Department of Mathematics
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Moduli Spaces of Riemannian Metrics of Positive Ricci and Non-Negative Sectional Curvature on 5, 7 and 15-dimensional Manifolds

THESIS

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Abstract

In this thesis, we study the topology of the moduli spaces of Riemannian metrics of non-negative sectional or positive Ricci curvature on certain 5, 7 and 15-dimensional manifolds.

Specifically, using the relative eta-invariant of the Atiyah-Patodi-Singer index theory, we show that the moduli space of non-negative sectional curvature metrics on orientable, closed, smooth non-spin 5-manifolds with universal cover $S^3 \times S^2$ and fundamental group $\mathbb{Z}_2$ has infinitely many path components.

Furthermore, we show that the moduli space of positive Ricci curvature metrics on the total space of linear S^7-bundles over S^8 which are rational cohomology 15-spheres has infinitely many path components. In addition, we carry out the diffeomorphism classification of certain homotopy $\mathbb{R}\mathrm{P}^7$ which arise as the quotient of a Milnor sphere by a specific involution to show that their moduli space of non-negative sectional curvature has infinitely many path components. Similarly, we show that there are only finitely many diffeomorphism types of certain homotopy $\mathbb{R}\mathrm{P}^{15}$ which arise as the quotient of a Shimada sphere by a specific involution to show that for those diffeomorphism types which can be described by an infinite family of pairwise diffeomorphic manifolds, the moduli space of positive Ricci curvature has infinitely many path components. These results are all obtained with the help of a relative index invariant due to Gromov and Lawson.

Zusammenfassung

In dieser Dissertation untersuchen wir die Topologie der Modulräume riemannscher Metriken mit nichtnegativer Schnittkrümmung oder positiver Riccikrümmung auf gewissen 5, 7 und 15-dimensionalen Mannigfaltigkeiten.

Insbesondere wird mit Hilfe der relativen Eta Invariante aus der Atiyah-Patodi-Singer Indextheorie bewiesen, dass der Modulraum von nichtnegativen Schnittkrümmungsmetriken auf 5-dimensionalen Mannigfaltigkeiten die orientierbar, geschlossen, glatt und nicht spin sind, deren universelle Überlagerung $S^3 \times S^2$ und Fundamentalgruppe $\mathbb{Z}_2$ ist, unendlich viele Zusammenhangskomponenten hat.

Ferner wird gezeigt, dass der Modulraum positiver Riccikrümmungsmetriken auf den Totalräumen linearer S^7-Bündel über S^8, die rationale Kohomologiesphären sind, unendlich viele Zusammenhangskomponenten hat. Zusätzlich wird die Diffeomorphieklassifikation von Mannigfaltigkeiten durchgeführt, die homotopieäquivalent zum $\mathbb{R}\mathrm{P}^7$ sind und als Quotienten von Milnorsphären unter einer gewissen Involution entstehen, um zu zeigen, dass der Modulraum nichtnegativer Schnittkrümmungsmetriken auf diesen Räumen unendlich viele Zusammenhangskomponenten hat. Ähnlich beweisen wir, dass es für Mannigfaltigkeiten, die homotopieäquivalent zum $\mathbb{R}\mathrm{P}^{15}$ sind und die als Quotienten von Shimadasphären unter einer gewissen Involution entstehen, nur endlich viele Diffeomorphietypen gibt und zeigen, dass für die Typen, die sich durch unendlich viele paarweise diffeomorphe Mannigfaltigkeiten beschreiben lassen, der Modulraum positiver Riccikrümmungsmetriken unendlich viele Zusammenhangskomponenten hat. Der Beweis dieser Resultate benutzt eine relative Indexinvariante von Gromov und Lawson.

Résumé

Dans cette thèse, nous étudions la topologie de l'espace de modules de métriques riemanniennes de courbure sectionelle non négative ou de courbure de Ricci positive sur certaines variétés de dimension 5, 7 et 15.

En particulier, en utilisant l'invariante éta relative de la théorie d'Atiyah-Patodi-Singer, nous montrons que l'espace de modules de métriques de courbure sectionnelle non négative sur les variétés orientables, closes, lisses, non spin, telles que leur revêtement universel est $S^3 \times S^2$ et leur groupe fondamentale est $\mathbb{Z}_2$, a une infinité de composantes connexes par arcs.

En outre, nous prouvons que l'espace de modules de métriques de courbure de Ricci positive sur l'espace fibré de fibre S^7, groupe de structure $SO(8)$ et base S^8, tel que le $8^{\text{ème}}$ groupe de cohomologie à coéfficients rationnels disparaît, a une infinité de composantes connexes par arcs. De plus, nous procédons à la classification des structures lisses sur les espaces homotopiquement équivalents à $\mathbb{R}\mathrm{P}^7$ et qui sont le quotient d'une sphère de Milnor par une certaine involution. Nous utilisons ce résultat pour montrer que l'espace de modules de métriques de courbure sectionnelle non négative sur ces espaces a une infinité de composantes connexes par arcs. De manière similaire, nous prouvons qu'il n'existe qu'un nombre fini de structures lisses non équivalentes sous difféomorphisme sur les espaces homotopiquement équivalents à $\mathbb{R}\mathrm{P}^{15}$ et qui sont le quotient d'une sphère de Shimada par une certaine involution. Ceci nous permet de montrer que l'espace de modules de métriques de courbure de Ricci positive sur ceux parmi ces espaces, qui peuvent être décrits par une infinité de variétés une à une difféomorphes, a une infinité de composantes connexes par arcs. Tous ces résultats sont obtenus à l'aide d'une invariante indicielle de Gromov et Lawson.

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Contents

Chapter 1

Introduction and main results

It is by now a firmly established fact that the interplay between topology and Riemannian geometry leads to many interesting results in both fields. To this day, questions pertaining to the curvature of Riemannian manifolds are a prolific domain of research. Certain cases are well understood, like for example the classification of closed[1] simply connected manifolds admitting a metric of positive scalar curvature (except in dimension 4), but much less is known about others. For instance, there are only very few known examples of manifolds with positive sectional curvature.

Once the existence of a metric with a certain curvature condition has been established on a manifold, the issue of characterizing different metrics satisfying said curvature condition arises. One natural way to think about this question is to ask whether all the possible metrics can be continuously deformed into one another, and if not, how many different classes of such metrics there are. The object which has asserted itself to quantify this distinction is the so-called moduli space of Riemannian metrics. It is defined as the quotient of the space of all Riemannian metrics by the group of diffeomorphisms of the manifold, which acts via pulling back metrics. Since isometric metrics give rise to essentially the same geometry on a manifold, this helps to focus the investigation.

In this thesis, we expand the list of examples where one is able to give a coarse quantification of the geometry of certain 5, 7 and 15-dimensional manifolds by studying the moduli space using topological methods.

We summarize some of the results related to moduli spaces leading up to this thesis. For a general introduction to the subject, see [TW15].

A major development in the study of moduli spaces has been the work of Kreck and Stolz [KS93]. The introduction of their s-invariant has given geometers an important tool to distinguish connected components, which

[1]Compact and without boundary.

they used to show that the moduli space of positive scalar curvature (psc) metrics on $(4k+3)$-dimensional closed spin manifolds with vanishing rational Pontrjagin classes has infinitely many path components. It has been used abundantly eversince, like the examination of the moduli space of positive Ricci curvature metrics on $(4k+3)$-dimensional spheres which bound parallelizable manifolds by Wraith [Wra11], non-negative sectional curvature metrics on the total space of S^1-bundles over $\mathbb{C}\mathrm{P}^{2n} \times \mathbb{C}\mathrm{P}^1$ by Dessai, Klaus and Tuschmann [DKT18] and on certain 7-dimensional manifolds by Goodman [Goo20b] to only name a few. However, the s-invariant has its limitations: the manifold must have vanishing real Pontrjagin classes, be spin and of dimension $4k+3$.

Another, slightly different approach has been used by Dessai [Des17] to prove that the moduli space of $sec \geq 0$ metrics on a Milnor sphere has infinitely many path components. It makes use of the so-called Gromov-Lawson invariant and works for $(4k+3)$-dimensional spin manifolds for which one can compute the Pontrjagin classes of a coboundary. In some sense this method is more elementary than the use of the s-invariant, since it does not require the eta-invariant from the Atiyah-Patodi-Singer index theory.

On the other hand, it has been noted early on that the *relative* eta-invariant emerging from the Atiyah-Patodi-Singer index theory could feature as an invariant of connected components of the space of psc metrics [APS75b]. In this case, there are the usual dimensional limitations of index theory, as well as necessitating a non-trivial fundamental group. Botvinnik and Gilkey [BG95] used this invariant to show that the moduli space of psc metrics on spin manifolds of odd dimension ≥ 5, whose fundamental group is non-trivial but finite and which admit a psc metric with $r_m(G) > 0$ (see [BG95, p.508] for the definition of $r_m(G)$), has infinitely many path components, and more recently, Dessai and González-Álvaro [DGA21] have investigated the moduli space of non-negative sectional curvature metrics on homotopy $\mathbb{R}\mathrm{P}^5$, and Goodman [Goo20a] has studied the moduli space of positive Ricci curvature metrics on certain 5-manifolds with fundamental group $\mathbb{Z}_2$ which are total spaces of principal S^1-bundles. In both cases, the conclusion is that the corresponding moduli spaces have infinitely many path components.

The first result of this thesis is a sort of completion of the above results by Dessai-González-Álvaro and Goodman. The family of manifolds which is used in [DGA21] are quotients of so-called Brieskorn manifolds, which arise as the intersection of the 7-sphere in $\mathbb{C}^4$ and the preimage of the origin under the polynomial $f_d(z) = z_0^d + z_1^2 + z_2^2 + z_3^2$. When d is odd, the Brieskorn manifolds are diffeomorphic to the 5-sphere and if we take the quotient by a fixed-point free involution, we get a homotopy $\mathbb{R}\mathrm{P}^5$. But when d is even, the Brieskorn manifold is diffeomorphic to $S^3 \times S^2$ and the quotients (which we will call *Brieskorn quotients*) are not pairwise homotopy equivalent. Su showed that there are five distinct diffeomorphism types in this family of manifolds [Su12]. With the same method that was employed by Dessai and González-Álvaro, one can study the moduli space of non-negative sectional curvature metrics of these spaces. On the other hand, some of the manifolds in [Goo20a] which are described as principal S^1-bundles over $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$, turn out to be quotients of $S^3 \times S^2$ by another class of involutions. One can also describe them as quotients of $S^3 \times S^3$ by an $S^1 \times \mathbb{Z}_2$-action which sits in a larger 2-torus action. By considering different actions, we will furthermore get manifolds which are the total space of principal S^1-bundles over $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$. Using a diffeomorphism classification result of Hambleton and Su [HS13], which has recently been corrected by Goodman (see [Goo20a]), we show that there are three distinct diffeomorphism types of the quotients which are S^1-bundles over $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$ and two distinct diffeomorphism types of the quotients which are S^1-bundles over $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$. With his constructions, Goodman focused on positive Ricci curvature metrics, but in the above cases the manifolds actually admit metrics of $sec \geq 0$. By combining these two categories of 5-manifolds, we get the following result.

Theorem A. *Let Q^5 be an orientable, closed, smooth non-spin 5-dimensional manifold with $\pi_1(Q) \cong \mathbb{Z}_2$, whose universal cover is $S^3 \times S^2$. Then the moduli space of non-negative sectional curvature metrics on Q has infinitely many path components. The same is true for the moduli space of positive Ricci curvature metrics on Q.*

Note that the proof of this result also implies that the corresponding moduli space of psc metrics has infinitely many path components.

In case $\pi_1(Q)$ acts trivially on higher homotopy groups (which corresponds to the principal S^1-bundles discussed above), this result emerged from joint work with Jackson McFeely Goodman.

Remark. *From Su's classification of free involutions on $S^3 \times S^2$ [Su12] it is apparent that the above result covers all the possible orientable non-spin quotients: there are 5 diffeomorphism types where the fundamental group acts trivially on the higher homotopy groups (the principal S^1-bundles) and also 5 oriented diffeomorphism types where it acts non-trivially (the Brieskorn quotients). When the quotient of $S^3 \times S^2$ by a free involution is orientable and spin (which includes for example $\mathbb{R}\mathrm{P}^3 \times S^2$), the eta-invariant vanishes for dimensional reasons and so we cannot use our methods to study their moduli space (see [DGA21, Remark 3.4] and [BG95]).*

The general idea of the proof of Theorem A is the following. One exhibits an infinite family of pairwise diffeomorphic 5-manifolds (which are the quotients of $S^3 \times S^2$ by an involution). These manifolds, their universal cover and a coboundary of the universal cover are all equipped with suitable metrics: the manifold is given a metric with the desired curvature condition and the metric on the coboundary needs to have non-negative scalar curvature everywhere, positive scalar curvature somewhere, and it needs to be of product form near its boundary (so that we can use an appropriate index theorem). Then one computes the relative eta-invariants of the manifolds using a formula which relates them to the equivariant eta-invariant of the universal cover. These equivariant eta-invariants are determined using an index theorem applied to the coboundary and by means of the topological consequences of positive scalar curvature. The relative eta-invariant of the manifold will then depend on the parameter(s) which indexes the infinite family of diffeomorphic manifolds. If there are infinitely many different values, this implies that there are infinitely many path components (the relative eta-invariant being constant on path components, see Proposition 2.3.23).

For the Brieskorn quotients, there is a slight complication in this sketch. The universal cover is defined as $M^5_\epsilon(d) := f_d^{-1}(\epsilon) \cap S^7$ and the coboundary by $W^6_\epsilon(d) := f_d^{-1}(\epsilon) \cap D^8$. To equip the universal cover (and subsequently the quotient) with a metric of $sec \geq 0$, we use the work of Grove and Ziller [GZ00], which requires a cohomogeneity one action with codimension 2 singular orbits. Such an action exists on $M^5_\epsilon(d)$ only if $\epsilon = 0$, but in this case $W^6_0(d)$ is not a manifold, only an algebraic variety, and so we cannot apply to it the aforementioned index theory to compute the relative eta-invariant. To cope with this, we construct a path in the space of psc metrics from the pullback of the Grove-Ziller metric to a metric on $M^5_\epsilon(d)$ for a certain $\epsilon \neq 0$, for which one can actually compute the relative eta-invariant (see Proposition 7.1.3).

Here is the outline of the steps for the Brieskorn quotients. In §4.1 the Brieskorn manifolds are defined and some of their topological properties are presented. The cohomogeneity one action and the involution which defines the Brieskorn quotients are defined in §4.1.1. Then, in §4.1.2 we present the diffeomorphism classification of the Brieskorn quotient, which is due to Su [Su12]. The metrics for these manifolds are constructed in §6.1.4, using mainly the work of Grove-Ziller as well as Cheeger deformations. In §7.1.1 the $Spin^c$ structures of the Brieskorn quotient, the Brieskorn manifold and its coboundary are defined, and the relative eta-invariant is computed. Theorem A is then proved for this class of manifolds, that is, the 5-manifolds which satisfy the conditions of Theorem A and whose fundamental group acts non-trivially on the second homotopy group.

For the other class of 5-manifolds (those whose fundamental group acts trivially on the second homotopy group), that is, the total spaces of principal S^1-bundles over $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ or $\mathbb{C}P^2 \# \mathbb{C}P^2$, the above proof idea applies straightforwardly. In §4.2, the classification of torus actions on $S^3 \times S^3$, which is due to DeVito [DeV11][DeV14], is first presented. Then the torus actions of interest will be defined (the so-called *non-exceptional* and *exceptional* torus actions), as well as the circle subactions which give rise to the universal covering of the 5-manifolds we are interested in. These universal coverings are diffeomorphic to total spaces of principal S^1-bundles, and by taking the total space of the two-fold tensor product of these bundles, we get the desired manifolds. In §4.2.3 we determine some properties of these spaces and bundles, which we will need for the diffeomorphism classification, the construction of the $Spin^c$ structures and eventually, the computation of the relative eta-invariants. First we determine the cohomology ring and some characteristic classes of the base spaces, then we compute the first Chern class of the S^1-bundles whose total space is the universal covering of our manifolds of interest and finally, we show that their second Stiefel-Whitney class is non-trivial (implying that they are not spin). Then, in §4.2.4 we present the diffeomorphism classification of the 5-manifolds with fundamental group $\mathbb{Z}_2$, which is due to Hambleton-Su [HS13] (but we will use Su's classification [Su12]). Suitable metrics are then constructed in §6.1.5, and in §7.1.2 the $Spin^c$ structures are defined and the relative eta-invariants computed. After that, the proof of Theorem A is completed.

The other results presented in this thesis all follow in the vein of Dessai [Des17]. In this paper, he showed that the moduli space of non-negative sectional curvature metrics on Milnor spheres, and more generally on the total space of linear S^3-bundles over S^4 such that it is a rational cohomology sphere, has infinitely many path components. Recall that a Milnor sphere is the total space of a linear S^3-bundle over S^4 which is a homotopy 7-sphere. The first new result we show is an application of the methods of this paper (which we will explain shortly) to the case of S^7-bundles over S^8.

Theorem B. *Let M^{15} be the total space of a linear S^7-bundle over S^8 and assume M^{15} is a rational homology sphere. The moduli space of positive Ricci curvature metrics on M has infinitely many path components.*

From the proof it also follows that the same is true for the moduli space of psc metrics on M.

The second result is an extension to the quotients of Milnor spheres by an involution which is induced by fiberwise antipodal maps. These quotients will be called *Milnor projective spaces* (see §5.1 for the precise definition). It is well known that there are 16 different oriented diffeomorphism types of Milnor spheres (see [EK62]). In this thesis, we show that there are also exactly 16 different oriented diffeomorphism types of Milnor projective spaces (see §5.3). Using this, we prove the following.

Theorem C. *The moduli space of metrics of non-negative sectional curvature of all Milnor projective spaces has infinitely many path components. The same is true for the moduli space of positive Ricci curvature metrics.*

Again, the proof implies that the moduli space of psc metrics has infinitely many path components as well.

The third theorem is the same idea applied to quotients of what we will call *Shimada spheres*. These are total spaces of the above S^7-bundles over S^8 which are homotopy 15-spheres. The quotients by an involution which is induced by fiberwise antipodal maps will be called *Shimada projective spaces* (see §5.1). We are not able to give a full diffeomorphism classification of these quotients, but we can prove that there are at least 4096 different oriented diffeomorphism types (see §5.3). We then finally prove the following.

Theorem D. *There exist finitely many and at least 4096 oriented diffeomorphism types of Shimada projective spaces whose moduli space of positive Ricci curvature metrics has infinitely many path components.*

Once more, the corresponding moduli spaces of psc metrics also has infinitely many path components.

The core idea of the proof of the three previous theorems is the following, which goes back to Gromov and Lawson [GL83] and has been used by Dessai [Des17]. For Theorem B, we start with an infinite family of pairwise diffeomorphic manifolds (the total spaces of S^7-bundles over S^8). We then construct metrics of positive Ricci curvature on these manifolds, which extend to a psc metric on the coboundary (which in this case is the disk bundle) and is of product form near the boundary. We then assume that there is a path between the equivalence classes of two such positive Ricci curvature metrics in the corresponding moduli space. This path gives a path in the space of psc metrics, which then allows to construct a spin manifold by taking a cylinder of the initial manifold and capping it off with two disk bundles with corresponding indices. The Gromov-Lawson invariant of the resulting manifold vanishes because of the positive curvature, which, together with Hirzebruch's signature theorem, puts some constraints on the values of certain Pontryagin numbers of the disk bundles. These numbers depend on the indices of the chosen metrics, leading to a contradiction: there can be no path connecting the two equivalence classes of metrics in the moduli space, and since they are indexed by an infinite set, the result follows.

In the proof of Theorem C, we start similarly with an infinite family of pairwise diffeomorphic manifolds. We equip these quotients with metrics of $sec \geq 0$ and $scal > 0$ which lift to the Grove-Ziller metrics on the covering Milnor spheres, which in turn extend to metrics of $\sec \geq 0$ on the disk bundles that are of product form near the boundary. We then assume as before that there is a path between the equivalence classes of two such metrics in the moduli space of non-negative sectional curvature metrics on the Milnor projective space. This path lifts to a path in the space of Riemannian metrics of $sec \geq 0$ on the quotient, which ultimately lifts to a path in the space of psc metrics on the covering Milnor sphere. We are now in a similar situation as in the proof of Theorem B, and an analoguous argument applies until we reach a contradiction.

Finally, the proof of Theorem D is analogous to the proof of Theorem C, the only major difference being that we don't have a complete diffeomorphism classification of the Shimada projective spaces, but only a finiteness result (see Proposition 5.3.14).

The main steps for the sphere bundles and their quotients are the following. In §5.1 we define the sphere bundles, discuss some of their topological properties and present the involution which allows to define the Milnor and Shimada projective spaces. In §5.2, we summarize the diffeomorphism classification of the total spaces of our sphere bundles, which is due to Crowley-Escher [CE03] for S^3-bundles over S^4 and Grey [Gre12] for S^7-bundles over S^8. The diffeomorphism classification of the Milnor and Shimada projective spaces is then carried out in §5.3. We first show that their Browder-Livesay invariant vanishes, then determine the normal invariants of the Milnor projective spaces and then compute the Eells-Kuiper invariant of both Milnor and Shimada projective spaces. The classification of Milnor projective spaces then follows, as well as the finiteness result for the Shimada projective spaces. The Grove-Ziller metrics of $sec \geq 0$ on the Milnor spheres and the corresponding quotients are constructed in §6.1.2, as well as the suitable metrics on the disk bundles. We construct metrics of positive Ricci curvature on the S^7-bundles over S^8 and the psc metrics on the corresponding disk bundles in §6.2.2. The positive Ricci curvature metrics on the Shimada projective spaces are defined in §6.2.3. Finally, the proof of the above theorems are given in §7.2, §7.3 and §7.4.

Remark. *Since the space of positive scalar curvature metrics and the space of positive Ricci curvature metrics are open in the space of all metrics, one can interchangeably speak about connected components and path components (and likewise for the corresponding moduli spaces). In general, this is not the case for the (moduli) space of metrics of non-negative sectional curvature. However, it has recently been showed by Belegradek and González-Álvaro that in the case of non-negative sectional curvature metrics which simultaneously have positive scalar curvature, this is true as well (see [BGA20, Theorem 1.2]).*

To end this introduction, we quickly discuss the overall structure of this thesis.

In Chapter 2, we give an overview of the definitions and results in spin geometry that constitute the tools with which we are going to analyze the moduli spaces. We start by giving a reminder on $Pin^{\pm}$, *Spin* and $Spin^c$ structures. We then present the 'most' general index theorem, i.e. Donnelly's equivariant index theorem, from which we deduce all the other ones. We then proceed to define the Dirac operators and present the (equivariant/twisted) index theorems in the three following cases: *Spin*, $Spin^c$ and signature. The eta-invariant of a covering is then discussed. The chapter ends with consequences of psc in the context of index theory: vanishing theorems are stated, the Gromov-Lawson invariant and the relative eta-invariant for $Spin^c$ structures are defined and their property as invariants of path components of the space of psc metrics is discussed.

Chapter 3 gives some of the tools of differential topology we are going to use in the diffeomorphism classification of Milnor and Shimada projective spaces. We start with the definition and discussion of the Eells-Kuiper invariant. Then we introduce some concepts from surgery theory, most notably the normal invariant, and after that, we give a reminder on the diffeomorphism classification of homotopy spheres. Finally, the diffeomorphism classification of homotopy projective spaces is discussed, which amounts to studying smooth involutions on homotopy spheres. This section culminates in the classification result of López de Medrano, which states that two homotopy projective spaces are diffeomorphic, up to connected sum with some homotopy sphere, if and only if their Browder-Livesay and normal invariants agree (see Theorem 3.4.5).

As we have discussed above, in Chapter 4 we give the definition of the 5-manifolds we are going to analyze, discuss some of their properties and present their diffeomorphism classification.

In Chapter 5, we define the sphere bundles and their quotients, discuss their topological properties and undertake their diffeomorphism classification.

Chapter 6 is dedicated to the construction of the different metrics and the definition of the moduli space.

Finally, the proofs of the main theorems are given in Chapter 7.

Remark. *Unless otherwise stated, all manifolds will assumed to be smooth. Furthermore, all manifolds which are orientable will be assumed oriented and every map from one oriented manifold to another will be assumed orientation preserving (except for involutions which can also be orientation reversing depending on the situation).*

Chapter 2

Spin geometry

Linking the index of a so-called Dirac operator on a manifold, which is an analytic object, to a certain topological index of the manifold, the celebrated Atiyah-Singer index theorem and its various extensions are some of the most powerful tools we have at our disposal. But index theory requires a large amount of definitions, concepts and results in order to be handled successfully. In this chapter, we only give the most essential definitions and then immediately proceed to listing the various index theorems we will need. The work of Lichnerowicz [Lic63], as well as Gromov and Lawson has unveiled the deep connection between this theory and positive scalar curvature metrics (see [LM89, Chapter IV] for an overview and references), and it is also in this theory that we find all of the known invariants related to the connected or path components of the (moduli) spaces of metrics, most notably the eta-invariant, but also the so-called Gromov-Lawson invariant.

We start with the definition of the $Pin^{\pm}$, $Spin$ and $Spin^c$ groups. The $Pin^{\pm}$ groups play an essential role in the diffeomorphism classification of the 5-manifolds we will be dealing with. $Spin$ and $Spin^c$ structures are then introduced. These exist on many manifolds and allow for a very natural construction of a Dirac operator. But before introducing these objects, we state the most general index theorem from which all the other ones we will need can be deduced as special applications to certain Dirac operators: Donnelly's equivariant index theorem. This is an index theorem for manifolds with boundary on which a group acts via isometries. We then apply this theorem to the aforementioned $Spin$ Dirac operator, the $Spin^c$ Dirac operator and the signature operator. The last is constructed on $4k$-dimensional manifolds, for which the signature is defined (and not always trivial). In the last part of this chapter, we discuss some of the consequences of positive scalar curvature in index theory and introduce the invariants we will use to study the moduli space of metrics.

2.1 Definition of the groups

We first start with some elementary definitions of index theory. For more details on the following definitions and results, see [LM89, Chapter I], [ABS64] and [KT90].

Let $\mathbb{R}^n$ be equipped with the quadratic form $q(x) = x_1^2 + ... + x_n^2$. The *Clifford algebra* of $(\mathbb{R}^n, q)$, denoted by $Cl^{\pm}(n)$, is the universal algebra generated by $\mathbb{R}^n$ subject to the relations

$$v \cdot w + w \cdot v = \begin{cases} -2q(v,w) & \text{for } Cl^+(n), \\ +2q(v,w) & \text{for } Cl^-(n). \end{cases}$$

Let $P^{\pm} := \{\phi \in Cl^{\pm}(n) | \exists \phi^{-1} \text{ s.t. } \phi \cdot \phi^{-1} = \phi^{-1} \cdot \phi = 1\}$ be the multiplicative group of units and define

$$Pin^{\pm}(n) := \{v_1 \cdot ... \cdot v_k \in P^{\pm} |\, |q(v_j)| = 1 \,\forall j\}$$

and

$$Spin(n) := \{v_1 \cdot ... \cdot v_k \in Pin^+(n) |\, k \text{ is even}\}.$$

These groups fit into the following short exact sequences:

$$0 \to \mathbb{Z}_2 \to Pin^{\pm}(n) \xrightarrow{\rho^{\pm}} O(n) \to 1,$$

$$0 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\rho} SO(n) \to 1.$$

Indeed, $Spin(n)$ is the non-trivial double-cover of $SO(n)$ and $Pin^{\pm}(n)$ correspond to topologically identical but, as groups, non-isomorphic double-covers of $O(n)$.
Furthermore, we define

$$Spin^c(n) := Spin(n) \times_{\mathbb{Z}_2} U(1)$$

where the division is by the element $(-1,-1) \in Spin(n) \times U(1)$ and there is a short exact sequence

$$0 \to \mathbb{Z}_2 \to Spin^c(n) \xrightarrow{\rho^c} SO(n) \times U(1) \to 1,$$

where $\rho^c([A,z]) = (\rho(A), z^2)$ for $A \in Spin(n)$ and $z \in U(1)$.

2.2 Structures on manifolds

2.2.1 $Pin^{\pm}$ structures

$Pin^{\pm}$ structures are needed for the diffeomorphism classification of the 5-manifolds we will be dealing with. See [ABS64] and [KT90] for more details on the following definitions, and [HS13] and [Su12] for their use in the aformentioned diffeomorphism classification.

Let M^n be a smooth manifold (possibly with non-empty boundary) of dimension n. Equip M with a Riemannian metric. Then the structure group of its tangent bundle TM reduces to $O(n)$. Let P_O be the bundle of orthonormal frames on M .

A $Pin^{\pm}$ *structure* on M is a principal $Pin^{\pm}(n)$-bundle $P_{Pin^{\pm}}$ over M together with a two-sheeted covering $\alpha : P_{Pin^{\pm}} \to P_O$ such that $\alpha(p \cdot g) = \alpha(p) \cdot \rho^{\pm}(g)$ for all $p \in P_{Pin^{\pm}}$ and $g \in Pin^{\pm}(n)$.

The manifold M admits a Pin^+ structure if and only if $w_2(M) = 0$. It admits a Pin^- structure if and only if $w_2(M) = w_1^2(M)$. The $Pin^{\pm}$ structures on M (if any exist) are in 1-to-1 correspondence with elements of $H^1(M; \mathbb{Z}_2)$ [HKT94, Lemma 1].

Let M_0^n and M_1^n be two Riemannian manifolds equipped with a $Pin^{\pm}$ structure and W^{n+1} a manifold with boundary $\partial W = M_0 \cup M_1$. If W can be given a $Pin^{\pm}$ structure which restricts to the ones on M_0 and M_1 respectively, then M_0 and M_1 are said to be $Pin^{\pm}$-cobordant. The $Pin^{\pm}$-cobordism group is denoted by $\Omega_n^{Pin^{\pm}}$.

We will need $\Omega_4^{Pin^+}$ for the diffeomorphism classification of 5-manifolds with fundamental group $\mathbb{Z}_2$ and universal covering $S^3 \times S^2$. In [KT90], Kirby and Taylor determine $\Omega_4^{Pin^+} \cong \mathbb{Z}_{16}$. By identifying an element with its additive inverse, we obtain a one-to-one correspondence between $\Omega_4^{Pin^+}/\pm$ and $\{0, 1, ..., 8\}$.

2.2.2 *Spin* structures

Spin structures allow us to construct the invariants we will need to give the diffeomorphism classification and study the moduli spaces of the sphere bundles over spheres and their quotients. See [LM89, Chapter II] for more details on *Spin* structures.

Assume that M^n is an oriented Riemannian manifold (possibly with non-empty boundary). Then $w_1(M) = 0$ and the structure group of its tangent bundle reduces to $SO(n)$. Let P_{SO} be the principal $SO(n)$-bundle of oriented orthonormal frames on M .

A *Spin structure* on M is a principal $Spin(n)$-bundle P_{Spin} over M together with a two-sheeted covering $\beta : P_{Spin} \to P_{SO}$ such that $\beta(p \cdot g) = \beta(p) \cdot \rho(g)$ for all $p \in P_{Spin}$ and $g \in Spin(n)$. We will say that a manifold is *spin* if it is oriented, Riemannian and it is equipped with a *Spin* structure.

Remark 2.2.1. *A Spin structure on M uniquely determines a Spin structure for any other metric on M, so that the Spin structure does not depend on the metric (see [LM89, p.86]).*

An oriented Riemannian manifold M admits a *Spin* structure if and only if $w_2(M) = 0$. The *Spin* structures on M (if any exist) are in 1-to-1 correspondence with elements of $H^1(M;\mathbb{Z}_2)$ [LM89, II. Theorem 2.1].

If W is a spin manifold with non-empty boundary $M = \partial W$, then the *Spin* structure on W reduces to a spin structure on M. Unless otherwise stated, we will always assume that the boundary of a spin manifold is equipped with this induced *Spin* structure.

Now let M be a spin manifold and suppose a compact Lie group G acts on M by orientation preserving isometries. Then the differential of any $g \in G$ induces an action on P_{SO} (see Appendix A). We say that the action of G on M *preserves the Spin structure* if it lifts to an action of G on the bundle P_{Spin} such that β is equivariant with respect to this action and the lifted action on P_{SO}. An individual isometry g on M is said to preserve the *Spin* structure if the closed group generated by g preserves the *Spin* structure.

For the definition and some properties regarding *equivariant Spin structures*, see Appendix A.

2.2.3 $Spin^c$ structures

$Spin^c$ structures allow us to construct the invariants we will need to study the moduli spaces on our 5-manifolds. See [LM89, Appendix D] for more details on these structures.

A *$Spin^c$ structure* on an oriented Riemannian manifold M (possibly with non-empty boundary) is a principal $Spin^c(n)$-bundle P_{Spin^c} and a principal $U(1)$-bundle $P_{U(1)}$, both over M, together with a bundle map $\gamma : P_{Spin^c} \to P_{SO} \times P_{U(1)}$ such that $\gamma(p \cdot g) = \gamma(p) \cdot \rho^c(g)$ for all $p \in P_{Spin^c}$ and $g \in Spin^c(n)$. We will say that a manifold M is a *spin^c* manifold if it is oriented, Riemannian and equipped with a $Spin^c$ structure. The first Chern class $c_1(P_{U(1)}) \in H^2(M;\mathbb{Z})$ (which corresponds to the first Chern class of the associated complex line bundle) is called the *canonical class* of the $Spin^c$ structure.

Remark 2.2.2. *For a fixed $P_{U(1)}$, a $Spin^c$ structure on M uniquely determines a $Spin^c$ structure for any other metric (see for example [DGA21, §3.2]).*

An oriented Riemannian manifold M admits a $Spin^c$ structure if and only if $w_2(M)$ is the mod 2 reduction of an element in $H^2(M;\mathbb{Z})$. In particular, given a principal $U(1)$-bundle $P_{U(1)}$ over M, there is a $Spin^c$ structure on M if and only if $w_2(M) \equiv c_1(P_{U(1)})$ mod 2. For $P_{U(1)}$ fixed, the different $Spin^c$ structures are in (non-canonical) 1-to-1 correspondence with $H^1(M;\mathbb{Z}_2)$. All in all, if any exist, the $Spin^c$ structures on M are in (non-canonical) 1-to-1 correspondence with $2H^2(M;\mathbb{Z}) \oplus H^1(M;\mathbb{Z}_2)$ [LM89, p.392].

We mention two important examples of manifolds on which a canonical $Spin^c$ structure is defined.

Example 2.2.3. *Any spin manifold carries a canonically determined $Spin^c$ structure. If P_{Spin} denotes the Spin structure, then*

$$P_{Spin^c} = P_{Spin} \times_{\mathbb{Z}_2} U_1$$

gives a $Spin^c$ structure, where $\mathbb{Z}_2$ acts diagonally via $(-1,-1)$ and U_1 denotes the trivial principal $U(1)$-bundle.

Example 2.2.4. *Any complex manifold carries a canonically determined $Spin^c$ structure. If we equip the tangent bundle, which has the structure of a complex vector bundle, with a hermitian metric, we can consider the principal $U(n)$-bundle of unitary frames $P_{U(n)}$. Then*

$$P_{Spin^c} = P_{U(n)} \times_j Spin^c(2n)$$

gives a $Spin^c$ structure, where $j : U(n) \to Spin^c(2n)$ denotes a canonical homomorphism (see [LM89, p. D.6]). The associated principal $U(1)$-bundle is then given by $P_{U(1)} = P_{U(n)} \times_{det} U(1)$.

If W is a *spin*^{c} manifold with non-empty boundary $M = \partial W$, then the *Spin*^{c} structure on W restricts to a *Spin*^{c} structure on M. Unless otherwise stated, we will always assume that the boundary of a *spin*^{c} manifold is equipped with this induced *Spin*^{c} structure.

Now let M be a *spin*^{c} manifold and suppose a compact Lie group G acts on M by orientation preserving isometries. We say that the action of G on M *preserves the Spin*^{c} *structure* if it lifts to an action of G on the bundle P_{Spin^c} such that γ is equivariant with respect to this action and the lifted action[1] on $P_{SO} \times P_{U(1)}$. An individual isometry g on M is said to preserve the *Spin*^{c} structure if the closed group generated by g preserves the *Spin*^{c} structure.

For the definition and some properties regarding *equivariant Spin*^{c} *structures*, see Appendix A.

2.3 (Equivariant) Index Theory

As we have mentioned at the beginning of this chapter, we begin with the statement of Donnelly's equivariant index theorem. All of the other index theorems follow as special cases of this result. It allows us to introduce all the concepts and objects we will need to give the more specific index theorems and the invariants we use to study the moduli spaces.

We say that a Riemannian manifold W with boundary M is *of product form near the boundary* if there is a neighborhood of the boundary M which is isometric to a product $I \times M$ for some interval I.

[1]Observe that in general, the action of G might not lift to $P_{U(1)}$, but a lift to P_{Spin^c} induces a G-action on $P_{U(1)}$.

Recall that if E is a vector bundle, $\Gamma(E)$ denotes the vector space of sections on E.

Condition 2.3.1 (APS boundary condition)**.** *Let W be a compact Riemannian manifold which is of product form near its boundary $M = \partial W$. Let $\mathcal{D}_W : \Gamma(E) \to \Gamma(F)$ be a linear first order elliptic differential operator for some vector bundles E and F over W, endowed with a smooth inner product. Denote by $\mathcal{D}_W^*$ its adjoint operator. Suppose that near the boundary, $\mathcal{D}_W$ takes the form*

$$\mathcal{D}_W = \sigma\Big(\frac{\partial}{\partial u} + \mathcal{D}_M\Big)$$

where u is the inward normal coordinate, $\sigma : E \to F$ is a bundle isomorphism and $\mathcal{D}_M : \Gamma(E') \to \Gamma(E')$ is a first order self-adjoint elliptic differential operator for E' the restriction of E to M. Let P be the spectral projection operator onto eigenvectors of non-negative eigenvalue of $\mathcal{D}_M$. If $Ps(-,0) = 0$, then the section s of E is said to satisfy the APS boundary condition *and we denote by $\Gamma(E, P)$ the space of such sections.*

Theorem 2.3.2. *[Don78, Theorem 1.2]*

Let W be a compact Riemannian manifold which is of product form near its boundary $M = \partial W$. Let $\mathcal{D}_W : \Gamma(E) \to \Gamma(F)$, $\mathcal{D}_W^*$, $\mathcal{D}_M$ and $\Gamma(E, P)$ be defined as in Condition 2.3.1 and satisfy the APS boundary condition.

Let G be a subgroup of the isometry group of W and assume that the action of G is a product near the boundary. Suppose furthermore that the action lifts to E and F and that the induced map on sections[2] commutes with $\mathcal{D}_W$. Then, for each $g \in G$, the equivariant index of $\mathcal{D}_W : \Gamma(E, P) \to \Gamma(F)$, defined by

$$\mathrm{index}(\mathcal{D}_W, g) := tr(g|_{ker\mathcal{D}_W}) - tr(g|_{ker\mathcal{D}_W^*}),$$

is given by

$$\mathrm{index}(\mathcal{D}_W, g) = \sum_{N \subset W^g} a(N) - \frac{\eta_g(\mathcal{D}_M) + h_g(\mathcal{D}_M)}{2} \tag{2.1}$$

where $N \subset W^g$ are the fixed-point components, $a(N)$ is a so-called local contribution[3]*, $h_g(\mathcal{D}_M) := tr(g|_{ker\mathcal{D}_M})$ and*

$$\eta_g(z) := \sum_{\lambda \neq 0} \frac{\mathrm{sign}(\lambda) tr(g_\lambda^\#)}{|\lambda|^z}, \tag{2.2}$$

[2] If $s \in \Gamma(E)$ is a section, the action of $g \in G$ on s is defined by $g \cdot s = \tilde{g} \circ s \circ g^{-1}$, where $\tilde{g}$ is the action induced by g on E.

[3] We do not give the precise definition of the local contribution in the most general case, since it is rather cumbersome and won't be needed. It can be found in [Don78]. For its formula in the case of the *Spin*, *Spin*^{c} and signature Dirac operators, we refer to the subsequent sections.

where $g_\lambda^\#$ is the map induced by g on the eigenspaces E'_λ of $\mathcal{D}_M$, defines the equivariant eta-invariant $\eta_g(\mathcal{D}_M) := \eta_g(0)$.

Remark 2.3.3. *Note that the series in Equation (2.2) converges for $Re(z)$ sufficiently large, such that there is a meromorphic continuation of $\eta_g(z)$ to the entire complex z-plane and $\eta_g(0)$ is finite (see [APS75a, p.56] and [APS75b, p.413]).*

Remark 2.3.4. *For $g = e = Id_W$,*

$$\text{index}(\mathcal{D}_W) := \text{index}(\mathcal{D}_W, e) = \dim(ker\mathcal{D}_W) - \dim(ker\mathcal{D}_W^*),$$

$$h(\mathcal{D}_M) := h_e(\mathcal{D}_M) = \dim(ker\mathcal{D}_M),$$

and

$$\eta(\mathcal{D}_M) := \eta_e(\mathcal{D}_M) = \eta(0)$$

is the eta-invariant *associated to $\mathcal{D}_M$, where*

$$\eta(z) = \sum_{\lambda \neq 0} \frac{\text{sign}(\lambda)}{|\lambda|^z} \tag{2.3}$$

is summed over the non-zero eigenvalues of $\mathcal{D}_M$. In particular, in this situation Theorem 2.3.2 reduces to the Atiyah-Patodi-Singer index theorem [APS75a, Theorem 3.10].

Remark 2.3.5. *Let W be a compact, oriented Riemannian manifold (possibly with boundary) and $g : W \to W$ an orientation preserving isometry. Consider a fixed point component $N \subset W^g$ and denote by ν its normal bundle in W. Then the differential of g induces a bundle isometry $dg : \nu \to \nu$ and from representation theory, it follows immediately that there is a direct sum decomposition*

$$\nu = \nu(\pi) \oplus \bigoplus_{0<\theta<\pi} \nu(\theta)$$

where $\nu(\pi)$ is real and $\nu(\theta)$ is complex for $0 < \theta < \pi$, dg acts on $\nu(\pi)$ via multiplication by -1 and on $\nu(\theta)$ via multiplication by $e^{i\theta}$ (see for example [LM89, pp.262-265]).

2.3.1 *Spin* Dirac operator

The *Spin* case is the first we discuss. The *Spin* Atiyah-Singer index theorem 2.3.6 for closed manifolds will be used in the proof of Theorems B, C and D in combination with the Gromov-Lawson invariant. The *Spin* Atiyah-Patodi-Singer index theorem 2.3.7 and the *Spin* equivariant index theorem will be used in the computation of the Eells-Kuiper invariant of the Milnor and Shimada projective spaces (see Theorem 5.3.7).

See [LM89, Chapter II] and [Nic07, Chapter 11] for more details on the *Spin* Dirac operator.

Let M^n be an n-dimensional compact spin manifold (possibly with non-empty boundary). Let $\mu : Spin(n) \to \mathrm{Aut}(\Delta_n)$ be the real spinor representation[4] and $S := P_{Spin} \times_\mu \Delta_n$ the associated spinor bundle. Equip S with the Riemannian connection ∇ induced by the canonical Riemannian connection on P_{SO}. The *Spin Dirac operator* of S at $x \in M$ is the first-order differential operator $D_M : \Gamma(S) \to \Gamma(S)$ defined by $D_M(\sigma) := \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \sigma$ where $\{e_1, ..., e_n\}$ is an orthonormal basis of $T_x M$ and "$\cdot$" denotes Clifford multiplication. It is well-known that this operator is elliptic and formally self-adjoint (see [LM89, II§5]). In particular, if M is a closed manifold, D_M being elliptic implies that $\dim(ker D_M)$ is finite.

If $n = 4k$, the spinor representation splits and there is a corresponding decomposition $S = S^+ \oplus S^-$. The *Spin* Dirac operator D_M preserves this $\mathbb{Z}_2$-grading and exchanges the factors. We may restrict the *Spin* Dirac operator to obtain operators $D_M^+ : \Gamma(S^+) \to \Gamma(S^-)$ and $D_M^- : \Gamma(S^-) \to \Gamma(S^+)$ which satisfy $(D_M^+)^* = D_M^-$. The operator D_M^+ will be called the *Spin*$^+$ *Dirac operator* of M.

If W^{4k} is a compact spin manifold with boundary $M = \partial W$, then the restriction of D_W^+ to M can be identified with the *Spin* Dirac operator D_M on M. From now on, this identification will always be understood.

Theorem 2.3.6 (*Spin* Atiyah-Singer index theorem)**.** *[AS68b, Theorem (5.3)] Let M^{4k} be a closed spin manifold and D_M^+ its Spin$^+$ Dirac operator. Then*

$$\mathrm{index}(D_M^+) = \langle \hat{A}(M), [M] \rangle$$

where $\mathrm{index}(D_M^+)$ *is defined in Remark 2.3.4, $\langle , \rangle$ denotes the Kronecker pairing and $[M] \in H_{4k}(M; \mathbb{Z})$ is the fundamental class of M.*

Here and in the following, $\hat{A}$ denotes the genus associated to the characteristic power series $(\sqrt{z}/2)/\sinh(\sqrt{z}/2)$ with corresponding multiplicative sequence $\{\hat{A}_k\}$ (see [MS74, §19.] and [LM89, III §11]). It is a power series in the Pontrjagin classes (or forms) of M^{4k}. In particular, for $k = 1, 2$ and 4 we have

[4] The representation theory of Clifford algebras and the representations they induce on the spin group are essential to index theory. But in order not to extend this thesis excessively, we do not discuss it here and refer the reader to [LM89, Chapter I] and [Nic07, Chapter 11] for thorough treatments.

$$\hat{A}_1(p_1) = -\frac{1}{24}p_1, \tag{2.4}$$

$$\hat{A}_2(p_1,p_2) = \frac{1}{5760}\Big(-4p_2 + 7p_1^2\Big), \tag{2.5}$$

$$\hat{A}_4(p_1,p_2,p_3,p_4) = \frac{1}{464486400}\Big(-192p_4 + 512p_3p_1 + 208p_2^2 - 904p_2p_1^2 + 381p_1^4\Big). \tag{2.6}$$

If x_i denote the formal roots of TM for $i = 1, ..., 2k$ (see [HBJ92, p.9] and [LM89, III §11]), then the Pontrjagin classes are given by the elementary symmetric functions[5] in the square of the formal roots, i.e. $p_i(M) = \sigma_i(x_1^2, ..., x_{2k}^2)$, and the $\hat{A}$-genus is given by

$$\hat{A}(M) = \prod_{i=1}^{2k} \frac{x_i/2}{\sinh(x_i/2)}. \tag{2.7}$$

When we consider spin manifolds with (possibly) non-empty boundaries, we obtain the following index theorem.

Theorem 2.3.7 (*Spin* Atiyah-Patodi-Singer index theorem)**.** *[APS75a, Theorem (4.2)] Let W^{4k} be a Riemannian spin manifold which is of product form near the boundary $M^{4k-1} = \partial W$. Assume that the Spin$^+$ Dirac operator D_W^+ and the restriction D_M to the boundary M satisfy the APS boundary condition 2.3.1. Then the index of $D_W^+ : \Gamma(S^+, P) \to \Gamma(S^-)$ is given by*

$$\mathrm{index}(D_W^+) = \int_W \hat{A}(W) - \frac{h(D_M) + \eta(D_M)}{2}$$

where $\mathrm{index}(D_W^+)$*,* $h(D_M)$ *and* $\eta(D_M)$ *are defined in Remark 2.3.4, and* $\hat{A}$ *is the* $\hat{A}$*-genus in the Pontrjagin forms of the Riemannian metric on* W*.*

If E is a real oriented rank $2k$-vector bundle with formal splitting $E = E_1 \oplus ... \oplus E_k$ into oriented 2-plane bundles and $y_j = e(E_j)$, we define

$$\hat{A}_\pi(E) := \frac{1}{(2i)^k} \prod_{j=1}^{k} \frac{1}{\cosh(y_j/2)}. \tag{2.8}$$

For a complex vector bundle F with formal splitting $F = l_1 \oplus ... \oplus l_k$ into complex line bundles and $x_j = c_1(l_j)$, let

$$\hat{A}_\theta(F) := \frac{1}{2^k} \prod_{j=1}^{k} \frac{1}{\sinh(\frac{1}{2}(x_j + i\theta))} \tag{2.9}$$

[5] Recall for example that $\sigma_1(z_1, ..., z_n) = \sum_{i=1}^n z_i$ and $\sigma_2(z_1, ..., z_n) = \sum_{i<j} z_i z_j$.

where $0 < \theta < \pi$.

Applying Theorem 2.3.2 to the *Spin*$^+$ Dirac operator, we obtain the following.

Theorem 2.3.8 (*Spin* equivariant index theorem)**.** *Let W^{4k} be a compact Riemannian spin manifold which is of product form near the boundary $M^{4k-1} = \partial W$. Suppose that the Spin$^+$ Dirac operator D_W^+ and the restriction D_M to the boundary M satisfy the APS boundary condition 2.3.1. Let $g : W \to W$ be an isometry preserving the Spin structure. Then the equivariant index of $D_W^+ : \Gamma(S^+, P) \to \Gamma(S^-)$ is given by*

$$\operatorname{index}(D_W^+, g) = \sum_{N \subset W^g} a_{spin}(N) - \frac{\eta_g(D_M) + h_g(D_M)}{2},$$

where $\operatorname{index}(D_W^+, g)$*, $\eta_g(D_M)$ and $h_g(D_M)$ are defined in Theorem 2.3.2, N denotes a fixed point component of the action of g on W and $a_{spin}(N)$ is the corresponding local contribution.*

If $N \subset W^g$ denotes a fixed point component without boundary, whose normal bundle splits as $\nu = \nu(\pi) \oplus \bigoplus_{0<\theta<\pi} \nu(\theta)$ in W (see Remark 2.3.5), then [LM89, III. Theorem 14.11]

$$a_{spin}(N) = (-1)^s \int_N \prod_{0<\theta\leq\pi} \hat{A}_\theta(\nu(\theta)) \cdot \hat{A}(N), \tag{2.10}$$

where $s \in \{0, 1\}$ depends on the action of g on the Spin structure (see [LM89, III. Remark 14.12]).

Remark 2.3.9. *In the situation of the above theorem, since g preserves the Spin structure, the action of g lifts to an action on the spinor bundles, and the Spin and Spin$^+$ Dirac operators commute with the induced action of g on the sections (see Appendix A). Hence Theorem 2.3.2 applies.*

2.3.2 *Spin*^{c} Dirac operator

Next we discuss the *Spin*^{c} case. Both the twisted *Spin*^{c} Atiyah-Patodi-Singer index theorem 2.3.10 and the *Spin*^{c} equivariant index theorem 2.3.11 are used to compute the eta-invariant which will allow us to distinguish the path components of the moduli space of metrics on our 5-manifolds. More specifically, the corresponding relative eta-invariants will be shown to be proportional to the local contributions we compute for the fixed point components of the involution which defines the 5-manifolds with fundamental group $\mathbb{Z}_2$.

See [LM89, Appendix D] and [Nic07, §11.2.3] for more details on the following constructions.

Let M^n be an n-dimensional compact *spin*^{c} manifold (possibly with non-empty boundary). Let $\nu : Spin^c(n) \to \operatorname{Aut}(\Delta_n^c)$ be a complex spinor representation. Then $S_c := P_{Spin^c} \times_\nu \Delta_n^c$ is a complex spinor bundle. Fix a unitary connection on $P_{U(1)}$. Together with the canonical Riemannian connection on P_{SO}, one can lift the product connection on $P_{SO} \times P_{U(1)}$ to P_{Spin^c} and this lifted connection induces a connection on S_c. We then obtain a corresponding *$Spin^c$-Dirac operator* $D_M^c : \Gamma(S^c) \to \Gamma(S^c)$ which is elliptic and formally self-adjoined (see [LM89, Appendix D] and [LM89, II§5]). If $n = 2l$, the spinor bundle decomposes $S_c = S_c^+ \oplus S_c^-$ and there is a restricted operator $D_M^{c,+} : \Gamma(S_c^+) \to \Gamma(S_c^-)$ with formal adjoint $D_M^{c,-} : \Gamma(S_c^-) \to \Gamma(S_c^+)$. We will call $D_M^{c,+}$ the *$Spin^{c,+}$ Dirac operator* of M. If E is a hermitian complex vector bundle over M equipped with a hermitian connection, then there also exist *twisted $Spin^c$* and $Spin^{c,+}$ Dirac operators $D_{M,E}^c : \Gamma(S^c \otimes E) \to \Gamma(S^c \otimes E)$ and $D_{M,E}^{c,+} : \Gamma(S_c^+ \otimes E) \to \Gamma(S_c^- \otimes E)$ (see [LM89, II. Proposition 5.10]).

If W has non-empty boundary $M = \partial W$, then the restriction of $D_W^{c,+}$ to the boundary can be identified with the $Spin^c$ Dirac operator D_M^c of M. This identification will always be understood from now on.

We can now also apply the Atiyah-Patodi-Singer index theorem [APS75a, Theorem 3.10] to the twisted $Spin^{c,+}$ Dirac operator to get an index formula (see also [APS75a, p.62]).

Theorem 2.3.10 (Twisted $Spin^c$ Atiyah-Patodi-Singer index theorem)**.** *Let W^{2l} be a compact spin^c manifold which is of product form near the boundary $M^{2l-1} = \partial W$. Suppose the unitary connection on $P_{U(1)}$ is constant in the normal direction near the boundary. Let $E \to W$ be a Hermitian complex vector bundle equipped with a Hermitian connection which is constant in the normal direction to the boundary and let $D_{W,E}^{c,+} : \Gamma(S_c^+ \otimes E) \to \Gamma(S_c^- \otimes E)$ be the twisted $Spin^{c,+}$ Dirac operator. Assume that $D_{W,E}^{c,+}$ and its restriction $D_{M,E'}^c$ to the boundary M (where E' is the restriction of E to M) satisfy the APS boundary condition 2.3.1. Then the index of $D_{W,E}^{c,+} : \Gamma(S_c^+ \otimes E, P) \to \Gamma(S_c^- \otimes E)$ is given by*

$$\mathrm{index}(D_{W,E}^{c,+}) = \int_W ch(E) e^{\frac{1}{2}c} \hat{A}(W) - \frac{h(D_{M,E'}^c) + \eta(D_{M,E'}^c)}{2} \tag{2.11}$$

where $\mathrm{index}(D_{W,E}^{c,+})$*,* $h(D_{M,E'}^c)$ *and* $\eta(D_{M,E'}^c)$ *are defined in Remark 2.3.4, $ch(E)$ is the Chern character for the Chern forms on $E \to W$, $c = c_1(P_{U(1)})$ is the canonical class of the $Spin^c$ structure on W and $\hat{A}(W)$ is the $\hat{A}$-genus with respect to the Pontrjagin forms on W.*

Applying Theorem 2.3.2 to the $Spin^{c,+}$ Dirac operator, we obtain the following.

Theorem 2.3.11 ($Spin^c$ equivariant index theorem)**.** *Let W^{2l} be a compact $Spin^c$ manifold which is of product form near the boundary $M^{2l-1} = \partial W$. Suppose the unitary connection on $P_{U(1)}$ is constant in the normal direction* *near the boundary. Assume that the $Spin^{c,+}$ Dirac operator $D_W^{c,+}$ and its restriction D_M^c to the boundary M satisfy the APS boundary condition 2.3.1. Let $g : W \to W$ be an isometry perserving the $Spin^c$ structure. Suppose furthermore that the unitary connection on $P_{U(1)}$ is invariant under the induced action of g. Then the equivariant index of $D_W^{c,+} : \Gamma(S_c^+, P) \to \Gamma(S_c^-)$ is given by*

$$\operatorname{index}(D_W^{c,+}, g) = \sum_{N \subset W^g} a_{spin^c}(N) - \frac{h_g(D_M^c) + \eta_g(D_M^c)}{2} \tag{2.12}$$

where $\operatorname{index}(D_W^{c,+}, g), h_g(D_M^{c,+})$ *and* $\eta_g(D_M^{c,+})$ *are defined in Theorem 2.3.2, N denotes a fixed point component of the action of g on W and $a_{spin^c}(N)$ is the corresponding local contribution.*

If $N \subset W^g$ denotes a fixed point component without boundary with normal bundle $\nu = \nu(\pi) \oplus \bigoplus_{0<\theta<\pi} \nu(\theta)$ in W (see Remark 2.3.5), then[6]

$$a_{spin^c}(N) = (-1)^s \int_N \lambda^{\frac{1}{2}} e^{\iota_N^*(c)/2} \prod_{0<\theta\leq\pi} \hat{A}_\theta(\nu(\theta)) \cdot \hat{A}(N), \tag{2.13}$$

where $s \in \{0, 1\}$ depends on the action of g on the $Spin^c$ structure, g acts via multiplication by λ on the restriction of $P_{U(1)}$ to N, $c := c_1(P_{U(1)})$ is the canonical class of the $Spin^c$ structure on W, $\iota_N : N \hookrightarrow W$ is the inclusion, and $\hat{A}_\theta$ is given by Equations (2.8) and (2.9).

Remark 2.3.12. *In the situation of the above theorem, since g preserves the $Spin^c$ structure and the connection on $P_{U(1)}$ is invariant under the induced action of g, the map g lifts to an action on the complex spinor bundles, and the $Spin^c$ and $Spin^{c,+}$ Dirac operators commute with the induced action of g on the sections (see Appendix A). Hence Theorem 2.3.2 applies.*

We now give the local contribution associated to the $Spin^{c,+}$ Dirac operator of an isolated fixed point of an involution. This will be used in the proof of Theorem A for the case of the Brieskorn quotients.

Proposition 2.3.13. *[DGA21, Proposition 3.5] Let W^{2l} be a compact complex spin^c manifold which is of product form near the boundary $M^{2l-1} = \partial W$. Let $\tau : W \to W$ be a holomorphic involution, acting by isometries and preserving the $Spin^c$ structure. Suppose τ has isolated fixed points which all lie in the interior of W. Then for every fixed point $p \in W$ the local contribution is*

$$a_{spin^c}(\{p\}) = 2^{-l}. \tag{2.14}$$

[6]The formula of the local contribution can for example be obtained by applying the "recipe" from [HBJ92, p.67] to the untwisted expression in the integral of Equation (2.11) (see also [Hat78] and [Des20]).

Proof. Since the $Spin^c$ structure on W is induced from a complex structure and the action of τ is holomorphic, the equivariant symbol of the $Spin^{c,+}$ Dirac operator is equal to the equivariant symbol of the Dolbeault operator on W (see [Dui11, Lemma 5.5]). In particular, the local contributions associated to the $Spin^{c,+}$ Dirac operator are then equal to the local contributions of the Dolbeault operator. This follows essentially from [AS68a, p.538], since the equivariant symbol determines an element in the equivariant K-theory of TW which is isomorphic to the equivariant K-theory of the TW^τ.

Hence, by applying the "recipe" from [HBJ92, p.67] to the Todd genus (see [HBJ92, p.61]), for an isolated fixed point $p \in W$ of τ we obtain

$$a_{spin^c}(\{p\}) = \int_{\{p\}} \prod_{j=1}^{l} \frac{1}{1 - \lambda e^{-x_j}}$$

where x_j are the formal complex roots of the tangent bundle TW for $j = 1, ..., l$ and $\lambda = -1$ is the eigenvalue of the action induced by τ on T_pW. The evaluation on the isolated fixed point p amounts to setting $x_j = 0$ for all $j = 1, ..., l$. The result now immediately follows. $\square$

2.3.3 Signature operator

Lastly, we present the index theorems for the signature case. The Hirzebruch signature theorem 2.3.14 will be used in various places, most notably in the proofs of Theorems B, C and D. The Signature Atiyah-Patodi-Singer index theorem 2.3.16 and the Signature equivariant index theorem 2.3.17 will be employed in the determination of the Eells-Kuiper invariant of the Milnor and Shimada projective spaces.

For a closed, oriented $4k$-dimensional manifold M, let $\operatorname{sign}(M)$ denote the signature of the non-degenerate quadratic form

$$H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \to \mathbb{R} : (\alpha, \beta) \mapsto \int_M \alpha \cup \beta. \tag{2.15}$$

The integer $\operatorname{sign}(M)$ will be called the *signature* of the closed manifold M.

Theorem 2.3.14 (Hirzebruch signature theorem). *[AS68b, Theorem (6.6)] Let M^{4k} be a closed, oriented manifold. Then*

$$\operatorname{sign}(M) = \langle L(M), [M] \rangle,$$

where $[M] \in H_{4k}(M;\mathbb{Z})$ is the fundamental class of M.

Here and in the following, L is the genus associated to the characteristic power series $\sqrt{z}/\tanh(\sqrt{z})$ with corresponding multiplicative sequence $\{L_k\}$

(see [MS74, §19.] and [LM89, III §11]). It is a power series in the Pontrjagin classes (or forms) of M. In particular, for $k = 1, 2$ and 4 we have

$$L_1(p_1) = \frac{1}{3}p_1, \tag{2.16}$$

$$L_2(p_1, p_2) = \frac{1}{45}\Big(7p_2 - p_1^2\Big), \tag{2.17}$$

$$L_4(p_1, p_2, p_3, p_4) = \frac{1}{14175}\Big(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4\Big). \tag{2.18}$$

If x_i, $i = 1, ..., 2k$, denote the formal roots of TM, then

$$L(M) = \prod_{i=1}^{2k} \frac{x_i}{\tanh(x_i)}. \tag{2.19}$$

Let W^{2k} be an oriented, compact manifold with boundary $\partial W = M$. If k is even, the *signature* of W is defined as the signature of the quadratic form defined on $\hat{H}^k(W;\mathbb{R}) := im(H^k(W, M;\mathbb{R}) \to H^k(W;\mathbb{R}))$ via the cup product. We will likewise denote it by $\mathrm{sign}(W)$.

The following result is due to Novikov.

Proposition 2.3.15. *[AS68b, Proposition (7.1)] Let M^{4k-1} be a compact, oriented manifold and let W and W' be oriented manifolds such that $\partial W = M$ and $\partial W' = -M$ (i.e. M with opposite orientation). Let $X = W \cup_M W'$. Then*

$$\mathrm{sign}(X) = \mathrm{sign}(W) + \mathrm{sign}(W').$$

For the following definitions and discussion, see also [APS75a, p.63].

Suppose W^{2k} is an oriented, compact Riemannian manifold which is of product form near the boundary $M^{2k-1} = \partial W$. Let $\Omega^p(W)$ be the space of p-forms on W and $\Omega(W) := \bigoplus_{p=0}^{2k} \Omega^p(W)$. Let $d : \Omega^p(W) \to \Omega^{p+1}(W)$ be the exterior derivative. The map $\tau : \Omega^*(W) \to \Omega^*(W)$ defined by $\tau(\omega) = i^{p(p-1)+k} \star \omega$ for $\omega \in \Omega^p(W)$ is an involution, where $\star$ is the Hodge star operator. Let $\Omega_\pm$ be the ±1-eigenspaces of τ applied to $\Omega(W)$. Then

$$A_W := d + d^* = d - \star d \star : \Omega_+ \to \Omega_-$$

is an elliptic operator which we will call the *signature operator*. Since the metric is of product form near the boundary, we have

$$A_W = \sigma\Big(\frac{\partial}{\partial u} + B_M\Big)$$

where

$$B_M\omega = (-1)^{k+p+1}(\epsilon \star d - d\star)\omega$$

for $\epsilon = 1$ if $\omega \in \Omega^{2p}(M)$ and $\epsilon = -1$ if $\omega \in \Omega^{2p-1}(M)$ (see [APS75a, p.63]). The operator B_M is formally self-adjoint and preserves the parity of the forms, so that there is a decomposition $B_M = B_M^{ev} \oplus B_M^{odd}$. The operator B_M^{ev} is sometimes called the *odd signature operator* of M.

Theorem 2.3.16 (Signature Atiyah-Patodi-Singer index theorem)**.** *[APS75a, Theorem (4·14)] Let W^{4l} be a compact oriented Riemannian manifold which is of product form near the boundary $M = \partial W$. Then*

$$\operatorname{sign}(W) = \int_W L(W) - \eta(B_M^{ev})$$

where $L(W)$ is the L-genus in the Pontrjagin forms of W and $\eta(B_M^{ev})$ is defined in Remark 2.3.4.

For the following definitions and discussion, see also [APS75b, pp.408-409].

Let W^{2k} be a compact, oriented, Riemannian manifold which is of product form near the boundary $M^{2k-1} = \partial W$, A_W the signature operator and $B_M = B_M^{ev} \oplus B_M^{odd}$ the operator from above.

Assume that G is a compact Lie group acting via orientation preserving isometries on W^{2k}. The induced action of G on sections commutes with the operator A_W. Furthermore, G acts on $\hat{H}^k(W;\mathbb{R})$ and preserves the quadratic form defining the signature of a manifold with boundary (see above) which is symmetric if k is even and skew-symmetric if k is odd. Complexify and consider the corresponding hermitian form. Now, any G-invariant inner product on $\hat{H}^k(W;\mathbb{C})$ will induce a G-invariant decomposition $\hat{H}^k(W;\mathbb{C}) = \hat{H}^k_+ \oplus \hat{H}^k_-$ such that the hermitian form is positive definite on $\hat{H}^k_+$ and negative definite on $\hat{H}^k_-$. The virtual representation $\operatorname{sign}(W,G) := \hat{H}^k_+ - \hat{H}^k_-$ will be called the *G-signature* of W and

$$\operatorname{sign}(W,g) := tr(g|_{\hat{H}^k_+}) - tr(g|_{\hat{H}^k_-}) \tag{2.20}$$

the *equivariant signature* of W with respect to $g \in G$.

Theorem 2.3.17 (Signature equivariant index theorem)**.** *[Don78, Theorem 2.1] Let W^{2k}, M^{2k-1}, G, A_W and $B_M = B_M^{ev} \oplus B_M^{odd}$ be as above and suppose they satisfy the APS boundary condition 2.3.1. Then for each $g \in G$,*

$$\operatorname{sign}(W,g) = \sum_{N \subset W^g} a_{sign}(N) - \eta_g(B_M^{ev}) \tag{2.21}$$

where $\eta_g(B_M^{ev})$ is defined using Equation (2.2).

If $N \subset W^g$ denotes a fixed point component without boundary, whose normal bundle splits as $\nu = \nu(\pi) \oplus \bigoplus_{0<\theta<\pi} \nu(\theta)$ in W (see Remark 2.3.5), then [LM89, III. Theorem 14.5]

$$a_{sign}(N) = \int_N \prod_{0<\theta\leq\pi} L_\theta(\nu(\theta)) \cdot L(N), \tag{2.22}$$

where for any oriented real vector bundle E,

$$L_\pi(E) := e(E)(L(E))^{-1}, \tag{2.23}$$

and for any complex vector bundle F with formal splitting $F = l_1 \oplus ... \oplus l_k$ into complex line bundles with $x_j = c_1(l_j)$,

$$L_\theta(F) := \prod_{j=1}^{k} \coth\left(x_j + \frac{i\theta}{2}\right), \tag{2.24}$$

for $0 < \theta < \pi$.

2.3.4 Eta-invariant of a covering

In this subsection we give a formula which relates the eta-invariant of a manifold to the eta-invariant of its cover. This formula will be needed in the next subsection to determine the relative eta-invariant in terms of the local contributions.

Let M^{2n+1} be a closed, oriented, Riemannian manifold and let $\pi : \tilde{M} \to M$ be a regular covering with finite covering group G. The metric on M lifts to a metric on $\tilde{M}$ and any elliptic self-adjoint operator $D_M : \Gamma(E) \to \Gamma(F)$ (where E and F are vector bundles on M with a smooth inner product) lifts to an elliptic self-adjoint operator $D_{\tilde{M}} : \Gamma(\pi^*E) \to \Gamma(\pi^*F)$ which is equivariant with respect to the action of G by deck transformation. For each irreducible unitary representation $\alpha : G \to U(k)$, there is a flat vector bundle $E_\alpha := \tilde{M} \times_\alpha \mathbb{C}^k \to M$ and a twisted operator $D_{M,E_\alpha} : \Gamma(E) \otimes E_\alpha \to \Gamma(F) \otimes E_\alpha$.

Theorem 2.3.18. *Let M, $\tilde{M}$, G, $D_{\tilde{M}}$ and D_{M,E_α} be as above. Then*

$$\eta(D_{M,E_\alpha}) = \frac{1}{|G|} \sum_{g \in G} \eta_g(D_{\tilde{M}}) \cdot \chi_\alpha(g) \tag{2.25}$$

where χ_α is the character of α, $\eta(D_{M,E_\alpha})$ is defined via Equation (2.3) and $\eta_g(D_{\tilde{M}})$ via Equation (2.2).

Proof. The statement in general follows from [APS75b, (2·14)] and the orthogonality relations of irreducible characters. For the special case of the signature operator, see also [Don78, Theorem 3.4.]. □

2.3.5 Applications to positive scalar curvature

We now come to the major consequences of positive scalar curvature applied to index theory. Since all our relevant metrics will in particular also have positive scalar curvature, we will be able to apply these results to distinguish path components in the various moduli spaces.

Vanishing theorems

We start with some of the earliest results on index theory with positive scalar curvature. For the following theorem, which is due to Lichnerowicz [Lic63], see also [LM89, II. Corollary 8.9] and [LM89, II. Theorem 8.11].

Theorem 2.3.19 (*Spin* vanishing theorem)**.** *Let M^{4k} be a closed spin manifold and D_M^+ its Spin$^+$ Dirac operator. If the Riemannian metric on M has non-negative scalar curvature everywhere and positive scalar curvature at some point, then $ker D_M = 0$ and consequently index$(D_M^+) = \hat{A}(M) = 0$.*

There also is a version for manifolds with boundary (see [APS75b, Theorem (3·9)]).

Theorem 2.3.20 (*Spin* vanishing theorem with boundary)**.** *Let W^{2l} be a compact spin manifold with boundary M^{2l-1}. Let D_W^+ be the Spin$^+$ Dirac operator on W and D_M the Spin Dirac operator on the boundary. If there is a Riemannian metric on W which is of product form near the boundary and which has non-negative scalar curvature everywhere and positive scalar curvature at some point on M, then* $\operatorname{index}(D_W^+) = 0$ *and* $\ker(D_M) = 0$.

The next result follows essentially by the same argument applied to the *Spin^c* Dirac operator (see [LM89, Corollary D.16.] and [DGA21, Theorem 3.1]).

Theorem 2.3.21 (*Spin^c* vanishing theorem)**.** *Let W^{2l} be a compact spin^c manifold with boundary $M^{2l-1} = \partial W$. Let $P_{U(1)}$ be the principal $U(1)$-bundle associated to the Spin^c structure on W and E a Hermitian complex vector bundle with Hermitian connection. Let $D^{c,+}_{W,E}$ be the twisted Spin$^{c,+}$ Dirac operator on W and $D^c_{M,E'}$ its restriction to M (where E' is the restriction of E to M). Suppose now that W has scal ≥ 0 everywhere and scal > 0 at some point on M, and that the connections on E and $P_{U(1)}$ are flat. Then* $\ker(D^c_{M,E'}) = 0$ *and* $\operatorname{index}(D^{c,+}_{W,E}) = 0$.

The Gromov-Lawson invariant

We now introduce the first of our invariants for (moduli) spaces of metrics, which will be used to prove Theorems B, C and D.

Let M^{4k-1} be a closed spin manifold and let W_0^{4k} and W_1^{4k} be two spin manifolds such that there exist orientation and *Spin* structure preserving isometries $\phi_0 : \partial W_0 \to M$ and $\phi_1 : \partial W_1 \to M$. Let g_0 and g_1 be two positive scalar curvature metrics on M and suppose $\phi_i^*(g_i)$ extends to a metric h_i of non-negative scalar curvature on W_i which is of product form near the boundary for $i = 0, 1$. Let $I = [0, a]$ for some $a \in \mathbb{R}_{>0}$ and g be any metric on $M \times I$ which restricts to g_0 on $M \times \{0\}$ and g_1 on $M \times \{a\}$. Define the Riemannian manifold

$$X^{4k} = W_0 \cup_{\phi_0} (M \times I) \cup_{\phi_1^{-1}} (-W_1).$$

Then X is spin as well and we can therefore consider its $Spin^+$ Dirac operator $D^+ : \Gamma(S^+) \to \Gamma(S^-)$. Similarly to Gromov and Lawson [GL83, pp.116-117] (see also [KS93, p.828]), we can define an invariant

$$i(g_0, g_1) := \mathrm{index}(D^+), \tag{2.26}$$

which we will call the *Gromov-Lawson invariant.* It can be shown that this invariant is well-defined (see [GL83] and [KS93]).

The following is an immediate consequence of the *Spin* vanishing theorem 2.3.19.

Proposition 2.3.22. *Let M^{4k-1} and X^{4k} be as above. If the metric g on $M \times I$ is of positive scalar curvature, then $i(g_0, g_1) = 0$.*

Relative $Spin^c$ eta-invariant

The relative $Spin^c$ eta-invariant is the invariant we will use to distinguish path components in the corresponding moduli space of metrics on our 5-manifolds.

Let M^{2l+1} be a closed connected $spin^c$ manifold with Riemannian metric g_M and suppose that the associated principal $U(1)$-bundle $P_{U(1)}$ is given a flat connection. Let D^c_M be the $Spin^c$ Dirac operator of M.

Let $\alpha : \pi_1(M) \to U(k)$ be a unitary representation and $E_\alpha := \tilde{M} \times_\alpha \mathbb{C}^k$ the flat complex rank k vector bundle associated to it (see [Kob87, (1.2.4)]), where $\tilde{M}$ is the universal covering of M. Let D^c_{M,E_α} be the twisted $Spin^c$ Dirac operator.

Now, to highlight the dependence on the metric, we set $\eta(M, g_M) := \eta(D^c_M)$ and $\eta_\alpha(M, g_M) := \eta(D^c_{M,E_\alpha})$ (see Remark 2.3.4). We can then define the *relative $Spin^c$ eta-invariant* of M by

$$\tilde{\eta}_\alpha(M, g_M) := \eta_\alpha(M, g_M) - k \cdot \eta(M, g_M). \tag{2.27}$$

If g_M has positive scalar curvature, this quantity is an invariant of the path component in the space of metrics of positive scalar curvature $\mathcal{R}_{scal>0}(M)$ (see p.70 for the definition).

Proposition 2.3.23. *[DGA21, Proposition 3.3] Let M^{2l+1} be a closed connected $spin^c$ manifold, $\alpha : \pi_1(M) \to U(k)$ a unitary representation and $E_\alpha = \tilde{M} \times_\alpha \mathbb{C}^k$ the associated flat complex vector bundle. Suppose that the principal $U(1)$-bundle $P_{U(1)}$ associated to the $Spin^c$ structure on M is given a flat connection. Let g_0 and g_1 be two metrics of $scal > 0$ which lie in the same path component in $\mathcal{R}_{scal>0}(M)$. Then $\tilde{\eta}_\alpha(M, g_0) = \tilde{\eta}_\alpha(M, g_1)$.*

Let W be a compact connected $spin^c$ manifold with boundary $\partial W = \tilde{M}$, equipped with the induced $Spin^c$ structure. Let τ be a smooth orientation preserving involution of W which preserves the $Spin^c$ structure and is fixed-point free on $\tilde{M}$. Let $M := \tilde{M}/\tau$ be equipped with the quotient $Spin^c$ structure (see Appendix A). Let $\alpha : \pi_1(M) \to U(1)$ be the non-trivial representation and $E_\alpha := \tilde{M} \times_\alpha \mathbb{C}$.

Making use of the consequences of positive scalar curvature in the above situation, we can express the relative eta-invariant of M in terms of local contributions.

Proposition 2.3.24. *Let W, $\tilde{M}$ and $M = \tilde{M}/\tau$ be as above, and let g_M be a metric of scal > 0 on M. Suppose that the lift $g_{\tilde{M}}$ of g_M to $\tilde{M}$ extends to a metric g_W on W which is τ-invariant, of product form near the boundary and with scal ≥ 0 everywhere. Then*

$$\tilde{\eta}_\alpha(M, g_M) = -2 \sum_{N \subset W^\tau} a_{spin^c}(N),$$

where the sum is over the fixed point components of the action of τ on W.

Proof. Recall that $\eta_\alpha(M, g_M) := \eta(D^c_{M,E_\alpha})$, where D^c_{M,E_α} is the twisted $Spin^c$ Dirac operator and $\eta(\tilde{M}, g_{\tilde{M}}) := \eta(D^c_{\tilde{M}})$. Applying Theorem 2.3.18 to D^c_{M,E_α} with $G = \mathbb{Z}_2 = \{Id, \tau\}$, we obtain

$$\begin{aligned}\eta_\alpha(M, g_M) &= \frac{1}{2}\Big(\eta_{Id}(\tilde{M}, g_{\tilde{M}}) \cdot \chi_\alpha(Id) + \eta_\tau(\tilde{M}, g_{\tilde{M}}) \cdot \chi_\alpha(\tau)\Big) \\ &= \frac{1}{2}\Big(\eta(\tilde{M}, g_{\tilde{M}}) - \eta_\tau(\tilde{M}, g_{\tilde{M}})\Big),\end{aligned}$$

where $\eta_\tau(\tilde{M}, g_{\tilde{M}}) := \eta_\tau(D^c_{\tilde{M}})$ is defined via Equation (2.2). Applying Theorem 2.3.18 with the trivial representation (whose character is always 1) instead of α, we get

$$\eta(M, g_M) = \frac{1}{2}\Big(\eta(\tilde{M}, g_{\tilde{M}}) + \eta_\tau(\tilde{M}, g_{\tilde{M}})\Big).$$

Therefore, the relative $Spin^c$ eta-invariant (see Equation (2.27)) is given by

$$\tilde{\eta}_\alpha(M, g_M) = \eta_\alpha(M, g_M) - 1 \cdot \eta(M, g_M) = -\eta_\tau(\tilde{M}, g_{\tilde{M}})$$

Now since the metrics on M and W both are τ-invariant, of *scal* ≥ 0 everywhere and of *scal* > 0 on M, we can apply the $Spin^c$ vanishing theorem 2.3.21 and thus $\mathrm{index}(D^{c,+}_W, \tau)$ and $h_\tau(D^c_{\tilde{M}})$ vanish. Therefore, Theorem 2.3.11 reduces to $\eta_\tau(D^c_{\tilde{M}}) = 2\sum_{N \subset W^\tau} a_{spin^c}(N)$ and the result follows. $\square$

Chapter 3

Differential topology

In this chapter, we are going to discuss the main tools we need to present the diffeomorphism classification of sphere bundles over spheres and to (partially) carry out the diffeomorphism classification of (Shimada) Milnor projective spaces. First, we present the Eells-Kuiper invariant, then we introduce some concepts of surgery theory and lastly we define the Browder-Livesay invariant and give the classification of smooth involutions on spheres.

3.1 Eells-Kuiper invariant

We start with the definition of the invariant introduced by Eells and Kuiper [EK62]. Then we give a formula which allows to determine this invariant in terms of eta-invariants, thus relating it to the index theory we have discussed in the previous chapter.

Let M^{4k-1} be a closed, oriented $(4k-1)$-dimensional manifold. Let W^{4k} be a compact, spin manifold with boundary $\partial W = M$. The spin structure on W restricts to a spin structure on M.

Suppose furthermore that the following holds.

Condition 3.1.1 (Condition μ). *1. The homomorphisms*

$$j^* : H^{4i}(W, M; \mathbb{Q}) \to H^{4i}(W; \mathbb{Q}) \qquad 0 < i < k$$
$$j^* : H^{2k}(W, M; \mathbb{Q}) \to H^{2k}(W; \mathbb{Q})$$

in the exact sequence of the pair (W, M) are isomorphisms.

2. The homomorphism $i^* : H^1(W; \mathbb{Z}_2) \to H^1(M; \mathbb{Z}_2)$ is surjective, where $i : M \to W$ denotes the inclusion.

Under these conditions, we can define

$$\overline{p}_i(W) := (j^*)^{-1}(p_i(W)) \in H^{4i}(W, M; \mathbb{Q}), \tag{3.1}$$

$0 < i < k$, where $p_i(W) \in H^{4i}(W;\mathbb{Q})$ are the rational Pontrjagin classes of W.

If M and W satisfy Condition 3.1.1, we can define the *Eells-Kuiper invariant* of M:

$$\mu(M) \equiv \frac{1}{a_k}\Big(\langle N_k(\overline{p}), [W, M]\rangle + t_k \mathrm{sign}(W)\Big) \bmod 1 \tag{3.2}$$

where

$$N_k(\overline{p}) := \hat{A}_k(\overline{p}_1(W), ..., \overline{p}_{k-1}(W), 0) - t_k L_k(\overline{p}_1(W), ..., \overline{p}_{k-1}(W), 0),$$

$a_k := 4/(3 + (-1)^k)$ and $t_k := \hat{A}_k(0, ..., 0, 1)/L_k(0, ..., 0, 1)$. Here $\hat{A}$ and L denote the respective genera from §2.3.1 and §2.3.3.

Proposition 3.1.2. *[EK62, §3.] The Eells-Kuiper invariant satisfies the following properties.*

1. *If M_1 and M_2 are orientation preserving diffeomorphic, then $\mu(M_1) = \mu(M_2)$.*
2. *If $-M$ denotes M with opposite orientation, then $\mu(-M) = -\mu(M)$.*
3. *$\mu(M_1 \# M_2) = \mu(M_1) + \mu(M_2)$.*

Now let M^{4k-1} be a closed spin manifold, equipped with a Riemannian metric g_M and suppose that $H^{4i}(M;\mathbb{R}) = 0$ for all $0 < i < k$. This means that there exist forms $\hat{p}_i(M) \in \Omega^{4i-1}(M)/\mathrm{Im}(d)$ such that $p_i(M) = d\hat{p}_i(M)$ where $p_i(M)$ are the Pontrjagin forms of M with respect to the metric g_M. Now let $\alpha(M) \in H^{4k-1}(M;\mathbb{R}) = \Omega^{4k-1}(M)/\mathrm{Im}(d)$ be defined as

$$\hat{A}_k(p_1, ..., p_{k-1}, 0) - t_k L_k(p_1, ..., p_{k-1}, 0),$$

with one factor $p_i(M)$ replaced by $\hat{p}_i(M)$ in each monomial[1].

Then there is the following formula for the Eells-Kuiper invariant, which is helpful in the case a spin coboundary cannot be found.

Theorem 3.1.3. *[Goe12, Theorem 4.8] Let M be as above, D_M denote its Spin Dirac operator (see §2.3.1) and B_M^{ev} its odd signature operator (see §2.3.3). Then*

$$\mu(M) = \frac{1}{a_k}\Big(\frac{\eta(D_M) + h(D_M)}{2} - t_k \eta(B_M^{ev}) - \int_M \alpha(M)\Big) \in \mathbb{Q}/\mathbb{Z},$$

where $\eta(D_M)$ and $\eta(B_M^{ev})$ are defined using Equation (2.3) with the corresponding operators and $h(D_M) = \dim(ker D_M)$.

Note in particular that the Eells-Kuiper invariant does not depend on the choice of a Riemannian metric.

[1]For example, if $\hat{A}_2(p_1(M), 0) - t_2 L_2(p_1(M), 0) = \frac{1}{2^7 \cdot 7} p_1(M) \wedge p_1(M)$ then $\alpha(M) = \frac{1}{2^7 \cdot 7} p_1(M) \wedge \hat{p}_1(M)$.

3.2 Surgery theory

In this section, we only introduce the concepts we will need and summarize some results that will help the comprehension of the following diffeomorphism classifications. For a general treatment of surgery theory, see [Bro72], [Ran02] and [CLM21]. Here, we generally follow the notation and conventions of López de Medrano [Med71, p. III.1], since we will mostly be using surgery theory in the context of smooth involutions on spheres.

Let M be a smooth manifold. The *smooth structure set*[2] $hS(M)$ of M is the set of equivalence classes of simple[3] homotopy equivalences $f : X^n \to M$ (sometimes called *homotopy smoothings*), where X^n is a smooth n-dimensional manifold. Two such simple homotopy equivalences $f_0 : X_0 \to M$ and $f_1 : X_1 \to M$ are *equivalent* if there exists a diffeomorphism $\phi : X_0 \to X_1$ such that $f_1 \circ \phi \simeq f_0$.

Let $f : X \to M$ be a map between two smooth manifolds. Let ν_X denote the stable normal bundle over X, ξ a stable vector bundle over M and $b : \nu_X \to \xi$ a bundle map covering f. If $b' : \nu_X \to \xi'$ is another bundle map with ξ' a stable vector bundle over M, then b and b' are *equivalent* if there exists a bundle isomorphism $c : \xi \to \xi'$ such that $c \circ b = b'$.

With this notation, a *normal map* is a pair $(f, [b])$ where $f : X^n \to M$ is a map of degree one and $[b]$ an equivalence class of bundle maps. Two normal maps $(f_i, [b_i]) : X_i \to M$, $i = 0, 1$, are called *normally cobordant* if there exists a map $F : Y^{n+1} \to M$ and a bundle map $B : \nu_Y \to \xi$ covering F, with Y^{n+1} a cobordism between X_0 and X_1, and such that $F|_{X_i} = f_i$ as well as $[B|\nu_{X_i}] = [b_i]$ for $i = 0, 1$. Then the pair $(F, [B])$ is called a *normal cobordism* between $(f_0, [b_0])$ and $(f_1, [b_1])$. The normal cobordism class of a manifold M is called its *normal invariant* and the set of normal invariants of M will be denoted by $\mathcal{N}(M)$.

Let $G_n := \{f : S^{n-1} \to S^{n-1} | deg(f) = \pm 1\}$, which is a topological monoid when equipped with the compact open topology, define the direct limit $G := \lim_{n\to\infty} G_n$ via suspension and consider the corresponding classifying space BG. Denote by BO the classifying space for stable linear bundles. Then there is a fibre map $BO \to BG$ whose fibre is denoted by G/O. For a smooth manifold M, the set of normal invariants $\mathcal{N}(M)$ is non-empty and it is in one-to-one correspondence with $[M, G/O]$ (see [MM79, Theorem 2.23]). From now on, we will identify the set of normal invariants with $[M, G/O]$ without further mention.

An element $[f] \in hS(M)$ determines a normal map in the following way. Let $g : M \to X$ be a homotopy inverse of $f : X \to M$. Taking $\xi = g^*\nu_X$, we get a stable vector bundle over M with a bundle map $b : \nu_X \to \xi$, and thus

[2] Also denoted by $\mathcal{S}^{Diff}(M)$.

[3] A homotopy equivalence $h : X \to M$ is called *simple* if its Whitehead torsion $\tau(h) \in Wh(\pi_1(X))$ vanishes (see [Ran02, Definition 8.12]). Note in particular that $Wh(\{1\})$ and $Wh(\mathbb{Z}_2)$ are trivial.

a normal invariant $\alpha(f)$ corresponding to f (we also denote it by $\alpha(X)$ if there can be no confusion). This gives a map $\alpha : hS(M) \to [M, G/O]$ (see [Med71, §III.1.3.]).

This map from the smooth structure set to the normal invariants of a manifold is crucial in the following fundamental result of surgery theory.

Theorem 3.2.1 (Surgery exact sequence)**.** *[Ran02, Theorem 1.18] Let M be an n-dimensional smooth manifold, $n \geq 5$. There is an exact sequence of pointed sets*

$$... \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to hS(M) \xrightarrow{\alpha} [M, G/O] \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(M)])$$

where $L_n(\mathbb{Z}[\pi_1(M)])$ and $L_{n+1}(\mathbb{Z}[\pi_1(M)])$ are the surgery obstruction groups and σ_* is the surgery obstruction map.

For more details on this result, see [Ran02, Chapter 13]. In particular, exactness of each map in the above sequence is explained in [Ran02, Theorem 13.2].

Remark 3.2.2. *There is a very similar theory for PL manifolds (see [Med71, Chapter III]). In this case, $hT(M)$ denotes the set of equivalence classes of homotopy equivalences $f : X \to M$, where X is a PL manifold. If BPL denotes the classifying space for stable PL bundles, there is a fibre map $BPL \to BG$ whose fibre is denoted by G/PL. Finally, there also is a map $\beta : hT(M) \to [M, G/PL]$.*

3.3 Diffeomorphism classification of spheres

Let Θ_n be the group of h-cobordism classes of n-dimensional homotopy spheres, where group addition is by connected sum. Note that $\Theta_n \cong hS(S^n)$ (see [Ran02, Lemma 13.20]) and so Θ_n can be determined using the surgery exact sequence with surgery obstruction groups in the simply connected case for $n \geq 5$. Let $bP_{n+1} \subset \Theta_n$ be the subgroup consisting of those elements which are the boundary of a parallelizable manifold.

Kervaire and Milnor studied these groups in [KM63] with early surgery methods and determined the order of the first few low-dimensional cases.

n	5	6	7	8	9	10	11	12	13	14	15
$\lvert\Theta_n\rvert$	1	1	28	2	8	6	992	1	3	2	16256
$\lvert bP_{n+1}\rvert$	1	1	28	1	2	1	992	1	1	1	8128

Remark 3.3.1. *We will mostly be interested in 7 and 15-dimensional homotopy spheres. In these dimensions, Eells and Kuiper showed that the smooth structure of a homotopy sphere which bounds a parallelizable manifold is completely determined by its μ-invariant (see [EK62, §6. and 9.]).*

3.4 Smooth involutions on spheres

In this section, we discuss smooth involutions on spheres, which ultimately leads to a diffeomorphism classification result for homotopy projective spaces.

A smooth involution on a smooth manifold M is a diffeomorphism $T : M \to M$ such that $T \circ T = Id_M$. Sometimes the pair (M, T) will also be called a (smooth) involution. Two involutions (M, T) and (N, T') are called equivalent if there exists a diffeomorphism $\phi : M \to N$ such that $\phi \circ T = T' \circ \phi$.

From now on, every involution on a closed manifold will be assumed to be smooth and fixed point free, unless otherwise stated. Note that we do not assume that involutions are orientation preserving in general.

Let M^n be a closed oriented smooth n-dimensional manifold and $T : M \to M$ a smooth fixed point free involution. A *characteristic submanifold* of (M, T) is a compact submanifold $C^{n-1} \subset M^n$ such that there exists a manifold with boundary A^n satisfying $C = A \cap T(A)$, $M = A \cup T(A)$ and $\partial A = C$. We will also say that $P := C/T$ is a characteristic submanifold for the quotient M/T.

For example, if $M = S^n$ and $T = a$ is the antipodal map on S^n, then any great circle $C = S^{n-1}$ is a characteristic submanifold of (S^n, a), where A can be chosen to be any of the two closed hemispheres with boundary S^{n-1}.

Remark 3.4.1. *[Med71, I.1.1 Lemma] Let (M, T) be a smooth fixed point free involution. Let $\pi : M \to M/T$ be the projection, $r : M/T \to \mathbb{R}\mathrm{P}^N$ the classifying map of the universal cover (N large) and suppose it is transverse to $\mathbb{R}\mathrm{P}^{N-1} \subset \mathbb{R}\mathrm{P}^N$. Then $\pi^{-1}(r^{-1}(\mathbb{R}\mathrm{P}^{N-1}))$ is a characteristic submanifold of (M, T). Equivalently, $r^{-1}(\mathbb{R}\mathrm{P}^{N-1})$ is a characteristic submanifold of M/T.*

We now focus on involutions on homotopy spheres.

Theorem 3.4.2. *[Bro67, (3.2) Proposition.] The set of equivalence classes of smooth fixed point free involutions on homotopy n-spheres is in one-to-one correspondence with the diffeomorphism classes of manifolds homotopy equivalent to $\mathbb{R}\mathrm{P}^n$.*

Recall that for M^n an oriented closed smooth manifold, the *intersection number* of $a \in H_p(M; \mathbb{Z})$ and $b \in H_q(M; \mathbb{Z})$ is given by

$$a \cdot b = \langle PD^{-1}(a) \cup PD^{-1}(b), [M] \rangle,$$

where $PD : H^l(M; \mathbb{Z}) \to H_{n-l}(M; \mathbb{Z})$ is the Poincaré duality isomorphism, $\cup$ denotes the usual cohomology cup product, $\langle , \rangle$ the Kronecker pairing and $[M]$ is the fundamental class.

In order to study involutions on homotopy spheres, we introduce the Browder-Livesay invariant. Let (Σ^{4k+3}, T), $k \geq 1$, be a homotopy $(4k+3)$-sphere with a smooth fixed point free involution T. Let $C \subset \Sigma^{4k+3}$ be a characteristic submanifold of this involution. For $x, y \in \ker(H_{2k+1}(C;\mathbb{Z}) \to H_{2k+1}(A;\mathbb{Z}))/\text{torsion}$, where the map is induced by the inclusion, we consider the bilinear form

$$B(x,y) := x \cdot T_*(y)$$

where the dot stands for the intersection number. The bilinear form B is even, symmetric and unimodular [Med71, p. I.1.3]. It follows that the index[4] of B, defined as the difference between the number of positive and negative values on the diagonal of a diagonalization of B, is a multiple of 8 (see for example [HM68, p.92 Korollar]).

We can now define the *Browder-Livesay invariant* by

$$\sigma(\Sigma^{4k+3}, T) := \frac{1}{8}\text{index}(B),$$

which by the above observation is an integer. It can be shown that this invariant is well-defined (i.e. it does not depend on the choice of the characteristic submanifold, see [BL73, Lemma. 3.2.]).

Remark 3.4.3. *If $n = 2k$, then we set $\sigma(\Sigma^{2k}, T) := 0$ for every involution (Σ^{2k}, T). If $n = 4k+1$, the Browder-Livesay invariant of an involution (Σ^{4k+1}, T) is defined using the Arf invariant of a quadratic form associated to B (see [Med71, p. I.1.3.]).*

Let $h : Q^n \to \mathbb{R}\mathrm{P}^n$ be a homotopy equivalence and (Σ^n, T) a smooth involution whose equivalence class is associated to the diffeomorphism class of Q via Theorem 3.4.2. Then $\sigma(Q) := \sigma(\Sigma^n, T)$ will be called the Browder-Livesay invariant of Q. It is straightforward to verify that this is well-defined.

We say that an involution (Σ^n, T) *desuspends* if there is a smoothly embedded $S^{n-1} \subset \Sigma^n$ such that $T(S^{n-1}) = S^{n-1}$. The involution *doubly desuspends* if there is also a smoothly embedded $S^{n-2} \subset S^{n-1} \subset \Sigma^n$ such that $T(S^{n-2}) = S^{n-2}$. The Browder-Livesay invariant gives a condition for an involution on a homotopy sphere to desuspend.

Theorem 3.4.4. *[Med71, I.1.3 Theorem] For $n \geq 6$, a smooth fixed point free involution (Σ^n, T) desuspends if and only if $\sigma(\Sigma^n, T) = 0$.*

Note that in even dimensions this makes sense, because every involution (Σ^{2k}, T) desuspends (see [BL73, Theorem 2.5.]).

We can now finally present the following classification result of homotopy real projective spaces.

[4]Sometimes also called the *signature* of the bilinear form.

Theorem 3.4.5. *[Med67, Theorem 4] Let Q^n be a smooth manifold and $h : Q^n \to \mathbb{RP}^n$ a homotopy equivalence, $n \geq 5$. Then the diffeomorphism class of Q^n is determined, up to connected sum with an element of bP_{n+1}, by its Browder-Livesay invariant and its normal invariant.*

Chapter 4

Five-dimensional manifolds

In this chapter, we begin the sections with the definition of some simply connected 5-manifolds. These will be the universal coverings of our 5-manifolds of interest, which we get after defining a suitable involution and going to the quotient. A coboundary of the universal covering is also defined. We discuss some topological properties of all these spaces and then proceed to present their diffeomorphism classification.

4.1 Quotients of Brieskorn manifolds

Our first class of manifolds is made up of quotients of spaces studied by Brieskorn [Bri66]. See [HM68], [Bre72] and [BG07, Chapter 9] for more details on the following spaces.

Let $D^8 := \{z \in \mathbb{C}^4 | |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 \leq 1\}$ and $S^7 := \{z \in \mathbb{C}^4 | |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$ be the unit disk and unit sphere in $\mathbb{C}^4$ respectively. Let $f_d : \mathbb{C}^4 \to \mathbb{C}$ be defined as $f_d(z) := z_0^d + z_1^2 + z_2^2 + z_3^2$, for $d \in \mathbb{N}_0$. For $\epsilon \in \mathbb{R}_{\geq 0}$, we define the *Brieskorn varieties*

$$W_\epsilon^6(d) := D^8 \cap f_d^{-1}(\epsilon),$$
$$M_\epsilon^5(d) := S^7 \cap f_d^{-1}(\epsilon).$$

For $\epsilon > 0$, $W_\epsilon^6(d)$ is a smooth complex manifold with boundary $\partial W_\epsilon^6(d) = M_\epsilon^5(d)$, whereas for $\epsilon = 0$, $M_0^5(d)$ is a smooth manifold but $W_0^6(d)$ is only a variety with an isolated singular point at $z = 0$. $M_\epsilon^5(d)$ comes with a natural orientation as a link and is sometimes also called a *Brieskorn manifold*. We summarize some properties of these spaces in the following theorem.

Theorem 4.1.1.

1. *$W_\epsilon^6(d)$ is homotopy equivalent to a bouquet $S^3 \vee ... \vee S^3$ with $d-1$ summands [HM68, p.83].*

2. *For ϵ sufficiently small, $M^5_\epsilon(d)$ is diffeomorphic to $M^5_0(d)$ [HM68, 14.3 Satz, p.103].*

3. *If $d = 2k+1$ is odd, then $M^5_0(d)$ is diffeomorphic to S^5 [HM68, 14.5 Satz, p.106]. In this case, we call $M^5_0(d)$ a* Brieskorn sphere.

4. *If $d = 2k$ is even, $M^5_0(d)$ is diffeomorphic to $S^3 \times S^2$ [GT98, Proposition 7].*

4.1.1 Cohomogeneity one action

An action of a compact Lie group G on a smooth manifold M is said to be of *cohomogeneity one* if the orbit space M/G is one-dimensional (see for example [AB15, §6.3] for more details on such actions).

For more details on the following cohomogeneity one action, see also [DGA21, §4.2] and [Gro+06, §1].

Let $S^1 \times O(3)$ act on $\mathbb{C}^4$ in the following way. For $(w, A) \in S^1 \times O(3)$ and $z \in \mathbb{C}^4$, we set

$$(w, A) \cdot z = (w^2 z_0, (A(w^d z_1, w^d z_2, w^d z_3)^T)^T).$$

This restricts to an action by $\mathbb{Z}_{2d} \times O(3)$ on $W^6_\epsilon(d)$ and $M^5_\epsilon(d)$ ($\epsilon \neq 0$), and by $S^1 \times O(3)$ on $W^6_0(d)$ and $M^5_0(d)$ which is of cohomogeneity one on $M^5_0(d)$. This can be seen by noticing that $|z_0|$ is invariant under this action and two points belong to the same orbit if and only if they have the same norm of the zero component. In the latter case, the principal isotropy is $\mathbb{Z}_2 \times O(1)$ and the singular isotropies are $S^1 \times O(1)$ and $\mathbb{Z}_2 \times O(2)$. The corresponding singular orbits are both of codimension two.

Now consider $\tau = (1, -Id) \in S^1 \times O(3)$. This gives a holomorphic, orientation preserving involution on $W^6_\epsilon(d)$, acting without fixed points on $M^5_\epsilon(d)$ for $0 \leq \epsilon < 1$. For $0 < \epsilon < 1$, the fixed points of the action of τ on $W^6_\epsilon(d)$ are $p_j = (\lambda_j, 0, 0, 0)$, $1 \leq j \leq d$, where λ_i is a complex d-root of ϵ for all j. These fixed points are all isolated and lie in the interior of $W^6_\epsilon(d)$. Furthermore, since the polynomial f_d is invariant under the action of τ, it descends to a map $\overline{f}_d : S^7/\tau \to \mathbb{C}$.

We call the quotient manifold $Q^5_\epsilon(d) := M^5_\epsilon(d)/\tau$ a *Brieskorn quotient*. Some important properties of these quotients are summarized in the following.

Theorem 4.1.2.

1. *For d odd, $Q^5_\epsilon(d)$ is homotopy equivalent to $\mathbb{R}\mathrm{P}^5$ (this follows from 4.1.1.3. and [Bro67, (3.1) Proposition.]).*

2. *For d even, the quotients $Q^5_\epsilon(d)$ are not all pairwise homotopy equivalent for different values of d [GT98, Remark (2), p.1180].*

3. *For all d and all ϵ sufficiently small, $Q^5_\epsilon(d)$ is diffeomorphic to $Q^5_0(d)$ (this is proved for example in [DGA21, Proposition 4.1] for d odd, but the proof equally applies to d even).*

   From now on, let d be even.

4. *$H_1(Q^5_\epsilon(d);\mathbb{Z}) \cong \pi_1(Q^5_\epsilon(d)) \cong \mathbb{Z}_2$ and $H_2(Q^5_\epsilon(d);\mathbb{Z}) = 0$ [GT98, Proposition 6].*

5. *$w_2(Q^5_0) \neq 0 \in H^2(Q^5_0;\mathbb{Z}_2)$ [GT98, Proposition 6].*

6. *$\tau_* : H_2(M^5_0(d);\mathbb{Z}) \to H_2(M^5_0(d);\mathbb{Z})$ is the non-trivial isomorphism (i.e. it exchanges the two generators) [GT98, Lemma 5].*

The following are immediate consequences. Recall that the fundamental group of a space X acts on the higher homotopy groups (see for example [Spa66, Chapter 7.3]).

Corollary 4.1.3. *Assume that d is even.*

1. *$H^2(Q^5_\epsilon(d);\mathbb{Z}) \cong \mathbb{Z}_2$ and $H^1(Q^5_\epsilon(d);\mathbb{Z}_2) \cong \mathbb{Z}_2$.*
2. *$\pi_1(Q^5_0(d))$ acts non-trivially on $\pi_2(Q^5_0(d)) \cong \mathbb{Z}$.*

Proof. The first statement follows by Theorem 4.1.2.4 and the universal coefficient theorem. The second statement follows by Theorem 4.1.2.6 and Hurewicz' theorem. $\square$

From now on, we will always assume that d is even, unless otherwise stated.

4.1.2 Diffeomorphism classification

Recall that for each fixed point free smooth involution (M,T), there exists a characteristic submanifold $P \subset M/T$ (see Remark 3.4.1).

Lemma 4.1.4. *[GT98, Lemma 9] Let Q^5 be a closed oriented smooth 5-dimensional manifold with $\pi_1(Q) = \mathbb{Z}_2$. Let P^4 be a characteristic submanifold of Q^5. If $w_2(Q) \neq 0$, then P admits a pair of Pin^+ structures. The Pin^+ cobordism class of this pair of structures does not depend on the choice of P.*

Note that the two Pin^+ structures are mutually inverses in $\Omega_4^{Pin^+}$ and are interchanged by the action of $w_1(P)$ (see [KT90, p.190]).

We can now state the following, which is part of Su's classification result for free involutions on $S^3 \times S^2$.

Theorem 4.1.5. *[Su12, Theorem 1.2.] Let Q^5 be an orientable, smooth 5-manifold with $\pi_1(Q) \cong \mathbb{Z}_2$ and universal cover $\tilde{Q} \cong S^3 \times S^2$. Assume furthermore that $\pi_1(Q)$ acts non-trivially on $\pi_2(Q) \cong \mathbb{Z}$ and $w_2(Q) \neq 0$. Then Q is diffeomorphic (orientation preserving or reversing) to a Brieskorn quotient $Q^5_0(d)$ for some $d = 0, 2, 4, 6, 8$. The quotients $Q^5_0(d)$ are classified by the Pin^+-cobordism class of their characteristic submanifold P_d with $[P_d] = d \in \Omega_4^{Pin^+}/\pm \approx \{0, 1, ..., 8\}$.*

The proof of this theorem uses Kreck's method of modified surgery. The idea is to determine the so-called normal 2-types of the considered 5-manifolds and compute the normal B-structure bordism groups (see [Su12, §3], [Kre85] and [Kre99] for more details).

Theorem 4.1.6. *For $d, d' \in \mathbb{N}$ even and $\epsilon \geq 0$ small enough, if $d \equiv d' \mod 16$ then $Q^5_\epsilon(d)$ and $Q^5_\epsilon(d')$ are orientation preserving diffeomorphic. Hence, for each $d = 0, 2, 4, 6, 8$, the set $\{Q^5_\epsilon(d + 16k)\}_{k \in \mathbb{N}_0}$ is an infinite family of manifolds all orientation preserving diffeomorphic to $Q^5_\epsilon(d)$.*

Proof. This follows immediately from Theorems 4.1.2.3 and 4.1.5. □

4.2 Principal S^1-bundles with fundamental group $\mathbb{Z}_2$

Our second family of 5-manifolds with fundamental group $\mathbb{Z}_2$ arises as quotients of an S^1-subaction sitting in a torus action on $S^3 \times S^3$.

4.2.1 Circle subactions of torus actions on $S^3 \times S^3$

We begin with the classification of certain torus actions on $S^3 \times S^3$ and then present the classification of some circle actions on $S^3 \times S^3$. These results are part of the classification of biquotients due to DeVito [DeV11] [DeV14]. We won't actually need the vocabulary of biquotients and so renounce defining them in this thesis.

Consider $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$, $T^2 = S^1 \times S^1$ and $S^3 = \{(p, q) \in \mathbb{C}^2 \,|\, |p|^2 + |q|^2 = 1\}$. Recall that $S^3 \cong SU(2)$.

An action of a group G on a set X is said to be *effectively free* if for all $g \in G$, if $g \cdot x = x$ for some $x \in X$, then $g \cdot y = y$ for all $y \in X$.

Proposition 4.2.1. *[DeV14, Proposition 3.1, 3.2, 3.3] Consider the action of T^2 on $S^3 \times S^3$ defined by*

$$(z, w) \cdot ((p_1, q_1), (p_2, q_2)) = ((z^a w^b p_1, z^c w^d q_1), (z^e w^f p_2, z^g w^h q_2))$$

with $gcd(a, c, e, g) = gcd(b, d, f, h) = 1$ and suppose this action is effectively free. Then there is a change of coordinates on T^2 for which this action has the form

$$(z,w)\cdot((p_1,q_1),(p_2,q_2)) = ((zp_1, z^{\alpha}w^{\beta}q_1),(wp_2, z^{\gamma}w^{\delta}q_2))$$

and this action is free if and only if $\alpha = \delta = 1$ and $|1-\beta\gamma| = 1$. We then have the following diffeomorphism types (the diffeomorphism might be orientation preserving or reversing):

$$B^4_{\beta,\gamma} := S^3 \times S^3/T^2 \cong \begin{cases} S^2 \times S^2 & \text{if } \gamma = 0 \text{ and } \beta \text{ even,} \\ \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2} & \text{if } \gamma = 0 \text{ and } \beta \text{ odd,} \\ \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2 & \text{if } \gamma = 2 \text{ and } \beta = 1. \end{cases} \tag{4.1}$$

Here $\overline{\mathbb{C}\mathrm{P}^2}$ denotes $\mathbb{C}\mathrm{P}^2$ with opposite orientation.

Proposition 4.2.2. *[DeV14, Proposition 4.1, Theorem 1.2] Let S^1 act on $S^3 \times S^3$ via*

$$z\cdot((p_1,q_1),(p_2,q_2)) = ((z^{a-c}p_1, z^{a+c}q_1),(z^{b-d}p_2, z^{b+d}q_2))$$

with $gcd(a,b,c,d) = 1$. Then this action is effectively free if and only if $g = gcd(a \pm c, b \pm d) = 1$ or 2. We then have the following diffeomorphism types (the diffeomorphism might be orientation preserving or reversing):

$$\mathcal{N}^5_{a,b,c,d} := S^3 \times S^3/S^1 \cong \begin{cases} S^3 \times S^2 & \text{if } g = 1, \\ S^3 \mathbin{\tilde{\times}} S^2 & \text{if } g = 2, \end{cases} \tag{4.2}$$

where $S^3 \mathbin{\tilde{\times}} S^2$ denotes the unique non-trivial S^3-bundle over S^2.

We are going to focus on two torus actions and corresponding circle subactions. For this, consider the inclusion

$$i_{k,l} : S^1 \to T^2 : z \mapsto (z^k, z^l).$$

Non-exceptional torus action

Let T^2 act on $S^3 \times S^3$ by

$$(z,w)\cdot((p_1,q_1),(p_2,q_2)) = ((zp_1, zw^{\beta}q_1),(wp_2, wq_2))$$

and consider the S^1-subaction

$$z\cdot((p_1,q_1),(p_2,q_2)) = ((z^k p_1, z^{k+\beta l}q_1),(z^l p_2, z^l q_2))$$

induced by $i_{k,l}(S^1) \subset T^2$. We identify the exponents from Proposition 4.2.2 $a = k + \beta l/2$, $c = \beta l/2$, $b = l$, $d = 0$. For all of these to be integers, βl must be even. From now on, we will assume β to be odd, l to be even and $gcd(k,l) = 1$ (hence k must be odd). By Proposition 4.2.1, we have

$$B^4_{\beta} := B^4_{\beta,0} \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$$

and by Proposition 4.2.2 we get

$$N^5_{k,l} := \mathcal{N}^5_{k+\beta l/2,l,\beta l/2,0} \cong S^3 \times S^2.$$

Denote by $S(L_{k,l})$ the principal bundle $S^1 \to N^5_{k,l} \to B^4_\beta$, where the S^1-action on $N^5_{k,l}$ is induced by $i_{-s,r}(S^1)$ where $kr+ls=1$. Here $L_{k,l}$ denotes the associated complex line bundle. We also define the associated disk bundle $D^2 \to W^6_{k,l} \to B_\beta$ where $W^6_{k,l} := N^5_{k,l} \times_{S^1} D^2$.

Exceptional torus action

Next, let T^2 act on $S^3 \times S^3$ via

$$(z,w)\cdot((p_1,q_1),(p_2,q_2)) = ((zp_1, zwq_1),(wp_2, z^2wq_2))$$

and consider the S^1-subaction

$$z\cdot((p_1,q_1),(p_2,q_2)) = ((z^kp_1, z^{k+l}q_1),(z^lp_2, z^{2k+l}q_2))$$

induced by $i_{k,l}(S^1) \subset T^2$. We identify $a = k+l/2$, $c = l/2$, $b = l+k$, $d = k$. Thus l must be even and if we assume $gcd(k,l) = 1$ (which we will do from now on) then k must be odd. From Proposition 4.2.1 it follows that

$$B^4 := B^4_{1,2} \cong \mathbb{C}P^2 \# \mathbb{C}P^2$$

and by Proposition 4.2.2 we have

$$\overline{N}^5_{k,l} := \mathcal{N}^5_{k+l/2,l+k,l/2,k} \cong S^3 \times S^2.$$

Denote by $S(\overline{L}_{k,l})$ the principal bundle $S^1 \to \overline{N}^5_{k,l} \to B$, where the S^1-action on $\overline{N}^5_{k,l}$ is induced by $i_{-s,r}(S^1)$ where $kr+ls=1$. Here $\overline{L}_{k,l}$ denotes the associated complex line bundle. The associated disk bundle $D^2 \to \overline{W}^6_{k,l} \to B$ is defined by $\overline{W}^6_{k,l} := \overline{N}^5_{k,l} \times_{S^1} D^2$.

4.2.2 Total spaces of principal S^1-bundles with fundamental group $\mathbb{Z}_2$

We now define the 5-manifolds whose moduli spaces we will be studying later on.

Let $X^5_{k,l,\beta}$ be the total space of the sphere bundle of the 2-fold tensor product $L^{\otimes 2}_{k,l} = L_{k,l} \otimes L_{k,l}$, that is $S(L^{\otimes 2}_{k,l})$ is the principal bundle $S^1 \to X^5_{k,l,\beta} \to B^4_\beta$. As such, $X^5_{k,l,\beta}$ is oriented once we choose an orientation on B^4_β (which we will do later on by fixing its fundamental class). We can also identify $X^5_{k,l,\beta}$ with the quotient space $N^5_{k,l}/\tau$, where τ is the involution on

$N^5_{k,l}$ induced by fiberwise antipodal maps on S^1 (see [Des20, p.5]). Hence, $N^5_{k,l} \cong S^3 \times S^2$ is a 2-fold universal covering of $X^5_{k,l,\beta}$ and so $\pi_1(X^5_{k,l,\beta}) \cong \mathbb{Z}_2$.

Similarly, we define $\overline{X}^5_{k,l}$ to be the total space of $S(\overline{L}^{\otimes 2}_{k,l})$ or equivalently the quotient $\overline{N}^5_{k,l}/\overline{\tau}$ where $\overline{\tau}$ is the involution on $\overline{N}^5_{k,l}$ induced by fiberwise antipodal maps on S^1. Thus $\pi_1(\overline{X}^5_{k,l}) \cong \mathbb{Z}_2$ as well.

Remark 4.2.3. *Since τ, respectively $\overline{\tau}$, restricts to multplication by $-1 \in S^1$ on the fibers, it follows that $\pi_1(X^5_{k,l})$ acts trivially on $\pi_*(X^5_{k,l})$ and likewise $\pi_1(\overline{X}^5_{k,l})$ acts trivially on $\pi_*(\overline{X}^5_{k,l})$.*

Proposition 4.2.4. *Let X be either $X^5_{k,l,\beta}$ or $\overline{X}^5_{k,l}$. Then*

$$H_1(X;\mathbb{Z}) \cong \mathbb{Z}_2, \qquad H_2(X;\mathbb{Z}) \cong \mathbb{Z}, \qquad H^2(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2,$$
$$H^1(X;\mathbb{Z}_2) \cong \mathbb{Z}_2, \qquad H^2(X;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Proof. Since $\pi_1(X)$ is abelian, we have $H_1(X;\mathbb{Z}) \cong \pi_1(X)$. For the second homology group, see [HS13, Proposition 6.1]. The rest follows from the universal coefficient theorem. □

4.2.3 Properties of the bundles and the associated spaces

In this subsection, we study some of the topological properties of the above spaces, which we will need in order to present the diffeomorphism classification, as well as for the computation of the relative eta-invariants later on.

Cohomology ring of the base space

In the following, we will identify $H^1(T^2;\mathbb{Z})$ with $\mathrm{Hom}(\Gamma,\mathbb{Z})$, where Γ is the unit lattice of $\mathbb{R}^2$ (see [BH58, §10]).

Non-exceptional case:

Lemma 4.2.5. *The integral cohomology ring of B^4_β is*

$$H^*(B^4_\beta;\mathbb{Z}) = \mathbb{Z}[u,v]/(v^2, u^2 + \beta uv). \tag{4.3}$$

Proof. We assume some familiarity with spectral sequences (see for example [Spa66, Chapter 9] for a classical treatment). Consider the non-exceptional torus bundle $T^2 \to S^3 \times S^3 \to B^4_\beta$. Let $(S^1)^2 \to (ES^1)^2 \to (BS^1)^2$ be the product of two universal S^1-bundles and $\phi : B^4_\beta \to (BS^1)^2$ the classifying map corresponding to $T^2 \to S^3 \times S^3 \to B^4_\beta$. Consider the fibration $S^3 \times S^3 \to B^4_\beta \to (BS^1)^2$. We can now use [BH58, Theorem 10.3] to determine the Euler classes of the two restricted S^3-bundles and then determine the cohomology ring via spectral sequences.

We have two representations

$$\rho_1 : T^2 \to T^2 \subset U(2) : (z, w) \mapsto (z, zw^\beta),$$

$$\rho_2 : T^2 \to T^2 \subset U(2) : (z, w) \mapsto (w, w).$$

The derivative of these maps are

$$d\rho_1 : \mathbb{R}^2 \to \mathbb{R}^2 : (a, b) \mapsto (a, a + \beta b)$$

$$d\rho_2 : \mathbb{R}^2 \to \mathbb{R}^2 : (a, b) \mapsto (b, b).$$

These give us the weights of the representation. Set $u' = -\tau_{ub}(a)$ and $v' = -\tau_{ub}(b)$, where $\tau_{ub} : H^1(T^2;\mathbb{Z}) \to H^2((BS^1)^2;\mathbb{Z})$ is the transgression[1]. Now, by [BH58, Theorem 10.3][2], the total Chern class of

$$\eta_i = \Big(U(2) \to (ES^1)^2 \times_{\rho_i} U(2) \to (BS^1)^2\Big),$$

for which the sphere bundle of the associated rank 2 complex vector bundle is $S^3 \to ES_i \to (BS^1)^2$ for $i = 1, 2$, is given by

$$c(\eta_1) = (1 + u')(1 + u' + \beta v'),$$

$$c(\eta_2) = (1 + v')(1 + v').$$

Hence the Euler classes are $e(\eta_1) = u'^2 + \beta u'v'$ and $e(\eta_2) = v'^2$. Set $u := \phi^*(u') \in H^2(B^4_\beta;\mathbb{Z})$ and $v := \phi^*(v') \in H^2(B^4_\beta;\mathbb{Z})$. Since the Euler classes form a regular sequence in the polynomial ring $H^*((BS^1)^2;\mathbb{Z})$, the cohomology ring of B^4_β follows from the spectral sequence of $S^3 \times S^3 \to B \to (BS^1)^2$. □

From now on we fix the fundamental class $[B^4_\beta] \in H_4(B^4;\mathbb{Z})$ such that $\langle uv, [B^4_\beta]\rangle = 1$.

Let us compute the characteristic classes we will need. Set $u_2 := u \bmod 2$ and $v_2 := v \bmod 2$.

Lemma 4.2.6. *We have*

$$p_1(B^4_\beta) = 0 \in H^4(B^4_\beta;\mathbb{Z}) \tag{4.4}$$

and

$$w_2(B^4_\beta) = v_2 \in H^2(B^4_\beta;\mathbb{Z}_2). \tag{4.5}$$

[1] See for example [Spa66, p.518] for the definition.

[2] Observe that $G = S^1 \times S^1$, $E_\xi = (ES^1)^2$, $B_\xi = (BS^1)^2$ and $\rho = Id$ in Borel and Hirzebruch's notation.

Proof. The first Pontrjagin class is immediately determined using the Hirzebruch signature theorem 2.3.14 with $L_1(p_1) = \frac{1}{3}p_1$ and $\operatorname{sign}(B^4_\beta) = 0$.

We know that $w_2^2(B^4_\beta) \equiv p_1(B^4_\beta) \bmod 2$ and so if $w_2(B^4) = au_2 + bv_2$, it follows that $a^2 \equiv 0 \bmod 2$. Thus a must be even and since $w_2(\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2})$ is non-trivial it follows that b must be odd. □

<ins>Exceptional case:</ins>

For the next result, see also [Tot02, p.404-405][3].

Lemma 4.2.7. *The integral cohomology ring of B^4 is*

$$H^*(B^4;\mathbb{Z}) = \mathbb{Z}[\overline{u}, \overline{v}]/(\overline{u}^2 + \overline{uv}, 2\overline{uv} + \overline{v}^2).$$

If we choose the fundamental class $[B^4] \in H_4(B^4;\mathbb{Z})$ such that $\langle \overline{u}^2, [B^4]\rangle = 1$, then $\operatorname{sign}(B^4) = 2$.

Proof. The argument of the proof of Lemma 4.2.5 applies to this case (with appropriate exponents) to give the cohomology ring of B^4.

The signature of B^4 can be determined using the intersection pairing $I : H^2(B^4;\mathbb{Z}) \times H^2(B^4;\mathbb{Z}) \to \mathbb{Z} : (x, y) \mapsto \langle xy, [B^4]\rangle$, namely, it is the signature of the symmetric bilinear form $I \otimes \mathbb{R}$. Hence, by the above and the assumption on the fundamental class, it is the signature of the matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

which has two positive eigenvalues.

□

From now on we fix the fundamental class $[B^4] \in H_4(B^4;\mathbb{Z})$ from above.

Let us compute the first Pontryagin class and the second Stiefel-Whitney class of B^4. Set $\overline{u}_2 := \overline{u} \bmod 2$ and $\overline{v}_2 := \overline{v} \bmod 2$.

Lemma 4.2.8. *We have*

$$p_1(B^4) = 6\overline{u}^2 \in H^4(B^4;\mathbb{Z}) \tag{4.6}$$

and

$$w_2(B^4) = \overline{v}_2 \in H^2(B^4;\mathbb{Z}_2). \tag{4.7}$$

Proof. The first statement follows immediately from the Hirzebruch signature theorem 2.3.14 and the fact that $\operatorname{sign}(B^4) = 2$.

If $w_2(B^4) = a\overline{u}_2 + b\overline{v}_2$, since $w_2^2(B^4) \equiv p_1(B^4) \bmod 2$ it follows that $a^2 \equiv 0 \bmod 2$. Thus a must be even and because $w_2(\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2)$ is non-trivial it follows that b must be odd. □

[3]Note that our torus action on $S^3 \times S^3$ differs from Totaro's by the automorphism $T^2 \to T^2 : (z, w) \mapsto (z, zw)$, leading to a different set of generators in the cohomology ring.

First Chern class of principal S^1-bundle

Proposition 4.2.9. *The first Chern class of $S^1 \to N^5_{k,l} \to B^4_\beta$ is*

$$c_1(S(L_{k,l})) = -lu + kv \in H^2(B^4_\beta;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}, \tag{4.8}$$

while the first Chern class of $S^1 \to \overline{N}^5_{k,l} \to B^4$ is

$$c_1(S(\overline{L}_{k,l})) = -l\overline{u} + k\overline{v} \in H^2(B^4;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{4.9}$$

Proof. Consider first the principal bundles $S^1 \to N^5_{k,l} \to B^4_\beta$ and $T^2 \to S^3 \times S^3 \to B^4_\beta$. Let $\tilde{\tau} : H^1(T^2;\mathbb{Z}) \to H^2(B^4_\beta;\mathbb{Z})$ and $\tau_{ub} : H^1(T^2;\mathbb{Z}) \to H^2((BS^1)^2;\mathbb{Z})$ be the corresponding transgressions and observe that $\tilde{\tau} = \phi^*\tau_{ub}$, where $\phi : B^4_\beta \to (BS^1)^2$ is the classifying map.

Let $pr : T^2 \to T^2/i_{k,l}(S^1)$ be the projection and consider

$$h : T^2/i_{k,l}(S^1) \to S^1 : [z_0, z_1] \mapsto z_0^{-l} z_1^k.$$

Now $\rho = h \circ pr : T^2 \to S^1 : (z_0, z_1) \mapsto z_0^{-l} z_1^k$ can be seen as a one-dimensional representation of T^2. Its differential is $d\rho : \mathbb{R}^2 \to \mathbb{R} : (a, b) \mapsto -la + kb$. If we set $u = -\tilde{\tau}(a) \in H^2(B^4_\beta;\mathbb{Z})$ and $v = -\tilde{\tau}(b) \in H^2(B^4_\beta;\mathbb{Z})$, then from [BH58, Theorem 10.3] it follows that the total Chern class of

$$S(L_{k,l}) = \left(S^1 \to (S^3 \times S^3) \times_\rho S^1 \to B^4_\beta\right)$$

is given by $c(S(L_{k,l})) = 1 + (-lu + kv)$ which proves the first statement.

The first Chern class of $S^1 \to \overline{N}^5_{k,l} \to B^4$ is determined similarly. $\square$

Observe that it follows immediately from the above and the additivity of the first Chern class of principal $U(1)$-bundles under tensoring that

$$c_1(S(L^{\otimes 2}_{k,l})) = 2c_1(S(L_{k,l})) = -2lu + 2kv \in H^2(B^4_\beta;\mathbb{Z})$$

and

$$c_1(S(\overline{L}^{\otimes 2}_{k,l})) = 2c_1(S(\overline{L}_{k,l})) = -2l\overline{u} + 2k\overline{v} \in H^2(B^4;\mathbb{Z}).$$

Second Stiefel-Whitney classes

As the total spaces of principal S^1-bundles, the tangent bundles of $X^5_{k,l,\beta}$ and $\overline{X}^5_{k,l}$ split in the following way. An action field trivializes the vertical bundle and the horizontal bundle is isomorphic to the pullback of the tangent bundle of the base and so we have $TX^5_{k,l,\beta} \cong \pi^*(TB^4_\beta) \oplus \epsilon$, where $\pi : X^5_{k,l,\beta} \to B^4_\beta$ is the projection and ϵ the trivial line bundle over $X^5_{k,l,\beta}$. Similarly, $T\overline{X}^5_{k,l} \cong \overline{\pi}^*(TB^4) \oplus \overline{\epsilon}$.

Recall that $w_2(\xi) \equiv c_1(\xi) \bmod 2$. Knowing the first Chern classes of $S(L_{k,l}^{\otimes 2})$ and $S(\overline{L}_{k,l}^{\otimes 2})$, we therefore have $w_2(S(L_{k,l}^{\otimes 2})) = 0$ and $w_2(S(\overline{L}_{k,l}^{\otimes 2})) = 0$. From the Gysin sequence with $\mathbb{Z}_2$-coefficients, it follows that the induced homomorphisms π^* and $\overline{\pi}^*$ are isomorphisms on the cohomology groups with $\mathbb{Z}_2$-coefficients of degree 2.

By Equations (4.7) and (4.5) we know that $w_2(B^4_\beta)$ and $w_2(B^4)$ are non-trivial. Hence, from the above, it follows that $w_2(X^5_{k,l,\beta})$ and $w_2(\overline{X}^5_{k,l})$ are non-trivial as well. Therefore, the manifolds $X^5_{k,l,\beta}$ and $\overline{X}^5_{k,l}$ do not admit *Spin* structures. But as we will see, they admit $Spin^c$ structures.

4.2.4 Diffeomorphism classification

To give the diffeomorphism classification of the $X^5_{k,l,\beta}$ and $\overline{X}^5_{k,l}$, we introduce another class of manifolds. See [HS13] and [Su12] for more details on these spaces.

Let $Q^5_0(d)$ be a Brieskorn quotient which is homotopy equivalent to $\mathbb{R}\mathrm{P}^5$ for $d = 1, 3, 5, 7$. Fix $d, d' \in \{1, 3, 5, 7\}$, denote by D the trivial oriented D^4-bundle over S^1 and choose embeddings of D into $Q^5_0(d)$ and $Q^5_0(d')$, representing a generator of $\pi_1(Q^5_0(d))$ and $\pi_1(Q^5_0(d'))$ respectively, such that the first embedding preserves the orientation and the second reverses it. We can then define

$$Q^5_0(d)\#_{S^1}Q^5_0(d') := (Q^5_0(d) \setminus D) \cup_\partial (Q^5_0(d') \setminus D).$$

Since $\pi_1(SO(4)) \cong \mathbb{Z}_2$, there are two possibilities to form this so-called *connected sum along a circle* (see [Su12, §2.3] and [HS13, §3]). Let $P \subset Q^5_0(d)$ and $P' \subset Q^5_0(d')$ be characteristic submanifolds. Then[4] $P\#_{S^1}P'$ is a characteristic submanifold of $Q^5_0(d)\#_{S^1}Q^5_0(d')$ and in order for the above manifold to be well-defined, we must fix a Pin^+ structure on P and P' (see [Su12, p.18-19]). The submanifold $P\#_{S^1}P'$ corresponds to the addition in the bordism group $\Omega_4^{Pin^+}$. Thus, we can choose $d, d' \in \{1, 3, 5, 7\}$ and appropriate Pin^+ structures on P and P' such that the manifold

$$X(q) := Q^5_0(d)\#_{S^1}Q^5_0(d'), \qquad q = 0, 2, 4, 6, 8$$

has a characteristic submanifold P_q whose Pin^+-cobordism class is $\pm q \in \Omega_4^{Pin^+}$. For instance, using different glueing maps: $X(0) = Q^5_0(1)\#_{S^1}Q^5_0(1)$ and $X(2) = Q^5_0(1)\#_{S^1}Q^5_0(1)$.

Let $q \in \{0, 2, 4, 6, 8\}$. By the Seifert-van Kampen theorem, it follows that $\pi_1(X(q)) \cong \mathbb{Z}_2$ and the Mayer-Vietoris sequence implies that $H_2(X(q); \mathbb{Z}) \cong \mathbb{Z}$. Furthermore, $w_2(X(q))$ is non-trivial and since the universal cover $\tilde{X}(q)$ of $X(q)$ is simply-connected, has trivial second Stiefel-Whitney class and

[4]See [HKT94, p.651] for the definition of $\#_{S^1}$ for non-orientable 4-manifolds with fundamental group $\mathbb{Z}_2$.

$H_2(\tilde{X}(q);\mathbb{Z}) \cong \mathbb{Z}$, it follows by the classification theorem of Smale that $\tilde{X}(q)$ is diffeomorphic to $S^3 \times S^2$ (see [Sma62]). Also, the action of $\pi_1(X(q))$ on $\pi_2(X(q))$ is trivial.

We can now state the classification result.

Theorem 4.2.10. *[Su12, Theorem 1.1] Let M be a smooth 5-dimensional orientable manifold with $\pi_1(M) \cong \mathbb{Z}_2$ and universal cover $S^3 \times S^2$. Suppose that $\pi_1(M)$ acts trivially on $\pi_2(M)$ and that $w_2(M) \neq 0$. Let $P \subset M$ be a characteristic submanifold. Then M is diffeomorphic (orientation preserving or reversing) to $X(q)$ for some $q \in \{0, 2, 4, 6, 8\}$, where $q = [P] \in \Omega_4^{Pin^+}/\pm$. The $X(q)$ are classified by the Pin^+-cobordism class of their characteristic submanifold.*

As we have seen, $X^5_{k,l,\beta}$ and $\overline{X}^5_{k,l}$ satisfy the above conditions. To classify them, it therefore suffices to determine the Pin^+-cobordism class of their characteristic submanifold.

Lemma 4.2.11. *Let $P \subset X^5_{k,l,\beta}$ and $P' \subset \overline{X}^5_{k,l}$ be characteristic submanifolds. Then*

$$q := [P] \equiv (1 + \frac{\epsilon}{2})(-\beta l^2 - 2kl) \bmod 16$$

and

$$q' := [P'] \equiv (1 + \frac{\epsilon}{2})(l^2 + 2kl + 2k^2) - \epsilon \bmod 16,$$

where $\epsilon = \pm 1$ is an unknown sign.

Proof. Recall that by Lemma 4.1.4, P and P' admit Pin^+ structures and the Pin^+-cobordism class of the pair does not depend on the choice of the characteristic submanifold. Now, from [Goo20a, Lemma 1.7], it follows that

$$q \equiv (1 + \frac{\epsilon}{2})\langle c_1(S(L_{k,l}))^2, [B^4_\beta]\rangle - \frac{\epsilon}{2}\mathrm{sign}(B^4_\beta) \bmod 16,$$

where $S(L_{k,l})$ is the principal S^1-bundle over B^4_β whose total space is $N^5_{k,l}$, the universal cover of $X^5_{k,l}$, and similarly

$$q' \equiv (1 + \frac{\epsilon}{2})\langle c_1(S(\overline{L}_{k,l}))^2, [B^4]\rangle - \frac{\epsilon}{2}\mathrm{sign}(B^4) \bmod 16.$$

The result now immediately follows from Equations (4.8), (4.9), (4.3), Lemma 4.2.7 and from the fact that the signatures are $\mathrm{sign}(B^4_\beta) = 0$ and $\mathrm{sign}(B^4) = 2$. $\square$

Recall that $gcd(k,l) = 1$, k is odd and l is even.

Proposition 4.2.12. *For each $q \in \{0, 4, 8\}$, there are infinitely many values of k and l such that $X^5_{k,l,\beta}$ is orientation preserving diffeomorphic to $X(q)$ (for β fixed). Similarly, for each $q' \in \{2, 6\}$, there are infinitely many values of k and l such that $\overline{X}^5_{k,l}$ is orientation preserving diffeomorphic to $X(q')$.*

Proof. This follows immediately from Theorem 4.2.10 and Lemma 4.2.11. Note that this result does not depend on whether $\epsilon = 1$ or -1. $\square$

Chapter 5

Sphere bundles over spheres and their quotients by involutions

In this chapter we first define linear sphere bundles over spheres and the involution under whose quotient we obtain Milnor and Shimada projective spaces. We discuss some of their topological properties and then present the diffeomorphism classification of the sphere bundles, which is due to Crowley and Escher [CE03] and Grey [Gre12]. After that, we carry out the diffeomorphism classification of Milnor projective spaces. We first determine the Browder-Livesay invariant and then the normal invariants. Lastly, we determine the Eells-Kuiper invariant of Milnor and Shimada projective spaces. This allows us to complete the diffeomorphism classification of Milnor projective spaces and give a finiteness result with a lower bound on the number of diffeomorphism types of Shimada projective spaces.

5.1 Definition

Let $n = 1, 2$ and fix a generator $\alpha \in H^{4n}(S^{4n}; \mathbb{Z})$. We use the same notation for the images of α under the isomorphisms $H^{4n}(S^{4n}; \mathbb{Z}) \cong H_{4n}(S^{4n}; \mathbb{Z}) \cong \pi_{4n}(S^{4n})$, where in homology α corresponds to the fundamental class and the second isomorphism is given by the Hurewicz map. Consider S^{4n-1}-bundles over S^{4n} with structure group $SO(4n)$. Equivalence classes of such bundles are in one to one correspondence with $\pi_{4n-1}(SO(4n)) \cong \mathbb{Z} \oplus \mathbb{Z}$ (see [Ste51, Theorem 18.5]). Let $\sigma : S^{4n-1} \to SO(4n)$ be defined by

$$\sigma(x)y := xy,$$

and $\rho : S^{4n-1} \to SO(4n-1) \subset SO(4n)$ by

$$\rho(x)y := xyx^{-1},$$

where $x \in S^{4n-1}$ is interpreted as a unit quaternion if $n = 1$ and a unit octonion if $n = 2$, with the corresponding multiplication. Then it can be shown that $\{[\sigma], [\rho]\}$ is a free generating set of $\pi_{4n-1}(SO(4n))$. Let $M^{8n-1}_{k,l}$ be the total space of the S^{4n-1}-bundle over S^{4n} determined by $k[\rho] + l[\sigma] \in \pi_{n-1}(SO(4n))$ and π_S its projection map. Hence, $M^{8n-1}_{k,l}$ can be identified with the quotient

$$D^{4n} \times S^{4n-1} \sqcup D^{4n} \times S^{4n-1} / \sim \tag{5.1}$$

where $(x, f(x)y) \sim (x, y) \in S^{4n-1} \times S^{4n-1}$ for the clutching function

$$f : S^{4n-1} \to SO(4n) : x \to (y \mapsto x^{k+l} y x^{-k}).$$

Let $D^{4n} \to W^{8n}_{k,l} \xrightarrow{\pi_D} S^{4n}$ be the associated disk bundle and denote by $\xi_{k,l}$ the associated vector bundle $\mathbb{R}^{4n} \to E_{k,l} \xrightarrow{\pi_E} S^{4n}$.

Since $W^{8n}_{k,l} \simeq S^{4n}$ we have $H^{4n}(W^{8n}_{k,l}; \mathbb{Z}) \cong \mathbb{Z}$. We orient $W^{8n}_{k,l}$ in such a way that $\operatorname{sign}(W^{8n}_{k,l}) = 1$ (see §2.3.3) and fix the induced orientation on the boundary $M^{8n-1}_{k,l}$.

Remark 5.1.1. *Note that a change of orientation of the base leads to a diffeomorphism $M^{8n-1}_{k,l} \cong -M^{8n-1}_{-k,-l}$, whereas a change of orientation in the fiber leads to $M^{8n-1}_{k,l} \cong -M^{8n-1}_{-k-l,l}$. Hence $M^{8n-1}_{k,-l} \cong M^{8n-1}_{k-l,l}$ and we can therefore focus on $l \geq 0$ from now on.*

We summarize some properties of these bundles and spaces in the following (see [CE03] and [Gre12]).

Theorem 5.1.2.

1. *The Euler class of $\xi_{k,l}$ is $e(\xi_{k,l}) = l\alpha \in H^{4n}(S^{4n}; \mathbb{Z})$.*
2. *The integer cohomology groups of $M^{8n-1}_{k,l}$ are*

$$\begin{aligned} & H^0(M^{8n-1}_{k,l}; \mathbb{Z}) \cong H^{8n-1}(M^{8n-1}_{k,l}; \mathbb{Z}) \cong \mathbb{Z}, \\ \textit{if } l \neq 0: \quad & H^{4n}(M^{8n-1}_{k,l}; \mathbb{Z}) \cong \mathbb{Z}_l, \\ \textit{if } l = 0: \quad & H^{4n-1}(M^{8n-1}_{k,0}; \mathbb{Z}) \cong H^{4n}(M^{8n-1}_{k,0}; \mathbb{Z}) \cong \mathbb{Z}, \\ & H^j(M^{8n-1}_{k,l}; \mathbb{Z}) = 0 \textit{ otherwise.} \end{aligned}$$

3. *Both $W^{8n}_{k,l}$ and $M^{8n-1}_{k,l}$ are spin and both have a unique Spin structure.*
4. *The only non-trivial Pontrjagin classes of $\xi_{k,l}$, $W^{8n}_{k,l}$ and $M^{8n-1}_{k,l}$ are*

$$\begin{aligned} p_n(\xi_{k,l}) &= (4n-2)(2k+l)\alpha \in H^{4n}(S^{4n}; \mathbb{Z}) \\ p_n(W^{8n}_{k,l}) &= (4n-2)(2k+l)\pi_D^*(\alpha) \in H^{4n}(W^{8n}_{k,l}; \mathbb{Z}) \\ p_n(M^{8n-1}_{k,l}) &= (4n-2)2k\pi_S^*(\alpha) \in H^{4n}(M^{8n-1}_{k,l}; \mathbb{Z}) \end{aligned}$$

   respectively.

Since $H^{4n}(M^{8n-1}_{k,l};\mathbb{Z}) \cong \mathbb{Z}_l$, it follows that $M^{8n-1}_{k,l}$ and $M^{8n-1}_{k',l'}$ cannot be homotopy equivalent if $l \neq l'$ (hence they cannot be diffeomorphic).

If $l = 0$, we see that $M^{8n-1}_{k,0}$ is not a rational homology sphere. As we will see, our argument depends on the fact that the above cohomology group with rational coefficients vanishes so that we can pull back the Pontryagin classes from the cohomology group of the disk bundle to the relative cohomology group. Hence, this case will be excluded from now on.

If $l = 1$, it was proved by Milnor [Mil56] for $n = 1$ and by Shimada [Shi57] for $n = 2$ that $M^{8n-1}_k := M^{8n-1}_{k,1}$ is homeomorphic but not always diffeomorphic to the standard $(8n-1)$-sphere[1]. Consequently, M^{8n-1}_k will be called a *Milnor sphere* if $n = 1$ and a *Shimada sphere* if $n = 2$.

Now consider the involution τ on M^{8n-1}_k which is induced by the fiberwise antipodal map on S^{4n-1}. Indeed, the antipodal map commutes with the action of the structure group $SO(4n)$ on the fibers and thus induces an action on the total space M^{8n-1}_k (see [Bre72, p. II.1.1]). Equivalently, τ is the map induced by $(x, y) \mapsto (x, -y)$ on $(x, y) \in D^{4n} \times S^{4n-1}$ from Equation (5.1), when descending to the quotient. For $n = 1$, the pair (M^7_k, τ) is called a *Hirsch-Milnor involution*. For both $n = 1, 2$, this involution is smooth, orientation preserving and fixed-point free. The quotient space $Q^{8n-1}_k := M^{8n-1}_k/\tau$ is homotopy equivalent to $\mathbb{RP}^{8n-1}$ (see [Bro67, (3.1) Proposition]) and will be called a *Milnor projective space* if $n = 1$ and a *Shimada projective space* if $n = 2$. Since being spin is a homotopy invariant (see [LM89, p.86-87]), it follows that Q^{8n-1}_k is spin for both $n = 1, 2$ and all k.

We also denote the involution induced by fiberwise antipodal maps on W^{8n}_k by τ. The fixed point set of this involution is the zero-section $S_0 \cong S^{4n}$.

Remark 5.1.3. *Suppose that W^{8n}_k is equipped with a τ-invariant metric which is of product form near the boundary M^{8n-1}_k. The fixed point set has even codimension in W^{8n}_k, and so by [AB68, Proposition 8.46] it follows in their terminology that τ is of* even *type. This means that the group action induced by $\mathbb{Z}_2 = \{Id, \tau\}$ lifts to a $\mathbb{Z}_2$-action on the Spin structure on W^{8n}_k (i.e. τ preserves the Spin structure), the complex spinor bundle and its space of sections which commutes with the Spin^+ Dirac operator D^+_W. Hence, the Spin structure on M^{8n-1}_k descends to a Spin structure on Q^{8n-1}_k (see Appendix A) and its Spin Dirac operator D_M (which is the restriction of D^+_W to the boundary) commutes with the induced action of τ on sections of the spinor bundle and thus descends to a Spin Dirac operator D_Q on Q^{8n-1}_k.*

[1]Note that Milnor [Mil56] and Shimada [Shi57] use different generators of $\pi_{4n-1}(SO(4n))$.

5.2 Diffeomorphism classification of sphere bundles over spheres

Let $M = M^{8n-1}_{k,l}$ be the total space of a S^{4n-1}-bundles over S^{4n} for $n = 1, 2$ and $W = W^{8n}_{k,l}$ the total space of the corresponding disk bundle. Let $x = \pi^*_D(\alpha) \in H^{4n}(W;\mathbb{Z}) \cong \mathbb{Z}$ and let y be a generator of $H^{4n}(W, M;\mathbb{Z}) \cong \mathbb{Z}$ such that $j^*(y) = lx$ where $j^* : H^{4n}(W, M;\mathbb{Z}) \to H^{4n}(W;\mathbb{Z})$ is the homomorphism from the long exact sequence of the pair. Hence $(j^*)^{-1}(x) = \frac{1}{l}y$ and from now on, we are dealing with rational coefficients. By the choice of the orientation on W (remember that $\mathrm{sign}(W) = 1$), we have $\langle y^2, [W, M]\rangle = l$. Therefore, using Theorem 5.1.2, we can compute

$$\langle \overline{p}_n^2(W^{8n}_{k,l}), [W^{8n}_{k,l}, M^{8n-1}_{k,l}]\rangle = (4n-2)^2\frac{(2k+l)^2}{l}, \tag{5.2}$$

where $\overline{p}_n(W^{8n}_{k,l}) = (j^*)^{-1}(p_n(W^{8n}_{k,l}))$. Using Equation (3.2), we obtain the following (see also p.57).

Lemma 5.2.1. *The Eells-Kuiper invariant of $M^{8n-1}_{k,l}$ is given by*

$$\mu(M^{8n-1}_{k,l}) \equiv \frac{1}{2^{4n-2}q_n}\frac{(2k+l)^2-l}{8l} \bmod 1,$$

where $q_1 = 7$ and $q_2 = 127$.

We can now present the diffeomorphism classification of the sphere bundles.

Theorem 5.2.2. *[CE03, Theorem 1.5][Gre12, Theorem 3.8.3] Let $M^{8n-1}_{k,l}$ and $M^{8n-1}_{k',l}$ be the total spaces of two S^{4n-1}-bundles over S^{4n} for $n = 1, 2$, $l > 0$. Then $M^{8n-1}_{k,l}$ is orientation preserving diffeomorphic to $M^{8n-1}_{k',l}$ if and only if*

$$\mu(M^{8n-1}_{k,l}) = \mu(M^{8n-1}_{k',l}) \quad \textit{and}$$
$$2k \equiv 2\gamma k' \bmod l$$

for some γ satisfying $\gamma^2 \equiv 1 \bmod l$.

In both cases, the proof makes use of the classification of highly connected manifolds in dimensions 7 and 15 from Crowley's PhD thesis [Cro02]. See [Cro02, Chapter 1] for an overview of this classification.

From the above classification theorem, we can immediately conclude the following.

Corollary 5.2.3. *Let $n = 1, 2$. For each total space $M^{8n-1}_{k,l}$ of an S^{4n-1}-bundle over S^{4n}, the set $\{M^{8n-1}_{k',l}\}_{m\in\mathbb{Z}}$, $k' = k + 2^{4n-2}lm \cdot q_n$, $q_1 = 7$ and $q_2 = 127$, is an infinite family of manifolds all orientation preserving diffeomorphic to $M^{8n-1}_{k,l}$.*

With the help of some modular arithmetic, one can deduce the number of different diffeomorphism types in case $l = 1$.

Corollary 5.2.4. *[EK62, §6. and 9.] There are 16 different oriented diffeomorphism types of Milnor spheres and 4096 different oriented diffeomorphism types of Shimada spheres.*

5.3 Diffeomorphism classification of quotients

Let M_k^{8n-1} be either a Milnor or Shimada sphere and consider the involution τ induced by fiberwise antipodal maps.

5.3.1 Browder-Livesay invariant of involution on Milnor and Shimada spheres

We begin by computing the Browder-Livesay invariant of (M_k^{8n-1}, τ). See §3.4 for the definition of the invariant and what it means for an involution to (doubly) desuspend.

Theorem 5.3.1. *For each $k \in \mathbb{Z}$, the involution τ on M_k^{8n-1} doubly desuspends if $n = 1$ and desuspends if $n = 2$. Hence, in particular $\sigma(M_k^{8n-1}, \tau) = 0$ for both $n = 1, 2$.*

Proof. For $n = 1$, the proof is due to Hirsch and Milnor (see [HM64, Lemma 1]). As we will see, the same argument applies to $n = 2$.

We use the following description of M_k^7 from [HM64].

$$M_k^7 = \mathbb{R}^4 \times S^3 \cup_\phi \mathbb{R}^4 \times S^3$$

where $\phi : \mathbb{R}^4 \setminus \{0\} \times S^3 \to \mathbb{R}^4 \setminus \{0\} \times S^3$ is the diffeomorphism

$$\phi(u, v) = \Big(\frac{u}{\|u\|^2}, \frac{u^{k+1} v u^{-k}}{\|u\|} \Big) = (u', v').$$

Let $g : M_k^7 \to \mathbb{R}$ be defined by

$$g([u, v]) = \frac{\mathrm{Re}(uv)}{\sqrt{1 + \|u\|^2}}, \qquad g([u', v']) = \frac{\mathrm{Re}(v')}{\sqrt{1 + \|u'\|^2}},$$

where $\mathrm{Re}(-)$ denotes the real part of the quaternion. This map is well-defined, smooth and only has two non-degenerate critical points. From the Morse lemma (see [Mil63, Lemma 2.2] for example), it follows that $D_0^7 := g^{-1}([0, \infty))$ is diffeomorphic to the standard 7-disk D^7 and therefore its boundary $S_0^6 := g^{-1}(0) = \partial D_0^7$ is diffeomorphic to the standard 6-sphere. Note that $\tau(S_0^6) = S_0^6$ and therefore S_0^6 is a characteristic submanifold of (M_k^7, τ), proving that τ desuspends.

Similarly, consider the function $f : S_0^6 \to \mathbb{R}$ defined by

$$f([u, v]) = \frac{\mathrm{Re}(v)}{\sqrt{1 + \|u\|^2}}, \qquad f([u', v']) = \frac{\mathrm{Re}(u'(v')^{-1})}{\sqrt{1 + \|u'\|^2}}.$$

By the same argument as above, one shows that $S_0^5 := f^{-1}(0)$ is diffeomorphic to S^5 and invariant under τ. Thus, the involution τ doubly desuspends.

As for M_k^{15}, we use the explicit description of Shimada [Shi57]. Consider $S^8 = \{(s, \sigma) \in \mathbb{O} \times \mathbb{R} \mid \|s\|^2 + (\sigma - \frac{1}{2})^2 = \frac{1}{4}, 0 \le \sigma \le 1\} \subset \mathbb{R}^9$, where $\mathbb{O} \cong \mathbb{R}^8$ denotes the octonions. Let $V_1 = S^8 \setminus \{(0, 0)\}$ and $V_0 = S^8 \setminus \{(0, 1)\}$. Then

$$M_k^{15} = V_1 \times S^7 \cup_\psi V_0 \times S^7$$

where $\psi : (V_1 \cap V_0) \times S^7 \to (V_1 \cap V_0) \times S^7$ is the diffeomorphism

$$\psi((s, \sigma, t)_1) = \Big(s, \sigma, \frac{s^{k+1} t s^{-k}}{\|s\|}\Big) = (s, \sigma, t')_0.$$

Define $h : M_k^{15} \to \mathbb{R}$ by

$$h([s, \sigma, t]) = \sqrt{\sigma}\mathrm{Re}(t), \qquad h([s, \sigma, t']) = \frac{\mathrm{Re}(\bar{s}t')}{\sqrt{1 - \sigma}}.$$

Then h has two non-degenerate critical points $(0, 1, \pm 1)$. Therefore, as above, it follows from Morse theory that $S_0^{14} := h^{-1}(0)$ is diffeomorphic to the standard 14-sphere. It is easy to see that S_0^{14} is invariant under τ and therefore this involution desuspends.

Finally, the last statement follows by Theorem 3.4.4.

□

Remark 5.3.2. *Hirsch and Milnor show in the case of M_3^7 that S_0^5/τ is not diffeomorphic to $\mathbb{R}\mathrm{P}^5$ (see [HM64, Lemma 2]). In [Kam81, Corollary 5.4.11], Kamishima associates these quotients to the Brieskorn quotients $Q_0^5(d)$ where d is odd (see also Lemma 5.3.4).*

5.3.2 Normal invariants of Milnor projective spaces

Next, we determine the normal invariants of Milnor projective spaces $Q_k^7 := M_k^7/\tau$.

Consider the following commuting diagram (see [Med71, p. IV.3.2]).

$$\begin{tikzcd}
hS(\mathbb{R}\mathrm{P}^5) \approx hT(\mathbb{R}\mathrm{P}^5) \arrow[r, "\beta_5", "\approx"'] \arrow[d, "\Sigma_5"', "\approx"] & {[\mathbb{R}\mathrm{P}^5, G/PL] \approx [\mathbb{R}\mathrm{P}^5, G/O]} \\
hS(\mathbb{R}\mathrm{P}^6) \approx hT(\mathbb{R}\mathrm{P}^6) \arrow[r, "\beta_6"] \arrow[d, "\Sigma_6"', "(inj.)"] & {[\mathbb{R}\mathrm{P}^6, G/PL] \approx [\mathbb{R}\mathrm{P}^6, G/O]} \arrow[u, "\iota_5"', "(surj.)"] \\
hT(\mathbb{R}\mathrm{P}^7) \arrow[r, "\beta_7"] & {[\mathbb{R}\mathrm{P}^7, G/PL] \approx [\mathbb{R}\mathrm{P}^7, G/O]} \arrow[u, "\iota_6"', "\approx"]
\end{tikzcd}$$

For $k = 5, 6$, Σ_k is a map induced by suspension, ι_k denotes the map induced by the inclusion $\mathbb{R}\mathrm{P}^k \hookrightarrow \mathbb{R}\mathrm{P}^{k+1}$ and $\beta_k : hT(\mathbb{R}\mathrm{P}^k) \to [\mathbb{R}\mathrm{P}^k, G/PL]$ is the map from Remark 3.2.2. For the injectivity and surjectivity of the above maps, see [Med71, IV.3.2, p.69 and IV.3.4 Theorem]. In partiacular, $hS(\mathbb{R}\mathrm{P}^5) \approx hT(\mathbb{R}\mathrm{P}^5) \approx [\mathbb{R}\mathrm{P}^5, G/PL] \approx \mathbb{Z}_4$ and $[\mathbb{R}\mathrm{P}^6, G/PL] \approx \mathbb{Z}_4 \oplus \mathbb{Z}_2$, where $\approx$ stands for bijective. Also, since PL/O is 6-connected and $\pi_7(G/O) = 0$ (see [MM79, Remark 4.21] and [Sul96]), by obstruction theory it follows that $[\mathbb{R}\mathrm{P}^k, G/O] \approx [\mathbb{R}\mathrm{P}^k, G/PL]$ for $k = 5, 6, 7$.

Hence, to get an understanding of the normal invariants of Milnor projective spaces, we can study the classification of smooth structures of $\mathbb{R}\mathrm{P}^5$, which correspond to the different oriented diffeomorphism types on homotopy $\mathbb{R}\mathrm{P}^5$'s.

Let $M_0^5(d)$ be a Brieskorn sphere (i.e. d is odd), τ the involution from §4.1.1 and $Q_0^5(d) := M_0^5(d)/\tau$ the resulting homotopy $\mathbb{R}\mathrm{P}^5$. We then have the following result.

Theorem 5.3.3. *[Kam81, Lemma 5.5.1. and (5.5.2)] The smooth structure set $hS(\mathbb{R}\mathrm{P}^5) \approx \mathbb{Z}_4$ is in one-to-one correspondence with $\{Q_0^5(d)\}_{d=1,3,5,7}$. Furthermore, $Q_0^5(d)$ is diffeomorphic to $Q_0^5(d+16i)$ and $Q_0^5(-d+16j)$ for $i, j \in \mathbb{Z}$ (provided $d + 16i \geq 0$ and $-d + 16j \geq 0$).*

Note that by the above diagram, the double desuspension of a homotopy $\mathbb{R}\mathrm{P}^7$ is unique. Kamishima showed that the double desuspension of Hirsch-Milnor involutions can be classified as follows.

Lemma 5.3.4. *[Kam81, Corollary 5.4.11] The double desuspension of (M_k^7, τ) is equivalent to $(M_0^5(2k-1), \tau)$ for $k > 0$ and to $(M_0^5(-2k+1), \tau)$ for $k \leq 0$.*

Let $\mathcal{N}_\alpha(Q^7)$ be the restriction of $\mathrm{Im}(\alpha_7)$ to Milnor projective spaces, where $\alpha_7 : hS(\mathbb{R}\mathrm{P}^7) \to [\mathbb{R}\mathrm{P}^7, G/O]$ is the map defined above Theorem 3.2.1. This corresponds to the set of normal invariants of Milnor projective spaces.

Lemma 5.3.5. *The map*

$$\begin{aligned} \Pi : \mathcal{N}_\alpha(Q^7) &\to hS(\mathbb{R}\mathrm{P}^5) \\ \alpha_7(Q_k^7) &\mapsto [Q_0^5(d)] \end{aligned}$$

is a bijection, where $2k-1 \equiv \pm d \bmod 16$ if $k > 0$ and $-2k+1 \equiv \pm d \bmod 16$ if $k \leq 0$.

Proof. Let $k_0, k_1 \in \mathbb{Z}$. As one sees from the above diagram, if $Q_{k_0}^7$ and $Q_{k_1}^7$ are normally cobordant, then their double desuspensions will be normally cobordant as well. By Theorem 5.3.3, Lemma 5.3.4 and since $\alpha_5 : hS(\mathbb{R}\mathrm{P}^5) \to [\mathbb{R}\mathrm{P}^5, G/O]$ is bijective, it follows that Π is well-defined. Taking for example $k = 1, 2, 3, 4$, we see that Π is surjective by Lemma 5.3.4. It has been shown that $\mathrm{Im}(\alpha_7) \approx \mathbb{Z}_4$ (see proof of [Med71, V.6 Theorem]) and therefore it follows that Π is bijective.

□

In fact, the normal invariant of a Milnor projective space only depends on the Eells-Kuiper invariant of its covering Milnor sphere.

Proposition 5.3.6. *The map*

$$\begin{aligned}\chi : \mathcal{N}_\alpha(Q^7) &\to \mathbb{Z}_4 \\ \alpha_7(Q^7_k) &\mapsto 28\mu(M^7_k) \bmod 4\end{aligned}$$

is a bijection, where we take $28\mu(M^7_k) \in \{0, 1, 2, ..., 27\}$.

Proof. Let $k_1, k_2 \in \mathbb{Z}$. If $\alpha_7(Q^7_{k_1}) = \alpha_7(Q^7_{k_2})$, then by Lemma 5.3.5 we have $(2k_1 - 1) \equiv \pm(2k_2 - 1) \bmod 16$. A quick computation then shows that $28\mu(M^7_{k_1}) \equiv 28\mu(M^7_{k_2}) \bmod 4$ and thus χ is well-defined. Surjectivity of χ is immediate (take for example $k = 1, 2, 3, 4$). Since by Lemma 5.3.5 the set $\mathcal{N}_\alpha(Q^7)$ has four elements, it follows that χ is bijective. □

Note that this result can also be deduced from the work of Mayer [May70].

5.3.3 Eells-Kuiper invariant of the Milnor and Shimada projective spaces

To complete the diffeomorphism classification of Milnor projective spaces, one can compute their Eells-Kuiper invariant. The computation was carried out by Tang and Zhang [TZ14] and we can apply the same argument to Shimada projective spaces.

Lemma 5.3.7. *The Eells-Kuiper invariant of* Q^{8n-1}_k*,* $n = 1, 2$*, is given by*

$$\mu(Q^{8n-1}_k) \equiv \Big(\frac{k(k+1)}{2^{4n} \cdot q_n} \pm \frac{(2k+1)}{2^{4n+1}}\Big) \bmod 1, \tag{5.3}$$

where $q_1 = 7$ *for Milnor projective spaces and* $q_2 = 127$ *for Shimada projective spaces.*

Proof. We can apply Theorem 3.1.3 to Q^{8n-1}_k for both $n = 1, 2$. Let D^+_W, D_M and D_Q denote the corresponding Dirac operators on $W := W^{8n}_k$, $M := M^{8n-1}_k$ and $Q := Q^{8n-1}_k$ from Remark 5.1.3. If B^{ev}_Q the odd signature operator of Q, then

$$\mu(Q) \equiv \frac{\eta(D_Q) + h(D_Q)}{2} - t_{2n}\eta(B^{ev}_Q) - c_{2n}\int_Q p_n(Q) \wedge \hat{p}_n(Q) \bmod 1$$

where $t_2 = -1/(2^5 \cdot 7)$, $c_2 = 1/(2^7 \cdot 7)$, $t_4 = -1/(2^9 \cdot 127)$ and $c_4 = 1/(2^{11} \cdot 3^2 \cdot 127)$. Remember that $\hat{p}_n(Q)$ is a $(4n-1)$-form satisfying $d\hat{p}_n(Q) = p_n(Q)$. Applying Theorem 2.3.18 to the covering $\pi : M \to Q$ with the trivial representation of $\pi_1(Q)$, we get

$$\eta(D_Q) = \frac{1}{2}\Big(\eta(D_M) + \eta_\tau(D_M)\Big),$$
$$\eta(B_Q^{ev}) = \frac{1}{2}\Big(\eta(B_M^{ev}) + \eta_\tau(B_M^{ev})\Big),$$

where B_M^{ev} is the lifted odd signature operator on M. Remember that the fixed point set of the action of τ on $W := W_k^{8n}$ is the zero-section S^{4n}. By Theorem 2.3.7, 2.3.8 and the above, we therefore obtain

$$\eta(D_Q) = -\mathrm{index}(D_W^+) + \int_W \hat{A}(p) - \frac{h(D_M)}{2} - \mathrm{index}(D_W^+, \tau) + a_{spin}(S^{4n}) - \frac{h_\tau(D_M)}{2}.$$

Similarly, by Theorem 2.3.16, 2.3.17 and the above,

$$\eta(B_Q^{ev}) = \frac{1}{2}\Big(- \mathrm{sign}(W) + \int_W L(p) - \mathrm{sign}(W,\tau) + a_{sign}(S^{4n})\Big).$$

Now, as we will see in Chapter 6, W can be equipped with a metric of non-negative scalar curvature everywhere, positive scalar curvature on the boundary M and which is of product form near the boundary (see Theorem 6.1.2 and 6.2.4). Therefore, it follows by the vanishing theorem 2.3.20 that $\mathrm{index}(D_W^+)$, $\mathrm{index}(D_W^+, \tau)$, $h(D_M)$ and $h_\tau(D_M)$ all vanish. By the definition of D_Q (see Remark 5.1.3), this implies that $h(D_Q)$ vanishes as well.

We know that $\mathrm{sign}(W) = 1$ (see §5.1). Since S^{4n} is the fixed point set of the action of τ on W, τ preserves the generator of $H^{4n}(S^{4n};\mathbb{Z})$, which is isomorphic to $H^{4n}(W;\mathbb{Z})$ via the bundle projection π_D^*, and thus we have $\mathrm{sign}(W,\tau) = 1$.

Let ν_k be the normal bundle of the zero section S^{4n} in W. Then $\nu_k \cong \xi_k$, where ξ_k is the vector bundle associated to the S^{4n-1}-bundle over S^{4n}. By Equation (2.10), identifying $N = S^{4n}$ and $\nu(\pi) = \xi_k$, we have

$$a_{spin}(S^{4n}) = \pm \int_{S^{4n}} \hat{A}_\pi(\xi_k),$$

since $\hat{A}(S^{4n}) = 1$. The sign depends on the action of τ on the $Spin$ structure. By Equation (2.8), we have

$$\hat{A}_\pi(\xi_k) = \frac{1}{(2i)^{2n}} \prod_{j=1}^{2n} \frac{1}{\cosh(y_j/2)}$$

where y_j are the formal roots of ξ_k (see §2.3.1). Since we integrate over S^{4n}, we only need to focus on the term of degree $4n$ in the series expansion of the above expression. Thus, using $\frac{1}{\cosh(x/2)} = 1 - \frac{x^2}{8} + \frac{5x^4}{384} - ...$ and $p_n = \sigma_n(y_1^2, ..., y_{2n}^2)$ (recall that σ_n denotes an elementary symmetric polynomial), it follows that

$$a_{spin}(S^4) = \pm\frac{1}{2^5}\int_{S^{4n}} p_1(\xi_k), \qquad a_{spin}(S^8) = \pm\int_{S^8}\Big(\frac{5}{2^{11}\cdot 3}p_1^2(\xi_k) - \frac{1}{2^9\cdot 3}p_2(\xi_k)\Big),$$

so that, using Theorem 5.1.2.4, we obtain

$$a_{spin}(S^{4n}) = \pm \frac{(2k+1)}{2^{4n}}.$$

Similarly, by Equation (2.22) and Theorem 5.1.2.1,

$$a_{sign}(S^{4n}) = \int_{S^{4n}} e(\xi_k) = 1.$$

We also have

$$\int_Q p_n(Q) \wedge \hat{p}_n(Q) = \frac{1}{2} \int_M p_n(M) \wedge \hat{p}_n(M)$$

since $\int_M \pi^*(\omega) = deg(\pi) \int_Q \omega$ for any $(8n-1)$-form ω. Finally, by [KS93, Lemma 2.7],

$$\int_W p_n(W) \wedge p_n(W) - \int_M p_n(M) \wedge \hat{p}_n(M) = \langle \overline{p}_n^2(W), [W, M] \rangle.$$

Applying Equation (5.2) and putting all of the above together, the result now follows. $\square$

Remark 5.3.8. *Observe that the sign in Equation (5.3) depends on the choice of the Spin structure on Q_k^{8n-1} (see [Mil65, p.58]). Indeed, we have $H^1(Q_k^{8n-1}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and so there are two different Spin structures on Q_k^{8n-1}. The Eells-Kuiper invariant $\mu(Q_k^{8n-1})$ therefore has to be interpreted as a pair of values, not as a singular value.*

To summarize, we have the following Eells-Kuiper invariants.

$$\mu(M_k^7) \equiv \frac{k(k+1)}{56} \bmod 1, \qquad \mu(Q_k^7) \equiv \frac{k(k+1)}{112} \pm \frac{2k+1}{32} \bmod 1,$$

$$\mu(M_k^{15}) \equiv \frac{k(k+1)}{16256} \bmod 1, \qquad \mu(Q_k^{15}) \equiv \frac{k(k+1)}{32512} \pm \frac{2k+1}{512} \bmod 1.$$

Proposition 5.3.9. *Let $Q_{k_i}^7 = M_{k_i}^7/\tau$ for $i = 0, 1$. Then $\mu(M_{k_0}^7) = \mu(M_{k_1}^7)$ implies $\mu(Q_{k_0}^7) = \mu(Q_{k_1}^7)$.*

Proof. Observe that $\mu(M_k^7) = \mu(M_{k+56m}^7)$ and $\mu(Q_k^7) = \mu(Q_{k+56m}^7)$ for $m \in \mathbb{Z}$. The result now follows by computing and comparing the different Eells-Kuiper invariants for $k = 0, 1, ..., 55$. $\square$

Proposition 5.3.10. *Let $Q_{k_i}^{15} = M_{k_i}^{15}/\tau$ for $i = 0, 1$. Then $\mu(M_{k_0}^{15}) = \mu(M_{k_1}^{15})$ implies $\mu(Q_{k_0}^{15}) = \mu(Q_{k_1}^{15})$. Furthermore, there are 4096 different pairs of values for $\mu(Q_k^{15})$.*

Proof. This is achieved through use of the C++ code in the Appendix B. $\square$

5.3.4 Classification of Milnor projective spaces

Theorem 5.3.11. *Let $Q^7_{k_i} = M^7_{k_i}/\tau$ be a Milnor projective space for $k_i \in \mathbb{Z}$, $i = 0, 1$. If $M^7_{k_0}$ is diffeomorphic to $M^7_{k_1}$, then $Q^7_{k_0}$ is diffeomorphic to $Q^7_{k_1}$.*

Proof. If $M^7_{k_0}$ is diffeomorphic to $M^7_{k_1}$, then $28\mu(M^7_{k_0}) = 28\mu(M^7_{k_1})$ and therefore by Proposition 5.3.6 their normal invariants are equal: $\alpha_7(Q^7_{k_0}) = \alpha_7(Q^7_{k_1})$. By Theorem 5.3.1, the Browder-Livesay invariant is $\sigma(M^7_k, \tau) = 0$ for all $k \in \mathbb{Z}$, hence by Theorem 3.4.5 it follows that $Q^7_{k_0}$ is diffeomorphic to $Q^7_{k_1} \# \Sigma^7$ for some sphere $\Sigma^7 \in bP_8$. Suppose $\mu(\Sigma^7) = \frac{l}{28} \bmod 1$, where $0 \le l < 28$ is an integer. By the properties of the Eells-Kuiper invariant (see Proposition 3.1.2), we have

$$\mu(Q^7_{k_0}) = \mu(Q^7_{k_1} \# \Sigma^7) \equiv \mu(Q^7_{k_1}) + \frac{l}{28} \bmod 1.$$

Proposition 5.3.9 now implies that $l = 0$. Therefore $\Sigma^7 \cong S^7$ and finally $Q^7_{k_1} \# \Sigma^7 \cong Q^7_{k_1}$. □

Theorem 5.3.12. *There are 16 different (oriented) diffeomorphism classes of Milnor projective spaces. All of the 16 diffeomorphism types can be realized by an infinite family of such quotients.*

Proof. The first statement follows from Corollary 5.2.4. The second statement follows from Theorem 5.2.2, 5.3.11 and Corollary 5.2.3. □

5.3.5 Diffeomorphism finiteness of Shimada projective spaces

As of the time of writing, the normal invariants of Shimada projective spaces are still unknown. Hence, the best we can do is to give a finiteness result and a lower limit for the number of diffeomorphism types of Shimada projective spaces.

Lemma 5.3.13. *There are only finitely many different oriented diffeomorphism types of Shimada projective spaces.*

Proof. If a_i is the order of $\pi_i(G/O)$ and b_i is the order of $\pi_i(G/O) \otimes \mathbb{Z}_2$, then it can be shown that the order of $[\mathbb{R}\mathrm{P}^{15}, G/O]$ is less than or equal to $\prod_{i=1}^{14} a_{15} b_i$ (see [Med71, p. V.1]). Now, since $a_{15} = 2$ and b_i is finite for all i (see [Sul96]), it follows that the order of $[\mathbb{R}\mathrm{P}^{15}, G/O]$ is finite. In particular, there are only finitely many distinct normal invariants that Shimada projective spaces can attain.

By Theorem 5.3.1 the Browder-Livesay invariant of every Shimada projective space vanishes. Therefore, by the above and Theorem 3.4.5, the result follows since $|bP_{16}| = 8128$ is finite. □

Proposition 5.3.14. *There are finitely many, but at least 4096 different oriented diffeomorphism types of Shimada projective spheres which can be realized by an infinite family of orientation preserving diffeomorphic manifolds.*

Proof. The first statement follows from Lemma 5.3.13. The second statement follows from Proposition 5.3.10. The last statement now follows by considering $\{Q^{15}_{k+130048m}\}_{m\in\mathbb{Z}}$, which all have the same Eells-Kuiper invariant as Q^{15}_k for any $k \in \mathbb{Z}$ (see Lemma 5.3.7). $\square$

Chapter 6

Riemannian geometry

The main objective of this chapter is to construct metrics of non-negative sectional curvature on the 5-manifolds and the Milnor projective spaces, as well as positive Ricci curvature metrics on the S^7-bundles over S^8 and the Shimada projective spaces. The metrics on the quotients should lift to positive scalar curvature metrics on the corresponding universal coverings and extend to the coboundary, such that it is of product form near the boundary.

The major construction which yields metrics of non-negative sectional curvature is due to Grove and Ziller [GZ00]. We give a short overview of the results we need. With this, we quickly get the desired metrics on Milnor projective spaces, the covering Milnor sphere and the corresponding disk bundle. In the case of the Brieskorn quotients, it is a bit more complicated. As we have explained in the introduction, we need to get away from the singularity in $W_0^6(d)$ to be able to compute the relative eta-invariants. In order to equip all the involved spaces with a suitable metric, we will use Cheeger deformations. First, this will allow us to obtain a metric which simultaneously has positive scalar and non-negative sectional curvature on $M_0^5(d)$ and $Q_0^5(d)$. It will then help us construct suitable metrics on $Q_\epsilon^5(d)$ and $W_\epsilon^6(d)$ for $\epsilon \neq 0$. After that, we give metrics of non-negative sectional curvature on the 5-manifolds which are the quotient of an isometric $S^1 \times \mathbb{Z}_2$ action on $S^3 \times S^3$. This is immediate by the Gray-O'Neill formula.

Next, we discuss positive Ricci curvature metrics. First we present a result by Böhm and Wilking, which will be needed in some of the proofs of the main theorems. More specifically, in the argument by contradiction, it will help us get a path of positive scalar curvature metrics on the manifold with endpoints the desired metrics. After that we give the construction of positive Ricci curvature metrics on our sphere bundles and finally on Shimada projective spaces. To end this chapter, we define the space and moduli space of Riemannian metrics and present a consequence of Ebin's slice theorem, which will be useful in the proofs of our main theorems.

6.0.1 Torpedo metrics

We first start by defining torpedo metrics, which are a special case of warped products (see [Pet16, pp.18]), and discuss some of their properties. They will be used to construct suitable metrics on disk bundles.

Let $f_\delta : (0,\infty) \to \mathbb{R}$ be a smooth function satisfying

1. $f_\delta(t) = \delta \sin(t/\delta)$ when t is near 0,
2. $f_\delta(t) = \delta$ when $t \geq \delta\pi/2$ and
3. $\ddot{f}_\delta(t) \leq 0$ for all t.

The function f_δ is known as a δ-*torpedo function*. By [Wal11, Lemma 2.3] (see also [Pet16, §1.4.4]), the metric $dr^2 + f_\delta(r)^2 ds^2_{n-1}$ on $(0,\infty) \times S^{n-1}$ extends smoothly to a metric of radius δ on $\mathbb{R}^n$. If we restrict this metric to $(0,b] \times S^{n-1}$ for some $b > \delta\pi/2$, then we obtain a so-called *torpedo metric* on the closed disk D^n. This metric is $O(n)$-invariant and is of product form near the boundary and restricts to the round metric of radius δ on $\partial D^n = S^{n-1}$. For $n > 2$, the metric has $scal \geq s > 0$ (where s can become arbitrarily large by choosing δ arbitrarily small) and for $n = 2$ it has $scal \geq 0$ (see [Wal11, §2.3]).

The precise value of the radius δ will often not be important in the following. Hence, we will usually simply denote a torpedo metric on D^n by g_{tor}.

6.1 Metrics of non-negative sectional curvature

6.1.1 Grove-Ziller metrics

Let M be a closed, connected smooth manifold and G a compact Lie group acting smoothly on M. Suppose the action of G on M is of cohomogeneity one, that is, the orbit space M/G is 1-dimensional. We then say that M is a *cohomogeneity one manifold*.

Let $\pi : M \to M/G$ be the quotient map, fix a G-invariant Riemannian metric g on M and assume that $M/G = I = [-1,1]$ is an interval. Consider a point $x_0 \in \pi^{-1}(0)$ and let $c : [-1,1] \to M$ the unique minimal geodesic with $c(0) = x_0$ and $\pi \circ c = Id_I$. Let $x_\pm = c(\pm 1)$. By the principal orbit theorem [AB15, Theorem 3.82], there are exactly two non-principal orbits $B_\pm = \pi^{-1}(\pm 1) = G \cdot x_\pm$ with corresponding isotropy groups $K_\pm = G_{x_\pm}$. Denote by $H = G_{x_0} = G_{c(t)}$, $-1 < t < 1$, the principal isotropy.

It follows from the slice theorem (sometimes also called the tubular neighborhood theorem [AB15, Theorem 3.57]) that

$$M \cong (G \times_{K_-} D^{l_-+1}) \cup_{G/H} (G \times_{K_+} D^{l_++1})$$

where $D^{l_\pm+1}$ is the so-called normal slice to $B_\pm$ at $x_\pm$. The exponent $l_\pm+1$ corresponds to the codimension of the singular orbits.

Theorem 6.1.1. *[GZ00, Theorem E] Let M be a closed, connected smooth manifold and G a compact Lie group acting smoothly and by cohomogeneity one on M. If the singular orbits are of codimension 2, then M admits a G-invariant Riemannian metric which has non-negative sectional curvature.*

6.1.2 Metrics on Milnor projective spaces

Consider principal $S^3 \times S^3$-bundles over S^4. These bundles are classified by elements of $\pi_3(SO(4)) = \mathbb{Z}\oplus\mathbb{Z}$. Let $P^{10}_{k,l}$ denote the total space corresponding to $[\alpha] \in \pi_3(SO(4))$ where[1] $\alpha(q)u = q^{k+l}uq^{-k}$.

The manifold $P^{10}_{k,l}$ admits a cohomogeneity one action by $S^3 \times S^3 \times S^3/\pm(1,1,1)$ with codimension 2 singular orbits [GZ00, Proposition 3.11]. By Theorem 6.1.1, it therefore admits an $S^3 \times S^3 \times S^3$-invariant metric with non-negative sectional curvature.

Let $S^3 \times S^3 \times 1$ be the subaction of $S^3 \times S^3 \times S^3$ corresponding to the principal $S^3 \times S^3$-bundle action on $P^{10}_{k,l}$. Note that $S^3 \times S^3 \times 1$ acts freely and isometrically on $P^{10}_{k,l}$ and that the quotient $P^*_{k,l} := P^{10}_{k,l}/(-1,-1)$ is the total space of the associated principal $SO(4)$-bundle over S^4. Let $S^3 \times S^3$ act on S^3 via $(q_1, q_2)\cdot v = q_1 v q_2^{-1}$, where quaternion multiplication is understood. Then $M^7_{k,l} := P^{10}_{k,l} \times_{S^3\times S^3} S^3$ is an S^3-bundle over S^4 with structure group $SO(4)$ and Euler class l. Hence, for $l = 1$, these spaces correspond to the Milnor spheres.

Theorem 6.1.2. *Fix $k \in \mathbb{Z}$ and let M^7_k be a Milnor sphere, W^8_k its associated disk bundle and $Q^7_k := M^7_k/\tau$ the associated Milnor projective space. Then M^7_k admits a metric that is simultaneously of non-negative sectional curvature and positive scalar curvature, which will be called its* Grove-Ziller metric *and denoted by $\tilde{g}^{GZ}_k$. This metric descends to a metric of non-negative sectional curvature on Q^7_k, which we likewise call its* Grove-Ziller metric *and denote by g^{GZ}_k. It satisfies $\tilde{g}^{GZ}_k = \pi_k^*(g^{GZ}_k)$, where $\pi_k : M^7_k \to Q^7_k$ is the projection. Furthermore, the Grove-Ziller metric on M^7_k extends to a metric h_k of non-negative sectional curvature on W^8_k which is of product form near the boundary.*

Proof. Equip $P^{10}_{k,1}$ with the metric of non-negative sectional curvature which is invariant under $S^3 \times S^3 \times S^3$ (Theorem 6.1.1). Give S^3 the round metric and equip $M^7_k := M^7_{k,1}$ with the metric which turns the projection $\pi : P^{10}_{k,1} \times S^3 \to M^7_k$ into a Riemannian submersion. This will be the Grove-Ziller metric $\tilde{g}^{GZ}_k$ of M^7_k.

[1] Note that Grove and Ziller [GZ00] use different indices.

By the Gray-O'Neill formula [Gra67][O'N66], this metric has non-negative sectional curvature. To show that it has positive scalar curvature, fix $z = (x, y) \in P^{10}_{k,1} \times S^3$. Then $\mathcal{V}_z := ker(d\pi_z) = T_z(S^3 \times S^3 \cdot z)$ is the vertical space and if $\mathcal{H}_z$ denotes the horizontal space with respect to the product metric on $P^{10}_{k,1} \times S^3$, it follows from a dimensional argument that the projection of $\mathcal{H}_z$ onto $T_y S^3$ is 3-dimensional. Since the tangent planes of S^3 all have the same constant positive sectional curvature, there is a plane $\mathcal{E} \subset \mathcal{H}_z$ that is of positive sectional curvature as well. Since π is a Riemannian submersion, i.e. the restriction $d\pi_z|_{\mathcal{H}_z} : \mathcal{H}_z \to T_{\pi(z)} M^7_k$ is an isometry, the plane $\mathcal{E}$ maps onto a plane of positive sectional curvature in $T_{\pi(z)} M^7_k$. Hence, the submersion metric on M^7_k has positive scalar curvature.

The involution τ at the end of §5.1 is induced by the action of $Id_P \times (-Id_{S^3})$ on $P^{10}_{k,1} \times S^3$. Therefore, the metric $\tilde{g}^{GZ}_k$ is invariant under τ and $Q^7_k := M^7_k/\tau$ inherits a metric g^{GZ}_k with non-negative sectional curvature, satisfying the required properties.

Finally, equip the closed disk D^4 with a torpedo metric (see §6.0.1). Then $W^8_k = P^*_{k,1} \times_{SO(4)} D^4$ inherits a submersion metric of non-negative sectional curvature. This metric is of product form near the boundary by the properties of the torpedo metric.

$\square$

Remark 6.1.3. *It is an open question whether $M^{15}_{k,l}$ admits a metric of non-negative sectional curvature for all $k, l \in \mathbb{Z}$. The above construction does not work in this case, essentially because S^8 does not admit a cohomogeneity one action with codimension 2 singular orbits (see [GZ00, p.350]).*

6.1.3 Cheeger deformation

In this subsection, we introduce some of the concepts related to Cheeger deformations we will need. This is by no means an extensive treatment, the interested reader is encouraged to consult for instance [AB15, §6.1] and [Zil09] for more details on this method.

Let G be a compact Lie group acting by isometries on a Riemannian manifold (M, g) and let q be a biinvariant metric on G. Equip $M \times G$ with the product metric $g + \frac{1}{t} q$, where $t \in \mathbb{R}_{>0}$. Let G act on $M \times G$ via $g_1 \cdot (x, g_2) = (g_1 \cdot x, g_1 \cdot g_2)$. This action is free and isometric, and the quotient $(M \times G)/G$ is diffeomorphic to M. The *Cheeger deformation* of g is the metric g_t on M which turns the projection

$$\pi_t : M \times G \to M : (x, a) \mapsto a^{-1} x$$

into a Riemannian submersion. The Lie group G acts by isometries on (M, g_t) for all $t > 0$. It can be shown that g_t varies smoothly in t and that it extends smoothly to $t = 0$ by setting $g_0 = g$ (see [AB15, Proposition 6.3]).

For $x \in M$, let G_x be the isotropy group and $\mathfrak{g}_x$ the corresponding Lie algebra. Denote by $\mathfrak{m}_x$ the q-orthogonal complement of $\mathfrak{g}_x$ in $\mathfrak{g}$, the Lie algebra of G. For $X \in \mathfrak{g}$ and $x \in M$, let

$$X^*_x = \frac{d}{dt}(\exp(tX) \cdot x)|_{t=0}$$

be the so-called *action field*. One can then identify the orthogonal complement with the tangent space of the G-orbit of x,

$$\mathfrak{m}_x \xrightarrow{\cong} T_x(Gx) : X \mapsto X^*.$$

Now, let $\mathcal{V}_x := T_x(Gx)$ and if we denote by $\mathcal{H}_x$ the orthogonal complement of $\mathcal{V}_x$ in T_xM with respect to g, we obtain a splitting $T_xM = \mathcal{V}_x \oplus \mathcal{H}_x$.

Consider the automorphism

$$P_x : \mathfrak{m}_x \to \mathfrak{m}_x$$

defined by the relation $q(P_x(X), Y) = g(X^*_x, Y^*_x)$ for all $Y \in \mathfrak{m}_x$.

For $X, Y \in \mathfrak{g}$, let $[X, Y]$ denote the Lie bracket of X and Y. Then one gets the following.

Proposition 6.1.4. *[DGA21, Proposition 5.2] Let (M, g) be a Riemannian manifold and G a compact Lie group acting on it by isometries. If $sec_g \geq 0$, then $sec_{g_t} \geq 0$ for all $t \geq 0$. If in addition either*

a) $scal_g > 0$, or

b) there exist $X, Y \in \mathfrak{m}_x$ such that $[P_x(X), P_x(Y)] \neq 0$ for all $x \in M$,

then $scal_{g_t} > 0$ for all $t > 0$.

Proposition 6.1.5. *[DGA21, Proposition 5.3] Let (M, g) be a Riemannian manifold, G a compact Lie group acting on it by isometries and $K \subset M$ a compact subset. Suppose that for all $x \in K$, there exist $X, Y \in \mathfrak{m}_x$ such that $[P_x(X), P_x(Y)] \neq 0$. Then there exists a $t_0 \geq 0$ such that the scalar curvature of (M, g_t) is positive on K for all $t > t_0$.*

6.1.4 Metrics on Brieskorn quotients

The following result is true for all d, but we will focus on d even (see also [DGA21, §5]).

Proposition 6.1.6. *The Brieskorn manifold $M^5_0(d)$ admits an $S^1 \times O(3)$-invariant metric with non-negative sectional curvature and positive scalar curvature. We will denote this metric by $\tilde{g}^{mGZ}_d$ and call it the* modified Grove-Ziller metric *of $M^5_0(d)$. This metric descends to a metric g^{mGZ}_d on the Brieskorn quotient $Q^5_0(d) := M^5_0(d)/\tau$ (which will likewise be called its* modified Grove-Ziller metric*) which has non-negative sectional curvature and positive scalar curvature.*

Proof. As we have seen in §4.1.1, $M_0^5(d)$ admits a cohomogeneity one action by $S^1 \times O(3)$ with singular orbits of codimension two. Therefore, by Theorem 6.1.1, $M_0^5(d)$ admits an $S^1 \times O(3)$-invariant metric g_d^{GZ} with non-negative sectional curvature.

To obtain a metric that also has positive scalar curvature, we will apply Proposition 6.1.4. The action by $S^1 \times O(3) = SO(2) \times O(3)$ on $M_0^5(d)$ has principal isotropy group $\mathbb{Z}_2 \times O(1)$ and singular isotropy groups $S^1 \times O(1)$ and $\mathbb{Z}_2 \times O(2)$. Since $S^1 \times O(3)$ is non-abelian and the Lie algebra of $\mathbb{Z}_2 \times O(1)$ is 0-dimensional, there exist non-commuting elements in this case. Now consider the singular isotropy groups, which are both 1-dimensional. It follows that the dimension of $\mathfrak{m}_p$ is 3 in both cases. By the classification of maximal tori of orthogonal groups, we know that $G = SO(2) \times O(3)$ has a 2-dimensional abelian maximal torus. Its Lie algebra $\mathfrak{g}$ is 4-dimensional and therefore splits into a 2-dimensional abelian and a 2-dimensional non-abelian part. P_p being an automorphism, this means that there must be non-commuting elements in its image. Thus the conditions of Proposition 6.1.4 are satisfied.

Fix some $t' > 0$. Then the Cheeger deformation metric $g_d^{mGZ} := (g_d^{GZ})_{t'}$ has all the required properties on $M_0^5(d)$.

Finally, since $\tau = (-1, Id) \in S^1 \times O(3)$, the metric $\tilde{g}_d^{mGZ}$ is invariant under τ and the Brieskorn quotient $Q_0^5(d)$ inherits a metric g_d^{mGZ} of non-negative sectional curvature and positive scalar curvature satisfying $\tilde{g}_d^{mGZ} = \pi_d^*(g_d^{mGZ})$, where $\pi_d : M_0^5(d) \to Q_0^5(d)$ is the projection. $\square$

Now fix $0 < \epsilon_0 \leq 1$ and let

$$Z(d) := \overline{f}_d^{-1}([0, \epsilon_0]) = \bigcup_{0 \leq \epsilon \leq \epsilon_0} Q_\epsilon^5(d), \tag{6.1}$$

where $\overline{f}_d : S^7/\tau \to \mathbb{C}$ is the map induced from the defining polynomial f_d (see §4.1.1). Equip $Q_0^5(d)$ with the modified Grove-Ziller metric g_d^{mGZ}. We can extend this metric to all of $Z(d)$ using a partition of unity argument and average it over $O(3)$. Denote this $O(3)$-invariant metric on $Z(d)$ by h.

The restriction of h to $Q_{\epsilon_0}^5(d)$ can be lifted to a metric on $M_{\epsilon_0}^5(d)$ via pullback. Extend this metric to all of $W_{\epsilon_0}^6(d)$ using a partition of unity argument such that it is of product form near the boundary. Observe that a neighborhood of a fixed point of the action of τ on $W_{\epsilon_0}^6(d)$ can be identified equivariantly with a hemisphere of the round sphere $S^6 = S(V \oplus \mathbb{R})$, where V is the $O(3)$-representation at the fixed point. Hence we can choose the metric on $W_{\epsilon_0}^6(d)$ in such a way that every fixed point of τ has an $O(3)$-invariant open neighborhood and the metric has positive sectional curvature when restricted to this neighborhood. If we average this metric over $O(3)$, we get an $O(3)$-invariant metric on $W_{\epsilon_0}^6(d)$ which we denote by k.

Proposition 6.1.7. *Let $Q_0^5(d)$ be equipped with the modified Grove-Ziller metric $g := g_d^{mGZ}$ from Proposition 6.1.6, $Z(d)$ with the $O(3)$-invariant metric h and $W_{\epsilon_0}^6$ with the $O(3)$-invariant metric k from above. Let g_t, h_t and k_t be the Cheeger deformations of these metrics under the action of $O(3)$. Then*

1. *for every $t > 0$, the metric g_t is of sec ≥ 0 and scal > 0 on $Q_0^5(d)$,*
2. *there exists a $t_0 \geq 0$ such that the metric $h_t^\epsilon := (h_t)|_{Q_\epsilon^5(d)}$ on $Q_\epsilon^5(d)$ has scal > 0 for all $t \geq t_0$ and $0 \leq \epsilon \leq \epsilon_0$, and*
3. *for all $t \geq t_0$, the metric k_t has scal > 0 on $W_{\epsilon_0}^6(d)$ and is of product form near the boundary.*

Proof. The first statement follows from Proposition 6.1.4.a). The second statement follows from Proposition 6.1.5. The third statement follows from [DGA21, Corollary 5.5]. □

6.1.5 Metrics on total spaces of principal S^1-bundle with fundamental group $\mathbb{Z}_2$

Recall that $N_{k,l}^5$ and $\overline{N}_{k,l}^5$ are quotients of a free S^1-action on $S^3 \times S^3$, while $X_{k,l,\beta}^5 := N_{k,l}^5/\tau$ and $\overline{X}_{k,l}^5 := \overline{N}_{k,l}^5/\overline{\tau}$ can be interpreted as quotients of a free $S^1 \times \mathbb{Z}_2$-action on $S^3 \times S^3$ (see §4.2).

Now give $S^3 \times S^3$ the product of round metrics. It can easily be verified that the non-exceptional and exceptional torus actions on $S^3 \times S^3$ act via isometries. Equip $N_{k,l}^5$ and $\overline{N}_{k,l}^5$ with the metrics g_N and $g_{\overline{N}}$ respectively, which turn $S^3 \times S^3 \to N_{k,l}^5$ and $S^3 \times S^3 \to \overline{N}_{k,l}^5$ into Riemannian submersions. Likewise, give $X_{k,l,\beta}^5$ and $\overline{X}_{k,l}^5$ the submersion metrics $g_{k,l,\beta}$ and $\overline{g}_{k,l}$ from the projections $S^3 \times S^3 \to X_{k,l,\beta}^5$ and $S^3 \times S^3 \to \overline{X}_{k,l}^5$. Observe that by construction, $N_{k,l}^5 \to X_{k,l,\beta}^5$ and $\overline{N}_{k,l}^5 \to \overline{X}_{k,l}^5$ are Riemannian universal coverings.

It follows immediately from the Gray-O'Neill formula [Gra67][O'N66] that $(N_{k,l}^5, g_N)$, $(\overline{N}_{k,l}^5, g_{\overline{N}})$, $(X_{k,l,\beta}^5, g_{k,l,\beta})$ and $(\overline{X}_{k,l}^5, \overline{g}_{k,l})$ all have *sec* ≥ 0. Furthermore, since the vertical spaces of all of the corresponding Riemannian submersions are 1-dimensional, we can find a plane of positive sectional curvature in the horizontal space of each point. By the Gray-O'Neill formula again, these planes get projected onto planes of positive sectional curvature and so it follows that all of the above Riemannian manifolds have *scal* > 0 as well.

Finally, consider the disk bundles $W_{k,l}^6$ and $\overline{W}_{k,l}^6$, where $\partial W_{k,l}^6 = N_{k,l}^5$ and $\partial \overline{W}_{k,l}^6 = \overline{N}_{k,l}^5$. Equip D^2 with a torpedo metric g_{tor} (see §6.0.1). If we give $N_{k,l}^5 \times D^2$ and $\overline{N}_{k,l}^5 \times D^2$ the product metrics $g_N + g_{tor}$ and $g_{\overline{N}} + g_{tor}$

respectively, we can equip $W^6_{k,l} = N^5_{k,l} \times_{S^1} D^2$ and $\overline{W}^6_{k,l} = \overline{N}^5_{k,l} \times_{S^1} D^2$ with the metrics which turn the projections $N^5_{k,l} \times D^2 \to W^6_{k,l}$ and $\overline{N}^5_{k,l} \times D^2 \to \overline{W}^6_{k,l}$ into Riemannian submersions. Denote these metrics by g_W and $g_{\overline{W}}$. They are of product form near the boundary (by property of the torpedo metric) and by the Gray-O'Neill formula, they have $sec \geq 0$.

6.2 Positive Ricci curvature metrics

6.2.1 A result of Böhm and Wilking

We cite the following result of Böhm and Wilking, which we will use in the following.

Proposition 6.2.1. *[BW07, Theorem A] Let M be a compact Riemannian manifold with metric g. Assume that g has $sec \geq 0$ and that M has finite fundamental group. Then the solution g_t to the Ricci flow has positive Ricci curvature for $t \in (0, \epsilon]$, where $\epsilon \in \mathbb{R}$ is small.*

Recall that the metric g_t is the solution of the (unnormalized) Ricci flow

$$\frac{\partial}{\partial t} g_t = -2Ric(g_t),$$

with initial metric $g_0 = g$. Böhm and Wilking show that g_t has positive Ricci curvature for $t \in (0, \epsilon]$ (see [BW07, pp.675]). Notice also that the isometry groups satisfy $\mathrm{Isom}(g) \subseteq \mathrm{Isom}(g_t)$ for all t.

6.2.2 Positive Ricci curvature metrics on bundles

The following result is due to Vilms [Vil70] (see also [GW09, Proposition 2.7.1]).

Theorem 6.2.2. *Let G be a compact Lie group and F be a smooth G-manifold. Let $\pi_P : P \to B$ be a principal G-bundle and $\pi : E \to B$ the associated bundle with fiber F, where $E := P \times_G F$. Given a Riemannian metric g_B on B, a G-invariant metric g_F on F and a connection θ on P, there exists a unique Riemannian metric g_E on E such that π is a Riemannian submersion with totally geodesic fibers isometric to (F, g_F) and horizontal distribution $\tilde{\mathcal{H}} := \rho_*(\mathcal{H} \times \{0\})$ where $\mathcal{H}$ is the horizontal distribution induced by θ and $\rho : P \times F \to P \times_G F$ denotes the projection.*

The metric g_E is constructed as follows. It is straightforward to see that $\pi_* \tilde{\mathcal{H}} = (\pi_P)_* \mathcal{H} = TB$, so that $TE = \ker(\pi_*) \oplus \tilde{\mathcal{H}}$. Then, if $\mathcal{V} = \ker(\pi_*)$ is the vertical distribution, the metric on E is

$$g_E(X, Y) := g_{F_x}(X^{\mathcal{V}}, Y^{\mathcal{V}}) + \pi^* g_B(X, Y)$$

for $X, Y \in T_xE$. Here $X^{\mathcal{V}}, Y^{\mathcal{V}} \in T_xF_x$ are the projections onto the tangent space at the fiber $F_x := \rho(x, F) = \pi^{-1}(\pi(x))$ and g_{F_x} is the Riemannian metric that turns $h_x : F \to F_x : f \mapsto \rho(x, f)$ into a Riemannian isometry (see the proof of [GW09, Proposition 2.7.1]). From now on, we will identify F and F_x through this isometry. The horizontal distribution $\tilde{\mathcal{H}}$ can be identified with the orthogonal complement of $\mathcal{V}$ with respect to g_E.

Theorem 6.2.3. *[GW09, Theorem 2.7.3]) Let B and F denote compact Riemannian manifolds with positive Ricci curvature and $\pi : E \to B$ a fiber bundle with fiber F and structure group G. If the metric on F is G-invariant, then E admits a metric with positive Ricci curvature.*

The idea of the proof is to use a canonical variation on the fibers (a special case of vertical warping, see [GW09, §2.1]). Let $P \to B$ be the principal G-bundle associated to $E \to B$, fix a connection on it and equip $E = P \times_G F$ with the metric g_E of Theorem 6.2.2. Let $\mathcal{V} := \ker(\pi_*)$ be the vertical distribution on E and $\mathcal{H}$ the orthogonal complement with respect to g_E, i.e. the horizontal distribution. The canonical variation then corresponds to setting $g_t|_{\mathcal{V}} := t \cdot g_E|_{\mathcal{V}}$ and $g_t|_{\mathcal{H}} := g_E|_{\mathcal{H}}$.

Let Ric_M be the Ricci curvature tensor of g_M on $M \in \{E, F, B\}$ and denote by $Ric^h_E(X, Y)$ the trace of the operator $Z \mapsto R_E(Z, X)Y$ projected onto $\mathcal{H}$, where R_E denotes the curvature tensor of the metric g_E.

One can then compute the Ricci curvature of the canonical variation metric (see the proof of [GW09, Theorem 2.7.3]):

$$\begin{aligned} Ric_{g_t}(W, W) =&(1 - t) \cdot Ric_B(\pi_*(X), \pi_*(X)) + t \cdot Ric_E(X, X) \\ &+ Ric_F(T, T) + t^2 \cdot Ric^h_E(T, T) + 2t \cdot Ric^h_E(X, T), \end{aligned}$$

where $W = X + T \in TE$, $X \in \mathcal{H}$ and $T \in \mathcal{V}$ (recall that we identify the fiber with (F, g_F)). For $t \to 0$, we have

$$Ric_{g_t}(W, W) \to Ric_B(\pi_*(X), \pi_*(X)) + Ric_F(T, T) > 0.$$

Hence, for small enough t, the metric g_t is of positive Ricci curvature.

We can now construct positive Ricci curvature metrics on S^{4n-1}-bundles over S^{4n} that extend to the corresponding disk bundle.

Proposition 6.2.4. *Let $M^{8n-1}_{k,l}$ be the total space of a linear S^{4n-1}-bundle over S^{4n} and $W^{8n}_{k,l}$ the total space of the associated disk bundle (see §5.1), $n = 1, 2$. There exists a metric $\tilde{g}_{k,l}$ on $W^{8n}_{k,l}$ which has positive scalar curvature, is of product form near the boundary $M^{8n-1}_{k,l}$ and such that $g_{k,l} = \tilde{g}_{k,l}|_{M^{8n-1}_{k,l}}$ has positive Ricci curvature.*

Proof. Let $P := P_{k,l}$, $M := M^{8n-1}_{k,l}$ and $W := W^{8n}_{k,l}$ be the total space of the associated principal $SO(4n)$-bundle, the S^{4n-1}-bundle and the D^{4n}-bundle over S^{4n} respectively. Equip S^{4n} with the round metric g_R and D^{4n} with a torpedo metric g_{tor} (see §6.0.1).

Fix a connection on $P \to S^{4n}$ and equip $W = P \times_{SO(4n)} D^{4n}$ with the metric of Theorem 6.2.2, which we denote by $\tilde{g}$. Then the restriction $g := \tilde{g}|_M$ on $M = P \times_{SO(4n)} S^{4n-1}$ corresponds to the metric of Theorem 6.2.2 applied to M. Let $\mathcal{V} := \ker(\pi_W)_*$ be the vertical distribution on W and set $\mathcal{H}$ as the orthogonal complement of $\mathcal{V}$ with respect to $\tilde{g}_W$. Consider the canonical variation

$$\tilde{g}_t|_{\mathcal{V}} := t \cdot \tilde{g}|_{\mathcal{V}}, \qquad \tilde{g}_t|_{\mathcal{H}} := \tilde{g}|_{\mathcal{H}} \qquad \text{and} \qquad \tilde{g}_t(\mathcal{V}, \mathcal{H}) := 0$$

on W, where $t \in \mathbb{R}_{\geq 0}$. By the discussion below Theorem 6.2.2 and a slight abuse of notation, this amounts to setting

$$\tilde{g}_t(X, Y) := t \cdot g_{tor}(X^{\mathcal{V}}, Y^{\mathcal{V}}) + \pi_W^* g_R(X, Y), \qquad \text{for } X, Y \in T_x W. \tag{6.2}$$

If we set $g_t := (\tilde{g}_t)|_M$ and restrict to $\mathcal{V}_M := \ker(\pi_M)_* \subset \mathcal{V}$ with its corresponding horizontal distribution $\mathcal{H}_M$, then this simultaneously corresponds to a canonical variation on $M = \partial W$:

$$g_t|_{\mathcal{V}_M} = t \cdot g|_{\mathcal{V}_M}, \qquad g_t|_{\mathcal{H}_M} = g|_{\mathcal{H}_M} \qquad \text{and} \qquad g_t(\mathcal{V}_M, \mathcal{H}_M) = 0.$$

Now by Theorem 6.2.3 and its proof applied to $E = M$, $B = S^{4n}$ and $F = S^{4n-1}$, there is an $0 < \epsilon << 1$ such that $g_M := g_\epsilon$ is of positive Ricci curvature.

Next we show that the metric on W has positive scalar curvature. By [Bes87, 9.70(d)], the scalar curvature of the canonical variation metric is given by

$$scal_{\tilde{g}_t} = \frac{1}{t} scal_{g_F} + scal_{g_B} \circ \pi_W - t|A|^2$$

where A is a tensor field on W and in our case, $g_F = g_{tor}$ and $g_B = g_R$. Obviously, $scal_{g_{tor}} > 0$ and $scal_{g_R} > 0$. Therefore, choosing ϵ to be even smaller if necessary, the metric $g_W := g_\epsilon$ is of positive scalar curvature everywhere and restricts to the positive Ricci curvature metric $g_M = g_W|_M$.

Finally, Equation (6.2) shows that g_W is of product form near the boundary. Indeed, by Theorem 6.2.2 the fibers on W are isometric to (D^{4n}, g_{tor}) and the canonical variation corresponds to shrinking the fibers, therefore respecting the product form near the boundary (see [Kor20, p.10]).

Hence, the desired metrics are $\tilde{g}_{k,l} := g_W$ and $g_{k,l} := g_M$.

□

6.2.3 Metrics on Shimada projective spaces

Let M_k^{15} be a Shimada sphere equipped with the metric g_k from Proposition 6.2.4 (remember that $M_k^{15} := M_{k,1}^{15}$). Since the fibers of the Riemannian submersion $\pi_S : M_k^{15} \to S^8$ are isometric to the round sphere (S^7, g_R) (see

Theorem 6.2.2), it follows that the involution τ on M_k^{15}, which is induced by fiberwise antipodal maps, is an isometry. Therefore, the induced metric g_k' on the quotient $Q_k^{15} := M_k^{15}/\tau$ is of positive Ricci curvature and satisfies $g_k = \pi_k^*(g_k')$ where $\pi_k : M_k^{15} \to Q_k^{15}$ is the canonical projection.

Remark 6.2.5. *The exact same argument applies to construct positive Ricci curvature metrics on Milnor projective spaces.*

6.3 Moduli spaces of Riemannian metrics

For more details on moduli spaces of Riemannian metrics, see [TW15].

Let M be a compact smooth manifold and $\mathcal{R}(M)$ the set of Riemannian metrics on M, equipped with the $\mathcal{C}^\infty$-topology of uniform convergence of all the derivatives. If we restrict to metrics with $sec \geq 0$, $Ric > 0$ and $scal > 0$, we get the corresponding sets $\mathcal{R}_{sec\geq 0}(M)$, $\mathcal{R}_{Ric>0}(M)$ and $\mathcal{R}_{scal>0}(M)$. The group of diffeomorphisms $\mathrm{Diff}(M)$ acts on $\mathcal{R}(M)$ by taking pullbacks of the metrics. The *moduli space of Riemannian metrics* of M is defined as the quotient space $\mathcal{M}(M) := \mathcal{R}(M)/\mathrm{Diff}(M)$ whose elements are isometry classes of Riemannian metrics. If we restrict to isometry classes of metrics of $sec \geq 0$, $Ric > 0$ and $scal > 0$ we get corresponding moduli spaces $\mathcal{M}_{sec\geq 0}(M)$, $\mathcal{M}_{Ric>0}(M)$ and $\mathcal{M}_{scal>0}(M)$.

We cite a consequence of Ebin's slice theorem [Ebi70] which will be used repeatedly in the following. See [CK19, Proposition 4.6] for a proof.

Proposition 6.3.1. *Let M be a compact Riemannian manifold and $\gamma : [a,b] \to \mathcal{M}(M)$ a path. Then for $g \in \mathcal{R}(M)$ such that $\gamma(a) = [g]$, there exists a path $\tilde{\gamma} : [a,b] \to \mathcal{R}(M)$ with $\tilde{\gamma}(a) = g$ and $\pi \circ \tilde{\gamma} = \gamma$, where $\pi : \mathcal{R}(M) \to \mathcal{M}(M)$ is the projection map.*

Note that this result also holds for all moduli spaces with curvature conditions.

Chapter 7

The Proofs

We are finally ready to give the proofs of our main results. The reader is referred to the introduction for an outline of the various proofs.

7.1 Proof of Theorem A

Recall Theorem A.

Theorem. *Let Q^5 be an orientable, closed, smooth non-spin 5-dimensional manifold with $\pi_1(Q) \cong \mathbb{Z}_2$, whose universal cover is $S^3 \times S^2$. Then the moduli space of non-negative sectional curvature metrics on Q has infinitely many path components. The same is true for the moduli space of positive Ricci curvature metrics on Q.*

First, we reduce the problem to showing that the quotient of the space of Riemannian metrics by the *Spin*^{c} structure preserving diffeomorphism group has infinitely many path components.

Let X^5 be a 5-dimensional *spin*^{c} manifold with $\pi_1(X) \cong \mathbb{Z}_2$. Assume that the principal $U(1)$-bundle associated to the *Spin*^{c} structure is equipped with a flat connection. Let $\mathrm{Diff}^c(X)$ be the set of diffeomorphisms of X which preserve the chosen *Spin*^{c} structure. Define $\mathcal{M}^c(X) := \mathcal{R}(X)/\mathrm{Diff}^c(X)$.

Lemma 7.1.1. *If $\mathcal{M}^c(X)$ has infinitely many path components, so does $\mathcal{M}(X)$.*

Proof. Since we assumed the connection on $P_{U(1)}$ to be flat, it follows that the canonical class c of the *Spin*^{c} structure must be torsion. The *Spin*^{c} structures on X are in one-to-one correspondence with $2H^2(X;\mathbb{Z}) \oplus H^1(X;\mathbb{Z}_2)$. Now if $\phi : X' \to X$ is a diffeomorphism, then the *Spin*^{c} structure on X is pulled back to a *Spin*^{c} structure on X' with canonical class $\phi^*(c) \in H^2(X';\mathbb{Z})$ which is torsion as well. Hence, since $Tor(H^2(X;\mathbb{Z})) \cong Tor(H_1(X;\mathbb{Z})) \cong \mathbb{Z}_2$, it follows that its *Spin*^{c} structure can only pull back to a finite number of $Spin^c$ structures and therefore $\mathrm{Diff}^c(X)$ has finite index in $\mathrm{Diff}(X)$. The result now follows. $\square$

Note that this result also holds for $\mathcal{M}^c_{sec\geq 0}(X)$ and $\mathcal{M}^c_{Ric>0}(X)$ (which are defined similarly) and the corresponding moduli spaces. Furthermore, there is an analogous consequence of Ebin's slice theorem (Proposition 6.3.1) for $\mathcal{M}^c(X)$ and the various analogs with curvature conditions.

7.1.1 1st case: π_1 acts non-trivially on π_2

$Spin^c$ structure and relative eta-invariant on Brieskorn manifolds

Let $W^6_\epsilon(d)$ and $M^5_\epsilon(d)$ be Brieskorn varieties for $\epsilon \geq 0$ and d even. Let $Q^5_\epsilon(d) = M^5_\epsilon(d)/\tau$ be the Brieskorn quotient, where τ is the involution defined in §4.1.1.

Let $0 < \epsilon_0 < 1$ and $t_0 \geq 0$ from Proposition 6.1.7. Equip $W^6_{\epsilon_0}(d)$ with the τ-invariant metric k_{t_0} and $Q^5_{\epsilon_0}(d)$ with the metric $h^{\epsilon_0}_{t_0}$ from Proposition 6.1.7. Fix the canonical $Spin^c$ structure on $W^6_{\epsilon_0}(d)$ induced by the complex structure (see Example 2.2.4) and a Hermetian metric on its tangent bundle, and suppose that this Hermitian metric is τ-invariant as well. It is easy to see from the definition of the involution and the complex $Spin^c$ structure that τ preserves the $Spin^c$ structure on $W^6_{\epsilon_0}(d)$. Therefore, the $Spin^c$ structure on $W^6_{\epsilon_0}(d)$ is $\mathbb{Z}_2$-equivariant (see Appendix A). In particular, τ preserves the $Spin^c$ structure on the boundary $M^5_{\epsilon_0}(d) = \partial W^6_{\epsilon_0}(d)$, which therefore descends to a $Spin^c$ structure on $Q^5_{\epsilon_0}(d) = M^5_\epsilon(d)/\tau$ (see Appendix A) we will call its *preferred* $Spin^c$ structure.

Now since by Theorem 4.1.1.1. we have $H^2(W^6_{\epsilon_0}(d);\mathbb{Z}) = 0$, any principal $U(1)$-bundle over $W^6_{\epsilon_0}(d)$ must be trivial (see for example [LM89, Appendix A]). Hence the restriction of that bundle to the boundary $M^5_{\epsilon_0}(d)$ of the base space is trivial as well. We can therefore identify the principal $U(1)$-bundle $P_{U(1)}$ of the preferred $Spin^c$ structure on $Q^5_{\epsilon_0}(d)$ with $M^5_{\epsilon_0}(d) \times_{\mathbb{Z}_2} U(1)$, where the non-trivial element of $\mathbb{Z}_2$ acts via $(\tau, -1)$ on $M^5_{\epsilon_0}(d) \times U(1)$. Equip the trivial principal $U(1)$-bundles over $W^6_{\epsilon_0}(d)$ with a flat unitary $\mathbb{Z}_2$-equivariant[1] connection which is constant in the normal direction near the boundary $M^5_{\epsilon_0}(d)$. The induced connection on $P_{U(1)}$ over $Q^5_{\epsilon_0}(d)$ (see Appendix A) is then flat as well. Let $E_\alpha := M^5_\epsilon(d) \times_{\mathbb{Z}_2} \mathbb{C}$ be the associated non-trivial flat complex line bundle.

Lemma 7.1.2. *Let $(W^6_{\epsilon_0}(d), k_{t_0})$, $(M^5_{\epsilon_0}(d), g_M)$ and $(Q^5_{\epsilon_0}(d), h^{\epsilon_0}_{t_0})$ be as above, where $g_M := (k_{t_0})|_M$ and $0 < \epsilon_0 < 1$. Then*

$$\tilde{\eta}_\alpha(Q^5_{\epsilon_0}(d), h^{\epsilon_0}_{t_0}) = -\frac{d}{4}.$$

[1] See Appendix A for the definition of an equivariant connection.

Proof. Let $Q := Q^5_\epsilon(d)$, $g_Q := h^{\epsilon_0}_{t_0}$ and $M := M^5_\epsilon(d)$. Recall that $\eta_\alpha(Q, g_Q) := \eta(D^c_{Q,E_\alpha})$, where D^c_{Q,E_α} is the twisted $Spin^c$ Dirac operator and $\eta(M, g_M) := \eta(D^c_M)$ (see §2.3.5). By Proposition 6.1.7, we can apply Proposition 2.3.24 to obtain

$$\tilde{\eta}_\alpha(Q, g_Q) = -2 \sum_{i=1}^{d} a_{spin^c}(\{p_i\}),$$

where the p_i are the isolated fixed points of the action of τ on $W := W^6_{\epsilon_0}(d)$ (see §4.1.1). Using Proposition 2.3.13 to compute $a_{spin^c}(\{p_i\}) = 2^{-l}$ with $l = 3$, we get the desired result.

□

We need to determine the relative $Spin^c$ eta-invariant of $(Q^5_0(d), g^{mGZ}_d)$, where g^{mGZ}_d is the modified Grove-Ziller metric from Proposition 6.1.6.

Now since $W^6_0(d)$ is not a manifold, it cannot be equipped with a $Spin^c$ structure and so we cannot define a $Spin^c$ structure on $M^5_0(d)$ and $Q^5_0(d)$ the way we did above for $\epsilon_0 \neq 0$. But by Theorem 4.1.2.3., there is a diffeomorphism $\phi_0 : Q^5_{\epsilon_0}(d) \to Q^5_0(d)$ and we can use its inverse to pull back the preferred $Spin^c$ structure on $Q^5_{\epsilon_0}(d)$ to $Q^5_0(d)$. The *preferred $Spin^c$ structure* of $Q^5_0(d)$ will be the one determined by this pulled back $Spin^c$ structure and the metric g^{mGZ}_d (see Remark 2.2.2). Observe that the principal $U(1)$-bundle of the preferred $Spin^c$ structure on $Q^5_0(d)$, which is the pullback from the one on $Q^5_{\epsilon_0}(d)$, is non-trivial and also has a flat unitary connection. We denote the associated non-trivial complex line bundle over $Q^5_0(d)$ by E_α as well.

Proposition 7.1.3. *Let $(Q^5_0(d), g^{mGZ}_d)$ be equipped with the preferred $Spin^c$ structure from above. Then,*

$$\tilde{\eta}_\alpha(Q^5_0(d), g^{mGZ}_d) = -\frac{d}{4}.$$

Proof. Set $Q_0 := Q^5_0(d)$, $Q_1 := Q^5_{\epsilon_0}(d)$, where $0 < \epsilon_0 < 1$, and $g := g^{mGZ}_d$. Let

$$Z(d) = \bigcup_{0 \le \epsilon \le \epsilon_0} Q^5_\epsilon(d)$$

be equipped with the extended metric h as in §6.1.4. Let g_t and h_t denote the Cheeger deformations of g and h respectively and $t_0 \geq 0$ the parameter from Proposition 6.1.7. Recall that $scal_{g_t} > 0$ for all t. Let $h^\epsilon_t = (h_t)|_{Q^5_\epsilon(d)}$ for $0 \le \epsilon \le \epsilon_0$. Let $\phi_\epsilon : Q_1 \to Q^5_\epsilon(d)$ be a smooth family of diffeomorphisms for $0 \le \epsilon \le \epsilon_0$ (see [DGA21, p.23]).

Using Proposition 6.1.7, we can now define two paths of metrics of positive scalar curvature on Q_1 in the following way.

$$\begin{aligned}\gamma_1 : [0, t_0] &\to \mathcal{R}_{scal>0}(Q_1)\\ t &\mapsto \phi_0^*(g_t)\end{aligned}$$

and

$$\begin{aligned}\gamma_2 : [0, \epsilon_0] &\to \mathcal{R}_{scal>0}(Q_1)\\ \epsilon &\mapsto \phi_\epsilon^*(h_{t_0}^\epsilon)\end{aligned}$$

Since $\gamma_1(t_0) = \phi_0^*(g_{t_0}) = \phi_0^*(h_{t_0}^0) = \gamma_2(0)$, we can concatenate these two paths to get a path γ in $\mathcal{R}_{scal>0}(Q_1)$ with endpoints $\phi_0^*(g)$ and $h_{t_0}^{\epsilon_0}$. Using the fact that the relative eta-invariant is preserved under pullbacks by diffeomorphisms, by Lemma 7.1.2 and Proposition 2.3.23 we obtain:

$$\tilde{\eta}_\alpha(Q_0, g) = \tilde{\eta}_\alpha(Q_1, \phi_0^*(g)) = \tilde{\eta}_\alpha(Q_1, h_{t_0}^{\epsilon_0}) = -\frac{d}{4}.$$

□

The proof

We can now prove Theorem A for the case when the fundamental group acts non-trivially on the second homotopy group.

Let X^5 be an orientable, smooth 5-manifold with $\pi_1(X) \cong \mathbb{Z}_2$, $w_2(X) \neq 0$, universal cover $\tilde{X} \cong S^3 \times S^2$ and suppose that $\pi_1(X)$ acts non-trivially on $\pi_2(X) \cong \mathbb{Z}$. Then by Theorem 4.1.5, there exists a $d \in \{0, 2, 4, 6, 8\}$ and an orientation on X such that it is orientation preserving diffeomorphic to $Q := Q_0^5(d)$. If the moduli space of metrics of Q has infinitely many path components, so does the moduli space of metrics of X, for any curvature conditions.

Now equip Q with its preferred $Spin^c$ structure. Let $d_0 \neq d_1 \in \mathbb{N}$ be such that $d_i \equiv d \bmod 16$ and such that there exist $Spin^c$ structure preserving diffeomorphisms[2] $\psi_i : Q \to Q_0^5(d_i)$ for $i = 0, 1$. Denote by $h_i := \psi_i^*(g_{d_i}^{mGZ})$ the pullback metric of the modified Grove-Ziller metric.

The proof is by contradiction. Assume that there is a path $\gamma : [0, 1] \to \mathcal{M}_{sec\geq 0}^c(Q)$ connecting $[h_0]$ and $[h_1]$. As a consequence of Ebin's slice theorem (Proposition 6.3.1), this path can be lifted to a path $\tilde{\gamma}$ in $\mathcal{R}_{sec\geq 0}(Q)$ such that $\tilde{\gamma}(0) = h_0$ and $\tilde{\gamma}(0) = \phi^*(h_1)$ for some $\phi \in \mathrm{Diff}^c(Q)$. By a result of Böhm and Wilking (Proposition 6.2.1), the path $\tilde{\gamma}$ evolves instantly to a path in $\mathcal{R}_{Ric>0}(Q)$ under the Ricci flow. If we concatenate this resulting path with the orbits of the endpoints of $\tilde{\gamma}$, we obtain a path γ' in $\mathcal{R}_{scal>0}(Q)$ with the same endpoints $\gamma'(0) = h_0$ and $\gamma'(1) = \phi^*(h_1)$.

[2]If the $Spin^c$ structure on $Q_0^5(d)$ happens to be such that there are only finitely many other d_i such that it is $Spin^c$ structure preserving diffeomorphic to $Q_0^5(d_i)$, simply replace d with an index $d' \equiv \pm d$ mod 16 for which there are infinitely many such manifolds.

Recall that E_α denotes the unique flat non-trivial complex line bundle over Q. Using Proposition 7.1.3, we can now compute the relative eta-invariants of the endpoints of γ' (note that the value of $\tilde{\eta}_\alpha$ remains unchanged by pullbacks via diffeomorphisms):

$$\tilde{\eta}_\alpha(Q, h_0) = -\frac{d_0}{4} \neq -\frac{d_1}{4} = \tilde{\eta}_\alpha(Q, \phi^*(h_1)).$$

Hence h_0 and $\phi^*(h_1)$ lie in different path components in $\mathcal{R}_{scal>0}(Q_0^5(d))$, which is a contradiction with the above.

Now by Theorem 4.1.6, there is an orientation preserving diffeomorphism between Q and $Q_0^5(d_i)$ for $i = 0, 1$. Since there are only two $Spin^c$ structures on Q (for a fixed orientation and fixed canonical class), there must be infinitely many d_i such that this diffeomorphism is $Spin^c$ structure preserving. Therefore, $\mathcal{M}^c_{\sec\geq 0}(Q)$ has infinitely many path components. By Lemma 7.1.1 and the above it follows that the moduli space $\mathcal{M}_{\sec\geq 0}(Q)$ has infinitely many path components. If we focus on the deformed metrics under the Ricci flow, we see that the same argument applies to $\mathcal{M}^c_{Ric>0}(Q)$ and hence $\mathcal{M}_{Ric>0}(Q)$ has infinitely many path components as well. ■

7.1.2 2nd case: π_1 acts trivially on π_2

This case started as joint work with Jan-Bernhard Kordaß. The details have later been worked out together with McFeely Jackson Goodman, who applied his methods from [Goo20a] to arrive at the same result. We give a different approach here, which is inspired by [Des20].

Spin^c structure and relative eta-invariant of principal S^1-bundles

In this section, we equip the manifolds $X_0 := X^5_{k,l,\beta}$, $N_0 := N^5_{k,l}$, $W_0 := W^6_{k,l}$ and $X_1 := \overline{X}^5_{k,l}$, $N_1 := \overline{N}^5_{k,l}$, $W_1 := \overline{W}^6_{k,l}$ (see §4.2) with appropriate $Spin^c$ structures and compute their relative eta-invariants. Let $L_0 := L_{k,l}$ and $L_1 := \overline{L}_{k,l}$ be the complex line bundles associated to N_0 and N_1, and $B_0 := B^4_\beta$ and $B_1 := B^4$ the base spaces.

Recall that we assume β and k to be odd, while l is even ($gcd(k, l) = 1$).

Suppose now that W_i is equipped with a Riemannian metric which is of product form near the boundary N_i for $i = 0, 1$.

The tangent bundle of the disk bundle satisfies $TW_i \cong \pi^*_W(TB_i) \oplus \pi^*_W(L_i)$ for $i = 0, 1$. Hence, since $w_2(\xi) \equiv c_1(\xi) \bmod 2$, by Equations (4.5), (4.8) and Equations (4.7), (4.9) it follows that $w_2(W_i)$ vanishes and so W_i has a $Spin$ structure. Denote by P_{Spin} this $Spin$ structure, which is unique since $H^1(W_i; \mathbb{Z}_2) = 0$.

Now consider the involutions $\tau_0 := \tau$ and $\tau_1 := \overline{\tau}$ which we have defined on N_i, $i = 0, 1$, respectively, to be induced by fiberwise antipodal maps on

S^1. Likewise, we will denote by τ_i the involution on W_i which reduces to multiplication by -1 on the fibers. Assume that W_i is equipped with a τ_i-invariant Riemannian metric. The action of $\mathbb{Z}_2 = \{Id, \tau_i\}$ is free on N_i, while the fixed point set of the corresponding $\mathbb{Z}_2$-action on W_i is the zero section B_i. In both cases, the $\mathbb{Z}_2$-action lifts to the oriented orthonormal frame bundles $P_{SO}(N_i)$ and $P_{SO} := P_{SO}(W_i)$ via differentials (see Appendix A).

Now let $P_{U(1)} := W_i \times U(1)$ be the trivial principal $U(1)$-bundle over W_i and $P_{Spin^c} := P_{Spin} \times_{\mathbb{Z}_2} P_{U(1)}$ the $Spin^c$ structure on W_i associated to the unique $Spin$ structure on W_i for $i = 0, 1$ (see Example 2.2.3). The $\mathbb{Z}_2$-action on W_i together with multiplication by ± 1 on $U(1)$ define a $\mathbb{Z}_2$-action on $P_{U(1)}$. Together with the $\mathbb{Z}_2$-action on P_{SO}, we get a $\mathbb{Z}_2$-action on $P_{SO} \times P_{U(1)}$.

Lemma 7.1.4. *Let $\gamma : P_{Spin^c} \to P_{SO} \times P_{U(1)}$ be the $Spin^c(n)$-equivariant map of the $Spin^c$ structure on W_i, $i = 0, 1$. Then the $\mathbb{Z}_2$-action on $P_{SO} \times P_{U(1)}$ lifts to a $\mathbb{Z}_2$-action on P_{Spin^c}.*

Proof. The $\mathbb{Z}_2$-action on $P_{SO} \times P_{U(1)}$ is defined by

$$t \cdot (p, w, z) = (d\tilde{\tau}_j(p), \tau_j(w), -z) \in P_{SO} \times (W_j \times U(1)),$$

where $d\tilde{\tau}_j$ denotes the involution on P_{SO} induced by the differential of τ_j on W_j for $j = 0, 1$. By [AB68, Proposition 8.46], the involution τ_j is of odd type, which means that the lifted action $\tilde{\tau}_j$ on P_{Spin} over W_i is of order 4 (with respect to the two-fold covering $P_{Spin} \to P_{SO}$). In particular, $\tilde{\tau}_j^2 = -1$. Likewise, if we consider the two-fold covering $P_{U(1)} \to P_{U(1)} : (w, z) \mapsto (w, z^2)$, then the involution $\tau'_j \cdot (w, z) = (\tau_j(w), -z) \in P_{U(1)} = W_i \times U(1)$ lifts to an action $\tilde{\tau}'_j \cdot (w, z) = (\tau_j(w), iz)$ which is of order 4 ($j = 0, 1$). Hence, the above $\mathbb{Z}_2$-action on $P_{SO} \times P_{U(1)}$ lifts to a $\mathbb{Z}_2$-action on $P_{Spin^c} = P_{Spin} \times_{\mathbb{Z}_2} P_{U(1)}$ defined by $T \cdot [q, w, z] = [\tilde{\tau}_j(q), \tau_j(w), iz]$ for $j = 0, 1$. $\square$

Hence, τ_i preserves the $Spin^c$ structure on W_i for $i = 0, 1$. If we consider the restricted $Spin^c$ structure on the boundary N_i, it therefore descends to a quotient $Spin^c$ structure on X_i (see Appendix A). From now on, we will equip X_i, N_i and W_i with these respective $Spin^c$ structures.

Now fix a flat unitary $\mathbb{Z}_2$-equivariant connection ∇^c on $P_{U(1)} = W_i \times U(1)$ over W_i which is constant in the normal direction near the boundary N_i, $i = 0, 1$. The induced connection $\tilde{\nabla}^c$ on $\tilde{P}_{U(1)} = N_i \times U(1)$ over N_i, which is simply the restriction of ∇^c to the boundary, is flat and $\mathbb{Z}_2$-equivariant. Passing to the quotient (see Appendix A), the connection $\overline{\nabla}^c$ on $\overline{P}_{U(1)}$ over X_i is flat as well. We can identify $\overline{P}_{U(1)}$ with $N_i \times_{\mathbb{Z}_2} U(1)$, where the non-trivial element of $\mathbb{Z}_2$ acts via $(\tau, -1)$ on $N_i \times U(1)$. This bundle is non-trivial, with the order of $c_1(\overline{P}_{U(1)})$ being 2. Recall that this characteristic class corresponds to the canonical class of the $Spin^c$ structure on X_i.

Let $E_{i,\alpha} := N_i \times_{\pi_1(X_i)} \mathbb{C}$ be the associated non-trivial flat complex line bundle over X_i, where τ_i acts via -1 on $\mathbb{C}$ for $i = 0, 1$. Recall that $X_0 = X^5_{k,l,\beta}$ and $X_1 = \overline{X}^5_{k,l}$.

Proposition 7.1.5. *Let g_{X_i} be a metric on X_i which is of scal > 0 and which lifts to a metric g_{N_i} on N_i such that this metric extends to a metric g_{W_i} on W_i which is τ_i-invariant, of product form near the boundary and has scal ≥ 0 everywhere for $i = 0, 1$. Then*

$$\tilde{\eta}_\alpha(X^5_{k,l,\beta}, g_{X_0}) = \pm\frac{1}{8}(l^2\beta + 2kl)$$

and

$$\tilde{\eta}_\alpha(\overline{X}^5_{k,l}, g_{X_1}) = \pm\frac{1}{8}(l^2 + 2kl + 2k^2 + 2).$$

Proof. By Proposition 2.3.24, we have

$$\tilde{\eta}_\alpha(X_j, g_{X_j}) = -2a_{spin^c}(B_j),$$

where B_j is the zero section of the disk bundle W_j, $j = 0, 1$. By Equations (2.13), (2.8) and (2.9), since the normal bundle of this fixed point set is $\nu = \nu(\pi) = L_j$ and τ_j acts via multiplication by -1 on $P_{U(1)}|_{B_j}$, we have

$$a_{spin^c}(B_j) = \pm i \int_{B_j} e^{\iota^*_{B_j}(c)/2} \hat{A}_\pi(L_j)\hat{A}(B_j),$$

where c is the canonical class of the $Spin^c$ structure on W_j and ι_{B_j} is the inclusion of B_j into W_j. The $\pm$-ambiguity does not affect the final result and so we will ignore it.

We first focus on $j = 0$. By Equations (2.4) and (4.4), we have $\hat{A}(B^4_\beta) = 1$ and by Equations (2.8) and (4.8), $\hat{A}_\pi(L_0) = \frac{1}{2i}\frac{1}{\cosh(c_1(L_0)/2)}$. Hence, we have

$$a_{spin^c}(B^4_\beta) = \pm\frac{1}{2}\int_{B^4_\beta} \frac{1}{\cosh(\frac{-lu+kv}{2})},$$

since $c = 0$ ($P_{U(1)}$ is trivial). By Lemma 4.2.5, integrating the above expression amounts to determining the coefficient of uv making use of the identities $v^2 = 0$ and $u^2 = -\beta uv$. Hence, using Taylor expansions, one computes

$$a_{spin^c}(B^4_\beta) = \pm\frac{1}{16}(-l^2\beta - 2kl).$$

Now take $j = 1$. In this case we still have $c = 0$ and $\hat{A}_\pi(L_1) = \frac{1}{2i}\frac{1}{\cosh(c_1(L_1)/2)}$. By Equation (4.6) the $\hat{A}$-genus is given by $\hat{A}(B^4) = 1 - \frac{6}{24}\overline{u}^2$, and therefore by Equation (4.9)

$$a_{spin^c}(B^4) = \pm\frac{1}{2}\int_{B^4} \frac{1}{\cosh(\frac{-l\overline{u}+k\overline{v}}{2})} \cdot (1 - \frac{1}{4}\overline{u}^2).$$

This time, by Lemma 4.2.7, evaluating the integral amounts to computing the coefficient of $\overline{u}^2$, making use of $\overline{u}^2 = -\overline{uv}$ and $\overline{v}^2 = 2\overline{u}^2$. Again using Taylor expansions, we obtain

$$a_{spin^c}(B^4) = \pm\frac{1}{16}(-l^2 - 2kl - 2k^2 - 2).$$

$\square$

The proof

We now prove Theorem A for the case when the fundamental group acts trivially on the second homotopy group.

Let $S^1 \to X^5_{k,l,\beta} \to B^4_\beta$ and $S^1 \to \overline{X}^5_{k,l} \to B^4$ be the principal bundles from §4.2, where l is even, k, β are odd and $gcd(k,l) = 1$. Recall that $\pi_1(X^5_{k,l,\beta}) \cong \mathbb{Z}_2$ acts trivially on $\pi_2(X^5_{k,l,\beta})$ (see Remark 4.2.3), $N^5_{k,l} \cong S^3 \times S^2$ is the universal covering of $X^5_{k,l,\beta}$ and $w_2(X^5_{k,l,\beta}) \neq 0$ (see §4.2.3). Similarly, $\pi_1(\overline{X}^5_{k,l}) \cong \mathbb{Z}_2$ acts trivially on $\pi_2(\overline{X}^5_{k,l})$, $\overline{N}^5_{k,l} \cong S^3 \times S^2$ is the universal covering of $\overline{X}^5_{k,l}$ and $w_2(\overline{X}^5_{k,l}) \neq 0$. Furthermore, let $W^6_{k,l}$ and $\overline{W}^6_{k,l}$ be the corresponding disk bundles.

Equip these spaces with their corresponding $Spin^c$ structures (see below Lemma 7.1.4).

<u>Non-exceptional case:</u>

Fix β odd and $q \in \{0, 4, 8\}$. By Proposition 4.2.12, there is an infinite family $\{X^5_{k,l,\beta}\}_{(k,l)\in Z_q}$, where Z_q is the set of all pairs (k,l) such that $X^5_{k,l,\beta}$ is orientation preserving diffeomorphic to $X(q)$. Fix $(k_0, l_0) \in Z_q$ and set $X := X^5_{k_0,l_0,\beta}$. Let $\phi_l : X \to X^5_{k_0,l,\beta}$ be such a diffeomorphism and assume furthermore that it preserves the $Spin^c$ structures. Define $g_l := \phi_l^*(g_{k_0,l,\beta})$ where $g_{k_0,l,\beta}$ is the metric on $X^5_{k_0,l,\beta}$ from §6.1.5 and pull back the $Spin^c$ structure on $X^5_{k_0,l,\beta}$ to X via the same diffeomorphism.

We now argue by contradiction. Suppose there is a path in $\mathcal{M}^c_{sec\geq 0}(X)$ connecting $[g_l]$ and $[g_{l'}]$, where $l \neq l'$. As a consequence of Ebin's slice theorem (Proposition 6.3.1), this path lifts to a path γ in $\mathcal{R}_{sec\geq 0}(X)$ with endpoints $\gamma(0) = g_l$ and $\gamma(1) = \psi^*(g_{l'})$, where $\psi \in \mathrm{Diff}^c(X)$. By Böhm and Wilking (Proposition 6.2.1), the path γ immediately evolves to a path in $\mathcal{R}_{Ric>0}(X)$ under the Ricci flow, and if we concatenate this resulting path the orbits of the endpoints, we get a path γ' in $\mathcal{R}_{scal>0}(X)$ with $\gamma'(0) = g_l$ and $\gamma'(1) = \psi^*(g_{l'})$.

Now let E_α denote the flat non-trivial complex line bundle over X. By Proposition 7.1.5, since the relative eta-invariant is preserved under diffeomorphisms, we have $\tilde{\eta}_\alpha(X, g_l) \neq \tilde{\eta}_\alpha(X, g_{l'}) = \tilde{\eta}_\alpha(X, \psi^*(g_{l'}))$. On the other hand, since the relative eta-invariant is constant on path components of $\mathcal{R}_{scal>0}(X)$ (see Proposition 2.3.23), by the above these quantities should be equal and thus we get a contradiction.

Now since there are only finitely many $Spin^c$ structures on X with torsion canonical class, there are infinitely many values of l and l' for which the above argument applies. It follows that $\mathcal{M}^c_{sec\geq 0}(X)$ has infinitely many path components and so by Lemma 7.1.1, $\mathcal{M}_{sec\geq 0}(X)$ has infinitely many path components as well.

An analogous argument shows that $\mathcal{M}_{Ric>0}(X)$ has infinitely many path components (this was already proved by Goodman [Goo20a]).

Exceptional case:

The argument is exactly the same as in the non-exceptional case. Fix $q \in \{2, 6\}$, k_0 odd and l_0 even. By Proposition 4.2.12, there is an infinite family $\{\overline{X}^5_{k,l}\}_{(k,l)\in Z'_q}$, where Z'_q is the set of all pairs (k_0, l) such that $X^5_{k_0,l}$ is orientation preserving diffeomorphic to $X(q)$ and to $\overline{X} := \overline{X}^5_{k_0,l_0}$. Let $\phi_l : \overline{X} \to \overline{X}^5_{k_0,l}$ be a $Spin^c$ structure preserving diffeomorphism, define $\overline{g}_l := \phi_l^*(\overline{g}_{k_0,l})$ where $\overline{g}_{k_0,l}$ is the metric on $\overline{X}^5_{k_0,l}$ from §6.1.5 and pull back the $Spin^c$ structure on $\overline{X}^5_{k_0,l}$ to $\overline{X}$ via the same diffeomorphism.

We argue by contradiction. Suppose there is a path in $\mathcal{M}^c_{sec\geq 0}(\overline{X})$ connecting $[\overline{g}_l]$ and $[\overline{g}_{l'}]$, where $l \neq l'$. As a consequence of Ebin's slice theorem, this path lifts to a path γ in $\mathcal{R}_{sec\geq 0}(\overline{X})$ with endpoints $\gamma(0) = \overline{g}_l$ and $\gamma(1) = \psi'^*(\overline{g}_{l'})$, where $\psi' \in \mathrm{Diff}^c(\overline{X})$. By Böhm and Wilking, the path γ immediately evolves to a path in $\mathcal{R}_{Ric>0}(\overline{X})$ under the Ricci flow, and if we concatenate this resulting path the orbits of the endpoints, we get a path γ' in $\mathcal{R}_{scal>0}(\overline{X})$ with $\gamma'(0) = \overline{g}_l$ and $\gamma'(1) = \psi'^*(\overline{g}_{l'})$.

Now let E'_α denote the flat non-trivial complex line bundle over $\overline{X}$. By Proposition 7.1.5 we have $\tilde{\eta}_\alpha(\overline{X}, \overline{g}_l) \neq \tilde{\eta}_\alpha(\overline{X}, \psi^*(\overline{g}_{l'}))$. On the other hand, since the relative eta-invariant is constant on path components of $\mathcal{R}_{scal>0}(X)$ (see Proposition 2.3.23), by the above these quantities should be equal and thus we get a contradiction.

Since there are only finitely many $Spin^c$ structures on $\overline{X}$ with torsion canonical class, there are infinitely many values of l and l' for which the above argument applies. It follows that $\mathcal{M}^c_{sec\geq 0}(\overline{X})$ has infinitely many path components and thus by Lemma 7.1.1, $\mathcal{M}_{sec\geq 0}(\overline{X})$ has infinitely many path components as well.

An analogous argument shows that $\mathcal{M}_{Ric>0}(\overline{X})$ has infinitely many path components.

■

This concludes the proof of Theorem A.

7.2 Proof of Theorem B

Recall Theorem B.

Theorem. *Let M^{15} be the total space of a linear S^7-bundle over S^8 and assume M^{15} is a rational homology sphere. The moduli space of positive Ricci curvature metrics on M has infinitely many path components.*

The proof follows the line of the proof of the main theorem in [Des17], which states that the moduli space of non-negative sectional curvature metrics on Milnor spheres (and more generally on the total space M of S^3-bundles over S^4 with $H^4(M;\mathbb{Q}) = 0$) has infinitely many path-components.

Fix $k \in \mathbb{Z}$, $l > 0$ and let $M := M^{15}_{k,l}$ be the total space of a linear S^7-bundle over S^8. Then, by Corollary 5.2.3, for $k(m) = k + 8128 \cdot 2lm$, $m \in \mathbb{Z}$, the set $\{M^{15}_{k(m),l}\}_{m\in\mathbb{Z}}$ is an infinite family of manifolds orientation preserving diffeomorphic to M.

We can now show the following.

Proposition 7.2.1. *The moduli space $\mathcal{M}_{Ric>0}(M)$ has infinitely many path components.*

Proof. Fix $m_0, m_1 \in \mathbb{Z}$ such that $|2k_0 + l| \neq |2k_1 + l|$, where $k_i := k + 8128 \cdot 2lm_i$ for $i = 0, 1$. Denote by $\phi_i : M \to M^{15}_{k_i,l}$ the diffeomorphism for $i = 0, 1$ (see Theorem 5.2.2). Equip $M_i := M^{15}_{k_i,l}$ with the metric $g_i := g_{k_i,l}$ from Proposition 6.2.4 and consider the metrics $h_i = \phi_i^*(g_i)$ on M.

The proof goes by contradiction. Assume there is a path γ in $\mathcal{M}_{Ric>0}(M)$ with endpoints $\gamma(0) = [h_0]$ and $\gamma(1) = [h_1]$. As a consequence of the Ebin slice theorem (Proposition 6.3.1), this path lifts to a path $\tilde{\gamma}$ in $\mathcal{R}_{Ric>0}(M)$ such that $\tilde{\gamma}(0) = h_0$ and $\tilde{\gamma}(1) = \psi^*(h_1)$ for some $\psi \in \mathrm{Diff}(M)$. If ψ is orientation reversing, we can replace g_1 by its pullback under an orientation reversing diffeomorphism of M_1 (the pullback of g_1 by this orientation reversing diffeomorphism still gives a representative of $[h_1]$ in $\mathcal{M}_{Ric>0}(M)$), in order to compensate. Hence, we can from now on assume (without loss of generality) that ψ is orientation preserving.

The path $\tilde{\gamma}$ in particular also lies in $\mathcal{R}_{scal>0}(M)$. We can reparametrize this path (and still denote it by $\tilde{\gamma}$) in such a way that it becomes constant near the endpoints $\tilde{\gamma}(0) = h_0$ and $\tilde{\gamma}(1) = \psi^*(h_1)$. According to Gromov and Lawson [GL80, Lemma 3], the product $M \times [0, a]$ equipped with $\gamma(t/a) + dt^2$ has positive scalar curvature for some $a \gg 0$.

Define the closed spin manifold

$$X^{16} := W_0 \cup_{\phi_0^{-1}} (M \times [0, a]) \cup_{\phi_1 \psi} (-W_1),$$

where $W_i := W^{16}_{k_i,l}$ is the total space of the disk bundle associated to M_i for $i = 0, 1$ (see §5.1), equipped with the positive scalar curvature metric which is of product form near the boundary from Proposition 6.2.4, in order to glue them to the cylinder $(M \times [0,a], \gamma(t/a) + dt^2)$. Note that X is diffeomorphic to $W_0 \cup_\alpha (-W_1)$, where $\alpha := \phi_1 \psi \phi_0^{-1}$.

We can now consider the Gromov-Lawson invariant of X

$$i(h_0, \psi^*(h_1)) = \operatorname{index}(D^+),$$

where D^+ is the $Spin^+$ Dirac operator on X. Since the metric on $M \times [0,a]$ is of positive scalar curvature, by Proposition 2.3.22 we have $i(h_0, \psi^*(h_1)) = 0$. But by the Atiyah-Singer index theorem 2.3.6, we have

$$\begin{aligned} \operatorname{index}(D^+) &= \langle \hat{A}(X), [X] \rangle && (7.1)\\ &= \langle \frac{-192p_4 + 512p_3p_1 + 208p_2^2 - 904p_2p_1^2 - 904p_1^4}{464486400}, [X] \rangle. && (7.2) \end{aligned}$$

By Hirzebruch's signature theorem 2.3.14, we have

$$\operatorname{sign}(X) = \langle L(X), [X] \rangle = \langle \frac{381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4}{14175}, [X] \rangle. \tag{7.3}$$

Recall that $\operatorname{sign}(\pm W_i) = \pm 1$ for $i = 0, 1$ (see §5.1). Therefore, since $X \cong W_0 \cup_\alpha (-W_1)$, by Proposition 2.3.15 the signature of X is

$$\operatorname{sign}(X) = \operatorname{sign}(W_0) + \operatorname{sign}(-W_1) = 1 - 1 = 0.$$

By the Mayer-Vietoris exact sequence, we have $H^4(X; \mathbb{Z}) = 0$. Therefore $p_1(X) = 0$ and so by the above, Equations (7.1) and (7.3) reduce to

$$\langle -192p_4 + 208p_2^2, [X] \rangle = 0,$$

$$\langle 381p_4 - 19p_2^2, [X] \rangle = 0.$$

It follows that both $\langle p_4, [X] \rangle$ and $\langle p_2^2, [X] \rangle$ must vanish. But using Equation (5.2) we compute

$$\begin{aligned} \langle p_2^2, [X] \rangle &= \langle \overline{p}_2^2(W_0), [W_0, M_0] \rangle - \langle \overline{p}_2^2(W_1), [W_1, M_1] \rangle \\ &= \frac{36}{l}\big((2k_0 + l)^2 - (2k_1 + l)^2\big). \end{aligned}$$

This is a contradiction, since we assumed $|2k_0 + l| \neq |2k_1 + l|$ at the beginning of the proof. Hence $[h_0]$ and $[h_1]$ cannot lie in the same path-component of $\mathcal{M}_{Ric>0}(M)$. The result now immediately follows. $\square$

Using Theorem 5.2.2, this completes the proof of Theorem B. $\blacksquare$

Remark 7.2.2. *If $l = 1$, i.e $M_{k,1}^{15}$ is a homotopy 15-sphere, this result was already proved by Wraith [Wra11] using a different method to construct suitable positive Ricci curvature metrics that extend to a coboundary (see also [Wra97]). Wraith's method can also be applied to the moduli space of more general S^7-bundles over S^8, leading to the same result.*

7.3 Proof of Theorem C

Recall Theorem C.

Theorem. *The moduli space of metrics of non-negative sectional curvature of all Milnor projective spaces has infinitely many path components. The same is true for the moduli space of positive Ricci curvature metrics.*

Like in the proof of Theorem B, the proof of Theorem C follows closely the main steps of [Des17].

Fix $k \in \mathbb{Z}$ and let $Q^7 := Q^7_k = M^7_k/\tau$ be a Milnor projective space. Equip Q^7 and the corresponding Milnor sphere $M^7 = M^7_k$ with the above Grove-Ziller metric $\tilde{g} := \tilde{g}^{GZ}_k$ and $g := g^{GZ}_k$ respectively (see Theorem 6.1.2).

Proposition 7.3.1. *The moduli space $\mathcal{M}_{sec\geq 0}(Q^7)$ has infinitely many path components. The same is true for $\mathcal{M}_{Ric>0}(Q^7)$.*

Proof. Fix $m_0, m_1 \in \mathbb{Z}$ such that $|2k_0+1| \neq |2k_1+1|$, where $k_i = k+56m_i$ for $i = 0, 1$. Then, by Theorem 5.2.2 and 5.3.11, there are orientation preserving diffeomorphisms $\Psi_i : Q^7 \to Q^7_{k_i}$ and τ-equivariant diffeomorphisms $\tilde{\Psi}_i : M^7 \to M^7_{k_i}$ for $i = 0, 1$. Equip $Q_i := Q^7_{k_i}$ with the Grove-Ziller metric $g_i := g^{GZ}_{k_i}$ and $M_i := M^7_{k_i}$ with the corresponding metric $\tilde{g}_i := \tilde{g}^{GZ}_{k_i}$ (see Theorem 6.1.2).

We then have the following commutative diagram

$$\begin{tikzcd} M_0 \arrow[d, "\pi_0"'] & M^7 \arrow[l, "\tilde{\Psi}_0"'] \arrow[r, "\tilde{\Psi}_1"] \arrow[d, "\pi"] & M^7_1 \arrow[d, "\pi_1"] \\ Q_0 & Q^7 \arrow[l, "\Psi_0"'] \arrow[r, "\Psi_1"] & Q_1 \end{tikzcd}$$

where π_i and π are the corresponding canonical projections for $i = 0, 1$.

The proof is by contradiction. Let $h_0 := \Psi_0^*(g_0)$ and $h_1 := \Psi_1^*(g_1)$. Consider $[h_0], [h_1] \in \mathcal{M}_{\mathrm{sec}\geq 0}(Q^7)$ and suppose that there is a path between them in this moduli space. As a consequence of the Ebin slice theorem (Proposition 6.3.1), this path lifts to a path $\widehat{\gamma}$ in $\mathcal{R}_{\mathrm{sec}\geq 0}(Q^7)$ such that $\widehat{\gamma}(0) = h_0$ and $\widehat{\gamma}(1) = \phi^*(h_1)$ for some $\phi \in \mathrm{Diff}(Q^7)$. If ϕ is orientation reversing, we can replace g_1 by its pullback under another orientation reversing diffeomorphism of Q^7_1, in order for the composition of the two diffeomorphisms to be orientation preserving. Hence, we can from now on assume (without loss of generality) that ϕ is orientation preserving.

By Böhm and Wilking (Proposition 6.2.1), the path $\widehat{\gamma}$ instantly evolves to a path $\widehat{\gamma}'$ in $\mathcal{R}_{\mathrm{Ric}>0}(Q^7)$ under the Ricci flow. If we denote the trajectories of the endpoints $\widehat{\gamma}(0)$ and $\widehat{\gamma}(1)$ under the Ricci flow by $\widehat{\gamma}'_0$ and $\widehat{\gamma}'_1$ respectively, then the concatenation of these trajectories with $\widehat{\gamma}'$ yields a path in $\mathcal{R}_{\mathrm{scal}>0}(Q^7)$, since it has $scal > 0$ at the endpoints and $Ric > 0$ in the interior of the interval. By a reparametrization and small perturbation that leaves the endpoints fixed, we obtain a smooth path $\gamma : [0,1] \to \mathcal{R}_{\mathrm{scal}>0}(Q^7)$ which is constant near the endpoints $\gamma(0) = h_0$ and $\gamma(1) = \phi^*(h_1)$.

Recall that π is a local isometry. Then, the pullback $\tilde{\gamma} := \pi^*(\gamma)$ is a path in $\mathcal{R}_{\mathrm{scal}>0}(M^7)$ starting at $\tilde{\gamma}(0) = \pi^*(h_0)$ and ending at $\tilde{\gamma}(1) = \pi^*\phi^*(h_1)$. Now as a special case of Theorem 3.4.2, there exists an orientation preserving diffeomorphism $\tilde{\phi} \in \mathrm{Diff}(M^7)$ which makes the following diagram commute.

$$\begin{tikzcd} M^7 \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi"] & M^7 \arrow[d, "\pi"] \\ Q^7 \arrow[r, "\phi"] & Q^7 \end{tikzcd}$$

By commutativity of the above diagrams, we see that we can rewrite the endpoints as $\tilde{\gamma}(0) = \tilde{\Psi}_0^*\pi_0^*(g_0) = \tilde{h}_0$ and $\tilde{\gamma}(1) = \tilde{\phi}^*\tilde{\Psi}_1^*\pi_1^*(g_1) = \tilde{\phi}^*(\tilde{h}_1)$ where $\tilde{h}_i := \tilde{\Psi}_i^*(\tilde{g}_i)$ for $i = 0, 1$.

We are now in a similar situation as in the proof of Theorem B and can replicate the argument.

According to Gromov and Lawson [GL80, Lemma 3], the product $M^7 \times [0,a]$ equipped with $\tilde{\gamma}(t/a) + dt^2$ has positive scalar curvature for some $a \gg 0$.

Define the following closed spin manifold:

$$X^8 := W_0 \cup_{\tilde{\Psi}_0^{-1}} (M^7 \times [0,a]) \cup_{\tilde{\phi}^{-1}\tilde{\Psi}_1^{-1}} (-W_1),$$

where $W_i := W^8_{h_i}$ is the total space of the disk bundle associated to M_i (see §5.1) for $i = 0, 1$, equipped with the metric of nonnegative sectional curvature which is of product form near the boundary from Theorem 6.1.2, in order to glue them to the cylinder $(M^7 \times [0,a], \tilde{\gamma}(t/a) + dt^2)$.

We can now consider the Gromov-Lawson invariant of X

$$i(\tilde{h}_0, \tilde{\phi}^*(\tilde{h}_1)) = \mathrm{index}(D^+),$$

where D^+ is the $Spin^+$ Dirac operator on X. Since the metric on $M \times [0,a]$ is of positive scalar curvature, by Proposition 2.3.22 we have $i(\tilde{h}_0, \tilde{\phi}^*(\tilde{h}_1)) = 0$. But by the Atiyah-Singer index theorem 2.3.6, we have

$$\mathrm{index}(D^+) = \langle \hat{A}(X), [X] \rangle = \langle \frac{-4p_2(X) + 7p_1^2(X)}{5760}, [X] \rangle.$$

By Hirzebruch's signature theorem 2.3.14,

$$\mathrm{sign}(X) = \langle L(X), [X] \rangle = \langle \frac{7p_2(X) - p_1^2(X)}{45}, [X] \rangle.$$

Furthermore, X is diffeomorphic to $W_0 \cup_{\tilde{\Psi}_1\tilde{\phi}\tilde{\Psi}_0^{-1}} (-W_1)$ and $\mathrm{sign}(\pm W_h) = \pm 1$ (see §5.1). Hence, by Proposition 2.3.15 we get

$$\mathrm{sign}(X) = \mathrm{sign}(W_0) + \mathrm{sign}(-W_1) = 1 - 1 = 0.$$

With the two preceeding constraints, it follows that $\langle p_1^2(X), [X]\rangle = 0$. But, using Equation (5.2), we also have

$$\begin{aligned}\langle p_1^2(X), [X]\rangle &= \langle(\overline{p}_1^2(W_0), [W_0, M_0]\rangle - \langle(\overline{p}_1^2(W_1), [W_1, M_1]\rangle \\ &= (2k_0+1)^2 - (2k_1+1)^2.\end{aligned}$$

See §3.1 (also §5.2) for the definition of $\overline{p}_1(W_i)$ for $i = 0, 1$.

This yields the desired contradiction, since we assumed $|2k_0+1| \neq |2k_1+1|$ at the beginning of the proof. Hence $[h_0]$ and $[h_1]$ cannot lie in the same path-component of $\mathcal{M}_{sec\geq 0}(Q^7)$. Therefore, since by Corollary 5.2.3 and Theorem 5.3.11 there are infinitely many values k_0 and k_1 satisfying the above conditions, $\mathcal{M}_{sec\geq 0}(Q^7)$ has infinitely many path components.

To show that $\mathcal{M}_{Ric>0}(Q^7)$ has infinitely many path components, let h_0' and h_1' be the evolved metrics of h_0 and h_1 under the Ricci flow. Assume by contradiction that there is a path in $\mathcal{M}_{Ric>0}(Q^7)$ connecting $[h_0']$ and $[h_1']$ (the square brackets now denote equivalence classes in the moduli space of positive Ricci curvature). Using Ebin's slice theorem, we lift this to a path $\overline{\gamma}$ in $\mathcal{R}_{Ric>0}(Q^7)$ with endpoints $\overline{\gamma}(0) = h_0'$ and $\overline{\gamma}(1) = \overline{\phi}^*(h_1')$, where $\overline{\phi} \in \mathrm{Diff}(Q^7)$ is orientation preserving (see the argument above). Observe that $\overline{\phi}^*(h_1')$ is the solution of the Ricci flow applied to $\overline{\phi}^*(h_1)$. If we now concatenate the trajectories of h_0 and $\overline{\phi}^*(h_1)$ under the Ricci flow with $\overline{\gamma}$, we are now in the same situation as above. That is, we can pull back the obtained path in $\mathcal{R}_{scal>0}(Q^7)$ to the cover M^7, construct the spin manifold X^8 and compute its Pontrjagin numbers to arrive at a contradiction.

□

Remark 7.3.2. *Instead of using the metrics we obtain through the Ricci flow, one can use the positive Ricci curvature metrics from Proposition 6.2.4 and Remark 6.2.5 to come to the same conclusion about* $\mathcal{M}_{Ric>0}(Q^7)$.

In view of the classification Theorem 5.3.12, this also concludes the proof of Theorem C. ■

7.4 Proof of Theorem D

Recall Theorem D.

Theorem. *There exist finitely many and at least 4096 oriented diffeomorphism types of Shimada projective spaces whose moduli space of positive Ricci curvature metrics has infinitely many path components.*

The idea of the proof is the same as the proof of Theorem C.

Fix $k \in \mathbb{Z}$ such that $\{Q^{15}_{k+130048m}\}_{m\in\mathbb{Z}}$ contains an infinite family all orientation preserving diffeomorphic to the Shimada projective space $Q := Q^{15}_k := M^{15}_k/\tau$ (which by Proposition 5.3.14 exists). Equip Q with the metric $g := g_k$ of positive Ricci curvature from §6.2.3 and $M := M^{15}_k$ with the corresponding metric from Proposition 6.2.4. Let $\pi : M \to Q$ denote the canonical projection.

Proposition 7.4.1. *The moduli space $\mathcal{M}_{Ric>0}(Q)$ has infinitely many path components.*

Proof. Let $m_0 \neq m_1 \in \mathbb{Z}$ be such that there exist orientation preserving diffeomorphisms $\Psi_i : Q \to Q^{15}_{k_i}$ where $k_i = k+130048m_i$ for $i = 0, 1$ and such that $|2k_0 + 1| \neq |2k_1 + 1|$. Then there exist τ-equivariant diffeomorphisms $\tilde{\Psi}_i : M \to M^{15}_{k_i}$. If $\pi_i : M^{15}_{k_i} \to Q^{15}_{k_i}$ is the canonical projection, it is straightforward to see that $\Psi_i \circ \pi = \pi_i \circ \tilde{\Psi}_i$. Equip $Q_i := Q^{15}_{k_i}$ with the metric $g_i := g_{k_i}$ of positive Ricci curvature from §6.2.3 and $M_i := M^{15}_{k_i}$ with the corresponding metric $\tilde{g}_i := g_{k_i,1}$ from Proposition 6.2.4. Note that $\tilde{g}_i = \pi_i^*(g_i)$. Consider the metrics $h_i = \Psi_i^*(g_i)$ on N.

The proof is by contradiction. Consider $[h_0], [h_1] \in \mathcal{M}_{Ric>0}(Q)$ and suppose there is a path connecting them in this moduli space. As a consequence of the Ebin slice theorem (Proposition 6.3.1), this path lifts to a path $\widehat{\gamma}$ in $\mathcal{R}_{Ric>0}(Q)$ such that $\widehat{\gamma}(0) = h_0$ and $\widehat{\gamma}(1) = \phi^*(h_1)$ for some $\phi \in \mathrm{Diff}(Q)$. If ϕ is orientation reversing, we can replace g_1 by its pullback under another orientation reversing diffeomorphism of Q_1, in order for the composition of the two diffeomorphisms to be orientation preserving. Hence, we can from now on assume (without loss of generality) that ϕ is orientation preserving.

In particular, the path $\widehat{\gamma}$ also lies in $\mathcal{R}_{scal>0}(Q)$. Hence, if $\pi : M \to Q$ is the canonical projection (which in particular is a local isometry), then $\gamma := \pi^*(\widehat{\gamma})$ is a path in $\mathcal{R}_{scal>0}(M)$ with endpoints $\gamma(0) = \pi^*(h_0) = \pi^*\Psi_0^*(g_0) = \tilde{\Psi}_0^*\pi_0^*(g_0) = \tilde{\Psi}_0^*(\tilde{g}_0)$ and $\gamma(1) = \pi^*\phi^*(h_1) = \tilde{\phi}^*\tilde{\Psi}_1^*(\tilde{g}_1)$, where $\tilde{\phi} : M \to M$ is a diffeomorphism satisfying $\phi \circ \pi = \pi \circ \tilde{\phi}$ (see Theorem 3.4.2).

We are now in the exact same situation as in the proof of Proposition 7.2.1. In the end, we get a contradiciton by considering the Gromov-Lawson invariant and computing the signature of a cylinder capped by the disk bundles $W^{16}_{k_i}$, whose boundary is $M^{15}_{k_i}$, for $i = 0, 1$. The result then follows since we get infinitely many values for k_0 and k_1 such that $[h_0]$ and $[h_1]$ lie in different path components of $\mathcal{M}_{Ric>0}(Q)$. □

In view of our classification of Shimada projective spaces (Proposition 5.3.14), this also completes the proof of Theorem D. ■

Chapter 8

Appendix

8.1 Appendix A. Equivariant bundles and structures

In this appendix, we discuss some defintions and results about equivariant bundles, equivariant connections and equivariant *Spin* and *Spin*^{c} structures. A reference for some of the elementary definitions is [Seg68], but the author was unable to find good references for most of the following material.

From now on, G is a compact Lie group, all spaces are smooth manifolds and all maps are assumed to be smooth.

A *G-equivariant vector bundle* is a vector bundle $p : E \to B$ such that both E and B are G-spaces, $p(g \cdot e) = g \cdot p(e)$ and $g : E_b \to E_{g \cdot b}$ is a vector space isomorphism for all $g \in G$, $e \in E$ and $b \in B$. If $\Gamma(E)$ is the space of sections, then G acts on $\Gamma(E)$ via $g * s$ for $g \in G$ and $s \in \Gamma(E)$, where $(g*s)(b) = g \cdot s(g^{-1} \cdot b)$ for all $b \in B$. A connection $\nabla : \mathfrak{X}(B) \times \Gamma(E) \to \Gamma(E)$ in E is called G-equivariant if $\nabla_{g*X}(g*s) = g * \nabla_X s$.

Proposition 8.1.1. *Let M be a closed manifold and τ a smooth orientation preserving fixed point free involution on M. Let $\mathbb{Z}_2 = \{Id_M, \tau\}$. Suppose that $p : E \to M$ is a vector bundle and that the action of τ on M lifts to an action $\tilde{\tau}$ on E such that p is a $\mathbb{Z}_2$-equivariant vector bundle. Then the quotient $\overline{p} : E/\tilde{\tau} \to M/\tau$ is a vector bundle and $E \cong \pi^*(E/\tilde{\tau})$, where $\pi : M \to M/\tau$ is the projection.*

Proof. First observe that $\tilde{\tau}$ acts without fixed points on E. Indeed, assume that there is an $e \in E$ such that $\tilde{\tau}(e) = e$, then $p(e) = p(\tilde{\tau}(e)) = \tau(p(e))$ by equivariance, but this is a contradiction since we assumed τ to be fixed point free.

We now show that $\overline{p} : E/\tilde{\tau} \to M/\tau : [e] \mapsto [p(e)]$ is a vector bundle. The map is well-defined since p is $\mathbb{Z}_2$-equivariant, and since p is surjective, $\overline{p}$ is surjective as well. Local triviality is established by using the $\mathbb{Z}_2$-equivariance of p and the fact that $\tilde{\tau} : E_x \to E_{\tau(x)}$ is an isomorphism.

An isomorphism between E and $\pi^*(E/\tilde{\tau}) = \{(x, [e]) \in M \times E/\tilde{\tau} \mid \pi(e) = \overline{p}([e])\}$ is given by $\phi(e) = (p(e), [e])$. The inverse is $\phi^{-1}(x, [e]) = e'$, where e' is the unique element in $[e] = \{e, \tilde{\tau}(e)\}$ such that $p(e') = x$. $\square$

Observe that one can apply this result to the tangent bundle of a smooth manifold M. Indeed, the involution τ induces an involution on TM via the differential $d\tau$, turning it into a $\mathbb{Z}_2$-equivariant vector bundle and the tangent bundle over M/τ is then precisely $TM/d\tau$.

Proposition 8.1.2. *Let $p : E \to M$, τ and $\overline{p} : E/\tilde{\tau} \to M/\tau$ be as above. Let ∇ be a $\mathbb{Z}_2$-equivariant connection on E. Then this connection induces a connection $\overline{\nabla}$ on $E/\tilde{\tau}$.*

Proof. Recall that $\pi : M \to M/\tau$ denotes the projection and let $\phi : E \to \pi^*(E/\tilde{\tau})$ be the isomorphism from the previous result. If $\overline{s}$ is a section on $E/\tilde{\tau}$, then $s(x) = \phi^{-1}(x, \overline{s}(\pi(x)))$, $x \in M$, defines a section on E, which is sometimes called the pullback section and denoted by $\pi^*\overline{s} = \overline{s} \circ \pi$. If we consider a vector field $\overline{X}$ on M/τ, we similarly get a 'pullback vector field' $X = \pi^*\overline{X}$ on M. Then $\overline{\nabla}_{\overline{X}}\overline{s}([x]) = [\nabla_{\pi^*\overline{X}}(\pi^*\overline{s})(x)]$, $x \in M$, defines a connection on $E/\tilde{\tau}$. $\square$

Let G and H be compact Lie groups. A *G-equivariant principal H-bundle* is a principal H-bundle $p : P \to M$ such that P and M are both G-spaces, the actions of G and H on P commute, i.e. $g \cdot (q \cdot h) = (g \cdot q) \cdot h$, and $p(g \cdot q) = g \cdot p(q)$ for all $g \in G$, $h \in H$ and $q \in P$. The action of G on the space of sections is defined like in the case of vector bundles. A connection on P, represented by the connection form ω, is said to be G-equivariant if $g_*\omega = \omega$ for all $g \in G$.

Proposition 8.1.3. *Let M be a closed manifold, H a compact group and τ a smooth orientation preserving fixed point free involution on M. Let $\mathbb{Z}_2 = \{Id_M, \tau\}$. Suppose that $p : P \to M$ is a principal H-bundle and that the action of τ on M lifts to an action $\tilde{\tau}$ on P which commutes with the principal H-action. Then the quotient $\overline{p} : P/\tilde{\tau} \to M/\tau$ is a principal H-bundle and $P \cong \pi^*(P/\tilde{\tau})$, where $\pi : M \to M/\tau$ is the projection.*

Proof. The proof is very similar to the proof of Proposition 8.1.1, the only major difference being that we have to establish that the action of H on $P/\tilde{\tau}$ is free. To do so, let $[q] = \{q, \tilde{\tau}(q)\} \in P/\tilde{\tau}$ and let $h \in H$ be such that $[q] \cdot h = [q]$. Thus either $q \cdot h = q$ or $\tilde{\tau}(q) \cdot h = q$. In the first case, since the action of H is free on P, we have $h = e$ and so the action on $P/\tilde{\tau}$ is indeed free. The second case impossible, since if we apply p to the equation, by commutativity and equivariance we get $p(\tilde{\tau}(q) \cdot h) = p(\tilde{\tau}(q \cdot h)) = \tau(p(q \cdot h)) = \tau(p(q)) = p(q)$, which cannot be because τ is fixed point free.

$\square$

Let M and τ be as above. Then the derivative $d\tau$ induces an action $d\tilde{\tau}$ on the principal bundle of orthonormal frames P_{SO}, turning it into a $\mathbb{Z}_2$-equivariant principal $SO(n)$-bundle. The principal bundle of orthonormal frames over M/τ is then $P_{SO}/d\tilde{\tau}$.

Proposition 8.1.4. *Let $p : P \to M$, τ and $\overline{p} : P/\tilde{\tau} \to M/\tau$ be as above. Let ω be a $\mathbb{Z}_2$-equivariant connection form on P, where $\mathbb{Z}_2 = \{Id_P, \tilde{\tau}\}$. Then this connection induces a connection $\overline{\omega}$ on $P/\tilde{\tau}$.*

Proof. Let $\mathcal{H}$ be the horizontal distribution defining the connection form ω, that is, if for $q \in P$, $\nu : T_qP = \mathcal{V}_q \oplus \mathcal{H}_q \to \mathcal{V}_q$ is the projection and $j_q : H \to P : h \mapsto q \cdot h$ gives the isomorphism $j_{q*} : \mathfrak{h} \to \mathcal{V}_q$, then $\omega_q(X_q) = j_{q*}^{-1}(\nu(X_q)) \in \mathfrak{h}$ for all $X_q \in T_qP$.

Now observe that an element of $T_{[q]}P/\tilde{\tau}$ is of the form $\overline{X}_{[q]} = \{X_q, \tilde{\tau}_{q*}(X_q)\}$. Then, since ω is $\mathbb{Z}_2$-equivariant, we have $\omega_{\tau(q)}(\tilde{\tau}_{q*}(X_q)) = \tilde{\tau}^*\omega_q(X_q) = \omega_q(X_q)$. Hence, by setting $\overline{\omega}_{[q]}(\overline{X}_{[q]}) := \omega_q(X_q)$ for all $[q] \in M/\tau$ and $\overline{X}_{[q]} \in T_{[q]}P/\tilde{\tau}$, we obtain a connection on $P/\tilde{\tau}$. □

Now let M be a closed Riemannian manifold on which a compact Lie group G acts. A *G-equivariant Spin structure* on M is a *Spin* structure, such that P_{Spin} is a G-equivariant principal $Spin(n)$-bundle and the map $\beta : P_{Spin} \to P_{SO}$ is G-equivariant.

Similarly, a *G-equivariant $Spin^c$ structure* on M is a $Spin^c$ structure, such that P_{Spin^c} is a G-equivariant principal $Spin^c(n)$-bundle and the map $\gamma : P_{Spin^c} \to P_{SO} \times P_{U(1)}$ is G-equivariant.

For the following result, see also [AB68, p.487].

Proposition 8.1.5. *Let M be a closed spin manifold and suppose that τ is a smooth orientation preserving fixed point free involution which is an isometry and preserves the Spin structure (i.e. τ is of* even *type in the language of [AB68]). Then the Spin structure is $\mathbb{Z}_2$-equivariant and the quotient M/τ inherits a Spin structure.*

Similarly, if M is a closed $Spin^c$ manifold, τ preserves the $Spin^c$ structure, then the $Spin^c$ structure is $\mathbb{Z}_2$-equivariant and the quotient M/τ inherits a $Spin^c$ structure.

Proof. We only prove the spin case, the $Spin^c$ case being similar.

Since τ preserves the *Spin* structure, it lifts to an action $\tilde{\tau}$ on P_{Spin}. By assumption $\tilde{\tau}^2$ is the identity, $\tilde{\tau}$ commutes with the action of $Spin(n)$ and therefore P_{Spin} is a $\mathbb{Z}_2$-equivariant principal $SO(n)$-bundle. As we have observed above, P_{SO} is a $\mathbb{Z}_2$-equivariant $Spin(n)$-bundle in this case and by assumption $\beta : P_{Spin} \to P_{SO}$ is $\mathbb{Z}_2$-equivariant. Thus the *Spin* structure is $\mathbb{Z}_2$-equivariant. Then, by Proposition 8.1.3, it follows $P_{Spin}/\tilde{\tau}$ gives a *Spin* structure on M/τ.

□

With all of the above, the next result now follows immediately.

Proposition 8.1.6. *Let M be a closed spin manifold and τ a smooth orientation preserving fixed point free involution which is an isometry and also preserves the Spin structure. The spinor bundle S is then a $\mathbb{Z}_2$-equivariant vector bundle, and the quotient bundle S/τ corresponds to the spinor bundle over M/τ associated to the induced Spin structure. Furthermore, the Spin Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is $\mathbb{Z}_2$-equivariant and descends to a Spin Dirac operator $\overline{D} : \Gamma(S/\tau) \to \Gamma(S/\tau)$.*

The same is true if we replace Spin by Spin^c.

8.2 Appendix B. C++ code

Unfortunately, it is unknown to the author whether there exist number theoretical methods to solve the arithmetics of Proposition 5.3.10. As a resort, the following C++ code[1] counts the number of different values of the Eells-Kuiper invariant of the Shimada projective spaces (and incidentally, the number of different values of the Eells-Kuiper invariant of the Milnor projective space, if one chooses to uncomment (remove //) and comment (add //) the appropriate lines).

```
#include <iostream>
#include <iomanip> // for setw
using namespace std;

int main() {

int counter;
counter=0;
int countermu, countermuquo, helpcountermu, helpcountermuquo;
countermu=0;
countermuquo=0;
int n, nn, m;
n=16255;
//n=56;

for (int i=0;i<n;i++){
    helpcountermu=0;
    helpcountermuquo=0;
 for (int k = i; k < n; k++) {

     int mui, muk, a, b, c, d;
```

[1]The author does not claim that this is the most efficient way to get to the answer, but it works.

```
     int muiquoplus, mukquoplus, muiquominus, mukquominus;

    mui=i*(i+1)%16256;
    muk=k*(k+1)%16256;

    //mui=i*(i+1)%56;
    //muk=k*(k+1)%56;

    a=2*i*(i+1)+127*(2*i+1);
    muiquoplus=a%65024;
    b=65024+2*i*(i+1)-127*(2*i+1);
    muiquominus=b%65024;

    //a=2*i*(i+1)+7*(2*i+1);
    //muiquoplus=a%224;
    //b=224+2*i*(i+1)-7*(2*i+1);
    //muiquominus=b%224;

    c=2*k*(k+1)+127*(2*k+1);
    mukquoplus=c%65024;
    d=65024+2*k*(k+1)-127*(2*k+1);
    mukquominus=d%65024;

    //c=2*k*(k+1)+7*(2*k+1);
    //mukquoplus=c%224;
    //d=224+2*k*(k+1)-7*(2*k+1);
    //mukquominus=d%224;

if (mui==muk) {
  if (k!=i) {

//If the Eells-Kuiper invariants of the Shimada sphere are equal,
//we don't count it as a "new" distinct value
//Thus, we increment helpcountermu and if it is non-zero,
//we don't increment countermu

      helpcountermu++;

//If the Eells-Kuiper invariants of the quotients are equal,
//we don't count it as a "new" distinct value
//Thus, we increment helpcountermu and if it is non-zero,
//we don't increment countermuquo

if (muiquoplus==mukquoplus && muiquominus==mukquominus)
```

```
  {helpcountermuquo++;}
else if (muiquoplus==mukquominus && muiquominus==mukquoplus)
  {helpcountermuquo++;}
    }
  }
}
 if (helpcountermu==0) {countermu++; }
 if (helpcountermuquo==0) {countermuquo++;}
}
//cout << counter << endl;
cout << "Number mu values (spheres): " << countermu << endl;
cout << "Number of mu values (quotients): "  << countermuquo << endl;
return 0;
}
```

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List of Symbols

$Pin^{\pm}(n)$	*Pin* groups, p. 10
$Spin(n)$	*Spin* group, p. 10
$Spin^c(n)$	$Spin^c$ group, p. 10
P_O	Bundle of orthonormal frames, p. 11
$P_{Pin^{\pm}}$	$Pin^{\pm}$ structure, p. 11
P_{SO}	Bundle of oriented orthonormal frames, p. 11
P_{Spin}	*Spin* structure, p. 11
P_{Spin^c}	$Spin^c$ structure, p. 12
$P_{U(1)}$	Principal $U(1)$-bundle associated to $Spin^c$ structure, p. 12
$\mathrm{index}(D_W, g)$	Equivariant index, p. 14
$h_g(D_M)$	Equivariant kernel, p. 14
$\eta_g(D_M)$	Equivariant η-invariant, p. 15
$\mathrm{index}(D_W)$	Index, p. 15
$h(D_M)$	Kernel, p. 15
$\eta(D_M)$	η-invariant, p. 15
S	Spinor bundle, p. 16
D_M	*Spin* Dirac operator, p. 16
S^+, S^-	Decomposition of spinor bundle, p. 16
D_M^+	$Spin^+$ Dirac operator, p. 16
$\widehat{A}$	$\widehat{A}$-genus in Pontrjagin classes (or forms), p. 16
$a_{spin}(N)$	*Spin* local contribution, p. 18

S_c	Complex spinor bundle, p. 19
D_M^c	$Spin^c$ Dirac operator, p. 19
S_c^+, S_c^-	Decomposition of complex spinor bundle, p. 19
$D_M^{c,+}$	$Spin^{c,+}$ Dirac operator, p. 19
$a_{spin^c}(N)$	$Spin^c$ local contribution, p. 20
$\mathrm{sign}(M)$	Signature, p. 21
L	L-genus in Pontrjagin classes (or forms), p. 21
B_M^{ev}	Odd signature operator, p. 23
$\mathrm{sign}(W, g)$	Equivariant signature, p. 23
$a_{sign}(N)$	Signature local contribution, p. 23
$i(g_0, g_1)$	Gromov-Lawson invariant, p. 26
$\tilde{\eta}_\alpha(M, g_M)$	Relative $Spin^c$ eta-invariant, p. 26
$\overline{p}_i(W)$	"Relative" Pontrjagin classes (or forms), p. 28
$\mu(M)$	Eells-Kuiper invariant, p. 29
$\widehat{p}_i(M)$	"Reduced" Pontrjagin form, p. 29
$hS(M)$	Smooth structure set, p. 30
$\mathcal{N}(M)$	Set of normal invariants, p. 30
G/O	Fiber of the fibration $BO \to BG$, p. 30
$L_n(\mathbb{Z}[\pi_1(M)])$	Surgery obstruction groups, p. 31
Θ_n	Group of h-cobordism classes of homotopy spheres, p. 31
bP_{n+1}	Subgroup of Θ_n, p. 31
PD	Poincaré duality isomorphism, p. 32
$\sigma(\Sigma, T)$	Browder-Livesay invariant, p. 33
$W_\epsilon^6(d), M_\epsilon^5(d)$	Brieskorn varieties, p. 35
$\tau, \overline{\tau}$	Involution, p. 36, p. 40, p. 50
$Q_\epsilon^5(d)$	Brieskorn quotient, p. 36
$N_{k,l}^5$, $\overline{N}_{k,l}^5$	Universal cover of $X_{k,l,\beta}^5$, $\overline{X}_{k,l}^5$, p. 40

$L_{k,l}, \overline{L}_{k,l}$	Line bundle associated to $N^5_{k,l}, \overline{N}^5_{k,l}$, p. 40
$S(L_{k,l}), S(\overline{L}_{k,l})$	Principal S^1-bundle associated to $L_{k,l}, \overline{L}_{k,l}$, p. 40
$W^6_{k,l}$, $\overline{W}^6_{k,l}$	Disk bundle associated to $N^5_{k,l}$, $\overline{N}^5_{k,l}$, p. 40
$X^5_{k,l,\beta}$, $\overline{X}^5_{k,l}$	Principal S^1-bundle with $\pi_1 = \mathbb{Z}_2$, p. 40
$M^{8n-1}_{k,l}$	S^{4n-1}-bundle over S^{4n} for $n = 1, 2$, p. 49
$W^{8n}_{k,l}$	Disk bundle associated to $M^{8n-1}_{k,l}$ for $n = 1, 2$, p. 49
$\xi_{k,l}$	Vector bundle associated to $M^{8n-1}_{k,l}$ for $n = 1, 2$, p. 49
Q^{8n-1}_k	Milnor or Shimada projective space, p. 50
$\mathcal{R}(M)$	Space of Riemannian metrics, p. 70
$\mathcal{M}(M)$	Moduli space of Riemannian metrics, p. 70
$\mathrm{Diff}^c(X)$	$Spin^c$ structure preserving diffeomorphism group, p. 71
$\mathcal{M}^c(X)$	Quotient of $\mathcal{R}(X)$ by $\mathrm{Diff}^c(X)$, p. 71

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Preprints

J. Wermelinger	*Moduli space of nonnegatively curved metrics on Milnor sphere quotients*, preprint on arXiv, (2020)
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Conferences Organized

11.2018	*Riemannian Topology Meeting*, Fribourg (CH)

Conferences Attended

08.2021	*Curvature and Global Shape*, Münster
01.2020	*Spaces and moduli spaces of Riemannian metrics with curvature bounds* - A-Fri-Ka, Karlsruhe
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10.2017	*Oberwolfach Seminar: Lower Curvature Bounds and Topology*, Oberwolfach

Talks Given

09.03.2021	*On the moduli space of positive Ricci metrics on 15-dimensional manifolds* Irish Geometry Seminar, online	
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24.01.2020	*Moduli spaces of metrics of nonnegative sectional curvature on homotopy RP7* Spaces and moduli spaces of Riemannian metrics with curvature bounds - A-Fri-Ka	
11.07.2019	*Milnor spheres, involutions and their quotients*, Oberseminar Topologie Fribourg	
08.07.2019	*The diffeomorphism classification of homotopy RP7*, Oberseminar Topologie Fribourg	
28.01.2019	*A short overview of Surgery Theory III*, Oberseminar Topologie Fribourg	
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26.02.2018	*Milnor's work on spheres and the Eells-Kuiper invariant II*, Oberseminar Topologie Fribourg	
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Teaching Experience

	University of Fribourg	
2020 – 2021	Teaching assistant:	- *Linear Algebra* I and II - *Mathematics* I and II for BSc_SI - *Linear Algebra* for propaedeutics
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2013 – 2016	Teaching assistant:	- *General physics* II, III and IV (for engineers) - *Probability and statistics* I and II (for engineers) - *Analysis* II (for engineers)

Popularization of Mathematics

25.04.2018 *Minimalflächen und Seifenfilmexperimente*, TecDay, Kantonschule Alpenquai, Luzern