Introduction to algebraic topology
Fall 2019

The following survey of the material covered in the course is incomplete and may contain typos. Please use your course notes.

1. Simplicial homology

1.1 Simplicial complexes and simplicial maps

Definition of a simplicial complex K (keywords: n-simplex σ, face, boundary $Bd(\sigma)$, subcomplex, p-skeleton $K^{(p)}$, underlying space/polytope $|K|$, triangulation of a space, topology of $|K|$, barycentric coordinates $t_v : |K| \to [0,1]$, ...)

Properties of $|K|$ ($|K|$ Hausdorff, K finite $\implies$ $|K|$ compact, ...)

Definition of a simplicial map and simplicial homeomorphism/isomorphism (keywords: star of v, link of v, vertex map, ...)

Definition of an abstract simplicial complex $\mathcal{S}$ (keywords: simplex, face, vertex set, subcomplex, isomorphic abstract simplicial complexes, vertex scheme, geometric realization of $\mathcal{S}$, ...), examples, minimal triangulations of the torus, ...

Lemma 1.18: Two simplicial complexes are isomorphic iff their vertex schemes are isomorphic as abstract simplicial complexes

1.1 Abelian groups and homology groups

Review of finitely generated abelian groups (keywords: basis, free abelian group, torsion subgroup, Fundamental theorem of finitely generated abelian groups 1.22, ...)

Definition of the group $C_p(K)$ of p-chains of K, examples

Definition of the Euler number $e(K)$, examples, classification of the regular polyhedrons

Definition of the simplicial complex of K and the simplicial homology groups $H_p(K)$ (keywords: boundary homomorphism ∂_p, $\partial_{p-1} \circ \partial_p = 0$, p-cycles, p-boundaries, $Z_p(K)$, $B_p(K)$,...), computations for the interval, triangle, n-gons, torus, Klein bottle, ...

1.3 Simplicial approximation of continuous maps

The aim of this subsection is to show that simplicial approximations exist (Theorem 1.42) and that the induced homomorphism in simplicial homology is independent of the choice of the simplicial approximation (Corollary

1.38). Keywords: simplicial map, induced chain map, star condition, contigous maps, simplicial approximation, chain homotopy, barycentric subdivision, stellar subdivision, cone construction, acyclic complexes, ...

1.4 Relative simplicial homology and excision

Definition of the group of relative p-chains $C_p(K, K_0)$ and relative simplicial homology $H_p(K, K_0)$, Excision theorem 1.51, ...

2. Singular homology

2.1 Singular chain complex

Definition of the singular chain complex and the (reduced) singular homology groups.

Keywords: standard p-simplex $\triangle_p$, singular p-simplex $T : \triangle_p \to X$, singular chain group $S_p(X)$, boundary homomorphism $\partial : S_p(X) \to S_{p-1}(X)$, $f : X \to Y$ continuous induces chain map $f_\sharp : S_p(X) \to S_p(Y)$, $f_\sharp \circ \partial = \partial \circ f_\sharp$, $\partial \circ \partial = 0$, singular chain complex $\mathcal{S}(X)$, singular p-cycles, singular p-boundaries, singular homology groups $H_p(X)$, induced homomorphism $f_* : H_p(X) \to H_p(Y)$, augmentation homomorphism ϵ, reduced singular homology groups $\widetilde{H}_p(X)$, functorial properties (Proposition 2.3 and the following corollary), computation of H_0 and $\widetilde{H}_0$ (Lemma 2.4), computation for a point (Lemma 2.5), group of relative singular p-chains $S_p(X, A)$, relative singular homology groups $H_p(X, A)$, ...

2.2 Excursion: The Zig-Zag Lemma

The aim of this subsection is to prove the Zig-Zag Lemma 2.6 and its Naturality 2.7.

Keywords: chain complexes, chain maps (=homomorphism between chain complexes), short exact sequence of chain complexes $0 \to \mathcal{C} \to \mathcal{D} \to \mathcal{E} \to 0$, homomorphism $\partial_* : H_p(\mathcal{E}) \to H_{p-1}(\mathcal{C})$, long exact sequence in homology, application to the singular homology of a pair (Theorem 2.8): there exists a homomorphism $\partial_* : H_p(X, A) \to H_{p-1}(A)$ and a long exact sequence in singular homology for the pair (X, A), ...

2.3 Homotopy invariance

The aim of this subsection is to prove that homotopic maps induce the same homomorphism in singular homology (Theorems 2.13, 2.14).

Keywords: star convex subsets, bracket operation, star convex $\implies$ acyclic, homotopy between maps, natural chain homotopy for the inclusions $i, j : X \to X \times I$, $f \simeq g \implies f_* = g_*$, consequences (Corollary 2.15, Exercises 2.16), Axiom of compact support 2.17.

2.4 The excision theorem

The aim of this subsection is to present the idea of the proof of the excision theorem in singular homology.

Keywords: barycentric subdivision operator sd_X, subgroup $S_p^{\mathcal{A}}(X)$ generated by $\mathcal{A}$-small singular p-simplices, subchain complex $\mathcal{S}^{\mathcal{A}}(X)$, long exact sequence induced from $0 \to \mathcal{S}^{\mathcal{A}}(X) \to \mathcal{S}(X) \to \mathcal{S}(X)/\mathcal{S}^{\mathcal{A}}(X) \to 0$, $H_*(\mathcal{S}^{\mathcal{A}}(X)) \cong H_*(X)$, relative version, proof of the 5-lemma, $H_*(\mathcal{S}^{\mathcal{A}}(X,B)) \cong H_*(X,B)$, proof of the excision theorem using $\mathcal{A} := \{X - U, A\}$, ...

2.5 Mayer-Vietoris sequences

Proof of the Mayer-Vietoris sequence 2.20 (keywords: excisive couple, zig-zag lemma, reduced version, ...)

3. First applications

Homology of the sphere 3.2, Corollary 3.3: $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$, Brouwer fixed point theorem 3.7, fixed point theorem 3.11, Corollary 3.12, application to vector fields on spheres 3.13, Fundamental theorem of algebra 3.16, Euler characteristic of the union, of the connected sum, of surfaces, ...

Keywords: retraction, deformation retraction, Mayer-Vietoris sequence for the union of upper and lower hemisphere, properties of the degree of a map $f : S^n \to S^n$, $H_i(B^n, S^{n-1})$, degree of the reflection map ρ_1, degree of the antipodal map a, local computations of the degree 3.14, Betti numbers, Euler characteristic, Lemma 3.18 on the rank for an exact sequence, applications to the Euler characteristic, ...

4. Cellular homology

The aim of this section is to introduce the cellular chain groups $\mathcal{C}_p(K)$ of a CW-complex K and the cellular chain complex

$$\ldots \xrightarrow{\partial} \mathcal{C}_p(K) \xrightarrow{\partial} \mathcal{C}_{p-1}(K) \xrightarrow{\partial} \ldots,$$

to prove that the cellular homology group $\mathcal{Z}_p(K)/\mathcal{B}_p(K)$ is isomorphic to the singular homology group $H_p(K)$ and to apply this to compute the homology of real and complex projective spaces. Also homology with coefficients in an abelian group G and the Bockstein sequence are discussed.

Keywords: CW-complex, cells, characteristic map, subcomplex, p-skeleton, proof of the long exact sequence for a triple (X, A, B), CW-structure for the real and complex projective space, $S_p(X; G)$, $H_p(X; G)$, zig-zag lemma, Bockstein sequence, ...

5. Cohomology

5.1 The Hom-functor

Discussion of basic properties of the contravariant functor $Hom(\quad, G)$ (Lemmas 5.1 and Proposition 5.2).

5.2 Singular cohomology

Definition of the (reduced) singular cohomology groups and the axiomatic description in Theorem 5.4.

Keywords: cochain group $S^p(X, A; G)$, dual operator $\delta := \widetilde{\partial}$, cocycles $Z^p(X, A; G)$, coboundaries $B^p(X, A; G)$, cohomology $H^p(X, A; G)$, long exact sequence in cohomology for the pair, augmentation map, reduced cohomology $\widetilde{H}^p(X, A; G)$, $f : X \to Y$ continuous induces chain map $f^\sharp := \widetilde{f}_\sharp : S^q(Y; G) \to S^p(X; G)$, $\delta \circ f^\sharp = f^\sharp \circ \delta$, induced map $f^* : H^p(Y; G) \to H^p(X; G)$, ...

5.3 The cohomology ring

Definition of the cup product

$$\cup : S^p(X; R) \times S^q(X; R) \to S^{p+q}(X; R)$$

for R a commutative ring with 1. Basic properties of $\cup$ are discussed in Proposition 5.7. These imply that $\cup$ defines a product in cohomology

$$\cup : H^p(X; R) \times H^q(X; R) \to H^{p+q}(X; R)$$

which is graded commutative. For $h : X \to Y$ continuous the induced map $h^* : H^*(Y; R) \to H^*(X; R)$ is a ring homomorphism.

Theorem of de Rham (without proof): There is an isomorphism $H^*_{dR}(M) \cong H^*(M; \mathbb{R})$ between de Rham cohomology and singular cohomology with real coefficients of a smooth manifold M.

5.4 Application to division algebras (not part of the exam)

A theorem of Hopf states that the existence of a continuous odd map

$$g : S^{n-1} \times S^{n-1} \to S^{n-1}$$

forces n to be a power of 2. A sketch of the proof is given using the exterior product, computation for the cohomology of the real projective space, the universal coefficient theorem and Poincaré duality. As an application the dimension of a finite dimensional real division algebra must be a power of 2 (actually Adams has proved that the only possibilities are $1, 2, 4$ or 8).