l)(\sigma) \cdot e(\nu_X), [X] \right\rangle.$$

Here  $\pm x_i$  (resp.  $\pm y_j$ ) denote the formal roots of  $X$  (resp. the normal bundle  $\nu_X$ ),  $ch(E_l)$  denotes the equivariant Chern character of  $E_l$ ,  $[X]$  the fundamental cycle and  $\langle \cdot, \cdot \rangle$  the Kronecker pairing.

Since  $e(\nu_X)$  is trivial  $sign_{S^1}(M, E_l)(\sigma) = 0$  for every  $l \geq 0$ . By Theorem 3.1  $sign(q, \mathcal{L}M)$  vanishes identically. Thus  $\Phi_0(M) = 0$  contradicting the initial assumption of the proof.  $\blacksquare$

**Remarks 4.4.** 1. *Theorem 4.3 also follows from Lemma 4.2 and Theorem 2.3.*

2. *The statement for  $r = 2$  in Theorem 4.3 can be strengthened to: If  $M$  admits a metric of positive curvature and symmetry rank  $\geq 3$  then the first three coefficients in the expansion  $\Phi_0(M)$  vanish, i.e.  $\hat{A}(M) = \hat{A}(M, TM) = \hat{A}(M, \Lambda^2(TM)) = 0$ .*

Under stronger assumptions on the bounds for the symmetry rank and the dimension the conclusion of Theorem 4.3 holds if one only assumes that  $M$  has positive  $k$ th Ricci curvature and an elementary  $p$ -abelian subgroup of the torus acts by isometries.

Recall that a manifold  $M$  has positive  $k$ th Ricci curvature (or  $k$ -positive Ricci curvature) if for any  $(k + 1)$  mutually orthogonal unit tangent vectors  $e, e_1, \dots, e_k$  (at any point of  $M$ ) the sum of curvatures  $\sum_{i=1}^k sec(e \wedge e_i)$  is positive [33]. Thus, 1-positive Ricci curvature is equivalent to positive curvature and  $(\dim M - 1)$ -positive Ricci curvature is equivalent to positive Ricci curvature.

Assume that  $M$  has positive  $k$ th Ricci curvature and assume that a torus  $T$  of rank  $R$  acts smoothly on  $M$  such that the induced action of the  $p$ -torus  $T_p \cong (\mathbb{Z}/p\mathbb{Z})^R$ ,  $p$  a prime, is isometric and effective. To keep the exposition simple we shall assume the generous bounds  $R \geq p \cdot r + \frac{k+1}{2}$  and  $\dim M \geq 6p \cdot r + (k - 1)$ .

**Proposition 4.5.** *For a connected Spin-manifold  $M$  as above the indices of twisted Dirac operators occurring as the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  vanish.*

**Sketch of proof:** First note that the intersection property for totally geodesic submanifolds in positive curvature (Theorem 2.1) extends to positive  $k$ th Ricci curvature [33]: *Two totally geodesic submanifolds  $N_1$  and  $N_2$  of a manifold  $M$  of positive  $k$ th Ricci curvature intersect if  $\dim N_1 + \dim N_2 \geq \dim M + (k - 1)$ .*

In particular, if  $F_1$  and  $F_2$  are two different connected fixed point components of an isometry  $\sigma$  then  $\dim F_1 + \dim F_2 < \dim M + (k - 1)$ .

Assume the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  do not vanish. Consider the action of  $\sigma \in T_p$  on  $M$ . By Corollary 3.5 the codimension of  $M^\sigma$  is  $\leq 2p \cdot r$ . Hence, a connected component  $F$  of  $M^\sigma$  has either ‘‘small codimension’’, i.e.  $\text{codim } F \leq 2p \cdot r$ , or ‘‘small dimension’’, i.e.  $\dim F < 2p \cdot r + (k - 1)$ .

Consider a  $T$ -fixed point  $pt \in M$  (which exists since  $\Phi(M) \neq 0$ ) and let  $F_\sigma \subset M^\sigma$  denote the component which contains  $pt$ . It is an elementary exercise

to show that the  $p$ -torus  $T_p$  has a basis  $\sigma_1, \dots, \sigma_R$  such that  $\dim F_{\sigma_i} \geq 2R - 2 \geq 2p \cdot r + (k - 1)$ . This implies that the codimension of  $F_{\sigma_i}$  is small. Since  $\dim M \geq 6p \cdot r + (k - 1)$  the codimension of  $F_{\sigma+\sigma'}$  is small provided this holds for  $F_\sigma$  and  $F_{\sigma'}$ . Hence, the codimension of  $F_\sigma$  is small for every  $\sigma \in T_p$ , i.e.  $\text{codim } F_\sigma \leq 2p \cdot r$  for every  $\sigma \in T_p$ . However, it follows from elementary linear algebra that for some  $\sigma \in T_p$  the codimension of  $F_\sigma$  is at least  $2R \geq 2p \cdot r + (k + 1)$ . This gives the desired contradiction. Hence, the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  vanish.  $\blacksquare$

The next result implies Theorem 1.2 stated in the introduction.

**Theorem 4.6.** *Let  $M$  be a closed 2-connected manifold of dimension  $> 8$  with effective smooth  $S^1$ -action. Assume  $M$  admits a metric of positive curvature such that the subgroup  $\mathbb{Z}/4\mathbb{Z} \subset S^1$  acts by isometries. Then  $\hat{A}(M)$  and  $\hat{A}(M, TM)$  vanish.*

**Proof:** We fix an orientation and a *Spin*-structure for  $M$ . Note that  $\hat{A}(M)$  vanishes by [35, 4]. Let  $\rho \in S^1$  be an element of order 4 and let  $\sigma := \rho^2$ . Assume  $\hat{A}(M, TM)$  does not vanish. Then  $M$  has dimension  $4k \geq 12$  and the fixed point manifold  $M^\sigma$  is the union of a connected component  $X$  of codimension 4 and (a possible empty set of) isolated fixed points by Corollary 2.2, Theorem 3.2 and Corollary 3.3. Next apply Theorem 3.4 (for  $o = 4$ ) to conclude that either  $X = X^\rho$  or the codimension of  $X^\rho$  in  $X$  is two.

We claim that  $X = X^\rho$ . If the codimension of  $X^\rho$  is two then the inclusion  $X^\rho \hookrightarrow X$  is  $(4k - 7)$ -connected by Theorem 2.3. This implies that  $b_4(X)$  is equal to  $b_2(X)$  (use Poincaré duality for  $X$  and  $X^\rho$ ). Note that  $X$  is 2-connected by Theorem 2.3. Hence,  $b_4(X)$  vanishes. In particular, the Euler class of the normal bundle of  $X$  in  $M$  vanishes rationally. Now argue as in the proof of Theorem 4.3 to conclude that  $\text{sign}(q, \mathcal{L}M) = \text{sign}_{S^1}(q, \mathcal{L}M)(\sigma) = 0$  contradicting the assumption on  $\hat{A}(M, TM)$ . Hence,  $X = X^\rho$ .

Since  $X$  is fixed by  $\rho$  the action of  $\rho$  on the normal bundle  $\nu_X$  induces a complex structure such that  $\rho$  acts by multiplication with  $i$ . We fix the orientation of  $X$  which is compatible with the orientations of  $\nu_X$  and  $M$ . Also the action of  $\rho$  induces a complex structure on the normal bundle of any of the isolated  $\sigma$ -fixed points. We shall now compute the local contributions in the Lefschetz fixed point formula for  $\text{sign}_{S^1}(q, \mathcal{L}M)(\rho)$ :

$$\text{sign}_{S^1}(q, \mathcal{L}M)(\rho) = \mu_X + \sum \mu_{p_j}$$

Here  $\mu_X$  (resp.  $\mu_{p_j}$ ) denotes the local contribution at  $X$  (resp. at an isolated fixed point  $p_j$ ). The term  $\mu_X$  is given by [3]

$$\mu_X = \langle \mathcal{T}_X \cdot \mathcal{N}_X, [X] \rangle$$

where

$$\mathcal{T}_X := \prod_i \left( x_i \cdot \frac{1 + e^{-x_i}}{1 - e^{-x_i}} \cdot \prod_{n=1}^{\infty} \frac{(1 + q^n \cdot e^{x_i}) \cdot (1 + q^n \cdot e^{-x_i})}{(1 - q^n \cdot e^{x_i}) \cdot (1 - q^n \cdot e^{-x_i})} \right)$$

and

$$\mathcal{N}_X := \prod_{j=1,2} \left( \frac{1 + e^{-y_j} \cdot (-i)}{1 - e^{-y_j} \cdot (-i)} \cdot \prod_{n=1}^{\infty} \frac{(1 + q^n \cdot e^{y_j} \cdot i) \cdot (1 + q^n \cdot e^{-y_j} \cdot (-i))}{(1 - q^n \cdot e^{y_j} \cdot i) \cdot (1 - q^n \cdot e^{-y_j} \cdot (-i))} \right)$$

Here  $\pm x_i$  (resp.  $y_1, y_2$ ) denote the formal roots of  $X$  (resp.  $\nu_X$ ). Since  $X$  is 2-connected the first Chern class  $y_1 + y_2$  of  $\nu_X$  vanishes. This implies

$$\mathcal{N}_X = \frac{1 + e^{-y_1} \cdot (-i)}{1 - e^{-y_1} \cdot (-i)} \cdot \frac{1 + e^{y_1} \cdot (-i)}{1 - e^{y_1} \cdot (-i)} = -1.$$

Hence, the expression for  $\mu_X$  simplifies to

$$\mu_X = -\langle \mathcal{T}_X, [X] \rangle = -\text{sign}(q, \mathcal{L}X).$$

The term  $\mu_{p_j}$  is given by [3]

$$\mu_{p_j} = \pm \left( \frac{1-i}{1+i} \prod_{n=1}^{\infty} \frac{(1+i \cdot q^n) \cdot (1-i \cdot q^n)}{(1-i \cdot q^n) \cdot (1+i \cdot q^n)} \right)^{2k} = \pm \left( \frac{1-i}{1+i} \right)^{2k} = \pm 1$$

By Theorem 3.1  $\text{sign}(q, \mathcal{L}M) = \text{sign}_{S^1}(q, \mathcal{L}M)(\rho) = -\text{sign}(q, \mathcal{L}X) + c$ , where  $c$  is the integer obtained by summing up  $\mu_{p_j}$ . Equivalently,

$$\Phi(M) = -\Phi(X) + c \tag{2}$$

Note that  $\Phi_0(M) \in q^{-k/2} \mathbb{C}[[q]]$ , whereas  $\Phi_0(X) \in q^{1/2} \cdot q^{-k/2} \mathbb{C}[[q]]$ . By comparing the expansions in the  $\hat{A}$ -cusp of both sides of (2) it follows that  $\Phi_0(M) \in \mathbb{Z}$  (in fact, depending on the parity of  $k$ ,  $\Phi_0(M)$  is equal to 0 or  $c$ ). Since  $\dim M > 8$  this implies that  $\hat{A}(M, TM) = 0$ .  $\blacksquare$

In the remaining part of this section we use Lemma 4.2 to study the Pontrjagin numbers of  $4k$ -dimensional *Spin*-manifolds with symmetry which have the same integral cohomology ring as a projective space. Such manifolds are either integral cohomology  $\mathbb{H}P^k$ 's or integral cohomology Cayley planes (for recent progress in the study of integral cohomology  $\mathbb{C}P^m$ 's with symmetry we refer to [11] and references therein). Note that an integral cohomology Cayley plane with smooth non-trivial  $S^1$ -action has the same Pontrjagin numbers as the Cayley plane since these are completely determined by the signature and the  $\hat{A}$ -genus (the latter vanishes by [4]). The same argument applies to an integral cohomology  $\mathbb{H}P^2$  with smooth non-trivial  $S^1$ -action. We shall now apply Lemma 4.2 to integral cohomology  $\mathbb{H}P^k$ 's for  $k > 2$ .

**Proposition 4.7.** *Let  $M$  be an integral cohomology  $\mathbb{H}P^k$ , i.e.  $H^*(M; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{k+1})$ , where  $x$  has degree 4. If a torus  $T$  of rank  $2r > 0$  acts effectively and smoothly on  $M$  and  $4k > 12r - 4$  then the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  vanish.*

**Proof:** Note that  $M$  is a *Spin*-manifold with restricted fixed point dimension for any prime. By Lemma 4.2 we may assume that for some  $S^1$ -subgroup of  $T$  the fixed point manifold  $M^{S^1}$  is connected of codimension 6. It is well known that  $M^{S^1}$  is an integral cohomology projective space of the form  $H^*(M^{S^1}; \mathbb{Z}) \cong \mathbb{Z}[y]/(y^{l+1})$  where  $y$  has even degree  $\leq 4$  (cf. [9], Chapter VII, Theorem 5.1). Also  $\chi(M) = \chi(M^{S^1})$  by the Lefschetz fixed point formula for the Euler characteristic. Since the codimension of  $M^{S^1}$  is equal to 6 it follows that  $\deg(y) = 2$  and  $k = l = 3$ . Hence,  $M$  is an integral cohomology  $\mathbb{H}P^3$ . For a 12-dimensional

manifold the elliptic genus is a linear combination of the signature and the  $\hat{A}$ -genus. The signature of  $M$  vanishes for trivial reasons. The  $\hat{A}$ -genus of  $M$  vanishes since  $M$  is a *Spin*-manifold with non-trivial  $S^1$ -action [4]. Hence,  $\Phi(M) = 0$ . This completes the proof. ■

Finally we point out the following consequence of Lemma 4.2.

**Proposition 4.8.** *Let  $T$  be a torus of rank  $2r > 0$  and let  $M$  be a *Spin*-manifold with  $T$ -action of dimension  $> 12r - 4$ . Assume the signature of  $M$  does not vanish. If  $M$  has restricted fixed point dimension for the prime 2 then the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  vanish.*

**Proof:** By the rigidity of the signature  $sign(M) = sign(M^{S^1})$  for any  $S^1$ -action. Since the signature of  $M$  does not vanish  $\dim M \equiv \dim M^{S^1} \equiv 0 \pmod{4}$  if  $M^{S^1}$  is connected. In particular,  $\text{codim } M^{S^1} \neq 6$  and the statement follows from Lemma 4.2. ■

## 5 Proof of the main Lemma

In this section we prove Lemma 4.2. Let  $T$  be a torus of rank  $2r > 0$  and let  $M$  be a *Spin*-manifold with effective  $T$ -action. Assume  $M$  has restricted fixed point dimension for the prime 2 and  $\dim M > 12r - 4$ . Assume also that the first  $(r + 1)$  coefficients in the expansion  $\Phi_0(M)$  do not vanish. This implies that  $\dim M = 4k \geq 12r$ .

Our goal is to show that for some subgroup  $S^1 \subset T$  with involution  $\sigma \in S^1$  the fixed point manifolds  $M^\sigma$  and  $M^{S^1}$  are orientable and connected of codimension 4 and 6, respectively. To this end we first examine the action of  $T$  at a fixed point  $pt$  (which exists since  $\Phi(M) \neq 0$ ).

The tangent space  $T_{pt}M$  of  $M$  at  $pt$  decomposes (non-canonically) into  $2k$  complex one-dimensional  $T$ -representations  $T_{pt}M = R_1 \oplus \dots \oplus R_{2k}$ . With respect to such a decomposition the action of  $T$  on  $T_{pt}M$  is given by a homomorphism  $T \hookrightarrow U(1)^{2k}$ . We denote by  $h$  the induced homomorphism of integral lattices  $I_T \rightarrow I_{U(1)^{2k}} = \mathbb{Z}^{2k}$ . Since the  $T$ -action is effective the mod  $p$  reduction  $I_T \otimes (\mathbb{Z}/p\mathbb{Z}) \rightarrow (\mathbb{Z}/p\mathbb{Z})^{2k}$  of  $h$  is injective for every  $p \geq 2$ .

For a finite covering homomorphism  $\hat{T} := S^1 \times \dots \times S^1 \rightarrow T$  we denote by  $\hat{h}$  the homomorphism of integral lattices  $I_{\hat{T}} = \mathbb{Z}^{2r} \rightarrow \mathbb{Z}^{2k}$  for the covering action on  $T_{pt}M$ . Note that an element  $\hat{h}(a)$ ,  $a \in I_{\hat{T}}$ , determines a homomorphism  $S^1 \xrightarrow{\iota} \hat{T}$ ,  $\exp(\theta) \mapsto \exp(\theta \cdot a)$ , and an  $S^1$ -subgroup  $\iota(S^1) \subset \hat{T}$ . Let  $c_1, \dots, c_{2r}$  be the image of the standard basis of  $I_{\hat{T}} = \mathbb{Z}^{2r}$  under  $\hat{h}$ .

**Claim 5.1.** *Let  $p$  be a prime. There exists a finite covering homomorphism  $\hat{T} := S^1 \times \dots \times S^1 \rightarrow T$  of degree coprime to  $p$  such that*

$$A := \begin{pmatrix} c_1 \\ \vdots \\ c_{2r} \end{pmatrix} = \begin{pmatrix} ? & 0 & 0 & 0 & \dots & 0 & ? & ? & ? & ? & \dots & ? \\ 0 & ? & 0 & 0 & \dots & 0 & ? & ? & ? & ? & \dots & ? \\ 0 & 0 & ? & 0 & \dots & 0 & ? & ? & ? & ? & \dots & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & ? & ? & ? & ? & \dots & ? \end{pmatrix} \equiv \begin{pmatrix} * & 0 & 0 & 0 & \dots & 0 & ? & ? & ? & \dots & ? \\ 0 & * & 0 & 0 & \dots & 0 & ? & ? & ? & \dots & ? \\ 0 & 0 & * & 0 & \dots & 0 & ? & ? & ? & \dots & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & * & ? & ? & ? & \dots & ? \end{pmatrix} \pmod{p}$$

where each  $*$  is coprime to  $p$ .

The claim is an elementary fact from linear algebra. For convenience we give a proof at the end of the section. In the following let  $p = 2$  and let  $\hat{T} = S^1 \times \dots \times S^1 \rightarrow T$  denote a finite covering action of odd degree with the properties described in Claim 5.1.

The next claim gives information on the action of involutions of  $\hat{T}$  on  $T_{pt}M$  (again the proof is postponed to the end of the section). This information will be used to exhibit involutions for which the fixed point manifold is connected of codimension 4 (since  $\hat{T} \rightarrow T$  has odd degree the same property holds for involutions of  $T$ ).

- Claim 5.2.**
1. For the involution  $\sigma_i \in \hat{T}$  corresponding to  $c_i$  ( $\sigma_i$  acts on  $T_{pt}M$  by  $\exp(c_i/2) \in U(1)^{2k}$ ) the component of  $M^{\sigma_i}$  which contains the  $\hat{T}$ -fixed point  $pt$  has codimension 4, i.e.  $c_i$  has two odd entries.
  2. The involution  $\sum_i \sigma_i$  acts trivially on each representation space  $R_i$ ,  $i > 2r$ , i.e. each of the last  $(2k - 2r)$  columns of  $A$  has an even number of odd entries.
  3. The 2-torus of  $\hat{T}$  (i.e. the subgroup generated by the involutions  $\sigma_i$ ) acts non-trivially on at most  $r$  of the representation spaces  $R_{2r+1}, \dots, R_{2k}$ , i.e. at most  $r$  of the last  $(2k - 2r)$  columns of  $A$  are non-zero modulo 2.

Hence, for every involution  $\sigma \in \hat{T}$  and every  $\hat{T}$ -fixed point the connected component of  $M^\sigma$  which contains the fixed point has codimension  $\leq 6r \leq \frac{1}{2} \dim M$ . Since  $M$  has restricted fixed point dimension for the prime 2 it follows that  $M^\sigma$  is the union of a connected component of codimension  $\leq 6r$  with  $\hat{T}$ -fixed point (in fact the codimension is  $\leq 4r$  by Theorem 3.2) and (a possible empty set of) components with fixed point free  $T$ -action. Recall that  $\text{codim } M^{\sigma_i} = 4$ . Since any isolated  $\sigma$ -fixed point is also fixed by  $\hat{T}$  it follows that  $M^{\sigma_i}$  is connected of codimension 4 for every  $i$ .

Let  $S_j$  denote the  $j$ th  $S^1$ -factor of  $\hat{T}$ . Below we shall use Theorem 3.4 to show

**Claim 5.3.** *The fixed point manifold  $M^{S_j}$  is connected of codimension 4. In particular,  $c_j$  has two odd entries and all other entries vanish.*

Note that Claim 5.1, Claim 5.2 and Claim 5.3 imply that the codimension of any connected component of  $M^{\hat{T}}$  is  $\leq 6r$ .

With this information at hand we shall now complete the proof of Lemma 4.2. By the second part of Claim 5.2 one can choose  $c_j$  such that

$$\begin{pmatrix} c_j \\ c_{2r} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & \text{odd} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \text{odd} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & \text{odd} & 0 & \dots & 0 & \text{odd} & 0 & \dots & 0 \end{pmatrix}$$

Consider the  $S^1$ -subgroup  $S \subset \hat{T}$  determined by  $2 \cdot c_j + c_{2r} \in \hat{h}(I_{\hat{T}})$ . The fixed point manifold  $M^\sigma$  of the involution  $\sigma \in S$  is equal to  $M^{\sigma_{2r}}$  which is connected of codimension 4. The  $S$ -fixed point manifold has codimension 2 in  $M^\sigma$ . Since  $M^\sigma$  has restricted fixed point dimension for the prime 2 the fixed point manifold  $M^S$  is connected. Also,  $M^S$  is orientable. Since  $M$  is a *Spin*-manifold  $M^\sigma$  is orientable as well (see for example [8], Lemma 10.1). Finally note that the same properties hold for the  $S^1$ -subgroup of  $T$  which is the image of  $S$  under the covering homomorphism  $\hat{T} \rightarrow T$ . ■

**Proof of Claim 5.1:** It is an elementary fact from linear algebra that  $h(I_T) \subset \mathbb{Z}^{2k}$  admits a basis  $b_1, \dots, b_{2r}$  such that

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{2r} \end{pmatrix} = \begin{pmatrix} ? & ? & ? & ? & \dots & ? & ? & ? & ? & \dots & ? \\ 0 & ? & ? & ? & \dots & ? & ? & ? & ? & \dots & ? \\ 0 & 0 & ? & ? & \dots & ? & ? & ? & ? & \dots & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & ? & ? & ? & ? & \dots & ? \end{pmatrix} \equiv \begin{pmatrix} * & 0 & 0 & 0 & \dots & 0 & ? & ? & ? & \dots & ? \\ 0 & * & 0 & 0 & \dots & 0 & ? & ? & ? & \dots & ? \\ 0 & 0 & * & 0 & \dots & 0 & ? & ? & ? & \dots & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & * & ? & ? & ? & \dots & ? \end{pmatrix} \pmod{p}$$

after permuting columns (i.e. after permuting the representation spaces  $R_i$ ) if necessary. Here each  $*$  is coprime to  $p$ . Note that the choice of a basis of  $h(I_T)$  is equivalent to the choice of an isomorphism  $S^1 \times \dots \times S^1 \rightarrow T$ . Using suitable row operations of the form  $b_i \rightsquigarrow \alpha \cdot b_i + \beta \cdot b_j$ , where  $i < j$ ,  $\alpha \not\equiv 0 \pmod{p}$  and  $\beta \equiv 0 \pmod{p}$ , one obtains a basis  $c_1, \dots, c_{2r}$  of  $h(I_T) \otimes \mathbb{Q}$  such that the matrix  $A = (c_1, \dots, c_{2r})^t$  has the properties given in the claim. Each  $c_j$  determines a homomorphism  $\iota_j : S^1 \rightarrow T$ . Since  $c_1, \dots, c_{2r}$  is a basis of  $h(I_T) \otimes \mathbb{Q}$  the homomorphism  $\hat{T} := S^1 \times \dots \times S^1 \xrightarrow{\iota_1 \cdots \iota_{2r}} T$  is a finite covering homomorphism. The induced homomorphism  $\hat{h} : I_{\hat{T}} \rightarrow \mathbb{Z}^{2k}$  of integral lattices maps the standard basis to  $c_1, \dots, c_{2r}$ . In view of the properties of the matrix  $A$  the covering homomorphism has degree coprime to  $p$ . ■

**Proof of Claim 5.2:** First note that by Theorem 3.2 and Corollary 3.3 for every involution  $\sigma \in \hat{T}$  the fixed point manifold  $M^\sigma$  has codimension  $\leq 4r$  and the dimension of each connected component of  $M^\sigma$  is divisible by 4. Since the dimension of  $M$  is  $\geq 8r$  and  $M$  has restricted fixed point dimension for the prime 2 it follows that for every connected component  $F$  of  $M^\sigma$  either  $\text{codim } F \leq 4r$  or  $\dim F \leq 4r - 4$ .

Let  $\hat{h}_1 : I_{\hat{T}} \otimes (\mathbb{Z}/2\mathbb{Z}) \rightarrow (\mathbb{Z}/2\mathbb{Z})^{2k}$  denote the mod 2 reduction of  $\hat{h}$ . For the binary linear code  $\mathcal{C}_1 := \text{im}(\hat{h}_1) \subset (\mathbb{Z}/2\mathbb{Z})^{2k}$  we conclude that each code word  $\sigma \in \mathcal{C}_1$  has either weight<sup>3</sup>  $wt(\sigma) \leq 2r$  or co-weight  $cowt(\sigma) := 2k - wt(\sigma) \leq 2r - 2$ . In particular, the mod 2 reduction of  $c_i$ , denoted by  $\sigma_i$ , has weight  $wt(\sigma_i) \leq 2r$ . Since the weight function is sublinear, i.e.  $wt(\sigma + \sigma') \leq wt(\sigma) + wt(\sigma')$ , and  $2k \geq 6r$  it follows that the subset of code words with weight  $\leq 2r$  is closed under addition. Hence  $wt(\sigma) \leq 2r$  for every  $\sigma \in \mathcal{C}_1$ . In particular, this inequality holds for  $\sum_j \sigma_j$  and  $\sum_{j \neq i} \sigma_j$  which implies  $wt(\sigma_i) = 2$  (i.e.  $c_i$  has two odd entries) and implies that each of the last  $(2k - 2r)$  columns of  $A$  has an even number of odd entries. Finally note that if  $l$  of the last  $(2k - 2r)$  columns of  $A$  are non-zero modulo 2 then the weight of some code word is  $\geq 2l$ . Hence  $l \leq r$ . This completes the proof of the claim. ■

**Proof of Claim 5.3:** To show that  $M^{S_j}$  is connected of codimension 4 it suffices to prove this property for all cyclic subgroups  $C_s \subset S_j$  of order  $2^s$ ,  $s \in \mathbb{N}$ . We know already that  $M^{C_1}$  is connected of codimension 4.

Assume  $M^{C_s}$  is connected of codimension 4. To show the corresponding property for  $C_{s+1}$  we will use the following consequence of Theorem 3.4 which we prove first:

*Let  $M$  be a connected Spin-manifold with  $S^1$ -action. Assume the cyclic subgroup  $C_s \subset S^1$  of order  $2^s$  acts non-trivially on  $M$  and the fixed point manifold  $M^{C_s}$  is*

<sup>3</sup>The weight of  $\sigma$  is defined as the number of entries equal to 1.

connected. If the first  $(r+1)$  coefficients in the expansion  $\Phi_0(M)$  do not vanish then the codimension of the submanifold  $M^{C_{s+1}}$  of  $M^{C_s}$  is  $\leq 4r-2$ .

Let  $o := 2^{s+1}$ . Note that for some connected component  $Y$  of the  $S^1$ -fixed point manifold  $m_o(Y) \leq r$  by Theorem 3.4. Let  $Z \subset M^{C_{s+1}}$  be the connected component which contains  $Y$ . Note that  $m_o(Y)$  is strictly larger than  $\text{codim}(Z \subset M^{C_s})/4$  since  $M^{C_s}$  is a proper connected submanifold of  $M$ . Hence the codimension of  $Z$  in  $M^{C_s}$  must be less than  $4r$  which implies that the codimension of the submanifold  $M^{C_{s+1}}$  of  $M^{C_s}$  is  $\leq 4r-2$ .

We shall now continue with the study of the action of  $S_j$ . By the statement above the codimension of the submanifold  $M^{C_{s+1}}$  of  $M^{C_s}$  is  $\leq 4r-2$ . Since  $M$  has restricted fixed point dimension for the prime 2 a connected component of  $M^{C_{s+1}}$  has either codimension  $\leq 4r-2$  in  $M^{C_s}$  or has dimension  $\leq 4r-4$ .

Next consider the action of the  $S^1$ -subgroup  $\tilde{S}$  of  $\hat{T}$  which is determined by  $\tilde{c} := c_j + 2^s \cdot \sum_i c_i \in \hat{h}(I_{\hat{T}})$ . Let  $\tilde{C}_s \subset \tilde{S}$  denote the cyclic subgroup of order  $2^s$ . By the statement above the codimension of the submanifold  $M^{\tilde{C}_{s+1}}$  of  $M^{\tilde{C}_s}$  is  $\leq 4r-2$ .

It follows that either the number of entries of  $c_j$  which are  $\equiv 2^s \pmod{2^{s+1}}$  is  $\leq 2r-1$  or the number of entries of  $c_j$  which are  $\equiv 0 \pmod{2^{s+1}}$  is  $\leq 2r-2$ . The same property holds for  $\tilde{c}$ . Also the mod  $2^{s+1}$ -reductions of  $c$  and  $\tilde{c}$  have the same last  $(2k-2r)$  entries by the second part of Claim 5.2.

This implies that the mod  $2^{s+1}$ -reduction of  $c_j$  has only two non-zero entries. In other words  $C_{s+1} \subset S_j$  acts trivially on the tangent bundle of  $M^{C_s}$  at the  $T$ -fixed point  $pt$ . Thus  $M^{C_{s+1}} = M^{C_s}$  and  $M^{C_{s+1}}$  is connected of codimension 4. This completes the induction step. It follows that the fixed point manifold  $M^{S_j}$  is connected of codimension 4.  $\blacksquare$

## 6 Ricci versus sectional curvature

Apparently the only known topological property which allows to distinguish positive Ricci curvature from positive curvature on simply connected manifolds is based on Gromov's Betti number theorem [19]: With respect to any field of coefficients the sum of Betti numbers of a positively curved  $n$ -dimensional manifold is less than a constant  $C(n)$  depending only on the dimension.

The bound in [19] which depends doubly exponentially on the dimension  $n$  has been improved by Abresch [1] who showed that the Betti number theorem holds for a bound  $C(n)$  depending exponentially on  $n^3$  (the bound  $C(n)$  may be chosen to satisfy  $\exp(5n^3 + 8n^2 + 4n + 2) \leq C(n) \leq \exp(6n^3 + 9n^2 + 4n + 4)$ ). Sha and Yang [49] constructed metrics of positive Ricci curvature on the  $k$ -fold connected sum of  $S^n \times S^m$ ,  $n, m > 1$ , for every  $k \in \mathbb{N}$ . By the Betti number theorem these manifolds do not admit positively curved metrics if  $k$  is sufficiently large.

In this section we present two methods to exhibit manifolds with "small" Betti numbers (i.e. the sum is less than  $C(n)$ ) which admit metrics of positive Ricci curvature but do not admit metrics of positive curvature under assumptions on the symmetry. The first method which relies on Theorem 1.1 leads to

**Theorem 6.1.** *For every  $r \geq 3$  and every  $d \in \mathbb{N}$  there exists a simply connected closed manifold  $M$  of dimension greater than  $d$  such that:*

1.  $M$  admits a metric of positive Ricci curvature with symmetry rank  $\geq 2r$ .
2.  $M$  does not admit a metric of positive curvature with symmetry rank  $\geq 2r$ .

The manifold  $M$  may be chosen to have small Betti numbers. The second method which relies on Theorem 3.4 and recent work of Wilking (see Theorem 2.3) gives stronger information. Recall from the introduction that an  $S^1$ -action is called finite-order-isometric of order  $o$  if the cyclic subgroup of order  $o$  acts effectively and isometrically.

**Theorem 6.2.** *For every  $d \in \mathbb{N}$  and every  $o \geq 2$  there exists a simply connected closed manifold  $M$  of dimension greater than  $d$  such that:*

1.  $M$  admits a metric of positive Ricci curvature with finite-order-isometric  $S^1$ -action of order  $o$ .
2.  $M$  does not admit a metric of positive curvature with finite-order-isometric  $S^1$ -action of order  $o$ .

Again the manifold  $M$  may be chosen to have small Betti numbers. The examples occurring in both theorems are given by Riemannian products of the form  $M_1 \times M_2$ . The first factor  $M_1$  is a *Spin*-manifold of positive Ricci curvature with large symmetry rank and non-vanishing elliptic genus (below we shall choose  $M_1$  to be a product of quaternionic projective planes). The second factor  $M_2$  is a *Spin*-manifold of positive Ricci curvature for which the index  $\hat{A}(M_2, TM_2)$  does not vanish. The next proposition shows that these properties hold for a hypersurface in  $\mathbb{C}P^{n+1}$  of degree  $n = 2m > 2$ .

**Proposition 6.3.** *A non-singular hypersurface  $V_n$  in  $\mathbb{C}P^{n+1}$  of degree  $n = 2m > 2$  has the following properties:*

1.  $V_n$  is *Spin* and admits a metric of positive Ricci curvature.
2. The Betti numbers of  $V_n$  are small.
3. The index  $\hat{A}(V_n, TV_n)$  does not vanish.

**Proof:** For a proof of some of the properties of hypersurfaces used below we refer to [27] and references therein. Let  $V^l$  denote a non-singular hypersurface in  $\mathbb{C}P^{n+1}$  of degree  $l$ . The tangent bundle of  $V^l$  is stably the restriction of  $(T\mathbb{C}P^{n+1} - \gamma^l)$  to  $V^l$ , where  $\gamma$  denotes the complex line bundle over  $\mathbb{C}P^{n+1}$  with first Chern class  $h \in H^2(\mathbb{C}P^{n+1}; \mathbb{Z})$  dual to  $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$  (in the following we shall denote the restriction of  $h$  to  $V^l$  also by  $h$ ). Note that  $\langle h^n, [V^l] \rangle = l$ . The total Chern class of  $V^l$  is given by

$$c(V^l) = (1 + h)^{n+2} \cdot (1 + l \cdot h)^{-1}.$$

In particular,  $c_1(V^l) = (n + 2 - l) \cdot h$ . Taking  $l = n$  we conclude that  $V_n$  is a *Spin*-manifold with positive first Chern class. By Yau's solution [59] of the Calabi conjecture  $V_n$  admits a metric of positive Ricci curvature.



The Euler characteristic of  $V_n$  is equal to the Chern number  $\langle c_n(V_n), [V_n] \rangle$  which can be computed via the formula for  $c(M)$  given above. It turns out that

$$\chi(V_n) = \frac{(n-1)^{n+2} - 1}{n} + (n+2).$$

By the Lefschetz theorem on hyperplane sections the odd Betti numbers of  $V_n$  vanish for any field of coefficients (cf. [27], §22). Hence the Betti numbers of  $V_n$  are small.

Finally we compute  $\hat{A}(V_n, TV_n)$ . By the cohomological version of the index theorem [3]  $\hat{A}(V^l, TV^l)$  is equal to  $\langle B, [V^l] \rangle$ , where

$$B := \left( \frac{h}{e^{h/2} - e^{-h/2}} \right)^{n+2} \cdot \frac{e^{lh/2} - e^{-lh/2}}{lh} \cdot ((n+2)e^h - e^{lh}).$$

It follows that  $\langle B, [V^l] \rangle$  is the residue of  $l \cdot B/h^{n+1}$  at  $h = 0$ . Changing variables,  $w := e^h - 1$ , one computes that  $\hat{A}(V^l, TV^l)$  is equal to the coefficient of  $w^{n+1}$  in

$$(1+w)^{(n-l)/2} \cdot ((1+w)^l - 1) \cdot ((n+2) \cdot (1+w) - (1+w)^l).$$

Hence,  $\hat{A}(V_n, TV_n) = n+2 - \binom{2n}{n+1}$  which does not vanish for  $n = 2m > 2$ . This completes the proof of the proposition.  $\blacksquare$

**Proof of Theorem 6.1:** Let  $M_1$  be the Riemannian product of  $(r-1)$ -copies of  $\mathbb{H}P^2$ , let  $M_2$  be a non-singular hypersurface in  $\mathbb{C}P^{n+1}$  of degree  $n = 2m \gg 0$  (equipped with a metric of positive Ricci curvature) and let  $M$  be the Riemannian product  $M_1 \times M_2$ . Since the symmetry rank of  $\mathbb{H}P^2$  is three and  $r \geq 3$  the symmetry rank of the positively Ricci curved *Spin*-manifold  $M$  is  $\geq 2r$ .

By Proposition 6.3  $\hat{A}(M_2, TM_2) \neq 0$ . Thus the expansion  $\Phi_0(M_2)$  has a pole of order  $(\dim M_2)/8 - 1 = (\dim M)/8 - r$ . For a homogeneous *Spin*-manifold the elliptic genus is equal to the signature [30]. In particular,  $\Phi(\mathbb{H}P^2) = 1$  which can also be shown by a direct computation. This gives

$$\Phi(M) = \Phi(\mathbb{H}P^2)^{r-1} \cdot \Phi(M_2) = \Phi(M_2).$$

It follows that  $\Phi_0(M)$  has a pole of order  $(\dim M)/8 - r$ . Thus, the first  $(r+1)$  coefficient in the expansion  $\Phi_0(M)$  do not vanish. By Theorem 1.1 the manifold  $M$  does not admit a metric of positive curvature with symmetry rank  $\geq 2r$ .  $\blacksquare$

**Proof of Theorem 6.2:** Let  $M$  be the Riemannian product of  $\mathbb{H}P^2$  and  $V_n$  (the latter shall be equipped with a metric of positive Ricci curvature). Since the positively Ricci curved manifold  $M$  has symmetry rank  $\geq 3$  it admits a finite-order-isometric  $S^1$ -action of any order.

Now assume  $M$  carries a metric of positive curvature with finite-order-isometric  $S^1$ -action of order  $o$ . We shall derive a contradiction for  $n = 2m \gg o$ . Let  $\sigma \in S^1$  be of order  $o$ . Since  $\Phi_0(M)$  has a pole of order  $(\dim M)/8 - 2$  the codimension of  $M^\sigma$  is bounded from above by a constant which depends on  $o$  but does not depend on  $\dim M = 2n + 8$  (see Corollary 3.5).

Let  $N \subset M^\sigma$  be a fixed point component of minimal codimension  $k$ . By Theorem 2.3 the inclusion  $N \hookrightarrow M$  is  $((2n+8) - (k+l))$ -connected, where

$l = k - 1$ . Using Poincaré duality for  $M$  and  $N$  it follows that  $H^i(M; \mathbb{Z})$  and  $H^{i+k}(M; \mathbb{Z})$  are isomorphic in the range  $l < i < (2n + 8) - (k + l)$ . Hence, for  $n \gg 0$  the cohomology groups  $H^{n-k}(M; \mathbb{Z})$  and  $H^n(M; \mathbb{Z})$  are isomorphic.

Recall from [27], §22, that the even Betti numbers  $b_{2i}(V_n)$  are one except for  $2i = n$  and the odd Betti numbers vanish. This implies  $b_{n-k}(M) \leq 3$ . However  $b_n(M)$  is  $> 3$  (compare with the formula for  $\chi(V_n)$  in the proof of the proposition above). This gives the desired contradiction. ■

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## References

- [1] U. Abresch, *Lower curvature bounds, Toponogov's theorem, and bounded topology. II.*, Ann. Sci. Ec. Norm. Super., IV. Ser. 20 (1987), 475-502
- [2] S. Aloff and N.L. Wallach, *An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. 81 (1975), 93-97
- [3] M.F. Atiyah and I.M. Singer, *The index of elliptic operators: III*, Ann. of Math. 87 (1968), 546-604
- [4] M.F. Atiyah and F. Hirzebruch, *Spin-Manifolds and Group Actions*, in: Essays on Topology and Related Topics. Memoires dédiés à Georges de Rham, Springer (1970), 18-28
- [5] Y. Bazaikin, *On one family of 13-dimensional closed Riemannian manifolds with positive curvature*, Sib. Math. J. 34 (1996), 1068-1085
- [6] L. Bérard Bergery, *Les variétés Riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive*, J. Math. Pures Appl. 55 (1976), 47-68
- [7] M. Berger, *Les variétés Riemanniennes homogènes normales simplement connexes à courbure strictement positive*, Ann. Scuola Norm. Sup. Pisa 15 (1961), 179-246
- [8] R. Bott and C.H. Taubes, *On the rigidity theorems of Witten*, J. of Amer. Math. Soc. 2 (1989), 137-186
- [9] G. Bredon, *Introduction to compact transformation groups*, Pure and applied math., Vol. 46, Academic Press (1972)
- [10] A. Dessai, *Rigidity theorems for Spin<sup>c</sup>-manifolds*, Topology 39 (2000), 239-258
- [11] A. Dessai, *Homotopy complex projective spaces with Pin(2)-action*, to appear in Topology and its Applications, available at the arXiv: <http://arxiv.org/abs/math.GT/0102061>
- [12] A. Dessai, *Cyclic actions and elliptic genera*, preprint, available at the arXiv: <http://arxiv.org/abs/math.GT/0104255>

- [13] A. Dessai, *Positively curved 8-manifolds with symmetry rank two*, in preparation
- [14] J.H. Eschenburg, *New examples of manifolds with strictly positive curvature*, Invent. Math. 66 (1982), 469-480
- [15] J.H. Eschenburg, *Inhomogeneous spaces of positive curvature*, Differential Geom. Appl. 2 (1992), 123-132
- [16] F. Fang and X. Rong, *Positively curved manifolds with maximal discrete symmetry rank*, preprint (2001)
- [17] T. Frankel, *Manifolds with positive curvature*, Pacific J. Math. 11 (1961), 165-174
- [18] T. Frankel, *On the fundamental group of a compact minimal submanifolds*, Ann. of Math. 83 (1966), 68-73
- [19] M. Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helvetici 56 (1981), 179-195
- [20] M. Gromov and H.B. Lawson, *The classification of simply-connected manifolds of positive scalar curvature*, Ann. of Math. 111 (1980), 209-230
- [21] K. Grove and C. Searle, *Positively curved manifolds with maximal symmetry-rank*, J. Pure Appl. Algebra 91 (1994), 137-142
- [22] K. Grove and C. Searle, *Differential topological restrictions by curvature and symmetry*, J. Diff. Geo. 47 (1997), 530-559
- [23] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. 152 (2000), 331-367
- [24] K. Grove and W. Ziller, *Cohomogeneity one manifolds with positive Ricci curvature*, preprint (2001)
- [25] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17 (1982), 255-306
- [26] H. and R. Herrera,  *$\hat{A}$ -genus on non-spin manifolds with  $S^1$ -actions and the classification of positive quaternion-Kähler 12-manifolds*, preprint IHES (2001)
- [27] F. Hirzebruch, *Topological methods in algebraic geometry* (3rd edition with appendices by R.L.E. Schwarzenberger and A. Borel), Grundlehren der mathematischen Wissenschaften 131, Springer (1966)
- [28] F. Hirzebruch, *Elliptic genera of level  $N$  for complex manifolds*, in: K. Bleuler and M. Werner (Eds.): Differential Geometrical Methods in Theoretical Physics (Como 1987), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 250, Kluwer (1988)
- [29] F. Hirzebruch, Th. Berger and R. Jung, *Manifolds and Modular Forms*, Aspects of Mathematics Vol. E20, Vieweg (1992)

- [30] F. Hirzebruch and P. Slodowy, *Elliptic Genera, Involutions and Homogeneous Spin Manifolds*, Geom. Dedicata 35 (1990), 309-343
- [31] N. Hitchin, *Harmonic spinors*, Adv. in Math. 14 (1974), 1-55
- [32] W. Hsiang and B. Kleiner, *On the topology of positively curved 4-manifolds with symmetry*, J. Diff. Geom. 30 (1989), 615-621
- [33] K. Kenmotsu and C. Xia, *Hadamard-Frankel type theorems for manifolds with partially positive curvature*, Pac. J. Math. 176 (1996), 129-139
- [34] P.S. Landweber (Ed.), *Elliptic Curves and Modular Forms in Algebraic Topology*, Proceedings Princeton 1986, Lecture Notes in Mathematics 1326, Springer (1988)
- [35] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris er. I Math 257 (1963), 7-9
- [36] K. Liu, *On modular invariance and rigidity theorems*, J. Diff. Geo. 41 (1995), 343-396
- [37] S.B. Myers, *Riemannian manifolds in the large*, Duke Math. J. 1 (1935), 39-49
- [38] T. Miyazaki, *Simply connected spin manifolds and positive scalar curvature*, Proc. Amer. Math. Soc. 93 (1985), 730-734
- [39] S. Ochanine, *Sur les genres multiplicatifs définis par des intégrales elliptiques*, Topology 26 (1987), 143-151
- [40] F. Podestà and L. Verdiani, *Totally geodesic orbits of isometries*, Ann. Global Anal. Geom. 16 (1998), 399-412; erratum ibid. 19 (2001), 207-208
- [41] F. Podestà and L. Verdiani, *Positively curved 7-dimensional manifolds*, Q. J. Math., Oxford 50 (1999), 497-504
- [42] T. Püttmann and C. Searle, *The Hopf conjecture for manifolds with low cohomogeneity or high symmetry rank*, Proc. Am. Math. Soc. 130 (2002), 163-166
- [43] X. Rong, *Positive curvature, local and global symmetry, and fundamental groups*, Am. J. Math. 121 (1999), 931-943
- [44] X. Rong, *Positively curved manifolds with almost maximal symmetry rank*, preprint 133, SFB Münster (2000)
- [45] R. Schoen and S.T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscr. Math. 28 (1979), 159-183
- [46] L. Schwachhöfer and W. Tuschmann, *Curvature and cohomogeneity one*, preprint (2001)
- [47] C. Searle, *Cohomogeneity and positive curvature in low dimensions*, Math. Z. 214 (1993), 491-498

- [48] C. Searle and D.G. Yang, *On the topology of nonnegatively curved simply connected 4-manifolds with continuous symmetry*, Duke Math. J. 74 (1994), 547-556
- [49] J.P. Sha and D.G. Yang, *Positive Ricci curvature on the connected sums of  $S^n \times S^m$* , J. Diff. Geom. 33 (1991), 127-137
- [50] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. 136 (1992), 511-540
- [51] S. Stolz, *A conjecture concerning positive Ricci curvature and the Witten genus*, Math. Ann. 304 (1996), 785-800
- [52] J.L. Synge, *On the connectivity of spaces of positive curvature*, Q. J. Math., Oxf. Ser. 7 (1936), 316-320
- [53] C.H. Taubes,  *$S^1$  Actions and Elliptic Genera*, Comm. Math. Phys. 122 (1989), 455-526
- [54] L. Verdiani, *Cohomogeneity one Riemannian manifolds of even dimension with strictly positive curvature, I*, preprint
- [55] N.R. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. of Math. 96 (1972), 277-295
- [56] B. Wilking, *Group actions on manifolds of positive sectional curvature*, talk at the conference "Differentialgeometrie im Großen", Oberwolfach, Germany (2001)
- [57] E. Witten, *The Index of the Dirac Operator in Loop Space*, in: [34], 161-181
- [58] D. Yang, *On the topology of nonnegatively curved simply connected 4-manifolds with discrete symmetry*, Duke Math. J. 74 (1994), 531-545
- [59] S.T. Yau, *On the Ricci curvature of a compact Kaehler manifold and the complex Monge-Ampère equation. I*, Commun. Pure Appl. Math. 31 (1978), 339-411

Anand Dessai  
 e-mail: [dessai@math.uni-augsburg.de](mailto:dessai@math.uni-augsburg.de)  
<http://www.math.uni-augsburg.de/geo/dessai/homepage.html>  
 Department of Mathematics, University of Augsburg, D-86159 Augsburg