15) Spaces and moduli spaces of Riemannian metrics with curvature bounds on compact and non-compact manifolds

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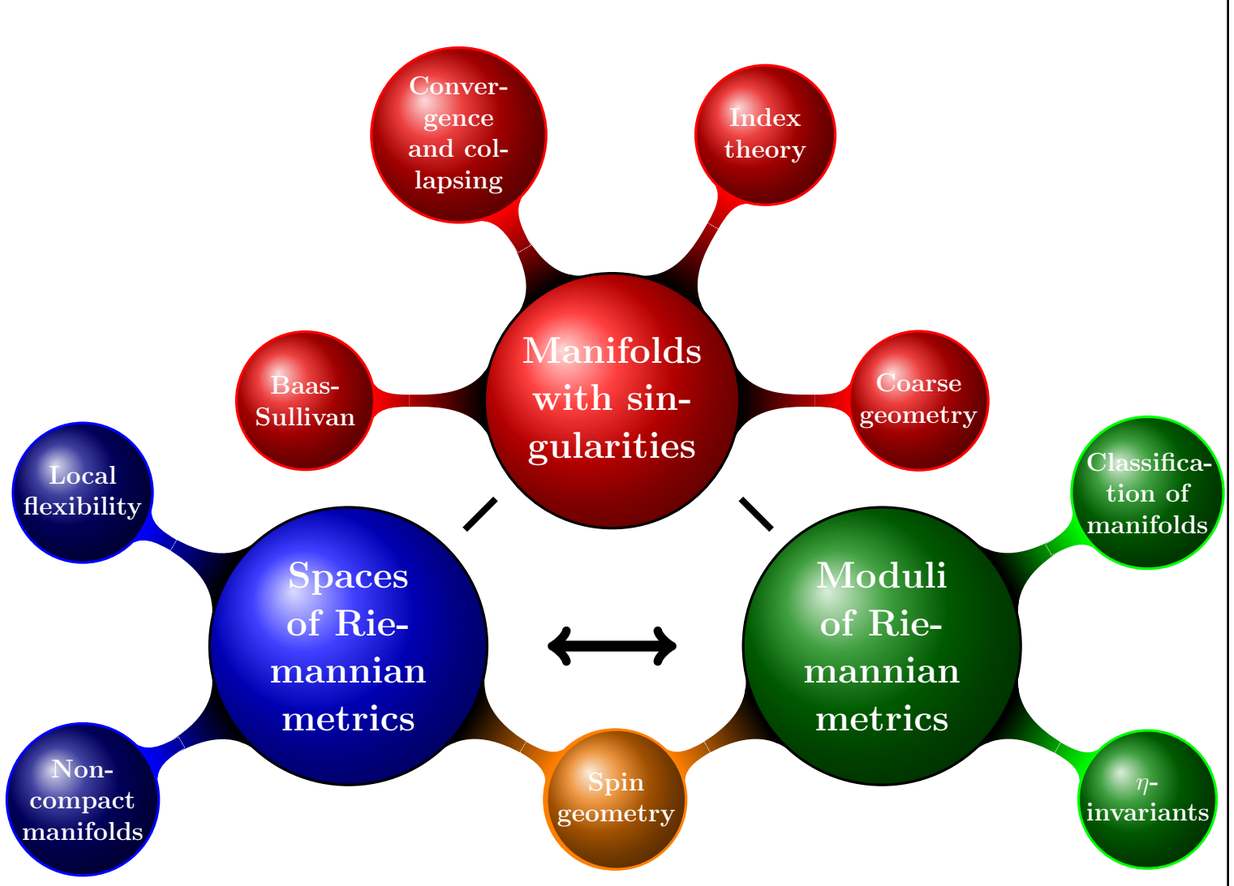
Spaces and moduli spaces of Riemannian metrics



Given a complete Riemannian manifold M with a certain curvature characteristic such as positivity of scalar or Ricci curvature, nonnegativity or negativity of sectional curvature, we ask:

How 'many' such metrics, and how 'many' different geometries of this kind exist on M?

Some activities within our project during the first SPP 2026 funding period:

- *Riemannian topology meeting: Spaces of riemannian metrics and related topics*, Fribourg, 8–9 November 2018, organized by M. Bustamante, D. González-Álvaro, J.-B. Kordaß and J. Wermelinger.
- *Australian-German workshop on differential geometry in the large*, Creswick/Melbourne, 4–15 February 2019, organized by D. Crowley, O. Dearricott, Th. Leistner, Y. Nikolaevsky and W. Tuschmann.
- *Geometry of Scalar Curvature 2019: Conference and Summer School*, Cortona, 8–12 July 2019, organized by B. Hanke, Ch. Mantoulidids, P. Piazza, Th. Schick and Ch. Sormani.

New directions in positive scalar curvature geometry (BH)

Main goals:

- Study obstructions to and constructions of positive scalar curvature metrics on manifolds of low dimensions, manifolds with finite fundamental groups, manifolds with Baas-Sullivan singularities and non-compact manifolds.
- Work on several versions of index theoretic obstructions on non-compact manifolds, relying on the coarse geometry approach of Roe and on the index theory of Callias type Dirac operators.
- Study spaces of positive scalar curvature metrics on manifolds with boundary.

The **classification of non-simply connected manifolds admitting positive scalar curvature metrics** is largely unknown, even for finite fundamental groups. In [Han19] we construct positive scalar curvature metrics on closed, non-spin, p-atoral manifolds M of dimensions at least 5 whose fundamental groups are abelian p-groups, where p is an odd prime, thus resolving a question of Botvinnik-Rosenberg (2002). Although our result deals with smooth manifolds our argument introduces and employs the concept of *positive scalar curvature metrics on manifolds with Baas-Sullivan singularities*. We plan to extend this result to the spin case, where we expect that Hitchin's α-invariant is the only obstruction to the existence of such a metric. This requires an analysis of the KO-homology of abelian p-groups for odd p.

Jian Wang proved **non-existence of positive scalar curvature metrics on some non-compact contractible 3-manifolds** in [Wan19] This argument is based on the minimal hypersurface method of Schoen-Yau. Together with Wang we plan to develop an index theoretic approach to this important result. We hope that the index theory of Callias type operators with potentials may lead to useful new insight. Questions of this type will also be pursued together with Christian Bär (Potsdam).

The paper [BH18] deals with **differential topology of jet spaces**. We show that local deformations, near closed subsets, of solutions to open partial differential relations can be extended to global deformations, provided all but the highest derivatives stay constant along the subset. This answers a question of Gromov. As an application we derive counterintuitive approximation results in Riemannian geometry. This technique is also used in [FH19] in order to show that in many cases closed simply connected manifolds in dimensions $n \geq 5$ and carrying metrics of positive scalar curvature allow scalar positive immersions into Euclidean space R^{2n-1}, which is the classical Whitney dimension bound for smooth immersions. We plan to apply local flexibility to study spaces of positive scalar curvature metrics, including manifolds with boundary. This will be done in cooperation with Alessandro Carlotto (ETH) and Christian Bär (Potsdam).

[BH18] Ch. Bär and B. Hanke, *Local flexibility for open partial differential relations* (2018), available at `arXiv:1809.05703`.

[FH19] L. Florit and B. Hanke, *Scalar positive immersions* (2019), available at `arxiv:1910.06290`.

[Han19] B. Hanke, *Positive scalar curvature on manifolds with odd order abelian fundamental groups* (2019), available at `arXiv:1908.00944`.

[Wan19] J. Wang, *Contractible 3-manifolds and positive scalar curvature (I)* (2019), available at `arXiv:1901.04605`.

Moduli spaces of metrics: topologies and compactifications (WT)

Main goals:

- Investigate moduli spaces of Riemannian metrics of non-negative Ricci curvature on closed and open manifolds, and, in particular, construct first examples of simply connected manifolds for which these moduli spaces exhibit higher non-trivial rational cohomology and homotopy groups.
- Given both a manifold M, or a certain class of such M, and a suitable set of curvature conditions $\mathcal{C}$, study the set of all isometry classes of such metrics satisfying $\mathcal{C}$, equipped with the (pointed) Gromov-Hausdorff or (pointed) C^k or $C^{k,\alpha}$ topologies, where $k = 0, 1, \ldots, \infty$.
- Determine, with respect to these topologies, the closures and compactifications of these spaces in comparison and distinction to the ones that have classically been considered here.

To construct first examples of simply connected manifolds for which the moduli spaces of non-negatively Ricci curved metrics have higher non-trivial rational cohomology and homotopy groups, we first want to investigate the moduli space $S := SO(3,19;\mathbb{Z})\backslash SO(3,19)/SO(3) \times SO(19)$ of Ricci flat Kähler metrics on K3 surfaces and extend the new tools and results obtained in [TW19a] to this situation. Understanding S and more general finite volume non-compact locally symmetric biquotient spaces showing up in this context will involve adopting techniques from Borel, Lück and others on arithmetic groups and classifying spaces of discrete groups, and should also yield new non-connectedness results complementing the ones obtained in [DKT18] and [TW19a] in new dimensions and the simply-connected category.
To investigate isometry classes of Riemannian metrics on manifolds M from the viewpoint of the Gromov-Hausdorff and $C^{k,\alpha}$ topologies, we will consider various curvature and other geometric conditions, described in detail in the proposal, that will together ensure precompactness in the Gromov-Hausdorff topology so that every metric space contained inside the corresponding closures can be approximated in a uniformly controlled way by a sequence of Riemannian metrics g_n on M. To understand the resulting neighbor and gate spaces, we intend to invoke Cheeger-Fukaya-Gromov theory in the case of bounded sectional curvature, and, when just a lower bound on sectional curvature is present, the theory of Alexandrov spaces and their gradient flows. In the realm of $C^{k,\alpha}$ convergence the results of Cheeger-Colding for convergence without collapse under lower bounds on Ricci curvature, and, in the collapsing case, the theory of metric measure spaces satisfying an $RCD(K,N)$ condition initiated by Lott-Villani and Sturm will come into play. Within this general framework we also seek to study questions like the Grove-Wilhelm stability conjecture and further extensions of first smooth stability results for closed manifolds with positive scalar curvature obtained in [TW19b],

We plan to work on these questions in cooperation with Igor Belegradek (Georgia, Atlanta), Thomas Farrell (Beijing/Karlsruhe), Anton Petrunin (Penn State), and Michael Wiemeler (Münster).

[DKT18] A. Dessai, S. Klaus, and W. Tuschmann, *Nonconnected Moduli Spaces of Nonnegative Sectional Curvature Metrics on Simply Connected Manifolds*, Bull. London Math Soc. **50** (2018), 96–107.

[TW19a] W. Tuschmann and M. Wiemeler, *On the topology of moduli spaces of non-negatively curved Riemannian metrics, v.5* (2019), available at `arxiv.org/abs/1712.07052v5`.

[TW19b] ———, *Smooth stability and sphere theorems for manifolds and Einstein manifolds with positive scalar curvature*, Communications in Analysis and Geometry **27** (2019), no. 2, 491–509.

Moduli spaces of metrics with lower curvature bounds (AD)

Main goals:

- Study the topology of moduli spaces of metrics of nonnegative sectional curvature or positive Ricci curvature for high-dimensional manifolds of dimension $\neq 4k-1$.
- Try to define Kreck-Stolz-type invariants for new classes of closed manifolds including simply connected manifolds of dimension $\neq 4k-1$.
- Apply equivariant index theory and rigidity phenomena to study moduli spaces of invariant metrics of nonnegative sectional curvature or positive Ricci curvature.

Only little is known about the topology of moduli spaces of metrics of positive sectional, nonnegative sectional or positive Ricci curvature. Examples of closed manifolds with finite fundamental group for which the moduli space has non-trivial topology have been exhibited (starting with work of Kreck-Stolz and subsequent work by Kapovitch-Petrunin-Tuschmann, Wraith, Dessai-Klaus-Tuschmann, Goodman, Dessai-González-Álvaro, Wermelinger, ...). However, all information is confined to non-connectedness properties of the moduli space and - until recently - based on the invariants of Gromow-Lawson and Kreck-Stolz for $(4k-1)$-dimensional manifolds.
First examples in dimension $4k+1$ were given in [DGÁ19]. In this paper it is shown, among other things, that for every homotopy $\mathrm{R}P^5$ the moduli space of metrics of nonnegative sectional curvature has infinitely many components.
More recently, we have studied certain $(4k+1)$-dimensional manifolds of nonnegative sectional curvature (for each $k > 1$). These manifolds can be described as quotients of twisted high-dimensional Witten manifolds. Our computations suggest that their moduli spaces of metrics of nonnegative sectional curvature have infinitely many components (work in progress, see [Des20]).
Relative η-invariants of $Spin^c$-Dirac operators are used to distinguish components of the respective moduli space for the $4k+1$-dimensional manifolds above and this requires that the fundamental group of the manifold is non-trivial. We plan to apply corresponding techniques for Pin^c-Dirac operators to exhibit corresponding features of the moduli space of certain *even-dimensional* non-simply connected manifolds.
The invariants of Gromow-Lawson and Kreck-Stolz are defined only for certain manifolds of dimension $4k-1$. One of our aims is to generalize these invariants and apply them to study the moduli space of metrics of nonnegative sectional or positive Ricci curvature for *simply connected* manifolds of dimension $\neq 4k-1$.
The aforementioned index theoretical invariants admit equivariant refinements in the presence of an isometric action of a compact Lie group. We would like to investigate how these equivariant invariants in combination with rigidity phenomena of classical operators and elliptic genera can be used to study moduli spaces of invariant metrics of nonnegative sectional or positive Ricci curvature.

We plan to carry out this subproject together with the postdoctoral researcher (Fribourg) in collaboration with Llohann Sperança (São Paulo) and David Wraith (Maynooth), among others.

[DGÁ19] A. Dessai and D. González-Álvaro, *Moduli space of metrics of nonnegative sectional or positive Ricci curvature on homotopy real projective spaces*, accepted for publication in Trans. AMS (2019), available at `arXiv:1902.08919`.

[Des20] A. Dessai, *Moduli space of nonnegatively curved metrics on manifolds of dimension* $4k+1$, pre-preprint (2020).

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