Master's Thesis:

Positive Scalar Curvature, Enlargeability and The Positive Energy Theorem

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Abstract. In this Master's thesis, we discuss the notion of enlargeability as obstruction to the existence of positive scalar curvature metrics on Riemannian manifolds. We will follow the approach of M. Gromov and H.B. Lawson, who used techniques from spin geometry to prove this result in '80. More specifically, they used a slightly modified version of the celebrated Lichnerowicz formula and showed that it contradicts the Atyiah-Singer index theorem in the presence of positive scalar curvature if the manifold at hand is enlargeable. We introduce some of the necessary prerequisites to be able to properly discuss the subject, such as: Riemannian Geometry, Spin Geometry, Characteristic classes and the Atiyah-Singer index theorem.

The last chapter is dedicated to the Positive Energy Theorem, a result that has its historical roots in General Relativity. We will use the non-existence of positive scalar curvature metrics on certain compact manifolds to discuss a proof of the Positive Energy Theorem that is due to J. Lohkamp.

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Zu gudder Lescht sinn ech menger Famill e grousse Merci schëlleg fir hir enorm Ënnerstëtzung; souwuel moralescher wéi och materieller. Méi spezifesch géif ech dës Masteraarbecht gäere mengen Eltere widmen dofir dass sie mir ëmmer de Fräiraum gelooss hunn, déi Richtungen am Liewen anzeschloen, déi ech wollt.

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Chapter 1

Introduction: Positive Scalar Curvature and Topology

This Master's thesis belongs to one of the numerous intersections of geometry and topology. Namely, we explore a certain relation of curvature, a notion from Riemannian geometry, to the topology of the space at hand. This is conceptually interesting in the following sense: Curvature is a description of the shape of the given space around a point. By this, we mean specifically that it is *local* in nature, i.e. it suffices to know what the space looks like in a small neighbourhood around the given point to be able to compute curvature. Furthermore, curvature is sensitive to small perturbations of the geometry of the space, provided these perturbations affect the neighbourhood of the point at which one measures. Topology on the other hand seems to be the opposite with respect to the two aforementioned properties in the sense that it is concerned with the *global* structure of a space. Small neighbourhoods of the given space carry no topological information, only the whole space does. Small perturbations (that fulfill some natural conditions) do not change the topological data associated to the space. These are the premises of topology, rather than deep mathematical insights.

Nonetheless, we will see that even seemingly weak conditions on curvature can have a dramatic effect on the global picture of the space.

Riemann surfaces. To illustrate what we mean, we will have a look at Riemann surfaces[1]. First of all, these are two-dimensional objects and thus suit the (limited) human imagination. Secondly they played a predominant historical role in the development of considerable parts of geometry and topology as we understand these disciplines today. For these reasons they provide adequate examples for the concepts we discuss here. Riemann surfaces are defined as smooth two-dimensional surfaces with a particularly nice additional structure[2]. For orientable surfaces, such a structure can always be chosen (although not unambiguously so) therefore

[1] not to be confused with Riemannian manifolds.
[2] They can succinctly be defined as one-dimensional complex manifolds.

we won't worry about this too much here.

By a celebrated result, namely the so-called Uniformization Theorem, Riemann surfaces can be put into three different categories: elliptic, parabolic and hyperbolic. These correspond to the three different curvature cases: positively curved (e.g. spheres), flat (e.g. the plane, cylinders, tori), negatively curved (e.g. surfaces of genus > 1). When we restrict ourselves to compact Riemann surfaces, the only remaining example for positive curvature is the 2-sphere. This is in stark contrast to the hyperbolic case, where there are infinitely many examples.

Positive curvature in general. This scarcity of positively curved manifolds seems to persist in higher dimensions, as it is actually an active field of research to find interesting examples of positively curved spaces for the different curvature notions[3]. We will restrict our attention to *scalar curvature*, which is somehow a weak notion of curvature since it only associates a number to every point of the space, whereas other curvatures are -roughly speaking- arrays of numbers. These concept are introduced and, to some extent, discussed in **chapter 2**. Nevertheless positive scalar curvature is a restrictive condition on topology. It is one of the goals of this master's thesis to illustrate this.

Spin Geometry. It took mathematicians quite a long time to get some kind of grip on this interaction of positive scalar curvature with topology. One reason for this is that virtually all results of this type rely on one crucial tool that requires a proper understanding of various mathematical disciplines: spin geometry. **Chapter 4** provides some overview of this subject with an emphasis on its differential geometric aspects. We completely omit the algebraic and representation theoretic prerequisites for which we refer to chapter I of the standard reference [LaMi].

Here we will briefly discuss the historic origins of spin geometry. The first mathematician who has properly understood the so-called spinors seems to have been Élie Cartan around the year 1910[4] in the context of representations of Lie groups. Similar ideas were put to use in theoretical physics in the 1920's in the wake of the emergence of quantum theory. Around 1928 Paul Dirac first wrote down the relativistically consistent quantum equation of motion which was subsequently named after him. In his efforts to achieve this, he invented the so-called Dirac operator, which is of fundamental importance for our purposes. He had the idea of trying to factorize a certain equation in which the Laplacian, a second order differential operator, occurred. He thus needed to take the (a priori ill-defined) *square root* of this Laplacian operator. Imposing that the resulting operator D be of first order, this means

[3]For dimension two most curvature notions coincide (up to constants) and thus there was no need to specify for our discussion of Riemann surfaces

[4]In 1913, his paper [Ca] was published.

solving the following system equations for the coefficients γ_k:

$$\begin{aligned} D &= \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} + ... + \gamma_n \frac{\partial}{\partial x_n} \\ D^2 &= \Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - ... - \frac{\partial^2}{\partial x_n^2} \end{aligned}$$

which yields the following relations:

$$\begin{aligned} \gamma_k \cdot \gamma_j + \gamma_j \cdot \gamma_k &= 0 \qquad\qquad \text{for } k \neq j \\ \gamma_k \cdot \gamma_k &= -1 \end{aligned}$$

This purely formal manipulation begs the question of what kind of mathematical object these γ_k actually are and how their product is defined. Dirac supposed them to be matrices equipped with the corresponding product and identified matrices that fulfill these relations in the context of space-time (i.e. in dimension four with different signs $(-,+,+,+)$ in the definition of the Laplacian). Abstractly, the above relations define a so-called *Clifford algebra*. These objects were already considered in the 1870's by their name-giving mathematician William K. Clifford, building on the works of William R. Hamilton and Hermann Grassmann, but apparently Dirac wasn't aware of this. In conclusion, this somehow naive attempt of taking the *square root* of a differential operator turned out to be something very profound: By doing this, one is forced to leave the classical vector-based realm for its Clifford-algebraic extension.

How scalar curvature enters. As it turns out, the fact that Dirac's trick works out is specific to the context where the underlying space is flat in an appropriate sense. In general, a curvature term will appear. This idea of relating curvature to second-order differential operators is sometimes called *Bochner's method* or *Weitzenböck formulae*. In the context of the Dirac operator, the corresponding formula goes back to André Lichnerowicz[5] and was proved in 1963. The curvature term in the so-called *Lichnerowicz formula* is one fourth of scalar curvature:

$$D^2 = \Delta + \frac{1}{4}\kappa$$

This formula is essential for our purposes. Its proof will be given in **chapter 4** and its applications will be discussed in **chapter 7**. The Dirac operator can be linked to an interesting topological invariant called the $\hat{A}$-genus by the Atiyah-Singer index theorem. In **chapter 6**, we will merely discuss the latter result, a proof of which would be beyond the scope of this thesis. In conclusion, the Lichnerowicz formula links scalar curvature to the Dirac operator, the index theorem links the Dirac operator to the $\hat{A}$-genus and thus the $\hat{A}$-genus is related to scalar curvature. More precisely, the existence of positive scalar curvature metrics for a

[5]Interestingly enough, already the physicist Erwin Schrödinger knew about this formula in the setting of space-time. He derived and applied it in his 1932 paper [Scr]. Some people therefore call this formula *Schrödinger-Lichnerowicz formula*.

certain manifold implies the vanishing of the corresponding $\hat{A}$-genus, thus it can be seen as an obstruction to the existence of positive scalar curvature metrics.

Enlargeability. Enlargeability is a topological property of manifolds which is strongly related to its fundamental group. It is one of the goals of this thesis (**chapter 7**) to show that enlargeability is an obstruction to the existence of positive scalar curvature metrics on spin manifolds. The idea is -roughly speaking- that the fundamental group of a positively curved space should be *small* in an appropriate sense[6]. This is again reflected in our basic example of Riemann surfaces: The 2-sphere is the only positively curved compact Riemann surface as well as the only one with trivial fundamental group. Enlargeability reflects the fact that the fundamental group of the space at hand is *large* by imposing that the space admits *arbitrarily large* covering spaces. In '79 R. Schoen and S.T. Yau published two papers ([SY1/2]) around this idea but in which they used minimal surface techniques and more specifically the regularity of minimal surfaces, a method that works properly only in dimensions ≤ 7. In '80 M. Gromov and H.B. Lawson employed arguments from spin geometry to prove results along the same lines (see [GrLa]). This is the approach we will adopt. One uses certain twisted spinor bundles for which there is a modified Lichnerowicz formula in which an additional curvature term appears (depending on the curvature of the vector bundle with which one twists). As it turns out, enlargeability is a suitable condition to gain some control over this additional term so that the Lichnerowicz argument can still be applied. Thus a modified $\hat{A}$-genus vanishes in the presence of positive scalar curvature. Computing the topological index and using the Atiyah-Singer index theorem shows that this vanishing doesn't occur.

The Positive Energy Theorem. Contrarily to other field theories, the formalism of General Relativity doesn't allow for a sensible definition of (local) energy density; only for a *global* numerical quantity (called total energy) which, in a sense, measures the asymptotic gravitational behaviour of the system[7]. The notion of total energy for a system in General Relativity was formalized by the physicists R. Arnowitt, S. Deser and C. Misner in their '61 paper [ADM]. It was conjectured that this total energy was positive for sufficiently reasonable space-time geometries, since positivity of energy is physically associated to the stability of the system. This is known as the *Positive Mass conjecture*. Again, this conjecture was proven in '79 by Schoen and Yau in a series of papers [SY3/4/5] by the use of minimal surface techniques which fail in dimensions ≥ 7. In '81 Witten presented another remarkable proof in [Wi] using spin geometry which can be applied to spin manifolds in arbitrary dimensions. The alternate proof for spin manifolds which we will present in **chapter 8** goes back to J. Lohkamp [Lo] and was published in '97. Lohkamp's idea can be roughly described as follows: After reducing the problem to a space-like hypersurface with positive scalar curvature (this was known before Lohkamp's paper) one can show that the assumption of negative energy leads to a contradiction. More specifically, this assumption can be used to modify the given metric so

[6]See the introduction to the fundamental paper [GrLa] for details on this idea.
[7]See the introduction to [Wi].

that the hypersurface is Euclidean outside of a compact which has itself non-negative (and somewhere strictly positive) scalar curvature. This contradicts the results from **chapter 7** in the sense that such a manifold could be *glued* into an otherwise flat compact manifold that cannot carry positive scalar curvature metrics, such as the torus for example.

Chapter 2

Some Aspects of Riemannian Geometry

In this first chapter we provide an introduction to the tools and concepts from Riemannian Geometry that are important in the context of Spin Geometry. Amongst other things this means that we work in the abstract setting of smooth vector bundles. The classical case can be extracted from this setting by choosing the tangent bundle as vector bundle over the base manifold. We do this on several occasions explicitly, since some results are specific to that context (the Levi-Civita connection for example). At the end of the chapter we discuss the various notions of curvature. Our approach is mainly based on [**Jos**], [**Pet16**] and the lecture notes [**Izm**] and [**Des**] (for the part on vector bundles).

2.1 Vector bundles

Let X be a smooth manifold.

Definition 2.1.1. Let E be a smooth manifold, and $\pi : E \to X$ a smooth map. A triple (E, X, π) is called a **smooth vector bundle** of rank n if the following hold:

1) $\pi^{-1}(\{x\})$ is an n-dimensional real vector space;

2) For every $x \in X$ there is a neighbourhood $U \subset X$ and a diffeomorphism:

$$\varphi : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n$$

with the additional property that for every $x' \in U$ the restriction:

$$\varphi|_{\pi^{-1}(\{x'\})} : \pi^{-1}(\{x'\}) \xrightarrow{\cong} \{x'\} \times \mathbb{R}^n$$

is a vector space isomorphism. ($\{x'\} \times \mathbb{R}^n$ is a vector space by dropping $\{x'\}$)

E is called *total space*, B *base space* and π *projection map.* The vector spaces $\pi^{-1}(\{x\})$ are called *fibres* and often denoted by E_x. The pairs (U, φ) are sometimes referred to as *bundle charts*, their collection as *bundle atlas.* In a context where there is no ambiguity over which base space the bundle is considered, the vector bundle is often just referred to as E.

Remark 2.1.2. There are different generalizations of the definition of a vector bundle. In the category of topological rather than smooth manifolds, one can just replace *smooth* by *continuous* in every occurrence to get the according definition. Furthermore, the model space for the fibres needn't be $\mathbb{R}^n$, it can for example be complex.

Example 2.1.3. The tangent bundle TX
The archetypal example for a smooth vector bundles is the tangent bundle (TX, X, π) associated to any smooth manifold X. By π, we denote the projection map of vectors to their base points. It is easy to see that the defining properties of vector bundles are met: The fibres are vector spaces and the bundle charts can be constructed from the manifold charts in the obvious manner.

Definitions 2.1.4. A **vector bundle morphism** between (E, X, π) and (E', X', π') is a pair of smooth maps $F : E \to E'$, $f : X \to X'$ s.th. the the following hold:

1) The following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

2) For every $x \in X$ the map $F|_x : E_x \to E'_{f(x)}$ is linear.

Vector bundle isomorphisms are defined correspondingly.

To describe the above commutative diagram, one says that F *covers* f. Property 2) is referred to as F being *linear in every fibre.* Very often the concept of vector bundle morphism is applied in the more restricted context where E and E' are bundles over the same base space and $f = id_X$. The corresponding diagram takes the following form:

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

In the case where furthermore $E' = E$, these maps are called *bundle endomorphisms* and their set is denoted by End$E$.

Definition 2.1.5. A vector bundle of rank n (E, X, π) is called **trivial** if it is isomorphic to the bundle $(X \times \mathbb{R}^n, X, \pi_1)$, where π_1 denotes the projection onto the first factor.

Because of this terminology, the condition **2.1.1.**2) is called *local triviality.* One of the crucial steps in properly understanding vector bundles is the clear distinction between *local triviality* on the one hand and *triviality of the bundle* on the other. In fact, the triviality or non-triviality of vector bundles over X is strongly related to the topology of X. Considerable parts of algebraic topology take bundles over X as geometric input to gain topological information about X. Examples of this idea are topological K-Theory (see for example the classic [**At**] or [**Ha09**]) and Characteristic classes (see Chapter **5**). Contractible spaces (i.e. topologically trivial spaces), for example, can be shown to admit only trivial vector bundles. We illustrate these ideas by the following (humble) example:

Example 2.1.6. $T\mathbb{S}^2$ is non-trivial
By **2.1.3**, the tangent bundle $T\mathbb{S}^2$ is a vector bundle over the 2-sphere. $T\mathbb{S}^2$ is non-trivial.
Proof: Suppose $T\mathbb{S}^2 \cong \mathbb{S}^2 \times \mathbb{R}^2$. This isomorphism can be used to construct a non-vanishing vector field on $\mathbb{S}^2$. This can be done by transporting for example $p \mapsto (p; (1, 0)) \in \mathbb{S}^2 \times \mathbb{R}^2$ over to $T\mathbb{S}^2$. The resulting vector field has to be non-vanishing, since the map is an isomorphism. This is a contradiction to the *hairy ball theorem* (see [**C.1.5**]).

$\square$

Next, we will review some methods of constructing new bundles from known ones. The first will be the so-called pull-back construction.

The needed data to carry out the pull-back construction is a rank-n bundle (E, X, π), a smooth manifold X' and a smooth map $f : X' \to X$. The basic idea is to turn X' into the base space of a new vector bundle by defining the fibre over the point $x' \in X'$ to be the fibre of E over $f(x)$. Define the following subset of $X' \times E$:

$$f^*E = \big\{(x', e) \in X' \times E \mid f(x') = \pi(e)\big\}$$

and the map (which is just projection onto the first factor):

$$\begin{array}{rcl} f^*\pi : f^*E & \longrightarrow & X' \\ (x'; e) & \longmapsto & x' \end{array}$$

Notice that by taking the projection on the second factor:

$$\begin{array}{rcl} F : f^*E & \longrightarrow & E \\ (x'; e) & \longmapsto & e \end{array}$$

we get the following commutative diagram:

$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d, "f^*\pi"] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

which means that F covers f and that we have thus a bundle morphism: $(f^*E, X', f^*\pi) \to (E, X, \pi)$ provided that we have indeed constructed a vector bundle. Furthermore the map F can easily be seen to be a fibre-wise vector space isomorphism (but obviously not necessarily a bundle isomorphism).

Claim: $(f^*E, X', f^*\pi)$ is a smooth vector bundle of rank n.
Proof: $X' \times E$ is a smooth product manifold of which f^*E is a submanifold by the implicit function theorem, furthermore $f^*\pi$ is clearly a smooth map. The fibre $f^*E_{x'}$ is by construction equal to the fibre $E_{f(x')}$ and thus an n-dimensional vector space. The local triviality requirement is met by defining the following bundle atlas: For all x' consider the chart (U, φ) around $f(x')$. We get a chart around x' by taking the pair:

$$(f^{-1}(U), \varphi \circ F|_{f^*\pi^{-1}(f^{-1}(U))})$$

where $\varphi \circ F|_{f^*\pi^{-1}(f^{-1}(U))}$ is a fibre-wise vector space isomorphism since both φ and F are.

$\square$

Definition 2.1.7. We call $(f^*E, X', f^*\pi)$ the **pull-back bundle of** (E, X, π) **under** f.

Properties 2.1.8. Basic properties of the pull-back

1) If $f : X' \to X$ is a diffeomorphism, then $(f^*E, X', f^*\pi)$ and (E, X, π) are isomorphic as vector bundles;

2) Given a sequence of smooth maps $Z \xrightarrow{g} Y \xrightarrow{f} X$ and a bundle E over X, we have the following *contravariance* property:

$$(f \circ g)^*E = g^*f^*E$$

Another method of constructing new bundles is applying functors to an existing vector bundle. Contrary to the pull-back construction this yields bundles over the same base space. Let **Vect** denote the category of vector spaces, and T a functor of the type:

$$T : \textbf{Vect} \times ... \times \textbf{Vect} \longrightarrow \textbf{Vect}$$

T may be co- or contravariant.

Definition 2.1.9. T is called **smooth functor** if for all $V, W \in$ **Vect**, the map (in the covariant case):

$$T : \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(T(V), T(W))$$

or (in the contravariant case):

$$T : \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(T(W), T(V))$$

is smooth. In the case where T takes multiple arguments from **Vect**, smoothness is defined correspondingly.

Examples 2.1.10. Examples of smooth functors on vector spaces

1) $\oplus : (V, W) \mapsto V \oplus W$ the direct sum,
2) $\otimes : (V, W) \mapsto V \otimes W$ the tensor product,
3) $^* : V \mapsto V^*$ dualization,
4) $\Lambda^i : V \mapsto \Lambda^i V$ i-th exterior product.

Proposition 2.1.11. If T is a smooth functor taking k arguments in **Vect** and $\{(E_i, X, \pi_i)\}_{i \in \{1,...,k\}}$ a collection of smooth vector bundles, then $(T(E_1, ..., E_k), X, \pi)$ is a smooth vector bundle as well (where the fibres are defined in the obvious manner: $T(E_1, ..., E_k)_x = T((E_1)_x, ..., (E_k)_x)$).

Applying the proposition to the examples from **2.1.10** yields a large set of examples of smooth vector bundles:

Examples 2.1.12. Let E and F be vector bundles over the base space X, then the following are vector bundles over X as well:

1) $E \oplus F$ the direct sum of vector bundles,
2) $E \otimes F$ the tensor product of vector bundles,
3) E^* the dual vector bundle,
4) $\Lambda^i E$ i-th exterior product of a vector bundle,
5) $T^r_s E = E \underset{r\text{-times}}{\otimes ... \otimes} E \otimes E^* \underset{s\text{-times}}{\otimes ... \otimes} E^*$ the tensor bundle of type (r, s), which is a combination of examples 2) and 3).

By $\Lambda^i X = \Lambda^i T^* X$ respectively $T^r_s X = T^r_s TX$ we will denote the result of these constructions applied to the cotangent bundle $T^*X = (TX)^*$, respectively the tangent bundle TX (see **2.1.3**). Next, we discuss sections of vector bundles.

Definition 2.1.13. For a smooth vector bundle (E, X, π) a smooth map $\sigma : X \to E$ is called **smooth section of** E if it is a right-inverse to π, i.e. if the following holds:

$$\pi \circ \sigma = id_X$$

The set of sections of E is denoted by $\Gamma(E)$.

The defining condition of sections is quite a natural requirement: If evaluated at some point x of the base manifold, we want σ to take its values in the fibre over the same base point, i.e. we require $\sigma|_x \in E_x$. $\Gamma(E)$ can easily be seen to be itself a real vector space. Many important objects from differential geometry occur as sections of appropriate vector bundles:

Examples 2.1.14. Examples of sections

1) $\Gamma(TX) = \mathfrak{X}(X)$ is the set of vector fields on X,

2) $\Gamma(\Lambda^i X) = \Omega^i X$ is the set of differential i-forms,

3) $\Gamma(T^r_s X) = \mathcal{T}^r_s X$ is the set of (r, s)-tensor fields.

In the spirit of the proof of **2.1.6**, one can show the following:

Proposition 2.1.15. A rank-n vector bundle is trivial if and only if it admits n sections that are linearly independent in every fibre.

As an example, one can show that if the base manifold is a Lie group G, then the tangent bundle is trivial, i.e.

$$TG \cong G \times T_1 G \cong G \times \mathfrak{g}$$

by choosing a basis $\{e_1, ..., e_n\}$ of $T_1 G$ and transporting it to every $T_g G$ by using group (left-) multiplication by the element g, denoted by L_g. This means that for all $i \in \{1, ..., n\}$, we define the sections of TG by:

$$\sigma_i|_g = dL_g(1)e_i$$

Since $L_g : G \to G$ is a diffeomorphism, every $dL_g(1) : T_1 G \to T_g G$ is a vector space isomorphism and the σ_i are thus linearly independent in every point.

2.2 Metrics and Connections

Let (E, X, π) be a smooth vector bundle over the n-dimensional base space X. To be able to measure lengths and angles in the fibres of E, we introduce the following concept:

Definition 2.2.16. A section

$$\langle \cdot, \cdot \rangle \in \Gamma(E^* \otimes E^*)$$

is called **(bundle) metric of** E if it is symmetric and positive definite in every fibre.

By a partition of unity, one can show that every vector bundle can be equipped with a bundle metric. For any point $x \in X$ this definition clearly yields an inner product on E_x, denoted by $\langle \cdot, \cdot \rangle_x$. $\{(E_x, \langle \cdot, \cdot \rangle_x)\}_{x \in X}$ is thus a family of Euclidean vector spaces *smoothly indexed* by the base manifold X. As such it is clear how lengths and angles can be measured. By non-degeneracy, this inner-product yields a canonical isomorphism between E and its dual, which is of fundamental importance in differential geometry:

$$\begin{array}{rcl} \iota : E & \xrightarrow{\cong} & E^* \\ (x, v) & \longmapsto & \langle v, \cdot \rangle_x \end{array}$$

Remark 2.2.17. Induced metrics
A metric on the bundle E induces a natural metric on its dual E^* in the following manner: Consider sections that locally form a basis of every fibre $\{\sigma_1, ..., \sigma_n\}$, these can be chosen orthonormal with respect to the given bundle metric by applying the Gram-Schmidt process. To this orthonormal frame, we associate its dual frame $\{s_1, ..., s_n\}$ (which are projections onto the σ_i). This dual frame is a local coordinate frame for E^*. The metric structure on E^* comes from declaring this dual frame to be orthonormal, i.e. to define $\langle \cdot, \cdot \rangle_{E^*}$, we put:

$$\langle s_i, s_j \rangle_{E^*} = \delta_{ij}$$

and require it to be linear. This construction is independent of the chosen basis. Another way to see this is by using the isomorphism ι described above. For any two sections $s, r \in \Gamma(E^*)$, we have:

$$\langle s, r \rangle_{E^*} = \langle \iota^{-1} \circ s, \iota^{-1} \circ r \rangle$$

The same idea yields an inner product on all tensor bundles derived from E. All of these are usually again denoted by $\langle \cdot, \cdot \rangle$ to simplify notation.

Putting $E = TX$ yields the most important special case of bundle metrics:

Definitions 2.2.18. A bundle metric $\langle \cdot, \cdot \rangle$ on TX is called **Riemannian metric on** X and is often denoted by $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$. The pair (X, g) is called **Riemannian manifold**.

Additionally to the fibre-wise measurements mentioned above, Riemannian metrics serve different basic purposes: They can be used to measure lengths of (piecewise) smooth curves $c : I \to X$ by integrating the norm of the velocity vector:

$$\mathrm{length}_g(c) = \int_I \|\dot{c}\|_g dt$$

By taking the infimum over all paths linking two given points on the manifold, this length measurement turns the manifold into a metric space (whose toplogy coincides with the manifold topology). Riemannian metrics also locally induce a volume form whose volume measurement is consistent with the Euclidean structure of the tangent spaces. This isn't surprising since the local construction of an orthonormal frame in the tangent bundle yields information on the volume of parallelepipeds by requiring that the unit n-cube have volume 1.

Next, we turn the related idea of a connection on a smooth vector bundle. The purpose of connections is to make sense of the directional derivative of sections. Given any vector bundle, there is no intrinsic way of forming a derivative of its sections. The problem is the following one: One would like to write down something of the form:

$$\lim_{t \to 0} \frac{\sigma_t - \sigma_0}{t}$$

where σ_0 is the value of the section in the point where we take the derivative and σ_t is the value along some integral curve c of the vector in whose direction we would like to take the derivative. Alas, the difference quotient we wrote down doesn't make sense in a canonical manner, since σ_0 and σ_t live in different fibres, namely $E_{c(0)}$ and $E_{c(t)}$. By prescribing a way of forming such derivatives, one *connects* a given fibre to its neighbouring fibres. This idea can be made rigorous by so-called parallel transport (which won't be discussed here, see for example **[Jos]** chapter 4).

Definition 2.2.19. A **connection** ∇ **on** (E, X, π) is an $\mathbb{R}$-linear map:

$$\begin{array}{rcl} \nabla : \Gamma(E) & \longrightarrow & \Gamma(T^*X \otimes E) \\ \sigma & \longmapsto & \nabla\sigma \end{array}$$

s.th. the following product rule holds for every function $f \in C^\infty(X)$ and every section $\sigma \in \Gamma(E)$:

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$

Some explanations are in order. Given a vector field V, one gets a map:

$$\nabla_V : \Gamma(E) \longrightarrow \Gamma(E)$$

by evaluating $\nabla\sigma$ in the T^*X-argument as follows:

$$\sigma \longmapsto \nabla\sigma(V) = \nabla_V\sigma$$

This is called the **covariant derivative in direction** V and explains why we have indeed constructed an analogue to the directional derivative. One can show that the quantity $\nabla_V\sigma|_x \in E_x$ depends only on the value $V|_x \in T_xX$, i.e. it has pointwise dependence in the vector component. The dependence on the section σ is local on the other hand, which means that one needs the values of σ in a neighbourhood of x to compute $\nabla_V\sigma|_x$. We get the simplest example for such an object by considering tangent vectors of $\mathbb{R}^n$, i.e. the bundle $(T\mathbb{R}^n, \mathbb{R}^n, \pi)$. A natural connection is given for all $Y \in \Gamma(T\mathbb{R}^n)$ by component-wise differentiation:

$$Y \mapsto \nabla Y = dY$$

which can easily be seen to meet all the requirements. Unfortunately, this example isn't adequate for a proper understanding of the full scope of the definition, since different fibres of $T\mathbb{R}^n$ are copies of $\mathbb{R}^n$ and can be canonically identified by Euclidean parallel transport, thus the above mentioned problem of deriving vector fields doesn't exist in the first place.

Once a connection ∇ is fixed on E, it can be extended to other vector bundles constructed from E. (For simplicity, these extensions will all be called ∇ as well)

Proposition 2.2.20. Extension of ∇

1) There is a unique connection on the dual bundle:

$$\nabla : \Gamma(E^*) \longrightarrow \Gamma(T^*X \otimes E^*)$$

s.th. the following rule holds for all $\sigma \in \Gamma(E), \sigma^* \in \Gamma(E^*), V \in \Gamma(TX)$:

$$d(\sigma^*(\sigma))V = \nabla_V\sigma^*(\sigma) + \sigma^*(\nabla_V\sigma)$$

2) Given a second bundle over the same base space equipped with a connection (F, X, π', ∇'), there is a unique connection ∇ on the tensor product $E \otimes F$ (see **2.1.10**):

$$\nabla : \Gamma(E \otimes F) \longrightarrow \Gamma(T^*X \otimes E \otimes F)$$

s.th. for all $\sigma \in \Gamma(E), \rho \in \Gamma(F)$, we have:

$$\nabla(\sigma \otimes \rho) = \nabla\sigma \otimes \rho + \sigma \otimes \nabla'\rho$$

The requirement in 1) makes sense, since $\sigma^*(\sigma) \in C^\infty(X)$. These formulae obviously all represent the occurrence of some kind of product rule. By combining 1) and 2), the connection can actually be extended to tensor fields of all types: Recall from **2.1.12** that all T^r_s are tensor products of E and its dual, thus we get a unique connection ∇:

$$\nabla : \Gamma(T^r_s E) \longrightarrow \Gamma(T^*X \otimes T^r_s E)$$

By the same principle, i.e. by combining known derivations and requiring some product rule, ∇ can be extended to so-called *vector-valued differential forms*. Since this extension makes use of the exterior derivative d (intrinsically) associated to X, it is denoted by d^∇. We will use this construction to define curvature (see **2.3.25**).

Definition 2.2.21. Extension of ∇ to vector-valued forms
Define the map:

$$d^\nabla : \Gamma(E \otimes \Lambda^k X) \longrightarrow \Gamma(E \otimes \Lambda^{k+1} X)$$

by putting for all k-forms $\omega \in \Lambda^k X$ and sections $\sigma \in \Gamma(E)$:

$$d^\nabla(\sigma \otimes \omega) = \nabla\sigma \wedge \omega + \sigma \otimes d\omega$$

and extending linearly.

The term $\nabla\sigma \wedge \omega$ requires an explanation, since $\nabla\sigma \in \Gamma(E \otimes \Lambda^k X)$ isn't a pure differential form. To make sense of the wedge product we ignore the $\Gamma(E)$-component of $\nabla\sigma$, i.e. for an elementary object of the tensor bundle $\sigma \otimes \omega \in \Gamma(E \otimes \Lambda^k X)$, and a one-form $\eta \in \Gamma(T^*X)$ we put:

$$\eta \wedge (\sigma \otimes \omega) = \sigma \otimes (\eta \wedge \omega)$$

and extend linearly to the whole of the tensor product bundle.

Up to this point, we have treated the concepts of metric and connection independently of each other. We will now see that on the tangent bundle of a Riemannian manifold, the connection is unique if we require some additional properties. Formally, this yields a canonical way to turn a pair (X, g) into a triple (X, g, ∇) where ∇ is a connection on TX.

Definitions 2.2.22. A connection ∇ on TX is called **torsion-free** if the following holds for all vector fields V and W:

$$\nabla_V W - \nabla_W V = [V, W]$$

A connection ∇ on any smooth vector bundle (E, X, π) equipped with a metric $\langle \cdot, \cdot \rangle$ is called **metric** if the following holds for all $V \in \Gamma(TM)$ and $\sigma, \tau \in \Gamma(E)$:

$$V(\langle \sigma, \tau \rangle) = \langle \nabla_V \sigma, \tau \rangle + \langle \sigma, \nabla_V \tau \rangle$$

Remark 2.2.23. As a consequence of **2.2.20** we have seen that a given connection can be extended to all tensor types. Applying this to the metric tensor $g \in \Gamma(T^*X \otimes T^*X)$, one can check that the covariant derivative ∇g satisfies:

$$(\nabla g)(V, W, U) = U(g(V, W)) - g(\nabla_U V, W) - g(V, \nabla_U W)$$

for all vector fields U, V, W. Comparing this to the defining property of the metric connection yields that being metric is equivalent to:

$$\nabla g = 0$$

In general, any tensor T s.th. $\nabla T = 0$ holds is called *parallel*.

Theorem 2.2.24. The Levi-Civita connection
Let (X, g) be a Riemannian manifold. There is a unique torsion-free metric connection on the tangent bundle of X.

This connection is called **Levi-Civita connection** or sometimes **Riemannian connection**. The proof of this theorem is computational and will not be given here (see for example **[Jos]** 4.3.). It relies on the fact that the following equality, called *Koszul formula*:

$$2g(\nabla_U V, W) = U(g(V, W)) - W(g(U, V)) + V(g(U, W)) - g(U, [V, W]) + g(W, [U, W]) + g(V, [W, U])$$

can be shown to hold for any torsion free metric connection ∇. On the other hand, this formula uniquely defines ∇, since the right-hand side is independent of the connection and g is non-degenerate. Notice how the Koszul formula is specific to the context of $E = TX$, since cyclic permutations can only by carried out if all of the arguments are of the same type (i.e. vector fields).

2.3 Curvature

To introduce the concept of curvature, we will adopt a top-down approach, by starting with an abstract definition and subsequently introducing the more concrete curvature quantities (sectional, Ricci and scalar curvature). Obviously, we put special emphasis on the concept of scalar curvature.

Given a smooth vector bundle with a connection (E, X, π, ∇), using **2.2.21**, we can define the following:

Definition 2.3.25. Define the **curvature operator** associated to ∇:

$$\begin{array}{rcl} R^\nabla : \Gamma(E) & \longrightarrow & \Gamma(E \otimes \Lambda^2 X) \\ \sigma & \longmapsto & R^\nabla \sigma \end{array}$$

by composition of the following maps:

$$\Gamma(E) \xrightarrow{d^\nabla} \Gamma(E \otimes \Lambda^1 X) \xrightarrow{d^\nabla} \Gamma(E \otimes \Lambda^2 X)$$

This operator has surprising properties. Contrarily, to d^∇ (which is a derivation), R^∇ can be shown to be $C^\infty(X)$-linear, i.e. we have:

$$R^\nabla(f\sigma) = fR^\nabla\sigma$$

for all functions f and sections σ. Thus it can be viewed as a section of the following bundle:

$$R^\nabla \in \Gamma(E^* \otimes E \otimes \Lambda^2 X) \cong \Gamma(\mathrm{End}E \otimes \Lambda^2 X)$$

Furthermore, the following formula can be derived by using the definition of d^∇:

Proposition 2.3.26. For all $V, W \in \Gamma(TX)$ and $\sigma \in \Gamma(E)$, we have:

$$(R^\nabla\sigma)(V, W) = \nabla_V \nabla_W \sigma - \nabla_W \nabla_V \sigma - \nabla_{[V,W]}\sigma \in \Gamma(E)$$

Here, the (V, W)-argument is interpreted as decomposable bivector $V \wedge W \in \Lambda^2 TX$, therefore the formula makes sense.

Given a Riemannian manifold (X, g), we will now considerably reduce the level of abstraction by considering these concepts on the tangent bundle to X equipped with the Levi-Civita connection. Since this will leave no ambiguity concerning the choice of connection, we will denote the curvature operator simply by R. Since now we have $E = TX$ and by omitting some of the symmetries, R can just be viewed as a $(1, 3)$-tensor:

$$R \in \Gamma(\mathrm{End}TX \otimes \Lambda^2 X) \subset \Gamma(T^*X \otimes TX \otimes T^*X \otimes T^*X) \cong \Gamma(T^1_3 X)$$

By use of the isomorphism $TX \cong T^*X$ induced by the metric, we can associate to R a $(0,4)$-tensor (again denoted by R):

$$R(V,W,U,Y) = g(R_{V,W}U, Y)$$

(Where we have adopted the classical notation $R_{V,W}U = (RU)(V,W)$.) The resulting element from $\Gamma(T_4^0 X)$ has considerable symmetry (part of which we already know about, but have omitted to get to this point of view):

Proposition 2.3.27. Symmetries of R and Bianchi identities
For all vector fields V, W, U, Y, the following identities hold:

1) $R(V,W,U,Y) = -R(W,V,U,Y) = -R(V,W,Y,U)$
2) $R(V,W,U,Y) = R(U,Y,V,W)$
3) $R_{V,W}U + R_{U,V}W + R_{W,U}V = 0$ (1st Bianchi)
4) $(\nabla_U R)(V,W)Y + (\nabla_V R)(W,U)Y + (\nabla_W R)(U,V)Y = 0$ (2nd Bianchi)

By these symmetries, the $(0,4)$-version of R can be viewed as a well-defined section:

$$R \in \Gamma((\Lambda^2 TM)^* \otimes (\Lambda^2 TM)^*)$$

Next, we will turn to the curvature quantities derived from the curvature tensor, the first of which is sectional curvature. Sectional curvature associates a number to every 2-plane in the tangent space, in other words it is a function on the bundle of 2-Grassmannians of TX.

Definition 2.3.28. The **sectional curvature** at $x \in X$ of the plane $\pi \subset T_x X$ spanned by $v, w \in T_x X$ is defined by:

$$\sec_x(\pi) = -\frac{R(v \wedge w, v \wedge w)}{g(v \wedge w, v \wedge w)} = \frac{g(R_{v,w}w, v)}{g(v,v)g(w,w) - g(v,w)^2}$$

The inner product in the denominator of the first form of $\sec_x(\pi)$ is induced by g on $\Lambda^2 TX$ (see the Remark **2.2.17**). The sectional curvature is well-defined on the space of planes since the space of bivectors associated to π is one-dimensional. To gain some intuition on this concept, we highlight the following link to classical differential geometry of surfaces:

Remark 2.3.29. Sectional curvature is Gauss curvature of certain 2-submanifolds of X in the following sense: Let π be a plane in T_xX. In a neighbourhood of x, the image of π under the exponential map is a 2-submanifold of X:

$$\Sigma_x(\pi) = \exp_x(\pi) \cap U$$

Gauss curvature of this submanifold in x equals the sectional curvature:

$$\sec_x(\pi) = K_{\Sigma_x(\pi)}(x)$$

In particular, if X is two-dimensional, then Gauss curvature equals sectional curvature.

Although sec is derived from R, the following holds true:

Theorem 2.3.30. The sectional curvature (for all points and with respect to all planes) determines the curvature tensor R completely.

The next curvature quantity we will discuss is Ricci curvature.

Definition 2.3.31. **Ricci curvature** at $x \in X$ for the vectors $v, w \in T_xX$ is defined as follows:

$$\mathrm{Ric}_x(v, w) = \sum_{j=1}^{n} R(e_j, v, w, e_j)$$

(where $\{e_1, ..., e_n\}$ is an orthonormal basis of T_xX)

This definition is independent of the chosen orthonormal basis. Furthermore, one can easily show, by using the symmetries of R, that the Ricci tensor is a symmetric section:

$$\mathrm{Ric} \in \Gamma(T^*X \otimes T^*X) = \Gamma(T_2^0X)$$

Just as taking an appropriate trace of $R \in \Gamma(T_4^0X)$ yields $\mathrm{Ric} \in \Gamma(T_2^0X)$, taking the trace again will yield a $(0,0)$-tensor, i.e. a function on the manifold. This function is called scalar curvature:

Definition 2.3.32. **Scalar curvature** at $x \in X$ is defined as the following number:

$$\kappa(x) = \mathrm{tr}(\mathrm{Ric}_x) = \sum_{j=1}^{n} \mathrm{Ric}_x(e_j, e_j)$$

(where $\{e_1, ..., e_n\}$ is an orthonormal basis of T_xX)

Scalar curvature in $x \in X$ can be interpreted as the average of all sectional curvatures of the coordinate 2-planes of T_xX:

$$\begin{aligned}\kappa(x) &= \sum_{j=1}^{n} \mathrm{Ric}_x(e_j, e_j) \\ &= \sum_{i,j=1}^{n} R(e_i, e_j, e_j, e_i) \\ &= \sum_{i,j=1}^{n} \sec_x(\pi_{ij}) \\ &= 2\sum_{i<j} \sec_x(\pi_{ij})\end{aligned}$$

where $\pi_{ij} = \mathrm{span}(\{e_i, e_j\})$ for some orthonormal basis $\{e_1, ..., e_n\}$ of T_xX. In particular, scalar curvature is twice Gauss curvature in the case where X is a 2-manifold.

Remark 2.3.33. (see **[GHL]** p.139) To gain some geometric insight on what scalar curvature measures, we note that the scalar curvature at some point $x \in X$ is related to the volume growth of balls centred around x compared to balls in flat space. Denote by $B_r^X(x)$ the ball of radius r around x in the manifold X, then we have the following expansion:

$$\frac{\mathrm{vol}_g(B_r^X(x))}{\mathrm{vol}_n(B_r^{\mathbb{R}^n}(0))} = 1 - \frac{\kappa(x)}{6(n+2)}r^2 + O(r^4)$$

Where the balls in X are determined by the metric space-structure induced by the Riemannian metric (see the remarks after **2.2.18**). Notice how the minus in the second-order term makes sense since for example the volume of balls on the sphere is smaller than in the Euclidean case, whereas hyperbolic balls have more volume that Euclidean ones.

Remark 2.3.34. Scalar curvature plays a prominent role in General Relativity. It appears as the Lagrangian of the so-called *Einstein-Hilbert* action:

$$\mathcal{A}_{EH}(g) = \int_X \kappa_g dvol_g$$

from which the Einstein field equations can be derived using the principle of least action. Note that General Relativity is not exactly cast in the framework of Riemannian Geometry, since its associated bilinear form is Lorentzian, i.e. has signature $(-, +, +, +)$ (or the opposite depending on convention) instead of being positive definite. All involved definitions can easily be adapted to this context.

Chapter 3

Principal bundles

Smooth principal bundles are essentially locally trivial bundles where some Lie Group G serves as model space for the fibres and that is equipped with a sufficiently nice G-action. The archetypal example of such a bundle is the so-called frame bundle of a smooth manifold, which consists of all bases of the tangent space at a given point (this works in any given vector bundle). Its *structure group* is the general linear group, since the latter operates transitively on the bases of the vector space to which it is associated. Interestingly, a number of additional geometric structures on smooth manifolds (orientation, Riemannian metrics, complex structures, symplectic structures, etc.) can be viewed in a unified way as so-called reductions of the structure group of the frame bundle. In later chapters, this idea will we applied to the spin group in order to produce a bundle that has Spin_n as structure group. Our treatment is mainly based on **[KNI]**, with Appendix B of **[Fri]** serving as secondary reference.

3.1 Principal bundles

Definition 3.1.1. Let P, X be smooth manifolds, G a Lie group and $\pi : P \to X$ a smooth map. The 4-tuplet (P, X, G, π) is called **G-principal bundle** if the following hold:

1) G acts freely on the right on P by a smooth group action:

$$\begin{array}{rcl} P \times G & \longrightarrow & G \\ (p, g) & \longmapsto & pg = R_g p \end{array}$$

2) The map π is surjective and for all $p, q \in P$ we have $\pi(p) = \pi(q)$ if and only if there is a $g \in G$ s.th. $q = pg$,

3) Every point $x \in X$ has a neighbourhood $U \subset X$ s.th. $\pi^{-1}(U)$ is trivial in the following sense: There is a pair of maps (Φ, φ)

$$\begin{aligned} \Phi : \pi^{-1}(U) &\xrightarrow{\cong} U \times G \\ \varphi : \pi^{-1}(U) &\longrightarrow G \end{aligned}$$

where Φ is a diffeomorphism of the form $\Phi = \pi \times \varphi$ and φ is G-equivariant, i.e. $\varphi(pg) = \varphi(p)g$ for all $p \in P$ and $g \in G$.

For simplicity, we will denote principal bundles (P, X, G, π) often by their so-called *total space* P. X is called the *base space*, π the *projection map* and G the *structure group* of the principal bundle. The set $P_x = \pi^{-1}(\{x\})$ is called *fibre over* x and one can deduce from the definition that $\pi^{-1}(\{x\})$ is diffeomorphic to G. Furthermore, one can show that $X \cong P/G$ with projection map π.

Example 3.1.2. The trivial bundle $X \times G$
The bundle $(X \times G, X, G, \pi)$ equipped with the obvious right-action and the projection onto the first factor is a G-principle bundle called the trivial G-principal bundle over X.

Example 3.1.3. The frame bundle FE
Given a smooth rank-n vector bundle (E, X, π), we can associate to it the so-called frame bundle (FE, X, GL_n, π) in the following way. We construct fibres over every point $x \in X$:

$$FE_x = \{(v_1, ..., v_n) \in (E_x)^n \mid v_1, ..., v_n \text{ are linearly independent}\}$$

whose elements are called *frames*. The corresponding total space:

$$FE = \bigcup_{x \in X} FE_x$$

with the obvious projection π. The free GL_n-action is defined as follows:

$$\begin{aligned} FE \times GL_n &\longrightarrow FE \\ ((x, v_1, ..., v_n), G) &\longmapsto \left(x, \sum G_{i1} v_i, ..., \sum G_{in} v_i\right) \end{aligned}$$

This action is clearly transitive. To show that FE is locally trivial, fix a point $x_0 \in X$ and take a bundle chart on some neighbourhood $U \subset X$ of x_0. This chart yields (by trivialization) a canonical basis for $E|_U$, which we will denote by the vector fields $X_1, ..., X_n \in \Gamma(E|_U)$. Every frame $(v_1, ..., v_n)$ over U, can be expressed in these vector fields as follows:

$$v_i = \sum_{j=1}^{n} v_i^j X_j$$

for all $i \in \{1, ..., n\}$. We can now define Φ by:

$$\begin{aligned} \Phi : \pi^{-1}(U) = FE|_U &\longrightarrow U \times GL_n \\ (x, v_1, ..., v_n) &\longmapsto (x, (v_i^j)_{ij}) \end{aligned}$$

The matrix $(v_i^j)_{ij}$ is non-degenerate, since it transforms one basis into another by definition. Φ can easily be seen to be a diffeomorphism and φ can be defined in the obvious way. Thus we have indeed constructed a GL_n-principal bundle. We will see that this bundle is of fundamental importance for our purposes and in differential geometry in general.

Remark 3.1.4. An alternative view of fibre bundles can be extracted from the definition as follows. Choose an open covering of $X = \bigcup U_\alpha$, s.th. the $\pi^{-1}(U_\alpha)$ are trivial with trivializing maps $\Phi_\alpha = \pi \times \varphi_\alpha$. By the G-equivariance property, one can show that the following maps are well-defined (where the product comes from the group structure of G):

$$\varphi_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1} : U_\alpha \cap U_\beta \longrightarrow G$$

These are called *transition functions* and they obviously have the following property on $U_\alpha \cap U_\beta \cap U_\gamma$

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta}\varphi_{\beta\gamma} \tag{$\clubsuit$}$$

As it turns out, conversly, knowing X, G and the data $(U_\alpha, \varphi_{\alpha\beta})_{\alpha,\beta}$ is sufficient to construct a principal bundle with transition functions $\varphi_{\alpha\beta}$ provided the data fulfills ($\clubsuit$). See **[KNI]** I.5.2. for details.

Definition 3.1.5. Let (P, X, G, π) and (P', X', G', π') be two principal bundles. A **morphism of principal bundles** is a pair of maps $F : P' \to P$ smooth; $f : G' \to G$ a group morphism s.th. for all $p \in P'$ and $g \in G'$:

$$F(pg) = F(p)f(g)$$

i.e. the following diagram commutes:

$$\begin{tikzcd} P' \times G' \arrow[r] \arrow[d, "F \times f"] & P' \arrow[d, "F"] \\ P \times G \arrow[r] & P \end{tikzcd}$$

Isomorphisms are defined correspondingly.

Since the respective structure groups act transitively on fibres, morphisms preserve fibres. In particular, they descend to the base spaces. This can be expressed by the following diagram:

$$\begin{tikzcd} P' \arrow[r, "F"] \arrow[d, "\pi'"] & P \arrow[d, "\pi"] \\ X' \arrow[r, "\underline{F}"] & X \end{tikzcd}$$

For simplicity, we will denote all maps of this triple $(F, \underline{F}, f)$ by the same letter.

Next, we will discuss so-called *reductions of the structure group*, which are of crucial importance in spin geometry.

Definition 3.1.6. Given two fibre bundles over the same base space (P', X, G', π') and (P, X, G, π) and a Lie group morphism $f : G' \to G$, an **f-reduction (of the structure group)** is a morphism of principal bundles (F, f) (i.e. the morphism on the structure groups is the prescribed f) so that F leaves the base space invariant: $\underline{F} = id_X$. This corresponds to the following commutative diagram:

$$\begin{tikzcd} P' \arrow[rr, "F"] \arrow[dr, "\pi'"] & & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

In the classical case, this definition is used if F is an embedding (in addition to the above requirements), i.e. $F : P' \to P$ an embedding and $f : G' \to G$ a group monomorphism. In the case where such a morphism exists, one says that *the structure group G can be reduced to G'*. This terminology is justified by the following result: (see **[KNI]** I.5.3.)
There is a reduction of the structure group G to $G' < G$ (Lie subgroup) if and only if there is a covering of X and transitions functions that have values in G' (See **3.1.4**).

We need the more general definition **3.1.6**, since we want to discuss Spin-principal bundles as reductions of SO-principal bundles. Semantically speaking, this is not a *reduction* of the structure group (nor would it be according to the classical definition discussed above), since Spin double-covers SO. It is nonetheless an occurrence of a structure group reduction as we defined it.

Example 3.1.7. The orthonormal frame bundle $F_O E$
Let (E, X, π) be a smooth vector bundle of rank n equipped with a bundle metric g. Similarly to **3.1.3**, we define the fibres of $F_O E$ to be:

$$F_O E_x = \{(v_1, ..., v_n) \in (E_x)^n \mid g_x(v_i, v_j) = \delta_{ij}\}$$

Which means that we require the frames to be orthonormal w.r.t. the Riemannian metric. In the same way as for FE, this can be checked to be an O_n-principal bundle. Let $f : O_n \hookrightarrow GL_n$ denote the canonical inclusion. $F_O E$ is the f-reduction of FE where we take as bundle morphism the natural fibre-wise inclusion:

$$F_O E \hookrightarrow FE$$

Since every orthonormal frame is a frame. This map obviously descends to the identity on the base space. This shows how the choice of a Riemannian metric leads to the reduction of the structure group from GL_n to O_n. Conversely, one can show that every choice of an f-reduction of FE induces a Riemannian metric on X (see **KNI** I.5.7.).

Example 3.1.8. The oriented frame bundle F_+E
Let (E, X, π) be an oriented vector bundle of rank n. Define the fibres of F_+E to be:

$$F_+E_x = \{(v_1, ..., v_n) \in (E_x)^n \mid v_1, ..., v_n \text{ is a basis with the preferred orientation}\}$$

This yields an GL_n^+-principal bundle. For the canonical inclusion $f : GL_n^+ \hookrightarrow GL_n$, F_+E is an f-reduction of FE. Again, the converse holds as well: (see [**Fri**] Appendix B.1.) Every f-reduction of FE induces an orientation on X.

Next, we discuss sections of principal bundles:

Definition 3.1.9. For a principal bundle (P, X, G, π) a smooth map $\sigma : X \to P$ is called **smooth section of** P if it is a right-inverse to π, i.e. if the following holds:

$$\pi \circ \sigma = id_X$$

Similarly to **2.1.15**, one can show the following criterion for trivial bundles:

Proposition 3.1.10. The bundle (P, X, G, π) is trivial (i.e. isomorphic to $X \times G$) if and only if it admits a global section.

This might seem surprising at first, since triviality is a strong requirement. But consider the following:

Example 3.1.11. The proposition infers that FE is trivial if and only if there is a global frame. But in terms of the vector bundle E this means precisely that there are n sections (of the rank n-vector bundle E) whose values are linearly independent in every fibre. Thus the triviality of the frame bundle (as principal bundle) is equivalent to the triviality of E as vector bundle (see **2.1.15**).

3.2 Associated bundles

The so-called associated bundle construction provides a way to associate vector bundles to a given principal bundle. The data needed to carry out this construction is the initial fibre bundle (P, X, G, π) and a representation of G in the general linear group of some vector space V, i.e. a homomorphism:

$$\rho : G \longrightarrow GL(V)$$

The resulting vector bundle will have copies of V as fibres. This construction is not restricted to vector spaces (see **[KNI]** I.5.), but this special case is sufficient for our purposes.

Under the above assumptions, G has the following free action on the product $P \times V$:

$$\begin{array}{rcl} G \times P \times V & \longrightarrow & P \times V \\ (g, p, v) & \longmapsto & (pg^{-1}, \rho(g)v) \end{array}$$

The quotient of this action will be denoted by $P \times_\rho V$ and can be checked to be a smooth vector bundle. The projection map π_ρ is defined by the following diagram:

$$\begin{tikzcd} P \times V \arrow[r, "\pi_1"] \arrow[d] & P \arrow[r, "\pi"] & X \\ P \times_\rho V \arrow[urr, "\pi_\rho"] & & \end{tikzcd}$$

where the vertical map is the canonical projection onto the quotient and π_1 is projection onto the first factor.

Definition 3.2.12. We call the so-obtained $(P \times_\rho V, X, \pi_\rho)$ the **vector bundle associated to** P via the representation ρ.

To illustrate this concept, consider the following examples:

Examples 3.2.13. tensor bundles as associated bundles

1) $TX = FX \times_{\rho_n} \mathbb{R}^n$ The tangent bundle is associated to the frame bundle via the canonical representation $\rho_n : GL_n \to GL(\mathbb{R}^n)$;

2) $T^*X = FX \times_{\rho_n^*} \mathbb{R}^n$ where $\rho_n^*(g) = \rho_n(g^{-1})^T$ denotes the dual representation;

3) $T_s^r X = FX \times_{T_s^r \rho_n} T_s^r \mathbb{R}^n$ where $T_s^r \rho_n$ is induced by the previous representations.

For a detailed discussion of how to view all tensor bundles as derived objects from the frame bundle, see **[Pet98]** Appendix B.4.

3.3 Connections on principal bundles

Let (P, X, G, π) be a smooth principal bundle. As the total space is locally a product space by definition, it is natural to ask whether the same holds for the tangent spaces. As it turns out, there is no such canonical decomposition. A connection is the *choice* of one such splitting. By G_p we will denote the subspace of vectors tangent to the fibre over p, i.e.

$$G_p = \left\{ v \in T_pP \,\middle|\, \exists \gamma : (-\varepsilon, \varepsilon) \to G \subset P_{\pi(p)} \text{ s.th. } v = \left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right\}$$

Although the inclusion $G \subset P_{\pi(p)}$ depends on the chosen trivialization around p, the existence of such a γ can easily be checked to be independent of this choice. Furthermore, one can show:

$$G_p \cong \mathfrak{g} \tag{$\clubsuit$}$$

isomorphic, by associating to every vector in the Lie Algebra $\mathfrak{g}$ its left invariant vector field in the corresponding fibre. For every $V \in \mathfrak{g}$, denote the corresponding vector field on P by V^*.

Definition 3.3.14. A **connection on** P is a smooth assignment $p \mapsto Q_p$ of subspaces $Q_p \subset T_pP$ for all $p \in P$ s.th. the following hold:

1) The tangent space of the total space splits: $T_p = G_p \oplus Q_p$,

2) The assignment behaves as follows under the action of G: $Q_{pg} = (R_g)_*Q_p$ for all $g \in G$.

The subspaces G_p are called *vertical*, since they correspond to the fibre-part of the tangent space, whereas the manifold-components Q_p are called *horizontal*. A related concept that contains the same information is the following one:

Definition 3.3.15. A **connection one-form on** P is an $\mathfrak{g}$-valued one-form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ s.th.

1) $\omega(V^*) = V$ for all $V \in \mathfrak{g}$,

2) $(R_g)^*\omega = \mathrm{ad}(g^{-1})\omega$ for all $g \in G$.

These concepts are equivalent in the following sense: (see **[KNI]** II.1.1.)

Proposition 3.3.16. Connections on P and connection one-forms on P are in bijective correspondence.

Given a connection on P, the corresponding form can be obtained by using the splitting **3.3.14.**1) to get a projection $T_pP \to G_p$ that yields ω by composing with the isomorphism (♣). Conversely, given a connection one-form ω, the distribution of tangent subspaces defined by $Q_p = \ker\omega_p$ is a connection. Note how this is geometrically consistent with viewing ω as the projection onto the vertical component of the tangent bundle of P.

We will discuss the connections of one bundle in particular, namely the bundle of oriented orthonormal frames. For this, let (E, X, π) be an oriented smooth vector bundle of rank $n > 2$ (see **[LaMi]** II.§1. for the other cases) with bundle metric g. The first thing we notice is that the structure group can be reduced from GL_n to SO_n by the presence of a metric and an orientation. Since $SO_n = O_n \cap GL_n^+$, this can be achieved by consecutively applying Examples **3.1.7** and **3.1.8** to FE. This yields the bundle of oriented orthonormal frames, which we will denote by $(F_{SO}E, X, SO_n, \pi)$. The Lie algebra associated to SO_n is the algebra of skew-symmetric $n \times n$-matrices, denoted by $\mathfrak{so}(n)$. The connection form is thus an $\mathfrak{so}(n)$-valued one-form or, equivalently, a skew-symmetric matrix of one-forms.

Connections of $F_{SO}E$ are strongly related to metric connections on (the vector bundle) E (see 2.2.19 and 2.2.22):

Proposition 3.3.17. Every connection 1-form ω on $F_{SO}E$ defines a unique metric connection on E and vice-versa. This correspondence is given by the following equation:

$$\nabla\sigma_i = \sum_{i=1}^{n} (\sigma^*\omega)_{ji} \otimes \sigma_j \tag{♠}$$

Where $\sigma_1, ..., \sigma_n \in \Gamma(E|_U)$ a pointwise positively oriented orthonormal basis on a coordinate neighbourhood U. Equivalently, $\sigma = (\sigma_1, ..., \sigma_n)$ is a section of $F_{SO}E|_U$.

The equation (♠) makes sense: Recall that $\nabla\sigma_i \in \Gamma(T^*U \otimes E)$ and here $\omega \in \Gamma(T^*F_{SO}E \otimes \mathfrak{so}(n))$, but $\sigma : U \to F_{SO}E|_U$, thus $(\sigma^*\omega)_{ji} \in \Gamma(T^*U)$. Moreover, one can show that this determines ∇ independently of the choice of the orthonormal frame σ. For a thorough discussion of the frame-dependent vector bundle-based approach and its invariant counterpart in fibre bundles, we refer to **[Pet98]** Appendix B.

In the case of $E = TX$, where we will denote the bundle of oriented orthonormal frames by $F_{SO}X$, **3.3.17** can be applied to the Levi-Civita connection (see **2.2.24**) to yield:

Proposition 3.3.18. Levi-Civita connection on $F_{SO}X$
There is a unique connection on $F_{SO}X$ so that its associated connection on E (by **3.3.17**) is torsion-free.

Remark 3.3.19. Connections on principal bundles induce unique connections on their associated vector bundles. For a short discussion of a special case, see [**LaMi**] II.§4.7. For a general treatment, see [**KNI**] Chapter III.

Chapter 4

Overview of Spin Geometry

Spin structures are certain reductions of the structure group of the frame bundle of a vector bundle to the spin group. Based on representations of the spin group, one can associate vector bundles to these principal Spin_n-bundles, called spinor bundles. These turn out to have an interesting Clifford-module structure which allows us to define a natural first-order differential operator, namely the Dirac operator. This construction is essential in the context of positive curvature geometry, since applying Bochner's method to the Dirac operator yields scalar curvature as difference between the square of the Dirac operator and the connection Laplacian. At the end, we discuss twisted spinor bundles, which are needed in **7.3**. **[LaMi]** served as main source for this chapter, **[Fri]** as secondary reference.

4.1 Spin structures and spin manifolds

Let $(F_{SO}E, X, SO_n, \pi)$ be the bundle of orthonormal frames associated to an oriented vector bundle E equipped with a bundle metric. Recall that Spin is the universal double-cover of SO (see **[LaMi]** I.§2.). Thus we have a Lie group morphism:

$$\lambda : \mathrm{Spin}_n \longrightarrow SO_n$$

Definition 4.1.1. The bundle $(F_{\mathrm{Sp}}E, X, \mathrm{Spin}_n, \pi')$ is called **spin structure** on E if it is a λ-reduction of $(F_{SO}E, X, SO_n, \pi)$ s.th. the associated bundle map

$$\Lambda : F_{\mathrm{Sp}}E \longrightarrow F_{SO}E$$

is a double-covering map.

By Definitions **3.1.5** and **3.1.6**, this construction can be summed up by the following commutative diagram:

$$\begin{tikzcd}
F_{\mathrm{Sp}}E \times \mathrm{Spin}_n \arrow[rr, "\Lambda\times\lambda"] \arrow[d] & & F_{SO}E \times SO_n \arrow[d] \\
F_{\mathrm{Sp}}E \arrow[rr, "\Lambda"] \arrow[dr, "\pi'"] & & F_{SO}E \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

A spin structure doesn't always exist. We mention the following topological constraint without discussing its proof (see **[LaMi]** II.§1. for details):

Theorem 4.1.2. Existence of spin structures on vector bundles
Let (E, X, π) be an oriented vector bundle with a bundle metric. There is a spin structure on E if and only if the second Stiefel-Whitney class vanishes: $w_2(E) = 0$.

For a short discussion of Stiefel-Whitney classes, see **5.1**.

As is often the case, one important special case of the above is $E = TX$:

Definition 4.1.3. Let X be an oriented Riemannian manifold. X is called **spin manifold** if there is a spin structure on TX.

Notice that this makes sense, since orientation and Riemannian structures for manifolds are exactly defined as orientation on TX and the existence of a bundle metric on TX respectively. Moreover, since by definition $w(X) = w(TX)$, the application of **4.1.2** directly yields:

Proposition 4.1.4. Existence of spin structures on manifolds
The oriented Riemannian manifold X is spin if and only if its second Stiefel-Whitney class vanishes: $w_2(X) = 0$.

4.2 Clifford and Spinor bundles

Using the associated bundle construction (see **3.2**), we will construct two bundles that are fundamental ingredients of spin geometry. The first one is called Clifford bundle and occurs as associated to the frame bundle. The second one on the other hand is associated to spin structures and its sections will be the so-called spinors. After their definition, we will discuss how they are interrelated by a module structure.

Let (E, X, π, g) be an oriented smooth rank n vector bundle equipped with a metric. The frame bundle can thus be reduced to $F_{SO}E$. Recall that every orthogonal transformation of $\mathbb{R}^n$ induces a unique automorphism of $C\ell(\mathbb{R}^n)$ (see **[LaMi]** I.§1.). This can be viewed as a representation:

$$c\ell(\rho_n) : SO_n \longrightarrow \mathrm{Aut}(C\ell(\mathbb{R}^n))$$

Definition 4.2.5. The **Clifford bundle** associated to (E, X, π, g) is the vector bundle $(C\ell(E), X, \pi')$ defined as follows:

$$C\ell(E) = F_{SO}E \times_{c\ell(\rho_n)} C\ell(\mathbb{R}^n)$$

Contrarily to the spinor bundles, the only topological obstruction to this construction is orientation. We get a bundle of algebras, i.e. every fibre of the obtained bundle is a Clifford algebra:

$$C\ell(E)_x = C\ell(E_x, \|\cdot\|_x)$$

For the construction of spinor bundles, one needs the following data: A vector bundle E as before with additionally $w_2(E) = 0$ s.th there is a spin structure $F_{\mathrm{Sp}}E$ and a $C\ell(\mathbb{R}^n)$-left module denoted by M. This module structure induces a metric on M with respect to which Clifford multiplication by vectors is orthogonal (see **[LaMi]** I.§5.)

Definitions 4.2.6. A **real spinor bundle** is associated to the spin structure $F_{\mathrm{Sp}}E$ by

$$S(E) = F_{\mathrm{Sp}}E \times_\mu M$$

where $\mu : \mathrm{Spin}_n \to SO(M)$ is the representation canonically yielded by the module structure of M, i.e. for every $\xi \in \mathrm{Spin}_n$ and $m \in M$, define:

$$\mu(\xi)m = \xi \cdot m$$

If $M_{\mathbb{C}}$ is a complex $C\ell(\mathbb{R}^n) \otimes \mathbb{C}$-module, define similarly **complex spinor bundles**:

$$S_{\mathbb{C}}(E) = F_{\mathrm{Sp}}E \times_\mu M_{\mathbb{C}}$$

Another way to view μ is as the composition of $\mathrm{Spin}_n \hookrightarrow C\ell_n \to \mathrm{End}(M)$, where the second comes from the module structure of M.

Example 4.2.7. As the most basic example, consider $M = C\ell(\mathbb{R}^n)$ for which μ can obviously just be chosen as Clifford left-multiplication denoted by ℓ:

$$C\ell_{\mathrm{Sp}}(E) = F_{\mathrm{Sp}}E \times_\ell C\ell(\mathbb{R}^n)$$

Proposition 4.2.8. The module structure of $S(E)$
The real spinor bundle $S(E)$ is a module over $C\ell(E)$ in the sense that μ descends to the quotient:

$$\begin{array}{ccc} F_{\mathrm{Sp}}E \times C\ell(E) \times M & \xrightarrow{\mu} & F_{\mathrm{Sp}}E \times M \\ \downarrow & & \downarrow \\ C\ell(E) \times S(E) & \xrightarrow{\mu} & S(E) \end{array}$$

Proof: It suffices to prove that μ is invariant under the operation of Spin_n on $F_{\mathrm{Sp}}E \times M$, i.e. that the following diagram commutes for all $\zeta \in \mathrm{Spin}_n$:

$$\begin{array}{ccc} F_{\mathrm{Sp}}E \times C\ell_n \times M & \xrightarrow{\mu} & F_{\mathrm{Sp}}E \times M \\ \downarrow{\scriptstyle \zeta\star} & & \downarrow{\scriptstyle \zeta\star} \\ F_{\mathrm{Sp}}E \times C\ell_n \times M & \xrightarrow{\mu} & F_{\mathrm{Sp}}E \times M \end{array}$$

Where the operation by ζ is denoted by $\zeta\star$. Recall that $C\ell_n$ is viewed here as $C\ell_n \to \mathrm{End}(M)$, thus the $C\ell_n$ factor is acted upon by conjugation. Compute:

$$(\mu \circ \zeta\star)(p, \xi, m) = \mu(p\zeta^{-1}, \zeta\xi\zeta^{-1}, \zeta m) = (p\zeta^{-1}, \zeta\xi m)$$

On the other hand:

$$((\zeta\star) \circ \mu)(p, \xi, m) = (\zeta\star)(p, \xi m) = (p\zeta^{-1}, \zeta\xi m)$$

Thus the diagram commutes.

□

A similar result holds in the complex case. Since we now have a map between bundles:

$$\mu : C\ell(E) \times S(E) \longrightarrow S(E)$$

This yields a map between corresponding sections:

$$\mu : \Gamma(C\ell(E)) \times \Gamma(S(E)) \longrightarrow \Gamma(S(E))$$

defined as:

$$\mu(\sigma, \tau)|_x = \mu(\sigma|_x, \tau|_x)$$

For any two sections σ and τ of the respective bundles, their product $\mu(\sigma, \tau)$ will be just denoted by $\sigma \cdot \tau$.

Next, we will briefly discuss connections on Clifford and spinor bundles. Let (E, X, π) be an oriented vector bundle with a bundle metric and a spin structure $\Lambda : F_{\mathrm{Sp}}E \to F_{SO}E$ with some spinor bundle $S(E)$ (the part on connections on Clifford bundles is obviously independent of any spin structures). Furthermore, we fix a connection on the bundle $F_{SO}E$. By **3.3.19**, this induces a connection on $C\ell(E)$, denoted simply by ∇. This ∇ can be shown to be well-behaved with respect to Clifford multiplication in the following sense:

Proposition 4.2.9. For any two $\sigma, \tau \in \Gamma(C\ell(E))$, the following product rule holds:

$$\nabla(\sigma \cdot \tau) = (\nabla\sigma) \cdot \tau + \sigma \cdot (\nabla\tau)$$

For the spinor bundle, we first lift the connection on $F_{SO}E$ to the double-covering $F_{\mathrm{Sp}}E$ via the map Λ. Again by **3.3.19**, this yields a connection on the associated spinor bundle $S(E)$, which is again just denoted by ∇. Since these two connections stem from the same connection on $F_{SO}E$, one can show that they are compatible with the module structure in the following sense:

Proposition 4.2.10. For any two $\sigma \in \Gamma(C\ell(E))$, $\tau \in \Gamma(S(E))$ the following product rule holds:

$$\nabla(\sigma \cdot \tau) = (\nabla\sigma) \cdot \tau + \sigma \cdot (\nabla\tau)$$

Notice that the above formula contains two different connections, although this is suppressed in the notation. The point in all of this is that there is one *basic* object, namely the connection on the frame bundle $F_{SO}E$, from which, once it is chosen, one canonically derives the connections on $C\ell(E)$ and on any spinor bundle $S(E)$. In particular, if we set $E = TX$, everything is canonical, since the natural choice on the frame bundle is the Levi-Civita connection (see **3.3.18**).

4.3 The Dirac operator and its Bochner formula

Let X be a spin n-manifold with a given spin structure on its tangent bundle. By $C\ell(X)$ we denote the Clifford bundle of TX. Furthermore, we take some spinor bundle $S(X)$, stemming from TX as well. By the discussion in **4.2**, we take the canonical connections (both denoted by ∇) inherited from the Levi-Civita connection. Under these circumstances, we can define the Dirac operator:

Definition 4.3.11. The **Dirac Operator** is defined as follows:

$$\begin{array}{rcl} D:\Gamma(S(X)) & \longrightarrow & \Gamma(S(X)) \\ \sigma & \longmapsto & \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\sigma \end{array}$$

where $e_1, ..., e_n$ is an orthonormal basis of the tangent space.

The definition is independent of the chosen basis: Let $\tilde{e_1}, ..., \tilde{e_n}$ be another orthonormal basis of the given tangent space. We then have (using $G \in O(n)$):

$$\tilde{e_j} = \sum_k G_{jk} e_k$$

for some $G \in O(n)$. Let $\tilde{D}$ denote the Dirac Operator defined by this second basis. For all $\sigma \in \Gamma(S(X))$, we have:

$$\begin{aligned} \tilde{D}\sigma &= \sum_j \tilde{e_j} \cdot \nabla_{\tilde{e_j}}\sigma \\ &= \sum_j \left(\sum_k G_{jk} e_k \right) \cdot \nabla_{\sum_l G_{jl} e_l}\sigma \\ &= \sum_{j,k,l} G_{jk} G_{jl} e_k \cdot \nabla_{e_l}\sigma \\ &= \sum_{k,l} \delta_{kl} e_k \cdot \nabla_{e_l}\sigma \\ &= D\sigma \end{aligned}$$

The Dirac operator is a differential operator of order one in the sense of **6.1.1**. Furthermore, one can compute the symbol of D and D^2 (see **6.1.2** for the definition of the symbol):

Proposition 4.3.12. The symbols of D and D^2

$$\begin{aligned}\sigma_\xi(D) &= i\xi\cdot \\ \sigma_\xi(D^2) &= \|\xi\|^2\end{aligned}$$

Thus the Dirac Operator and its square are elliptic (see **6.1.5**).

Furthermore, the Dirac operator is formally self-adjoint. Given a bundle metric on $\Gamma(S(X))$, we can define an L^2-inner product on $\Gamma(S(X))$ (more specifically for compactly supported sections of $S(X)$) by defining:

$$\langle\sigma_1,\sigma_2\rangle_{L^2} = \int_X \langle\sigma_1,\sigma_2\rangle dvol$$

Proposition 4.3.13. (see **[LaMi]** II.§5.) The Dirac operator is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{L^2}$, i.e. for all compactly supported sections $\sigma_1,\sigma_2\in\Gamma(S(X))$ the following holds:

$$\langle D\sigma_1,\sigma_2\rangle_{L^2} = \langle\sigma_1,D\sigma_2\rangle_{L^2}$$

The following is a direct consequence: (assuming finite-dimensionality, which is given by the fact that D and D^2 are elliptic)

Proposition 4.3.14.

$$\ker D = \ker D^2$$

Before discussing the Bochner identity, we need to define the connection Laplacian. Let (E,X,π,∇) be a smooth vector bundle with bundle metric and connection.

Definition 4.3.15. For any two vector fields $V,W\in\Gamma(TX)$, define the **invariant second derivative**:

$$\begin{aligned}\nabla^2_{V,W}:\Gamma(E) &\longrightarrow \Gamma(E) \\ \sigma &\longmapsto \nabla^2_{V,W}\sigma = \nabla_V\nabla_W\sigma - \nabla_{\nabla_V W}\sigma\end{aligned}$$

Where we used the Levi-Civita connection on TX.

One can show that $\nabla^2\sigma$ is tensorial in both arguments, i.e. that $\nabla^2\sigma \in \Gamma(T^*X \otimes T^*X \otimes E)$. Furthermore, the invariant second derivative is related to the curvature operator of E by:

$$\nabla^2_{V,W} - \nabla^2_{W,V} = R_{V,W} \tag{$\blacklozenge$}$$

Definition 4.3.16. The **connection Laplacian** is defined as the negative trace of the invariant second derivative:

$$\begin{aligned} \nabla^*\nabla : \Gamma(E) &\longrightarrow \Gamma(E) \\ \sigma &\longmapsto -\sum_{j=1}^{n} \nabla^2_{e_j,e_j}\sigma \end{aligned}$$

for a basis orthonormal basis $e_1, ..., e_n$ of the tangent space.

The connection Laplacian has the same symbol as the square of the Dirac operator and is thus elliptic. Furthermore it can be shown to be symmetric and formally self-adjoint. This implies that $\nabla^*\nabla$ is positive in the following sense:

$$\langle \nabla^*\nabla\sigma, \sigma\rangle_{L^2} = \langle \nabla\sigma, \nabla\sigma\rangle_{L^2} = \|\nabla\sigma\|^2_{L^2} \geq 0$$

We return to the initial setting of a given spin n-manifold, over which we take some spinor bundle $S(X)$ endowed with its connection coming from the Levi-Civita connection on TX. This yields a corresponding connection Laplacian, which is linked to the square of the Dirac operator by the following identity (which holds in a more general context, see **[LaMi]** II.§8.2.):

Theorem 4.3.17. Bochner identity

$$D^2 = \nabla^*\nabla + \mathfrak{R}$$

Where $\mathfrak{R}$ is defined in terms of the curvature operator of $S(E)$ (denoted by R^s) by:

$$\mathfrak{R}\sigma = \frac{1}{2}\sum_{i,j}^{n} e_i \cdot e_j \cdot R^s_{e_i,e_j}\sigma$$

We have three operators of the same type:

$$D^2, \nabla^*\nabla, \mathfrak{R} : \Gamma(S(X)) \longrightarrow \Gamma(S(X))$$

and thus the formula makes sense. Furthermore, one can easily check that $\mathfrak{R}$ is linear as opposed to the other two operators.
Proof: (Bochner identity)
Around a given point $x \in X$, choose a local orthonormal frame field $E_1, ..., E_n \in \Gamma(TX|_U)$ s.th. $\nabla E_j|_x = 0$ for all indices j. Then we can compute at $x \in X$:

$$\begin{aligned}
D^2\sigma &= \sum_i E_i \cdot \nabla_{E_i}\left(\sum_j E_j \cdot \nabla_{E_j}\sigma\right) \\
&= \sum_{i,j} E_i \cdot E_j \cdot \nabla_{E_i}\nabla_{E_j}\sigma \\
&\overset{\nabla E_j|_x=0}{=} \sum_{i,j} E_i \cdot E_j \cdot \nabla^2_{E_i,E_j}\sigma \\
&= -\sum_i \nabla^2_{E_i,E_i}\sigma + \sum_{i<j} E_i \cdot E_j \cdot \left(\nabla^2_{E_i,E_j}\sigma - \nabla^2_{E_j,E_i}\sigma\right) \\
&\overset{(\blacklozenge)}{=} \nabla^*\nabla\sigma + \mathfrak{R}\sigma
\end{aligned}$$

□

The proof thus shows that **4.3.17** reflects the decomposition of D^2 into a *trace component* $\nabla^*\nabla$ and an *off-trace component* $\mathfrak{R}$.

In our specific case (see the setup in **7.2**), the curvature term $\mathfrak{R}$ can be identified with one fourth of scalar curvature (see **2.3.32**). This remarkable fact is known as *Lichnerowicz formula.*

Theorem 4.3.18. Lichnerowicz formula

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}\kappa$$

Proof:
We need some facts on the involved curvature operators. By R we denote the curvature operator of the tangent bundle TX and by R^s the curvature operator of the spinor bundle $S(X)$. These are related by the following formula (see **[LaMi]** II.§4.15.):

$$R^s_{V,W}\sigma = \frac{1}{2}\sum_{i<j}\langle R_{V,W}e_i, e_j\rangle\, e_i \cdot e_j \cdot \sigma \qquad (\star)$$

where $V, W \in T_xX$ and $e_1, ..., e_n \in T_xX$ an orthonormal basis, if we consider the equation in the point $x \in X$. Another crucial ingredient are the following two identities which are specific to the context where the Levi-Civita connection is chosen on TX (see **2.3.27**.2) and 3)):

$$\begin{aligned}
\langle R_{V,W}U, Y\rangle - \langle R_{U,Y}V, W\rangle &= 0 \\
R_{V,W}U + R_{U,V}W + R_{W,U}V &= 0
\end{aligned}$$

for all tangent vectors V, W, U, Y at all base points. Let x be a point on the base manifold X and $e_1, ..., e_n$ an orthonormal basis of T_xX. Using **4.3.17**, we only need to identify the curvature term $\mathfrak{R}$:

$$\begin{aligned}
\mathfrak{R}\sigma &= \frac{1}{2}\sum_{i,j} e_i \cdot e_j \cdot R^s_{e_i,e_j}\sigma \\
&\overset{(\star)}{=} \frac{1}{8}\sum_{i,j,k,l} \langle R_{e_i,e_j}e_k, e_l\rangle e_i \cdot e_j \cdot e_k \cdot e_l \cdot \sigma
\end{aligned}$$

This fourfold sum can be separated in terms where i, j, k are distinct:

$$\frac{1}{8}\sum_l \left(\frac{1}{3}\sum_{i,j,k \text{ distinct}} \langle R_{e_i,e_j}e_k + R_{e_k,e_i}e_j + R_{e_j,e_k}e_i, e_l\rangle e_i \cdot e_j \cdot e_k\right) \cdot e_l \cdot \sigma$$

And in terms where at least two of the indices i, j, k coincide. The first term vanishes, due to the first Bianchi identity (see above or **2.3.27**.3)). The latter term can be rewritten as:

$$\begin{aligned}
&\frac{1}{8}\sum_l \left(\sum_{i,j}\langle R_{e_i,e_j}e_i, e_l\rangle e_i \cdot e_j \cdot e_i + \sum_{i,j}\langle R_{e_i,e_j}e_j, e_l\rangle e_i \cdot e_j \cdot e_j\right) \cdot e_l \cdot \sigma \\
= \;&\frac{1}{8}\sum_l \left(\sum_{i,j}\langle R_{e_i,e_j}e_i, e_l\rangle e_j - \sum_{i,j}\langle R_{e_i,e_j}e_j, e_l\rangle e_i\right) \cdot e_l \cdot \sigma
\end{aligned}$$

since $e_i \cdot e_j \cdot e_i = e_j$ and $e_i \cdot e_j \cdot e_j = -e_i$. Thus (recalling **2.3.31**) we have:

$$\begin{aligned}
\mathfrak{R}\sigma &= \frac{1}{4}\sum_{i,j,l}\langle R_{e_i,e_j}e_i, e_l\rangle e_j \cdot e_l \cdot \sigma \\
&= -\frac{1}{4}\sum_{j,l}\mathrm{Ric}(e_j, e_l) e_j \cdot e_l \cdot \sigma \\
&= \frac{1}{4}\kappa\sigma
\end{aligned}$$

□

Remark 4.3.19. The Lichnerowicz formula yields a lot of information about spin manifolds with positive scalar curvature. More specifically, it implies that $\ker \not{D} = \{0\}$ in the presence of positive scalar curvature by a short computation and by positivity of the connection Laplacian. Via the Atiyah-Singer index theorem, this yields the vanishing of the so-called $\hat{A}$-genus. See **5.5** for the definition of the $\hat{A}$-genus and **7.2** for a discussion on how it relates to positive scalar curvature.

4.4 Twisted spinor bundles

For our purposes, a slight generalization of the Lichnerowicz formula is required. Let X be a compact Riemannian spin manifold and $S(X)$ one of its spinor bundles endowed with the connection inherited from the Levi-Civita connection. Additionally, let E be a smooth vector bundle over X, equipped with a bundle metric and a metric connection. We want to consider the product bundle $S(X) \otimes E$, on which Clifford multiplication is defined as follows:

$$\xi \cdot (\sigma \otimes \tau) = (\xi \cdot \sigma) \otimes \tau$$

for all $\xi \in C\ell(X)$, $\sigma \in S(X)$ and $\tau \in E$. Furthermore, we equip the product bundle with the product metric and the product connection, i.e. for the given connections ∇^S on $S(X)$ and ∇^E on E, define:

$$\nabla(\sigma \otimes \tau) = (\nabla^S \sigma) \otimes \tau + \sigma \otimes (\nabla^E \tau)$$

The associated connection Laplacian will be denoted by $\nabla^* \nabla$, the Dirac operator by D_E. The appropriate modification for the Lichnerowicz formula is exactly what one might expect: The general Bochner curvature term splits in two parts, the first corresponding to the spinor-part of the bundle and thus turning out to be one fourth of scalar curvature, the other one corresponding to the E-part in the product bundle and thus depending on the curvature of E.

Theorem 4.4.20. Lichnerowicz formula for twisted spinor bundles

$$D_E^2 = \nabla^* \nabla + \frac{1}{4}\kappa + \mathfrak{R}^E$$

Where the last summand is defined by:

$$\mathfrak{R}^E(\sigma \otimes \tau) = \frac{1}{2} \sum_{i,j}^{n} (e_i \cdot e_j \cdot \sigma) \otimes R^E_{e_i, e_j} \tau$$

for all $\sigma \in S(X)$, $\tau \in E$ and extended linearly to all elements of the tensor product.

Proof: One can show that the curvature operator of the product connection (simply denoted by R) operates as follows on tensor products of simple type:

$$R(\sigma \otimes \tau) = (R^S \sigma) \otimes \tau + \sigma \otimes (R^E \tau)$$

Again starting with the Bochner identity **4.3.17**, we want to identify $\mathfrak{R}$:

$$\begin{aligned}
\mathfrak{R}(\sigma \otimes \tau) &= \frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \cdot R_{e_i,e_j}(\sigma \otimes \tau) \\
&= \left(\frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \cdot R^S_{e_i,e_j} \sigma \right) \otimes \tau + \frac{1}{2} \sum_{i,j}^{n} (e_i \cdot e_j \cdot \sigma) \otimes R^E_{e_i,e_j} \tau \\
&= \frac{1}{4} \kappa(\sigma \otimes \tau) + \mathfrak{R}^E(\sigma \otimes \tau)
\end{aligned}$$

Where the last line uses the proof of the classical Lichnerowicz formula.

□

Chapter 5

Characteristic classes, the Chern character and the $\hat{A}$-genus

Characteristic classes associate to a given vector bundle an element in the cohomology ring of the base space of the vector bundle, i.e. for a generic characteristic class denoted by c and a (not necessarily smooth) vector bundle (E, B, π) we have $c(E) \in H^*(B, R)$, where R is some coefficient ring (in our case $R \in \{\mathbb{Z}_2, \mathbb{Z}\}$). All characteristic classes vanish on trivial bundles, thus they can be viewed as a measure of *non-triviality* of vector bundles. Stiefel-Whitney classes are important in our context since they occur as an obstruction to the existence of spin structures (see **4.1.2**); Chern and Pontryagin classes will be used in the later chapters. We will use a purely axiomatic approach and omit the construction of the characteristic classes as well as most proofs. Detailed accounts of the subject can be found in the classic [**MiSt**] as well as [**Ha09**].
The Chern character is *a priori* a formal power series of the first Chern class of complex line bundles (the so-called formal roots). It can be extended to general complex vector bundles by use of the splitting principle. To discuss it properly, one needs furthermore some facts about symmetric polynomials (see **Appendix A**). The main interest of the Chern character is the fact that it yields an isomorphism from K-Theory tensored with $\mathbb{Q}$ to rational cohomology. Our treatment of this subject is based on [**LaMi**] III. §11. The $\hat{A}$ genus is of utmost importance in the study of the topology of manifolds with positive scalar curvature. Via the Atiyah-Singer index theorem it is the index of the Dirac operator and can be shown to be an obstruction to the existence of scalar positive metrics on a given manifold. See **chapter 7** for details.

(As opposed to our treatment up until this point, this chapter is set in the continuous rather than the smooth category.)

5.1 Stiefel-Whitney classes

Stiefel-Whitney classes are defined for real not necessarily oriented vector bundles. The first Stiefel-Whitney w_1 class actually measures orientability in the sense that $w_1(E) = 0$ if and only if E is orientable.

Theorem 5.1.1. Existence and uniqueness of Stiefel-Whitney classes.
Given a base space B, there is a unique map

$$\begin{aligned} w : \{\text{real vector bundles over B}\} &\longrightarrow H^*(B, \mathbb{Z}_2) \\ (E, B, \pi) &\longmapsto w(E) \end{aligned}$$

which we will denote by $w = 1 + w_1 + w_2 + ...$ with $w_i \in H^i(B, \mathbb{Z}_2)$, s.th. the following hold:

1) $w(f^*E) = f^*w(E)$;
2) $w(E \oplus F) = w(E)w(F)$;
3) $w_i(E) = 0 \quad \forall i > \text{rank}E$;
4) $w_1(\gamma^1)$ (where we denote by γ^1 the canonical line bundle over $\mathbb{RP}^1$) is the generator of $H^*(\mathbb{RP}^1, \mathbb{Z}_2)$

The first property is called *naturality* and is shared by all characteristic classes, it immediately implies that the classes are invariant under vector bundle isomorphisms and the fact that $w(E) = 1$ if E is trivial. The formulation of the second property is shorthand notation for the following:

$$w_i(E \oplus F) = \sum_{j+k=i} w_j(E)w_k(F)$$

where the product on the right-hand side is to be understood as the cup product of the cohomology ring. Furthermore we have that $w(E \oplus \varepsilon) = w(E)$ for all trivial bundles ε. This property is called *stability* and is a direct consequence of 2). The fourth property serves as normalization. Note also how the trivial prescription $w \equiv 1$ (up to the rank of the vector bundle) would qualify as Chern class if 4) was to be omitted. The canonical line bundle over $\mathbb{RP}^1$ has the following total space:

$$\gamma^1 = \{([x_0 : x_1], y_0, y_1) \in \mathbb{RP}^1 \times \mathbb{R}^2 \mid (y_0, y_1) \text{ lies on the line spanned by } (x_0, x_1)\}$$

with projection map:

$$\begin{aligned} \pi_{\gamma^1} : \gamma^1 &\longrightarrow \mathbb{RP}^1 \\ ([x_0 : x_1], y_0, y_1) &\longmapsto [x_0 : x_1] \end{aligned}$$

which is to say that the fibre over a given point $p \in \mathbb{RP}^1$ consists of 0 and all points in $\mathbb{R}^2 \setminus \{0\}$ that project to p under the quotient map $\mathbb{R}^2 \setminus \{0\} \twoheadrightarrow \mathbb{RP}^1$.

5.2 Chern classes

Chern classes are only defined for complex vector bundles. Their coefficient ring is $\mathbb{Z}$, which is linked to the fact that $\mathbb{C}$-vector bundles carry a canonical orientation on their underlying real bundles. For a detailed discussion of orientation and its influence on the coefficient ring, see **[MiSt]** §9.

Theorem 5.2.2. Existence and uniqueness of Chern classes.
Given a base space B, there is a unique map

$$\begin{aligned} c : \{\text{complex vector bundles over B}\} &\longrightarrow H^{2*}(B, \mathbb{Z}) \\ (E, B, \pi) &\longmapsto c(E) \end{aligned}$$

which we will denote by $c = 1 + c_1 + c_2 + ...$ with $c_i \in H^{2i}(B, \mathbb{Z})$, s.th. the following hold:

1) $c(f^*E) = f^*c(E)$;
2) $c(E \oplus F) = c(E)c(F)$;
3) $c_i(E) = 0 \quad \forall i > \mathrm{rank} E$;
4) $c_1(\gamma^1_{\mathbb{C}})$ (where we denote by $\gamma^1_{\mathbb{C}}$ the canonical line bundle over $\mathbb{CP}^1$) is a generator of $H^*(\mathbb{CP}^1, \mathbb{Z})$

Where the canonical line bundle $\gamma^1_{\mathbb{C}}$ is defined analogously to γ_1 in **5.1**.

Remark 5.2.3. Relation to the Euler class.
Denote by $E_{\mathbb{R}}$ the $2n$-dimensional real bundle associated to E, which is obtained by omitting the complex structure. $E_{\mathbb{R}}$ has a canonical orientation (see **[MiSt]** §14.) thus it has an Euler class $e(E_{\mathbb{R}}) \in H^{2n}(B, \mathbb{Z})$ which is equal to the n-th Chern class:

$$c_n(E) = e(E_{\mathbb{R}})$$

It is possible to take this equality as starting point to define the Chern classes by top-down recursion, an approach that is carried out in **[MiSt]** §14. The idea is roughly to consider a new $n-1$-dimensional bundle over $E \setminus \{0\}$ and to extract a suitable isomorphism from the Gysin sequence to carry the information over to the cohomology of the original base space.

A complex bundle of dimension n can be viewed as a real $2n$-bundle together with an integrable *complex structure* $J : E \to E$, that is, a bundle endomorphism s.th. J^2 is fibrewise the negative of the identity map. Complex multiplication in the fibres is determined by the equation

$$(a + ib)v = av + bJ(v)$$

By *reversing* the complex structure, one can construct the *conjugate bundle* associated to E, denoted by $\overline{E}$:

Definition 5.2.4. The **conjugate bundle** associated to E is the bundle $\overline{E}$ with the same underlying real bundle $E_{\mathbb{R}} = (\overline{E})_{\mathbb{R}}$ and the following complex multiplication

$$(a + ib)v = av - bJ(v)$$

The Chern classes of the conjugate bundle are determined by those of the original bundle:

Lemma 5.2.5. Chern classes of the conjugate bundle

$$c_k(\overline{E}) = (-1)^k c_k(E)$$

5.3 Pontryagin classes

Pontryagin classes are defined for real vector bundles and are derived from Chern classes by complexification. For a vector space V, its complexification is given by $V \otimes \mathbb{C}$, which is a complex vector space.

Definition 5.3.6. Let $E \to B$ be a real n-dimensional vector bundle. Let the **complexification of** E be the complex bundle $E \otimes \mathbb{C} \to B$ with fibres $(E \otimes \mathbb{C})_p = E_p \otimes \mathbb{C}$.

For vector spaces, one has $V \otimes \mathbb{C} = V \oplus iV$, which is by the definition of the complexification the pointwise picture for bundles. Thus we have:

$$(E \otimes \mathbb{C})_{\mathbb{R}} = E \oplus E$$

Before turning to the Pontryagin classes themselves, we note the following isomorphisms of vector bundles which will be of use later on:

Proposition 5.3.7.

1. For a real vector bundle E we have:
$$E \otimes \mathbb{C} \cong \overline{E \otimes \mathbb{C}}$$

2. For a complex vector bundle E we have:
$$E_{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus \overline{E}$$

3. For an oriented real vector bundle E:
$$(E \otimes \mathbb{C})_{\mathbb{R}} \cong E \oplus E$$
where the isomorphism preserves orientation when $\frac{n(n-1)}{2}$ is even and reverses it if not.

The first isomorphism is simply given by:

$$\begin{array}{rcl} E \otimes \mathbb{C} & \to & \overline{E \otimes \mathbb{C}} \\ (p, v + iw) & \mapsto & (p, v - iw) \end{array}$$

For the remaining ones, we refer to **[MiSt]** §15. pp. 176-178.

In order to define the Pontryagin classes for a real vector bundle E, combine **5.2.5** and **5.3.7.**1. to see:

$$\begin{array}{rcl} c(E \otimes \mathbb{C}) & = & 1 + c_1(E \otimes \mathbb{C}) + c_2(E \otimes \mathbb{C}) + ... + c_n(E \otimes \mathbb{C}) \\ = c(\overline{E \otimes \mathbb{C}}) & = & 1 - c_1(E \otimes \mathbb{C}) + c_2(E \otimes \mathbb{C}) + ... + (-1)^n c_n(E \otimes \mathbb{C}) \end{array}$$

The odd Chern classes are thus uninteresting since they are all of order 2. This leads to the following definition:

Definition 5.3.8. The Pontryagin classes
For a real n-dimensional bundle E, the i-th Pontryagin class is defined by:

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(B, \mathbb{Z})$$

The total Pontryagin class is defined correspondingly:

$$p(E) = 1 + p_1(E) + ... + p_m(E)$$

where m is the largest integer $m \leq \frac{n}{2}$.

Naturality and stability under addition of trivial bundles for Pontryagin classes is directly inherited from Chern classes. The product rule is however replaced by the following:

$$p(E \oplus F) \equiv p(E)p(F)$$

modulo elements of order 2. Or equivalently:

$$2p(E \oplus F) = 2p(E)p(F)$$

This is clearly due to the fact that in the definition of Pontryagin classes we ignored the Chern classes of order 2. (see **[MiSt]** §15. p.175)
Next we will see that in the special case where the initial bundle E is complex, the Pontryagin classes are determined by the Chern classes.

Proposition 5.3.9. Let E be complex, then:

$$1 - p_1 + p_2 - ... + (-1)^n p_n = (1 - c_1 + c_2 - ... + (-1)^n c_n)\,(1 + c_1 + c_2 + ... + c_n)$$

or coefficient-wise:

$$p_k = c_k^2 - 2c_{k-1}c_{k+1} + 2c_{k-2}c_{k+2} - ... - (-1)^{k+1} 2c_1 c_{2k-1} + (-1)^k 2c_{2k}$$

By c_i we denote $c_i(E)$ and by p_i we denote the corresponding Pontryagin class of the underlying real vector bundle: $p_i(E_{\mathbb{R}})$. Notice how in the first formula all odd Chern classes on the right-hand side cancel and the equality thus makes sense.

proof:

$$\begin{aligned} p(E_{\mathbb{R}}) = c(E_{\mathbb{R}} \otimes \mathbb{C}) &\overset{5.3.7.2}{=} c(\overline{E} \oplus E) \\ &= c(\overline{E})c(E) \\ &\overset{5.2.5}{=} (1 - c_1 + c_2 - ...)(1 + c_1 + c_2 + ...) \end{aligned}$$

□

In order to showcase the usefulness of the tools that have been introduced, we give a proof of the well-known fact that $\mathbb{S}^4$ has no almost complex structure:

Application 5.3.10. $\mathbb{S}^4$ has no almost complex structure.
Proof: An almost complex structure on M is a complex structure on TM. Suppose that $\mathbb{S}^4$ is almost complex, i.e. that $T\mathbb{S}^4$ is a complex vector bundle. First:

$$\begin{aligned} c(T\mathbb{S}^4) &= 1 + c_1 + c_2 \\ &= 1 + e(T\mathbb{S}^4) \end{aligned}$$

where we have used the fact that $c_1 \in H^2(\mathbb{S}^4, \mathbb{Z}) = \{0\}$ and 5.2.3. But the Euler number of the sphere is 2, therefore:

$$c(T\mathbb{S}^4) = 1 + 2\alpha$$

where $\alpha \in H^4(\mathbb{S}^4, \mathbb{Z})$ is the generator preferred by the orientation. Since $T\mathbb{S}^4$ is complex, we can use **5.3.9** to compute the Pontryagin class:

$$\begin{aligned} p(T\mathbb{S}^4) &= (1 + 2\alpha)(1 + 2\alpha) \\ &= 1 + 4\alpha \end{aligned}$$

But since the normal bundle $N\mathbb{S}^4$ of the 4-sphere $\mathbb{S}^4 \subset \mathbb{R}^5$ as well as the direct sum $N\mathbb{S}^4 \oplus T\mathbb{S}^4$ are trivial, we have:

$$p(T\mathbb{S}^4) = 1$$

5.4 The Chern character

Definition 5.4.11. Let ℓ be a complex line bundle over a manifold X. Using the Chern class $c_1(\ell) \in H^2(X, \mathbb{Z})$ we define the **Chern character** to be:

$$\mathrm{ch}(\ell) = e^{c_1(\ell)} = \sum_{k=0}^{\infty} \frac{c_1(\ell)^k}{k!} \in H^{2*}(X, \mathbb{Q})$$

For a sum of line bundles $\ell_1 \oplus ... \oplus \ell_k$ define correspondingly:

$$\mathrm{ch}(\ell_1 \oplus ... \oplus \ell_k) = e^{c_1(\ell_1)} + ... + e^{c_1(\ell_k)} \in H^{2*}(X, \mathbb{Q})$$

In order to define the Chern character for arbitrary complex vector bundles, we will use the following result:

Theorem 5.4.12. The Splitting Principle
For any n-dimensional complex vector bundle $E \to X$ over a manifold, there is a space FE and a map $p : FE \to X$ s.th.

1) $p^* : H^*(X, \mathbb{Z}) \to H^*(FE, \mathbb{Z})$ is injective,

2) The pullback bundle splits as sum of line bundles:

$$p^*E = \ell_1 \oplus ... \oplus \ell_n$$

proof: Associate to E its corresponding projective bundle $p : \mathbb{P}E \to X$, given by pointwise projectivization, i.e. where a given fibre E_x is replaced by its projective space $\mathbb{P}E_x$. Now pull E back to yield a bundle $p^*E \to \mathbb{P}E$. This bundle contains a canonical line bundle given over every point in $\mathbb{P}E$ (i.e. every line in E) by the same line. Call the obtained line bundle ℓ_1. For any hermitian metric, we thus get a splitting

$$p^*E = \ell_1 \oplus \ell_1^\perp$$

This process can obviously by iterated until the remaining complement is itself a line bundle. Injectivity of $p^* : H(X; \mathbb{Z}) \to H(\mathbb{P}E; \mathbb{Z})$ is a consequence of the Leray-Hirsch theorem. See **[LaMi]** Appendix C for details.

□

Thus we can extend the definition of the **Chern character** to all complex vector bundles:

$$p^*\mathrm{ch}(E) := \mathrm{ch}(p^*E)$$

Before further investigating the Chern character, we apply the splitting principle to the Chern classes. By injectivity of the by p induced map on cohomology, we can look at $p^*c(E)$ instead of $c(E)$ without loosing information. Using the axiomatic properties of the Chern classes (see **5.2.2**) we can write for any complex vector bundle E:

$$\begin{aligned} p^*c(E) &= c(p^*E) \\ &= c(\ell_1 \oplus ... \oplus \ell_n) \\ &= c(\ell_1) \cdot ... \cdot c(\ell_n) \\ &= (1 + c_1(\ell_1)) \cdot ... \cdot (1 + c_n(\ell_n)) \\ &=: (1 + x_1) \cdot ... \cdot (1 + x_n) \end{aligned}$$

The $x_i = c_1(\ell_i)$ are called **Chern roots** or **formal roots** of the vector bundle E. With the splitting principle in mind, we write from now on $c(E) = p^*c(E)$, thus by writing out the product in the last line:

$$c(E) = (1 + x_1) \cdot ... \cdot (1 + x_n) = \sum_{k=0}^{n} \sigma_k(x_1, ..., x_n)$$

Equivalently:

$$c_k(E) = \sigma_k(x_1, ..., x_n)$$

Where we use the elementary symmetric polynomials σ_k (see **A.0.2**). The k-th Chern class of E is thus the k-th symmetric polynomial evaluated in the Chern roots of E. Together with the fundamental theorem on symmetric polynomials (see **A.0.3**), this yields the following:

Proposition 5.4.13. If $p^*E = \ell_1 \oplus ... \oplus \ell_n$, then:
Any symmetric polynomial with coefficients in $\mathbb{Q}$ of the Chern roots $x_i = c_1(\ell_i)$ depends only on the Chern classes of E. Furthermore it is a polynomial expression of the latter and thus an element in the ring $H^{2*}(X, \mathbb{Q})$.

We will now return to our original object of study, the Chern character, by showing that **5.4.13** applies:

$$\begin{aligned}
\mathrm{ch}(E) &= e^{c_1(\ell_1)} + ... + e^{c_1(\ell_n)} \\
&= e^{x_1} + ... + e^{x_n} \\
&= n + \sum_{i=1}^{n} x_i + \frac{1}{2}\sum_{i=1}^{n} x_i^2 + ... \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{i=1}^{n} x_i^k\right)
\end{aligned}$$

Using the polynomials $q_k(x_1, ..., x_n) = \sum_i x_i^k$, (see A.0.2), rewrite:

$$\begin{aligned}
\mathrm{ch}(E) &= \sum_{k=0}^{\infty} \frac{1}{k!} q_k(x_1, ..., x_n) \\
&=: \sum_{k=0}^{\infty} \mathrm{ch}_k(E)
\end{aligned}$$

Since the q_k are symmetric, the principle applies. In fact, one can give the explicit expression of q_k in terms of σ_k using polynomials s_k (These formulae are called *Newton's identities*, see **A.0.4**). Therefore:

$$\begin{aligned}
\mathrm{ch}_k(E) &= \frac{1}{k!} s_k\left(\sigma_1(x_1, ..., x_n), ..., \sigma_k(x_1, ..., x_n)\right) \\
&= \frac{1}{k!} s_k(c_1(E), ..., c_k(E))
\end{aligned}$$

Proposition 5.4.14. (see **[LaMi]** III.§11.) The Chern character is additive and multiplicative, i.e. for two bundles E and E' over X, we have:

$$\begin{aligned}\mathrm{ch}(E \oplus E') &= \mathrm{ch}(E) + \mathrm{ch}(E') \\ \mathrm{ch}(E \otimes E') &= \mathrm{ch}(E)\mathrm{ch}(E')\end{aligned}$$

Example 5.4.15. complex vector bundles on $\mathbb{S}^{2n}$
Let E be a complex vector bundle over an even-dimensional sphere. Computing the Chern character of such a bundle turns out to be fairly easy. This is mainly due to the fact that all $H^i(\mathbb{S}^{2n}, \mathbb{Q})$ are trivial except for $i \in \{0, 2n\}$. Since $\mathrm{ch}_k \in H^{2k}(\mathbb{S}^{2n}, \mathbb{Q})$ this directly implies that only ch_0 and ch_n are of interest. By the above:

$$\mathrm{ch}_0 = \dim_{\mathbb{C}} E$$

For ch_n we take a look at the Newton identities. Since all $\mathrm{ch}_k = (1/k!)\, s_k = 0$ if $k \in \{1, ..., n-1\}$, the recursion formula simplifies to: $s_k = (-1)^{k-1} k \sigma_k$, therefore:

$$\begin{aligned}\mathrm{ch}_n(E) &= \frac{1}{n!}\, s_n \\ &= \frac{1}{n!}(-1)^{n-1} n \sigma_n \\ &= \frac{(-1)^{n-1}}{(n-1)!} c_n(E)\end{aligned}$$

In conclusion:

$$\mathrm{ch}(E) = \dim_{\mathbb{C}}(E) + \frac{(-1)^{n-1}}{(n-1)!} c_n(E)$$

5.5 The $\hat{A}$-genus

Define the following power series:

$$\hat{a}(x) = \frac{x/2}{\sinh(x/2)} = \frac{x}{e^{x/2} - e^{-x/2}} = 1 - \frac{1}{24}x^2 + \frac{7}{2^7 3^2 5}x^4 + ...$$

which has only terms of even order since $x/2$ as well as $\sinh(x/2)$ are odd functions. Given a smooth oriented vector bundle (E, X, π) of rank $2n$, by the use of its formal roots $x_1, ..., x_n$ (see below), we can formally define:

$$\hat{\mathbf{A}}(E) = \prod_{i=1}^{n} \hat{a}(x_i) = \prod_{i=1}^{n} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}$$

Similarly, we denote by $\hat{\mathbf{A}}(X)$ the case where E is the tangent bundle. Since $\hat{a}$ is even, $\hat{\mathbf{A}}$ is a polynomial in the squares $x_1^2, ..., x_n^2$. Furthermore, it is easy to see that this polynomial is symmetric. In the spirit of **5.4.13**, it is thus a polynomial expression of $\sigma_k(x_1^2, ..., x_n^2)$. Analogously to the proof of the fact that the elementary symmetric polynomials in the x_i are equal to the corresponding Chern classes, we will now prove that the $\sigma_k(x_1^2, ..., x_n^2)$ correspond to Pontryagin classes. This proves that $\hat{\mathbf{A}}(E) \in H^{4*}(X, \mathbb{Q})$.

Theorem 5.5.16. Modified Splitting Principle
For any $2n$-dimensional oriented real vector bundle $E \to X$ over a manifold, there is a space FE and a map $p : FE \to X$ s.th.

1) $p^* : H^*(X, \mathbb{Z}) \to H^*(FE, \mathbb{Z})$ is injective,

2) The pullback bundle splits as sum of line bundles:

$$p^*(E \otimes \mathbb{C}) = \ell_1 \oplus \overline{\ell_1} \oplus ... \oplus \ell_n \oplus \overline{\ell_n}$$

By this modified splitting principle, we can compute using **5.2.5**:

$$\begin{aligned} c(E \otimes \mathbb{C}) &= c(\ell_1 \oplus \overline{\ell_1} \oplus ... \oplus \ell_n \oplus \overline{\ell_n}) \\ &= c(\ell_1)c(\overline{\ell_1}) \cdot ... \cdot c(\ell_n)c(\overline{\ell_n}) \\ &= (1 + x_1)(1 - x_1) \cdot ... \cdot (1 + x_n)(1 - x_n) \\ &= (1 - x_1^2) \cdot ... \cdot (1 - x_n^2) \end{aligned}$$

But by the definition of Pontryagin classes (**5.3.8**), this yields:

$$p(E) = (1 + x_1^2) \cdot ... \cdot (1 + x_n^2)$$

And thus we have the desired identity:

$$p_i(E) = \sigma_k(x_1^2, ..., x_n^2)$$

For a full discussion of these matters using general multiplicative sequences, we refer to **[LaMi]** III.§11. The $\hat{A}$-genus of an even dimensional oriented manifold X is defined as $\hat{\mathbf{A}}(X)$ evaluated on the fundamental class of X.

Definition 5.5.17. The **$\hat{A}$-genus**

$$\hat{A}(X) = \hat{\mathbf{A}}(X)[X] \in \mathbb{Q}$$

This genus turns out to be an integer for spin manifolds. This is explained by the Atiyah-Singer index theorem, which identifies it with the index of the Dirac operator associated to the given spin manifold. We will need a slightly more general notion, namely the twisted $\hat{A}$-genus which turns out to be the index of the twisted Dirac operator (see **4.4**). The **twisted $\hat{A}$-genus** depends on the tangent bundle of a manifold X (with the same properties as before) and some other vector bundle E over X and is defined as:

$$\hat{A}(X,E) = \Big(\mathrm{ch}E \cdot \hat{\mathbf{A}}(X)\Big)[X]$$

Chapter 6

The Atiyah-Singer index theorem

In this chapter, we briefly discuss the Atiyah-Singer index theorem. Proved in '63, this result establishes a strong link between analysis and topology. More concretely, it relates the index of differential operators of sections over some compact manifold to the topological index of the same manifold. This is best understood in the setting of K-Theory in the sense of Appendix **B**. The proof is omitted here, since it is quite involved and since we are mainly interested in the application of the theorem to Dirac operators. Moreover we don't discuss the functional analytic aspects of differential operators. The lecture notes [**Des**] along with [**LaMi**] served as main references. Further applications can be found in [**HBJ**].

Preliminary remark: All base spaces in this chapter are taken compact and without boundary, however we want to refer to the K-Theory of not necessarily compact spaces (vector bundles for instance). Let Y therefore be a locally compact space. In what follows, we will denote by $K(Y)$ the reduced K-Theory of the one-point compactification of Y, i.e. $K(Y) = \tilde{K}(Y^+)$.

6.1 The analytic index for elliptic differential operators

Let E and F be smooth complex vector bundles of dimensions p and q respectively over a given space X.

Definition 6.1.1. A linear map $D : \Gamma(E) \to \Gamma(F)$ is called **differential operator (of order** m**)** if it can be written around every point $x \in X$ as

$$D = \sum_{|\alpha| \leq m} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_\alpha}$$

for A^α $p \times q$-matrices of smooth $\mathbb{C}$-valued functions, which are not all zero in the highest-order term, i.e. for $|\alpha| = m$.

Some explanations might be in order: We are allowed to write down partial derivatives since this is a local condition: For a given $x \in X$ there is an open coordinate neighbourhood $U \ni x$ i.e. we have trivializations $E|_U \cong U \times \mathbb{C}^p$ and $F|_U \cong U \times \mathbb{C}^q$ and the partial derivatives are to be interpreted with respect to these. D is thus an operator that is locally given by a matrix with polynomial entries in partial derivatives of order $\leq m$ which is non-trivial in the top degree.

To every differential operator we associate its symbol:
Consider the cotangent bundle:

$$\pi : T^*X \to X$$

Via π, the preexisting vector bundles can be pulled back to T^*X to yield bundles $\pi^*E \to T^*X$ and $\pi^*F \to T^*X$.

Definition 6.1.2. The **symbol of a differential operator** D is for all $\xi \in T_x^*X$ a map:

$$\sigma_\xi(D) : \pi^*E|_\xi \longrightarrow \pi^*F|_\xi$$

that is locally given by:

$$\sigma_\xi(D) = i^m \sum_{|\alpha|=m} A_\alpha(x)\xi_\alpha$$

for $\xi = \sum \xi_k dx_k$. This means that we replace the partial derivatives $\partial/\partial x_k$ in the local expression by $i\xi_k$.

Example 6.1.3. The Laplace operator Δ
Let E and F be two trivial $\mathbb{C}$-line bundles over X. The Laplace operator thus takes a $\mathbb{C}$-valued function on X into another one. It is defined by:

$$\begin{aligned} \Delta : \Gamma(E) = C^\infty_{\mathbb{C}}(X) &\longrightarrow \Gamma(F) = C^\infty_{\mathbb{C}}(X) \\ f &\longmapsto \sum_j \frac{\partial^2 f}{\partial x_j^2} \end{aligned}$$

(This is a global definition by triviality of the bundles.)
For some $\xi = \sum \xi_j dx_j \in T^*X$ the corresponding symbol is given by:

$$\sigma_\xi(\Delta) = \sum_j (i\xi_j)^2 = -\sum_j \xi_j^2 = -||\xi||^2$$

Which is to be understood as multiplication by the number $-||\xi||^2$ and is thus an isomorphism provided ξ is non-zero. The Laplace operator is thus elliptic, by the definition given below.

Example 6.1.4. The exterior derivative d
Let X be a smooth n-dimensional manifold, and $T^*X \xrightarrow{\pi} X$ its cotangent bundle. Define the complexified differential forms $\Omega^i_{\mathbb{C}}(X) = \Gamma(\Lambda^i(T^*X \otimes \mathbb{C}))$. The exterior derivative d is a map increasing the degree of a given form by one:

$$d : \Gamma(E) = \Omega^i_{\mathbb{C}}(X) \longrightarrow \Gamma(F) = \Omega^{i+1}_{\mathbb{C}}(X)$$

defined in local coordinates by:

$$\begin{aligned} d\left(\sum_{|I|=k} a_I dx_I\right) &= \sum_{|I|=k}\sum_{j=1}^{n} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I \\ &= \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} dx_j\right) \wedge \left(\sum_{|I|=k} a_I dx_I\right) \end{aligned}$$

Where the last form is particularly useful to compute the symbol of d (recall that the symbol is obtained by replacing $\partial/\partial x_k$ by $i\xi_k$):

$$\sigma_\xi(d)\eta = i\xi \wedge \eta$$

Definition 6.1.5. A differential operator $D : \Gamma(E) \to \Gamma(F)$ is **elliptic** if $\sigma_\xi(D)$ is an isomorphism for all non-zero $\xi \in T^*X$.

For elliptic D, we can thus consider $\pi^*E \xrightarrow{\sigma(D)} \pi^*F$ which is an isomorphism when restricted to $T^*X \setminus X$ i.e. for all $\xi \neq 0$. Thus this map defines an element in $\mathcal{L}_1(T^*X, T^*X \setminus X) = K(T^*X)$ (See Appendix **B**).

Theorem 6.1.6. Properties of elliptic operators

1. If D is elliptic, then $\ker D$ and $\operatorname{coker} D$ are finite-dimensional;
2. The number $\dim\ker D - \dim\operatorname{coker} D$ only depends on the symbol $\sigma(D)$.

By this theorem we can define for any elliptic differential operator D:

Definition 6.1.7. Let a-Ind(D) be the quantity $\dim\ker D - \dim\operatorname{coker} D \in \mathbb{Z}$ called the **analytic index of** D. Since $\sigma(D) \in K(T^*X)$, we have:

$$\text{a-Ind} : K(T^*X) \longrightarrow \mathbb{Z}$$

6.2 The analytic index for elliptic complices

We recall the result $\mathcal{L}_n \cong \mathcal{L}_1 \cong K$ (see **Appendix B**) and thus we can adapt the above construction to sequences of vector bundles and differential operators.

Definition 6.2.8. Let $E_0, ..., E_m$ be smooth complex vector bundles over X and $D_i : \Gamma(E_i) \to \Gamma(E_{i+1})$ be differential operators of fixed degree p with $D_{i+1} \circ D_i = 0$. Then

$$0 \to \Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} ... \xrightarrow{D_{m-1}} \Gamma(E_m) \to 0$$

is called **elliptic complex** if the associated sequence pulled back to T^*X

$$0 \to \pi^*E_0 \xrightarrow{\sigma(D_0)} \pi^*E_1 \xrightarrow{\sigma(D_1)} ... \xrightarrow{\sigma(D_{m-1})} \pi^*E_m \to 0$$

is exact on $T^*X \setminus \{0\}$.

Thus the differential forms and the exterior derivative (discussed in **6.1.4**) form an elliptic complex. Such an elliptic complex D thus defines an element $\sigma(D) \in \mathcal{L}_{m+1}(T^*X, T^*X\setminus\{0\}) \cong K(T^*X)$ (again, see **Appendix B**). One can prove that the associated cohomology groups $H^j = \mathrm{ker}D_j/\mathrm{im}D_{j-1}$ are finite-dimensional. Similarly to the case of only two vector bundles, we define the analytic index:

Definition 6.2.9.

$$\begin{aligned} \text{a-ind} : K(T^*X) &\to \mathbb{Z} \\ \sigma(D) &\mapsto \sum_{j=1}^{n} (-1)^j \mathrm{dim} H^j \end{aligned}$$

6.3 The topological index of a smooth manifold

Before writing down the definition of the topological index, some preparations are necessary. Let $\pi : E \to X$ be a complex vector bundle of dimension n. We want to associate to E an exact sequence of vector bundles over X. We therefore consider $E_k := \pi^*(\Lambda^k E) \subset E \times \Lambda^k E$ the exterior powers of E pulled back to X. We get a sequence

$$0 \to E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow ... \longrightarrow E_n \to 0$$

by taking the wedge product with the base point in the fibres. More precisely, we can look at the following pointwise operation:

$$\begin{array}{rcl}\{v\} \times \Lambda^k E|_v & \to & \{v\} \times \Lambda^{k+1} E|_v \\ \omega & \mapsto & v \wedge \omega\end{array}$$

Since this sequence is exact on $E \setminus \{0\}$, it defines an element $\lambda_E \in \mathcal{L}_{n+1}(E, E \setminus \{0\}) = \mathcal{L}_{n+1}(B(E), S(E)) = K(B(E), S(E)) = \tilde{K}(B(E)/S(E)) = \tilde{K}(E^+) = K(E)$, where we used the following notation:

$$\begin{array}{rcl}S(E) & = & \{v \in E| \, ||v|| = 1\} \\ B(E) & = & \{v \in E| \, ||v|| \leq 1\}\end{array}$$

The space $B(E)/S(E)$ is sometimes called *Thom space* and denoted by $\mathrm{Th}(E)$.

Theorem 6.3.10. Thom isomorphism (see **[At]** 2.7.)

$$K(X) \xrightarrow{\lambda_E \otimes} K(B(E), S(E)) = K(E)$$

is an isomorphism.

Let $f : X \hookrightarrow \mathbb{R}^N$ be a proper embedding and N its normal bundle. The complexification $N_{\mathbb{C}} = N \otimes \mathbb{C} \cong N \oplus N$ is the normal bundle of $f_* : TX \to T\mathbb{R}^N$ where the first N is viewed as normal component to the manifold and the second one as normal to the respective tangent space. The normal bundle of TX is diffeomorphic to a regular neighbourhood U of TX in $T\mathbb{R}^N = \mathbb{R}^{2N}$ (see **[MiSt]** 11.1 p.115), which yields in K-Theory

$$K(N_{\mathbb{C}}) \xrightarrow{\cong} K(U) \to K(T\mathbb{R}^N)$$

Precomposing this map with the Thom isomorphism

$$K(TX) \xrightarrow{\lambda_{N_{\mathbb{C}}} \otimes} K(N_{\mathbb{C}}) \longrightarrow K(T\mathbb{R}^N)$$

we see that any proper embedding $X \hookrightarrow \mathbb{R}^N$ thus yields a covariant map on the level of the corresponding K-Theories. Since $K(T\mathbb{R}^N) = K(\mathbb{R}^{2N}) \cong K(\{\mathrm{pt.}\}) = \mathbb{Z}$ also by the Thom isomorphism, we finally get a map

$$\text{t-ind} : K(TX) \longrightarrow \mathbb{Z}$$

6.4 The Atiyah-Singer index theorem

Theorem 6.4.11. Atiyah-Singer index theorem
Let D be an elliptic complex with symbol $\sigma(D) \in K(T^*X)$. Then:

$$\text{a-ind}(\sigma(D)) = \text{t-ind}(\sigma(D))$$

See **[LaMi]** III.§13. for details and a proof. In light of this theorem, we will write from now on $\text{ind}(D) = \text{a-ind}(\sigma(D)) = \text{t-ind}(\sigma(D))$.

Example 6.4.12. The de Rham complex
Let X be a compact n-dimensional manifold. The exterior derivative d yields a complex, since $d^2 = 0$

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} ... \xrightarrow{d} \Omega^n(X) \to 0$$

The corresponding symbol complex

$$0 \to \pi^*\Omega^0(X) \longrightarrow \pi^*\Omega^1(X) \longrightarrow ... \longrightarrow \pi^*\Omega^n(X) \to 0$$

In the fibre over ξ, these maps are given by $\xi\wedge$ (see **6.1.4**), which makes the complex exact for $\xi \neq 0$. We are thus in the above defined situation of an elliptic complex. Since the cohomology associated to this complex is de Rham cohomology, the Atiyah-Singer index theorem yields for this case:

$$\begin{aligned} \text{t-ind}(\sigma(d)) &= \sum_{j=0}^{n}(-1)^j \dim H^j_{dR} \\ &= \sum_{j=0}^{n}(-1)^j b_j \\ &= e(X) \end{aligned}$$

where H^j_{dR} denotes the j-th de Rham cohomology group, b_j the corresponding Betti number and $e(\cdot)$ the Euler characteristic.

Remark 6.4.13. The cohomological formula for the topological index (see **[HBJ] 5.2.**)
Given an elliptic complex over an n-dimensional X:

$$0 \to \Gamma(E_0) \longrightarrow \Gamma(E_1) \longrightarrow ... \longrightarrow \Gamma(E_m) \to 0$$

with symbol $\sigma(D)$, one can express the index in purely cohomological terms via the Chern character of the bundles E_i and the formal roots $x_1, ..., x_n$ of the tangent bundle (see **chapter 5** for both concepts):

$$\text{ind}(D) = \left(\left(\sum_{i=0}^{m}(-1)^i \text{ch}(E_i)\right)\prod_{j=1}^{n}\left(\frac{x_j}{1-e^{-x_j}}\frac{1}{1-e^{x_j}}\right)\right)[X]$$

Chapter 7

Enlargeability and Positive Scalar Curvature

In this chapter we introduce the concept of enlargeability of manifolds and discuss its adverse relation to positive scalar curvature. The prototypical example of an enlargeable manifold is the torus. Thus, $\mathbb{T}^n$ can't carry a metric of positive scalar curvature. Furthermore enlargeability has some nice stability properties: It is stable under taking products between enlargeable manifolds and it is stable under direct sum with arbitrary manifolds (see **[LaMi]** IV. Theorem 5.3.). Thus we can expect enlargeability to be applicable to a wide range of manifolds. The proof of the fact that enlargeability excludes the existence of positive scalar curvature relies on the Atiyah-Singer index theorem and on spin geometry, namely it uses the Dirac operator of twisted complex spinor bundles and its Bochner formula (**4.4**):

$$\not{D}_E^2 = \nabla^*\nabla + \frac{1}{4}\kappa + \mathfrak{R}^E$$

The difference to the usual Lichnerowicz formula is obviously the E-curvature term $\mathfrak{R}^E$. As it turns out, enlargeability is a sufficient condition to gain some control over this *error term* and to be able to apply a Lichnerowicz-type argument to show the vanishing of the index of $\not{D}_E$ in the presence of positive scalar curvature. We will show that this contradicts the Atyiah-Singer index theorem.

7.1 Enlargeability

Definition 7.1.1. Let $\varepsilon > 0$. A smooth map:

$$f : (X, g_X) \longrightarrow (Y, g_Y)$$

between Riemannian manifolds is called **ε-contracting** if one of the following (equivalent) statements hold:

$$\begin{aligned} &\|f_*(p)v\|_Y \leq \varepsilon\|v\|_X \quad \forall (p,v) \in TX \\ \Leftrightarrow\ &\text{length}_Y(f(\gamma)) \leq \varepsilon \cdot \text{length}_X(\gamma) \quad \forall \gamma : [0;1] \to X \text{ piecewise } C^1\text{-curve} \\ \Leftrightarrow\ &\|f_*(p)\|_{X,Y} \leq \varepsilon \ \ \forall p \in X \ (\text{where } \|\cdot\|_{X,Y} \text{ denotes the operator norm}) \end{aligned}$$

Note that the existence of ε-contracting maps between two given manifolds contains no geometric information, since such maps always exist (constant maps are ε-contracting for all given $\varepsilon > 0$). The notion is also not intrinsic to the manifold, i.e. depends on the Riemannian metric: consider that in the compact case every given f can be made ε-contracting for every given ε by suitably scaling the metric on X up or the metric on Y down. In order to *compare manifolds in size* in a meaningful way, we will use the notion of degree of a map (see **Appendix C**). Non-zero degree for a map $f : X \to Y$ means, roughly speaking, that X is *wrapped around Y by f at least once*. Thus if there is an ε-contracting map of non-zero degree form X onto Y, we can say that *Y is smaller than X by an order of at least ε*. Note that this still depends on the chosen metric, by the same argument as above. Therefore let us define the notion of enlargeability by comparing (coverings of) Riemannian manifolds to the round n-sphere of unit radius. This will turn out to be a notion that is independent of the chosen metric.

Definition 7.1.2. A compact n-dimensional Riemannian manifold (X, g) is **enlargeable** if for all $\varepsilon > 0$ there is an orientable covering $(\tilde{X}_\varepsilon, \tilde{g}_\varepsilon)$ of X and an ε-contracting map

$$f_\varepsilon : (\tilde{X}_\varepsilon, \tilde{g}_\varepsilon) \to (\mathbb{S}^n, g_{st.})$$

which is constant outside of a compact set and has non-zero degree. If for each of the $\tilde{X}_\varepsilon$ we can choose finite coverings, (X, g) is called **compactly enlargeable**.

Remarks 7.1.3.

1. On the coverings, we chose the lifted Riemannian metrics: Let $\pi : \tilde{X} \to X$ be a covering. Define the **lifted metric** $\tilde{g}$ of g by:

$$\tilde{g}|_p(v, w) := g|_{\pi(p)}(\pi_*(p)v, \pi_*(p)w)$$

2. The maps f_ε need to be constant outside of a compact set, since otherwise the notion of degree would not be well-defined. See **C.4.10** for details.

3. The existence of degree one maps onto the sphere isn't a constraint: For a given manifold M, take a coordinate neighbourhood U and a ball $B \subset U$. The map $M \twoheadrightarrow M/(M \setminus B) \approx \mathbb{S}^n$ has degree one.

Example 7.1.4. the n-torus $\mathbb{T}^n$
The flat n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is enlargeable. Consider therefore the universal covering space $\mathbb{R}^n$. Although the fact that $\mathbb{R}^n$ has ε-contracting maps onto the unit sphere might be intuitively clear, one can write out the following argument:

Proof:
We consider here $(\mathbb{S}^n, g_{st.}) \subset (\mathbb{R}^{n+1}, g_{eucl.})$. Define the following map:

$$\begin{aligned} \exp : \mathbb{R}^n &\longrightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1} \\ x &\longmapsto \cos(\|x\|)e_{n+1} + \sin(\|x\|)\frac{x}{\|x\|} \end{aligned}$$

(We view $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ by $x \mapsto (x, 0)$)
As the name suggests, this is nothing else than the exponential map [1] for the unit sphere with the north pole as base point: $\exp(\cdot) = \exp(e_{n+1}, \cdot)$ when we identify $T_{e_{n+1}}\mathbb{S}^n$ with $\mathbb{R}^n$. Thus, exp maps the open ball of radius π diffeomorphically to $\mathbb{S}^n \setminus \{-e_{n+1}\}$ and collapses $\partial B_\pi(0)$ to the south pole $-e_{n+1}$. This suggests to define the following map:

$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{S}^n$$

by defining Φ to be equal to exp on the open ball $B_\pi(0)$ and equal to the south pole everywhere else. By the fact mentioned above, this yields a continuous map.

We now replace Φ by a smooth map $\tilde{\Phi}$ and show that these maps both have degree one. For this we want to apply the differential topological approach to the degree (see **C.6.12**) of counting preimages of regular points. To define the smooth map $\tilde{\Phi}$ we apply the Whitney approximation theorem for manifolds[2] (see **[BJ]** Theorem 14.8). Choose as closed subset $A = B_{\pi/2}(0) \cup (\mathbb{R}^n \setminus \overline{B_{2\pi}(0)}) \subset \mathbb{R}^n$, Φ is clearly smooth on A, since it is a diffeomorphism on $B_\pi(0)$ and constant outside of $\overline{B_\pi(0)}$. Applying the Whitney approximation theorem yields a map $\tilde{\Phi}$ that is homotopic relative A to Φ. By the fact that the degree is a homotopy invariant, we have:

$$\deg \Phi = \deg \tilde{\Phi}$$

We thus only need to show that $\tilde{\Phi}$ has degree one. By the fact that $\tilde{\Phi}$ can be chosen arbitrarily close to Φ, we can choose a point $p \in \tilde{\Phi}(B_{\pi/2}(0)) \subset \mathbb{S}^n$ that has one single point as preimage

[1] The exponential map is defined in most texts on Riemannian geometry. See for example **[DCa]** chapter 3 Example 2.11

[2] **Whitney approximation theorem:** Let $f : X \to Y$ be a continuous map between smooth manifolds that is smooth on a closed $A \subset X$. Then there is a smooth $\tilde{f} : X \to Y$ s.th. $\tilde{f}$ homotopic to f relative A and arbitrarily close to f.

(Recall that $B_{\pi/2}(0) \subset A$, and Φ is a diffeomorphism on this set). Since $\tilde{\Phi} = \Phi$ on A, p is a regular point. Up to reversal of orientation, we thus have:

$$\deg \Phi_\varepsilon = \deg \Phi = \deg \tilde{\Phi} = 1$$

Claim: $\tilde{\Phi}$ is ε_0- contracting for some $\varepsilon_0 > 0$.
Let $n(p) := \|\tilde{\Phi}_*(p)\|_{\mathbb{R}^n,\mathbb{S}^n}$ denote the value of the operator norm $\forall p \in \mathbb{S}^n$. Since $n \equiv 0$ outside of $B_{2\pi}(0)$ (everything being collapsed to one point), we can define ε_0 to be the maximum of n on the compact set $\overline{B_{2\pi}(0)}$.

For a given $\varepsilon > 0$ define $\Phi_\varepsilon(x) = \tilde{\Phi}\,(\varepsilon/\varepsilon_0\; x)\ \forall x \in \mathbb{R}^n$. By the fact that multiplication by a constant c has operator norm c (note $(c\cdot)_* = c\cdot$ since it is linear) and that $\|\Psi \circ \Xi\| \leq \|\Psi\|\|\Xi\|$ for all operator norms, compute:

$$\|(\Phi_\varepsilon)_*\|_{\mathbb{R}^n,\mathbb{S}^n} = \|\tilde{\Phi}_* \circ ((\varepsilon/\varepsilon_0)\cdot)_*\|_{\mathbb{R}^n,\mathbb{S}^n} \leq \varepsilon_0 \frac{\varepsilon}{\varepsilon_0} = \varepsilon$$

Thus Φ_ε is an ε-contracting map from $\mathbb{R}^n$ onto the unit sphere.

Furthermore Φ_ε is constant outside of $\overline{B_{2\pi\varepsilon_0/\varepsilon}(0)}$ (the complement being collapsed to the south pole).

$\square$

The torus is even compactly enlargeable. For this, consider the tori $\mathbb{R}^n/(k\mathbb{Z})^n$ for $k \in \mathbb{N}$ as covering spaces, which yield k^n-fold coverings and can be checked to admit themselves ε-contracting maps onto $\mathbb{S}^n$ if k is chosen large enough.

Proposition 7.1.5. Properties of enlargeability for compact manifolds

1) Let Y be enlargeable and $f : X \to Y$ of non-zero degree. Then X is enlargeable as well.

2) Enlargeability is invariant under homotopy- equivalences. In particular: Enlargeability is intrinsic to the manifold, i.e. doesn't depend on the metrics.

Proofs:
1) Let ε_0 be an upper bound on the operator norm of f_* (given by compactness), i.e.

$$\|f_*(x)\|_{X,Y} < \varepsilon_0 \quad \forall x \in X$$

Given $\varepsilon > 0$, by enlargeability of Y choose a covering space $\tilde{Y}$ of Y and an $\varepsilon/\varepsilon_0$-contracting h of non-zero degree onto the n-sphere and constant outside of a compact set. We obtain a suitable covering space of X using a pullback construction: Define:

$$\tilde{X} := \{(x, \tilde{y}) \in X \times \tilde{Y} |\ f(x) = \pi_Y(\tilde{y})\}$$

along with the projection: $\pi_X : \tilde{X} \to X$ by $\pi_X(x, \tilde{y}) := x$. One can check that $\pi_X : \tilde{X} \to X$ is a covering. Furthermore one gets the induced map on the bundles $\tilde{f} : \tilde{X} \to \tilde{Y}$ by setting $\tilde{f}(x, \tilde{y}) := \tilde{y}$. This map is easily seen to be proper. By the defining condition of $\tilde{X}$, we get

$$\pi_Y \circ \tilde{f} = f \circ \pi_X$$

Thus we get a commuting diagram:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\pi_X"] & \tilde{Y} \arrow[r, "h"] \arrow[d, "\pi_Y"] & \mathbb{S}^n \\ X \arrow[r, "f"] & Y & \end{tikzcd}$$

By equipping the coverings with their respective lifted metrics, π_X,π_Y become local isometries. In particular, their operator norms equal 1. Therefore, by commutativity, the following holds locally:

$$\tilde{f}_* = ((\pi_Y)_*)^{-1} \circ f_* \circ (\pi_X)_* \Rightarrow \|\tilde{f}_*\|_{\tilde{X},\tilde{Y}} \leq \|((\pi_Y)_*)^{-1}\|_{Y,\tilde{Y}} \|f_*\|_{X,Y} \|(\pi_X)_*\|_{\tilde{X},X} = \varepsilon_0$$

Which means that $\tilde{f}$ is also ε_0-contracting. Thus the composition $h \circ \tilde{f}$ yields an $\varepsilon_0 \cdot \varepsilon/\varepsilon_0 = \varepsilon$-contracting map from $\tilde{X}$ onto the unit sphere.
Next, we prove $\deg(f) = \deg(\tilde{f})$ by counting preimages of the map $\tilde{f}$:

$$\begin{aligned} \tilde{f}^{-1}(\{\tilde{y}_0\}) &= \{\tilde{x} \in \tilde{X} | \tilde{f}(\tilde{x}) = \tilde{y}_0\} \\ &= \{(x, \tilde{y}_0) \in X \times \tilde{Y} | f(x) = \pi_Y(\tilde{y}_0)\} \\ &= f^{-1}(\{\pi_Y(\tilde{y}_0)\}) \end{aligned}$$

The orientation around $\pi_Y(\tilde{y}_0)$ corresponds to the orientation around $\tilde{y}_0$, thus $\tilde{f}$ is also a degree-one map.

1) $\square$

2) Let X and Y be homotopy equivalent, i.e. there are two maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id$ and $g \circ f \simeq id$. Then, by functoriality and since $\deg(\cdot) \in \mathbb{Z}$:

$$H_n(f \circ g) = H_n(f) \circ H_n(g) = id \Rightarrow \deg(f)\deg(g) = 1 \Rightarrow \deg(f) = \deg(g) \in \{-1, 1\}$$

Thus homotopy equivalence provides non-zero degree maps in both directions and we can apply 1)

2) $\square$

Theorem 7.1.6. Any compact manifold X that can carry a metric with non-positive sectional curvature (sec ≤ 0) is enlargeable.

Proof:
Let $\tilde{X}$ be the universal covering of X. By Cartan- Hadamard (see for example **[DCa]** 7.3.), $\exp_x : T_x\tilde{X} \to \tilde{X}$ is a global diffeomorphism and its inverse is distance decreasing or in other words we obtain a 1-contracting map:

$$\exp_x^{-1} : \tilde{X} \longrightarrow T_x\tilde{X} \cong (\mathbb{R}^n, g_{eucl.})$$

By the fact that the exponential map is a diffeomorphism, it has degree[3] ± 1. The composition $\Phi_\varepsilon \circ exp_x^{-1}$ (see **7.1.4**) is thus ε-contracting, has non-zero degree and is constant outside of a compact set.

$\square$

7.2 The setup in spin geometry

Before turning to the relation between enlargeability and positive scalar curvature, we will discuss the spin geometric setup needed in the proof of **7.3.7**. In order to make our account more readable, all aspects of this setting will be discussed here instead of the different (thematically more appropriate) earlier chapters.

Let (X, g) be an Riemannian spin manifold of dimension $4m = n$ whose tangent bundle will serve as initial vector bundle on which all of the following constructions will be carried out. Recall that, in order to associate a spinor bundle to some spin structure we choose on X, we need a representation of the corresponding Clifford algebra. As it turns out, there is a unique complex irreducible such representation in even dimensions (see **[LaMi]** I.§5.). The restriction of this representation to the group $\mathrm{Spin}_n \subset C\ell_n$ yields a (complex) representation of the respective spin group:

$$\Delta_{\mathbb{C}} : \mathrm{Spin}_n \longrightarrow GL_{\mathbb{C}}(M)$$

[3] Strictly speaking, Appendix **C** doesn't treat the topological degree in the case where both manifolds under consideration are non-compact. To remedy the problem, one can use one-point-compactification of $\tilde{X}$ and $\mathbb{R}^n$ (since the exponential map is a diffeomorphism, the one-point-compactifications will be spheres) which will yield a diffeomorphism between spheres. Alternatively, the local degree (see C.2) can be used to define a global degree in the case of proper maps. This is carried out in **[Do]** VIII.4.

For some complex $C\ell(\mathbb{R}^n) \otimes \mathbb{C}$-module M. This can also be done in the real case, but the complex case has the following additional property when n is even: The representation splits into two different (in the sense that they are not equivalent by conjugation), irreducible representations (This comes essentially from the $\mathbb{Z}_2$-grading of Clifford algebras):

$$\Delta_{\mathbb{C}} = \Delta_{\mathbb{C}}^{+} \oplus \Delta_{\mathbb{C}}^{-}$$

The crucial point in all of this is the following one: The so-obtained representations don't descend to SO_n under the double-covering map $\lambda : \mathrm{Spin}_n \to SO_n$. Recall from **3.2** that tensor bundles come from representations of the general linear group (respectively SO_n in the oriented Riemannian case), therefore the fact that the spin representations don't descend means that we obtain *new* bundles by this construction, i.e. bundles that aren't tensor bundles (see also the discussion in **[Pet98]** Appendix C.2.). Denote these bundles as follows:

$$\$_{\mathbb{C}}^{(\pm)} = F_{\mathrm{Sp}}E \times_{\Delta_{\mathbb{C}}^{(\pm)}} M$$

They will be equipped with the canonical connection stemming from the Levi-Civita connection. One can show that the splitting carries over to the vector bundles:

$$\$_{\mathbb{C}} = \$_{\mathbb{C}}^{+} \oplus \$_{\mathbb{C}}^{-}$$

The Dirac operator of the bundle $\$_{\mathbb{C}}$ will be denoted by $\not{D}$. Since by the definition of the Dirac operator the corresponding spinor is Clifford-multiplied with a vector (i.e. the $\mathbb{Z}_2$-grading is interchanged), the Dirac operator inverts the splitting. Therefore we can define restricted operators:

$$\begin{aligned} \not{D}^{+} : \Gamma(\$_{\mathbb{C}}^{+}) &\longrightarrow \Gamma(\$_{\mathbb{C}}^{-}) \\ \not{D}^{-} : \Gamma(\$_{\mathbb{C}}^{-}) &\longrightarrow \Gamma(\$_{\mathbb{C}}^{+}) \end{aligned}$$

The application of the Atiyah-Singer index theorem to this operator yields:

Proposition 7.2.7.

$$\mathrm{ind}\not{D}^{+} = \hat{A}(X)$$

See **5.5** for a discussion of the $\hat{A}$-genus. For our purposes, we need the twisted version the spinor bundle $\$_{\mathbb{C}} \otimes E$ (see **4.4**), for which we have a corresponding restricted Dirac operator:

$$\not{D}_{E}^{+} : \Gamma(\$_{\mathbb{C}}^{+} \otimes E) \longrightarrow \Gamma(\$_{\mathbb{C}}^{-} \otimes E)$$

In this case, the index theorem yields the following:

Proposition 7.2.8.

$$\mathrm{ind} \not{D}_E^+ = \hat{A}(X,E) = \left(\mathrm{ch}E \cdot \hat{\mathbf{A}}(X)\right)[X]$$

This formula can be proved by the use of the cohomological formula **6.4.13**. For this we use the fact that the formal roots of the bundles $\$_{\mathbb{C}}^+$ and $\$_{\mathbb{C}}^-$ can be expressed using the formal roots $x_1, ..., x_n$ of the tangent bundle in the following manner (see **[AS]** section 3):

$$\frac{1}{2}(\pm x_1 \pm x_2 \pm ... \pm x_n)$$

With an even number of minuses for $\$_{\mathbb{C}}^+$ and an odd number for $\$_{\mathbb{C}}^-$. Using the definition of the Chern character **5.4.11**, we want to develop the expression:

$$\mathrm{ch}(\$_{\mathbb{C}}^+) - \mathrm{ch}(\$_{\mathbb{C}}^-) = \sum_{\#\{\text{minuses}\}\,\text{even}} e^{\frac{1}{2}(\pm x_1 \pm x_2 \pm ... \pm x_n)} - \sum_{\#\{\text{minuses}\}\,\text{odd}} e^{\frac{1}{2}(\pm x_1 \pm x_2 \pm ... \pm x_n)}$$

In order to carry out computations, we interpret the signs as elements of $\mathbb{Z}_2 = \{-1, +1\}$:

$$\begin{aligned}
\mathrm{ch}(\$_{\mathbb{C}}^+) - \mathrm{ch}(\$_{\mathbb{C}}^-) &= \sum_{(\sigma_1,...,\sigma_n)\in(\mathbb{Z}_2)^n} \sigma_1\sigma_2 \cdot ... \cdot \sigma_n e^{\frac{1}{2}(\sigma_1 x_1 + \sigma_2 x_2 + ... + \sigma_n x_n)} \\
&= \left(e^{\frac{x_1}{2}} - e^{-\frac{x_1}{2}}\right) \sum_{(\sigma_2,...,\sigma_n)\in(\mathbb{Z}_2)^{n-1}} \sigma_2 \cdot ... \cdot \sigma_n e^{\frac{1}{2}(\sigma_2 x_2 + ... + \sigma_n x_n)} \\
&= \left(e^{\frac{x_1}{2}} - e^{-\frac{x_1}{2}}\right)\left(e^{\frac{x_2}{2}} - e^{-\frac{x_2}{2}}\right) \sum_{(\sigma_3,...,\sigma_n)\in(\mathbb{Z}_2)^{n-2}} \sigma_3 \cdot ... \cdot \sigma_n e^{\frac{1}{2}(\sigma_3 x_3 + ... + \sigma_n x_n)} \\
&= ... \\
&= \prod_{i=1}^{n} \left(e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}\right)
\end{aligned}$$

Substituting this in **6.4.13** (Notice that $\not{D}_E^+ : \Gamma(\$_{\mathbb{C}}^+ \otimes E) \longrightarrow \Gamma(\$_{\mathbb{C}}^- \otimes E)$ can be interpreted as an elliptic complex of length 2.) and using the multiplicativity of the Chern character (see

5.4.14) yields:

$$
\begin{aligned}
\operatorname{ind}(\not{D}_E^+) &= \left(\left(\operatorname{ch}(\not{\$}_{\mathbb{C}}^+ \otimes E) - \operatorname{ch}(\not{\$}_{\mathbb{C}}^- \otimes E)\right) \prod_{j=1}^n \left(\frac{x_j}{1-e^{-x_j}} \frac{1}{1-e^{x_j}}\right)\right)[X] \\
&= \left(\operatorname{ch}E \left(\operatorname{ch}(\not{\$}_{\mathbb{C}}^+) - \operatorname{ch}(\not{\$}_{\mathbb{C}}^-)\right) \prod_{j=1}^n \frac{x_j}{(1-e^{-x_j})(1-e^{x_j})}\right)[X] \\
&= \left(\operatorname{ch}E \prod_{i=1}^n \left(e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}\right) \prod_{j=1}^n \frac{x_j}{\left(e^{\frac{x_j}{2}} - e^{-\frac{x_j}{2}}\right)\left(e^{-\frac{x_j}{2}} - e^{\frac{x_j}{2}}\right)}\right)[X] \\
&= \left(\operatorname{ch}E \prod_{j=1}^n \frac{x_j}{e^{-\frac{x_j}{2}} - e^{\frac{x_j}{2}}}\right)[X] \\
&= \left(\operatorname{ch}E\ \hat{\mathbf{A}}(X)\right)[X]
\end{aligned}
$$

Where we have used that n is even in the last line in order to get the right sign. This proves **7.2.8** (and the untwisted case **7.2.7** by leaving out the vector bundle E).

The following isn't needed *per se* in our applications, but since the ideas are very similar to the techniques used in the proof of **7.3.7**, we will briefly discuss them: The above applications of the index theorem combined with the Lichnerowicz formula yield an interesting obstruction to the existence of positive scalar curvature metrics on manifolds, namely:

Theorem 7.2.9. If the compact spin manifold X admits a metric of positive scalar curvature, then $\hat{A}(X) = 0$

This theorem holds for arbitrary dimensions when one considers a more refined invariant, denoted by $\hat{\mathcal{A}}(\cdot)$ (see **[LaMi]** II.§8.12). We will prove it for dimensions divisible by 4.

Proof: By the Atiyah-Singer index theorem we have $\operatorname{ind}\not{D}^+ = \hat{A}(X)$. Thus it suffices to prove that $\ker\not{D}^+ - \operatorname{coker}\not{D}^+ = 0$. Furthermore, since $\not{D}$ is self-adjoint, $\ker\not{D}^- = \operatorname{coker}\not{D}^+$, we will prove that:

$$\ker\not{D}^2 = \{0\}$$

in the case where $\kappa \geq 0$ everywhere and $\kappa > 0$ in at least one point. This is by **4.3.14** equivalent to the triviality of the kernel of $\not{D}$. Suppose $\psi \in \Gamma(\not{\$}_{\mathbb{C}})$ so that $\not{D}\psi = 0$. The

Lichnerowicz formula implies:

$$\begin{aligned} & \nabla^*\nabla\psi + \frac{1}{4}\kappa\psi = 0 \\ \Rightarrow \quad & \langle \nabla^*\nabla\psi, \psi\rangle + \frac{1}{4}\kappa\langle\psi,\psi\rangle = 0 \\ \Rightarrow \quad & \langle \nabla^*\nabla\psi, \psi\rangle_{L^2} + \frac{1}{4}\int_X \kappa\|\psi\|^2 = 0 \\ \Rightarrow \quad & \|\nabla\psi\|^2_{L^2} + \frac{1}{4}\int_X \kappa\|\psi\|^2 = 0 \end{aligned}$$

Since both terms must vanish, we have $\nabla\psi = 0$ and since this implies that $\|\psi\|$ is constant, this constant must be zero by evaluation at a point where $\kappa > 0$.

□

7.3 Positive scalar curvature

In this section we will discuss and prove the fact that enlargeability is an obstruction to the existence of positive scalar curvature. This type of result was first proved by Schoen and Yau for dimensions ≤ 7 (see [**ScYa**] and [**ScYa2**]) in '79 by the use of techniques from minimal surface theory. In '80 Gromov and Lawson proved the result for spin manifolds in arbitrary dimensions using spin geometry (see [**GrLa**]). The second approach will be discussed here and we closely follow the exposition of [**LaMi**] IV. §5.

Theorem 7.3.7. Compactly enlargeable spin manifolds manifolds cannot carry a metric of positive scalar curvature.

Remarks 7.3.8.

1) Recall that this theorem is stated in the context of compact manifolds, since compactness is contained in the definition of enlargeability.

2) The theorem holds for (not necessarily compactly) enlargeable manifolds, but its proof requires more refined techniques (see [**LaMi**] IV. §6. p.316f for details).

7.3.7 combined with **7.3.8.**2) and **7.1.6** yield the following result:

Theorem 7.3.9. Exclusion theorem
Compact manifolds which can carry a metric with non-positive sectional curvature don't admit positive scalar curvature metrics.

Proof of 7.3.7: Step 0: Preparations

Let X be a compactly enlargeable manifold with $\tilde{X}_\varepsilon$ the corresponding spin covering manifold, which admits an ε-contracting map f_ε of non-zero degree onto the sphere of the same dimension. Suppose it carries a metric of positive scalar curvature, i.e.

$$\kappa > 0$$

everywhere. If X is odd-dimensional, replace it by $X \times \mathbb{S}^1$; it can be easily checked that all relevant properties remain unchanged: enlargeability is stable under taking cartesian products and the product metric on $X \times \mathbb{S}^1$ has positive scalar curvature. Thus we assume from here on out that X is even-dimensional. By compactness of X we can choose κ_0 s.th.

$$\kappa \geq \kappa_0 > 0$$

Step 1: Choose a suitable bundle over the sphere and pull it back to $\tilde{X}_\varepsilon$

Since $\mathrm{ch} : K(\mathbb{S}^{2n}) \to H^*(S^{2n}; \mathbb{Z})$ is an isomorphism (see **[AH]**) and we have computed in **5.4.15**:

$$\mathrm{ch}(E) = \dim_{\mathbb{C}}(E) + \frac{(-1)^{n-1}}{(n-1)!} c_n(E)$$

it is possible to choose a complex vector bundle E_0 over $\mathbb{S}^{2n}$ s.th. $c_n(E_0) \neq 0$. Take a unitary connection ∇^{E_0} on E_0 and denote by R^{E_0} the corresponding curvature tensor. We can now pull the bundle back along with its connection by defining:

$$(E, \nabla^E) = (f_\varepsilon^* E_0, f_\varepsilon^* \nabla^{E_0})$$

Step 2: Preparations to the Lichnerowicz argument

Consider the complex spinor bundle $\$_{\mathbb{C}}$ (see **7.2**) endowed with its canonical connection $\nabla^{\$_{\mathbb{C}}}$ (stemming from the Levi-Civita connection) over $\tilde{X}_\varepsilon$. We will work with the twisted spinor bundle $\$_{\mathbb{C}} \otimes E$ (see section **4.4**) and its corresponding Dirac operator:

$$\not{D}_E : \Gamma(\$_{\mathbb{C}} \otimes E) \longrightarrow \Gamma(\$_{\mathbb{C}} \otimes E)$$

for which the following modified Lichnerowicz formula holds (see **4.4.20**):

$$\not{D}_E^2 = \nabla^* \nabla + \frac{1}{4}\kappa + \mathfrak{R}^E \tag{$\clubsuit$}$$

where we denote by ∇ the natural connection on the tensor product defined by the following relation:

$$\nabla(\sigma \otimes v) = \left(\nabla^{\$_{\mathbb{C}}} \sigma\right) \otimes v + \sigma \otimes \left(\nabla^E v\right)$$

and by $\mathfrak{R}^E$:

$$\mathfrak{R}^E(\sigma \otimes v) = \sum_{j<k} (e_j e_k \sigma) \otimes \left(R^E_{e_j, e_k} v \right)$$

Which means that $\mathfrak{R} \in \Gamma(\mathrm{End}(\$_{\mathbb{C}} \otimes E))$ is a symmetric section of the bundle of endomorphisms of $\$_{\mathbb{C}} \otimes E$. In order to be able to apply the classical Lichnerowicz-argument, we have to gain some control over the curvature term $\mathfrak{R}^E$. More precisely, we will show that the following holds for a suitable choice of ε:

$$\|\mathfrak{R}^E\| < \frac{1}{4}\kappa_0 \tag{$\star$}$$

On $\Gamma(\mathrm{End}(\$_{\mathbb{C}} \otimes E))$ we take the operator norm with respect to the given norm on $\$_{\mathbb{C}} \otimes E$. To prove $(\star)$, we show the following inequalities:

$$\|\mathfrak{R}^E\| \overset{(1)}{\leq} k_n \|R^E\| \overset{(2)}{\leq} \varepsilon^2 k_n \|R^{E_0}\|$$

where k_n is a constant depending on dimension. Thus one can choose:

$$\varepsilon < \sqrt{\frac{\kappa_0}{4k_n \max\{\|R^{E_0}\|\}}}$$

which proves the inequality $(\star)$. To show (1), compute (at a given point $p \in \tilde{X}_\varepsilon$):

$$\begin{aligned}
\|\mathfrak{R}^E\| &= \max_{\|\sigma \otimes v\|=1} \left\{ \|\mathfrak{R}^E(\sigma \otimes v)\| \right\} \\
&= \max_{\|\sigma\| \cdot \|v\|=1} \left\{ \left\| \sum_{j<k} e_j e_k \sigma \otimes R^E_{e_j,e_k} v \right\| \right\} \\
&\overset{\|e_k\|=1}{\leq} \max_{\|\sigma\| \cdot \|v\|=1} \left\{ \sum_{j<k} \|\sigma\| \|R^E_{e_j,e_k}\| \|v\| \right\} \\
&= \sum_{j<k} \|R^E_{e_j,e_k}\| \\
&\leq \frac{n(n-1)}{2} \|R^E\|
\end{aligned}$$

and thus $k_n = (n(n-1))/2$. We have used the following inequality $\|R^E\| = (\sum \|R^E_{e_j,e_k}\|^2)^{1/2} \geq \|R^E_{e_j,e_k}\|$.

In order to prove (2) choose, at a given point $p \in \tilde{X}_\varepsilon$, an orthonormal basis $\{\psi_1, ..., \psi_{k_n}\}$ of $\Lambda^2 T_p \tilde{X}_\varepsilon$ which diagonalizes the symmetric bilinear form:

$$(\psi, \psi') \mapsto \langle (f_\varepsilon)_* \psi; (f_\varepsilon)_* \psi' \rangle$$

i.e. we have

$$\langle (f_\varepsilon)_*\psi_j ; (f_\varepsilon)_*\psi_k \rangle = \lambda_j \delta_{jk}$$

Where $f_* : \Lambda^2 T_p \tilde{X}_\varepsilon \to \Lambda^2 T_{f(p)} \mathbb{S}^{2n}$ is the induced map on bivectors. By the definition of enlargeability, f_ε is an ε-contraction on vectors and thus an ε^2-contraction on bivectors. We therefore have the following:

$$\lambda_j = \langle (f_\varepsilon)_*\psi_j ; (f_\varepsilon)_*\psi_j \rangle = \|(f_\varepsilon)_*\psi_j\| \leq \varepsilon^2 \|\psi_j\| = \varepsilon^2$$

Define for all $1 \leq j \leq k_n$:

$$\zeta_j = \frac{1}{\lambda_j} f_* \psi_j$$

which yields an orthonormal basis for $\Lambda^2 T_{f(p)} \mathbb{S}^{2n}$ the space of bivectors on $\mathbb{S}^{2n}$ based at $f(p)$. We can now compute using the fact that (E, ∇^E) is the pullback of (E_0, ∇^{E_0}) under f:

$$\begin{aligned} \|R^E\|_p^2 &= \sum_{j=1}^{k_n} \|R^E_{\psi_j}\|^2 \\ &= \sum_{j=1}^{k_n} \|R^{E_0}_{f_*\psi_j}\|^2 \\ &= \sum_{j=1}^{k_n} \lambda_j^2 \|R^{E_0}_{\psi_j}\|^2 \\ &\leq \varepsilon^4 \|R^{E_0}\|^2 \end{aligned}$$

Which proves (2) and thus ($\star$).

Step 3: The Lichnerowicz argument

In order to show that $\text{index}(\not{D}_E^+) = 0$, it suffices to show:

$$\ker(\not{D}_E) = \ker(\not{D}_E^2) = \{0\} \in \Gamma(\not{S}_\mathbb{C} \otimes E)$$

To prove this, we need some preparations. Let $p \in \tilde{X}$ and $\xi \in \Gamma(\not{S}_\mathbb{C} \otimes E)$ a twisted complex spinor. We apply the Cauchy-Schwarz inequality and ($\star$):

$$\left| \left\langle \mathfrak{R}_p^E(\xi_p) ; \xi_p \right\rangle_p \right| \leq \left\| \mathfrak{R}_p^E(\xi_p) \right\|_p \|\xi_p\|_p \leq \frac{1}{4}\kappa_0 \|\xi_p\|_p^2 \qquad (\dagger)$$

Define the operator $A \in \Gamma(\text{End}(\not{S}_\mathbb{C} \otimes E))$ by:

$$A = \mathfrak{R}^E + \frac{1}{4}\kappa$$

where the last term is multiplication by the number κ. Still looking at the pointwise picture, we use (†):

$$\langle A_p(\xi_p); \xi_p \rangle_p = \left\langle \mathfrak{R}^E_p(\xi_p); \xi_p \right\rangle_p + \frac{1}{4}\kappa(p) \langle \xi_p; \xi_p \rangle_p \geq \frac{1}{4}(\kappa(p) - \kappa_0) \|\xi_p\|^2_p \qquad (‡)$$

We prove that A is a *positive* operator, i.e. that for the L^2-inner product we have $\langle A(\xi); \xi \rangle \geq 0$. Notice that contrary to (†) and (‡), this is a global rather than a pointwise statement, since it involves an integral over the manifold. Recall that $\kappa(p) - \kappa_0 \geq 0$ everywhere and combine this with (‡) to yield:

$$\begin{aligned} \langle A(\xi); \xi \rangle_{L^2} &= \int_{\tilde{X}_\varepsilon} \langle A_p(\xi_p); \xi_p \rangle_p \, dvol_{\tilde{g}_\varepsilon} \\ &\geq \int_{\tilde{X}_\varepsilon} \frac{1}{4}(\kappa(p) - \kappa_0) \langle \xi_p; \xi_p \rangle_p \, dvol_{\tilde{g}_\varepsilon} \\ &\geq \frac{1}{4} \langle \xi; \xi \rangle_{L^2} \\ &\geq 0 \end{aligned}$$

Notice that the integral over $\tilde{X}_\varepsilon$ makes sense since X is compactly enlargeable. Now rewrite (♣) as:

$$\not{D}^2_E(\xi) = \nabla^*\nabla\xi + A(\xi)$$

and take $\xi_0 \in \ker \not{D}^2_E$ which implies:

$$\left\langle \not{D}^2_E \xi_0; \xi_0 \right\rangle_{L^2} = 0 = \langle \nabla^*\nabla\xi_0 + A(\xi_0); \xi_0 \rangle_{L^2} = \underbrace{\langle \nabla\xi_0; \nabla\xi_0 \rangle_{L^2}}_{\geq 0} + \underbrace{\langle A(\xi_0); \xi_0 \rangle_{L^2}}_{\geq \langle \xi_0; \xi_0 \rangle \geq 0}$$

Therefore, we have $\nabla\xi_0 = 0$ and $\langle \xi_0; \xi_0 \rangle_{L^2} = 0$ which implies that ξ_0 itself vanishes everywhere. The kernel of $\not{D}^2_E$ is thus trivial and the index of $\not{D}^+_E$ vanishes as well.

Step 4: Contradiction to the index theorem
By use of the Atiyah-Singer index theorem (see **7.2.8**) and the computation in **5.4.15** , we

compute the index:

$$\begin{aligned}
\mathrm{index}(\not{D}_E^+) &= \hat{A}(\tilde{X}_\varepsilon, E) \\
&= \Big(\mathrm{ch}E \cdot \hat{\mathbf{A}}(\tilde{X}_\varepsilon)\Big)[\tilde{X}_\varepsilon] \\
&= \left(\left(\dim_{\mathbb{C}}(E) + \frac{(-1)^{n-1}}{(n-1)!}c_n(E)\right) \cdot \hat{\mathbf{A}}(\tilde{X}_\varepsilon)\right)[\tilde{X}_\varepsilon] \\
&= \dim_{\mathbb{C}}(E)\hat{A}(\tilde{X}_\varepsilon) + \frac{(-1)^{n-1}}{(n-1)!}c_n(E)[\tilde{X}_\varepsilon] \\
&= \frac{(-1)^{n-1}}{(n-1)!}c_n(f_\varepsilon^* E_0)[\tilde{X}_\varepsilon] \\
&= \frac{(-1)^{n-1}}{(n-1)!}f_\varepsilon^*(c_n(E_0))[\tilde{X}_\varepsilon] \\
&= \frac{(-1)^{n-1}}{(n-1)!}\deg(f_\varepsilon)c_n(E_0)[\mathbb{S}^{2n}] \\
&\neq 0
\end{aligned}$$

Since by the choice of E_0, we have $c_n(E_0) \neq 0$ and $\deg(f_\varepsilon) \neq 0$ by enlargeability.

$\square$

Chapter 8

The Positive Energy Theorem

For the discussion of the Positive Energy Theorem, we will closely follow J. Lohkamp (see **[Lo]**). The version of this theorem that occurs in General Relativity consists of a claim about a *space-time*, i.e. a Lorentzian four-manifold. The problem can be reduced to questions about *space-like* hypersurfaces of positive scalar curvature. The restriction of the Lorentzian metric to a space-like hypersurface is positive definite, thus one can work in a purely Riemannian setting. Considerations from physics show that one can further restrict the context to so-called asymptotically flat manifolds with some additional asymptotic condition on scalar curvature. This will thus be the context for this chapter. Lohkamp proved that for these specific manifolds, the condition of negative energy can be interpreted purely geometrically: The metric on the manifold can be modified so that it becomes Euclidean outside of a compact without losing positive scalar curvature. If the manifold is spin, this leads to a contradiction by the results we proved in **chapter 7**.

8.1 The setup

Let (X, g) be Riemannian manifold with positive scalar curvature.

Definition 8.1.1. A Riemannian n-manifold (X, g) for $n > 2$ is called **asymptotically flat** if there is a compact $K \subset X$ s.th. the following hold:

1) There is a diffeomorphism $\phi : X \setminus K \longrightarrow \mathbb{R}^n \setminus \overline{\mathbb{D}^n} = \{x \in \mathbb{R}^n | \ \|x\| > 1\}$

2) With respect to the *chart* ϕ, the metric g satisfies the following conditions for some $p > \frac{n-2}{2}$: (i) The metric is Euclidean (denoted by g_0) up to an asymptotically vanishing contribution in the sense that:

$$g = g_0 + h$$

with $h_{ij}(x) = O(\|x\|^{-p})$ for all i, j. The derivatives of the metric are controlled as follows:

$$\|x\| \left| \frac{\partial g_{ij}}{\partial x_k}(x) \right| + \|x\|^2 \left| \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(x) \right| = O(\|x\|^{-p})$$

Furthermore, in order to be able to define the energy of this type of manifold, we need to assume that its scalar curvature is controlled as follows:

$$\kappa_g(x) = O(\|x\|^{-q}) \tag{$\clubsuit$}$$

for some $q > n$.

Definition 8.1.2. The total energy of an asymptotically flat manifold satisfying ($\clubsuit$) is defined by the expression:

$$E(X, g) = b_n \lim_{\rho \to \infty} \int_{S_\rho} \sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_j} \right) \nu_j \, dvol_{S_\rho}$$

Where S_ρ is the Euclidean sphere of radius ρ, $dvol_{S_\rho}$ is the volume form induced on this sphere by the inclusion $S_\rho \subset \mathbb{R}^n$, ν is the outwards pointing unit normal vector of S_ρ and the constant b_n is defined as:

$$b_n = \frac{1}{4(n-2)\mathrm{vol}_{n-1}\mathbb{S}^{n-1}}$$

Energy can be shown to be independent of which chart ϕ is chosen for the asymptotic flatness structure. We will only show that the limit makes sense.

Proof: (Existence of the limit)
Under the asymptotic assumptions, one can show (by a direct computation) that scalar curvature has the following form (see **[Sch]**):

$$\kappa_g(x) = \sum_{i,j=1}^{n} \left(\frac{\partial^2 g_{ij}}{\partial x_i \partial x_j} - \frac{\partial^2 g_{ii}}{\partial x_j \partial x_j} \right) + O(\|x\|^{-2p-2}) \tag{$\spadesuit$}$$

The left-hand side can be integrated by ($\clubsuit$) and $O(\|x\|^{-2p-2})$ can be integrated since $p > \frac{n-2}{2}$, thus the sum on the right-hand side is integrable as well. For any $\varepsilon > 0$, we can choose ρ_0

large enough so that for every ρ_1, ρ_2 satisfying $\rho_2 > \rho_1 > \rho_0$ we have:

$$\begin{aligned}
& \left| \int_{S_{\rho_2}} \sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_j} \right) \nu_j \, dvol_{S_\rho}^{n-1} - \int_{S_{\rho_1}} \sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_j} \right) \nu_j \, dvol_{S_\rho}^{n-1} \right| \\
= & \int_{B_{\rho_2} \setminus B_{\rho_1}} \sum_{i,j=1}^{n} \operatorname{div} \left(\frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_j} \right) dvol^n \\
= & \int_{B_{\rho_2} \setminus B_{\rho_1}} \sum_{i,j=1}^{n} \left(\frac{\partial^2 g_{ij}}{\partial x_i \partial x_j} - \frac{\partial^2 g_{ii}}{\partial x_j \partial x_j} \right) dvol^n \\
< & \ \varepsilon
\end{aligned}$$

Where we have used the divergence theorem, the asymptotic behaviour of scalar curvature prescribed in (♣), together with (♠) and where B_r denotes the Euclidean ball of radius r. The the limit defining the energy exists.

□

8.2 The Positive Energy Theorem

We will mention some reductions that go back to R. Schoen and S.T. Yau without giving all the details (they can be found in [**Sch**] section 4). More precisely, these reductions show that it is sufficient to consider metrics that:

1) Have vanishing scalar curvature,

2) Are conformally flat near infinity,

And this, of course, without losing asymptotic flatness and with total energy arbitrarily close to the total energy of the initial metric. This is essentially Proposition 4.1. from [**Sch**].

For this type of metric $\overline{g}$, we will prove the positive energy theorem:

Theorem 8.2.3. Positive energy theorem

$$E(X, \overline{g}) \geq 0$$

Before turning to Lohkamp's proof of the positive energy theorem, we will roughly describe the above mentioned reductions. Let (X, g) be a Riemannian manifold as in the previous section. First, the condition of scalar flatness can be deduced from the properties of the so-called **conformal Laplacian**, which is defined as the following modification of the usual Laplace-Beltrami operator Δ:

$$\begin{array}{rcl} L : C^\infty(X) & \longrightarrow & C^\infty(X) \\ f & \longmapsto & -c_n \Delta f + \kappa_g \cdot f \end{array}$$

Where c_n is a dimensional constant $c_n = 4\frac{n-1}{n-2}$. This operator has the advantage that it describes the transformation behaviour of scalar curvature under conformal changes of the metrics nicely. Recall the transformation behaviour of scalar curvature under conformal changes:

Proposition 8.2.4. Scalar curvature under conformal changes
Define $\tilde{g} = \lambda g$ for a strictly positive function λ. Then we have:

$$\kappa_{\tilde{g}} = \frac{1}{\lambda}\left(\kappa_g - c_n \lambda^{-\frac{n-2}{4}} \Delta \lambda^{\frac{n-2}{4}}\right)$$

By a direct computation, one can deduce that under the conformal change $\tilde{g} = \lambda^{\frac{4}{n-2}} g$, scalar curvature behaves as follows:

$$\kappa_{\tilde{g}} = \lambda^{-\frac{n+2}{n-2}} L\lambda$$

One can show that the kernel of L contains a strictly positive function u that tends to 1 at infinity (see **[Ba]** for this). Using this, define the conformally modified metric:

$$g_1 = u^{\frac{4}{n-2}} g$$

which is obviously scalar flat since $Lu = 0$. For the conformal flatness at infinity, one defines an appropriate cut-off function φ that is equal to one around the compact K (see **8.1.1**) and vanishes near infinity. Using this, one can define:

$$g_2 = \varphi g + (1 - \varphi) g_0$$

Where g_0 is the Euclidean metric given by the chart for $X \setminus K$ (again, see **8.1.1**). These two modifications of the metric can be suitably combined to yield the desired metric denoted by $\overline{g}$.

We need one last remark before turning to the actual proof of the positive energy theorem. Since $(X, \overline{g})$ is asymptotically flat and conformally flat near infinity, the metric can be written as:

$$\overline{g} = h^{\frac{4}{n-2}} g_0$$

close to infinity such that $h(x)$ tends to one as $x \to \infty$. By the fact that $\kappa_{\overline{g}} = 0$ everywhere, h is harmonic (again by the conformal behaviour of scalar curvature) and one can show that it has the following expansion (see **[Lo]**):

Proposition 8.2.5.

$$h(x) = 1 + \frac{E(X, \overline{g})}{4(n-1)\|x\|^{2-n}} + O(\|x\|^{n-1})$$

Thus, the total energy of the manifold can be seen as a coefficient in the expansion of the metric close to infinity. This is the form of the energy that we will use from now on (in **[Lo]** this is used as the definition of the total energy).

8.3 Proof of the Positive Energy Theorem

We will now discuss Lohkamp's proof of the Positive Energy Theorem for the case where (X, g) is a spin manifold. First, we will prove the (already above mentioned) fact that the function h is harmonic. For this, apply **8.2.4** to our case, where $\overline{g} = h^{\frac{4}{n-2}} g_0$:

$$\kappa_{\overline{g}} = h^{\frac{n-2}{4}} \left(\kappa_{g_0} + c_n h^{-1} \Delta h \right)$$

But, by construction, we have $\kappa_{g_0} = \kappa_{\overline{g}} = 0$, therefore h is harmonic. The proof of the Positive Energy Theorem is roughly divided into two steps: The first consists of showing that under the assumption that energy is negative, one can choose another metric on X so that the manifold is isometric to flat Euclidean space outside of a compact without destroying positive scalar curvature inside of that compact set. The second step is showing that this is contradictory to what we already know about the topology of manifolds with positive scalar curvature.

Step 1:

Proposition 8.3.6. The geometry of (X, g) for negative energy
If $E(M, \overline{g}) < 0$, then there is another complete metric g' on X so that:

1) $\kappa_{g'} \geq 0$ everywhere and $\kappa_{g'} > 0$ somewhere,

2) There is a compact set $K \subset X$ and a radius $R > 0$ so that the following are isometric:

$$(X \setminus K, g') \cong (\mathbb{R}^n \setminus B_R, g_0)$$

(Where B_r denotes the Euclidean ball of radius r centred at the origin.)

For the proof of this we will use the following somewhat technical proposition:

Proposition 8.3.7. For $S = \overline{B_6} \setminus B_1 \subset (\mathbb{R}^n, g_0)$ if the function $u \in C^\infty(S)$ is harmonic and of the form $u = \frac{1}{\|x\|^{n-2}} + f$ with f arbitrarily small $|f| < \delta$, then there is a strictly positive $H \in C^\infty(S)$ so that:

1) $\Delta H \geq 0$ with strict inequality somewhere in the interior of S,

2) $H = u$ in a neighbourhood of ∂B_1,

3) H equal to some strictly positive constant in a neighbourhood of ∂B_6.

The proof of this proposition is entirely constructive:

Proof of 8.3.7: The idea of proof is the following: First we will deal with the special case where $f = 0$. This simplifies the situation considerably by noticing that u is then rotationally symmetric, since $\frac{1}{\|x\|^{n-2}}$ is. By the fact that rotationally symmetric functions $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ are in bijection to functions in one (positive) variable $G \in C^\infty(\mathbb{R}^{>0})$ via the relation $F(x) = G(\|x\|)$. Thus, we will try to find some $h_1 \in C^\infty(\mathbb{R}^{>0})$ whose associated rotationally symmetric function will satisfy conditions 1)-3). Afterwards, we will generalize to the situation where f is non-zero. First we will compute the Laplacian of F in terms of G to gain some insight on which conditions to impose on h_1.

$$\Delta F(x) = G''(\|x\|) + \frac{n-1}{\|x\|} G'(\|x\|) \tag{$\blacklozenge$}$$

Recall that the Euclidean norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ has partial derivatives:

$$\frac{\partial \|\cdot\|}{\partial x_i}(x) = \frac{x_i}{\|x\|}$$

Therefore:

$$\begin{aligned}
\Delta F(x) = \Delta G(\|x\|) &= \sum_{i=1}^n \frac{\partial^2 (G \circ \|\cdot\|)}{\partial x_i^2}(x) \\
&= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(G'(\|x\|) \frac{x_i}{\|x\|} \right) \\
&= \sum_{i=1}^n \left[G''(\|x\|) \frac{x_i^2}{\|x\|^2} + G'(\|x\|) \left(\frac{1}{\|x\|} - \frac{x_i^2}{\|x\|^3} \right) \right] \\
&= G''(\|x\|) + \frac{n-1}{\|x\|} G'(\|x\|)
\end{aligned}$$

Define the function $g \in C^\infty(\mathbb{R} \setminus \{0\}, \mathbb{R})$ by $g(t) = \frac{1}{t^{n-2}}$. Notice that this is the one-variable function corresponding to $\frac{1}{\|x\|^{n-2}}$ under the above described bijection. We want to construct another function $h_1 \in C^\infty(\mathbb{R}^{>0}, \mathbb{R}^{>0})$ which has the following properties (as mentioned above, this will correspond to the H we want to construct in the rotationally symmetric picture):

i) $h_1 = g$ on the interval $]0, 2]$,

ii) h_1 constant and strictly larger than 0 on $[5, \infty)$,

iii) $h_1''(t) + \frac{n-1}{t} h_1'(t) \geq 0$ on $]2, 5[$ and strictly positive on some interval in $[3, 4]$.

To achieve this, we define auxiliary functions $f_{d,s} \in C^\infty(\mathbb{R}, \mathbb{R}^{\geq 0})$ depending on parameters $d, s > 0$ as:

$$f_{d,s}(t) = se^{-\frac{d}{5-t}}$$

for t strictly smaller than 5 and constantly vanishing elsewhere. It is easy to see that this will indeed yield a smooth function for all choices of d and s. We claim that the parameters $d, s > 0$ can be chosen so that:

i)' $f''_{d,s} + (n-1) f'_{d,s} > 0$ on the interval $]1, 5[$,

ii)' $f''_{d,s}(4) > g''(4) = (2-n)(1-n)4^{-n}$,
$f'_{d,s}(4) > g'(4) = (2-n)4^{1-n}$ and
$f_{d,s} < \frac{1}{2} g$ on $]1, 5[$.

To prove this claim, compute:

$$\begin{aligned} f'_{d,s}(t) &= -s\frac{d}{(5-t)^2} e^{-\frac{d}{5-t}} \\ f''_{d,s}(t) &= s\left(\frac{d^2}{(5-t)^4} - \frac{2d}{(5-t)^3}\right) e^{-\frac{d}{5-t}} \end{aligned}$$

For any real k, we have furthermore:

$$(f''_{d,s} + kf'_{d,s})(t) = sd\left(\frac{d - 2(5-t) - k(5-t)^2}{(5-t)^4}\right) e^{-\frac{d}{5-t}}$$

Therefore one can choose d_0 large enough (depending on k) so that $f''_{d,s} + kf'_{d,s} > 0$ on $]1, 5[$ for any $s > 0, d \geq d_0$. The condition (i)' thus is satisfied for $k \geq n - 1$.

Next, choose some k large enough so that

$$k \geq 2\frac{g''(4)}{g'(4)} = \frac{1-n}{2} \tag{I}$$

For any $d > d_0(k)$, we may choose the scaling $s > 0$ so that

$$f''_{d,s}(4) = 2g''(4) = 2(2-n)(1-n)4^{-n} \tag{II}$$

Which immediately implies the first condition of (ii)'. The second condition can be verified by computing using $f''_{d,s}(4) + kf'_{d,s}(4) > 0$ and the above (I) and (II):

$$f'_{d,s}(4) > -\frac{f''_{d,s}(4)}{k} = -\frac{2g''(4)}{k} \geq \frac{2g''(4)g'(4)}{2g''(4)} = g'(4)$$

The third condition can simply be fulfilled by taking d large enough (g is bounded below on the interval $]1,5[$), which is compatible with all previous choices. This proves the claim.

We call f_0 some function $f_{d,s}$ which fulfils (i)' and (ii)'. The next steps consists in recovering from f_0 and g a function h_1 fulfilling (i)-(iii). For this, we will define yet another smooth auxiliary function $k_1 > 0$ that *interpolates* between g'' on $[1,3]$ and f''_0 on $[4,\infty)$ in a suitable sense, we mean by this that k_1 meets the following conditions:

a) $k_1 = g''$ on $[1,3]$,

b) $k_1 = f''_0$ on $[4,\infty)$,

c) $k_1 \geq g''$ on $[3,4]$

This is possible by property ii)' of f_0. We define h_1 to be the following integral (with integration constants $= 0$).

$$h_1(z) = \int_1^z \left[\left(\int_1^y k_1(x)dx\right) + g'(1)\right] dy + g(1)$$

One can easily see that $h_1 = g$ on $[1,3]$ and $h_1 = f_0$ on $[4,\infty)$, thus i) and ii) are satisfied (recall that $f_0 = 0$ on $[5,\infty)$). The condition iii) is obviously satisfied on $[1,3]$ since $h_1 = g$ on this interval and g is associated to an harmonic function. For the interval $[4,\infty)$, this property is true by i)' combined with the fact that $f' < 0$. On $]3,4[$ we use the fact that $h''_1 = k_1 \geq g''$ which yields $h'_1 \geq g'$ and combine it with $g''(t) + \frac{n-1}{t}g'(t) = 0$ to get the corresponding inequality. This shows the Proposition for the special case where $f = 0$ as one can easily check that taking H_1 to be the rotationally symmetric function associated to h_1 yields a function satisfying 1)-3). We will now deduce the general case.

Choose $\psi \in C^\infty(\mathbb{R}^n, [0,1])$ to be cut-off function so that $\psi = 1$ on the ball $B_{3.4}$ and $= 0$ outside of the slightly larger $B_{3.6}$. We are now in position to define the desired function:

$$H = h_1 + \psi f$$

Properties 2) and 3) obviously remain true under this modification of h_1. Thus we only need to check that 1) holds, for which we need some control over $\Delta(\psi f)$. Here we use the hypothesis $|f| < \delta$. Namely, since f is harmonic, its higher partial derivatives are bounded by the supremum of f times a constant (depending on dimension, the order of the partial derivatives as well as the radius of the ball which is considered)[1]. Therefore, we can compute:

$$\begin{aligned}
|\Delta(\psi f)| &\leq |\psi\Delta f| + |f\Delta\psi| + 2|\langle\nabla f, \nabla\psi\rangle| \\
&\leq \sup_{\overline{B_{3.6}}\setminus B_{3.4}}(|f|)|\Delta\psi| + 2c(1,n,d)\sup_{\overline{B_{3.6}}\setminus B_{3.4}}(|f|)\sqrt{\sum_i \frac{\partial\psi}{\partial x_i}}
\end{aligned}$$

Since we consider all of this on the annulus $\overline{B_{3.6}} \setminus B_{3.4}$, the partial derivatives of ψ as well as the constant $c(|\alpha|, n, d)$ are bounded. Thus for sufficiently small δ, we have $|\Delta(\psi f)| < \varepsilon$ for all $\varepsilon < 0$. Since h_1 was constructed so that $\Delta H_1 \geq c > 0$ on some annulus containing $\overline{B_{3.6}} \setminus B_{3.4}$, condition 1) will hold for a small enough ε.

□

Proof of 8.3.6: Recall the setup from **8.2**: $M \setminus K$ is equipped with the metric $\overline{g} = h^{\frac{4}{n-2}} g_0$, where g_0 is the Euclidean metric given by the chart coming from asymptotic flatness. Furthermore, h is harmonic and has the following expansion:

$$h(x) = 1 + \frac{E(X, \overline{g})}{4(n-1)\|x\|^{n-2}} + f$$

For some f vanishing of order $n-1$, i.e. $\sup(|f|\|x\|^{n-1}) = c_0 < \infty$ for some constant c_0. The idea is to show, that f can be taken arbitrarily small $|f| < \delta$ in order to be able to apply **8.3.7**. This can be done by the following rescaling argument. For some $\lambda > 0$, define:

$$\begin{aligned}
s_\lambda : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\
x &\longmapsto \lambda x
\end{aligned}$$

For which $s_\lambda^*(g_0) = \lambda^2 g_0$. Consider the rescaled metric

$$g_\lambda = \frac{1}{\lambda^2} s_\lambda^* \overline{g} = h_\lambda^{\frac{4}{n-2}} g_0$$

where the corresponding h_λ can easily be seen to be given by:

$$h_\lambda(x) = s_\lambda^* h(x) = h(\lambda x)$$

[1]See **[GT]** section 2.7.: If f is harmonic on some Ω which has a compact subset $\Omega' \subset \Omega$, then higher partial derivative of multi-index α is controlled by (where $|\alpha|$ denotes the length of the multi-index α and $d = \mathrm{dist}(\Omega', \partial\Omega)$):

$$\sup_{\Omega'}|\partial^\alpha f| \leq \left(\frac{n|\alpha|}{\mathrm{dist}(\Omega', \partial\Omega)}\right)^{|\alpha|} \sup_{\Omega}|f| = c(|\alpha|, n, d)\sup_{\Omega}|f|$$

By the expansion of h, we have:

$$\begin{aligned} h_\lambda(x) &= 1 + \frac{E}{4(n-1)\lambda^{n-2}\|x\|^{n-2}} + f(\lambda x) \\ &= 1 + \frac{E_\lambda}{4(n-1)\|x\|^{n-2}} + f_\lambda(x) \end{aligned}$$

Where we have thus determined the behaviour of the energy and f under rescalings: $f_\lambda = f \circ s_\lambda$, $E_\lambda = \lambda^{2-n}E$. Thus, using the asymptotic behaviour of f:

$$\begin{aligned} |f_\lambda(x)|\|x\|^{n-1} &= \frac{|f(\lambda x)|\|\lambda x\|^{n-1}}{\lambda^{n-1}} \\ &< \frac{c_0}{\lambda^{n-1}} \end{aligned}$$

Which implies, by the transformation behaviour of E, that the term f_λ vanishes faster than the energy term, i.e. for $\lambda \to \infty$ we have that:

$$\left| \frac{4(n-1)f_\lambda}{E_\lambda} \right| \to 0$$

and can thus be chosen arbitrarily small. By **8.3.7** applied to $u(x) = \frac{1}{\|x\|^{n-2}} + \frac{4(n-1)}{E}f$, we can define a function:

$$\hat{h} = 1 + \frac{E}{4(n-1)}H$$

and a metric:

$$\hat{g} = \hat{h}^{\frac{4}{n-2}} g_0$$

which can be easily seen to be isometric to the Euclidean metric outside of $K \cong \overline{B_1}$. Furthermore, $\hat{g}$ has positive scalar curvature (and strictly somewhere) by application of **8.2.4** and by $\Delta H \geq 0$ (with strict inequality somewhere):

$$\begin{aligned} \kappa_{\hat{g}} &= -c_n \overline{h}^{\frac{n-6}{4}} \Delta \overline{h} \\ &= -4\left(\frac{n-1}{n-2}\right) \overline{h}^{\frac{n-6}{4}} \Delta \left(1 + \frac{E}{4(n-1)}H\right) \\ &= -\left(\frac{E}{n-2}\right) \overline{h}^{\frac{n-6}{4}} \Delta H \\ &\geq 0 \end{aligned}$$

Under the assumption of negative energy, with strict inequality somewhere.

$\square$

Step 2:

Here, we need to assume that X is spin.

Recall that under the negative energy assumption we have constructed a non-negative scalar curvature metric g' on X that is Euclidean outside of some compact set K and has strictly positive scalar curvature somewhere in K. Take now some large enough Euclidean square that contains K and identify opposite edges in the same way as for a torus. The notion of Euclidean square makes sense in $X \setminus K$ due to the fact that X is Euclidean outside of K. This yields a manifold homeomorphic to the connected sum of $\hat{X}$ (the one-point compactification of X) and the n-torus $\mathbb{T}^n$ equipped with a non-negative curvature metric (that is positive somewhere). Since X is spin and enlargeability of the torus implies enlargeability of $\hat{X} \# \mathbb{S}^n$ (if one of the summands is enlargeable, the connected sum is enlargeable, see **[LaMi]** IV. Theorem 5.3.), we have an enlargeable spin manifold with positive scalar curvature metrics. This is a contradiction to **7.3.7**.

□

Appendices

Appendix A

Symmetric polynomials

The purpose of this appendix is to discuss some facts about symmetric polynomials, in particular the corresponding fundamental theorem. These facts are used in Chapter **5** to construct the Chern character. See **[La]** IV.6. for details on symmetric polynomials.

Let R be a commutative ring and $R[x_1, ..., x_n]$ the associated ring of polynomials in the n variables $x_1, ..., x_n$. By S_n we denote the symmetric group, i.e. the group of permutations on n distinct elements. S_n then operates in the following manner on $R[x_1, ..., x_n]$:

$$\begin{array}{rcl} S_n \times R[x_1, ..., x_n] & \longrightarrow & R[x_1, ..., x_n] \\ (\pi, P(x_1, ..., x_n)) & \longmapsto & (\pi P)(x_1, ..., x_n) = P(x_{\pi(1)}, ..., x_{\pi(n)}) \end{array}$$

Definition A.0.1. We call **symmetric polynomials** the elements of $R[x_1, ..., x_n]$ which are invariant under the above operation, i.e. which satisfy: $\pi P = P \;\; \forall \pi \in S_n$

Examples A.0.2. Examples for symmetric polynomials

1) Symmetric polynomials can be non-homogeneous, e.g. $x_1 + x_2 + 1$ is symmetric;

2) The elementary symmetric polynomials $\{\sigma_k(x_1, ..., x_n)\}_{k \leq n}$ are symmetric. The polynomial σ_k in n variables is defined as the sum of all possible terms $x_{i_1}...x_{i_k}$, where all i_j differ. For $n = 3$ one would get for example:

$$\begin{array}{rcl} \sigma_0(x_1, x_2, x_3) & = & 1 \\ \sigma_1(x_1, x_2, x_3) & = & x_1 + x_2 + x_3 \\ \sigma_2(x_1, x_2, x_3) & = & x_1x_2 + x_1x_3 + x_2x_3 \\ \sigma_3(x_1, x_2, x_3) & = & x_1x_2x_3 \end{array}$$

By the fundamental theorem of symmetric polynomials mentioned below, the elementary symmetric polynomials are in some sense the building blocks of all symmetric polynomials;

3) The polynomials $q_k(x_1, ..., x_n) = x_1^k + ... + x_n^k$ are symmetric.

Theorem A.0.3. Fundamental theorem of symmetric polynomials
Every symmetric polynomial $P(x_1, ..., x_n)$ can be uniquely written as a polynomial expression in the elementary symmetric polynomials, i.e. $\exists!\ Q(y_1, ..., y_n) \in R[y_1, ..., y_n]$

$$P(x_1, ..., x_n) = Q(\sigma_1(x_1, ..., x_n), ..., \sigma_n(x_1, ..., x_n))$$

Example A.0.4. Newton's identities for the q_k
As mentioned above, the q_k are symmetric polynomials, thus by the theorem, they can be expressed using the elementary symmetric polynomials: $\exists!\, s_k$ s.th.

$$q_k = s_k(\sigma_1, ..., \sigma_k)$$

The corresponding polynomials can be computed recursively by formulae called *Newton's identities*:

$$s_k = (-1)^{k-1} k\sigma_k + \sum_{i=1}^{k-1} (-1)^{k+i-1} s_i \sigma_{k-i}$$

Which yields for the first few k:

$$\begin{aligned} s_1 &= \sigma_1 \\ s_2 &= \sigma_1 s_1 - 2\sigma_2 = \sigma_1^2 - 2\sigma_2 \\ s_3 &= \sigma_1 s_2 - \sigma_2 s_1 + 3\sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \end{aligned}$$

Appendix B

The Atiyah-Bott-Shapiro construction

The goal of this appendix is to review an alternate approach to relative K-theory that was developed by Atiyah, Bott and Shapiro in '63 in [**ABS**] which will is used in the formulation of the Atiyah-Singer index theorem. Elements of $K(X,A)$ will essentially be seen as equivalence classes of sequences of vector bundles over the subspace A. We will closely follow [**ABS**], but the lecture notes [**Des**] served as secondary reference.

B.1 Sequences of vector bundles

Definitions B.1.1. Basic definitions

1) A set of vector bundles $\{E_n, ..., E_0\}$ over a common base space X together with a collection of homomorphisms $\alpha_i : E_i \to E_{i-1}$ for $i \in \{1, ..., n\}$ belongs to the set $\mathcal{C}_n(X,A)$ (for $A \subset X$) if the sequence:

$$0 \to E_n|_A \xrightarrow{\alpha_n|_A} E_{n-1}|_A \xrightarrow{\alpha_{n-1}|_A} \quad ... \quad \xrightarrow{\alpha_2|_A} E_1|_A \xrightarrow{\alpha_1|_A} E_0|_A \to 0$$

is exact. Write in shorthand notation $E = (E_i, \alpha_i) \in \mathcal{C}_n(X,A)$.

2) A **morphism** φ between two objects $E = (E_i, \alpha_i), F = (F_i, \beta_i)$ in this category is given by a collection of bundle morphisms $\varphi_i : E_i \to F_i$ so that $\beta_i \circ \varphi_i = \varphi_{i-1} \circ \alpha_i$ (Isomorphisms are defined correspondingly). Equivalently the following diagram commutes:

$$\begin{tikzcd} ... \arrow[r, "\alpha_{i+1}"] & E_i \arrow[r, "\alpha_i"] \arrow[d, "\varphi_i"] & E_{i-1} \arrow[r, "\alpha_{i-1}"] \arrow[d, "\varphi_{i-1}"] & ... \\ ... \arrow[r, "\beta_{i+1}"] & F_i \arrow[r, "\beta_i"] & F_{i-1} \arrow[r, "\beta_{i-1}"] & ... \end{tikzcd}$$

3) $E \in \mathcal{C}_n(X, A)$ is called **elementary** if it is of the form:

$$0 \to 0 \to ... \to E_j \overset{id}{\to} E_{j-1} \to ... \to 0 \to 0$$

4) Define $E \oplus F = (E_i, \alpha_i) \oplus (F_i, \beta_i) = (E_i \oplus F_i, \alpha_i \oplus \beta_i)$ the **direct sum for** $\mathcal{C}_n(X, A)$.

5) $E, F \in \mathcal{C}_n(X, A)$ are called **equivalent** (denote $E \sim F$) if they are isomorphic up to addition of elementary objects, i.e.

$$E \sim F \Leftrightarrow E \oplus Q_1 \oplus ... \oplus Q_r \cong F \oplus P_1 \oplus ... \oplus P_s$$

for $P_i, Q_j \in \mathcal{C}_n(X, A)$ elementary.

The set of equivalence classes under the relation defined in 5) is denoted by $\mathcal{L}_n(X, A)$. Since one can check that the direct sum is well defined on the equivalence classes, $\mathcal{L}_n$ is equipped with the structure of a semi-group. Furthermore, one has inclusions:

$$\mathcal{C}_n(X, A) \hookrightarrow \mathcal{C}_{n+1}(X, A)$$

by adding a zero bundle at the end of the sequence. These descend to homomorphisms of the quotients:

$$\mathcal{L}_n(X, A) \to \mathcal{L}_{n+1}(X, A)$$

Our goal will now be to show that the grading in this set of equivalence classes is irrelevant and that they are isomorphic to the relative K-theory $K(X, A)$, i.e. that we have:

$$\mathcal{L}_1(X, A) \cong \mathcal{L}_n(X, A) \cong K(X, A)$$

B.2 Relation to K-Theory

Lemma B.2.2.

$$\mathcal{L}_1(X, \varnothing) \cong K(X, \varnothing) \cong K(X)$$

Proof: Elementary objects in $\mathcal{C}_1(X, \varnothing)$ are of the form $0 \to Q \overset{id}{\to} Q \to 0$, therefore $E \sim F$ with:

$$E = \left(0 \to E_1 \overset{\cong}{\to} E_0 \to 0\right) \text{ and } \quad F = \left(0 \to F_1 \overset{\cong}{\to} F_0 \to 0\right)$$

if and only if there are elementary Q and P s.th.

$$E \oplus Q \cong F \oplus P \Rightarrow E_1 \oplus Q \cong F_1 \oplus P \text{ and } E_0 \oplus Q \cong F_0 \oplus P$$

Define the map:

$$\begin{array}{rcl}\chi_1 : \mathcal{L}_1(X,\varnothing) & \to & K(X) \\ \left(E_1 \stackrel{\cong}{\to} E_0\right) & \mapsto & E_0 - E_1\end{array}$$

The map is well defined, since χ_1 is a morphism and yields 0 when applied to elementary objects, thus $E \sim F \Rightarrow \chi_1(E) = \chi_1(F)$. Furthermore it is surjective, since elements of K-Theory can always be represented as $E_0 - E_1$. To show injectivity, assume $\chi_1(E) = 0$ and thus $E_0 \oplus G \cong E_1 \oplus G$ for some vector bundle G over X by definition of $K(X)$. Therefore $E_1 \oplus G \to E_0 \oplus G \cong 0 \to 0$ in $\mathcal{L}_1(X,\varnothing)$, which proves injectivity.

□

We generalize the map χ_1 used in the proof above:

Definition B.2.3. A transformation of functors $\chi_n : \mathcal{L}_n(X,A) \to K(X,A)$ is called **Euler characteristic for** $\mathcal{L}_n(X,A)$ if for $A = \varnothing$ the following holds:

$$\chi_n(E_n \to ... \to E_0) = \sum_{i=0}^{n}(-1)^i E_i$$

One can show that the above lemma holds without the restriction $A = \varnothing$ with any Euler characteristic χ_1 as isomorphism, i.e.

$$\chi_1 : L_1(X,A) \stackrel{\cong}{\to} K(X,A)$$

Lemma B.2.4. χ_1 exists and is unique.

Proof: Uniqueness
Let χ_1' be a second Euler characteristic for $\mathcal{L}_1$. Define:

$$\vartheta = \chi_1' \circ \chi_1^{-1} : K(X,A) \to K(X,A)$$

For $A = \varnothing$, we have $\vartheta = id$ by the definition of the Euler characteristic. The general case can be reduced to this by considering the following diagram:

$$\begin{array}{ccc} K(X,A) = \tilde{K}(X/A) & \longrightarrow & K(X/A) \\ \downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle \vartheta = id} \\ K(X,A) = \tilde{K}(X/A) & \longrightarrow & K(X/A) \end{array}$$

The canonical horizontal maps are injective and the right vertical map is the identity by what preceded. Thus the left one is as well.

□

Proof: Existence
We will explicitly construct χ_1 to show its existence. Given an element in $\mathcal{L}_1(X, A)$ it can be represented by $E_1 \xrightarrow{\alpha} E_0$. A bundle over the new base space $Y = X_0 \cup_A X_1$ (where X_0 and X_1 are disjoint copies of X) is given by a glueing construction: We think of the vector bundles E_0 and E_1 as $E_0 \to X_0$ and $E_1 \to X_1$ respectively. To view them as a bundle over Y, identify them over A by the map α, i.e.

$$\alpha|_A : E_1|_A \xrightarrow{\cong} E_0|_A$$

This yields a vector bundle that we will denote by $[E_1, \alpha, E_0] \to Y$.

Next, we will look at the situation in K-Theory. Since $X_0/A \approx Y/X_1$, the inclusion map $(X_0, A) \hookrightarrow (Y, X_1)$ induces an isomorphism $K(Y, X_1) \to K(X_0, A)$. Furthermore, since there are natural retractions $\pi_i : Y \to X_i$ whose induced maps split the short exact sequence of the pair (Y, X_i), $K(Y)$ can be viewed as direct sum $K(Y) = K(Y, X_i) \oplus K(X_i)$. Define χ_1 now by viewing $E_1 \xrightarrow{\alpha} E_0 \in \mathcal{L}_1(X, A)$ as $[E_1, \alpha, E_0] \in K(Y)$ and by applying the following sequence of maps:

$$K(Y) \xrightarrow{pr} K(Y, X_1) \cong K(X_0, A) \cong K(X, A)$$

One can show furthermore that for $A = \varnothing$ the image of this map equals $E_1 - E_0$.

□

In a next step we will show that the maps $\mathcal{L}_n \to \mathcal{L}_{n+1}$ discussed above are isomorphisms. Before doing so, we need an auxiliary result on the extension of monomorphisms of vector bundles:

Lemma B.2.5. Let E and F be vector bundles over X with $Y \subset X$ and $f|_Y : E|_Y \to F|_Y$ a monomorphism. If $\dim F > \dim E + \dim X$ then f can be extended to a global monomorphism and this extension is unique up to homotopy.

Proof: The proof relies on a general extension result for fibre bundles. Namely, we consider the fibre bundle $\mathrm{Mon}(E, F)$ of monomorphisms over X. To be more precise:

$$\mathrm{Mon}(E, F)|_x = \{E_x \xrightarrow{i} F_x | \ i \text{ injective}\}$$

One can show that the fibres of this bundle are homeomorphic to $GL(n)/GL(n - m)$ for $n = \dim F$ and $m = \dim E$. This can be seen by identifying $F_x = \mathbb{R}^n$ and $E_x = \mathbb{R}^m$, then

$F_x = \mathrm{im} f \oplus \mathrm{im} f^\perp$, where $\dim(\mathrm{im} f) = m$ and thus $\dim(\mathrm{im} f^\perp) = n - m$. A homeomorphism can be defined:

$$\begin{array}{rcl} \mathrm{Mon}(E,F)|_x & \longrightarrow & GL(n)/GL(n-m) \\ i & \mapsto & [(i(e_1), ..., i(e_m), f_1, ..., f_{n-m})] \end{array}$$

where $f_1, ..., f_{n-m}$ complete the image vectors to a basis. This map can easily be seen to have a well-defined inverse, since $GL(n-m)$ acts transitively on the set of bases for $\mathrm{im} f^\perp$. The fibres are thus $(n-m-1)$-connected[1] and can be extended uniquely up to homotopy if $\dim X \leq n - m - 1$ by a general result on sections of fibre bundles (see **[Hu]** ch. 2 Theorem 7.1). A section of $\mathrm{Mon}(E,F)$ is a global monomorphism between the two vector bundles.

$\square$

Lemma B.2.6.

$$L_n(X,A) \xrightarrow{\cong} L_{n+1}(X,A)$$

This immediately implies the goal of this section: $\mathcal{L}_1(X,A) \cong \mathcal{L}_n(X,A) \cong K(X,A)$ by induction on n

Proof: (During the proof, the pair (X,A) remains fixed, thus we will drop it in the notation.) In order to be able to use the above Lemma, we restrict our attention to the subset:

$$\mathcal{C}'_{n+1} = \{E \in \mathcal{C}_{n+1} \mid \dim E_n \overset{(\dagger)}{>} \dim E_{n+1} + \dim X\} \subset \mathcal{C}_{n+1}$$

This can be done since every element in $\mathcal{L}_{n+1}$ can be represented by an element in $\mathcal{C}'_{n+1}$, which can be seen by adding an elementary object to E if the condition $(\dagger)$ isn't met.

One can choose $E \in \mathcal{C}'_{n+1}$ and extend α_{n+1} to a monomorphism on X α'_{n+1} by the above lemma. We will try to rewrite $E = E' \oplus P$, for E of length n and P an elementary sequence. A natural candidate for P is

$$0 \to E_{n+1} \xrightarrow{id} E_{n+1} \to 0$$

Define $E'_n = \mathrm{coker}\, \alpha'_{n+1} = E_n / \mathrm{im}\, \alpha'_{n+1}$. The rest of E' is defined by the lower sequence of the following diagram

$$\begin{tikzcd} 0 \arrow[r] & E_{n+1} \arrow[r, "\alpha'_{n+1}"] & E_n \arrow[r, "\alpha_n"] \arrow[d, "p"] & E_{n-1} \arrow[r, "\alpha_{n-1}"] & ... \arrow[r, "\alpha_1"] & E_0 \arrow[r] & 0 \\ & 0 \arrow[r] & E'_n \arrow[ur, "\rho'"] & & & & \end{tikzcd}$$

[1] See **[Hu]** 7.5. combined with the fact that $GL(n)/GL(n-m) \approx V_m(\mathbb{R}^n)$, where $V_k(\mathbb{R}^n)$ is a Stiefel variety (again, see **[Hu]** 7.1.)

Where p denotes the projection map to the quotient and ρ' is defined by commutativity of the diagram. Choosing a metric on E_n induces a splitting of the exact sequence

$$0 \to E_{n+1} \xrightarrow{\alpha'_{n+1}} E_n \xrightarrow{p} E'_n \to 0$$

Which proves

$$E \cong E' \oplus P$$

Let us show that this yields a well-defined map from the isomorphism classes in $\mathcal{C}'_{n+1}$ to those in $\mathcal{C}_n$. If α''_{n+1} another extension of α_{n+1} then we know by the previous lemma that $\alpha''_{n+1} \simeq \alpha'_{n+1}$ homotopic. This implies $E'_n \cong E''_n$ by an isomorphism that makes the following diagram commutative:

$$\begin{tikzcd} E'_n \arrow[r, "\rho'"] \arrow[d, "\cong"] & E_{n-1} \\ E''_n \arrow[ur, "\rho''"] & \end{tikzcd}$$

Hence $E' \cong E''$.

Moreover, we will show that the class of E' in $\mathcal{L}_n$ depends only on the class of E in $\mathcal{L}_{n+1}$ i.e. that the map:

$$\begin{array}{rcl} \mathcal{C}'_{n+1} & \to & \mathcal{C}_n \\ E & \mapsto & E' \end{array}$$

descends to a map $\mathcal{L}_{n+1} \to \mathcal{L}_n$. Adding an elementary sequence to any two terms other than E_{n+1} and E_n is easily seen to have no effect on the map $E \mapsto E'$, i.e. we have:

$$(E \oplus P)' = E' \oplus P$$

for any such elementary sequence P. Assume now that P is added to the first two terms, which means we have to consider:

$$0 \to E_{n+1} \oplus P \xrightarrow{\alpha_{n+1} \oplus id_P} E_n \oplus P \to E_{n-1} \to ... \to 0$$

The first term of $(E \oplus P)'$ will thus be $(E \oplus P)'_n = \operatorname{coker}(\alpha_{n+1} \oplus id_P) = (E_n \oplus P)/\operatorname{im}(\alpha_{n+1} \oplus id_P) = E_n/\operatorname{im}\alpha_{n+1}$. This means that we have determined

$$(E \oplus P)' = E'$$

for this case.
Finally, we notice that this map is inverse to the map $\mathcal{L}_n \to \mathcal{L}_{n+1}$ defined above.

$\square$

One can furthermore show the following:

Lemma B.2.7. There is a unique Euler characteristic χ_n on $\mathcal{L}_n(X, A)$ s.th. the following diagram commutes:

$$\begin{tikzcd} \mathcal{L}_1(X, A) \arrow[rr, "\cong"] \arrow[dr, "\chi_1"'] & & \mathcal{L}_n(X, A) \arrow[dl, "\chi_n"] \\ & K(X, A) & \end{tikzcd}$$

χ_n is in particular an isomorphism, since χ_1 is.

Appendix C

The topological degree

The concept of the degree of a map is rather common in topology, since it has a formulation in terms of algebraic topology as well as differential topology. The degree of a map $f : X \to Y$ measures, roughly speaking, how often X is wrapped around Y by f. In differential topology this idea is carried out by counting preimages of regular points with multiplicity, in algebraic topology one looks at the map induced by f on the top (co-)homology groups.
Our exposition of the subject is tailored to the use of the degree in the definition of enlargeability (see 7.1.2). In that context, one wants to make sense of the degree of maps of the type:

$$f : X \to \mathbb{S}^n$$

Where X is not necessarily compact but f is constant outside of a compact set. To acheive this, we use cohomology with compact support. We will also discuss how this approach relates to differential forms and the differential topological approach. In order to illuminate the matter unhindered by the technical complications of the more general cases, we discuss as an introduction the case where the two manifolds are compact. **[MiSt]** Appendix A and **[BoTu]** served as references for our exposition. Other sources from algebraic topology include **[Ha01]** 2.2, **[Do]** IV.§4/§5 / VIII.§4. A purely differential topological approach is carried out in **[Mi]** 4./5.

If not explicitely stated otherwise, (co-)homology classes are taken over $\mathbb{Z}$.

C.1 Introduction: The compact case

Let X be a compact, connected topological n-manifold.

Definition C.1.1. X is called **orientable** if $H_n(X) = \mathbb{Z}$

The choice of a generator $\mu \in H_n(X)$ is called an **orientation**. We denote an **oriented manifold** by (X, μ).

See **[Ha01]** section 3.3. for geometric explanations on how this is related to the orientation of vector spaces. In the context of *differentiable* manifolds, there are a number of different approaches to orientation: A differentiable manifold is orientable if it can be equipped with a differentiable structure whose transition maps have positive determinant. This approach is discussed in most books on differential geometry (see e.g. **[DCa]** p.18). One can check that this is equivalent to the existence of a non-vanishing n-form on the manifold.

Let (X, μ) and (Y, ν) be two closed, connected, oriented n-manifolds. To a continuous map $f : X \to Y$ one can associate the degree as follows:

Definition C.1.2. The **degree of** f, denoted by deg$f$, is an integer defined by the following diagram:

$$\begin{array}{ccc} H_n(X) & \xrightarrow{f_*} & H_n(Y) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z} & \xrightarrow{(\deg f)\cdot} & \mathbb{Z} \end{array}$$

where f_* denotes the induced map on the top homology class. This makes sense, since any morphism $\mathbb{Z} \to \mathbb{Z}$ is multiplication by an integer.

Properties C.1.3. Basic properties of the degree

1) Homotopy equivalences have degree 1 or -1;

2) The degree is multiplicative in the following sense: $\deg(f \circ g) = \deg f \deg g$;

3) The degree is a homotopy invariant: $f \simeq g \Rightarrow \deg f = \deg g$. Interestingly, one can show that the converse also holds in the context of maps $X \to \mathbb{S}^n$, where X any closed, oriented n-manifold. In this context the degree is thus a *complete* homotopy invariant. See **[Hi]** p.129 for details.

   Proofs: Apply functoriality and homotopy invariance of H_n. $\square$

Example C.1.4. Using complex multiplication from $\mathbb{S}^1 \subset \mathbb{C}$, define:

$$\begin{array}{rcl} f_k : \mathbb{S}^1 & \to & \mathbb{S}^1 \\ z & \mapsto & z^k \end{array}$$

Then f_k has degree k. This can easily be seen using the differential topological approach of counting preimages of regular points (see **C.6**). The same holds for the maps $w \mapsto w^k$ on $\mathbb{S}^3$ with multiplication induced by the inclusion $\mathbb{S}^3 \subset \mathbb{H}$.

Application C.1.5. The *Hairy ball theorem*

Every smooth vector field on $\mathbb{S}^{2n}$ vanishes somewhere.

Proof: Suppose $X \in \Gamma(T\mathbb{S}^{2n})$ is non-vanishing. For every point p of the sphere there is a unique plane Π_p containing p, the vector $X|_p$ and the centre of the sphere. Define a homotopy H_t by mapping p to p rotated by $t\pi$ in the plane Π_p in the sense prescribed by X_p. Then $H_0 = id$ and $H_1 = -id$. Let's show $\deg(id) = 1$ and $\deg(-id) = -1$, which finishes the proof by homotopy invariance by deg.

The first statement is a direct consequence from functoriality of H_n. For the second statement, consider that $-id$ is an orientation reversing homotopy equivalence ($\mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ and $-id$ reverses orientation in odd dimensions). For details on this, see [**Do**] section IV.4.

$\square$

C.2 Orientation

Let X be a connected n-manifold. Since it is not necessarily compact, the treatment from **1.1** doesn't apply. We will use relative homology to construct a *localized* version of orientation and cohomology with compact support to define the degree. First notice that for all $x \in X$

$$H_i(X, X \setminus x) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$$

Therefore the only non-zero homology group of $(X, X \setminus x)$ is $H_n(X, X \setminus x) \cong \mathbb{Z}$. This allows us to define:

Definition C.2.6. $\mu_x \in H_n(X, X \setminus x)$ is a **local orientation at** x if it is a generator of $H_n(X, X \setminus x) \cong \mathbb{Z}$.

Before transferring this concept to the whole of the manifold, we need some remarks:

Let U be some neighbourhood of the point x. The natural inclusion:

$$\iota_x : (X, X \setminus U) \hookrightarrow (X, X \setminus x)$$

induces a map ι_{x*} on the n-th homology classes that will be of interest. For example, if $U = B$ a ball with respect to a coordinate chart containing x, then ι_x is an isomorphism. This means that choosing a local orientation μ_x induces an orientation for every other point in B. More precisely, define μ_y for $y \in B$ by:

$$\mu_y = (\iota_{y*} \circ \iota_{x*}^{-1})\mu_x$$

We are now in a position to define orientation:

Definition C.2.7. The prescription $\mu : X \ni p \mapsto \mu_p \in H_n(M, M \setminus p)$, where μ_p is a local orientation, is an **orientation** on M if for all $p \in M$ there is a compact neighbourhood K and a class $\mu_K \in H_n(M, M \setminus K)$ so that $\iota_{p*}(\mu_K) = \mu_p$.
The pair (M, μ) is called **oriented manifold**.

One can show that prescribing a local orientation at a point x prescribes a unique orientation in every compact neighbourhood of x, i.e. there is a unique $\mu_K \in H_n(M, M \setminus K)$ s.th. $\mu_K \mapsto \mu_x$ under the map induced by inclusion. μ_K is sometimes called the *fundamental class* of K.

Using this homological approach, one can define the **local degree** of maps between two oriented manifolds (X, μ) and (Y, ν) in the following manner: Given a continuous $f : X \to Y$ and a compact $K \subset Y$ s.th. $f^{-1}(K) \subset X$ compact as well, define $\deg_K f$ by the equation

$$f_*(\mu_{f^{-1}(K)}) = \deg_K f \cdot \nu_K \tag{$\clubsuit$}$$

This can be used to define the degree of f by putting $\deg f = \deg_K f$ for some compact K, which can be shown to be independent of the choice of K and thus well-defined. See [**Do**] VIII.4. for details. The crucial problem is that f needs to be proper for this approach to make sense, since we need $f^{-1}(K)$ to be compact. This is generally not true in the context in which we want to apply the concept of the degree (see for example 7.1.4). We will therefore formulate similar ideas in terms of cohomology with compact support.

C.3 Cohomology with compact support

In this section, we briefly introduce cohomology with compact support and formulate the corresponding Poincaré duality result. For details and proofs, see [**MiSt**] Appendix A. [**BoTu**] §1 introduces the same concept on $\mathbb{R}^n$ in terms of de Rham cohomology.

Definitions C.3.8.

1) A cochain $c \in C^i(X)$ is said to have **compact support** if there is a compact set $K \subset X$ s.th. c vanishes outside on $C_i(X \setminus K)$ i.e. c belongs to the submodule $C^i(X, X \setminus K)$,

2) Denote by $C^i_{cpt}(X) \subset C^i(X)$ the submodule formed by cochains with compact support,

3) Denote by $H^i_{cpt}(X)$ the cohomology groups associated to the chain complex $\{C^i_{cpt}(X)\}$.

One can show that:

$$H^i_{cpt}(X) \cong \varinjlim \left(H^i(X, X \setminus K)\right)$$

where the direct limit is taken over all compact $K \subset X$. Using this isomorphism, we can construct a morphism

$$H^n_{cpt}(X) \to \mathbb{Z}$$

in the case of oriented (X, μ) by evaluation on the fundamental class:

$$\alpha \mapsto \alpha'[\mu_K] = \langle \alpha'; \mu_K \rangle$$

where $\alpha' \in H^n(X, X \setminus K)$ represents α for some K by the above isomorphism. This is independent of the choice of α'

Using this evaluation morphism one can define a bilinear pairing, called **cap product**:

$$\begin{array}{rcl} \cap : C^i(X) \times C_n(X) & \to & C_{n-i}(X) \\ (\alpha, \xi) & \mapsto & \alpha \cap \xi \end{array}$$

where $\alpha \cap \xi$ is the unique element in $C_{n-i}(X)$ so that the following holds:

$$\langle \beta; \alpha \cap \xi \rangle = \langle \beta\alpha; \xi \rangle$$

The cap product can be used to construct a Poincaré duality isomorphism:

Theorem C.3.9. Poincaré duality in the compactly supported case

$$\begin{array}{rcl} H^i_{cpt}(X) & \longrightarrow & H_{n-i}(X) \\ \alpha & \longmapsto & \alpha' \cap \mu_K \end{array}$$

(where $\alpha' \in H^n(X, X \setminus K)$ represents α) is an isomorphism.

C.4 The cohomological degree

Let X be a connected, oriented n-manifold and f a continuous map:

$$f : X \to \mathbb{S}^n$$

that is constant outside of a compact subset $K \subset X$. One could replace $\mathbb{S}^n$ by some other compact n-manifold, but taking the n-sphere is sufficient for our purposes. As a corollary to **C.3.9** and since X is connected, we can compute:

$$H^n_{cpt}(X) \cong H_0(X) \cong \mathbb{Z}$$

Definition C.4.10. The **degree of** f, denoted by deg$f$ is defined by the following diagram:

$$\begin{tikzcd} H^n(\mathbb{S}^n) \arrow[r, "f^*"] \arrow[d, "\cong"] & H^n_{cpt}(X) \arrow[d, "\cong"] \\ \mathbb{Z} \arrow[r, "(\deg f)\cdot"] & \mathbb{Z} \end{tikzcd}$$

We need to show that $f^* : H^n(\mathbb{S}^n) \to H^n_{cpt}(X)$ is well-defined, i.e. that for all $\alpha \in H^n(\mathbb{S}^n)$ the element $f^*\alpha$ has compact support:

Proof: Let $\xi \in C_n(M \setminus K)$ be an element with support outside of K. We will show that it is annihilated by $f^*\alpha$. First define the canonical inclusion maps $i : M \setminus K \hookrightarrow M$ and $j : \{*\} \hookrightarrow \mathbb{S}^n$ where $*$ denotes the point in $\mathbb{S}^n$ to which $M \setminus K$ is mapped by f. In other words, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] & \mathbb{S}^n \\ M \setminus K \arrow[u, "i"] \arrow[r, "f"] & \{*\} \arrow[u, "j"'] \end{tikzcd}$$

We can now compute:

$$\begin{aligned} \langle f^*\alpha; i_*\xi\rangle &= \langle i^* f^*\alpha; \xi\rangle \\ &= \langle f^* j^*\alpha; \xi\rangle \\ &\overset{j^*\alpha=0}{=} 0 \end{aligned}$$

Thus we have $f^*\alpha \in H^n_{cpt}(M)$.

$\square$

C.5 The degree in the smooth setting using differential forms

For an account of cohomology with compact support purely in the language of de Rham theory and the definition of the degree in that context, see **[Pet98]** Appendix A.5/A.6.

For the rest of **Appendix C**, we will assume X to be a connected, oriented *smooth* n-manifold and $f : X \to \mathbb{S}^n$ to be a smooth map. Using the following de Rham isomorphisms (here we take coefficients in $\mathbb{R}$):

$$\begin{aligned} H^i(\mathbb{S}^n;\mathbb{R}) &\cong H^i_{dR}(\mathbb{S}^n) \\ H^i_{cpt}(X;\mathbb{R}) &\cong H^i_{dR,cpt}(X) \end{aligned}$$

we can reformulate the definition of the topological degree in terms of differential forms. Any given $\omega \in \Omega^n(\mathbb{S}^n)$ defines an element in $H^n_{dR}(\mathbb{S}^n)$, since forms of the top degree are closed. The corresponding pull-back under f is closed (for the same reason) and has compact support, since: $f|_{M\setminus K}$ is constant and thus $f^*\omega|_{M\setminus K} \equiv 0$. Therefore we have $[f^*\omega] \in H^n_{dR,cpt}(X)$. This point of view yields the following computational formula:

Proposition C.5.11. For $\omega \in \Omega^n(\mathbb{S}^n)$ s.th. $\int_{\mathbb{S}^n} \omega \neq 0$, the following holds:

$$\deg f = \frac{\int_X f^*\omega}{\int_{\mathbb{S}^n} \omega}$$

Proof: Using the above de Rham isomorphisms, consider the following commutative diagram:

$$\begin{tikzcd}
H^n_{dR}(\mathbb{S}^n) \arrow[r, "\cong"] \arrow[dr, "\int_{\mathbb{S}^n}"'] & H^n(\mathbb{S}^n,\mathbb{R}) \arrow[r, "f^*"] \arrow[d, "\cong"] & H^n_{cpt}(X,\mathbb{R}) \arrow[d, "\cong"] & H^n_{dR,cpt}(X) \arrow[l, "\cong"'] \arrow[dl, "\int_X"] \\
& \mathbb{R} \arrow[r, "(\deg f)\cdot"] & \mathbb{R} &
\end{tikzcd}$$

Therefore $\int_X f^*\omega = \deg f \cdot \int_{\mathbb{S}^n} \omega$

□

C.6 The differential topological degree

In this section, we will reformulate our definition of the degree in terms of notions from differential topology. In this setting, the degree is essentially viewed as a count of preimages of a regular value weighted by whether orientation is preserved or reversed around the pair formed by the regular value and the preimage under consideration. This point of view is useful for computations.

Let $p \in \mathbb{S}^n$ be a regular value and $f^{-1}(\{p\}) \subset X$ the set of its preimages. For any $x \in f^{-1}(\{p\})$, by the inverse function theorem, we can choose neighbourhoods $U \subset \mathbb{S}^n$ of p and $V_x \subset X$ of x s.th. V_x is mapped diffeomorphically onto U by $f|_{V_x}$.

The **local degree** ε_x around the pair (p, x) is defined by the following diagram:

$$\begin{tikzcd} H^n_{cpt}(U) \arrow[r, "f^*"] \arrow[d, "\cong"] & H^n_{cpt}(V_x) \arrow[d, "\cong"] \\ \mathbb{Z} \arrow[r, "(\varepsilon_x)\cdot"] & \mathbb{Z} \end{tikzcd}$$

Where $\varepsilon_x = \pm 1$ depending on whether the orientation around these points is preserved or reversed. Note how this is dual to (♣).

Proposition C.6.12. For $f : X \to \mathbb{S}^n$ a smooth map between smooth manifolds and p a regular value of f, we have:

$$\deg f = \sum_{x \in f^{-1}(\{p\})} \varepsilon_x$$

independently of the choice of p.

Proof: Consider the following diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "(\deg f)\cdot"] \arrow[d, "\cong"] & \mathbb{Z} \arrow[d, "\cong"] & & \\ H^n(\mathbb{S}^n) \arrow[r, "f^*"] \arrow[d, "\cong"] & H^n_{cpt}(X) & & \\ H^n(U, U \setminus \{p\}) \arrow[r, "f^*"] & H^n(f^{-1}(U), f^{-1}(U \setminus \{p\})) \arrow[u, "\bigoplus \varepsilon_x"] \arrow[r, "\cong"] & \bigoplus_{x \in f^{-1}(\{p\})} H^n(V_x, V_x \setminus x) \cong \bigoplus_{x \in f^{-1}(\{p\})} \mathbb{Z} \arrow[uul, "\sum_{x \in f^{-1}(\{p\})}"'] & \end{tikzcd}$$

□

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