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Master's Thesis

The signature of an oriented manifold and Ochanine's Theorem

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Abstract

The signature is an integer invariant of oriented manifolds of dimensions divisible by four. In 1981, Ochanine proved that the signature of a smooth, closed, oriented, spin manifold of dimension $8k + 4$ is a multiple of 16. A few years later, the development of the theory of elliptic genera led to another and simpler proof of Ochanine's Theorem. This thesis aims to expose Ochanine's Theorem and the notions used for the latter proof. As a first step, we study characteristic classes and multiplicative genera in order to expose the Hirzebruch Signature Theorem, which tells us how to express the signature in terms of Pontryagin numbers. Then we focus on the key instrument of the proof, namely Ochanine's elliptic genus, which takes values in a certain ring of modular forms.

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Contents

1 Introduction

The signature is an important integer invariant of oriented manifolds of dimensions divisible by four. More precisely, it is an example of a multiplicative genus, which are oriented cobordism invariants. In the 1950s, Hirzebruch developed a way to express multiplicative genera in terms of other oriented cobordism invariants, namely Pontryagin numbers. He showed that the signature is equal to the L-genus, which is associated to the power series expansion of $\frac{x}{\tanh x}$. This important result is known as the Hirzebruch Signature Theorem.

The signature has proven to play a significant role in some important results of topology. To cite some examples, it was used to prove the existence of exotic 7-spheres[1], i.e. of smooth manifolds that are homeomorphic, but not diffeomorphic, to S^7. The signature also appears in surgery theory[2] – more precisely in the Fundamental Surgery Theorem – which provides important tools for the classification of high-dimensional manifolds. As a more direct consequence of the results we will present in this thesis, Rokhlin's Theorem (see below) allowed Freedman[3] to show that some 4-manifolds admit no smooth structure by showing the existence of oriented, spin 4-manifolds having signature 8.

We will mostly focus on the signature of manifolds of dimension $8k + 4$ possessing a spin structure. In 1952, Rokhlin already showed that the signature of a closed, spin, smooth manifold of dimension 4 is divisible by 16. In 1981, Ochanine generalised this result to every closed, spin, smooth manifold of dimension $8k + 4$ in the article [Och81]. Later, a group of topologists and physicists – including Ochanine – developed the theory of elliptic genera. This theory led to another proof of Ochanine's Theorem which was published in 1988 in an article by Landweber, "Elliptic cohomology and modular forms" [Lan88], and in 1994 by Hirzebruch, Berger and Jung in their book "Manifolds and modular forms" [HBJ94].

The aim of this thesis is to present the necessary notions and results to understand Ochanine's statement and the proof presented by Hirzebruch, Berger and Jung. From their book, we tried to sort out and collect only the material that is needed for this proof. Emphasis was put on multiplicative genera and Hirzebruch's formalism, as well as on the subject of modular forms and Ochanine's elliptic genus. As for the results coming from spin geometry and index theory, they will only be briefly explained as it was not our purpose to immerse into these theories. More details can be found in [LJM89]. Reading this paper requiring some knowledge of algebraic topology and simplicial homology and cohomology, we refer for example to [Hat01] for an introduction.

Chapter 2 is dedicated to the definition of the signature and its properties. In chapter 3, we give a brief exposition of the main types of characteristic classes, namely the Stiefel-Whitney classes, the Chern classes and the Pontryagin classes. The Pontryagin numbers are introduced in this chapter. We also present the splitting principle as well as the Chern roots and Chern character, which are

[1] See [MS74, §20].
[2] See for example [Bro72].
[3] See [F$^+$82].

necessary for the understanding of twisted genera in the next chapter. Chapter 4 is devoted to defining the notions of oriented cobordism and multiplicative genera and to explaining Hirzebruch's formalism, which enables us to express multiplicative genera as a linear combination of Pontryagin numbers. We will present two important examples of multiplicative genera, namely the L-genus and the $\hat{A}$-genus, as well as their "twisted" versions. Hirzebruch Signature Theorem states that the signature is actually equal to the L-genus, implying that the latter takes integral values. As for the $\hat{A}$-genus, some results of index theory and spin geometry will be necessary to make inferences about its integrality and parity. More precisely, the $\hat{A}$-genus of a spin manifold M is an integer because it is equal to the index of the so-called Dirac operator. If additionally M has dimension $8k+4$, its $\hat{A}$-genus is even. At this stage we will be able to prove Rokhlin's Theorem. Chapter 5 is an introduction to modular forms. We will particularly be concerned with the structure of the ring of modular forms for the specific congruence subgroup $\Gamma_0(2) \subset \mathrm{SL}_2(\mathbb{Z})$. We will get to the heart of the subject in chapter 6, in which we will define and study the elliptic genus. This genus takes its values in the ring of modular forms for $\Gamma_0(2)$, studied in the preceding chapter. It represents the crucial tool for the proof of Ochanine's Theorem, which is finally exposed in the last chapter.

2 The signature

2.1 Some recalls and notations about smooth manifolds

Here we briefly present some recalls about properties of smooth manifolds as well as their homology and cohomology groups. For a more complete review, we refer to Appendix A in [MS74] and Chapters 2 and 3 in [Hat01].

Definition 2.1. A smooth manifold M is said to be *closed* if it is compact and without boundary.

Definition 2.2. Let M be a closed, connected, smooth n-manifold. It is said to be *orientable* if

$$H_n(M;\mathbb{Z}) \cong \mathbb{Z}.$$

In this case, an *orientation* on M is defined by a choice of one of the two generators of $H_n(M;\mathbb{Z})$. The preferred generator is called the *fundamental homology class of* M and will be denoted by $[M]$. We will write $-M$ to denote the oriented manifold M considered with the opposite orientation.

Remark 2.3. When M is not connected, it is orientable if each of its connected components is orientable.

Remark 2.4. For a connected, orientable, smooth n-manifold M, Poincaré duality[4] implies that the n^{th} cohomology group $H^n(M;\mathbb{Z})$ is also isomorphic to $\mathbb{Z}$.

[4] See [Hat01, section 3.3].

If not stated otherwise, all the manifolds we will consider throughout this paper are closed, oriented, smooth manifolds. For such a manifold M of dimension n and for a cohomology class $\alpha \in H^n(M;\mathbb{Z})$, we use the notation $\alpha[M]$ to denote the evaluation of α on the fundamental homology class $[M] \in H_n(M;\mathbb{Z})$.

In the rest of this chapter, it will be useful to consider the cohomology of a manifold M with coefficients in the field of rational numbers. By the Universal Coefficient Theorem[5],

$$H^p(M;\mathbb{Q}) \cong H^p(M;\mathbb{Z}) \otimes \mathbb{Q}, \quad \forall p \in \mathbb{N}.$$

Taking the tensor product with $\mathbb{Q}$ kills the torsion of the cohomology group and provides the structure of a $\mathbb{Q}$-vector space.

2.2 Definition of the signature and properties

Let V be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}$) and let B and q denote a symmetric bilinear form on V and its associated quadratic form. Let us consider a basis $v_1, \dots, v_r$ of V such that the matrix associated to B is diagonal, i.e.

$$A := (B(v_i, v_j))_{1 \le i,j \le r} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r \end{pmatrix}, \quad \lambda_i \in \mathbb{K}.$$

We set

$$\begin{aligned} q^+ &:= \#\{\lambda_i \mid i \in \{1, \dots, r\},\ \lambda_i > 0\}, \\ q^- &:= \#\{\lambda_i \mid i \in \{1, \dots, r\},\ \lambda_i < 0\}. \end{aligned}$$

The integers q^+ and q^- do not depend[6] on the chosen basis for V. Therefore we can make the following definition.

Definition 2.5. The *signature* of the quadratic form q is defined as

$$\operatorname{sign}(q) := q^+ - q^-.$$

Now let M denote a closed, oriented, smooth manifold of doubly even dimension, i.e. $\dim M = 4k$ for some $k \in \mathbb{N}$. We consider the *middle cohomology group* $H^{2k}(M;\mathbb{Z})$ and the associated $\mathbb{Q}$-vector space

$$H^{2k}(M;\mathbb{Q}) \cong H^{2k}(M;\mathbb{Z}) \otimes \mathbb{Q}.$$

The *intersection form* is defined by

$$\begin{aligned} H^{2k}(M;\mathbb{Q}) \times H^{2k}(M;\mathbb{Q}) &\longrightarrow \mathbb{Q} \\ (\alpha, \beta) &\longmapsto (\alpha \smile \beta)[M]. \end{aligned}$$

Recall that the cup product is bilinear and graded commutative, i.e. for classes $\alpha \in H^p(M;\mathbb{Q})$ and $\beta \in H^q(M;\mathbb{Q})$, we have $(\alpha \smile \beta) = (-1)^{pq}(\beta \smile \alpha)$. Here $p = q = 2k$, hence the intersection form is symmetric and bilinear. Let us denote its associated quadratic form by q_M.

[5] See [Hat01, section 3.1].
[6] See for example [Gab96, pp. 396-397].

Definition 2.6. The signature of a closed, oriented, smooth manifold M is defined to be zero if the dimension of M is not a multiple of 4. If $\dim M = 4k$, then the signature of M is defined to be the signature of the quadratic form q_M, i.e.

$$\operatorname{sign}(M) := \operatorname{sign}(q_M).$$

Remark 2.7. Poincaré duality implies that for an oriented n-manifold M, the cup product pairing

$$\begin{aligned} H^k(M;\mathbb{Q}) \times H^{n-k}(M;\mathbb{Q}) &\longrightarrow \mathbb{Q} \\ (\alpha,\beta) &\longmapsto (\alpha \smile \beta)[M] \end{aligned}$$

is non-singular, i.e. the induced maps

$$\begin{aligned} H^k(M;\mathbb{Q}) &\longrightarrow \operatorname{Hom}(H^{n-k}(M;\mathbb{Q}),\mathbb{Q}), \\ \alpha &\longmapsto (\alpha \smile \cdot)[M] \end{aligned}$$

and

$$\begin{aligned} H^{n-k}(M;\mathbb{Q}) &\longrightarrow \operatorname{Hom}(H^{k}(M;\mathbb{Q}),\mathbb{Q}), \\ \beta &\longmapsto (\cdot \smile \beta)[M] \end{aligned}$$

are isomorphisms[7]. This also holds for cohomology with coefficients in any other field. For $\mathbb{Z}$-cohomology groups, the assertion holds after factoring out the torsion in $H^*(M;\mathbb{Z})$. Considering the middle cohomology group of a $4k$-manifold M, we get an isomorphism

$$\begin{aligned} H^{2k}(M;\mathbb{Q}) &\longrightarrow \operatorname{Hom}(H^{2k}(M;\mathbb{Q}),\mathbb{Q}) \\ \alpha &\longmapsto (\alpha \smile \cdot)[M]. \end{aligned} \tag{1}$$

Let us now consider a basis $\alpha_1,\dots,\alpha_r$ of $H^{2k}(M;\mathbb{Q})$ such that the matrix associated to the intersection form is diagonal, and call the i^{th} element of the diagonal λ_i. Assume there is an i_0 with $\lambda_{i_0} = 0$, i.e. $(\alpha_{i_0}^2)[M] = 0$. Since the associated matrix is diagonal, $(\alpha_{i_0} \smile \alpha_i)[M] = 0$ for every $i \in \{1,\dots,r\}$, hence $(\alpha_{i_0} \smile \beta)[M] = 0$ for every class $\beta \in H^{2k}(M;\mathbb{Q})$. We get a contradiction, since the map given by (1) is an isomorphism. So the quadratic form q_M associated to the intersection form is non-degenerate.

The signature is a topological invariant of oriented manifolds. Indeed, let M and N be two oriented $4k$-manifolds and let $f : M \to N$ be an orientation preserving homeomorphism – i.e. a homeomorphism so that the induced isomorphism $f_* : H_{4k}(M;\mathbb{Z}) \to H_{4k}(N;\mathbb{Z})$ maps the fundamental homology class $[M]$ to $[N]$. Suppose that $\alpha_1,\dots,\alpha_r$ is a set of generators of $H^{2k}(N;\mathbb{Q})$ so that the matrix associated to the intersection form of N is diagonal. We consider the generators $\beta_1,\dots,\beta_r$ of $H^{2k}(M;\mathbb{Q})$, defined by $\beta_i := f^*(\alpha_i)$. Then

$$\begin{aligned} (\beta_i \smile \beta_j)[M] &= (f^*(\alpha_i) \smile f^*(\alpha_j))(f_*^{-1}([N])) \\ &= (f^*(\alpha_i \smile \alpha_j))(f_*^{-1}([N])) \\ &= (\alpha_i \smile \alpha_j)[N]. \end{aligned}$$

[7]For a proof of the non-singularity, see [Hat01, pp. 249-250].

So the matrix associated to the intersection form of M with respect to $\beta_1, \dots, \beta_r$ is also diagonal and $\operatorname{sign}(q_M) = \operatorname{sign}(q_N)$, so $\operatorname{sign}(M) = \operatorname{sign}(N)$.

Theorem 2.8. *The signature satisfies the following properties:*

(i) *If* $\dim M = \dim N$, *then* $\operatorname{sign}(M \sqcup N) = \operatorname{sign}(M) + \operatorname{sign}(N)$, *where* $\sqcup$ *denotes a disjoint union,*

(ii) $\operatorname{sign}(-M) = -\operatorname{sign}(M)$,

(iii) $\operatorname{sign}(M \times N) = \operatorname{sign}(M)\operatorname{sign}(N)$,

(iv) *If M is the boundary of some compact, oriented manifold with boundary, then* $\operatorname{sign}(M) = 0$.

Proof. Proofs of (i) and (ii) follow directly from

$$H^*(M \sqcup N; \mathbb{Q}) \cong H^*(M; \mathbb{Q}) \oplus H^*(N; \mathbb{Q})$$

and $[-M] = -[M]$. Showing (iii) is a bit more technical and requires the use of the Künneth formula[8]

$$H^{2k}(M \times N; \mathbb{Q}) \cong \bigoplus_{s=0}^{2k} H^s(M; \mathbb{Q}) \otimes H^{2k-s}(N; \mathbb{Q})$$

and to consider separately two components of this direct sum. As for (iv), its proof involves Poincaré-Lefschetz duality[9] and a convenient commutative diagram. For more details of these proofs, we refer to [Hir78, pp. 84-86]. □

2.3 Simple examples

The $4k$-spheres are uninteresting examples since the middle cohomology group $H^{2k}(S^{4k}; \mathbb{Z})$ is trivial and therefore $\operatorname{sign}(S^{4k}) = 0$ for every $k \geq 1$. In order to find simple but non-trivial examples, let us consider the complex and quaternionic projective spaces.

Theorem 2.9. *Let $n \in \mathbb{N}$. There is an element $\alpha \in H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ such that the cohomology ring of $\mathbb{C}\mathrm{P}^n$ is the truncated ring of polynomials in α of degree less or equal than n, i.e.*

$$H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1}).$$

Similarly, there is an element $\beta \in H^4(\mathbb{H}\mathrm{P}^n; \mathbb{Z})$ such that

$$H^*(\mathbb{H}\mathrm{P}^n; \mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^{n+1}).$$

Proof. We refer to [Hat01, pp. 220-222]. □

[8] See for example [Hat01, section 3.2].

[9] Again, we refer to [Hat01, section 3.3].

Remark 2.10. Seeing $\mathbb{C}\mathrm{P}^n$ as a complex manifold induces a natural orientation on the real tangent bundle $T\mathbb{C}\mathrm{P}^n$. As usual, let $[\mathbb{C}\mathrm{P}^n] \in H_{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ denote its fundamental homology class. It can be shown[10] that for a well chosen generator α of $H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$,

$$\alpha^n[\mathbb{C}\mathrm{P}^n] = 1. \tag{2}$$

For every integer $k \geq 0$, $\mathbb{C}\mathrm{P}^{2k}$ is a closed and oriented $4k$-manifold. Let us consider a generator α of $H^2(\mathbb{C}\mathrm{P}^{2k};\mathbb{Z})$ satisfying property (2). Theorem 2.9 implies that α^k is a generator of $H^{2k}(\mathbb{C}\mathrm{P}^{2k};\mathbb{Z})$. Then the signature of $\mathbb{C}\mathrm{P}^{2k}$ is given by the sign of $(\alpha^k \smile \alpha^k)[\mathbb{C}\mathrm{P}^{2k}] = \alpha^{2k}[\mathbb{C}\mathrm{P}^{2k}] = 1$. So

$$\mathrm{sign}(\mathbb{C}\mathrm{P}^{2k}) = 1, \quad \forall k \geq 0. \tag{3}$$

The quaternionic projective space $\mathbb{H}\mathrm{P}^k$ is also a closed and oriented $4k$-manifold. If k is odd, then its middle cohomology group $H^{2k}(\mathbb{H}\mathrm{P}^k;\mathbb{Z})$ is trivial by Theorem 2.9, hence

$$\mathrm{sign}(\mathbb{H}\mathrm{P}^k) = 0, \quad \forall k \geq 0,\ k \text{ odd}.$$

When k is even, then the signature of $\mathbb{H}\mathrm{P}^k$ is given by the sign of $\beta^k[\mathbb{H}\mathrm{P}^k]$, for a generator β of $H^4(\mathbb{H}\mathrm{P}^k;\mathbb{Z})$, so

$$\mathrm{sign}(\mathbb{H}\mathrm{P}^k) = \mathrm{sgn}(\beta^k[\mathbb{H}\mathrm{P}^k]) = \pm 1, \quad \forall k \geq 0,\ k \text{ even},$$

where sgn denotes the sign function.

We will see in chapter 4 that the signature is an example of what is called a multiplicative genus and that it can be expressed as a combination of invariants called the Pontryagin numbers. This requires the study of characteristic classes.

3 Characteristic classes

The aim of this chapter is to introduce the main types of characteristic classes, namely the Stiefel-Whitney classes, the Chern classes and the Pontryagin classes. Although they are of great importance in the expression of multiplicative genera (see chapter 4), it is not our purpose here to prove all presented results. A more complete study of characteristic classes can be found in [MS74] or in [Hat09, chapter 3].

3.1 Basic notions

Let us expand the framework in this chapter and consider more generally topological spaces rather than only smooth manifolds. All the maps we will consider here are continuous. We will use $E \to B$ to denote a vector bundle[11] over a topological space B and with total space E. We will often shorten the notation by simply denoting such a vector bundle by E.

[10] See [MS74, pp. 169-170 and p. 177]. Note that the proofs make use of characteristic classes, which we will study in the next chapter.

[11] Should the reader not be familiar with vector bundles and their associated constructions, we refer to [Hat09, chapter 1] or [MS74, chapters 2 and 3].

Definition 3.1. A *characteristic class* is a function x assigning to vector bundles over a topological space B a cohomology class in $H^i(B; R)$, where $i \in \mathbb{N}$ and R generally denotes $\mathbb{Z}$ or $\mathbb{Z}_2$, such that the class $x(E)$ only depends on the isomorphism type of E and such that the following *naturality property* is satisfied: for any map $f : B' \to B$ and any vector bundle E over B,

$$x(f^*E) = f^*(x(E)), \tag{4}$$

where f^*E denotes the pullback of E induced by f.

Remark 3.2. The naturality property implies that the value of a characteristic class x of a trivial bundle T is zero if $x(T) \in H^i(B; R)$ with $i > 0$. Indeed, let $T = B \times \mathbb{K}^n$ be a trivial n-dimensional $\mathbb{K}$-vector bundle ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) over B. Let $\{pt\}$ denote a one point space, p the projection $p : B \to \{pt\}$ and let us consider the trivial n-dimensional bundle $\mathbb{K}^n = E \to \{pt\}$. Then clearly $T \cong p^*E$. Consequently,

$$x(T) = x(p^*E) \overset{(4)}{=} p^*(x(E)).$$

But $x(E) \in H^i(\{pt\}; R)$, which is trivial when $i > 0$. Hence $x(T) = 0$. So the vanishing of all characteristic classes of a vector bundle is a necessary condition for it to be trivial.

3.2 Stiefel-Whitney classes and Chern classes

3.2.1 Stiefel-Whitney classes

The Stiefel-Whitney classes are defined on real vector bundles and take their values in $\mathbb{Z}_2$-cohomology groups. Here we provide an axiomatic definition.

Theorem 3.3. *There exists a unique sequence of characteristic classes $\{\omega_i\}_{i\geq 1}$ assigning to each real vector bundle $E \to B$ of dimension n a class*

$$\omega_i(E) \in H^i(B; \mathbb{Z}_2)$$

such that

(i) $\omega_i(E) = 0$ *if* $i > n$,

(ii) $\omega(E_1 \oplus E_2) = \omega(E_1) \smile \omega(E_2)$, *where* $\omega := 1 + \omega_1 + \omega_2 + \dots$,

(iii) $\omega_1(L_{\mathbb{R}})$ *is a generator of* $H^1(\mathbb{R}\mathrm{P}^1; \mathbb{Z}_2)$, *where* $L_{\mathbb{R}}$ *denotes the* canonical line bundle over $\mathbb{R}\mathrm{P}^1$, *defined by the total space*

$$L_{\mathbb{R}} = \{([x_0 : x_1], v) \mid [x_0 : x_1] \in \mathbb{R}\mathrm{P}^1,\ v \in \mathbb{R}(x_0, x_1)\} \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}^2$$

and the natural projection $\pi : ([x_0 : x_1], v) \mapsto [x_0 : x_1] \in \mathbb{R}\mathrm{P}^1$.

Proof. See [Hat09, pp. 77-81] or [MS74, chapter 8]. $\square$

Definition 3.4. For a real vector bundle $E \to B$, the cohomology class $\omega_i(E)$ is called the i^{th} *Stiefel-Whitney class of* E. The *total Stiefel-Whitney class of* E is defined as $\omega(E) := 1 + \omega_1(E) + \omega_2(E) + \omega_3(E) + \ldots \in H^*(B; \mathbb{Z}_2)$.

Remark 3.2 implies that for a trivial vector bundle T over B, $\omega(T) = 1$. Together with property (ii) in Theorem 3.3, this yields that

$$\omega(E \oplus T) = \omega(E)$$

for any real vector bundle E over B. Therefore the Stiefel-Whitney classes are said to be *stable*.

There is a close relation between the first Stiefel-Whitney class and the orientability of a vector bundle.

Definition 3.5. An *orientation* for a $\mathbb{K}$-vector bundle $\pi : E \to B$ of dimension n is a function assigning to each fibre F an orientation such that the following *local compatibility condition* is satisfied: for every element $b_0 \in B$ there exist a neighbourhood U of b_0 and a local trivialisation $h : U \times \mathbb{K}^n \to \pi^{-1}(U)$ such that for every $b \in U$, the isomorphism between $\mathbb{K}^n$ and the fibre over b induced by h is orientation preserving. If such a function exists, the vector bundle E is said to be *orientable*.

Proposition 3.6. *Provided that B is homotopy equivalent to a CW complex, a real vector bundle $E \to B$ is orientable if and only if its first Stiefel-Whitney class $\omega_1(E)$ is zero.*

Proof. See [Hat09, p. 87]. □

Note that a smooth manifold can always be given the structure of a CW complex. Furthermore, it can be shown that a manifold M is orientable if and only if its tangent bundle TM is orientable[12]. Hence the orientability of M is equivalent to the vanishing of $\omega_1(TM)$.

3.2.2 Chern classes

The axiomatic definition of the Chern classes is very similar to the one of the Stiefel-Whitney classes, but Chern classes are defined for complex vector bundles and take their values in $\mathbb{Z}$-cohomology rings.

Theorem 3.7. *There exists a unique sequence of characteristic classes $\{c_i\}_{i \geq 1}$ assigning to each complex vector bundle $E \to B$ of dimension n a class*

$$c_i(E) \in H^{2i}(B; \mathbb{Z})$$

such that

(i) $c_i(E) = 0$ *if* $i > n$,

(ii) $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$, *where* $c := 1 + c_1 + c_2 + \ldots$,

[12] See for example [MS74, p. 122].

(iii) *$c_1(L_{\mathbb{C}})$ is a fixed generator of $H^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z})$, where $L_{\mathbb{C}}$ denotes the* canonical line bundle over $\mathbb{C}\mathrm{P}^1$, *defined by the total space*

$$L_{\mathbb{C}} = \{([z_0 : z_1], w) \mid [z_0 : z_1] \in \mathbb{C}\mathrm{P}^1,\ w \in \mathbb{C}(z_0, z_1)\} \subset \mathbb{C}\mathrm{P}^1 \times \mathbb{C}^2$$

and the natural projection $\pi : ([z_0 : z_1], w) \mapsto [z_0 : z_1] \in \mathbb{C}\mathrm{P}^1$.

Proof. Again, see [Hat09, pp. 77-81], where the real and complex cases (i.e. Stiefel-Whitney and Chern classes) are treated similarly. □

Definition 3.8. For a complex vector bundle $E \to B$, the cohomology class $c_i(E)$ is called the i^{th} *Chern class of* E. The *total Chern class of* E is defined as $c(E) := 1 + c_1(E) + c_2(E) + c_3(E) + \ldots \in H^*(B;\mathbb{Z})$.

As in the case of Stiefel-Whitney classes, property (ii) implies the stability of Chern classes, i.e.

$$c(E \oplus T) = c(E)$$

when T is a trivial vector bundle.

Remark 3.9. Alternatively, the Chern classes can be defined inductively by using the Euler class e. The latter is defined for oriented, real vector bundles $E \to B$ and $e(E) \in H^n(B;\mathbb{Z})$ for $n = \dim E$. For a complex vector bundle $E \to B$ of dimension n, let $E_{\mathbb{R}}$ denote the underlying real oriented vector bundle, which has dimension $2n$. We set

$$c_n(E) := e(E_{\mathbb{R}}) \in H^{2n}(B;\mathbb{Z}).$$

Then we define the lower Chern classes by constructing a complex vector bundle of dimension $n-1$ over the base space

$$E_0 := \bigcup_{b \in B} F_b \backslash \{0_b\},$$

where F_b denotes the fibre of E over b, and by using the Gysin sequence. For more details, we refer to [MS74, chapter 14].

For a complex vector bundle $E \to B$, let $\overline{E} \to B$ denote its conjugate vector bundle, i.e. the complex vector bundle yielding the same underlying real vector bundle as E, but considered with the opposite complex structure. More precisely, if the scalar multiplication in a fibre F of E is given by $(z, v) \to z \cdot v$ for $z \in \mathbb{C}$ and $v \in F$, then the scalar multiplication in the corresponding fiber $\overline{F}$ of $\overline{E}$ is given by $(z, v) \to \overline{z} \cdot v$. Then the Chern classes of $\overline{E}$ can be expressed in terms of the ones of E as follows:

Lemma 3.10. *For every $k \in \mathbb{N}$,*

$$c_k(\overline{E}) = (-1)^k c_k(E).$$

Proof. See [MS74, p. 168]. □

3.3 Pontryagin classes and Pontryagin numbers

3.3.1 Pontryagin classes

The Chern classes are used to construct other characteristic classes – the Pontryagin classes – defined on real vector bundles. In order to do so, we need an operation that enables us to construct a complex vector bundle out of a real one.

Definition 3.11. The *complexification* of a real vector space V is the complex vector space given by the tensor product $V \otimes \mathbb{C}$ together with the scalar multiplication $\lambda(v \otimes z) := v \otimes (\lambda z)$. Note that $V \otimes \mathbb{C}$ is isomorphic to the direct sum $V \oplus iV$.

Now let $E \to B$ be a real vector bundle of dimension n.

Definition 3.12. The *complexification of* E, denoted $E \otimes \mathbb{C}$, is the n-dimensional complex vector bundle over B obtained after complexifying every fibre of E.

Definition 3.13. The i^{th} *Pontryagin class* of E is defined by

$$p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(B; \mathbb{Z}).$$

The *total Pontryagin class* of E is then given by

$$p(E) := 1 + p_1(E) + \cdots + p_{\lfloor n/2 \rfloor}(E) \in H^*(B; \mathbb{Z}).$$

Note that $p_i(E) = 0$ if $i > n/2$ since $c_{2i}(E \otimes \mathbb{C}) = 0$ for $2i > n$.

The invariance of Pontryagin classes under vector bundle isomorphisms and their naturality follow directly from properties of Chern classes.

Lemma 3.14. *The complexification $E \otimes \mathbb{C}$ of a real vector bundle $E \to B$ is isomorphic to its conjugate bundle $\overline{E \otimes \mathbb{C}}$.*

Proof. The map

$$\begin{aligned} f : E \otimes \mathbb{C} &\longrightarrow \overline{E \otimes \mathbb{C}} \\ v \otimes z &\longmapsto v \otimes \overline{z} \end{aligned}$$

is a homeomorphism between the two total spaces and is linear in each fibre since

$$f(z' \cdot (v \otimes z)) = f(v \otimes z'z) = v \otimes \overline{z'z} = \overline{z'} \cdot (v \otimes \overline{z}) = \overline{z'} \cdot f(v \otimes z).$$

□

Lemma 3.15. *For an n-dimensional complex vector bundle $E \to B$, let us denote by $E_{\mathbb{R}}$ its underlying real vector bundle of dimension $2n$. Then*

$$E_{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus \overline{E}.$$

Proof. See [MS74, pp. 176-177]. □

Lemmas 3.10 and 3.15 allow us to express the Pontryagin classes $p_i(E_{\mathbb{R}})$ in terms of the Chern classes of E. We have

$$\begin{aligned} c(E_{\mathbb{R}} \otimes \mathbb{C}) &\overset{3.15}{=} c(E \oplus \overline{E}) \\ &= c(E) \smile c(\overline{E}) \\ &\overset{3.10}{=} (1 + c_1(E) + c_2(E) + \cdots) \smile (1 - c_1(E) + c_2(E) \mp \cdots) \\ &= 1 - c_1^2(E) + 2c_2(E) + c_2^2(E) - 2c_1(E)c_3(E) + 2c_4(E) + \cdots . \end{aligned}$$

Then we have for example

$$p_1(E_{\mathbb{R}}) = -c_2(E_{\mathbb{R}} \otimes \mathbb{C}) = c_1^2(E) - 2c_2(E). \tag{5}$$

3.3.2 Pontryagin numbers

From now on, we will mostly consider characteristic classes of tangent bundles. For a smooth manifold M, we will then shorten the notations as follows:

$$\begin{aligned} \omega_i(M) &:= \omega_i(TM), & \omega(M) &:= \omega(TM), \\ c_i(M) &:= c_i(TM_{\mathbb{C}}), & c(M) &:= c(TM_{\mathbb{C}}), \\ p_i(M) &:= p_i(TM), & p(M) &:= p(TM), \end{aligned}$$

where $TM_{\mathbb{C}} := TM \otimes \mathbb{C}$.

Let M be a closed, oriented, smooth $4k$-manifold. We will assign M some integral numbers by using its Pontryagin classes.

Definition 3.16. For every partition $I = (i_1, \ldots, i_r)$ of k – i.e. $i_1, \ldots, i_r$ are integers such that $i_1 + \cdots + i_r = k$ – the I^{th} *Pontryagin number of* M is defined as

$$p_I[M] := (p_{i_1}(M) \smile p_{i_2}(M) \smile \cdots \smile p_{i_r}(M))[M] \in \mathbb{Z}.$$

The $\smile$-notation looking a bit heavy for long products and since there is no risk of confusion with any other product in cohomology in this paper, let us simply drop it from now on. We will then write

$$p_I[M] = (p_{i_1}(M) \cdots p_{i_r}(M))[M].$$

Suppose that $f : M \to N$ is a diffeomorphism preserving the orientation between two oriented $4k$-manifolds. Then f together with its differential provide an isomorphism between the vector bundles TM and TN. Since TM is clearly isomorphic to the pullback $f^*(TN)$ and by naturality of Pontryagin classes,

$$p_i(M) = p_i(f^*(TN)) = f^*(p_i(N)).$$

Thus for a partition $I = (i_1, \ldots, i_r)$ of k,

$$\begin{aligned} p_I[M] &= (p_{i_1}(M) \cdots p_{i_r}(M))[M] \\ &= f^*(p_{i_1}(N) \cdots p_{i_r}(N))(f_*^{-1}([N])) \\ &= (p_{i_1}(N) \cdots p_{i_r}(N))[N] \\ &= p_I[N]. \end{aligned}$$

Therefore Pontryagin numbers are topological invariants of oriented, smooth manifolds. Actually, Pontryagin numbers are even oriented cobordism invariants[13].

3.4 Chern roots and Chern character

The following result allows one to simplify some questions by reducing them to the case of line bundles:

Theorem 3.17 (**Splitting Principle**). *Let $E \to M$ be an n-dimensional complex vector bundle over a manifold M. Then there exist a space F_E and a map*

$$p : F_E \longrightarrow M$$

such that

(i) for any ring R, the induced homomorphism $p^* : H^*(M; R) \to H^*(F_E; R)$ is injective,

(ii) the pullback bundle $p^*E \to F_E$ splits as a direct sum

$$p^*E \cong L_1 \oplus \cdots \oplus L_n,$$

where the L_i are complex line bundles over F_E.

$$\begin{array}{ccc} p^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ F_E & \xrightarrow{p} & M \end{array}$$

Proof. See for example [Hat09, section 3.1]. □

By the naturality of Chern classes and the product formula, we have

$$\begin{aligned} p^*(c(E)) = c(p^*E) &= c(L_1 \oplus \cdots \oplus L_n) \\ &= c(L_1) \cdots c(L_n) \\ &= (1 + c_1(L_1)) \cdots (1 + c_1(L_n)) \\ &= (1 + x_1) \cdots (1 + x_n), \end{aligned}$$

where $x_i := c_1(L_i)$. Thus the j^{th} Chern class of p^*E is equal to the j^{th} elementary symmetric function in the x_i, i.e.

$$c_j(p^*E) = \sigma_j(x_1, \ldots, x_n) := \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j}.$$

The injectivity of p^* allows us to formally write

$$c(E) = (1 + x_1) \cdots (1 + x_n).$$

[13] See section 4.1.1 for the definition of the oriented cobordism ring. A proof of the multiplicativity of Pontryagin numbers is included in the proof of Theorem 4.9 in section 4.1.2.

The classes $x_1, \ldots, x_n$ are called the *Chern roots* of E. By the Fundamental Theorem of symmetric functions, any expression $f(x_1, \ldots x_n)$ that is symmetric in the x_i corresponds to an expression in the Chern classes $c_j(E)$, therefore it makes sense to evaluate it on a homology class of M.

In the case where M is a $4k$-manifold, let us consider the complexified tangent bundle $TM_{\mathbb{C}}$. By the splitting principle, there exist a space $F_{TM_{\mathbb{C}}}$ and a map $p : F_{TM_{\mathbb{C}}} \to M$ such that p^* is injective and

$$p^*(TM_{\mathbb{C}}) \cong L_1 \oplus \cdots \oplus L_{4k},$$

where the L_i are complex line bundles over $F_{TM_{\mathbb{C}}}$. By Lemma 3.14, we know that $TM_{\mathbb{C}} \cong \overline{TM_{\mathbb{C}}}$, therefore

$$\begin{aligned} L_1 \oplus \cdots \oplus L_{4k} &\cong p^*(TM_{\mathbb{C}}) \\ &\cong p^*(\overline{TM_{\mathbb{C}}}) \\ &\cong \overline{p^*(TM_{\mathbb{C}})} \\ &\cong \overline{L_1 \oplus \cdots \oplus L_{4k}} \\ &\cong \overline{L_1} \oplus \cdots \oplus \overline{L_{4k}}. \end{aligned}$$

So the total Chern class of $TM_{\mathbb{C}}$ has a formal factorisation of the form

$$c(M) = (1 + x_1) \cdots (1 + x_{2k})(1 - x_1) \cdots (1 - x_{2k}).$$

We will call $x_1, \ldots, x_{2k}$ the *Chern roots of* M. The Pontryagin classes of TM can be written, still formally, as

$$p_j(M) = (-1)^j c_{2j}(M) = (-1)^j \sigma_{2j}(x_1, \ldots, x_{2k}, -x_1, \ldots, -x_{2k}).$$

It is not hard to check that it is equal to the j^{th} elementary symmetric function in $x_1^2, \ldots, x_{2k}^2$, i.e.

$$p_j(M) = \sigma_j(x_1^2, \ldots, x_{2k}^2),$$

and

$$p(M) = (1 + x_1^2) \cdots (1 + x_{2k}^2).$$

Let $E \to M$ be an n-dimensional complex vector bundle with formal factorisation

$$c(E) = (1 + x_1) \cdots (1 + x_n).$$

Definition 3.18. The *Chern character of* E is defined by

$$\operatorname{ch}(E) := \sum_{i=1}^{n} e^{x_i} = \sum_{k=0}^{\infty} \frac{x_1^k + \cdots + x_n^k}{k!}.$$

The Chern character has the following nice properties:

Proposition 3.19. *For any complex vector bundles $E_1 \to M$ and $E_2 \to M$,*

$$\begin{aligned} \operatorname{ch}(E_1 \oplus E_2) &= \operatorname{ch}(E_1) + \operatorname{ch}(E_2), \\ \operatorname{ch}(E_1 \otimes E_2) &= \operatorname{ch}(E_1) \operatorname{ch}(E_2). \end{aligned}$$

Proof. Let us denote the dimensions of E_1 and E_2 by n_1, respectively n_2. By the splitting principle, there exist a space F_{E_1} and a map

$$p_1 : F_{E_1} \longrightarrow M$$

so that the homomorphism $p_1^* : H^*(M; R) \to H^*(F_{E_1}; R)$ is injective and the pullback $p_1^* E_1 \to F_{E_1}$ splits as a direct sum of n_1 line bundles $L_1, \dots, L_{n_1}$.

$$\begin{array}{ccc} L_1 \oplus \cdots \oplus L_{n_1} \cong p_1^* E_1 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ F_{E_1} & \xrightarrow{p_1} & M \end{array}$$

Let us consider now the pullback $p_1^* E_2 \to F_{E_1}$

$$\begin{array}{ccc} p_1^* E_2 & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ F_{E_1} & \xrightarrow{p_1} & M \end{array}$$

and apply the splitting principle again. We get a space F_{E_1,E_2} and a map

$$p_2 : F_{E_1,E_2} \longrightarrow F_{E_1}$$

such that $p_2^* : H^*(F_{E_1}; R) \to H^*(F_{E_1,E_2}; R)$ is injective and the pullback $p_2^* p_1^* E_2$ splits as a direct sum of n_2 line bundles $L_1^{(2)}, \dots, L_{n_2}^{(2)}$.

$$\begin{array}{ccc} L_1^{(2)} \oplus \cdots \oplus L_{n_2}^{(2)} \cong p_2^* p_1^* E_2 & \longrightarrow & p_1^* E_2 \\ \downarrow & & \downarrow \\ F_{E_1,E_2} & \xrightarrow{p_2} & F_{E_1} \end{array}$$

Then the composition $p := p_1 \circ p_2 : F_{E_1,E_2} \to M$ gives an injective homomorphism

$$p^* : H^*(M; R) \longrightarrow H^*(F_{E_1,E_2}; R)$$

and it enables us to pull both E_1 and E_2 back to a direct sum of line bundles. By setting $p_2^*(L_i) := L_i^{(1)}$, we have

$$\begin{aligned} p^* E_1 &\cong p_2^*(L_1) \oplus \cdots \oplus p_2^*(L_{n_1}) = L_1^{(1)} \oplus \cdots \oplus L_{n_1}^{(1)}, \\ p^* E_2 &\cong L_1^{(2)} \oplus \cdots \oplus L_{n_2}^{(2)}. \end{aligned}$$

The pullback of the direct sum $E_1 \oplus E_2$ is then given by

$$p^*(E_1 \oplus E_2) \cong p^*(E_1) \oplus p^*(E_2) = L_1^{(1)} \oplus \cdots \oplus L_{n_1}^{(1)} \oplus L_1^{(2)} \oplus \cdots \oplus L_{n_2}^{(2)}.$$

If the formal factorisations of $c(E_1)$ and $c(E_2)$ are given by

$$\begin{aligned} c(E_1) &= (1 + x_1) \cdots (1 + x_{n_1}), \\ c(E_2) &= (1 + y_1) \cdots (1 + y_{n_2}), \end{aligned}$$

then by the product formula of Chern classes

$$c(E_1 \oplus E_2) = (1 + x_1) \cdots (1 + x_{n_1})(1 + y_1) \cdots (1 + y_{n_2}).$$

Therefore $\operatorname{ch}(E_1 \oplus E_2) = \operatorname{ch}(E_1) + \operatorname{ch}(E_2)$.

As for the multiplicativity, we have

$$\begin{aligned} p^*(E_1 \otimes E_2) &\cong p^*(E_1) \otimes p^*(E_2) \\ &\cong (L_1^{(1)} \oplus \cdots \oplus L_{n_1}^{(1)}) \otimes (L_1^{(2)} \oplus \cdots \oplus L_{n_2}^{(2)}) \\ &\cong \bigoplus_{\substack{1 \le i \le n_1 \\ 1 \le j \le n_2}} L_i^{(1)} \otimes L_j^{(2)}, \end{aligned}$$

therefore the tensor product $E_1 \otimes E_2$ is also pulled back to a direct sum of line bundles. For complex line bundles L_1 and L_2, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$[14]. Consequently, $c(E_1 \otimes E_2)$ is formally factorised as

$$c(E_1 \otimes E_2) = \prod_{\substack{1 \le i \le n_1 \\ 1 \le j \le n_2}} (1 + (x_i + y_j)),$$

thus

$$\operatorname{ch}(E_1 \otimes E_2) = \sum_{\substack{1 \le i \le n_1 \\ 1 \le j \le n_2}} e^{x_i} e^{y_j} = \operatorname{ch}(E_1) \operatorname{ch}(E_2).$$

□

For a complex vector bundle $E \to M$ of dimension n, let us consider the following associated vector bundles:

- $\Lambda^k E \to M$, obtained by replacing each fibre F of E by its k^{th} exterior power. Its dimension is then equal to $\binom{n}{k}$.
- $S^k E \to M$, obtained by replacing each fibre F of E by its k^{th} symmetric power. The latter has dimension $\binom{n+k-1}{k}$.

If E has Chern roots $x_1, \ldots, x_n$, then we have the following formal factorisations[15]:

$$\begin{aligned} c(\Lambda^k E) &= \prod_{1 \le i_1 < \cdots < i_k \le n} (1 + (x_{i_1} + \cdots + x_{i_k})), \\ c(S^k E) &= \prod_{1 \le i_1 \le \cdots \le i_k \le n} (1 + (x_{i_1} + \cdots + x_{i_k})). \end{aligned}$$

[14] See [Hat09, p. 86].

[15] These results follow from a study of the associated vector bundle ρE induced by a representation $\rho : U(n) \to U(m)$. For more details, see [HBJ94, pp. 7-13].

We define the formal power series

$$\Lambda_t E := \sum_{k=0}^{\infty} (\Lambda^k E) t^k,$$
$$S_t E := \sum_{k=0}^{\infty} (S^k E) t^k.$$

Note that $\Lambda_t E$ is finite because $\Lambda^k E = 0$ for $k > n$. The Chern character being additive and multiplicative (Proposition 3.19), the following definitions make sense:

$$\mathrm{ch}(\Lambda_t E) := \sum_{k=0}^{\infty} \mathrm{ch}(\Lambda^k E) t^k,$$
$$\mathrm{ch}(S_t E) := \sum_{k=0}^{\infty} \mathrm{ch}(S^k E) t^k.$$

Lemma 3.20. *For a complex vector bundle $E \to M$ with Chern roots $x_1, \dots, x_n$, we have*

$$\mathrm{ch}(\Lambda_t E) = \prod_{i=1}^{n} (1 + t e^{x_i}),$$
$$\mathrm{ch}(S_t E) = \prod_{i=1}^{n} \frac{1}{1 - t e^{x_i}}.$$

Proof. We easily compute

$$\begin{aligned}\prod_{i=1}^{n} (1 + t e^{x_i}) &= 1 + t \left(\sum_{1 \le i \le n} e^{x_i} \right) + t^2 \left(\sum_{1 \le i_1 < i_2 \le n} e^{x_{i_1} + x_{i_2}} \right) + \cdots + t^n e^{x_1 + \cdots + x_n} \\ &= \sum_{k=0}^{\infty} \mathrm{ch}(\Lambda^k E) t^k \\ &= \mathrm{ch}(\Lambda_t E).\end{aligned}$$

As for the second equality, we have

$$\begin{aligned}\prod_{i=1}^{n} \frac{1}{1 - t e^{x_i}} &= \prod_{i=1}^{n} \sum_{k=0}^{\infty} (t e^{x_i})^k \\ &= \prod_{i=1}^{n} (1 + t e^{x_i} + t^2 e^{2x_i} + t^3 e^{3x_i} + \cdots) \\ &= 1 + t \left(\sum_i e^{x_i} \right) + t^2 \left(\sum_i e^{2x_i} + \sum_{i_1 < i_2} e^{x_{i_1} + x_{i_2}} \right) + \cdots \\ &= \sum_{k=0}^{\infty} t^k \left(\sum_{k_1 + \cdots + k_r = k} \sum_{i_{k_1} < \cdots < i_{k_r}} e^{k_1 x_{i_{k_1}} + \cdots + k_r x_{i_{k_r}}} \right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} t^k \left(\sum_{i_1 \leq \cdots \leq i_k} e^{x_{i_1} + \cdots + x_{i_k}} \right) \\
&= \sum_{k=0}^{\infty} \mathrm{ch}(S^k E) t^k \\
&= \mathrm{ch}(S_t E).
\end{aligned}
$$

□

When $E = TM_{\mathbb{C}}$ for a $4k$-manifold M, we have seen that its Chern roots have the form $x_1, \ldots, x_{2k}, -x_1, \ldots, -x_{2k}$. Lemma 3.20 implies

$$\mathrm{ch}(\Lambda_t TM_{\mathbb{C}}) = \prod_{i=1}^{2k} (1 + te^{\pm x_i}), \tag{6}$$

$$\mathrm{ch}(S_t TM_{\mathbb{C}}) = \prod_{i=1}^{2k} \frac{1}{1 - te^{\pm x_i}}, \tag{7}$$

where to shorten the notation we use $1 + te^{\pm x_i} := (1 + te^{x_i})(1 + te^{-x_i})$ and $1 - te^{\pm x_i} := (1 - te^{x_i})(1 - te^{-x_i})$.

4 Multiplicative genera and the Signature Theorem

In this chapter we introduce the notion of multiplicative genera and present Hirzebruch's formalism. It describes how multiplicative genera, power series and multiplicative sequences are closely related and how multiplicative genera can be expressed in terms of Pontryagin numbers. Following Hirzebruch's formalism, we will contruct two genera, namely the L-genus and the $\hat{A}$-genus, and show that the signature is equal to the L-genus.

4.1 Multiplicative genera and Hirzebruch's formalism

4.1.1 The oriented cobordism ring

We begin by defining the oriented cobordism ring and present some of its properties, without getting into details of proofs. The interested reader is referred to [MS74, chapters 17 and 18] for more details. Recall that unless we state it otherwise, we consider manifolds that are smooth, closed and oriented. Let M and M' be two such n-manifolds.

Definition 4.1. M and M' are said to be *cobordant* (denoted $M \simeq M'$) if there exists a compact, oriented manifold with boundary $X^{n+1} = X$ so that ∂X is diffeomorphic to $M \sqcup (-M')$ under a diffeomorphism that preserves orientation.

This definition induces an equivalence relation on the set of all closed, oriented n-manifolds. Let Ω_n^{SO} denote the set of its equivalence classes. Then Ω_n^{SO} together with disjoint union forms an additive group with zero element $[\emptyset]$ –

which is equal to the set of all n-dimensional boundaries – and in which the inverse of $[M]$ is $[-M]$. Moreover, the cartesian product of two manifolds induces a well-defined operation

$$\begin{aligned} \Omega_n^{SO} \times \Omega_m^{SO} &\longrightarrow \Omega_{n+m}^{SO} \\ ([N^n],[M^m]) &\longmapsto [N^n \times M^m]. \end{aligned}$$

We get a graded ring

$$\Omega_*^{SO} := \bigoplus_{k=0}^{\infty} \Omega_k^{SO},$$

called the *oriented cobordism ring*.

It follows from Thom's work[16] that Ω_k^{SO} is finite when k is not divisible by 4 and that Ω_{4k}^{SO} is finitely generated for every $k \geq 0$. Taking the tensor product with $\mathbb{Q}$ allows us to kill the torsion parts and we get the $\mathbb{Q}$-algebra

$$\Omega_*^{SO} \otimes \mathbb{Q} = \bigoplus_{k\geq 0}^{\infty} \Omega_k^{SO} \otimes \mathbb{Q} = \bigoplus_{k\geq 0}^{\infty} \Omega_{4k}^{SO} \otimes \mathbb{Q}.$$

Theorem 4.2. *The group $\Omega_{4k}^{SO} \otimes \mathbb{Q}$ is generated by the equivalence classes of*

$$\mathbb{C}\mathrm{P}^{2i_1} \times \cdots \times \mathbb{C}\mathrm{P}^{2i_r},$$

with $i_1, \ldots, i_r$ positive integers such that $i_1 + \cdots + i_r = k$.

Proof. See [MS74, p. 216]. □

Definition 4.3. A *multiplicative genus* (or shortly *genus*) is a map associating to every closed, oriented, smooth manifold an element in a commutative $\mathbb{Q}$-algebra with unit A, in such a way that it induces a $\mathbb{Q}$-algebra homomorphism

$$\varphi : \Omega_*^{SO} \otimes \mathbb{Q} \to A.$$

In other words, φ satisfies:

(i) $\varphi(M \sqcup N) = \varphi(M) + \varphi(N)$,

(ii) $\varphi(M \times N) = \varphi(M)\varphi(N)$,

(iii) $\varphi(M) = 0$ if M is diffeomorphic to the boundary ∂X of some compact, oriented manifold with boundary X.

Theorem 2.8 tells us that the signature is a multiplicative genus. Later we will present some other examples.

By Theorem 4.2, the $\mathbb{Q}$-algebra $\Omega_*^{SO} \otimes \mathbb{Q}$ is equal to the polynomial ring $\mathbb{Q}[[\mathbb{C}\mathrm{P}^2], [\mathbb{C}\mathrm{P}^4], [\mathbb{C}\mathrm{P}^6], \ldots]$. Thus a multiplicative genus is uniquely determined by its values on the classes of the complex projective spaces $\mathbb{C}\mathrm{P}^{2k}$. Before continuing our investigation on multiplicative genera, it is useful to know how to express

[16] See [MS74, p. 216].

the Pontryagin classes of these spaces. Let L^n denote the canonical line bundle over $\mathbb{C}\mathrm{P}^n$, i.e. the bundle with total space

$$L^n = \{([z_0 : \dots : z_n], v) \mid [z_0 : \dots : z_n] \in \mathbb{C}\mathrm{P}^n,\ v \in \mathbb{C}(z_0, \dots, z_n)\}$$

and natural projection

$$\begin{aligned} \pi : L^n &\longrightarrow \mathbb{C}\mathrm{P}^n \\ ([z_0 : \dots : z_n], v) &\longmapsto [z_0 : \dots : z_n]. \end{aligned}$$

It can be shown[17] that its first Chern class is

$$c_1(L^n) = -\alpha,$$

where α is a generator of $H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ satisfying $\alpha^n[\mathbb{C}\mathrm{P}^n] = 1$ and where $[\mathbb{C}\mathrm{P}^n]$ is the fundamental class corresponding to the natural orientation on $T\mathbb{C}\mathrm{P}^n$. The total Pontryagin class of $T\mathbb{C}\mathrm{P}^n$ is given by[18]

$$p(\mathbb{C}\mathrm{P}^n) = (1 + \alpha^2)^{n+1}. \tag{8}$$

4.1.2 Multiplicative sequences

Our next goal is to find a one-to-one correspondence between multiplicative genera and even power series. We will do it here for multiplicative genera taking rational values, respectively even power series with coefficients in $\mathbb{Q}$. However, all the results of this section also hold for multiplicative genera taking their values in A, where A denotes a commutative $\mathbb{Q}$-algebra with unit (all the proofs are exactly the same). This is worth noting, as later we will construct genera taking values in a ring of modular forms.

Let $K_1(x_1), K_2(x_1, x_2), K_3(x_1, x_2, x_3), \dots$ be a sequence of polynomials with coefficients in $\mathbb{Q}$ such that after assigning weight i to the variable x_i, the polynomial $K_n(x_1, \dots, x_n)$ is homogeneous of weight n. Let $A = \bigoplus_{k \geq 0} A_k$ be a graded commutative $\mathbb{Q}$-algebra with unit $1 \in A_0$. Then for every element of the form

$$a = 1 + a_1 + a_2 + a_3 + \cdots, \quad a_i \in A_i,$$

we define

$$K(a) := 1 + K_1(a_1) + K_2(a_1, a_2) + K_3(a_1, a_2, a_3) + \cdots,$$

which is again an element of A.

Definition 4.4. The sequence of polynomials $\{K_n\}_{n \geq 1}$ is called a *multiplicative sequence* if for every commutative graded $\mathbb{Q}$-algebra with unit $A = \bigoplus_{k \geq 0} A_k$ and for every pair of elements of the form

$$\begin{aligned} a &= 1 + a_1 + a_2 + a_3 + \cdots, \quad a_i \in A_i, \\ b &= 1 + b_1 + b_2 + b_3 + \cdots, \quad b_i \in A_i, \end{aligned}$$

$$K(ab) = K(a)K(b).$$

[17] See [MS74, pp. 169-170].
[18] See [MS74, pp. 177-178].

Example 4.5. For any $\lambda \in \mathbb{Q}$, the polynomials $K_n(x_1, \ldots, x_n) := \lambda^n x_n$ form a multiplicative sequence. Indeed, for elements $a, b \in A$ as above, we have

$$ab = \sum_{k=0}^{\infty} \sum_{i=0}^{k} a_i b_{k-i}$$

where we define $a_0 = b_0 := 1 \in A_0$. Then

$$\begin{aligned} K(ab) &= 1 + \lambda(a_1 + b_1) + \lambda^2(b_2 + a_1 b_1 + a_2) + \cdots \\ &= \sum_{k=0}^{\infty} \lambda^k \left(\sum_{i=0}^{k} a_i b_{k-i} \right), \end{aligned}$$

and

$$\begin{aligned} K(a)K(b) &= (1 + \lambda a_1 + \lambda^2 a_2 + \cdots) \cdot (1 + \lambda b_1 + \lambda^2 b_2 + \cdots) \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \lambda^k a_i b_{k-i} \\ &= \sum_{k=0}^{\infty} \lambda^k \left(\sum_{i=0}^{k} a_i b_{k-i} \right) \\ &= K(ab). \end{aligned}$$

The following result is due to Hirzebruch:

Theorem 4.6. *For every formal even power series of the form*

$$Q(x) = 1 + \lambda_2 x^2 + \lambda_4 x^4 + \lambda_6 x^6 + \cdots, \quad \lambda_{2i} \in \mathbb{Q},$$

there exists a unique multiplicative sequence $\{K_n\}_{n \geq 1}$ *such that*

$$Q(x) = K(1 + x^2), \tag{9}$$

i.e. such that the coefficient of x_1^n *in* $K_n(x_1, \ldots, x_n)$ *is equal to* λ_{2n}.

Note that the $\mathbb{Q}$-algebra we are considering here is the $\mathbb{Q}$-algebra of polynomials in x^2 with rational coefficients, i.e. $A = \mathbb{Q}[x^2]$.

Proof. We refer to [MS74, pp. 221-222] or [Hir78, pp. 9-10]. Our investigation in the rest of this section will justify why we adapted the statement to the case of even power series. □

Definition 4.7. The sequence $\{K_n\}_{n \geq 1}$ satisfying (9) is called the *multiplicative sequence associated to* Q.

Example 4.8. The multiplicative sequence we have introduced in Example 4.5 is the multiplicative sequence associated to the power series

$$Q(x) = 1 + \lambda x^2.$$

Let $Q(x)$ be an even power series with rational coefficients and starting with 1 and let $\{K_r\}_{r\geq 1}$ be its associated multiplicative sequence. Then

$$\begin{aligned} Q(x_1)\cdots Q(x_n) &= K(1+x_1^2)\cdots K(1+x_n^2) \\ &= K((1+x_1^2)\cdots(1+x_n^2)) \\ &= K(1+p_1+p_2+\cdots+p_n), \end{aligned}$$

where $p_j := \sum_{i_1<\cdots<i_j} x_{i_1}^2 \cdots x_{i_j}^2$ is the j^{th} elementary symmetric function in the x_i^2. Like we have seen in section 3.4, if $n = 2k$ and $x_1, \dots, x_{2k}$ denote the Chern roots of a $4k$-manifold M, then the total Pontryagin class of TM is formally factorized as

$$p(M) = p(TM) = (1+x_1^2)\cdots(1+x_{2k}^2),$$

and the j^{th} elementary symmetric function p_j corresponds to the j^{th} Pontryagin class of TM. So we define

$$K[M] := \begin{cases} K_k(p_1,\dots,p_k)[M] \in \mathbb{Q} & \text{if } \dim M = 4k, \\ 0 & \text{if } \dim M \not\equiv 0 \ (4), \end{cases}$$

which is equal to a combination of Pontryagin numbers of M. In other words, when $\dim M = 4k$,

$$K[M] = \left(\prod_{i=1}^{2k} Q(x_i)\right)[M],$$

where $x_1, \dots, x_{2k}$ are the Chern roots of M.

Theorem 4.9. *For every multiplicative sequence $\{K_r\}_{r\geq 1}$, $K[\,\cdot\,]$ defines a multiplicative genus taking its values in $\mathbb{Q}$, called the K-*genus*.*

Proof. The additivity of $K[\,\cdot\,]$ follows directly from the fact that for two manifolds of same dimension M and N, $H^*(M \sqcup N;\mathbb{Z}) \cong H^*(M;\mathbb{Z}) \oplus H^*(N;\mathbb{Z})$ and that the class $p_i(M \sqcup N)$ corresponds to $p_i(M) + p_i(N)$.

Now let M and N denote two manifolds of dimensions m and n respectively. It is not hard to see that

$$T(M\times N) \cong \pi_M^*(TM) \oplus \pi_N^*(TN),$$

where π_M and π_N denote the natural projections

$$\begin{aligned} \pi_M &: M\times N \longrightarrow M, \\ \pi_N &: M\times N \longrightarrow N. \end{aligned}$$

Then the total Pontryagin class of $M \times N$ is

$$p(M\times N) = p(\pi_M^*(TM)\oplus\pi_N^*(TN)) = \pi_M^*(p(M)) \smile \pi_N^*(p(N)).$$

Since $\{K_r\}_{r\geq 1}$ is a multiplicative sequence,

$$K(p(M\times N)) = K(\pi_M^*(p(M))) \smile K(\pi_N^*(p(N))). \tag{10}$$

To simplify the notations, let us set

$$p_i := p_i(M \times N), \quad p_i' := p_i(M), \quad p_i'' := p_i(N).$$

Equation (10) then becomes

$$\begin{aligned} K(p(M \times N)) =& 1 + K_1(p_1) + K_2(p_1, p_2) + \cdots \\ =& (1 + K_1(\pi_M^*(p_1')) + K_2(\pi_M^*(p_1'), \pi_M^*(p_2')) + \cdots) \\ & \smile (1 + K_1(\pi_N^*(p_1'')) + K_2(\pi_N^*(p_1''), \pi_N^*(p_2'')) + \cdots) \\ =& \pi_M^*(1 + K_1(p_1') + K_2(p_1', p_2') + \cdots) \\ & \smile \pi_N^*(1 + K_1(p_1'') + K_2(p_1'', p_2'') + \cdots). \end{aligned}$$

Furthermore, we have the following equality:

$$(\pi_M^*(\alpha) \smile \pi_n^*(\beta))[M \times N] = \alpha[M]\beta[N], \quad \forall \alpha \in H^*(M; \mathbb{Z}),\ \beta \in H^*(N; \mathbb{Z}). \tag{11}$$

The K-genus of the cartesian product is then

$$\begin{aligned} K[M \times N] &= K_{m+n}(p_1, \ldots, p_{m+n})[M \times N] \\ &= \sum_{i+j=m+n} (\pi_M^*(K_i(p_1', \ldots, p_i')) \smile \pi_N^*(K_j(p_1'', \ldots, p_j'')))[M \times N] \\ &\overset{(11)}{=} \sum_{i+j=m+n} K_i(p_1', \ldots, p_i')[M] \cdot K_j(p_1'', \ldots, p_j'')[N]. \end{aligned}$$

The terms $K_i(p_1', \ldots, p_i')[M]$ and $K_j(p_1'', \ldots, p_j'')[N]$ are zero unless $i = \frac{m}{4}$, respectively $j = \frac{n}{4}$. Let us then distinguish cases. If $m = 4k_1$ and $n = 4k_2$, then

$$K[M \times N] = K_{k_1}(p_1', \ldots, p_i')[M] \cdot K_{k_2}(p_1'', \ldots, p_j'')[N] = K[M]K[N].$$

If m and/or n is not divisible by 4, then $K[M \times N] = K[M]K[N] = 0$. So $K[\,\cdot\,]$ is indeed multiplicative.

Let now M be a $4k$-manifold such that M is the boundary of an oriented $(4k+1)$-manifold with boundary X, i.e. $M = \partial X$. We wish to prove that its K-genus is zero. We will actually show that all its Pontryagin numbers are zero. Let us denote by $[X]$ the fundamental homology class of X in $H_{4k+1}(X, M; \mathbb{Z})$[19]. Then

$$\partial_*([X]) = [M], \tag{12}$$

where ∂_* denotes the homomorphism in the long exact sequence in homology

$$\cdots \longrightarrow H_{4k+1}(X, M; \mathbb{Z}) \xrightarrow{\partial_*} H_{4k}(M; \mathbb{Z}) \longrightarrow \cdots.$$

[19]For more explanations about orientability of a manifold with boundary and its fundamental class, see [Hat01, pp. 252-254].

Let us now consider the inclusion $i : M \hookrightarrow X$ and the pullback $i^*(TX)$ over M. This pullback is isomorphic to the restriction of TX to the subspace $M \subset X$, i.e.

$$i^*(TX) \cong TX|_M \,. \tag{13}$$

Given a Euclidean metric on TX, there is a unique outward pointing normal vector field along M, and this vector field spans a trivial line bundle T on M so that

$$TX|_M \cong TM \oplus T.$$

The naturality and stability of Pontryagin classes imply that

$$\begin{aligned} i^*(p_j(TX)) &\overset{(13)}{=} p_j(TX|_M) \\ &= p_j(TM \oplus T) \\ &= p_j(M). \end{aligned} \tag{14}$$

Finally, let us consider the long exact sequence in cohomology

$$\cdots \longrightarrow H^{4k}(X;\mathbb{Z}) \xrightarrow{i^*} H^{4k}(M;\mathbb{Z}) \xrightarrow{\delta^*} H^{4k+1}(X,M;\mathbb{Z}) \longrightarrow \cdots ,$$

which implies that

$$\delta^*(p_j(M)) \overset{(14)}{=} \delta^*(i^*(p_j(TX))) = 0 \tag{15}$$

for every $j \in \mathbb{N}$ by exactness of the sequence. Thus for a partition $I = (i_1, \ldots, i_r)$ of k,

$$\begin{aligned} p_I[M] = p_{i_1} \cdots p_{i_r}[M] &= (p_{i_1}(M) \cdots p_{i_r}(M))[M] \\ &\overset{(12)}{=} (p_{i_1}(M) \cdots p_{i_r}(M))(\partial_*[X]) \\ &= (\delta^*(p_{i_1}(M) \cdots p_{i_r}(M)))[X] \\ &= (\delta^*(p_{i_1}(M)) \cdots \delta^*(p_{i_r}(M)))[M] \\ &\overset{(15)}{=} 0. \end{aligned}$$

Obviously, the vanishing of all Pontryagin numbers of M implies the vanishing of $K[M]$. □

Definition 4.10. Let $Q(x)$ be an even power series with rational coefficients and starting with 1 and let $\{K_n\}_{n\geq 1}$ denote its associated multiplicative sequence. We will also denote the genus $K[\,\cdot\,]$ by φ_Q and will call it the *genus corresponding to* Q.

We will now show that there is a one-to-one correspondence between even power series starting with 1 and multiplicative genera. Let then Q be an even rational power series starting with 1. The formal power series $f(x) := \frac{x}{Q(x)}$ is odd and starts with x, i.e. it is of the form

$$f(x) = x + a_3 x^3 + a_5 x^5 + \cdots .$$

Then there is a formal power series g with $g(f(x)) = x$. The latter is called the *logarithm of* φ_Q.

Lemma 4.11. *The logarithm of φ_Q is given by*

$$g(y) = \sum_{n=0}^{\infty} \frac{\varphi_Q(\mathbb{C}\mathrm{P}^n)}{n+1} y^{n+1}.$$

Proof. Let us consider the total Pontryagin class of $\mathbb{C}\mathrm{P}^n$, which is given by $p = (1+\alpha^2)^{n+1}$ according to (8). Since $\{K_r\}_{r\geq 1}$ is the multiplicative sequence associated to Q, we have

$$K(p) = K((1+\alpha^2)^{n+1}) = K((1+\alpha^2))^{n+1} = Q(\alpha)^{n+1}.$$

Then

$$\begin{aligned} \varphi_Q(\mathbb{C}\mathrm{P}^n) = K[\mathbb{C}\mathrm{P}^n] &= \{\text{term of weight } 2n \text{ in } K(p)\}[\mathbb{C}\mathrm{P}^n] \\ &= \{\text{term in } \alpha^n \text{ in } Q(\alpha)^{n+1}\}[\mathbb{C}\mathrm{P}^n] \\ &= \{\text{coefficient of } \alpha^n \text{ in } Q(\alpha)^{n+1}\}, \end{aligned}$$

where the last equality comes from $\alpha^n[\mathbb{C}\mathrm{P}^n] = 1$.

The function $\frac{1}{f(x)} = \frac{Q(x)}{x}$ has a simple pole at $x = 0$, thus $(\frac{1}{f(x)})^{n+1}$ has a Laurent expansion of the form

$$\sum_{k=-n-1}^{\infty} a_k x^k.$$

The coefficient of x^n in the series $Q(x)^{n+1} = (\frac{x}{f(x)})^{n+1}$ is then equal to a_{-1}, the residue of $(\frac{1}{f(x)})^{n+1}$ at the pole $x = 0$. Substituting $y = f(x)$ gives

$$\begin{aligned} \varphi_Q(\mathbb{C}\mathrm{P}^n) &= \mathrm{res}_0 \left(\frac{1}{f(x)}\right)^{n+1} \\ &= \frac{1}{2\pi i}\int_\kappa \left(\frac{1}{f(x)}\right)^{n+1} dx \\ &= \frac{1}{2\pi i}\int_{f(\kappa)} \left(\frac{1}{y}\right)^{n+1} g'(y)dy \\ &= \mathrm{res}_0 \left(\frac{g'(y)}{y^{n+1}}\right), \end{aligned}$$

where κ denotes a circle around 0. Hence $\varphi_Q(\mathbb{C}\mathrm{P}^n)$ is equal to the coefficient of y^n in $g'(y)$. Note that the integral does not make any sense if f is not convergent, however the formula $\mathrm{res}_0(\frac{1}{f(x)^{n+1}}) = \mathrm{res}_0(\frac{g'(y)}{y^{n+1}})$ still holds. So we have

$$g'(y) = \sum_{n=0}^{\infty} \varphi_Q(\mathbb{C}\mathrm{P}^n) y^n,$$

and since g has no constant term (otherwise $g(f(x)) \neq x$),

$$g(y) = \sum_{n=0}^{\infty} \frac{\varphi_Q(\mathbb{C}\mathrm{P}^n)}{n+1} y^{n+1}.$$

□

Remark 4.12. Note that $\mathbb{C}\mathrm{P}^0$ is a one point space, so by multiplicativity of the genus $\varphi_Q(\mathbb{C}\mathrm{P}^0) = 1$. Therefore the logarithm of φ_Q is of the form

$$g(y) = y + \frac{\varphi_Q(\mathbb{C}\mathrm{P}^2)}{3} y^3 + \frac{\varphi_Q(\mathbb{C}\mathrm{P}^4)}{5} y^5 + \cdots .$$

Conversely, for a fixed genus φ, let us consider the power series given by

$$g(y) = \sum_{n=0}^{\infty} \frac{\varphi(\mathbb{C}\mathrm{P}^n)}{n+1} y^{n+1}.$$

Then the odd power series $f(x) = x + a_3x^3 + a_5x^5 + \cdots$ so that $g(f(x)) = x$ as well as the even power series $Q(x) = \frac{x}{f(x)}$ are uniquely determined, thus we have a one-to-one correspondence between genera and even power series that start with 1. Therefore, any genus φ is equal to a K-genus for a multiplicative sequence $\{K_r\}_{r\geq 1}$, and for every manifold M, $\varphi(M)$ can be expressed as combination of Pontryagin numbers of M.

4.2 The L-genus and $\hat{A}$-genus

Let us consider the function $\frac{x}{\tanh x}$ with its power series expansion

$$Q_L(x) := \frac{x}{\tanh x} = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n},$$

where the B_{2n} denote the Bernoulli numbers. The multiplicative sequence associated to Q_L is denoted by $\{L_r\}_{r\geq 1}$ and the first polynomials of the sequence are

$$\begin{aligned} L_1(x_1) &= \frac{1}{3}x_1, \\ L_2(x_1, x_2) &= -\frac{1}{45}x_1^2 + \frac{7}{45}x_2, \\ L_3(x_1, x_2, x_3) &= \frac{2}{945}x_1^3 - \frac{13}{945}x_1x_2 + \frac{62}{945}x_3. \end{aligned} \tag{16}$$

Hirzebruch's formalism then yields the L-genus, the genus corresponding to Q_L. The function $f_L(x) = \frac{x}{Q_L(x)}$ is equal to $\tanh x$, so the logarithm of the L-genus is given by

$$g_L(y) = \operatorname{arctanh} y = \sum_{n=0}^{\infty} \frac{1}{2n+1} y^{2n+1}.$$

Then Lemma 4.11 gives

$$L[\mathbb{C}\mathrm{P}^{2n}] = 1, \quad \forall n \in \mathbb{N}. \tag{17}$$

We now consider the even function

$$Q_{\hat{A}}(x) := \frac{x/2}{\sinh(x/2)} = 1 - \frac{1}{24}x^2 + \frac{7}{5760}x^4 + \cdots .$$

The first two polynomials of the corresponding multiplicative sequence are

$$\hat{A}_1(x_1) = -\frac{1}{24}x_1, \tag{18}$$
$$\hat{A}_2(x_1, x_2) = \frac{7}{5760}x_1^2 - \frac{4}{5760}x_2.$$

The logarithm of the corresponging genus, called the $\hat{A}$-*genus*, is then given by $g_{\hat{A}} = f_{\hat{A}}^{-1}$ for $f_{\hat{A}}(x) = \frac{x}{Q_{\hat{A}}(x)} = 2\sinh(x/2)$, hence

$$g_{\hat{A}}(y) = 2\,\mathrm{arcsinh}(y/2) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n (2n)!}{2^{4n}(n!)^2} \right) \frac{1}{2n+1} y^{2n+1}.$$

The $\hat{A}$-genus of $\mathbb{C}\mathrm{P}^{2n}$ is then

$$\hat{A}[\mathbb{C}\mathrm{P}^{2n}] = \frac{(-1)^n(2n)!}{2^{4n}(n!)^2}.$$

4.3 Hirzebruch Signature Theorem

As we pointed out earlier, the signature is a genus. The question is, what is its corresponding power series, respectively its corresponding multiplicative sequence. Hirzebruch showed that the signature is equal to the L-genus.

Theorem 4.13 (**Hirzebruch Signature Theorem**)**.** *For every closed, oriented, smooth manifold M,*

$$\mathrm{sign}(M) = L[M].$$

Proof. The signature and the L-genus both induce a $\mathbb{Q}$-algebra homomorphism $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$. It is then sufficient to check the assertion for the generators of the oriented cobordism ring, i.e. the complex projective spaces $\mathbb{C}\mathrm{P}^{2k}$ (by Theorem 4.2). We have seen in chapter 2 (see (3)) that $\mathrm{sign}(\mathbb{C}\mathrm{P}^{2k}) = 1$ for every $k \in \mathbb{N}$. And as we have seen in (17), the L-genus of $\mathbb{C}\mathrm{P}^{2k}$ is also equal to 1. $\square$

A direct consequence of the signature Theorem is that the L-genus of a manifold is always an integer. This enables us to make some inferences about divisibility of Pontryagin numbers. For example, let us consider a 4-manifold M and the generator α of $H^4(M;\mathbb{Z})$ given by the orientation of M, i.e. such that $\alpha[M] = 1$. The first Pontryagin class of M can be written as

$$p_1 = n\alpha$$

for some $n \in \mathbb{Z}$, and so the Pontryagin number $p_1[M]$ is equal to n. Now recall that the first polynomial of $\{L_r\}_{r\geq 1}$ – the multiplicative sequence that induces the L-genus – is given by

$$L(x_1) = \frac{1}{3}x_1.$$

We have

$$L_1(p_1)[M] = \frac{n}{3}\alpha[M] = \frac{n}{3} \in \mathbb{Z}.$$

Thus the Pontryagin number $p_1[M]$ must be a multiple of 3.

4.4 Twisted genera

Let M be a closed, oriented, smooth manifold of dimension $4k$, and let us write the formal factorisation of $c(M) = c(TM_{\mathbb{C}})$ as

$$c(M) = (1 + x_1) \cdots (1 + x_{2k})(1 - x_1) \cdots (1 - x_{2k}).$$

By the Hirzebruch Signature Theorem 4.13,

$$\operatorname{sign}(M) = L[M] = \left(\prod_{i=1}^{2k} \frac{x_i}{\tanh x_i} \right) [M].$$

Definition 4.14. Let E denote a complex vector bundle over M. Then we define the *signature of M with values in E* as

$$\operatorname{sign}(M, E) := \left(\operatorname{ch}(E) \prod_{i=1}^{2k} \frac{x_i}{\tanh(x_i/2)} \right) [M].$$

Let r denote the dimension of E and let us write the formal factorisation of its total Chern class as

$$c(E) = (1 + y_1) \cdots (1 + y_r).$$

It is possible to construct another associated vector bundle, denoted by $\Psi_2(E)$, with Chern roots $2y_1, \ldots 2y_r$, i.e.

$$\operatorname{ch}(\Psi_2(E)) = e^{2y_1} + \cdots + e^{2y_r}.$$

To make the relation with the signature more explicit, we can rewrite $\operatorname{sign}(M, E)$ as

$$\begin{aligned} \operatorname{sign}(M, E) &= \frac{1}{2^{2k}} \left((e^{2y_1} + \cdots + e^{2y_r}) \prod_{i=1}^{2k} \frac{2x_i}{\tanh x_i} \right) [M] \\ &= \left(\operatorname{ch}(\Psi_2(E)) \prod_{i=1}^{2k} \frac{x_i}{\tanh x_i} \right) [M]. \end{aligned}$$

Remark 4.15. The signature of M is actually equal to the index of some elliptic operator[20] called the signature operator. Using E, we can construct a twisted elliptic operator whose index is $\operatorname{sign}(M, E)$. The index of an operator being defined as a difference between two finite dimensions, this infers that $\operatorname{sign}(M, E)$ is integral.

Now we want to make the same process for the $\hat{A}$-genus. Recall that

$$\hat{A}(M) = \left(\prod_{i=1}^{2k} \frac{x_i/2}{\sinh(x_i/2)} \right) [M].$$

[20] For the definition of an elliptic operator and its index, the reader is referred to [HBJ94, chapter 5].

Definition 4.16. For a given complex vector bundle E over M, we define the *twisted $\hat{A}$-genus* of M as

$$\hat{A}(M,E) := \left(\mathrm{ch}(E)\prod_{i=1}^{2k}\frac{x_i/2}{\sinh(x_i/2)}\right)[M].$$

Note that if T is a trivial complex line bundle over M, then $c(T) = 1$ and $\mathrm{ch}(T) = 1$. Hence

$$\mathrm{sign}(M,T) = \mathrm{sign}(M) \text{ and } \hat{A}(M,T) = \hat{A}(M).$$

Remark 4.17. In general, $\hat{A}(M)$ is simply a rational number. But there are certain smooth manifolds whose $\hat{A}$-genus is always an integer, for example the *spin manifolds.* These are oriented, smooth manifolds that possess what is called a spin structure[21]. Equivalently, a smooth manifold M is a spin manifold if and only if its first and second Whitney-Stiefel classes vanish, i.e.

$$\omega_1(M) = \omega_2(M) = 0,$$

the condition $\omega_1(M) = 0$ corresponding to the orientability of M. For an oriented, spin manifold M, $\hat{A}(M)$ is equal to the index of an operator called the *Dirac operator*[22]. Similarly, for a complex vector bundle E over M, $\hat{A}(M,E)$ is the index of a twisted Dirac operator. Consequently, $\hat{A}(M)$ and $\hat{A}(M,E)$ are integral. For spin manifolds of dimension $\dim M \equiv 4\ (8)$, there is a stronger result, namely that $\hat{A}(M,E)$ is an even integer[23]. To sum up, we have the following theorem:

Theorem 4.18. *For every closed spin manifold M and for every complex vector bundle E on M,*

$$\hat{A}(M,E) \in \mathbb{Z}.$$

Furthermore, if $\dim M = 8k+4$, *then*

$$\hat{A}(M,E) \equiv 0\ (2).$$

4.5 Rokhlin's Theorem

We are now able to prove Ochanine's Theorem in dimension 4. This result was proven by Rokhlin in 1950 – using different methods than the ones we will use here.

Theorem 4.19 (Rokhlin, 1950). *Let M be a 4-dimensional closed, oriented, spin smooth manifold. Then*

$$\mathrm{sign}(M) \equiv 0\ (16).$$

[21] We will not enter into details of the definition of a spin structure in this paper. We refer the interested reader to [LJM89].

[22] An introduction to the Dirac operator can also be found in [LJM89].

[23] See [HBJ94] and references therein.

Proof. By the Hirzebruch Signature Theorem 4.13 and by (16)

$$\operatorname{sign}(M) = L[M] = L_1(p_1)[M] = \frac{1}{3}p_1[M].$$

On the other hand, the $\hat{A}$-genus of M is given by (recall (18))

$$\hat{A}(M) = \hat{A}_1(p_1)[M] = -\frac{1}{24}p_1[M].$$

Thus $\operatorname{sign}(M) = -8\hat{A}(M)$. But Theorem 4.18 tells us that $\hat{A}(M)$ is an even integer. Therefore $\operatorname{sign}(M)$ is a multiple of 16. $\square$

5 Modular forms

5.1 Modular forms for congruence subgroups

We introduce here basic notions of congruence subgroups and modular forms. More details can be found in [Kob93, chapter III].

The functions we will consider in this chapter are defined on the *upper-half plane*

$$\mathcal{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}.$$

The group $\mathrm{SL}_2(\mathbb{R}) = \{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}(2;\mathbb{R}) \mid \det A = 1\}$ acts on $\mathcal{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau := \frac{a\tau + b}{c\tau + d}.$$

Indeed, the result stays in the upper-half plane since its imaginary part is given by

$$\frac{\operatorname{Im}(\tau)(ad - bc)}{(c\operatorname{Re}(\tau) + d)^2 + c^2\operatorname{Im}(\tau)^2}, = \frac{\operatorname{Im}(\tau)}{(c\operatorname{Re}(\tau) + d)^2 + c^2\operatorname{Im}(\tau)^2},$$

which is positive. We will be particularly interested in the induced action of the subgroup

$$\Gamma := \mathrm{SL}_2(\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \ \mathrm{SL}_2(\mathbb{R}) \mid a, b, c, d \in \mathbb{Z}\}$$

on $\mathcal{H}$. If $i\infty$ denotes the point at infinity (i.e. $\operatorname{Im}(\tau) = \infty$) then, after introducing the conventions

$$\frac{a(i\infty) + b}{c(i\infty) + d} := \frac{a}{c}$$
$$\frac{a(\frac{-d}{c}) + b}{c(\frac{-d}{c}) + d} := i\infty,$$

Γ also acts on $\mathbb{Q} \cup \{i\infty\}$.

Let N be a positive integer.

Definition 5.1. The *principal congruence subgroup of level N* is defined by

$$\Gamma(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid a \equiv d \equiv 1\ (N), b \equiv c \equiv 0\ (N)\},$$

shortly written

$$\Gamma(N) = \{A \in \Gamma \mid A \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\ (N)\}.$$

Note that $\Gamma(N)$ is the kernel of the natural projection

$$\begin{aligned}\pi : \Gamma = \mathrm{SL}_2(\mathbb{Z}) &\to \mathrm{SL}_2(\mathbb{Z}_N)\\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\mapsto \left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right),\end{aligned}$$

where for an integer n, $\overline{n}$ denotes its congruence class modulo N. Hence $\Gamma(N)$ is a normal subgroup of Γ.

Definition 5.2. A subgroup Γ' of Γ is called a *congruence subgroup of level* N if it contains $\Gamma(N)$.

Remark 5.3. If Γ' is a congruence subgroup of level N, then it is also a congruence subgroup of level N' for any multiple N' of N, since $\Gamma(N') \subset \Gamma(N)$.

Like Γ, any congruence subgroup acts on the set $\mathbb{Q} \cup \{i\infty\}$. Bézout's identity implies that the action of Γ is transitive, however this is not the case for other congruence subgroups in general.

Definition 5.4. Let Γ' denote a congruence subgroup. The orbits of $\mathbb{Q} \cup \{i\infty\}$ under the action of Γ' are called the *cusps of* Γ'.

In general, we choose a convenient representative to denote a cusp. We say for example that Γ has "a single cusp at $i\infty$". By abuse of language we will often say "the cusp $s \in \mathbb{Q} \cup \{i\infty\}$" to denote the cusp to which s belongs.

Let Γ' be a congruence subgroup of (minimal) level N and let f be a holomorphic function on $\mathcal{H}$. In all that follows, we assume there exists an integer k so that

$$f(A\tau) = (c\tau + d)^k f(\tau), \quad \forall A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'. \tag{19}$$

In particular for $T_N = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma'$, f satisfies

$$f(T_N\tau) = f(\tau + N) = f(\tau).$$

Therefore it is periodic with period N and has a Fourier expansion of the form

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i \tau n/N}.$$

From now on, it will be convenient to use the variable q to denote $e^{2\pi i\tau}$ for $\tau \in \mathcal{H}$. By expanding this change of variable to $\mathbb{Q} \cup \{i\infty\}$, we get a map

$$\begin{aligned}\mathcal{H} \cup \mathbb{Q} \cup \{i\infty\} &\to \overline{D}\\ \tau &\mapsto q := e^{2\pi i\tau},\end{aligned}$$

where D denotes the open unit disk in $\mathbb{C}$. Points of $\mathcal{H}$ are mapped to points of D (excluding 0), points of $\mathbb{Q}$ are mapped to the boundary of D and $i\infty$ to 0.

Let us now come back to our function f. We can now write its Fourier expansion as

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^{n/N}. \tag{20}$$

Definition 5.5. The function f is said to be *holomorphic at* $i\infty$ if for all negative $n \in \mathbb{Z}$, the coefficient a_n in (20) is zero.

In other words, holomorphicity at $i\infty$ indicates that f has a "good behaviour" when $|\tau| \to \infty$. We would like to find a way to describe the same "good behaviour" at the cusps of Γ'. For that purpose, let us first introduce the following "$|_k \Gamma$" notation. For a matrix $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we set

$$f|_k B(\tau) := (c\tau + d)^{-k} f(B\tau).$$

Property (19) can be rephrased as

$$f|_k A = f, \quad \forall A \in \Gamma'. \tag{21}$$

A simple computation shows that for every $B_1, B_2 \in \Gamma$,

$$(f|_k B_1)|_k B_2 = f|_k (B_1 B_2). \tag{22}$$

Equalities (21) and (22) imply that for every $B \in \Gamma$ and every $A \in \Gamma'$

$$(f|_k B)|_k B^{-1}AB = f|_k AB = (f|_k A)|_k B = f|_k B.$$

Therefore, for every $C = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \in B^{-1}\Gamma' B$,

$$f|_k B(C\tau) = (g\tau + h)^k \, f|_k B(\tau).$$

In particular, $T_N = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$ belongs to $B^{-1}\Gamma' B$ for all $B \in \Gamma$ because $\Gamma(N)$ is a normal subgroup of Γ'. Thus

$$f|_k B(T_N \tau) = f|_k B(\tau + N) = f|_k B(\tau).$$

Consequently, $f|_k B$ also has period N and has a Fourier expansion of the form

$$f|_k B(\tau) = \sum_{n=-\infty}^{\infty} b_n q^{n/N}. \tag{23}$$

For a cusp $s \in \mathbb{Q} \cup \{i\infty\}$, let $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ be such that $Bi\infty = s$. Roughly speaking, the behaviour of $f|_k B(\tau) = (c\tau + d)^{-k} f(B\tau)$ when $\tau \to i\infty$ reflects the behaviour of $f(\tau)$ when $\tau \to s$.

Definition 5.6. We say that f is *holomorphic at the cusp* s if for every negative $n \in \mathbb{Z}$, the coefficient b_n in the expansion (23) is zero, i.e. if the function $f|_k B$ is holomorphic at $i\infty$.

Remark 5.7. This definition depends neither on the choice of the representative of the cusp s nor on the choice of the matrix $B \in \Gamma$ such that $Bi\infty = s$. To see that, let us consider another representative $t = As$ for some $A \in \Gamma'$, as well as a matrix $B' \in \Gamma$ with $B' i\infty = t$. We have $B' i\infty = As = ABi\infty$. Hence the matrix $B^{-1}A^{-1}B' \in \Gamma$ preserves the point $i\infty$. But then it must be of the form

$$B^{-1}A^{-1}B' = \left(\begin{smallmatrix} \pm 1 & \pm j \\ 0 & \pm 1 \end{smallmatrix}\right) = \pm T^j,$$

for some $j \in \mathbb{Z}$, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, $B' = AB(\pm T^j)$ and

$$\begin{aligned} f|_k B' &= f|_k AB(\pm T^j) \overset{(22)}{=} (f|_k A)|_k B(\pm T^j) \\ &\overset{(21)}{=} f|_k B(\pm T^j) \overset{(22)}{=} (f|_k B)|_k (\pm T^j). \end{aligned}$$

But

$$\begin{aligned} (f|_k B)|_k (\pm T^j)(\tau) &= (\pm 1)^{-k} f|_k B\left(\tfrac{\pm(\tau+j)}{\pm 1}\right) = (\pm 1)^{-k} f|_k B(\tau + j) \\ &= (\pm 1)^{-k} \sum_{n=-\infty}^{\infty} b_n e^{2\pi i(\tau+j)n/N} = \sum_{n=-\infty}^{\infty} c_n q^{n/N}, \end{aligned}$$

where the coefficient $c_n := (\pm 1)^{-k} e^{2\pi i j n/N} b_n$ is 0 if and only if $b_n = 0$. Therefore $f|_k B'$ is holomorphic at $i\infty$ if and only if $f|_k B$ is.

We are now able to understand the definition of a modular form. Let $\Gamma' \subset \Gamma$ be a congruence subgroup and let k be an integer.

Definition 5.8. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called a *modular form of weight k for Γ'* if it satisfies

$$f(A\tau) = (c\tau + d)^k f(\tau)$$

for every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'$ and if it is holomorphic at every cusp of Γ'.

It is easily checked that modular forms of weight k for Γ' form a complex vector space, which we will denote by $M_k(\Gamma')$. Furthermore, multiplying a modular form of weight k for Γ' with one of weight l gives a modular form of weight $k + l$ for Γ'. Hence the direct sum

$$M_*(\Gamma') := \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma')$$

is a graded commutative ring with unit $1 \in M_0(\Gamma')$.

In the second part of this chapter, we will focus on a particular congruence subgroup of level 2.

5.2 The congruence subgroup $\Gamma_0(2)$ and its ring of modular forms

The aim of this section is to find generators for the ring of modular forms for the congruence subgroup of level 2

$$\Gamma_0(2) := \{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c \equiv 0 \ (2)\}.$$

If a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to $\Gamma_0(2)$, then clearly c is even by definition, but we can also immediately deduce that a and d are odd, otherwise $\det A \neq 1$. Let us now investigate the cusps of $\Gamma_0(2)$.

Proposition 5.9. *The congruence subgroup $\Gamma_0(2)$ possesses exactly two cusps, one at $i\infty$ and one at 0.*

Proof. First, we see that there is no $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ such that $Ai\infty = 0$, because this would imply $a = 0$. So there are at least two cusps.

Let us now consider a rational number $s = \frac{n}{m}$ with coprime integers n, m. Then Bézout's identity ensures the existence of integers u, v so that

$$nu + mv = 1. \tag{24}$$

Two cases can be distinguished:

- Case 1: m is even (and then clearly n is odd). Then the matrix $A = \left(\begin{smallmatrix} -u & -v \\ m & -n \end{smallmatrix}\right)$ lies in $\Gamma_0(2)$ and satisfies

$$As = \frac{-u(\frac{n}{m}) - v}{m(\frac{n}{m}) - n} = i\infty.$$

- Case 2: m is odd. Here we consider the matrix $A = \left(\begin{smallmatrix} m & -n \\ u & v \end{smallmatrix}\right)$, which clearly maps $s = \frac{n}{m}$ to 0. It remains to ensure that u is even. Actually, making the following modification to (24) allows us to assume without loss of generality that u is even:

$$n(u + km) + m(v - kn) = 1, \quad \forall k \in \mathbb{Z}.$$

□

The matrix $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ maps $\tau \in \mathcal{H}$ to $\frac{-1}{\tau}$, so $Si\infty = 0$. For a modular form $f \in M_k(\Gamma_0(2))$, we will denote its expansion at the 0 cusp by

$$f^0(\tau) := f|_k\, S(\tau) = \tau^{-k} f(\tfrac{-1}{\tau}).$$

The matrix $-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ belongs to $\Gamma_0(2)$, consequently, a modular form f of weight k for $\Gamma_0(2)$ satisfies

$$f(-I\tau) = f(\tau) = (-1)^k f(\tau).$$

Therefore $M_k(\Gamma_0(2)) = \{0\}$ if k is odd and

$$M_*(\Gamma_0(2)) = \bigoplus_{k\in\mathbb{Z}} M_{2k}(\Gamma_0(2)).$$

In order to determine generators for $M_*(\Gamma_0(2))$, we begin by considering the Weierstrass $\wp$-function

$$\begin{aligned} \wp : \mathcal{H} \times \mathbb{C} &\to \overline{\mathbb{C}}, \\ (\tau, x) &\mapsto \frac{1}{x^2} + \sum_{\gamma \in L_\tau \setminus \{0\}} \left(\frac{1}{(x-\gamma)^2} - \frac{1}{\gamma^2} \right), \end{aligned}$$

where $\overline{\mathbb{C}}$ denotes the Riemann sphere $\mathbb{C} \cup \{i\infty\}$ and L_τ denotes the lattice $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. For a fixed τ, the function $\wp(\tau, \cdot)$ is elliptic of order 2 with respect to L_τ, with poles of order 2 at the lattice points.[24]

[24] The reader is referred to [Kob93, pp. 14-18] for the definition of an elliptic function and the study of the Weierstrass $\wp$-function.

Proposition 5.10. *For every* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$\wp(A\tau, \tfrac{x}{c\tau+d}) = (c\tau+d)^2 \wp(\tau, x), \quad \forall (\tau, x) \in \mathcal{H} \times \mathbb{C}.$$

Proof. We have

$$\begin{aligned}
\wp(A\tau, \tfrac{x}{c\tau+d}) &= \frac{(c\tau+d)^2}{x^2} + \sum_{\tilde{\gamma} \in L_{A\tau} \setminus \{0\}} \left(\frac{1}{(\frac{x}{c\tau+d} - \tilde{\gamma})^2} - \frac{1}{\tilde{\gamma}^2} \right) \\
&= \frac{(c\tau+d)^2}{x^2} + \sum_{\tilde{\gamma} \in L_{A\tau} \setminus \{0\}} \left(\frac{(c\tau+d)^2}{(x - \tilde{\gamma}(c\tau+d))^2} - \frac{(c\tau+d)^2}{(\tilde{\gamma}(c\tau+d))^2} \right) \\
&= (c\tau+d)^2 \left(\frac{1}{x^2} + \sum_{\tilde{\gamma} \in L_{A\tau} \setminus \{0\}} \left(\frac{1}{(x - \tilde{\gamma}(c\tau+d))^2} - \frac{1}{(\tilde{\gamma}(c\tau+d))^2} \right) \right) \\
&\overset{(*)}{=} (c\tau+d)^2 \left(\frac{1}{x^2} + \sum_{\gamma \in L_{\tau} \setminus \{0\}} \left(\frac{1}{(x - \gamma)^2} - \frac{1}{\gamma^2} \right) \right) \\
&= (c\tau+d)^2 \wp(\tau, x),
\end{aligned}$$

where equality $(*)$ holds because the map $\tilde{\gamma} \mapsto \tilde{\gamma}(c\tau+d)$ defines a bijection from $L_{A\tau} \setminus \{0\}$ to $L_\tau \setminus \{0\}$. Indeed, any $\tilde{\gamma} \in L_{A\tau} \setminus \{0\}$ can be written in the form

$$\tilde{\gamma} = 2\pi i (n_1 \tfrac{a\tau+b}{c\tau+d} + n_2) = 2\pi i \frac{n_1(a\tau+b) + n_2(c\tau+d)}{c\tau+d}$$

with $(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Then

$$\begin{aligned}
\tilde{\gamma}(c\tau+d) &= 2\pi i (n_1(a\tau+b) + n_2(c\tau+d)) \\
&= 2\pi i (\tilde{n}_1 \tau + \tilde{n}_2) \in L_\tau \setminus \{0\},
\end{aligned}$$

where $\left(\begin{smallmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \left(\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right) = A^t \left(\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right)$. Bijectivity of the map follows from the fact that $\det A^t = \det A = 1$. □

We now define the following functions:

$$\begin{aligned}
e_1(\tau) &:= \wp(\tau, \pi i), \\
e_2(\tau) &:= \wp(\tau, \pi i \tau), \\
e_3(\tau) &:= \wp(\tau, \pi i (\tau+1)).
\end{aligned}$$

These functions are actually the first terms in the Taylor expansion of $\wp(\tau, x)$ about the points $\pi i, \pi i \tau$ and $\pi i(\tau+1)$ respectively. The following result will then help us show that e_1, e_2 and e_3 satisfy some modular properties:

Lemma 5.11. *Let* $\alpha, \beta \in \mathbb{R}$ *so that* $2\pi i(\alpha\tau + \beta) \notin L_\tau$ *and let us write the Taylor expansion of* $\wp(\tau, \cdot)$ *about the point* $2\pi i(\alpha\tau+\beta)$ *as*

$$\wp(\tau, x) = \sum_{n=0}^{\infty} g_n(\tau)(x - 2\pi i(\alpha\tau+\beta))^n.$$

Then the coefficients $g_n(\tau)$ satisfy

$$g_n|_{n+2}\,A = g_n$$

for every matrix $A \in \Gamma$ such that $(\alpha, \beta)A \equiv (\alpha, \beta)\ (\mathbb{Z}^2)$.

Proof. The proof makes use of the Cauchy integral formula and Proposition 5.10. We refer to [HBJ94, p.128] for more details. □

The preceding Lemma enables us to show that e_1 transforms like a modular form of weight 2 for $\Gamma_0(2)$, i.e.

$$e_1|_2\,A = e_1, \quad \forall A \in \Gamma_0(2). \tag{25}$$

Since $e_1(\tau) = \wp(\tau, \pi i)$ is the constant term in the Taylor expansion of $\wp(\tau, \cdot)$ about the point $\pi i = 2\pi i(0\tau + 1/2)$, Lemma 5.11 implies that $e_1|_2\,A = e_1$ for every matrix $A \in \Gamma$ satisfying $(0, 1/2)A \equiv (0, 1/2)\ (\mathbb{Z}^2)$. If we write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, this is equivalent to saying that c is even and d is odd, i.e. that $A \in \Gamma_0(2)$. Similar arguments also show that

$$e_2|_2\,A = e_2, \quad \forall A \in \Gamma^0(2), \tag{26}$$

$$e_3|_2\,A = e_3, \quad \forall A \in \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma_0(2)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right), \tag{27}$$

where

$$\Gamma^0(2) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid b \equiv 0\ (2)\}.$$

We will show that e_1, e_2 and e_3 are modular forms. To do that, we will need to consider their Fourier expansions.

Proposition 5.12. *For $|q| = |e^{2\pi i\tau}| < \min\{|e^x|, |e^{-x}|\}$, $\wp(\tau, x)$ is equal to*

$$\frac{1}{(e^{x/2} - e^{-x/2})^2} + \sum_{n=1}^{\infty}\left(\sum_{d|n} d(e^{dx} + e^{-dx})\right) q^n + \frac{1}{12}\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right),$$

where $\sigma_1(n) := \sum_{d|n} d$.

Proof. See [HBJ94, p.129]. □

Applying the preceding result for $x = \pi i, \pi i\tau, \pi i(\tau + 1)$ yields the Fourier expansions of e_1, e_2 and e_3 at $i\infty$:

Proposition 5.13. *For every $\tau \in \mathcal{H}$,*

$$e_1(\tau) = -\frac{1}{6}\left(1 + 24\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n\right), \tag{28}$$

$$e_2(\tau) = \frac{1}{12}\left(1 + 24\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}q^{n/2}\right), \tag{29}$$

$$e_3(\tau) = \frac{1}{12}\left(1 + 24\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(-1)^n q^{n/2}\right), \tag{30}$$

where $\sigma_1^{\mathrm{odd}} := \sum_{d|n,\ d \text{ odd}} d$.

Theorem 5.14. *The functions e_1, e_2 and e_3 are modular forms, more precisely:*

$$e_1 \in M_2(\Gamma_0(2)), \tag{31}$$
$$e_2 \in M_2(\Gamma^0(2)), \tag{32}$$
$$e_3 \in M_2\left(\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma_0(2)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)\right). \tag{33}$$

Proof. Recall that

$$\begin{aligned} e_1(\tau) &= \wp(\tau, \pi i), \\ e_2(\tau) &= \wp(\tau, \pi i\tau), \\ e_3(\tau) &= \wp(\tau, \pi i(\tau+1)), \end{aligned}$$

where $\wp$ is meromorphic and for every $\tau \in \mathcal{H}$, $\wp(\tau, \cdot)$ is holomorphic at the points $\pi i, \pi i\tau$ and $\pi i(\tau+1)$. Thus the three functions are holomorphic on $\mathcal{H}$.

The fact that they transform like modular forms has already been seen in (25), (26) and (27).

It remains to see that they are holomorphic at all cusps. The Fourier expansions seen in Proposition 5.13 already show that they are holomorphic at $i\infty$. Now we wish to see that for every $A \in \Gamma$, $e_i|_2 A$ is holomorphic at $i\infty$ for $i \in \{1,2,3\}$. In fact, the "$|_2\,\Gamma$"-operation permutes e_1, e_2, e_3, i.e.

$$e_i|_2 A \in \{e_1, e_2, e_3\} \quad \forall A \in \Gamma, \quad \forall i \in \{1,2,3\}. \tag{34}$$

To show that, let us set

$$\wp_{(\alpha,\beta)}(\tau) := \wp(\tau, \alpha\tau + \beta), \quad \forall \alpha, \beta \in \mathbb{R}.$$

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have

$$\begin{aligned} \wp_{(\alpha,\beta)}\big|_2 A(\tau) &= (c\tau+d)^{-2}\wp_{(\alpha,\beta)}(A\tau) \\ &= (c\tau+d)^{-2}\wp(A\tau, \alpha(A\tau)+\beta) \\ &= (c\tau+d)^{-2}\wp(A\tau, \tfrac{\alpha(a\tau+b)+\beta(c\tau+d)}{c\tau+d}) \\ &\overset{5.10}{=} \wp(\tau, \alpha'\tau+\beta') \\ &= \wp_{(\alpha',\beta')}(\tau), \end{aligned}$$

where $(\alpha', \beta') = (\alpha, \beta)A$.

Our functions can be written as

$$e_1 = \wp_{(0,\pi i)}, \quad e_2 = \wp_{(\pi i, 0)}, \quad e_3 = \wp_{(\pi i, \pi i)}.$$

Using the fact that $\wp(\tau, \cdot)$ is elliptic with respect to $L_\tau = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$, we find

$$e_1|_2 A(\tau) = \wp_{(c\pi i, d\pi i)}(\tau) = \begin{cases} \wp(\tau, \pi i) = e_1(\tau) & \text{if } c \text{ is even, } d \text{ is odd} \\ \wp(\tau, \pi i\tau) = e_2(\tau) & \text{if } c \text{ is odd, } d \text{ is even} \\ \wp(\tau, \pi i(\tau+1)) = e_3(\tau) & \text{if } c, d \text{ are odd.} \end{cases} \tag{35}$$

Note that the case where c and d are both even cannot occur because $\det A = 1$. Similarly,

$$e_2|_2\, A(\tau) = \wp_{(a\pi i, b\pi i)}(\tau) = \begin{cases} \wp(\tau, \pi i) = e_1(\tau) & \text{if } a \text{ is even, } b \text{ is odd} \\ \wp(\tau, \pi i\tau) = e_2(\tau) & \text{if } a \text{ is odd, } b \text{ is even} \\ \wp(\tau, \pi i(\tau+1)) = e_3(\tau) & \text{if } a, b \text{ are odd.} \end{cases} \tag{36}$$

and

$$e_3|_2\, A(\tau) = \wp_{((a+c)\pi i, (b+d)\pi i)}(\tau) = \begin{cases} e_1(\tau) & \text{if } a+c \text{ is even, } b+d \text{ is odd} \\ e_2(\tau) & \text{if } a+c \text{ is odd, } b+d \text{ is even} \\ e_3(\tau) & \text{if } a+c, b+d \text{ are odd.} \end{cases} \tag{37}$$

Since e_1, e_2 and e_3 are holomorphic at $i\infty$ and by (34), they are also holomorphic at all cusps. □

We will now use the functions e_1, e_2, e_3 in order to construct modular forms for $\Gamma_0(2)$.

Theorem 5.15. *The following functions are modular forms for $\Gamma_0(2)$:*

$$\delta := -\frac{3}{2}e_1 = \frac{1}{4} + 6 \cdot \sum_{n=1}^{\infty} \sigma_1^{\text{odd}}(n) q^n \in M_2(\Gamma_0(2)), \tag{38}$$

$$\varepsilon := (e_1 - e_2)(e_1 - e_3) = \frac{1}{16} - q + 7q^2 - 28q^3 \pm \cdots \in M_4(\Gamma_0(2)), \tag{39}$$

$$\tilde{\delta} := \frac{3}{4}e_1 = -\frac{1}{8} - 3 \cdot \sum_{n=1}^{\infty} \sigma_1^{\text{odd}}(n) q^n \in M_2(\Gamma_0(2)), \tag{40}$$

$$\tilde{\varepsilon} := \frac{1}{16}(e_2 - e_3)^2 = q + 8q^2 + 28q^3 + \cdots \in M_4(\Gamma_0(2)). \tag{41}$$

Proof. (38) and (40) are clear because we know that $e_1 \in M_2(\Gamma_0(2))$.

The fact that ε and $\tilde{\varepsilon}$ are holomorphic on $\mathcal{H}$ and at all cusps is obvious since e_1, e_2 and e_3 are modular forms. We still need to check that they transform like modular forms. So let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\Gamma_0(2)$. Then

$$\begin{aligned} \varepsilon(A\tau) &= (e_1(A\tau) - e_2(A\tau)) \cdot (e_1(A\tau) - e_3(A\tau)) \\ &\overset{(31)}{=} ((c\tau+d)^2 e_1(\tau) - e_2(A\tau)) \cdot ((c\tau+d)^2 e_1(\tau) - e_3(A\tau)) \\ &= (c\tau+d)^4 \cdot (e_1(\tau) - (c\tau+d)^{-2} e_2(A\tau)) \cdot (e_1(\tau) - (c\tau+d)^{-2} e_3(A\tau)) \\ &= (c\tau+d)^4 \cdot (e_1(\tau) - e_2|_2\, A(\tau)) \cdot (e_1(\tau) - e_3|_2\, A(\tau)). \end{aligned}$$

According to (36) and (37), we need to distinguish two cases. If b is even, then $e_2|_2\, A = e_2$ and $e_3|_2\, A = e_3$, so

$$\varepsilon(A\tau) = (c\tau+d)^4 (e_1(\tau) - e_2(\tau))(e_1(\tau) - e_3(\tau)) = (c\tau+d)^4 \varepsilon(\tau).$$

If b is odd, then $e_2|_2\, A = e_3$ and $e_3|_2\, A = e_2$, hence

$$\varepsilon(A\tau) = (c\tau+d)^4 (e_1(\tau) - e_3(\tau))(e_1(\tau) - e_2(\tau)) = (c\tau+d)^4 \varepsilon(\tau),$$

so (39) is proven. The proof of (41) is similar. □

Our goal is now to show that δ and ε are generators of $M_*(\Gamma_0(2))$. To do that, a result called the Valence formula will be used. Before stating it, let us make the following observation about the index of congruence subgroups.

Proposition 5.16. *Let $\Gamma' \subset \Gamma$ be a congruence subgroup of level N. Then its index in Γ is finite, i.e.*

$$[\Gamma : \Gamma'] = \#\{A\Gamma' \mid A \in \Gamma\} < \infty.$$

Proof. Recall that for a group G and subgroups K, H such that $K \subset H \subset G$, the index satisfies

$$[G : K] = [G : H][H : K].$$

Since $\Gamma(N) \subset \Gamma'$, it is then sufficient to show that $\Gamma(N)$ is of finite index in Γ.

Let A, B be matrices in Γ. We claim that $A\Gamma(N) = B\Gamma(N)$ if and only if $A \equiv B\ (N)$, i.e. if the components of the two matrices are the same modulo N. Indeed, suppose $A\Gamma(N) = B\Gamma(N)$, i.e. $A = BC$ for some $C \in \Gamma(N)$. Let us consider the homomorphism

$$\begin{aligned} \pi : \Gamma &\longrightarrow \mathrm{SL}_2(\mathbb{Z}_N) \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\longmapsto \left(\begin{smallmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{smallmatrix}\right). \end{aligned}$$

We have

$$\pi(A) = \pi(B)\pi(C) = \pi(B)I = \pi(B),$$

hence $A \equiv B\ (N)$.

Conversely, suppose that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $B = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \in \Gamma$ are the same modulo N. Then

$$B^{-1}A = \begin{pmatrix} h & -f \\ -g & e \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ah - cf & bh - df \\ -ag + ce & -bg + de \end{pmatrix}.$$

But $A \equiv B\ (N)$ implies that $ag \equiv ce\ (N)$ and $bh \equiv df\ (N)$, therefore

$$-ag + ce \equiv 0\ (N), \quad bh - df \equiv 0\ (N).$$

Furthermore, $ah - cf \equiv -bg + de \equiv \det A \equiv \det B \equiv 1\ (N)$, hence $B^{-1}A \in \Gamma(N)$ and $A\Gamma(N) = B\Gamma(N)$. So

$$[\Gamma : \Gamma(N)] = \#\,\mathrm{SL}_2(\mathbb{Z}_N),$$

which is finite. □

We now need to introduce and explain some notations. Let Γ' denote a congruence subgroup of level N. Then we define

$$\mathrm{P}\Gamma' := \begin{cases} \Gamma' & \text{if } -I \notin \Gamma' \\ \Gamma'/\{\pm I\} & \text{if } -I \in \Gamma'. \end{cases}$$

For every $A \in \Gamma$ and for every $\tau \in \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}$, $(-A)\tau = A\tau$. Consequently $\mathrm{P}\Gamma'$ acts on $\mathcal{H}$ and on $\mathbb{Q} \cup \{i\infty\}$. Let $\mathrm{P}\Gamma'_s$ and $\mathrm{P}\Gamma'_\tau$ denote the stabilizer of $s \in \mathbb{Q} \cup \{i\infty\}$, respectively $\tau \in \mathcal{H}$, under the action of $\mathrm{P}\Gamma'$.

For a modular form f in $M_k(\Gamma')$ and for a cusp $s \in \mathbb{Q} \cup \{i\infty\}$, let $B \in \Gamma$ be so that $Bi\infty = s$. Then the *order of f at the cusp s* is defined to be

$$\operatorname{ord}_s(f) := n_0/N,$$

where n_0 is the smallest integer with $a_{n_0} \neq 0$ in the Fourier expansion

$$f|_k B(\tau) = \sum_{n=0}^{\infty} a_n q^{n/N}.$$

Similarly, for an element τ_0 in the upper-half plane, let $\operatorname{ord}_{\tau_0}(f)$ denote the smallest integer m_0 so that $b_{m_0} \neq 0$ in the Taylor expansion of f about τ_0 $f(\tau) = \sum_{m=0}^{\infty} b_m(\tau - \tau_0)^m$. Note that $\operatorname{ord}_s(f)$ and $\operatorname{ord}_{\tau_0}(f)$ only depend on the equivalence class of s and τ_0 in the quotient sets $\mathbb{Q} \cup \{i\infty\}/_{\Gamma'}$, respectively $\mathcal{H}/_{\Gamma'}$[25].

Theorem 5.17 (**Valence formula**)**.** *Let $\Gamma' \subset \Gamma$ be a congruence subgroup and let $f \in M_k(\Gamma')$, $f \neq 0$. Then*

$$\sum_{s \in \mathbb{Q} \cup \{i\infty\}/_{\Gamma'}} [\mathrm{P}\Gamma_s : \mathrm{P}\Gamma'_s] \operatorname{ord}_s(f) + \sum_{\tau \in \mathcal{H}/_{\Gamma'}} \frac{1}{\#\mathrm{P}\Gamma'_\tau} \operatorname{ord}_\tau(f) = \frac{k}{12}[\mathrm{P}\Gamma : \mathrm{P}\Gamma'].$$

Proof. See [HBJ94, pp. 134-135] and references therein. □

Remark 5.18. If $-I \in \Gamma'$, then

$$[\mathrm{P}\Gamma : \mathrm{P}\Gamma'] = [\Gamma : \Gamma'].$$

Indeed, it is not hard to see that the map

$$\begin{aligned} \Gamma/_{\Gamma'} &\to \mathrm{P}\Gamma/_{\mathrm{P}\Gamma'} \\ A\Gamma' &\mapsto \{\pm A\}\mathrm{P}\Gamma' \end{aligned}$$

is an isomorphism.

Corollary 5.19. *For a congruence subgroup $\Gamma' \subset \Gamma$, we have*

$$M_k(\Gamma') = \{0\}, \quad \forall k < 0, \tag{42}$$

$$\dim M_k(\Gamma') \leq 1 + \frac{k}{12}[\mathrm{P}\Gamma : \mathrm{P}\Gamma'], \quad \forall k \geq 0. \tag{43}$$

Proof. Equality (42) follows directly from Theorem 5.17 since the left hand side in the Valence formula is non-negative.

Let N be an integer such that $N > \frac{k}{12}[\mathrm{P}\Gamma : \mathrm{P}\Gamma']$. We consider elements $\tau_1, \ldots, \tau_N \in \mathcal{H}$ pairwise inequivalent modulo Γ' and so that their stabilizers are trivial, i.e. $\mathrm{P}\Gamma'_{\tau_i} = \{\{\pm I\}\}$ (or $\mathrm{P}\Gamma'_{\tau_i} = \{I\}$ if $-I \notin \Gamma'$) for every $i \in \{1, \ldots, N\}$. The map

$$\begin{aligned} \psi : M_k(\Gamma') &\to \mathbb{C}^N \\ f &\mapsto (f(\tau_1), \ldots, f(\tau_N)) \end{aligned}$$

[25] See [HBJ94, p. 123].

is clearly a homomorphism. Let us show that it is injective. To do that, let f be a function in $M_k(\Gamma')\backslash\{0\}$. Then by the Valence formula 5.17,

$$\sum_{s\in\mathbb{Q}\cup\{i\infty\}/\Gamma'} [\mathrm{P}\Gamma_s : \mathrm{P}\Gamma'_s] \operatorname{ord}_s(f) + \sum_{\tau\in\mathcal{H}/\Gamma'} \frac{1}{\#\mathrm{P}\Gamma'_\tau} \operatorname{ord}_\tau(f) < N.$$

We denote the left hand side of this inequality by (LHS). Then obviously

$$\text{(LHS)} \geq \sum_{\tau\in\mathcal{H}/\Gamma'} \frac{1}{\#\mathrm{P}\Gamma'_\tau} \operatorname{ord}_\tau(f) \geq \sum_{i=1}^{N} \operatorname{ord}_{\tau_i}(f),$$

so

$$\sum_{i=1}^{N} \operatorname{ord}_{\tau_i}(f) < N,$$

which means that there is an $i_0 \in \{1,\dots,N\}$ with $\operatorname{ord}_{\tau_{i_0}}(f) = 0$. By definition of the order, this implies that the constant term in the Taylor expansion of f at τ_{i_0} is non-zero, hence that $f(\tau_{i_0}) \neq 0$. We have then shown that $\ker\psi = \{0\}$, so ψ is injective and $\dim M_k(\Gamma') \leq N$. In particular, choosing $N = 1 + \frac{k}{12}[\mathrm{P}\Gamma : \mathrm{P}\Gamma']$ proves (43). □

We now get back to the congruence subgroup $\Gamma_0(2)$:

Corollary 5.20. *For $k \in \mathbb{N}$,*

$$\dim M_k(\Gamma_0(2)) \leq 1 + \frac{k}{4}.$$

Proof. By Remark 5.18, we know that $[\mathrm{P}\Gamma : \mathrm{P}\Gamma_0(2)] = [\Gamma : \Gamma_0(2)]$. Let us consider the matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(2)$. As is easily checked, we have

$$\begin{aligned} S\Gamma_0(2) &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c \equiv 1 \ (2), a \equiv 0 \ (2)\} \\ (TS)\Gamma_0(2) &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c \equiv a \equiv 1 \ (2)\}. \end{aligned}$$

Then clearly $\Gamma = \Gamma_0(2) \sqcup S\Gamma_0(2) \sqcup (TS)\Gamma_0(2)$, therefore

$$[\mathrm{P}\Gamma : \mathrm{P}\Gamma_0(2)] = [\Gamma : \Gamma_0(2)] = 3.$$

From (43), we get

$$\dim M_k(\Gamma_0(2)) \leq 1 + \frac{k}{4}.$$

□

Remark 5.21. The modular forms $\delta \in M_2(\Gamma_0(2))$ and $\varepsilon \in M_4(\Gamma_0(2))$ are algebraically independent. To see this, we use the fact[26] that if $\tau \in \mathcal{H}$ has a non-trivial stabilizer with respect to the action of $\mathrm{P}\Gamma_0(2)$, i.e. if $\mathrm{P}\Gamma_0(2)_\tau \neq \{\{\pm I\}\}$,

[26]See [HBJ94, p. 160].

then τ is equivalent to $e := \frac{1+i}{2}$ modulo $\Gamma_0(2)$, and the stabilizer of e is equal to $\mathrm{P}\Gamma_0(2)_e = \{\{\pm I\}, \{\pm\left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)\}\}$. A simple computation shows that

$$[\mathrm{P}\Gamma_0 : \mathrm{P}\Gamma_0(2)_0] = 2, \quad [\mathrm{P}\Gamma_\infty : \mathrm{P}\Gamma_0(2)_\infty] = 1.$$

Therefore the Valence formula for $\Gamma_0(2)$ becomes

$$2\,\mathrm{ord}_0(f) + \mathrm{ord}_\infty(f) + \frac{1}{2}\,\mathrm{ord}_e(f) + \sum_{\substack{\tau \in \mathcal{H}/\Gamma_0(2) \\ \tau \not\sim e\ (\Gamma_0(2))}} \mathrm{ord}_\tau(f) = \frac{k}{4} \tag{44}$$

for every non-trivial $f \in M_k(\Gamma_0(2))$. From the Fourier expansions (38) and (39), we know that $\mathrm{ord}_\infty(\delta), \mathrm{ord}_\infty(\varepsilon) = 0$. Furthermore, the definition of ε and relations (35), (36) and (37) yield $\mathrm{ord}_0(\varepsilon) = \frac{1}{2}$. So equality (44) gives $\mathrm{ord}_e(\varepsilon) = 0$, $\mathrm{ord}_e(\delta) = 1$ and $\mathrm{ord}_0(\delta) = 0$, hence

$$\varepsilon(e) \neq 0, \quad \delta(e) = 0.$$

Now assume that δ and ε are algebraically independent, i.e. that there exists a non-trivial homogeneous polynomial P of weight $2n$ with $P(\delta, \varepsilon) = 0$. Let us suppose that P is of minimal weight. We can write

$$P(\delta, \varepsilon) = a\varepsilon^{n/2} + b\delta^n + \varepsilon\delta Q(\delta, \varepsilon),$$

where Q is a homogeneous polynomial of weight smaller than $2n$. Evaluating $P(\delta, \varepsilon)$ at e and at the 0 cusp yields $a = b = 0$. Then $Q(\delta, \varepsilon) = 0$, which is a contradiction since we have chosen P of minimal weight.

We are now able to describe exactly the ring $M_*(\Gamma_0(2))$.

Theorem 5.22. *The ring of modular forms for $\Gamma_0(2)$ is given by*

$$M_*(\Gamma_0(2)) = \mathbb{C}[\delta, \varepsilon].$$

Proof. We already know that $M_k(\Gamma_0(2)) = \{0\}$ for $k \leq 0$ and k odd. Recall that $\delta \in M_2(\Gamma_0(2))$ and $\varepsilon \in M_4(\Gamma_0(2))$. So the ring $\mathbb{C}[\delta, \varepsilon]$ is clearly included in $M_*(\Gamma_0(2))$ and it suffices to show that the component of weight k – with k even – of $\mathbb{C}[\delta, \varepsilon]$ has the same dimension as $M_k(\Gamma_0(2))$. Let then A_k denote the component of weight k in $\mathbb{C}[\delta, \varepsilon]$, i.e.

$$A_k = \left\{ \sum_{2l+4m=k} \alpha_{l,m}\delta^l\varepsilon^m \mid \alpha_{l,m} \in \mathbb{C} \right\}.$$

So the set

$$\{\delta^l\varepsilon^m \mid l, m \in \mathbb{Z}, 4m + 2l = k\} = \{\delta^l\varepsilon^m \mid l, m \in \mathbb{Z}, 2m + l = k/2\}$$

builds a basis for A_k. It contains exactly $1 + \lfloor k/4 \rfloor$ elements (one for each integer $0 \leq m \leq \lfloor k/4 \rfloor$), hence

$$\dim A_k = 1 + \lfloor k/4 \rfloor.$$

Corollary 5.20 implies that $A_k \subset M_k(\Gamma_0(2))$ has maximal dimension, hence $A_k = M_k(\Gamma_0(2))$. We conclude that

$$M_*(\Gamma_0(2)) = \mathbb{C}[\delta, \varepsilon]$$

$\square$

Remark 5.23. The same proof shows for example that

$$\begin{aligned} M_*(\Gamma_0(2)) &= \mathbb{C}[\tilde{\delta}, \tilde{\varepsilon}] \\ &= \mathbb{C}[8\delta, \varepsilon] \\ &= \mathbb{C}[8\tilde{\delta}, \tilde{\varepsilon}]. \end{aligned}$$

We will now see that there is a close relation between $\tilde{\delta}$, $\tilde{\varepsilon}$ and the expansion of δ, respectively ε, at the 0 cusp. In order to do so, let us first observe a relation between $\Gamma_0(2)$ and $\Gamma^0(2)$, which is easily checked by direction computation:

Lemma 5.24. *For $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T_2 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $T_{1/2} = \left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have*

$$S\Gamma_0(2)S^{-1} = \Gamma^0(2), \tag{45}$$

$$T_2\Gamma_0(2)T_{1/2} = \Gamma^0(2). \tag{46}$$

Remark 5.25. If we consider the action of Γ on $\mathcal{H}$ and $\mathbb{Q} \cup \{i\infty\}$, the matrix S corresponds to the map $\tau \mapsto \frac{-1}{\tau}$ and T_2, $T_{1/2}$ correspond to $\tau \mapsto 2\tau$, respectively $\tau \mapsto \tau/2$. Note that $T_{1/2} = T_2^{-1}$.

Corollary 5.26. *If f is a modular form of weight k for $\Gamma^0(2)$, then the function g given by $g(\tau) = f(2\tau)$ is a modular form of weight k for $\Gamma_0(2)$*

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$. Relation (46) tells us that the matrix B given by $B = T_2 A T_{1/2} = \left(\begin{smallmatrix} a & 2b \\ c/2 & d \end{smallmatrix}\right)$ is an element of $\Gamma^0(2)$. We have then

$$\begin{aligned} g(A\tau) &= g((T_{1/2}BT_2)\tau) \\ &= f(2(T_{1/2}BT_2)\tau) \\ &= f(B(T_2\tau)) \\ &= (\tfrac{c}{2}(T_2\tau) + d)^k f(2\tau) \\ &= (c\tau + d)^k g(\tau). \end{aligned}$$

$\square$

Theorem 5.27. *The functions δ^0 and ε^0 satisfy*

$$\delta^0(2\tau) = \tilde{\delta}(\tau), \tag{47}$$

$$\varepsilon^0(2\tau) = \tilde{\varepsilon}(\tau). \tag{48}$$

Proof. Since $\delta = -\frac{3}{2}e_1$ and $\tilde{\delta} = \frac{3}{4}e_1$, our goal is to show that $e_1^0(2\tau) = -\frac{1}{2}e_1(\tau)$. By our computations from (35),

$$e_1^0 = e_1\big|_2 S = e_2.$$

We know that $e_2 \in M_2(\Gamma^0(2))$, therefore $e_1^0(2\tau) = e_2(2\tau) \in M_2(\Gamma_0(2))$ by Corollary 5.26. Corollary 5.20 implies that $\dim M_2(\Gamma_0(2)) = 1$, thus there is a constant λ so that $e_1^0(2\tau) = e_2(2\tau) = \lambda e_1(\tau)$ for every $\tau \in \mathcal{H}$. The Fourier expansions in Proposition 5.13 allow us compute the limits

$$\lim_{\tau\to\infty} e_2(2\tau) = \frac{1}{12},$$
$$\lim_{\tau\to\infty} \lambda e_1(\tau) = -\lambda\frac{1}{6}.$$

Hence $\lambda = -\frac{1}{2}$ and $e_1^0(2\tau) = -\frac{1}{2}e_1(\tau)$, which proves (47).

Let us now compute

$$\begin{aligned}\varepsilon^0(\tau) = \varepsilon\big|_4 S(\tau) &= \tau^{-4}(e_1(\tfrac{-1}{\tau}) - e_2(\tfrac{-1}{\tau}))\cdot(e_1(\tfrac{-1}{\tau}) - e_3(\tfrac{-1}{\tau}))\\ &= (\tau^{-2}e_1(\tfrac{-1}{\tau}) - \tau^{-2}e_2(\tfrac{-1}{\tau}))\cdot(\tau^{-2}e_1(\tfrac{-1}{\tau}) - \tau^{-2}e_3(\tfrac{-1}{\tau}))\\ &= (e_1\big|_2 S(\tau) - e_2\big|_2 S(\tau))\cdot(e_1\big|_2 S(\tau) - e_3\big|_2 S(\tau))\\ &= (e_2(\tau) - e_1(\tau))\cdot(e_2(\tau) - e_3(\tau)),\end{aligned} \tag{49}$$

where the last equality comes from (35),(36) and (37). We now claim that $\varepsilon^0 \in M_4(\Gamma^0(2))$. Indeed, let $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ be in $\Gamma^0(2)$. Then

$$\begin{aligned}\varepsilon^0\big|_4 A(\tau) &\overset{(49)}{=} (c\tau+d)^{-4}(e_2(A\tau) - e_1(A\tau))\cdot(e_2(A\tau) - e_3(A\tau))\\ &= (e_2\big|_2 A(\tau) - e_1\big|_2 A(\tau))\cdot(e_2\big|_2 A(\tau) - e_3\big|_2 A(\tau)).\end{aligned}$$

Here we need to distinguish cases where c is even or odd, but in both cases relations (35),(36) and (37) give us

$$\varepsilon^0\big|_4 A(\tau) = \varepsilon^0(\tau).$$

Since the functions e_i $(i = 1,2,3)$ are holomorphic on $\mathcal{H}$ and at all cusps, then so is $\varepsilon^0 = (e_2 - e_1)(e_2 - e_3)$. Consequently $\varepsilon^0 \in M_4(\Gamma^0(2))$. Corollary 5.26 now tells us that

$$\varepsilon^0(2\tau) \in M_4(\Gamma_0(2)).$$

By Remark 5.23, it can be written as

$$\varepsilon^0(2\tau) = \lambda\tilde{\delta}^2(\tau) + \mu\tilde{\varepsilon}(\tau).$$

Let us compare the q-expansions:

$$\begin{aligned}\tilde{\varepsilon}(\tau) &= q + 8q^2 + \cdots\\ \tilde{\delta}^2(\tau) &= (-\frac{1}{8} - 3q - 3q^2)^2\\ &= \frac{1}{64} + \frac{3}{4}q + \cdots\\ \varepsilon^0(2\tau) &\overset{(49)}{=} (e_2(2\tau) - e_1(2\tau))\cdot(e_2(2\tau) - e_3(2\tau))\\ &= q + 8q^2 + \cdots.\end{aligned}$$

There are no constant term in the q-expansions of $\varepsilon^0(2\tau)$ and $\tilde{\varepsilon}(\tau)$, but there is a non-zero one in the expansion of $\tilde{\delta}^2(\tau)$. Therefore the coefficient λ must be 0. Comparing the first terms of $\varepsilon^0(2\tau)$ and $\tilde{\varepsilon}(\tau)$ yields $\mu = 1$, so (48) is proven. $\square$

6 The elliptic genus

Our purpose is now to introduce what will be called the elliptic genus, a multiplicative genus taking its values in the ring of modular forms $M_*(\Gamma_0(2))$. It will constitute our principal tool for the proof of Ochanine's Theorem in section 7.

Following Hirzebruch's formalism (see section 4.1), we will first construct an even power series. Particular attention must be paid to convergence issues as we shall make use of infinite products in our constructions.

Definition 6.1. A series of meromorphic functions $\sum_{n=1}^{\infty} f_n(x)$ *converges normally* if for every compact subset K there exists an $n \in \mathbb{N}$ such that for all $n \geq N$, f_n is holomorphic on K and the series $\sum_{n=N}^{\infty} \sup_{x\in K} |f_n(x)|$ converges.

If the series converges normally, then there exists a meromorphic function $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If all the f_n are holomorphic, then so is f.

Definition 6.2. Let $\prod_{n=1}^{\infty}(1+f_n(x))$ be an infinite product where the functions f_n are meromorphic. It is said to be normally convergent if the sum $\sum_{n=1}^{\infty} f_n(x)$ is.

In that case, $\prod_{n=1}^{\infty}(1 + f_n(x))$ converges to a meromorphic function $f(x)$. If all the f_n are holomorphic, then so is f. Normal convergence implies absolute convergence and allows us to rearrange the terms of the series as well to differentiate term by term.

6.1 The elliptic genus at the signature cusp

Consider the function Φ defined on $\mathcal{H} \times \mathbb{C}$ by

$$\Phi(\tau, x) := (e^{x/2} - e^{-x/2}) \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2}, \tag{50}$$

where q denotes $e^{2\pi i\tau}$ as usual. Let K be a compact subset of $\mathcal{H} \times \mathbb{C}$ and let $a := \max_{(\tau,x)\in K} |q|$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \sup_{(\tau,x)\in K} |-q^n e^x| &\leq \sup_{(\tau,x)\in K} |e^x| \cdot \sum_{n=1}^{\infty} \sup_{(\tau,x)\in K} |q^n| \\
&= \sup_{(\tau,x)\in K} |e^x| \cdot \sum_{n=1}^{\infty} a^n, \\
\sum_{n=1}^{\infty} \sup_{(\tau,x)\in K} |-q^n e^{-x}| &\leq \sup_{(\tau,x)\in K} |e^{-x}| \cdot \sum_{n=1}^{\infty} \sup_{(\tau,x)\in K} |q^n| \\
&= \sup_{(\tau,x)\in K} |e^{-x}| \cdot \sum_{n=1}^{\infty} a^n,
\end{aligned}$$

which both converge because e^x and e^{-x} are bounded in K and $a < 1$. Using the inequalities

$$|1-q^n| \geq |1-|q|^n| \geq |1-|q|| \geq 1-a,$$

we get

$$\begin{aligned}\sum_{n=1}^{\infty} \sup_{(\tau,x)\in K} \left|\frac{1}{1-q^n}-1\right| &= \sum_{n=1}^{\infty} \sup_{(\tau,x)\in K} \left|\frac{q^n}{1-q^n}\right| \\ &\leq \sum_{n=1}^{\infty} \frac{\sup_{(\tau,x)\in K}|q^n|}{\inf_{(\tau,x)\in K}|1-q^n|} \\ &\leq \frac{1}{1-a}\cdot\sum_{n=1}^{\infty} a^n,\end{aligned}$$

which converges too. Consequently, the infinite products $\prod_{n=1}^{\infty}(1-q^n e^{\pm x})$ and $\prod_{n=1}^{\infty}(1-q^n)^{-1}$ are normally convergent, and so is Φ.

The function Φ presents the following invariance properties:

Lemma 6.3. *For every matrix $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and for all integers λ, μ, Φ satisfies*

$$\Phi(A\tau, \tfrac{x}{c\tau+d}) = (c\tau+d)^{-1}\cdot\Phi(\tau,x)\cdot\exp\left(\tfrac{cx^2}{4\pi i(c\tau+d)}\right) \tag{51}$$

$$\Phi(\tau, x+2\pi i(\lambda\tau+\mu)) = \Phi(\tau,x)\cdot q^{-\frac{\lambda^2}{2}}e^{-\lambda x}(-1)^{\lambda+\mu}. \tag{52}$$

Proof. See [HBJ94, pp. 145-146]. Note that our approach differs from that of [HBJ94], where the Φ function is defined as $\sigma(\tau,x)\cdot\exp(-G_2(\tau)x^2)$. Here σ denotes the Weierstrass sigma-function and $G_2(\tau) = -\frac{B_2}{4}+\sum_{n=1}^{\infty}\sigma_1(n)q^n$. The preceding theorem then follows from the invariance properties of these functions. □

Let us now define the meromorphic function

$$\begin{aligned}&\varphi : \mathcal{H}\times\mathbb{C} \longrightarrow \overline{\mathbb{C}} \\ &\varphi(\tau,x) := \frac{1}{2}\cdot\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}}\cdot\prod_{n=1}^{\infty}\frac{(1+q^n e^{\pm x})(1-q^n)^2}{(1-q^n e^{\pm x})(1+q^n)^2},\end{aligned}$$

where we have used $(1+q^n e^{\pm x})$ to denote $(1+q^n e^x)(1+q^n e^{-x})$ (respectively $(1-q^n e^{\pm x})$ for $(1-q^n e^x)(1-q^n e^{-x})$).

Lemma 6.4. *$\varphi(\tau,x)$ converges normally on $\mathcal{H}\times\mathbb{C}$ and we have*

$$\varphi(\tau,x) = \frac{\Phi(\tau,x-\pi i)}{\Phi(\tau,x)\Phi(\tau,-\pi i)}.$$

Proof. An analysis similar to the one we made for Φ allows us to show the normal convergence. The equality can be proven by a direct computation. □

We know that Φ satisfies some invariance properties, so we can expect that φ shows a similar behaviour. It is indeed the case with respect to the congruence subgroup $\Gamma_0(2)$.

Proposition 6.5. *For every matrix* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$\varphi(A\tau, \tfrac{x}{c\tau+d}) = (c\tau+d)\varphi(\tau, x).$$

Proof. Lemma 6.4 and equality (51) yield

$$\begin{aligned}
\varphi(A\tau, \tfrac{x}{c\tau+d}) &\overset{6.4}{=} \frac{\Phi(A\tau, \frac{x}{c\tau+d} - \pi i)}{\Phi(A\tau, \frac{x}{c\tau+d})\Phi(A\tau, -\pi i)} \\
&= \frac{\Phi(A\tau, \frac{x-\pi i(c\tau+d)}{c\tau+d})}{\Phi(A\tau, \frac{x}{c\tau+d})\Phi(A\tau, \frac{-\pi i(c\tau+d)}{c\tau+d})} \\
&\overset{(51)}{=} (c\tau+d)\exp\left(-\tfrac{cx}{2}\right) \frac{\Phi(\tau, x - \pi i\tau c - \pi i d)}{\Phi(\tau, x)\Phi(\tau, -\pi i\tau c - \pi i d)}.
\end{aligned}$$

Since $A \in \Gamma_0(2)$, there exist integers λ and μ such that $c = 2\lambda$ and $d = 2\mu + 1$. Then (52) gives

$$\begin{aligned}
\frac{\Phi(\tau, x - \pi i\tau c - \pi i d)}{\Phi(\tau, -\pi i\tau c - \pi i d)} &= \frac{\Phi(\tau, x - \pi i + 2\pi i(-\lambda\tau - \mu))}{\Phi(\tau, -\pi i + 2\pi i(-\lambda\tau - \mu))} \\
&\overset{(52)}{=} \exp(\lambda x)\frac{\Phi(\tau, x - \pi i)}{\Phi(\tau, -\pi i)} \\
&= \exp(\tfrac{cx}{2})\frac{\Phi(\tau, x - \pi i)}{\Phi(\tau, -\pi i)},
\end{aligned}$$

which concludes the proof by Lemma 6.4. □

It is not hard to check that for a fixed τ, $\varphi(\tau, \cdot)$ is an odd elliptic function of order 2 with respect to the lattice $L = 2\pi i(2\mathbb{Z}\tau + \mathbb{Z})$, with simple zeros at the points $x \equiv \pi i, \pi i + 2\pi i\tau$ (L) and simple poles at the points $x \equiv 0, 2\pi i\tau$ (L). Thus the function

$$Q_\tau(x) := x\varphi(\tau, x)$$

is even and holomorphic with respect to x in a neighbourhood of 0. A simple computation shows that $\lim_{x\to 0} Q_\tau(x) = 1$. It may then be written in the form of an even power series as follows:

$$Q_\tau(x) = x\varphi(\tau, x) = \sum_{n=0}^{\infty} a_{2n}(\tau)x^{2n}, \quad a_0 \equiv 1, \tag{53}$$

Hirzebruch's formalism now enables us to make the following definition:

Definition 6.6. The genus defined for all closed, oriented $4k$-manifolds M by

$$\begin{aligned}
\varphi(M) &:= \left(\prod_{i=1}^{2k} Q_\tau(x_i)\right)[M] \\
&= \left(\prod_{i=1}^{2k} \frac{x_i}{2} \cdot \frac{e^{x_i/2} + e^{-x_i/2}}{e^{x_i/2} - e^{-x_i/2}} \cdot \prod_{n=1}^{\infty} \frac{(1+q^n e^{\pm x_i})(1-q^n)^2}{(1-q^n e^{\pm x_i})(1+q^n)^2}\right)[M]
\end{aligned}$$

is called the *elliptic genus*. Here $x_1, \ldots, x_{2k}$ denote the Chern roots of M as usual. Obviously, we define $\varphi(M)$ to be 0 when the dimension is not divisible by 4.

The product in the parentheses being holomorphic in a neighbourhood of $q = 0$, the elliptic genus takes its values in the ring of rational power series $\mathbb{Q}[[q]]$. Actually, our results from section 3.4 give us a more precise idea of the form of its q-expansion and show a relation with the signature of M. Indeed, equalities (6) and (7) and the multiplicativity of the Chern character imply

$$\begin{aligned}\operatorname{ch}\left(\bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}TM_{\mathbb{C}}\right) &= \prod_{n=1}^{\infty} \operatorname{ch}(S_{q^n}TM_{\mathbb{C}}) \operatorname{ch}(\Lambda_{q^n}TM_{\mathbb{C}}) \\ &= \prod_{n=1}^{\infty}\prod_{i=1}^{2k} \frac{1+q^n e^{\pm x_i}}{1-q^n e^{\pm x_i}}.\end{aligned}$$

Thus

$$\begin{aligned}\varphi(M)(\tau) =& 2^{-2k}\prod_{n=1}^{\infty}\frac{(1-q^n)^{4k}}{(1+q^n)^{4k}} \\ & \cdot \left(\operatorname{ch}\left(\bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}TM_{\mathbb{C}}\right) \cdot \prod_{i=1}^{2k}\frac{x_i}{\tanh(x_i/2)}\right)[M] \\ =& 2^{-2k}\prod_{n=1}^{\infty}\frac{(1-q^n)^{4k}}{(1+q^n)^{4k}} \cdot \operatorname{sign}\left(M, \bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}TM_{\mathbb{C}}\right) \\ \stackrel{(*)}{=}& 2^{-2k}\prod_{n=1}^{\infty}\frac{(1-q^n)^{4k}}{(1+q^n)^{4k}} \cdot (\operatorname{sign}(M) + 2q\cdot\operatorname{sign}(M, TM_{\mathbb{C}}) + \cdots), \qquad (54)\end{aligned}$$

where equality $(*)$ follows from $\operatorname{sign}(M, E_1 \oplus E_2) = \operatorname{sign}(M, E_1) + \operatorname{sign}(M, E_2)$, which is a direct consequence of the additivity of the Chern character.

The rest of this section will be devoted to showing that $\varphi(M^{4k})$ is a modular form of weight $2k$ for the group $\Gamma_0(2)$. The strategy of our proofs will be to show that the functions $a_{2n}(\tau)$ that appear in the expansion of $Q_\tau(x)$ in (53) are modular forms of weight $2n$ for $\Gamma_0(2)$.

To begin our investigation, let us try to write the elliptic genus differently in order to make Pontryagin numbers appear. Recall that we have written

$$x\varphi(\tau, x) = Q_\tau(x) = \sum_{n=0}^{\infty} a_{2n}(\tau)x^{2n},$$

where $a_0 \equiv 1$. Now let us consider the product

$$\prod_{i=1}^{2k} Q_\tau(x_i) = \prod_{i=1}^{2k}\left(\sum_{n=0}^{\infty} a_{2n}(\tau)x_i^{2n}\right).$$

It is symmetric in the x_i^2, hence can be rewritten as a polynomial in $p_1, p_2, \ldots,$ where $p_j = \sum_{i_1<\cdots<i_j} x_{i_1}^2 \cdots x_{i_j}^2$ is the j^{th} elementary symmetric function in the

x_i^2. In the case where the x_i are the Chern roots of an oriented $4k$-manifold M, i.e. when its total Chern class is formally factorised as

$$c(M) = c(TM_{\mathbb{C}}) = (1 + x_1) \cdots (1 + x_{2k})(1 - x_1) \cdots (1 - x_{2k}),$$

p_j corresponds to the j^{th} Pontryagin class of TM. Since the whole expression will be evaluated on M, we are only interested in the term of weight $4k$ (the x_i being of weight 2). Using our multi-index notation

$$p_I = p_{i_1} \cdots p_{i_j}$$

for $I = (i_1 \leq \cdots \leq i_j)$, we can write

$$\varphi(M^{4k})(\tau) = \sum_{\substack{I=(i_1\leq\cdots\leq i_j)\\ i_1+\cdots+i_j=k}} b_I(\tau) p_I[M^{4k}], \tag{55}$$

where, after assigning weight $4n$ to a_{2n}, each b_I is a homogeneous polynomial of weight $4k$ in the a_{2n}.

From now on, we will always assume that our manifolds have dimension $\dim M = 4k$ for some $k \in \mathbb{N}$.

Proposition 6.7. *The elliptic genus $\varphi(M)$ is holomorphic on the upper-half plane $\mathcal{H}$.*

Proof. For some fixed and well chosen x (eg. $x = \frac{\pi i}{2}$), the function $x\varphi(\cdot, x)$ is holomorphic and nontrivial on $\mathcal{H}$ (because $\varphi(\tau, x)$ never has a pole or a zero at $x = \frac{\pi i}{2}$). It follows from (53) that the functions a_{2n} are holomorphic too. Consequently, the polynomials b_I are holomorphic on $\mathcal{H}$ and equality (55) concludes our proof. $\square$

The next step is to check that $\varphi(M)$ transforms like a modular form of weight $2k$ for $\Gamma_0(2)$. This will follow easily from the invariance property of φ.

Proposition 6.8. *For every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,*

$$\varphi(M)(A\tau) = (c\tau + d)^{2k} \varphi(M)(\tau)$$

Proof. We wish to show that for every $n \in \mathbb{N}$, a_{2n} transforms like a modular form of weight $2n$ for $\Gamma_0(2)$, i.e.

$$a_{2n}(A\tau) = (c\tau + d)^{2n} a_{2n}(\tau), \quad \forall A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2).$$

This will imply that for every multi-index $I = (i_1 \leq \cdots \leq i_j)$ with $i_1 + \cdots + i_j = k$, b_I transforms like a modular form of weight $2k$ for $\Gamma_0(2)$, and so will $\varphi(M)$ by (55).

By equality (53) and Proposition 6.5, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n}(A\tau)(c\tau+d)^{-2n}x^{2n} &= \sum_{n=0}^{\infty} a_{2n}(A\tau)(\tfrac{x}{c\tau+d})^{2n} \\ &\overset{(53)}{=} \tfrac{x}{c\tau+d}\varphi(A\tau, \tfrac{x}{c\tau+d}) \\ &\overset{6.5}{=} x\varphi(\tau,x) \\ &\overset{(53)}{=} \sum_{n=0}^{\infty} a_{2n}(\tau)x^{2n}.\end{aligned}$$

Comparing coefficients in both series leads to

$$a_{2n}(A\tau) = (c\tau+d)^{2n}a_{2n}(\tau).$$

□

It remains to check that $\varphi(M)(\tau)$ is holomorphic at the two cusps of $\Gamma_0(2)$, namely 0 and $i\infty$. Holomorphicity at the $i\infty$ cusp is quickly verified. Indeed, we have

$$\sum_{n=0}^{\infty} a_{2n}(\tau)x^{2n} = x\varphi(\tau,x) = \frac{x}{2}\cdot\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}}\cdot\prod_{n=1}^{\infty}\frac{(1+q^n e^{\pm x})(1-q^n)^2}{(1-q^n e^{\pm x})(1+q^n)^2},$$

which is holomorphic at $q=0$, i.e. at $\tau = i\infty$, for x close to 0. Hence, the functions $a_{2n}(\tau)$ are holomorphic at the $i\infty$ cusp, therefore so is $\varphi(M)$ by (55).

In order to verify the holomorphicity at $i\infty$, we will first construct another genus, $\tilde{\varphi}(M)$, which will be closely related to the expansion of $\varphi(M)$ at 0.

6.2 The elliptic genus at the Â-genus cusp

This subsection will develop very similarly to the preceding one in order to define a new multiplicative genus. Starting with $\varphi(\tau,x)$, let us define another meromorphic function:

$$\begin{aligned}&\tilde{\varphi} : \mathcal{H}\times\mathbb{C} \longrightarrow \overline{\mathbb{C}}, \\ &\tilde{\varphi}(\tau,x) := \tau^{-1}\varphi(\tfrac{-1}{\tau}, \tfrac{x}{\tau}).\end{aligned}$$

We have:

Lemma 6.9.

$$\tilde{\varphi}(\tau,x) = \frac{\Phi(\tau, x-\pi i\tau)}{\Phi(\tau,x)\Phi(\tau,-\pi i\tau)}\cdot e^{-x/2}, \tag{56}$$

$$\tilde{\varphi}(\tau,x) = \frac{1}{e^{x/2}-e^{-x/2}}\prod_{n=1}^{\infty}\left(\frac{(1-q^{n/2})^2}{(1-q^{n/2}e^{\pm x})}\right)^{(-1)^n}. \tag{57}$$

Proof. Equality (56) follows directly from Lemmas 6.4 and 6.3. As for (57), we need to substitute the definition of Φ (see (50)) in (56). Doing so, $\tilde{\varphi}(\tau, x)$ is equal to

$$
\begin{aligned}
& e^{-x/2}\frac{(e^{x/2}q^{-1/4}-e^{-x/2}q^{1/4})\prod_{n=1}^{\infty}(1-q^{n-1/2}e^{x})(1-q^{n+1/2}e^{-x})(1-q^{n})^2}{(e^{x/2}-e^{-x/2})(q^{-1/4}-q^{1/4})\prod_{n=1}^{\infty}(1-q^{n}e^{\pm x})(1-q^{n\pm 1/2})} \\
= & \frac{1}{(e^{x/2}-e^{-x/2})}\frac{(q^{-1/4}-q^{1/4}e^{-x})}{(q^{-1/4}-q^{1/4})}\prod_{n=1}^{\infty}\frac{(1-q^{n})^2}{(1-q^{n\pm 1/2})}\prod_{n=1}^{\infty}\frac{(1-q^{n\pm 1/2}e^{\mp x})}{(1-q^{n}e^{\pm x})} \\
= & \frac{1}{(e^{x/2}-e^{-x/2})}\frac{(1-q^{1/2}e^{-x})}{(1-q^{1/2})}\prod_{n=1}^{\infty}\frac{(1-q^{n})^2}{(1-q^{n\pm 1/2})}\prod_{n=1}^{\infty}\frac{(1-q^{n\pm 1/2}e^{\mp x})}{(1-q^{n}e^{\pm x})}.
\end{aligned}
$$

Let us transform this expression piece by piece. First, the product

$$
\frac{1}{(1-q^{1/2})}\prod_{n=1}^{\infty}\frac{(1-q^{n})^2}{(1-q^{n\pm 1/2})}
$$

can be written as

$$
\begin{aligned}
& \frac{1}{(1-q^{1/2})}\prod_{n=1}^{\infty}(1-q^{n})^2\prod_{n=1}^{\infty}\frac{1}{(1-q^{n-1/2})}\prod_{n=1}^{\infty}\frac{1}{(1-q^{n+1/2})} \\
= & \frac{1}{(1-q^{1/2})}\prod_{\substack{n=1\\ n\text{ even}}}^{\infty}(1-q^{n/2})^2\prod_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{(1-q^{n/2})}\prod_{\substack{n=3\\ n\text{ odd}}}^{\infty}\frac{1}{(1-q^{n/2})} \\
= & \prod_{\substack{n=1\\ n\text{ even}}}^{\infty}(1-q^{n/2})^2\prod_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{(1-q^{n/2})}\prod_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{(1-q^{n/2})} \\
= & \prod_{\substack{n=1\\ n\text{ even}}}^{\infty}(1-q^{n/2})^2\prod_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{(1-q^{n/2})^2} \\
= & \prod_{n=1}^{\infty}\left((1-q^{n/2})^2\right)^{(-1)^n}.
\end{aligned}
$$

Second, the product

$$
(1-q^{1/2}e^{-x})\prod_{n=1}^{\infty}\frac{(1-q^{n-1/2}e^{x})(1-q^{n+1/2}e^{-x})}{(1-q^{n}e^{\pm x})}
$$

is equal to

$$
\begin{aligned}
& (1-q^{1/2}e^{-x})\prod_{n=1}^{\infty}\frac{1}{1-q^{n}e^{\pm x}}\prod_{n=1}^{\infty}(1-q^{n-1/2}e^{x})\prod_{n=1}^{\infty}(1-q^{n+1/2}e^{-x}) \\
= & (1-q^{1/2}e^{-x})\prod_{\substack{n=1\\ n\text{ even}}}^{\infty}\frac{1}{1-q^{n/2}e^{\pm x}}\prod_{\substack{n=1\\ n\text{ odd}}}^{\infty}(1-q^{n/2}e^{x})\prod_{\substack{n=3\\ n\text{ odd}}}^{\infty}(1-q^{n/2}e^{-x})
\end{aligned}
$$

$$= \prod_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{1}{1-q^{n/2}e^{\pm x}} \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (1-q^{n/2}e^{x}) \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (1-q^{n/2}e^{-x})$$

$$= \prod_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{1}{1-q^{n/2}e^{\pm x}} \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (1-q^{n/2}e^{\pm x})$$

$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-q^{n/2}e^{\pm x}}\right)^{(-1)^n}.$$

All in all,

$$\tilde{\varphi}(\tau, x) = \frac{1}{e^{x/2}-e^{-x/2}} \prod_{n=1}^{\infty} \left(\frac{(1-q^{n/2})^2}{(1-q^{n/2}e^{\pm x})}\right)^{(-1)^n}.$$

□

Recall the congruence subgroup

$$\Gamma^0(2) = \left\{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv 0 \ (2)\right\}$$

and its relation with $\Gamma_0(2)$ given in Lemma 5.24. Similarly to $\varphi(\tau, x)$, $\tilde{\varphi}(\tau, x)$ presents an invariance property, but this time with respect to the congruence subgroup $\Gamma^0(2)$. The relation between the two congruence subgroups allows us to make only a small change - namely, multiplying τ by 2 - to get an invariance property with respect to $\Gamma_0(2)$.

Lemma 6.10. *For matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ and $B = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \in \Gamma^0(2)$,*

$$\tilde{\varphi}(B\tau, \tfrac{x}{g\tau+h}) = (g\tau+h)\tilde{\varphi}(\tau, x), \tag{58}$$

$$\tilde{\varphi}(2(A\tau), \tfrac{x}{c\tau+d}) = (c\tau+d)\tilde{\varphi}(2\tau, x). \tag{59}$$

Proof. Let us begin with (58). By (45), there exists a matrix $B' \in \Gamma_0(2)$ such that $B = SB'S^{-1}$. Then

$$B' = S^{-1}BS = \begin{pmatrix} h & -g \\ -f & e \end{pmatrix}.$$

We get

$$\begin{aligned}
\tilde{\varphi}(B\tau, \tfrac{x}{g\tau+h}) &= (B\tau)^{-1}\varphi(\tfrac{-1}{B\tau}, \tfrac{x}{B\tau(g\tau+h)}) \\
&= \frac{g\tau+h}{e\tau+f} \cdot \varphi(S^{-1}B\tau, \tfrac{x}{e\tau+f}) \\
&= \frac{g\tau+h}{e\tau+f} \cdot \varphi(B'(\tfrac{-1}{\tau}), \tfrac{(1/\tau)x}{(1/\tau)(e\tau+f)}) \\
&= \frac{g\tau+h}{e\tau+f} \cdot \varphi(B'(\tfrac{-1}{\tau}), \tfrac{(1/\tau)x}{(-f(-1/\tau)+e)}) \\
&\overset{(*)}{=} \frac{g\tau+h}{e\tau+f} (f(\tfrac{-1}{\tau})+e)\varphi(\tfrac{-1}{\tau}, \tfrac{x}{\tau}) \\
&= (g\tau+h)\tau^{-1}\varphi(\tfrac{-1}{\tau}, \tfrac{x}{\tau}) \\
&= (g\tau+h)\tilde{\varphi}(\tau, x),
\end{aligned}$$

where $(*)$ follows from Proposition 6.5.

Now let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ and let $B := T_2 A T_{1/2} = \left(\begin{smallmatrix} a & 2b \\ c/2 & d \end{smallmatrix}\right) \in \Gamma^0(2)$ according to (46). Then

$$\begin{aligned} \tilde{\varphi}(2(A\tau), \tfrac{x}{c\tau+d}) &= \tilde{\varphi}((T_2 A)\tau), \tfrac{x}{c\tau+d}) \\ &= \tilde{\varphi}(B(T_2\tau)), \tfrac{x}{c\tau+d}) \\ &= \tilde{\varphi}(B(2\tau)), \tfrac{x}{\frac{c}{2}(2\tau)+d}) \\ &\overset{(58)}{=} (c\tau+d)\tilde{\varphi}(2\tau, x), \end{aligned}$$

which proves (59). □

Since $\varphi(\tau, x)$ is odd with respect to x, then so is $\tilde{\varphi}(\tau, x)$. When $\tau \in \mathcal{H}$ is fixed, it is an elliptic function with respect to the lattice $\tilde{L} = 2\pi i(\mathbb{Z}\tau + 2\mathbb{Z})$, with simple poles at points $x \equiv 0, 2\pi i\ (\tilde{L})$ and simple zeros at $x \equiv \pi i\tau, 2\pi i + \pi i\tau\ (\tilde{L})$. Following the same ideas as previously, it becomes natural to make the following definitions. Let $\tilde{Q}_\tau$ denote the function

$$\tilde{Q}_\tau(x) := x\tilde{\varphi}(2\tau, x),$$

which has an expansion of the form

$$\tilde{Q}_\tau(x) = x\tilde{\varphi}(2\tau, x) = \sum_{n=0}^{\infty} \tilde{a}_{2n}(\tau)x^{2n}, \quad \tilde{a}_0 \equiv 1. \tag{60}$$

We now define for all closed, oriented smooth manifolds M,

$$\begin{aligned} \tilde{\varphi}(M) &:= \left(\prod_{i=1}^{2k} \tilde{Q}_\tau(x_i)\right)[M] \\ &\overset{(57)}{=} \left(\prod_{i=1}^{2k} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \prod_{n=1}^{\infty} \left(\frac{(1-q^n)^2}{(1-q^n e^{\pm x_i})}\right)^{(-1)^n}\right)[M] \end{aligned}$$

when $\dim M = 4k$, and $\tilde{\varphi}(M) = 0$ otherwise.

Recall that the q-expansion of $\varphi(M)$ is related to $\operatorname{sign}(M)$ (see (54)). This time a relation with the $\hat{A}$-genus appears. Using again equalities (6) and (7) as well as the multiplicativity of the Chern character, we find that the Chern character

$$\operatorname{ch}\left(\bigotimes_{\substack{n=1 \\ n \text{ even}}}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \Lambda_{-q^n} TM_{\mathbb{C}}\right)$$

is equal to

$$\begin{aligned}
&\prod_{\substack{n=1\\ n \text{ even}}}^{\infty} \mathrm{ch}(S_{q^n} TM_{\mathbb{C}}) \prod_{\substack{n=1\\ n \text{ odd}}}^{\infty} \mathrm{ch}(\Lambda_{-q^n} TM_{\mathbb{C}}) \\
= &\prod_{\substack{n=1\\ n \text{ even}}}^{\infty} \prod_{i=1}^{2k} \frac{1}{(1-q^n e^{\pm x_i})} \cdot \prod_{\substack{n=1\\ n \text{ odd}}}^{\infty} \prod_{i=1}^{2k} (1-q^n e^{\pm x_i}) \\
= &\prod_{n=1}^{\infty} \prod_{i=1}^{2k} \left(\frac{1}{(1-q^n e^{\pm x_i})}\right)^{(-1)^n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{\varphi}(M)(\tau) =& 2^{2k} \prod_{n=1}^{\infty} (1-q^n)^{(-1)^n 4k} \\
&\cdot \left(\mathrm{ch} \left(\bigotimes_{\substack{n=1\\ n \text{ odd}}}^{\infty} \Lambda_{-q^n} TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1\\ n \text{ even}}}^{\infty} S_{q^n} TM_{\mathbb{C}} \right) \prod_{i=1}^{2k} \frac{x_i/2}{\sinh{(x_i/2)}} \right) [M] \\
=& 2^{2k} \prod_{n=1}^{\infty} (1-q^n)^{(-1)^n 4k} \cdot \hat{A} \left(M, \bigotimes_{\substack{n=1\\ n \text{ odd}}}^{\infty} \Lambda_{-q^n} TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1\\ n \text{ even}}}^{\infty} S_{q^n} TM_{\mathbb{C}} \right) \\
=& 2^{2k} \prod_{n=1}^{\infty} (1-q^n)^{(-1)^n 4k} \cdot (\hat{A}(M) - q \cdot \hat{A}(M, TM_{\mathbb{C}}) + \cdots). \qquad (61)
\end{aligned}$$

Proposition 6.11. *The function $\tilde{\varphi}(M)(\tau)$ is holomorphic on $\mathcal{H}$ and transforms like a modular form of weight $2k$ for $\Gamma_0(2)$.*

Proof. We proceed as we did for $\varphi(M)$. We have

$$\prod_{i=1}^{2k} \tilde{Q}_\tau(x_i) = \prod_{i=1}^{2k} \left(\sum_{n=0}^{\infty} \tilde{a}_{2n}(\tau) x_i^{2n} \right), \quad \tilde{a}_0 \equiv 1,$$

and we assign weight $4n$ to $\tilde{a}_{2n}$. Then $\tilde{\varphi}(M)(\tau)$ can be rewritten as

$$\tilde{\varphi}(M)(\tau) = \sum_{\substack{I=(i_1 \le \cdots \le i_j)\\ i_1+\cdots+i_j=k}} \tilde{b}_I(\tau) p_I[M], \qquad (62)$$

with $\tilde{b}_I$ homogeneous polynomials of weight $4k$ in the $\tilde{a}_{2n}$. It is then sufficient to show that the functions $\tilde{a}_{2n}$ are holomorphic on $\mathcal{H}$ and transform like modular forms of weight $2n$ for $\Gamma_0(2)$.

For some fixed and well chosen x (eg. $x = \pi i$), $x\tilde{\varphi}(2(\cdot), x)$ is nontrivial and holomorphic on $\mathcal{H}$. Then by equality (60), the $\tilde{a}_{2n}$ are holomorphic on $\mathcal{H}$.

Now let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix in $\Gamma_0(2)$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} \tilde{a}_{2n}(A\tau)(c\tau+d)^{-2n}x^{2n} &= \sum_{n=0}^{\infty} \tilde{a}_{2n}(A\tau)(\tfrac{x}{c\tau+d})^{2n} \\ &\overset{(60)}{=} \tfrac{x}{c\tau+d}\tilde{\varphi}(A(2\tau), \tfrac{x}{c\tau+d}) \\ &\overset{(59)}{=} x\tilde{\varphi}(2\tau, x) \\ &\overset{(60)}{=} \sum_{n=0}^{\infty} \tilde{a}_{2n}(\tau)x^{2n}.\end{aligned}$$

Comparing coefficients finishes the proof. □

Like we have seen for the functions a_{2n}, the $\tilde{a}_{2n}$ are holomorphic at the $i\infty$ cusp, because

$$\sum_{n=0}^{\infty} \tilde{a}_{2n}(\tau)x^{2n} = x\tilde{\varphi}(2\tau, x) = \frac{x}{e^{x/2} - e^{-x/2}} \prod_{n=1}^{\infty} \left(\frac{(1-q^n)^2}{(1-q^n e^{\pm x})} \right)^{(-1)^n},$$

which is holomorphic at $q = 0$.

Our last goal is to check holomorphicity of both $\varphi(M)$ and $\tilde{\varphi}(M)$ at the 0 cusp. As we did throughout this chapter, we will first prove this assertion for the functions a_{2n} and $\tilde{a}_{2n}$, i.e. we wish to show that the expansions at the 0 cusp a^0_{2n} and $\tilde{a}^0_{2n}$ are holomorphic at $i\infty$. For that purpose, we compare:

$$\begin{aligned}\sum_{n=0}^{\infty} a^0_{2n}(\tau)x^{2n} &= \sum_{n=0}^{\infty} \tau^{-2n}a_{2n}(\tfrac{-1}{\tau})x^{2n} = \sum_{n=0}^{\infty} a_{2n}(\tfrac{-1}{\tau})(\tfrac{x}{\tau})^{2n} \\ &= \tfrac{x}{\tau}\varphi(\tfrac{-1}{\tau}, \tfrac{x}{\tau}) = x\tilde{\varphi}(\tau, x) = \sum_{n=0}^{\infty} \tilde{a}_{2n}(\tfrac{\tau}{2})x^{2n}.\end{aligned}$$

This implies that for all $\tau \in \mathcal{H}$,

$$a^0_{2n}(\tau) = \tilde{a}_{2n}(\tfrac{\tau}{2}), \tag{63}$$

which is holomorphic at $i\infty$. Doing the same for $\tilde{a}^0_{2n}$ yields $\tilde{a}^0_{2n}(\tau) = 4^{-n}a_{2n}(\frac{\tau}{2})$, which is also holomorphic at $i\infty$. Hence, the functions a_{2n} and $\tilde{a}_{2n}$ are modular forms of weight $2n$ for $\Gamma_0(2)$ and equalities (55) and (62) imply that $\varphi(M)$, respectively $\tilde{\varphi}(M)$, are modular forms of weight $2k$ for the same congruence subgroup.

Additionnally, (63) shows that

$$\begin{aligned}\varphi(M)^0(2\tau) &= \sum_{\substack{I=(i_1\leq\cdots\leq i_j) \\ i_1+\cdots+i_j=k}} b^0_I(2\tau)p_I[M] \\ &= \sum_{\substack{I=(i_1\leq\cdots\leq i_j) \\ i_1+\cdots+i_j=k}} \tilde{b}_I(\tau)p_I[M] \\ &= \tilde{\varphi}(M)(\tau).\end{aligned}$$

All in all we have proven the following theorem:

Theorem 6.12. *The elliptic genus of a closed, oriented smooth $4k$-manifold M satisfies*

$$\varphi(M) \in M_{2k}(\Gamma_0(2)),$$
$$\tilde{\varphi}(M) \in M_{2k}(\Gamma_0(2)).$$

Furthermore, for all $\tau \in \mathcal{H}$,

$$\varphi(M)^0(2\tau) = \tilde{\varphi}(M)(\tau).$$

Remark 6.13. Theorem 6.12 and relations (54) and (61) provide us a motivation to call $\varphi(M)$ and $\tilde{\varphi}(M)$ the *elliptic genus at the signature cusp*, respectively *at the $\hat{A}$-genus cusp.*

7 Ochanine's Theorem

Having introduced the elliptic genus and discussed results from index theory and spin geometry, we are now in a position to prove Ochanine's Theorem, following [HBJ94], pp. 117-120.

Theorem 7.1 (**Ochanine**). *Let M be a compact, oriented, spin, smooth manifold of dimension $\dim M = 4k \equiv 4\ (8)$. Then its signature is divisible by 16.*

Proof. Let us denote the formal factorisation of the total Pontryagin and Chern classes of the tangent bundle TM, respectively its complexification $TM_{\mathbb{C}}$, by

$$p(M) = p(TM) = (1 + x_1^2)\cdots(1 + x_{2k}^2)$$
$$c(M) = c(TM_{\mathbb{C}}) = (1 + x_1)\cdots(1 + x_{2k})(1 - x_1)\dots(1 - x_{2k}).$$

We begin by considering the elliptic genus of M at the $\hat{A}$-genus cusp. We know (recall (61)) that

$$\begin{aligned}\tilde{\varphi}(M)(\tau) &= 2^{2k}\prod_{n=1}^{\infty}(1-q^n)^{(-1)^n 4k}\cdot \hat{A}\left(M, \bigotimes_{\substack{n=1\\ n \text{ odd}}}^{\infty}\Lambda_{-q^n}TM_{\mathbb{C}}\otimes\bigotimes_{\substack{n=1\\ n \text{ even}}}^{\infty}S_{q^n}TM_{\mathbb{C}}\right)\\ &= 2^{2k}\prod_{n=1}^{\infty}(1-q^n)^{(-1)^n 4k}\cdot(\hat{A}(M) - q\cdot\hat{A}(M, TM_{\mathbb{C}}) + \cdots),\end{aligned}$$

the last parenthesis being a power series in q where the coefficients are twisted $\hat{A}$-genera and the constant term $\hat{A}(M)$. But M is a spin manifold of dimension $4k$ with k odd, so by Theorem 4.18 all the coefficients in the q-expansion of $\tilde{\varphi}(M)(\tau)$ are even integers.

Since $\tilde{\varphi}(M)$ is a modular form of weight $2k$ for the congruence subgroup $\Gamma_0(2)$, then according to Remark 5.23 we may write it as a homogeneous polynomial of weight $2k$ in $8\tilde{\delta}$ and $\tilde{\varepsilon}$:

$$\begin{aligned}
\tilde{\varphi}(M) &= P(8\tilde{\delta}, \tilde{\epsilon}) \\
&= \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \lambda_{a,b}(8\tilde{\delta})^a(\tilde{\varepsilon})^b \\
&= \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \lambda_{a,b}(-1-24q-24q^2-\cdots)^a(q+8q^2+\cdots)^b \\
&= \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \lambda_{a,b}((-1)^a q^b + \{\text{terms of higher powers}\}),
\end{aligned}$$

where the coefficients $\lambda_{a,b}$ are a priori only complex numbers.

But, as we have just seen, the coefficients in the q-expansion of $\tilde{\varphi}(M)$ are even integers. Then the coefficients $\lambda_{a,b}$ must be even integers too. This can be proven by induction on b. The coefficient $-\lambda_{k,0}$ is the constant term of the series, hence it must be an even integer. Now assume that for some $b \in \mathbb{N}$, $\lambda_{k,0}, \lambda_{k-2,1}, \ldots, \lambda_{k-2b,b}$ are even integers. The coefficient of q^{b+1} being a linear combination of $\lambda_{k,0}, \lambda_{k-2,1}, \ldots, \lambda_{k-2b,b}$ and $\lambda_{k-2(b+1),b+1}$, the latter must be an even integer. We deduce that

$$\tilde{\varphi}(M) = P(8\tilde{\delta}, \tilde{\varepsilon}) \in 2\mathbb{Z}[8\tilde{\delta}, \tilde{\varepsilon}].$$

Let us now consider the elliptic genus at the signature cusp and write it as a homogeneous polynomial of weight $2k$ in 8δ and ε:

$$\varphi(M) = \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \mu_{a,b}(8\delta)^a\varepsilon^b, \quad \mu_{a,b} \in \mathbb{C}.$$

Using Theorem 6.12, we have

$$\begin{aligned}
\tilde{\varphi}(M)(\tau) &= \varphi^0(M)(2\tau) \\
&= (2\tau)^{-2k}\varphi(M)\left(\tfrac{-1}{2\tau}\right) \\
&= (2\tau)^{-2k} \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \mu_{a,b}\left(8\delta(\tfrac{-1}{2\tau})\right)^a \left(\varepsilon(\tfrac{-1}{2\tau})\right)^b \\
&= \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \mu_{a,b}\left(8(2\tau)^{-2}\delta(\tfrac{-1}{2\tau})\right)^a \left((2\tau)^{-4}\varepsilon(\tfrac{-1}{2\tau})\right)^b \\
&= \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \mu_{a,b}\left(8\delta^0(2\tau)\right)^a \left(\varepsilon^0(2\tau)\right)^b \\
&= \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \mu_{a,b}\left(8\tilde{\delta}(\tau)\right)^a (\tilde{\varepsilon}(\tau))^b.
\end{aligned}$$

where the last equality comes from Theorem 5.27. This infers that the coefficients $\lambda_{a,b}$ and $\mu_{a,b}$ coincide. Therefore

$$\varphi(M) = P(8\delta, \varepsilon) \in 2\mathbb{Z}[8\delta, \varepsilon].$$

Let us now consider

$$\mathrm{sign}(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM_{\mathbb{C}}),$$

which has an integral q-expansion with constant term $\mathrm{sign}(M)$ and other coefficients in the form of twisted signatures. We know from (54) that it is equal to

$$\begin{aligned}
2^{2k}\varphi(M) \prod_{n=1}^{\infty} \frac{(1+q^n)^{4k}}{(1-q^n)^{4k}} &= 2^{2k} P(8\delta, \varepsilon) \prod_{n=1}^{\infty} \frac{(1+q^n)^{4k}}{(1-q^n)^{4k}} \\
&= 2^{2k} \prod_{n=1}^{\infty} \frac{(1+q^n)^{4k}}{(1-q^n)^{4k}} \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \lambda_{a,b} (8\delta)^a (\varepsilon)^b \\
&= \prod_{n=1}^{\infty} \frac{(1+q^n)^{4k}}{(1-q^n)^{4k}} \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \lambda_{a,b} 2^{2a} (8\delta)^a 2^{4b} (\varepsilon)^b \\
&= \prod_{n=1}^{\infty} \frac{(1+q^n)^{4k}}{(1-q^n)^{4k}} \sum_{\substack{a,b\in\mathbb{N} \\ 2a+4b=2k}} \lambda_{a,b} (32\delta)^a (16\varepsilon)^b \\
&= P(32\delta, 16\varepsilon) \prod_{n=1}^{\infty} \frac{(1+q^n)^{4k}}{(1-q^n)^{4k}}.
\end{aligned}$$

We see that

$$64\delta = 64(\tfrac{1}{4} + 6q + 6q^2 + \cdots) = 16(1 + 24q + 24q^2 + \dots)$$

divides $P(32\delta, 16\varepsilon)$. Indeed, k being odd, there is no $b \in \mathbb{N}$ such that $4b = 2k$, thus the factor 32δ appears in every term of the polynomial. The extra 2 factor comes from the fact that the coefficients $\lambda_{a,b}$ are even. All in all we get

$$\begin{aligned}
\mathrm{sign}(M) + 2q \cdot \mathrm{sign}(M, TM_{\mathbb{C}}) + \cdots &= P(32\delta, 16\varepsilon) \prod_{n=1}^{\infty} \frac{(1+q^n)^{4k}}{(1-q^n)^{4k}} \\
&= 16 \cdot (\text{integral } q\text{-expansion}).
\end{aligned}$$

Thus 16 divides all coefficients on the left hand side, and in particular $\mathrm{sign}(M)$. □

As a final remark, it is worth noting that Ochanine's result cannot be improved as there exist compact, oriented, spin manifolds of dimension $8k + 4$ whose signature is 16. A typical example is the following 4-manifold, called the *$K3$-surface*, or *Kummer surface*, and defined by

$$K3 := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \right\} \subset \mathbb{C}\mathrm{P}^3.$$

This is a complex manifold, so let us use $c(K3)$ to denote the total Chern class of its complex tangent bundle of dimension 2 and $\omega(K3)$, $p(K3)$ to denote the total

Stiefel-Whitney class, respectively the total Pontryagin class, of the underlying real tangent bundle $TK3$ of dimension 4.

Let α be the generator of $H^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z})$ so that $\alpha^3[\mathbb{C}\mathrm{P}^3] = 1$, where $[\mathbb{C}\mathrm{P}^3]$ is the fundamental class for the natural orientation of $\mathbb{C}\mathrm{P}^3$. Let $\tilde{\alpha}$ denote its restriction to $H^2(K3;\mathbb{Z})$, i.e.

$$\tilde{\alpha} = i^*(\alpha)$$

for the inclusion $i : K3 \hookrightarrow \mathbb{C}\mathrm{P}^3$. Seen as a complex manifold, $K3$ has total Chern class[27]

$$\begin{aligned} c(K3) &= (1+\tilde{\alpha})^4(1+4\tilde{\alpha})^{-1} \\ &= (1+\tilde{\alpha})^4(1-4\tilde{\alpha}+16\tilde{\alpha}^2) \\ &= 1+6\tilde{\alpha}^2, \end{aligned}$$

so

$$c_1(K3) = 0 \quad \text{and} \quad c_2(K3) = 6\tilde{\alpha}^2.$$

The vanishing of the first Chern class implies[28] that $\omega_2(K3) = 0$, therefore $K3$ is a spin manifold.

Relation (5) implies

$$p_1(K3) = c_1^2(K3) - 2c_2(K3) = -12\tilde{\alpha}^2.$$

The evaluation of $\tilde{\alpha}^2$ on a fundamental class $[K3]$ is $\tilde{\alpha}^2[K3] = \pm 4$, depending on the chosen orientation. By the Hirzebruch Signature Theorem, we finally have

$$\operatorname{sign}(K3) = \frac{1}{3}p_1(K3)[K3] = -4\tilde{\alpha}^2[K3] = \pm 16.$$

[27]See for example [Hir54], p. 463.

[28]This is because the coefficient homomorphism $H^*(K3;\mathbb{Z}) \to H^*(K3;\mathbb{Z}_2)$ maps the total Chern class $c(K3)$ to the total Stiefel-Whitney class $\omega(K3)$. See [MS74] p. 171.

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