University of Fribourg

Master's Thesis

Lawson-Yau Theorem on positive scalar curvature through Cheeger deformations

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Abstract

In the early 70's, Cheeger used an *isometric* action of a *compact* Lie group G with a biinvariant metric b on a Riemannian manifold (M, h) to create a parametrized family of metrics $(h_t^G)_{t>0}$ on M which shrinks the orbits $G \cdot p$. According to the Gray-O'Neill Formula applied to the orbital submersions $\rho : (M \times G, h + \frac{1}{t}b) \to (M, h_t^G)$, the Cheeger metrics don't carry a lower sectional curvature than the respective ones on $M \times G$. This construction discloses new non-negatively or even positively curved manifolds.

This thesis first details the technical aspects of the Cheeger deformation following Müter's approach and, as an illustration of this process, we explore the example of the rotation of $\mathbb{C}$ through an S^1-action. We then expose some properties of the sectional curvatures $sec_{h_t^G}$ compared to the initial one sec_h. In the last chapter, we discuss the Lawson-Yau Theorem (1974) stating that a *compact* manifold with a *non-abelian symmetry* always admits a Riemannian metric of strictly positive scalar curvature. In 2018, Cavenaghi and Sperança found a more intuitive proof of this result by using Cheeger deformations. A concrete formula for the scalar curvatures $scal_{h_t^G}$ developped through all the accumulated knowledge plays a crucial role in their argumentation.

Acknowledgements

About five years ago began this long and eventful love story with Mathematics, the real ones with more abstract statements than real computations. With my fellow students, we faced numerous technical proofs with the necessary but insufficient skills of cooperation and rigor learnt from talented assistants like Dr. Jordane Granier.

Already in the first months, one field attracted me particularly : Algebra and later Geometry. By some miracle, I had the opportunity to deepen my knowledge with the more conscientious and patient of all : Prof. Anand Dessai. Semester after semester, he always accepted to answer my relentless questions and to welcome me in his office, especially during this last decisive month. All the conjectures in the world wouldn't be enough to express my gratitude towards him.

I also profusely thank my ex-colleagues, Jonathan Wermelinger and David González Álvaro, who unlocked more than one problem during the quiet moments of Linear Algebra exercise class. A wink to Joé for his improvised explanation on fiber bundles in the train.

But this productive journey wouldn't have been possible without the moral support of my relatives and friends. I dedicate this Master thesis to my mother who always offered me a listening ear, even when she hadn't the slightest idea what I was talking about.

Contents

Introduction

This thesis covers two relatively recent discoveries in Riemannian Geometry, namely the *Cheeger deformations* - a detection method of some new nonnegatively curved metrics -, and the *Lawson-Yau Theorem* - proving the existence of a metric of positive scalar curvature on compact and connected manifolds on which a compact, connected and non-abelian Lie group acts smoothly. But let's first briefly explain the different notions of curvatures.

A manifold M always admits a Riemannian metric h, allowing convenient measures of angles and lengths of curves on M. A goal of Riemannian Geometry is to compare the geometrical aspects of (M, h) with the ones of the euclidean space $\mathbb{R}^n$ equipped with the standard inner product h_{euc}. This can be done through the *sectional curvature* sec_h, which catches some information about the "acceleration" of h compared to h_{euc}. This interpretation is illustrated by the so-called *geodesics*, the curves linking two points in the most direct way, that is by minimizing the distances. By definition, their trajectory is entirely determined by the chosen metric. Roughly speaking, on a manifold with a positively curved metric, geodesics with slight different directions get closer, while with a negatively curved metric, these curves would move away from each other. Riemannian manifolds of positive and negative sectional curvature can be depicted through the sphere $S^2 \subset \mathbb{R}^3$ endowed with the round metric $\iota^* h_{euc}$ and the hyperbolic manifold $(\mathbb{H}^2, h_{hyp})$, respectively.

Note that these two examples are so-called *space forms*, i.e. manifolds of constant sectional curvature. However, for more general manifolds, this quantity is only a local property in the sense that its amplitude and even its sign can differ from a point to another, but of course in a smooth manner. In addition, the sectional curvature is defined on 2-planes which means that it may be positive for some pairs of tangent vectors and negative for others when the manifold dimension exceeds 2.

Some general results have been formulated on metrics of negative sectional curvature or negative Ricci curvature, a notion derived from *sec*. As an example, the Cartan-Hadamard Theorem states that the Euclidean space is the universal cover of any complete Riemannian manifold of nonpositive sectional curvature via the exponential map [Car92, Chapter 7, Theorem 3.1]. In this situation, the fundamental group contains all topological information. More recently, Lohkamp proved that manifolds of dimension at least 3 always admit a metric of negative Ricci curvature [Loh94].

Contrary to the above statements, results on positive and nonnegative sectional curvature often need some very specific assumptions and therefore concern a limited number of manifolds. One can ask what is the topological significance of metrics of positive sectional curvature. For example, the Bonnet-Schoenberg-Myers Theorem (1935) states that a complete Riemannian manifold with this property is automatically compact [Ber03, Theorem 62]. However, as remarked by Yau in 1982, it is not known whether a compact and simply connected manifold endowed with a metric of nonnegative sectional curvature also admits a metric of positive sectional curvature [Ber03, Fact 325]. Gromoll and Meyer also demonstrated an interesting result : if a manifold M admits a non-compact complete and positively curved metric h, then a diffeomorphism exists between $\mathbb{R}^n$ and M [GW69].

One sometimes uses a well-known Riemannian manifold $(\hat{M}, \hat{h})$ to study the possible metrics that another one, M, can admit. A convenient tool in this direction is the *Gray-O'Neill Formula* (1966-67) which studies the relation between the sectional curvatures of the total and the base spaces of a Riemannian submersion $\pi : (\hat{M}, \hat{h}) \to (M, h)$:

$$sec(V, W) = sec(\hat{V}, \hat{W}) + \frac{3}{4}\|[\hat{V}, \hat{W}]^{\mathcal{V}}\|^2_{\hat{h}},$$

where $V, W \in \mathfrak{X}(M)$ are orthonormal vector fields with horizontal lifts $\hat{V}, \hat{W} \in \mathfrak{X}(\hat{M})$. Hence, the base space carries not lower curvature than the total space.

Based on this formula and inspired by the Berger spheres, Cheeger developed in 1973 a method allowing some Riemannian manifold (M, h) to - possibly - increase its sectional curvature whenever a compact Lie group G acts isometrically on it [Che73]. The idea consists in the construction of a new isometric G-action on the product manifold $(M \times G, h + \frac{1}{t}b)$, where $t > 0$ is a variating parameter and b is a biinvariant metric. The resulting orbital submersions[1], $\rho : (M \times G, h + \frac{1}{t}b) \to (M, h_t^G) \cong ({}^{M \times G}/_{G}, \bar{h}_t^G)$, create a smooth variation of Riemannian metrics $\{h_t^G\}_{t>0}$ on M. This *Cheeger deformation* $h \to h_t^G$ has the interesting property to shrink along the orbits - the vertical directions - while letting horizontal spaces unchanged. The point of this method is to discover new nonnegatively curved metrics by using the key property that the biinvariant metrics carry nonnegative sectional curvature.

The first part of this thesis details the Cheeger method in three chapters by following the 1987's Phd thesis of Müter [Mü87]. To ease understanding of quite specialized concepts, basic tools of Riemannian Geometry and Lie groups Theory are recalled in Appendices A and B, respectively. The notions defined in this part are mainly based on the taken notes of the lectures [Gon17], [Bau16] and [Des18] given at the University of Fribourg.

Chapter 1 is dedicated to concepts related to the Gray-O'Neill Formula, as well as its implications on the homogeneous manifolds. In Chapter 2, we explore the essential link between the manifold at hand and the Lie group acting on it. Indeed, elements x of the Lie algebra $\mathfrak{g}$ generate vector fields X^* on M through the G-action. These are called the *action fields* on M and are tangent to the orbits. Each tangent space T_pM can also split into a *vertical space* $\mathcal{V}_p$ - made up of the tangent vectors of the form $X^*(p)$ - and its orthogonal component, the *horizontal space* $\mathcal{H}_p$. The relation between the metrics b and h are highlighted by the *orbit tensor* S. At last, the construction of Cheeger properly speaking only appears in Chapter 3 but a concrete example of the rotation of $\mathbb{C}$ through an S^1-action illustrates all abstract notions introduced so far.

In Chapter 4, we present some of the curvature characteristics of the Cheeger metrics h_t^G, notably the existence of a lower bound for $sec_{h_t^G}$ but also some conditions for a positive scalar curvature $scal_{h_t^G}$. Proofs of this section come from [DG19, Section 5.1]. We then undertake a more advanced analysis on the additional term ζ_t appearing in the computation

[1] The orbital submersions are the Riemannian submersions built from the quotient map of the considered action.

of $sec_{h_t^G}$ as a function of the original sectional curvature sec_h. Several years after Müter's work, Wolfgang Ziller wrote a concise summary of the Cheeger construction and exposed main conclusions, as well as some further applications and specific results [Zil06].

In the sequel, we'll rather focus on the scalar curvature in the Cheeger metrics h_t^G, that's why we briefly define this notion here. The *scalar curvature scal* of a Riemannian manifold (M, h) assigns a real number to each point p of M by taking some sort of averaging of the sectional curvature over the different directions. To be precise :

$$scal(p) := 2 \sum_{1 \leqslant i < j \leqslant n} sec(e_i, e_j),$$

where $(e_1, ..., e_n)$ is an orthonormal basis of T_pM.

In a sense, *scal* loses some information compared to *sec*, but it possess some useful utilization to measure how the volume of the ball $B_p(r) \subset M$ centered at p and with radius $r > 0$ differs from the one of the euclidean unit ball $B^e(1) \subset \mathbb{R}^n$ [GHL04, Theorem 3.98] :

$$vol\big(B_p(r)\big) = r^n \cdot vol\big(B^e(1)\big) \cdot \big(1 - \frac{scal(p)}{6(n+2)} \cdot r^2 + o(r^2)\big).$$

If we think again of the sphere S^2 which has positive scalar curvature, the volume of a calotte - a portion of the sphere - has some missing "matter" compared to a disc in $\mathbb{R}^2$, since it grows slower with the radius r than in the flat space. Intuitively, if we try to flatten this shape, it would tear at some places.

The final aim of this thesis, developed in Chapter 5, is to prove the Lawson-Yau Theorem [LY74], which states that if a compact, connected and non-abelian Lie group G acts effectively and isometrically on a compact and connected Riemannian manifold (M, h), then M admits a metric of positive scalar curvature.

Cavenaghi and Speranças's article [CS18] gives an alternative proof of this statement through Cheeger deformations. We follow their approach to explore a concrete formula of the scalar curvature for the Cheeger metrics h_t^G. Each of the three components involved in this formula plays a central role in the proof of the Lawson-Yau Theorem to find a Cheeger metric of positive scalar curvature. The first component can be bounded from below by a compactness argument, while the other two diverge either if we look at singular or regular points. Hence, for both kinds of points, there exists a $t_p > 0$ with $scal_{h_t^G}(p) > 0$ for all $t > t_p$, $p \in M$. Again, by a compactness argument, a common lower bound exists for the parameters t to obtain $scal_{h_t^G} > 0$.

Chapter 1

Isometric actions on Riemannian manifolds

This first chapter introduces some basic elements entailed in the Cheeger construction, requiring two Riemannian manifolds related by a special submersion.

1.1 Riemannian submersions and Gray-O'Neill formula

We first consider a submersion $\pi : \hat{M} \to M$ between two connected manifolds, i.e. the following holds :

(i) π is smooth ;

(ii) its differential $d\pi_{\hat{p}} : T_{\hat{p}}\hat{M} \to T_{\pi(\hat{p})}M$ is surjective $\forall \hat{p} \in \hat{M}$.

As a consequence, $\dim \hat{M} \geqslant \dim M$.

Remark 1.1. In this thesis, we will always assume a submersion to be surjective.

Several notions are related to this map :

Definitions 1.2 - Fiber & vertical/horizontal space

For $p \in M$, the **fiber at p** $\mathcal{F}_p := \pi^{-1}(p)$ forms a submanifold of $\hat{M}$ by the *Submersion Theorem.* All together the fibers form the **foliation** $\mathcal{F} := (\mathcal{F}_p)_{p\in M}$.

We now endow $\hat{M}$ with a metric $\hat{h}$, which allows us to break each tangent space $T_{\hat{p}}\hat{M}$, $\hat{p} \in \hat{M}$, down into two components :

- the (π-)**vertical space at $\hat{\mathbf{p}}$** : $\mathcal{V}_{\hat{p}} := T_{\hat{p}}\mathcal{F}_{\pi(\hat{p})} = \ker(d\pi_{\hat{p}})$;
- the (π-)**horizontal space at $\hat{\mathbf{p}}$** : $\mathcal{H}_{\hat{p}} := \mathcal{V}_{\hat{p}}^{\perp} := \left\{\hat{w} \in T_{\hat{p}}\hat{M} \mid \hat{h}_{\hat{p}}(\hat{v}, \hat{w}) = 0 \ \forall \hat{v} \in \mathcal{V}_{\hat{p}}\right\}$.

Hence,

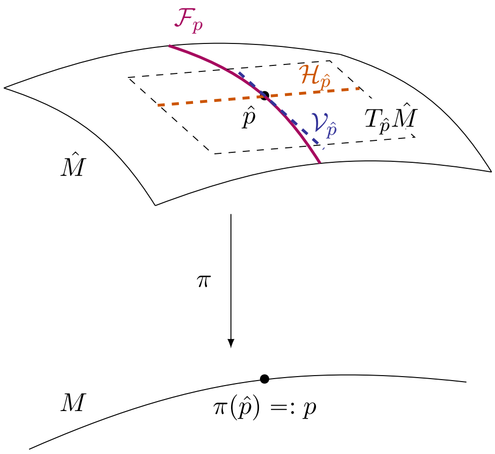
$$T_{\hat{p}}\hat{M} = \mathcal{V}_{\hat{p}} \oplus \mathcal{H}_{\hat{p}}.$$

and each tangent vector $\hat{v} \in T_{\hat{p}}\hat{M}$ splits into its vertical and horizontal components :

$$\hat{v} = \hat{v}^{\mathcal{V}} + \hat{v}^{\mathcal{H}} \text{ with } \hat{v}^{\mathcal{V}} \in \mathcal{V}_{\hat{p}} \text{ and } \hat{v}^{\mathcal{H}} \in \mathcal{H}_{\hat{p}}.$$

This distinction forms :

- the (π-)**vertical distribution** : $\mathcal{V} := (\mathcal{V}_{\hat{p}})_{\hat{p}\in\hat{M}}$;
- the (π-)**horizontal distribution** $\mathcal{H} := (\mathcal{H}_{\hat{p}})_{\hat{p}\in\hat{M}}$.



We denote by $\mathfrak{X}(\cdot)$ the **Lie algebra of the vector fields** of a manifold.

Definition 1.3 - Horizontal lift

Since the restriction $d\pi_{\hat{p}}|_{\mathcal{H}_{\hat{p}}}$ is an isomorphism for all $\hat{p} \in \hat{M}$, to every $X \in \mathfrak{X}(M)$ corresponds a **horizontal lift** $\hat{X} \in \mathfrak{X}(\hat{M})$, i.e. for all $\hat{p} \in \hat{M}$:

- $\hat{X}$ is *horizontal* in all points : $\hat{X}(\hat{p}) \in \mathcal{H}_{\hat{p}}$;
- $\hat{X}$ and X are *π-related* : $d\pi_{\hat{p}}\big(\hat{X}(\hat{p})\big) = X\big(\pi(\hat{p})\big)$.

The same result exists for curves in M.

Let's now also endow the base space M with a metric h. Then, there may exist a stronger relation between the Riemannian manifolds $(\hat{M}, \hat{h})$ and (M, h) than a mere submersion :

Definition 1.4 - Riemannian submersion

One calls a map $\pi : (\hat{M}, \hat{h}) \to (M, h)$ a **Riemannian submersion** if

(i) $\pi : \hat{M} \to M$ is a *submersion* ;

(ii) the restricted differential $d\pi_{\hat{p}}\,|_{\mathcal{H}_{\hat{p}}}$ is an *isometry* for all $\hat{p} \in \hat{M}$, i.e. if we denote by $\hat{X}$ and $\hat{Y}$ the horizontal lifts of $X, Y \in \mathfrak{X}(M)$, then :

$$\hat{h}_{\hat{p}}(\hat{X}, \hat{Y}) = h_{\pi(\hat{p})}(X, Y).$$

In this case, h is called **submersion metric**.

We denote by $\hat{\nabla}$ and ∇ the Levi-Civita connections for $(\hat{M}, \hat{h})$ and (M, h), respectively.

Let's construct two vector fields tensors on $\hat{M}$:

Definitions 1.5 - Tensors related to a Riemannian submersion

Let $\hat{X}, \hat{Y} \in \mathfrak{X}(\hat{M})$.

We define :

- the **second fundamental tensor of the fibers** T :

$$T_{\hat{X}}\hat{Y} := \left(\hat{\nabla}_{\hat{X}^{\mathcal{V}}}\hat{Y}^{\mathcal{V}}\right)^{\mathcal{H}} + \left(\hat{\nabla}_{\hat{X}^{\mathcal{V}}}\hat{Y}^{\mathcal{H}}\right)^{\mathcal{V}};$$

- the **O'Neill tensor** A :

$$A_{\hat{X}}\hat{Y} := \left(\hat{\nabla}_{\hat{X}^{\mathcal{H}}}\hat{Y}^{\mathcal{V}}\right)^{\mathcal{H}} + \left(\hat{\nabla}_{\hat{X}^{\mathcal{H}}}\hat{Y}^{\mathcal{H}}\right)^{\mathcal{V}}.$$

Proposition 1.6

Let $\pi : (\hat{M}, \hat{h}) \to (M, h)$ be a Riemannian submersion ;
$\hat{X}, \hat{Y} \in \mathfrak{X}(\hat{M})^{\mathcal{H}}$, where $\mathfrak{X}(\hat{M})^{\mathcal{H}}$ denotes the set of horizontal vector fields on $\hat{M}$.

Then the following identity holds :

$$A_{\hat{X}}\hat{Y} = \frac{1}{2}\left[\hat{X}, \hat{Y}\right]^{\mathcal{V}}.$$

To prove it, we first need some more basic properties :

Lemma 1.7

We consider a Riemannian submersion $\pi : (\hat{M}, \hat{h}) \to (M, h)$;
a vertical vector field $\hat{V} \in \mathfrak{X}(\hat{M})^{\mathcal{V}}$;
some $X, Y \in \mathfrak{X}(M)$ with horizontal lifts $\hat{X}, \hat{Y} \in \mathfrak{X}(\hat{M})^{\mathcal{H}}$.

Then :

(i) $[\hat{V}, \hat{X}]$ is vertical ;

(ii) The Lie derivative[1] of $\hat{h}$ in the direction $\hat{V}$ vanishes at $(\hat{X}, \hat{Y})$:

$$(\mathcal{L}_{\hat{V}}\hat{h})(\hat{X}, \hat{Y}) = \hat{V}\hat{h}(\hat{X}, \hat{Y}) = 0;$$

(iii) $\hat{h}\left([\hat{X}, \hat{Y}], \hat{V}\right) = 2\hat{h}\left(\hat{\nabla}_{\hat{X}}\hat{Y}, \hat{V}\right) = -2\hat{h}\left(\hat{\nabla}_{\hat{V}}\hat{X}, \hat{Y}\right) = 2\hat{h}\left(\hat{\nabla}_{\hat{Y}}\hat{V}, \hat{X}\right)$;

(iv) $\hat{\nabla}_{\hat{X}}\hat{Y} = \widehat{\nabla_X Y} + \frac{1}{2}[\hat{X}, \hat{Y}]^{\mathcal{V}}$, with $\widehat{\nabla_X Y}$ horizontal lift of $\nabla_X Y$.

[1] See Definition A.15

Proof 1.7:

Ad (i) : We know that $d\pi(\hat{V}) = 0$ since $\hat{V}$ vertical.
Thus, by Naturality of Lie brackets (Proposition A.9):

$$d\pi_{\hat{p}}\left([\hat{X},\hat{V}](\hat{p})\right) = \left[d\pi_{\hat{p}}(\hat{X}(\hat{p})), d\pi_{\hat{p}}(\hat{V}(\hat{p}))\right] = \left[X\left(\pi(\hat{p})\right), 0_{\hat{p}}\right] = 0_p.$$

$\Rightarrow [\hat{X},\hat{V}](\hat{p}) \in \ker(d\pi_{\hat{p}}) = \mathcal{V}_{\hat{p}}\ \forall \hat{p} \in \hat{M}$.
$\Rightarrow [\hat{X},\hat{V}] \in \mathfrak{X}(\hat{M})^{\mathcal{V}}$.

✓

Ad (ii) :

$$\mathcal{L}_{\hat{V}}\hat{h}(\hat{X},\hat{Y}) := \hat{V}\hat{h}(\hat{X},\hat{Y}) - \underbrace{\hat{h}\left([\hat{V},\hat{X}],\hat{Y}\right)}_{\substack{=0 \text{ by } (i) \text{ and} \\ \text{orthogonality}}} - \underbrace{\hat{h}\left(\hat{X},[\hat{V},\hat{Y}]\right)}_{\substack{=0 \text{ by } (i) \text{ and} \\ \text{orthogonality}}} = \hat{V}\hat{h}(\hat{X},\hat{Y}).$$

Now, since π is a Riemannian submersion, we have :

$$\hat{h}(\hat{X},\hat{Y})(\hat{p}) = h(X,Y)(p) \qquad \forall p \in M \text{ and } \forall \hat{p} \in \pi^{-1}(p).$$

$\Rightarrow \hat{h}(\hat{X},\hat{Y})$ *stays constant in vertical directions.*
$\Rightarrow \mathcal{L}_{\hat{V}}\hat{h}(\hat{X},\hat{Y}) = \hat{V}\hat{h}(\hat{X},\hat{Y}) = 0$.

✓

Ad (iii) : We simply use the Koszul formula :

$$\begin{aligned} 2\hat{h}(\hat{\nabla}_{\hat{X}}\hat{Y},\hat{V}) &= \hat{X}\ \underbrace{\hat{h}(\hat{Y},\hat{V})}_{\substack{=0 \text{ by} \\ \text{orthogonality}}} + \hat{Y}\ \underbrace{\hat{h}(\hat{V},\hat{X})}_{\substack{=0 \text{ by} \\ \text{orthogonality}}} - \underbrace{\hat{V}\hat{h}(\hat{X},\hat{Y})}_{\overset{(ii)}{=}0} \\ &\quad + \hat{h}\left([\hat{X},\hat{Y}],\hat{V}\right) - \underbrace{\hat{h}\left([\hat{Y},\hat{V}],\hat{X}\right)}_{\substack{=0 \text{ by } (i) \text{ and} \\ \text{orthogonality}}} + \underbrace{\hat{h}\left([\hat{V},\hat{X}],\hat{Y}\right)}_{\substack{=0 \text{ by } (i) \text{ and} \\ \text{orthogonality}}} \\ &= \hat{h}\Big([\hat{X},\hat{Y}],\hat{V}\Big). \end{aligned}$$

✓

We compute the other equalities similarly.

Ad(iv) :

$$\hat{h}(\hat{\nabla}_{\hat{X}}\hat{Y},\hat{V}) \overset{(iii)}{=} \frac{1}{2}\hat{h}\left([\hat{X},\hat{Y}],\hat{V}\right) = \frac{1}{2}\hat{h}\left([\hat{X},\hat{Y}]^{\mathcal{V}},\hat{V}\right) \qquad \forall\, \hat{V} \in \mathfrak{X}(\hat{M})^{\mathcal{V}}.$$

$\Rightarrow \frac{1}{2}[\hat{X},\hat{Y}]^{\mathcal{V}}$ *is the vertical component of $\hat{\nabla}_{\hat{X}}\hat{Y}$.*
Moreover, by definition $\widehat{\nabla_X Y}$ is horizontal.

Claim : $\widehat{\nabla_X Y}$ *is the horizontal component of* $\hat{\nabla}_{\hat{X}}\hat{Y}$.

Proof of the claim :

Let $Z \in \mathfrak{X}(M)$;
$\hat{Z} \in \mathfrak{X}(\hat{M})^{\mathcal{H}}$ *its horizontal lift ;*
$\hat{p} \in \hat{M}$.

Then

$$\hat{X}\hat{h}(\hat{Y},\hat{Z})(\hat{p}) = d\left(\hat{h}(\hat{Y},\hat{Z})\right)\left(\hat{X}(\hat{p})\right) \overset{\pi \text{ isometry}}{=} Xh(Y,Z)\big(\pi(\hat{p})\big). \qquad (\star)$$

Furthermore, by isometry and the Naturality of Lie Brackets A.9:

$$\hat{h}\left([\hat{X},\hat{Y}],\hat{Z}\right)(\hat{p}) = h([X,Y],Z)\big(\pi(\hat{p})\big). \qquad (\star\star)$$

This leads us to :

$$\begin{aligned}
2\hat{h}(\hat{\nabla}_{\hat{X}}\hat{Y},\hat{Z})(\hat{p}) &\overset{\substack{\text{Koszul}\\ \text{formula}}}{=} \hat{X}\hat{h}(\hat{Y},\hat{Z})(\hat{p}) + \hat{Y}\hat{h}(\hat{Z},\hat{X})(\hat{p}) - \hat{Z}\hat{h}(\hat{X},\hat{Y})(\hat{p})\\
&\quad + \hat{h}\left([\hat{X},\hat{Y}],\hat{Z}\right)(\hat{p}) - \hat{h}\left([\hat{Y},\hat{Z}],\hat{X}\right)(\hat{p})\\
&\quad + \hat{h}\left([\hat{Z},\hat{X}],\hat{Y}\right)(\hat{p})\\
&\overset{(\star)+(\star\star)}{=} Xh(Y,Z)\left(\pi(\hat{p})\right) + Yh(Z,X)\left(\pi(\hat{p})\right)\\
&\quad - Zh(X,Y)\left(\pi(\hat{p})\right) + h\left([X,Y],Z\right)\left(\pi(\hat{p})\right)\\
&\quad - h\left([Y,Z],X\right)\left(\pi(\hat{p})\right) + h\left([Z,X],Y\right)\left(\pi(\hat{p})\right)\\
&\overset{\substack{\text{Koszul}\\ \text{formula}}}{=} 2h(\nabla_X Y, Z)\left(\pi(\hat{p})\right)\\
&\overset{\pi \text{ isometry}}{=} 2\hat{h}(\widehat{\nabla_X Y},\hat{Z})(\hat{p}).
\end{aligned}$$

$\Rightarrow \hat{h}(\hat{\nabla}_{\hat{X}}\hat{Y},\hat{Z})(\hat{p}) = \hat{h}(\widehat{\nabla_X Y},\hat{Z})(\hat{p})$.

Since it holds for any $\hat{Z} \in \mathfrak{X}(\hat{M})^{\mathcal{H}}$, *we conclude that* $\widehat{\nabla_X Y}$ *is the horizontal component of* $\hat{\nabla}_{\hat{X}}\hat{Y}$.

$\overset{\text{Claim}}{\blacksquare}$

$\Rightarrow \hat{\nabla}_{\hat{X}}\hat{Y} = \widehat{\nabla_X Y} + \frac{1}{2}[\hat{X},\hat{Y}]^{\mathcal{V}}$.

✓

■

Proof 1.6:

Using Lemma 1.7 (iv), it is straightforward to show :

$$\begin{aligned}
\tfrac{1}{2}[\hat{X},\hat{Y}]^{\mathcal{V}} &= \hat{\nabla}_{\hat{X}}\hat{Y} - \widehat{\nabla_X Y}\\
&= (\hat{\nabla}_{\hat{X}}\hat{Y})^{\mathcal{V}} + 0\\
&\overset{\substack{\hat{X} \text{ and } \hat{Y}\\ \text{horizontal}}}{=} \left(\hat{\nabla}_{\hat{X}^{\mathcal{H}}}\hat{Y}^{\mathcal{H}}\right)^{\mathcal{V}} + \underbrace{\left(\hat{\nabla}_{\hat{X}^{\mathcal{H}}}\underbrace{\hat{Y}^{\mathcal{V}}}_{=0}\right)^{\mathcal{H}}}_{=0}\\
&= A_{\hat{X}}\hat{Y}.
\end{aligned}$$

■

Using the previous identities leads to the following essential result of Riemannian Geometry, which links the sectional curvatures of the two Riemannian manifolds related by a Riemannian submersion :

Theorem 1.8 - Gray-O'Neill Formula

Let $\pi : (\hat{M}, \hat{h}) \to (M, h)$ be a Riemannian submersion ;
$\hat{k}$ and k be the numerator maps involved in the sectional curvatures[2] of 2-planes for $(\hat{M}, \hat{h})$ and (M, h), respectively ;
$X, Y \in \mathfrak{X}(M)$;
$\hat{X}, \hat{Y} \in \mathfrak{X}(\hat{M})$ the horizontal lifts of X and Y, respectively.

Then,

$$k(X \wedge Y) = \hat{k}(\hat{X} \wedge \hat{Y}) + 3\hat{h}(A_{\hat{X}}\hat{Y}, A_{\hat{X}}\hat{Y})$$

$$\overset{\text{Prop. 1.6}}{\Leftrightarrow} k(X \wedge Y) = \hat{k}(\hat{X} \wedge \hat{Y}) + \frac{3}{4}\left\|[\hat{X}, \hat{Y}]^{\mathcal{V}}\right\|^2_{\hat{h}}.$$

In particular, for $p \in M$ and an orthonormal basis $\big(X(p), Y(p)\big)$ of a 2-plane σ in T_pM, we have :

$$sec(\sigma) \geqslant sec(\hat{\sigma}),$$

where $\hat{\sigma} := \text{span}\big(\hat{X}(p), \hat{Y}(p)\big) \subseteq T_{\hat{p}}\hat{M}$ is the horizontal lift of σ at $\hat{p} \in \pi^{-1}(p)$.

It implies that the basis space (M, h) carries not lower curvature than the total space $(\hat{M}, \hat{h})$. Observe then that if $G := \hat{M}$ is a compact Lie group equipped with a biinvariant metric $b := \hat{h}$, by Corollary B.15, (M, h) also has non-negative curvature.

Proof 1.8:

Let again $V \in \mathfrak{X}(\hat{M})^{\mathcal{V}}$.

Recall the following identites from Lemma 1.7 (iii) and (iv):

$$\hat{h}\left(\hat{\nabla}_{\hat{V}}\hat{X}, \hat{Y}\right) = -\frac{1}{2}\hat{h}\left([\hat{X}, \hat{Y}]^{\mathcal{V}}, \hat{V}\right) \; ; \qquad (\clubsuit)$$

$$\hat{h}\left(\hat{\nabla}_{\hat{X}}\hat{V}, \hat{Y}\right) = -\frac{1}{2}\hat{h}\left([\hat{X}, \hat{Y}]^{\mathcal{V}}, \hat{V}\right) \; ; \qquad (\blacklozenge)$$

$$\hat{\nabla}_{\hat{X}}\hat{Y} = \widehat{\nabla_X Y} + \frac{1}{2}[\hat{X}, \hat{Y}]^{\mathcal{V}}. \qquad (\spadesuit)$$

Observe also that the horizontal lift of $d\pi\left([\hat{X}, \hat{Y}]^{\mathcal{H}}\right) = d\pi\left([\hat{X}, \hat{Y}]\right)$ is $\widehat{[X, Y]}$, by Naturality of Lie Brackets.

Then it suffices to compute :

$$\begin{aligned}\hat{k}(\hat{X} \wedge \hat{Y}) &= \hat{h}\left(\hat{R}(\hat{X}, \hat{Y})\hat{Y}, \hat{X}\right) \\ &= \hat{h}\left(\hat{\nabla}_{\hat{X}}\hat{\nabla}_{\hat{Y}}\hat{Y}, \hat{X}\right) - \hat{h}\left(\hat{\nabla}_{\hat{Y}}\hat{\nabla}_{\hat{X}}\hat{Y}, \hat{X}\right) - \hat{h}\left(\hat{\nabla}_{[\hat{X}, \hat{Y}]}\hat{Y}, \hat{X}\right)\end{aligned}$$

[2] See Definition A.29

$$
\begin{aligned}
\overset{(\spadesuit)}{=} \quad & \hat{h}\left(\hat{\nabla}_{\hat{X}}\widehat{\nabla_Y Y}, \hat{X}\right) + \frac{1}{2}\underbrace{\hat{h}\left(\hat{\nabla}_{\hat{X}}[\hat{Y},\hat{Y}]^{\mathcal{V}}, \hat{X}\right)}_{=0} - \hat{h}\left(\hat{\nabla}_{\hat{Y}}\widehat{\nabla_X Y}, \hat{X}\right) \\
& - \frac{1}{2}\hat{h}\left(\hat{\nabla}_{\hat{Y}}[\hat{X},\hat{Y}]^{\mathcal{V}}, \hat{X}\right) - \hat{h}\left(\hat{\nabla}_{[\hat{X},\hat{Y}]^{\mathcal{H}}}\hat{Y}, \hat{X}\right) - \hat{h}\left(\hat{\nabla}_{[\hat{X},\hat{Y}]^{\mathcal{V}}}\hat{Y}, \hat{X}\right) \\
\overset{\substack{(\clubsuit)\\(\blacklozenge),(\spadesuit)}}{=} \quad & \hat{h}\left(\widehat{\nabla_X\nabla_Y Y}, \hat{X}\right) + \frac{1}{2}\underbrace{\hat{h}\left([\hat{X},\widehat{\nabla_Y Y}]^{\mathcal{V}}, \hat{X}\right)}_{=0 \text{ by orthogonality}} \\
& - \hat{h}\left(\widehat{\nabla_Y\nabla_X Y}, \hat{X}\right) - \frac{1}{2}\underbrace{\hat{h}\left([\hat{Y},\widehat{\nabla_X Y}]^{\mathcal{V}}, \hat{X}\right)}_{=0 \text{ by orthogonality}} \\
& - \frac{1}{4}\hat{h}\left([\hat{X},\hat{Y}]^{\mathcal{V}}, [\hat{X},\hat{Y}]^{\mathcal{V}}\right) - \hat{h}\left(\nabla_{\widehat{[X,Y]}}\hat{Y}, \hat{X}\right) \\
& - \frac{1}{2}\underbrace{\hat{h}\left(\left[[\hat{X},\hat{Y}]^{\mathcal{H}}, \hat{Y}\right]^{\mathcal{V}}, \hat{X}\right)}_{=0 \text{ by orthogonality}} - \frac{1}{2}\hat{h}\left([\hat{X},\hat{Y}]^{\mathcal{V}}, [\hat{X},\hat{Y}]^{\mathcal{V}}\right) \\
\overset{\pi \text{ isometry}}{=} \quad & h(\nabla_X\nabla_Y Y, X) - h(\nabla_Y\nabla_X Y, X) - h(\nabla_{[X,Y]}Y, X) \\
& - \frac{3}{4}\hat{h}\left([\hat{X},\hat{Y}]^{\mathcal{V}}, [\hat{X},\hat{Y}]^{\mathcal{V}}\right) \\
= \quad & h\left(R(X,Y)Y, X\right) - \frac{3}{4}\|[\hat{X},\hat{Y}]^{\mathcal{V}}\|^2 \\
= \quad & k(X \wedge Y) - \frac{3}{4}\|[\hat{X},\hat{Y}]^{\mathcal{V}}\|^2.
\end{aligned}
$$

■

1.2 Orbital submersions

Many notions used in this section are discussed in Appendix B. We construct now a special type of Riemannian submersion, using a Lie group G acting on a Riemannian manifold (M, h) through $\mu : G \times M \to M$.

Note 1.9. From now on, all Lie groups are assumed to be *compact*. Then so are all the quotient spaces ${}^{G}\!/_{G_p}$, $p \in M$.

Theorem 1.10 - Homogeneous Space Characterization Theorem

For any $p \in M$, the following diffeomorphism links the orbit $G \cdot p$ of $p \in M$ with the left coset space of the isotropy group G_p at p :

$$
\begin{array}{rccc}
\tau_p : & G/G_p & \overset{\cong}{\to} & G \cdot p \\
& gG_p & \mapsto & g \cdot p.
\end{array}
$$

Proof 1.10: *See [Bre72, Corollary 1.3 p.303], [Kaw91, Theorem 3.43] or [Lee12, Theorem 21.18].*

■

Furthermore note that each left coset space G/G_p is diffeomorphic to the corresponding quotient set ${}^{G}/_{G_p}$. Theorem 1.10 entails that the orbits $G \cdot p$ inherit from the compactness of ${}^{G}/_{G_p}$ and are *closed* due to the Hausdorff property of M.

Characterization of orbital submersions

We equip ${}^{\hat{M}}/_{G}$ with the quotient topology, i.e. the finest topology such that the quotient map $\pi : \hat{M} \to {}^{\hat{M}}/_{G}$ is continuous :

$$U \subset {}^{\hat{M}}/_{G} \text{ open} \Leftrightarrow \pi^{-1}(U) \subset \hat{M} \text{ open.}$$

Proposition 1.11

Suppose G acts *freely* and *isometrically* on $(\hat{M}, \hat{h})$.

Then :

(i) The orbital space ${}^{\hat{M}}/_{G}$ has a smooth manifold structure ;

(ii) There exists a canonical metric h on ${}^{\hat{M}}/_{G}$ such that the quotient map $\pi : (\hat{M}, \hat{h}) \to ({}^{\hat{M}}/_{G}, h)$ is a Riemannian submersion, called **orbital submersion**.

To prove this proposition we'll need two results :

Theorem 1.12 - Quotient Manifold Theorem

Let G be a Lie group acting *smoothly*, *freely* and *properly* on a smooth manifold M.

Then the quotient space ${}^{M}/_{G}$ forms a smooth manifold with a unique smooth structure making the quotient map $\pi : M \to {}^{M}/_{G}$ into a submersion.

Proof 1.12: *See [Lee12, Theorem 21.10].*

■

Lemma 1.13

A smooth action of a *closed* subgroup of $\mathrm{Iso}(M, h)$ on a Riemannian manifold (M, h) with *closed orbits* is *proper*.

Proof 1.13: *See [AB15, Proposition 3.62].*

■

Proof 1.11:

Ad (i) : Remember that a free action is effective. That's why the homomorphism $G \to Diff(M), g \mapsto \mu_g$, has a trivial kernel, where $\mu_g(p) := \mu(g, p) = g \cdot p$. Since G acts isometrically on $(\hat{M}, \hat{h})$ and is compact, it is isomorphic to a closed subgroup $\tilde{G}$ of $Iso(\hat{M}, \hat{h})$.

By Lemma 1.13, the action is also proper. Thus, by using Theorem 1.12, we conclude that $\hat{M}/G$ *has a smooth structure and that the quotient map* π *forms a submersion.*

✓

Ad (ii) : We construct a submersion metric h *on* $\hat{M}/G$.

Let $p \in \hat{M}/G$;
$\hat{p} \in \pi^{-1}(p) \subset \hat{M}$;
$v, w \in T_p \hat{M}/G$.

π *submersion* $\Rightarrow$ $\exists \hat{v}, \hat{w} \in \mathcal{H}_{\hat{p}}$ *horizontal lifts of* v *and* w.

Let simply define h *such that* $d\pi_{\hat{p}}\big|_{\mathcal{H}_{\hat{p}}}$ *becomes an isometry :*

$$h_p(v, w) := \hat{h}_{\hat{p}}(\hat{v}, \hat{w}).$$

It remains to show the well-definition of h.
Claim : $h_p(v, w)$ *is independent of the chosen element of* $\pi^{-1}(p)$.
Proof of the claim :

Let $\hat{p}_1, \hat{p}_2 \in \pi^{-1}(p)$.

By definition it means that $\hat{p}_1 \sim \hat{p}_2$ *regarding the* G*-action on* $(\hat{M}, \hat{h})$. *Thus, there exists* $g \in G$ *with*

$$\hat{p}_2 = g \cdot \hat{p}_1 =: \phi_g(\hat{p}_1).$$

Let $\hat{v}_i, \hat{w}_i \in T_{\hat{p}_i}\hat{M}$ *for* $i = 1, 2$.

W.l.o.g. $(d\phi_g)_{\hat{p}_1}(\hat{v}_1) = \hat{v}_2$ *and* $(d\phi_g)_{\hat{p}_1}(\hat{w}_1) = \hat{w}_2$ *by the previous statement.*

Then,

$$\begin{aligned} \hat{h}_{\hat{p}_1}(\hat{v}_1, \hat{w}_1) &\overset{\phi_g \text{ isometry}}{=} \hat{h}_{\phi_g(\hat{p}_1)}\left((d\phi_g)_{\hat{p}_1}(\hat{v}_1), (d\phi_g)_{\hat{p}_1}(\hat{w}_1)\right) \\ &= \hat{h}_{\hat{p}_2}(\hat{v}_2, \hat{w}_2). \end{aligned}$$

$\blacksquare_{\text{Claim}}$

$\blacksquare$

A straightforward example of orbital submersion appears in the context of homogeneous manifolds, related to a geometric concept we will first introduce : the principal bundles.

Fiber and principal bundles

The key idea behind fiber bundles consists of a smooth manifold which we can locally see as a product of two other manifolds.

Definitions 1.14 - Fiber bundle

Let E, B and F be smooth manifolds ;
$\pi : E \to B$ be a smooth surjective map.

To be called a **(smooth) fiber bundle** the 4-tuplet (E, B, π, F) must satisfy the **local triviality condition** :

For all $b \in B$ there exists an open neighborhood $U_b \subset B$ of b such that a diffeomorphism $\phi_b : \pi^{-1}(U_b) \to U_b \times F$ exists and for which the following diagram is commutative :

$$\begin{tikzcd} \pi^{-1}(U_b) \arrow[r, "\phi_b"] \arrow[d, "\pi"] & U_b \times F \arrow[dl, "\mathrm{proj}_1"] \\ U_b & \end{tikzcd}$$

We have the following terminology :

- E = the **total space** ;
- B = the **base space** ;
- F = the **fiber** ;
- π = the **bundle projection** ;
- U_b = a **trivializing neighborhood** ;
- $\{(U_b, \phi_b)\}_{b \in B}$ = the **local trivialization of the bundle** ;
- $\psi_b := \phi_b^{-1} : U_b \times F \to \pi^{-1}(U_b)$ = a **bundle chart** ;
- for $b \in B$, $\pi^{-1}(\{b\}) \cong F$ = the **fiber over b**.

If there exists a global trivialization, one speaks of a **trivial fiber bundle**.

Examples 1.15

(i) Product manifold

A product manifold $M_1 \times M_2$ is canonically a trivial fiber bundle when projected onto M_1 or M_2 with global trivialization $id_{M_1 \times M_2}$ and that we can denote by $(M_1 \times M_2, M_1, \mathrm{proj}_1, M_2)$:

$$\begin{tikzcd} M_1 \times M_2 \arrow[r, "id_{M_1 \times M_2}"] \arrow[d, "\mathrm{proj}_1"'] & M_1 \times M_2 \arrow[dl, "\mathrm{proj}_1"] \\ M_1 & \end{tikzcd}$$

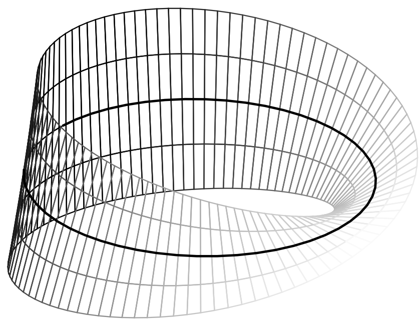
(ii) **Möbius strip**

A Möbius band has local similarities with a product space but differs by its global shape. There exist several equivalent definitions to construct it. Here is one of them[3]:

Let $E := [0,1] \times \mathbb{R}$ on which we introduce the following equivalence relation $\sim$:

$$(0,t) \sim (1,-t).$$

We simply define the **Möbius strip** as the quotient space $M := {}^{E}/_{\sim}$:



M is the total space of a fiber bundle $(M, \mathbb{S}^1, \pi, \mathbb{R})$[4].

In comparison with previous example, the "normal band" corresponding to Möbius strip - i.e. without twist - would be simply the infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$, which is a trivial fiber bundle. But for M, only local trivializations exist. Therefore the Möbius bundle is non-trivial.

Often some fiber bundles have peculiarities as to how local trivializations stick together, using smooth actions by a Lie group :

Definitions 1.16 - G-bundle & principal bundle

To the elements introduced for a fiber bundle (E, B, π, F) let's add a Lie group G, the **structure group**, acting smoothly and freely from the right on the fiber F, and trivially from the right on the B-component of $U_b \times F$.

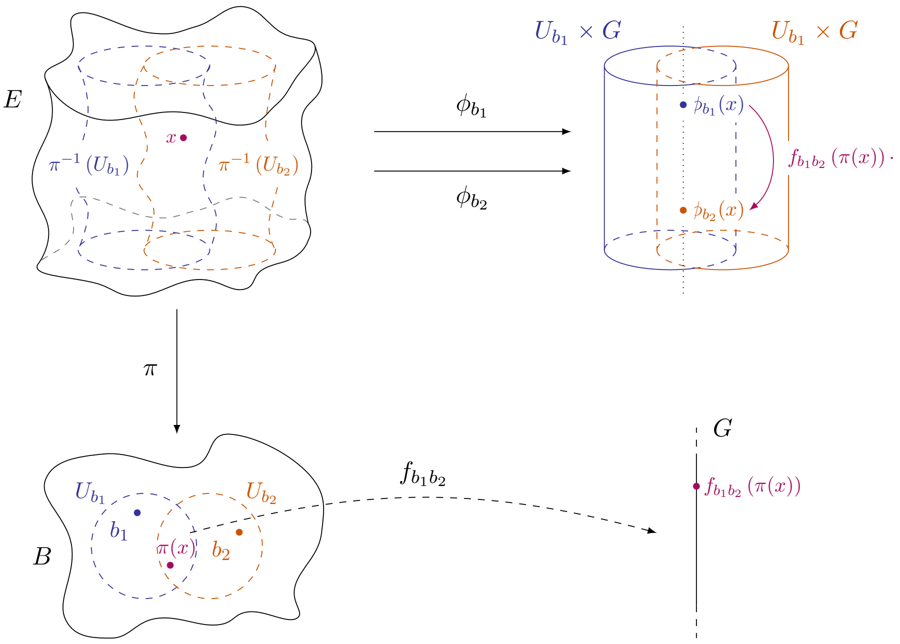
(E, B, π, F) is named **G-bundle** if additionally for all $b_1, b_2 \in B$, there exists a smooth map $f_{b_1 b_2} : U_{b_1} \cap U_{b_2} \to G$, called **transition map**, such that the following holds for all $x \in \pi^{-1}\left(U_{b_1} \cap U_{b_2}\right)$:

$$\phi_{b_2}(x) = \phi_{b_1}(x) \cdot f_{b_1 b_2}\big(\pi(x)\big).$$

Furthermore, one speaks of **principal G-bundle** (E, B, π, G) if among all other conditions the fiber F is G, or equivalently if G acts freely and transitively on F.

[3] See [GHL04, Exercise 1.11]. A different and very complete description of the Möbius bundle can be found in [Lee12, Example 10.3].

[4] See [Kaw91, Example 2 p.65]



The matching condition between local trivializations of a *principal* G-bundle can be expressed in terms of bundle charts :

Lemma 1.17 - Matching condition equivalence (principal bundles)

Let's use the notations of Definition 1.16.

We fix $x \in \pi^{-1}(U_{b_1}) \cap \pi^{-1}(U_{b_2})$.

Let $\quad b := \pi(x) \in B$;
$\qquad g_1, g_2 \in G$ such that $\phi_{b_i}(x) = \psi_{b_i}^{-1}(x) = (b, g_i)$, $i = 1, 2$.

Then

$$\phi_{b_2}(x) = \phi_{b_1}(x) \cdot f_{b_1 b_2}(b) \tag{$\star$}$$

$$\Leftrightarrow$$

$$\psi_{b_1}(b, g_1) = \psi_{b_2}\big(b, g_1 \cdot f_{b_1 b_2}(b)\big). \tag{$\star\star$}$$

Proof 1.17:

$\underline{\Rightarrow}$: $x = x \Leftrightarrow \psi_{b_1}(b, g_1) = \psi_{b_2}(b, g_2) \overset{(\star)}{=} \psi_{b_2}\big(b, g_1 \cdot f_{b_1 b_2}(b)\big)$. $\checkmark$

$\underline{\Leftarrow}$: $\phi_{b_2}(x) = \psi_{b_2}^{-1}\left(\psi_{b_1}(b, g_1)\right) \overset{(\star\star)}{=} (b, g_1 \cdot f_{b_1 b_2}(b)) = \phi_{b_1}(x) \cdot f_{b_1 b_2}(b)$. $\checkmark$

■

We commonly use the alternative notation $G \to E \xrightarrow{\pi} B$ for a principal G-bundle, which suggests a G-action on E. We even observe the following equivalence: a fiber bundle (E, B, π, G) becomes principal with structure group G if and only if its fibers correspond to the orbits of a free proper left action on E. In this case, the orbit space is simply B.

Proposition 1.18 - Underlying action of a principal bundle

A principal G-bundle $G \to E \xrightarrow{\pi} B$ admits an underlying *free proper* G-action from the left : $\mu : G \times E \to E$ whose orbits coincide with the fibers of the bundle.

Proof 1.18: *Based on [AB15, Proposition 3.33].*

Let $x \in E$.

Define $b := \pi(x) \in B$ and $g \in G$ such that $x = \psi_\alpha(b, g) \in \pi^{-1}(U_\alpha)$ for a local trivialization $(U_\alpha, \psi_\alpha^{-1})$.

Recall that G acts freely and properly on $U_\alpha \times G$ from the right by

$$(y, g_1) \cdot g_2 := (y, g_1 \cdot g_2),$$

because right multiplication is trivially free and proper.

Let's simply construct $\mu : G \times E \to E$ as follows :

$$\boxed{\mu(\overline{g}, x) := \psi_\alpha\left(\overline{g} \cdot \psi_\alpha^{-1}(x)\right) = \psi_\alpha(b, \overline{g} \cdot g)}$$

Claim: The definition of μ is independent of the bundle chart.

Consider another suitable local trivialization $(U_\beta, \psi_\beta^{-1})$ and define $\tilde{g} \in G$ such that $x = \psi_\beta(b, \tilde{g}) \in \pi^{-1}(U_\beta)$.

By the matching conditions required for a principal bundle, g and $\tilde{g}$ are related by right multiplication with $f_{\beta\alpha}(b)$ and then :

$$(b, g) = \big(b, \tilde{g} \cdot f_{\beta\alpha}(b)\big). \qquad (\star)$$

Finally :

$$\begin{aligned}
\psi_\beta\big(\overline{g} \cdot \psi_\beta^{-1}(x)\big) &= \psi_\beta(b, \overline{g} \cdot \tilde{g}) \\
&\overset{\text{Lemma}}{\underset{1.17}{=}} \psi_\alpha\big(b, (\overline{g} \cdot \tilde{g}) \cdot f_{\beta\alpha}(b)\big) \\
&= \psi_\alpha\Big(\overline{g} \cdot \big(b, \tilde{g} \cdot f_{\beta\alpha}(b)\big)\Big) \\
&\overset{(\star)}{=} \psi_\alpha\big(\overline{g} \cdot (b, g)\big) \\
&= \psi_\alpha\big(\overline{g} \cdot \psi_\alpha^{-1}(x)\big) \\
&= \mu(x, \overline{g}).
\end{aligned}$$

Claim ■

The facts that G acts properly and freely on E and that the orbits coincide with the fibers of π come directly from the definition of μ.

■

Conversely :

Theorem 1.19

Given a free proper left (or right) action $\mu : G \times M \to M$, the orbit space ${M}/{G}$ admits a smooth structure which gives rise to the principal G-bundle $G \to M \xrightarrow{\pi} {M}/{G}$.

Proof 1.19: *See [AB15, Theorem 3.34] for a sketch.*

■

A special notion appears in the context of principal fiber bundle : the twisted spaces.

Twisted spaces

First, for a Lie group G, we write $\triangle G := \{(g,g) \in G \times G\} \cong G$.

Definition 1.20 - Twisted space

Let $G \to E \overset{\pi_1}{\to} B$ be a principal G-bundle with underlying free proper left G-action $\mu_1 : G \times E \to E$;
F be a smooth manifold with a right G-action $\mu_2 : F \times G \to F$.

From μ_1 and μ_2, we construct the **diagonal action** $\mu : \triangle G \times (E \times F) \to E \times F$ by :

$$\mu\big(g,(x,f)\big) := \big(\mu_1(g^{-1},x), \mu_2(f,g)\big).$$

The orbit space of μ is called the **twisted space** that we write :

$$E \times_G F := \big\{[x,f] \mid x \in E, f \in F\big\}.$$

Theorem 1.21

The twisted product $E \times_G F$ admits a smooth structure and generates a G-bundle $F \to E \times_G F \overset{\pi}{\to} B$ called the **associated bundle with fiber F**.

Proof 1.21: *See [AB15, Theorem 3.51] for a sketch.*

■

Lemma 1.22

Let M be a smooth manifold ;
G be a Lie group, acting *freely* and *properly* on M and by right translation on itself.

Then we obtain the following diffeomorphism :

$$M \times_G G \cong M.$$

Proof 1.22:
Consider the following map :

$$\begin{array}{cccc} \bar{\rho} & : & M \times_G G & \to & M \\ & & [m,g] = [g^{-1} \cdot m, e] & \mapsto & g^{-1} \cdot m. \end{array}$$

We easily check that $\bar{\rho}$ is surjective, injective and smooth. Moreover, the inverse map $\bar{\rho}^{-1}$: $m \mapsto [m,e]$ has these same properties.

■

Homogeneous manifolds

Recall from Definition B.9 that if a Lie group G acts *transitively* and *by isometries* on a Riemannian manifold (M, h), we call it **homogeneous**. This subsection aims to construct such a manifold and to use it to illustrate the Gray-O'Neill formula.

Let (G, b) be a Lie group together with a biinvariant metric ;
$H \subset G$ be a *closed* subgroup acting freely and properly on G by right translation ;
$\pi : G \to {}^{G}/_{H}$ the canonical projection.

Remark 1.23. H is a Lie subgroup by the *Closed Subgroup Theorem*[5].

Remark 1.24. According to Theorem 1.19, the right translation of H on G gives rise to a principal bundle $H \to G \xrightarrow{\pi} {}^{G}/_{H}$.

If we refer to Proposition 1.11, there exists a metric $\tilde{b}$ on ${}^{G}/_{H}$ making

$$\pi \ : \ (G, b) \ \to ({}^{G}/_{H}, \tilde{b}),$$

into an orbital submersion.

We now introduce the left translation of G over G :

$$\begin{array}{rccl} \psi & : & G \times G & \to & G \\ & & (g_1, g_2) & \mapsto & g_1 \cdot g_2. \end{array}$$

ψ induces a left action on ${}^{G}/_{H}$:

$$\begin{array}{rccl} \bar{\psi} & : & G \times ({}^{G}/_{H}, \tilde{b}) & \to & ({}^{G}/_{H}, \tilde{b}) \\ & & (g_1, \underbrace{[g_2]}_{=g_2 H}) & \mapsto & \underbrace{[g_1 \cdot g_2]}_{=(g_1 \cdot g_2)H}. \end{array}$$

Lemma 1.25

${}^{G}/_{H}$ is a homogeneous G-manifold, i.e. G acts by isometries on $({}^{G}/_{H}, \tilde{b})$.

Proof 1.25:

The left translation ψ being smooth and transitive, $\bar{\psi}$ inherits these properties. Furthermore, $G \curvearrowright \left({}^{G}/_{H}, \tilde{b}\right)$ isometrically since ψ is an isometry and π is a Riemannian submersion.

■

Definition 1.26 - Normal homogeneous

We call $\tilde{b}$ **normal homogeneous** when it is not only homogeneous but it also results from a biinvariant metric b.

[5] See [Lee12, Theorem 20.12]

As mentioned after the Gray-O'Neill Formula 1.8 :

Proposition 1.27

A normal homogeneous metric has non-negative curvature.

Proof 1.27:

Consider the previous orbital submersion $\pi : (G, b) \to ({}^{G}/_{H}, \tilde{b})$.

Let $\tilde{X}, \tilde{Y} \in \mathfrak{X}\left({}^{G}/_{H}\right)$ *and denote by* X, Y *their horizontal lift in* $\mathfrak{X}(G)$.

Then, by the Gray-O'Neill Formula 1.8:

$$\begin{aligned} \tilde{k}\left(\tilde{X} \wedge \tilde{Y}\right) &= k(X \wedge Y) + 3 \cdot b\left(A_X Y, A_X Y\right) \\ &\overset{\substack{\text{Prop. B.14} \\ \text{and 1.6}}}{=} \tfrac{1}{4}\|[X, Y]\|^2 + \tfrac{3}{4}\|[X, Y]^{\mathcal{V}}\|^2 \\ &\geqslant 0. \end{aligned}$$

■

Suppose now that a Lie group G acts isometrically and transitively on a Riemannian manifold (M, h), that is it acts *homogeneously*.

We fix a point $p \in M$ and observe that the orbit map $\mu^p : G \to M, g \mapsto g \cdot p$, is a submersion by Theorem 1.10 since π is:

$$\begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[drr, "\mu^p"] & & \\ {}^{G}/_{G_p} \arrow[r, dashed, "\cong"] & G/G_p \arrow[rr, dashed, "\cong", "\tau_p"'] & & G \cdot p = M \end{tikzcd}$$

Remark 1.28. This little example shows that any homogeneous G-manifold can be represented through a non-unique orbital space.

Chapter 2

Action fields

2.1 Lie exponential map

Let G be a fixed Lie group.

Lemma 2.1

Let $x \in \mathfrak{g} = T_eG$.

Then, there exists a unique 1-parameter subgroup $\gamma_x : \mathbb{R} \to G$ being the integral curve of the left-invariant vector field $X_L \in \mathfrak{X}(G)^L$ generated by x, i.e. the following holds for all $t \in \mathbb{R}$:

$$\dot{\gamma_x}(t) = X_L\big(\gamma_x(t)\big),$$

with

$$X_L(g) := (dL_g)_e(x),$$

where $L_g : G \to G, \widetilde{g} \mapsto g \cdot \widetilde{g}$ is the left translation by $g \in G$.

Proof 2.1:

<u>*Existence and uniqueness of the integral curve*</u> *:*

Due to the compactness of G, the left-invariant vector field X_L has a global flow[1] *:*

$$\begin{array}{rcccl} \tilde{\theta} & : & (\mathbb{R}, +) & \to & (\mathit{Diff}(G), \circ) \\ & & t & \mapsto & \begin{array}{rcccl} \theta_t & : & G & \to & G \\ & & g & \mapsto & \theta(t,g). \end{array} \end{array}$$

Let's recall the main property of θ :

$$\frac{\partial}{\partial t}\Big|_{t=0} \theta(t,g) = X_L(g) \qquad \forall g \in G. \tag{$\star$}$$

We define

$$\gamma_x(t) := \theta(t,e),$$

[1] See Definition A.6

and verify for $t_0 \in \mathbb{R}$:

$$\begin{aligned}
\dot{\gamma_x}(t_0) &= \left.\tfrac{\partial}{\partial t}\right|_{t=t_0} \theta(t,e) &&= \left.\tfrac{\partial}{\partial s}\right|_{s=0} \theta(t_0+s,e) \\
&= \left.\tfrac{\partial}{\partial s}\right|_{s=0} \theta\big(s,\theta(t_0,e)\big) &&\overset{(\star)}{=} X_L\big(\theta(t_0,e)\big) \\
&= X_L\big(\gamma_x(t_0)\big).
\end{aligned}$$

The unicity comes from the Fundamental Theorem for Autonomous ODE's[2] *if we look the initial value problem in local charts of G.*

<u>γ_x *being a 1-parameter subgroup*</u> *:*

Let's analyze the following curves for any $s \in \mathbb{R}$:

$$\begin{array}{rcccl} \xi_1 & : & \mathbb{R} & \to & G \\ & & t & \mapsto & \gamma_x(s+t) \end{array} \quad \textit{and} \quad \begin{array}{rcccl} \xi_2 & : & \mathbb{R} & \to & G \\ & & t & \mapsto & \gamma_x(s)\cdot\gamma_x(t) \end{array}$$

We denote by L_g the left translation map on G by $g \in G$.
We then obtain for all $t \in \mathbb{R}$:

$$\dot{\xi_1}(t) = \tfrac{d}{dt}\gamma_x(s+t) = X\big(\gamma_x(s+t)\big) = X\big(\xi_1(t)\big),$$

and

$$\begin{aligned}
\dot{\xi_2}(t) &= \tfrac{d}{dt}\gamma_x(s)\cdot\gamma_x(t) &&\overset{\substack{\textit{chain}\\\textit{rule}}}{=} \big(dL_{\gamma_x(s)}\big)_{\gamma_x(t)} \circ \tfrac{d}{dt}\gamma_x(t) \\
&= \big(dL_{\gamma_x(s)}\big)_{\gamma_x(t)}\left(X(\gamma_x(t))\right) &&\overset{\substack{X\,\textit{left-}\\\textit{invariant}}}{=} X\left(\gamma_x(s)\cdot\gamma_x(t)\right) \\
&= X\big(\xi_2(t)\big).
\end{aligned}$$

Since ξ_1 and ξ_2 are both integral curves for X with the same initial condition $\xi_1(0) = \xi_2(0) = \gamma_x(s)$, we conclude that $\xi_1 = \xi_2$.

■

This curve γ_x is used to define a map linking a Lie group G with its Lie algebra $\mathfrak{g}$:

Definition 2.2 - Lie exponential map

The **Lie exponential map of G** is defined as :

$$\begin{array}{rcccl} \exp & : & \mathfrak{g} & \to & G \\ & & x & \mapsto & \gamma_x(1). \end{array}$$

Properties 2.3 - Lie exponential map

The following holds for all $X \in \mathfrak{g}$ and for all $t \in \mathbb{R}$:

(i) $\exp(tx) = \gamma_x(t)$;

(ii) $\exp(-tx) = \exp(tx)^{-1}$;

(iii) $\exp(t_1x + t_2x) = \exp(t_1x)\cdot\exp(t_2x)$;

(iv) exp is a local diffeomorphism, i.e. $\exists$ open neighborhood U of $0_\mathfrak{g} \in T_eG = \mathfrak{g}$ with $e \in \exp(U)$ such that $\exp|_U \colon U \to \exp(U)$ is a diffeomorphism.

[2] See [Lee12, Theorem D.1 (b)]

Proof 2.3: *See [AB15, Proposition 1.30].*

The property (iv) is due to the smoothness of exp *and the fact that the differential* $(d\exp)_{0_{\mathfrak{g}}} = id_{\mathfrak{g}}$ *after identification of* $T_{0_{\mathfrak{g}}}(T_eG)$ *with* $T_eG = \mathfrak{g}$.

■

2.2 Action fields

Let's come back to the context of a smooth G-action on a manifold M : $\mu : G \times M \to M$. The next definition shows that each left-invariant vector field on G induces a smooth vector field on M, coherent with the action structure by using the exponential map :

Definitions 2.4 - Action field & infinitesimal generator

Every $x \in \mathfrak{g}$ induces a smooth vector field $X^* \in \mathfrak{X}(M)$, called **action field**, defined at each point $p \in M$ as follows :

$$X^*(p) := \frac{d}{dt}\Big|_{t=0} \mu\big(\exp(tx), p\big).$$

We define the **infinitesimal generator of** μ as the map assigning the action field to every $x \in \mathfrak{g}$:

$$\begin{array}{cccc} * & : & \mathfrak{g} & \to & \mathfrak{X}(M) \\ & & x & \mapsto & X^*. \end{array}$$

Remark 2.5. Action fields are examples of Killing vector fields for Riemannian manifolds (M, h), as defined in Section B.5. That's why we sometimes call it the **corresponding Killing vector field to x** $\in \mathfrak{g}$.

There exists another definition (e.g. in [Vog15]), involving orbit maps[3] μ^p. However, these two expressions are equivalent.

Proposition 2.6

For $x \in \mathfrak{g}$ and for $p \in M$:

$$X^*(p) = (d\mu^p)_e(x).$$

Proof 2.6: *Adapted from [SA08, Proposition 2.10].*

Let $g \in G$;
$x \in \mathfrak{g}$;
$X_R \in \mathfrak{X}(G)$ *the right-invariant vector field generated by* x*, which is defined by* $X_R(\widetilde{g}) := dR_{\widetilde{g}}(x)$ *for all* $\widetilde{g} \in G$*, where* $R_{\widetilde{g}}$ *is the right translation map by* $\widetilde{g}$;
$p \in M$.

[3] See Definitions B.8

Then

$$\begin{aligned}
(d\mu^p)_g (X_R(g)) &= d\mu_{(g,p)}(dR_g(x), 0_p) \\
&= d\mu_{(g,p)} \left[\left.\frac{d}{dt}\right|_{t=0} (\exp(tx)\cdot g, p) \right] \\
&\overset{\substack{\text{Chain}\\\text{rule}}}{=} \left.\frac{d}{dt}\right|_{t=0} \mu(\exp(tx)\cdot g, p) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mu(\exp(tx), g\cdot p) \\
&= X^*(g\cdot p).
\end{aligned}$$

In particular, for $g = e$:

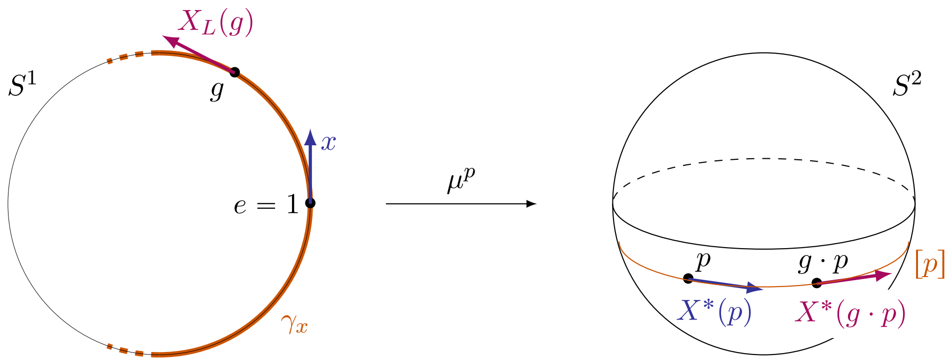
$$(d\mu^p)_e (X_R(e)) = (d\mu^p)_e (x) = X^*(p).$$

∎

Let's illustrate this new correspondance between vector fields in G and in M :

Example 2.7 - Action field

We consider the natural action $\mu : S^1 \times S^2 \to S^2$ by rotation :



Proposition 2.8

The **infinitesimal generator of** μ, $* : \mathfrak{g} \to \mathfrak{X}(M)$, is a Lie algebra antihomomorphism, i.e. :

$$*([x,y]) =: [X,Y]^* = -[X^*, Y^*],$$

for all $x, y \in \mathfrak{g}$.

Proof 2.8: *Based on [Lee12, Theorems 20.15 and 20.18].*

Let $x, y \in \mathfrak{g}$ *which generate the right-invariant vector fields* $X_R, Y_R \in \mathfrak{X}(G)^R$;
$p \in M$ *and its orbit map* $\mu^p : G \to M, \tilde{g} \mapsto \tilde{g}\cdot p$;
$g \in G$ *and the right translation map* $R_g : G \to G, \tilde{g} \mapsto \tilde{g}\cdot g$;
$q := g\cdot p = \mu^p(g)$ *and its orbit map* μ^q.

① *Claim : $*$ is a linear map.*

Define the curve

$$\begin{array}{rcccl} \gamma & : & \mathbb{R} & \to & G \\ & & t & \mapsto & \exp(tx), \end{array}$$

whose initial velocity is

$$\gamma'(0) = \frac{d}{dt}\Big|_{t=0} \exp(tx) = \underbrace{(d\exp)_0}_{=id_{\mathfrak{g}}} \circ \underbrace{\frac{d}{dt}\Big|_{t=0} tx}_{=x} = x.$$

Hence,

$$X^*(p) = \frac{d}{dt}\Big|_{t=0} \exp(tx) \cdot p = (\mu^p \circ \gamma)'(0) \underset{\gamma'(0)=x}{\overset{\gamma(0)=e}{=}} (d\mu^p)_e(x).$$

Since a differential is linear, $X^*(p)$ depends linearly on x, for any $p \in M$. Thus $*$ is linear.

✓

② *Claim : X_R and X^* are μ^p-related.*

$$\begin{array}{rrcll} & (\tilde{g} \cdot g) \cdot p & = & \tilde{g} \cdot q & \forall \tilde{g} \in G \\ \Leftrightarrow & \mu^p \circ R_g(\tilde{g}) & = & \mu^q(\tilde{g}) & \forall \tilde{g} \in G \\ \Leftrightarrow & \mu^p \circ R_g & = & \mu^q & \end{array}$$

Hence,

$$\begin{array}{rcl} X^*(\mu^p(g)) & = & X^*(q) \\ & \overset{Prop.\ 2.6}{=} & (d\mu^q)_e(x) \\ & = & \big(d(\mu^p \circ R_g)\big)_e(x) \\ & \overset{\substack{Chain \\ rule}}{=} & (d\mu^p)_g \circ (dR_g)_e(x) \\ & = & (d\mu^p)_g\big(X_R(g)\big). \end{array}$$

✓

③ *Claim : $[X_R, Y_R]$ is μ^p-related to $[X^*, Y^*]$.*

This result comes directly from the Naturality of Lie Brackets A.9. Since X_R and Y_R are μ^p-related to X^* and Y^*, respectively, then :

$$(d\mu^p)_g \left[X_R, Y_R\right](g) = \left[X^*, Y^*\right](g \cdot p) \qquad\qquad \forall g \in G.$$

✓

④ *Claim : $\left[X^*, Y^*\right](p) = -[X, Y]^*(p)$.*

$$\begin{array}{rcl} \left[X^*, Y^*\right](p) & \overset{③}{=} & (d\mu^p)_e \left[X_R, Y_R\right](e) \\ & \overset{Prop.\ B.7}{=} & -(d\mu^p)_e\big(\left[X_L, Y_L\right](e)\big) \\ & = & - * \big(\left[X_L, Y_L\right](e)\big)(p) \\ & = & -[X, Y]^*(p) \end{array}$$

✓

■

2.3 Splitting of tangent spaces and orbit tensors

Consider a smooth action μ of a Lie group G on a Riemannian manifold (M, h).

In a similar way as for Riemannian submersions, we proceed to an orthogonal decomposition of each tangent space:

Definition 2.9 - Vertical and horizontal spaces

We split the tangent space T_pM, $p \in M$ into :

- The **vertical space** = the tangent space of the orbit at p :

$$\mathcal{V}_p := T_p(G \cdot p) = \{X^*(p) \mid x \in T_eG\};$$

- The **horizontal space** = its orthogonal complement :

$$\mathcal{H}_p := T_pM \ominus \mathcal{V}_p = \{\xi \in T_pM \mid h(\xi, \mathcal{V}_p) = 0\}.$$

Consequently, for each tangent vector $v \in T_pM$ there exists some $x \in \mathfrak{g} = T_eG$ and a unique $\xi \in \mathcal{H}_p$ with $v = \underbrace{X_p^*}_{=:v^{\mathcal{V}}} + \underbrace{\xi}_{=:v^{\mathcal{H}}}$. Furthermore, if μ is free, x is uniquely determined by v.

Remark 2.10. The dimensions of orbits and thus of their tangent spaces may vary with $p \in M$. As a result, we can't define vertical and horizontal distributions.

We now equip G with a *biinvariant* metric b and suppose the action μ to be *isometric*. We introduce a link between metrics b on G and h on M :

Definition 2.11 - Orbit tensor

We define the **orbit tensor S of the action of $(\mathbf{G}, \mathbf{b})$ on $(\mathbf{M}, \mathbf{h})$** by

$$b\big(S(p)x, y\big) = h\big(X^*(p), Y^*(p)\big),$$

for all $p \in M$, $x, y \in \mathfrak{g}$, and with $S(p) : \mathfrak{g} \to \mathfrak{g}$ a self-adjoint homomorphism.

Let $\mathfrak{g}_p \subseteq \mathfrak{g}$ denote the Lie algebra of the isotropy group $G_p \subseteq G$ and $\mathfrak{m}_p$ its orthogonal complement in $\mathfrak{g}$ with respect to b, for all $p \in M$.

Remark 2.12. $\mathfrak{g}_p = \{x \in \mathfrak{g} \mid X^* = 0\}$.

Lemma 2.13

$S(p)\Big|_{\mathfrak{m}_p} : \mathfrak{m}_p \to \mathfrak{m}_p$ is a self-adjoint automorphism, for all $p \in M$.

Proof 2.13:

Let $p \in M$.

$S(p)(\mathfrak{m}_p) \subseteq \mathfrak{m}_p$: Let $x \in \mathfrak{m}_p$ and $y \in \mathfrak{g}_p$.

Then $b\big(S(p)x, y\big) = h\big(X^*(p), Y^*(p)\big) = h\big(X^*(p), 0_p\big) = 0.$

Self-adjoint: By symmetry of h.

Linearity: Because $*$ is linear.

Injectivity:
$$\begin{aligned}
\ker\left(S(p)\big|_{\mathfrak{m}_p}\right) &= \{x \in \mathfrak{m}_p \mid S(p)x = 0_{\mathfrak{g}}\} \\
&= \big\{x \in \mathfrak{m}_p \mid b\big(S(p)x, y\big) = h\big(X^*(p), Y^*(p)\big) = 0 \ \forall y \in \mathfrak{g}\big\} \\
&= \{x \in \mathfrak{m}_p \mid X^*(p) = 0_p\} \\
&= \mathfrak{m}_p \cap \mathfrak{g}_p \\
&= \{0_{\mathfrak{g}}\}.
\end{aligned}$$

Surjectivity: Comes from the injectivity and the finite dimension of $\mathfrak{m}_p \subseteq \mathfrak{g}$.

∎

Remark 2.14. The restriction $*\big|_{\mathfrak{m}_p} : \mathfrak{m}_p \to \mathcal{V}_p$ is an isomorphism for all $p \in M$.We conclude that each $X^*(p) \in \mathcal{V}_p$ determines a unique $S(p)x \in \mathfrak{m}_p$, for $x \in \mathfrak{m}_p$, which allows us to define the following map (also denoted by $S(p)$) :

Lemma 2.15

For any $p \in M$, the well-defined map

$$\begin{array}{ccccc} S(p) & : & \mathcal{V}_p & \to & \mathcal{V}_p \\ & & X^*(p) & \mapsto & (SX)^*(p) := *\left(S(p)x\right)(p), \end{array}$$

is an isomorphism.

Proof 2.15:

Linearity: Trivial since the maps $S(p) : \mathfrak{m}_p \to \mathfrak{m}_p$ and $* : \mathfrak{g} \to \mathfrak{X}(M)$ both are linear.

Injectivity:
$$\begin{aligned}
\ker\left(S(p)\right) &= \big\{X^*(p) \in \mathcal{V}_p \mid (Sx)^*(p) = 0_p\big\} \\
&= \{X^*(p) \in \mathcal{V}_p \mid h\left((SX)^*(p), Y^*(p)\right) = 0 \ \forall y \in \mathfrak{m}_p\} \\
&\underset{\textit{symmetric}}{\overset{h \textit{ and } b}{=}} \{X^*(p) \in \mathcal{V}_p \mid b\left(S(p)x, S(p)y\right) = 0 \ \forall y \in \mathfrak{m}_p\} \\
&= \{X^*(p) \in \mathcal{V}_p \mid S(p)x = 0_{\mathfrak{g}}\} \\
&\underset{\textit{injective}}{\overset{S(p):\mathfrak{m}_p \to \mathfrak{m}_p}{=}} \{X^*(p) \in \mathcal{V}_p \mid x = 0_{\mathfrak{g}}\} \\
&= \{0_p\}.
\end{aligned}$$

Surjectivity: Results from the injectivity of $S(p)$ and the finite dimension of $\mathcal{V}_p$.

∎

Chapter 3

Cheeger deformations

Throughout this chapter, let (M, h) denote a Riemannian manifold on which a compact Lie group (G, b) equipped with a biinvariant metric acts *isometrically* through μ. Recall that such an action is *proper* and has *closed orbits*.

The two previous chapters presented key notions to define the so-called *Cheeger construction* which transforms the initial metric h of a manifold M into new ones $\left(h_t^G\right)_{t \geqslant 0}$ "shrinking along the orbits" and enlarging the curvature as t grows. We now present this process in details.

3.1 Construction

From the action $G \curvearrowright M$, we define $G \curvearrowright (M \times G, h + \frac{1}{t}b)$ for all $t \in \mathbb{R}_{>0}$ by using left translations on G :

$$\boxed{\begin{array}{cccc} \psi & : & G \times (M \times G) & \mapsto & M \times G \\ & & \big(g, (p, \tilde{g})\big) & \mapsto & \big(\mu_g(p), g \cdot \tilde{g}\big), \end{array}}$$

where $h + \frac{1}{t}b$ is a product metric.[1]

Lemma 3.1

ψ is *free* and *isometric* with respect to $h + \frac{1}{t}b$ for any $t \in \mathbb{R}_{>0}$.

Proof 3.1:

Free: Trivial since the left translations $g\cdot$ on G are free, for all $g \in G$.

Isometric: Let $t > 0$, $p \in M$, $g, \tilde{g} \in G$ and $(v_1, x_1), (v_2, x_2) \in T_{(p,\tilde{g})}(M \times G)$.

Remember that $T_{(p,\tilde{g})}(M \times G) \cong T_pM \times T_{\tilde{g}}G$.

[1] See Definition A.34

Then,

$$\begin{aligned}
\left(h + \tfrac{1}{t}b\right)_{(p,\tilde{g})}\left((v_1, x_1), (v_2, x_2)\right) &= h_p(v_1, v_2) + \tfrac{1}{t} b_{\tilde{g}}(x_1, x_2) \\
&\overset{(\star)}{=} h_{\mu_g(p)}\left((d\mu_g)_p(v_1), (d\mu_g)_p(v_2)\right) \\
&\quad + \tfrac{1}{t} b_{g\cdot\tilde{g}}\Big((dL_g)_{\tilde{g}}(x_1), (dL_g)_{\tilde{g}}(x_2)\Big) \\
&= (h + \tfrac{1}{t}b)_{g\cdot(p,\tilde{g})}\big((d\psi_g)_{(p,\tilde{g})}(v_1, x_1), \\
&\qquad\qquad (d\psi_g)_{(p,\tilde{g})}(v_2, x_2)\big).
\end{aligned}$$

where the equality $(\star)$ *holds since* G *acts on* (M,h) *by isometries and* b *is a biinvariant metric.*

■

According to Proposition 1.11, for all $t \in \mathbb{R}_{t>0}$ the twisted space $M \times_G G$ inherits a metric $\bar{h}_t^G$ such that the quotient map $\pi : (M \times G, h + \frac{1}{t}b) \to \left(M \times_G G, \bar{h}_t^G\right)$ is an orbital submersion.

Similarly to Lemma 1.22, we can prove that $M \times_G G \cong M$ by the diffeomorphisms $\bar{\rho} : {(M \times G)}/{G} \to M, \bar{\rho}([p, g]) = g^{-1} \cdot p$. By turning $\bar{\rho}$ into isometries, for all $t \in \mathbb{R}_{t>0}$ we get new metrics h_t^G on M, called the **Cheeger metrics**,

$$\left(h_t^G\right)_p(v, w) := \left(\bar{h}_t^G\right)_{[p,e]}\left(d\bar{\rho}^{-1}(v), \bar{\rho}^{-1}(w)\right),$$

for all $p \in M$ and for all $v, w \in T_pM$.

In the end, we have a parametrized family of orbital submersions :

$$\left(\begin{array}{cccc} \rho & : & \left(M \times G, h + \frac{1}{t}b\right) & \to & \left(M, h_t^G\right) \\ & & (p, g) & \mapsto & g^{-1} \cdot p \end{array}\right)_{t>0}.$$

Now consider the centralizer of G in the isometry group $\mathrm{Iso}(M, h)$ [2] :

$$C(G) := Z_{\mathrm{Iso}(M,h)}(G) = \{\phi \in \mathrm{Iso}(M, h) \mid \phi \circ \mu_g \circ \phi = \mu_g \ \forall g \in G\} \supseteq Z\big(\mathrm{Iso}(M, h)\big).$$

Lemma 3.2

$C(G) \times G$ acts isometrically on $\left(M, h_t^G\right)$, for all $t \in \mathbb{R}_{>0}$.

Proof 3.2:

Let $t > 0$*;*
$\phi \in C(G)$*;*
$g, \tilde{g} \in G$*;*
$p \in M$.

$C(G)$ *gives rise to a new parametrized family of isometric actions :*

$$\begin{array}{cccc} \tilde{\theta} & : & (C(G) \times G) \times \left(M \times G, h + \frac{1}{t}b\right) & \to & \left(M \times G, h + \frac{1}{t}b\right) \\ & & \big((\phi, g), (p, \tilde{g})\big) & \mapsto & \left(\phi(p), \tilde{g} \cdot g^{-1}\right). \end{array}$$

[2] Since G acts on (M, h) isometrically, $\{\mu_g : (M, h) \to (M, h) \mid g \in G\}$ is considered as a Lie subgroup of $\mathrm{Iso}(M, h)$.

<u>*Claim :* $(\rho \circ \tilde{\theta})\big((\phi, g), (p, \tilde{g})\big) = (\phi \circ \mu_g \circ \rho)(p, \tilde{g})$</u>

$$
\begin{aligned}
(\rho \circ \tilde{\theta})\big((\phi, g), (p, \tilde{g})\big) \quad &= \quad \big(\tilde{g} \cdot g^{-1}\big)^{-1} \cdot (\phi(p)) \\
&\overset{\phi \in \underline{C}(G)}{=} \quad \phi\Bigg(\underbrace{\big(g \cdot \tilde{g}^{-1}\big) \cdot p}_{\mu_g(\tilde{g}^{-1} \cdot p)}\Bigg) \\
&= \quad (\phi \circ \mu_g \circ \rho)(p, \tilde{g})
\end{aligned}
$$

$\blacksquare_{\textit{Claim}}$

Finally $C(G) \times G \curvearrowright \big(M, h_t^G\big)$ *by composition of isometries:*

$$\theta\left((\phi, g), p\right) := (\phi \circ \mu_g \circ \rho)(p, e)$$

$\blacksquare$

The following commutative diagram summarizes the whole situation :

$$
\begin{tikzcd}
(M, h) \arrow[loop above, "\mu \text{ isometric } G\text{-action}"] \arrow[r, "\iota"', "\text{injective}"] \arrow[dr, dotted] & \big(M \times G, h + \frac{1}{t} b\big) \arrow[loop above, "\psi \text{ isometric and free } G\text{-action}"] \arrow[d, "\pi"', "\text{orb. subm.}"] \arrow[dr, "\rho \text{ orb. subm.}"] & \\
& \Big((M \times G)/G, \bar{h}_t^G\Big) \arrow[r, "\bar{\rho}", "\text{isometry}"'] & \big(M, h_t^G\big) \arrow[loop below, "\theta \text{ isometric } (C(G) \times G)\text{-actions}"]
\end{tikzcd}
$$

where

$$
\begin{array}{cccccc}
\iota & : & M & \to & M \times G & \\
 & & p & \mapsto & (p, e) & ,
\end{array}
$$

is smooth and injective.

3.2 Metric tensor C_t of h_t^G with respect to h

We now explore the relation between the initial metric h and the Cheeger metrics $h_t^G, t \in \mathbb{R}_{>0}$. This investigation requires the understanding of the vertical and horizontal spaces in $T(M \times G)$ with respect to $h + \frac{1}{t} b$ for the action ψ. We'll describe characteristics of the points (p, e) for all $p \in M$ only, since any other point $(p, g) \in M \times G$ can be studied with $\psi\left(g, (g^{-1} \cdot p, e)\right)$.

For $p \in M$ and $x \in T_e G$, we denote by :

- $X^*(p)$ the action field on M at p generated by x through the action $\mu : G \times M \to M$;
- $\mathcal{H}_p \subset T_p M$ the horizontal space at p with respect to h for μ. As we'll see later, $\mathcal{H}_p$ is also the horizontal space at p with respect to h_t^G, for any $t > 0$.

Proposition 3.3

Let $t > 0$;
$p \in M$ and $v \in T_pM$;
$g \in G$;
$x \in T_eG = \mathfrak{g}$.

Then :

(i) The *action field* generated by x on $M \times G$ is

$$X^*(p, g) = (X^*(p), X_R(g)),$$

where X_R is the right-invariant vector field on G induced by x.

The *vertical space* $\mathcal{V}_{(p,g)}$ is thus made of vectors of this form ;

(ii) The *horizontal space* with respect to $h + \frac{1}{t}b$ is determined by the vertical components of T_pM :

$$\mathcal{H}^t_{(p,e)} = \{(Y^*(p) + \xi, -tS(p)y) \mid y \in \mathfrak{g}, \xi \in \mathcal{H}_p\},$$

where $S(p) : T_eG \to T_eG$ is the orbit tensor of h with respect to b ;

(iii) The *differential of the orbital submersion* $\rho : (M \times G, h + \frac{1}{t}b) \to (M, h^G_t)$ is :

$$d\rho_{(p,e)}(v, x) = v - X^*(p) ;$$

(iv) If $x \in T_eG$ is such that $X^*(p) = v^{\mathcal{V}}$, then the *differential of* ρ *at the horizontal vector* $(v, -tS(p)x) \in \mathcal{H}^t_{(p,e)}$ is

$$d\rho_{(p,e)}\underbrace{(v, -tS(p)x)}_{=:f(v)} = (Id + tS)\, v^{\mathcal{V}} + v^{\mathcal{H}},$$

where $Sv^{\mathcal{V}} := (SX)^*(p)$.

Proof 3.3:

Ad (i):

$$\begin{aligned} X^*(p, g) &\overset{def.}{=} \left.\frac{d}{ds}\right|_{s=0} \psi(\exp(sx), (p, g)) \\ &= \left.\frac{d}{ds}\right|_{s=0} \Big(\mu(\exp(sx), p), \underbrace{\exp(sx) \cdot g}_{=R_g(\exp(sx))}\Big) \\ &\overset{(\star)}{=} (X^*(p), dR_g(x)) \\ &= (X^*(p), X_R(g)), \end{aligned}$$

where $(\star)$ *comes from the result* $(d\exp)_0 = id_{T_eG}$.

✓

Ad (ii): Every tangent vector $w \in T_pM$ *has a vertical part* $Y^*(p) \in \mathcal{V}_p$ *and a h-horizontal component* $\xi \in \mathcal{H}_p$*. That's why an element* $(w, z) \in T_{(p,e)}(M \times G)$ *can be written as* $(Y^*(p) + \xi, z)$*.*

Suppose now (w, z) *to be* $\big(h + \frac{1}{t}b\big)$*-horizontal.*

Then, for any $\chi^*(p,e) = (\chi^*(p), \chi) \in T_{(p,e)}(M \times G)$,

$$\begin{aligned}
0 &= (h + \tfrac{1}{t}b)_{(p,e)}\left((X^*(p) + \xi, z), \chi^*(p,e)\right) \\
&= \underbrace{h(X^*(p) + \xi, \chi^*(p))}_{=h(X^*(p),\chi^*(p))} + \tfrac{1}{t}b(z,\chi) \\
&= b(S(p)x, \chi) + b\left(\tfrac{1}{t}z, \chi\right) \\
&= b\left(S(p)x + \tfrac{1}{t}z, \chi\right).
\end{aligned}$$

By nondegeneracy of the inner product, we conclude that $S(p)x + \frac{1}{t}z = 0$, and thus :

$$(w,z) = (Y^*(p) + \xi, -tS(p)x) \in \mathcal{H}^t_{(p,e)}.$$

✓

<u>Ad (iii)</u>: We use the following curves defined on a commun open interval $(a,b) \subset \mathbb{R}$ to calculate the differential of ρ :

- *$\alpha : (a,b) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = v$;*
- *$\beta : (a,b) \to (M \times G), s \mapsto (\alpha(t), \exp(s \cdot x))$, satisfying $\beta(0) = (p,e)$ and $\beta'(0) = (v,x)$;*
- *$\gamma : (a,b) \to G \times M, s \to (\exp(-s \cdot x), \alpha(t))$.*

We'll need the following differentials :

- *$d\mu_{(e,p)}(0,w) = w$ for all $w \in T_pM$;*
- *$d\mu_{(e,p)}(y,0) = Y^*(p)$;*
- *$(d\gamma)_0(1) = \left(\frac{d}{ds}\Big|_{s=0} exp(-s \cdot x), \alpha'(0)\right) = (-x, v)$.*

We finally calculate :

$$\begin{aligned}
d\rho_{(p,e)}(v,x) &= \frac{d}{ds}\Big|_{s=0}(\rho \circ \beta)(s) \\
&= \frac{d}{ds}\Big|_{s=0}\mu\Big(\underbrace{\exp(s \cdot x)^{-1}}_{=\exp(-s\cdot x)}, \alpha(s)\Big) \\
&= \frac{d}{ds}\Big|_{s=0}(\mu \circ \gamma)(s) \\
&\overset{\text{Chain rule}}{=} \left(d\alpha_{(e,p)} \circ d\gamma_0\right)(1) \\
&= d\alpha_{(e,p)}(-x, v) \\
&= v - X^*(p).
\end{aligned}$$

✓

<u>Ad (iv)</u>:

$$\begin{aligned}
d\rho_{(p,e)}(v, -tS(p)x) &\overset{(iii)}{=} v - (-tSX)^*(p) \\
&= \left(v^{\mathcal{V}} + v^{\mathcal{H}}\right) + t(SX)^*(p) \\
&= (Id + tS)v^{\mathcal{V}} + v^{\mathcal{H}}.
\end{aligned}$$

✓

■

The following family of tensors will characterize the Cheeger construction $h \underset{G}{\to} h_t^G$ by linking the original metric to the new ones. Note that we'll also consider the case $t = 0$.

Definition 3.4 - Metric tensor of h_t^G with respect to h

Let $t \geqslant 0$.

We define $C_t : T_pM \to T_pM$ by

$$C_t v := (Id + tS(p))^{-1} v^{\mathcal{V}} + v^{\mathcal{H}},$$

for all $p \in M$, $v \in T_pM$.

Lemma 3.5

C_t is an *invertible* tensor field.

Proof 3.5:

For t = 0: $C_0 = Id_{TM}$.

For t > 0: We suppose by contradiction, that $(Id + tS)$ is not invertible and so not injective.

$\Rightarrow$ $\exists x \in T_eG - \{0\}$ *such that* $tSx = -x$.

Then, $0 > -\frac{1}{t} b(x, x) = b(Sx, x) = h(X^*, X^*) \geqslant 0$. $\lightning$

$\Rightarrow$ $Id + tS$ *injective and thus invertible.*

$\blacksquare$

Let's define $h_0^G := h$.

We now prove that the name given to C_t is well chosen :

Theorem 3.6

Let $t \geqslant 0$;
$p \in M$;
$v, w \in T_pM$;
$x := *^{-1}\big(v^{\mathcal{V}}\big) \in \mathfrak{m}_p$ and $y := *^{-1}\big(w^{\mathcal{V}}\big) \in \mathfrak{m}_p$.

We define

$$\begin{array}{rcccc} f & : & T_pM & \to & \mathcal{H}^t_{(p,e)} \\ & & \tilde{v} & \mapsto & \big(\tilde{v}, -tS(p)\tilde{x}\big), \end{array}$$

where $\widetilde{x} := *^{-1}\big(\widetilde{v}^{\mathcal{V}}\big) \in \mathfrak{m}_p$.

Then,

(i) $d\rho_{(p,e)} \circ f = C_t^{-1}$, i.e. $f(v)$ and $f(w)$ are the horizontal lifts of $C_t^{-1}v$ and $C_t^{-1}w$, respectively ;

(ii) $h_t^G(v, w) = h\big(C_t v, w\big)$.

Proof 3.6:

For $t = 0$, the statements are clear. So consider the case $t > 0$.

Ad (i) : Direct consequence of Proposition 3.3 (iv). By the following commutative diagram, $f(v)$ and $f(w)$ are the ρ-horizontal lifts of $C_t^{-1}v$ and $C_t^{-1}w$, respectively :

$$\begin{tikzcd} H^t_{(p,e)} \ni & \underset{(v,-tS(p)x)}{\overset{f(v)}{=}} \arrow[r, "d\rho_{(p,e)}"] & \underset{(Id+tS)v^{\mathcal{V}}+v^{\mathcal{H}}}{\overset{C_t^{-1}v}{=}} & \in T_pM \\ & & v \arrow[ul, "f"] \arrow[u, "C_t^{-1}"'] & \in T_pM \end{tikzcd}$$

Ad (ii) : The idea is to work with C_t^{-1} by using the statement (i) :

$$\begin{aligned} h_t^G\left(C_t^{-1}v, C_t^{-1}w\right) &= h_t^G\left(d\rho_{(p,e)}(f(v)), d\rho_{(p,e)}(f(w))\right) \\ &\overset{\rho \text{ Riem. subm.}}{=} \left(h + \tfrac{1}{t}b\right)\left((v, -tS(p)x), (w, -tS(p)y)\right) \\ &= h(v,w) + b\left(S(p)x, tS(p)y\right) \\ &= h(v,w) + h\left(X^*(p), t(SY)^*(p)\right) \\ &\overset{h\left(v^{\mathcal{H}}, t(SY)^*(p)\right)=0}{=} h\left(v, w + t(SY)^*(p)\right) \\ &= h\left(v, C_t^{-1}w\right). \end{aligned}$$

The invertibility of C_t leads to the desired result.

■

Remark 3.7. The horizontal vectors in T_pM with respect to h and h_t^G coincide, because C_t is the identity on $\mathcal{H}_p$, and by the second statement of Theorem 3.6. That's why no "t" indice is required in the notation $\mathcal{H}_p$.

The last theorem permits to link the orbit tensor S_t of the action (G, b) on (M, h_t^G), $t \geqslant 0$, with the initial one $S =: S_0$:

Corollary 3.8

Let $t \geqslant 0$.

The orbit tensor S_t of $(G, b) \curvearrowright (M, h_t^G)$ satisfies
$$S_t = S(Id + tS)^{-1}.$$

Proof 3.8:

Let $x, y \in T_eG$;
$p \in M$.

Then,

$$\begin{aligned} h_t^G\left(X^*(p), Y^*(p)\right) &\overset{\text{Theorem 3.6}}{=} h\left(C_tX^*(p), Y^*(p)\right) \\ &= h\left((Id + tS)^{-1}X^*(p), Y^*(p)\right) \\ &= b\big(\underbrace{S(p)(Id + tS)^{-1}}_{=:S_t(p)}x, y\big). \end{aligned}$$

■

3.3 Shrinking along the orbits of $(h_t^G)_{t\geqslant 0}$

Theorem 3.6 implies that the family of Cheeger metrics $(h_t^G)_{t\geqslant 0}$ forms a *differential variation of* h, meaning that the following maps are smooth, for all $X, Y \in \mathfrak{X}(M)$:

$$\begin{array}{rcccc} h^G & : & \mathbb{R}_{\geqslant 0} & \to & C^\infty(M) \\ & & t & \mapsto & h_t^G(X,Y) \end{array}$$

This leads to the question of knowing how the Cheeger metrics h_t^G change with t, which is answered by the following result :

Theorem 3.9

Let $0 < t_1 =: t < t_2$;
$p \in M$;
$v, w \in T_pM$.

(i) The metrics h_t^G are independent of t in horizontal components, i.e. for vectors perpendicular to the orbits :

$$h_t^G\left(v^{\mathcal{H}}, w\right) = h(v^{\mathcal{H}}, w).$$

(ii) The Cheeger metrics decrease with t in direction tangent to the orbits :

$$\left|h\left(v^{\mathcal{V}}, w\right)\right| \geqslant \left|h_{t_1}^G\left(v^{\mathcal{V}}, w\right)\right| \geqslant \left|h_{t_2}^G\left(v^{\mathcal{V}}, w\right)\right|.$$

Proof 3.9:

Ad (i):

Step ① : All eigenvalues of $S(p)$ are positive

Let $\lambda \in \mathbb{R}$ *be an eigenvalue of* $S(p)$;
$x \in \mathfrak{m}_p$ *an eigenvector of* $S(p)$ *corresponding to* λ.

$x \neq 0 \Rightarrow X^*(p) \neq 0$, *since* $x \notin \mathfrak{g}_p$.

Then,

$$0 < h\left(X^*(p), X^*(p)\right) = b(S(p)x, x) = b(\lambda x, x) = \lambda \underbrace{b(x,x)}_{>0}.$$

$\Rightarrow \ \lambda > 0$, *which means that* $S(p)$ *is positive definite.*

Step ② : Eigenspaces are orthogonal with respect to b, h and h_t^G, respectively

Consider $x_1, x_2 \in \mathfrak{m}_p$ *eigenvectors of* $S(p)$ *corresponding to the eigenvalues* $\lambda_1 \in \mathbb{R}$ *and* $\lambda_2 \in \mathbb{R}$, *respectively, with* $\lambda_1 \neq \lambda_2$.

The orbit tensor $S(p) : \mathfrak{m}_p \to \mathfrak{m}_p$ *being self-adjoint implies that* $b(x_1, x_2) = 0$. *The consequences for* h *and* h_t^G *are :*

$$h\big(X_1^*(p), X_2^*(p)\big) = b\big(S(p)x_1, x_2\big) = \lambda_1 b(x_1, x_2) = 0,$$

and

$$\begin{aligned} h_t^G\big(X_1^*(p), X_2^*(p)\big) &= h\big(X_1^*(p), X_2^*(p)\big) + th\big((S^{-1}X_1)^*(p), X_2^*(p)\big) \\ &= 0 + tb(x_1, x_2) \\ &= 0. \end{aligned}$$

This also shows that for $\lambda_1 = \lambda_2$ *and* $b(x_1, x_2) = 0$ *:*

$$h\left(X_1^*(p), X_2^*(p)\right) = h_t^G\left(X_1^*(p), X_2^*(p)\right) = 0.$$

Step ③ : Effect of the Cheeger deformation h_t^G *for any vectors of* V_p

Let $\tilde{v}^{\mathcal{V}} := C_t\left(v^{\mathcal{V}}\right), \tilde{w}^{\mathcal{V}} := C_t\left(w^{\mathcal{V}}\right) \in V_p$ *;*
$x := *^{-1}\left(\tilde{v}^{\mathcal{V}}\right) \in \mathfrak{m}_p$ *and* $y := *^{-1}\left(\tilde{w}^{\mathcal{V}}\right) \in \mathfrak{m}_p$ *;*
$d := \dim V_p$ *;*
$(x_1, ..., x_d)$ *an orthonormal basis of* $*^{-1}\left(\mathcal{V}_p\right) = \mathfrak{m}_p \subseteq T_eG$ *with respect to* b *with corresponding eigenvalues* $(\lambda_1, ...\lambda_2)$ *;*
$\tilde{\lambda} := \min\{\lambda_i \mid i = 1, ..., d\}$.

There exist $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, ..., d$ *such that* $x = \sum_{i=1}^d \alpha_i x_i$ *and* $y = \sum_{i=1}^d \beta_i x_i$ *and by linearity of the infinitesimal generator* $*$ *of the action* μ, $X^*(p) = \sum_{i=1}^d \alpha_i X_i^*(p)$ *and* $Y^*(p) = \sum_{i=1}^d \beta_i X_i^*(p)$.

It suffices to compute the initial metric :

$$\begin{aligned} h\left(v^{\mathcal{V}}, w\right) &= h\left(v^{\mathcal{V}}, w^{\mathcal{V}}\right) \\ &= h\left(C_t^{-1}\tilde{v}^{\mathcal{V}}, C_t^{-1}\tilde{v}^{\mathcal{V}}\right) \\ &= \textstyle\sum_{i,j=1}^d \alpha_i\beta_j h\big(C_t^{-1}X_i^*(p), C_t^{-1}X_j^*(p)\big) \\ &\overset{X_i^*(p)\perp X_j^*(p)}{\underset{if\ i\neq j}{=}} \textstyle\sum_{i=1}^d \alpha_i\beta_i\Big[h\big(X_i^*(p), X_i^*(p)\big) + 2th\big(X_i^*(p), (SX_i)^*(p)\big) \\ &\qquad + t^2 h\big((SX_i)^*(p), (SX_i)^*(p)\big)\Big] \\ &= \textstyle\sum_{i=1}^d \alpha_i\beta_i\Big[b\big(S(p)x_i, x_i\big) + 2tb\big(S(p)x_i, S(p)x_i\big) \\ &\qquad + t^2 b\big(S(p)^2x_i, S(p)x_i\big)\Big] \\ &\overset{b(x_i,x_i)=1}{=} \textstyle\sum_{i=1}^d \alpha_i\beta_i\lambda_i(1+\lambda_i t)^2, \end{aligned}$$

and the Cheeger metric :

$$\begin{aligned} h_t^G\left(v^{\mathcal{V}}, w\right) &= h_t^G\left(v^{\mathcal{V}}, w^{\mathcal{V}}\right) \\ &= h_t^G\left(C_t^{-1}\tilde{v}^{\mathcal{V}}, C_t^{-1}\tilde{v}^{\mathcal{V}}\right) \\ &= \textstyle\sum_{i,j=1}^d \alpha_i\beta_j h_t^G\big(C_t^{-1}X_i^*(p), C_t^{-1}X_j^*(p)\big) \\ &\overset{X_i^*(p)\perp X_j^*(p)}{\underset{if\ i\neq j}{=}} \textstyle\sum_{i=1}^d \alpha_i\beta_j h\Big(X_i^*(p), C_t^{-1}X_i^*(p)\Big) \\ &= \textstyle\sum_{i=1}^d \alpha_i\beta_i\Big[h\big(X_i^*(p), X_i^*(p)\big) + th\big(X_i^*(p), (SX_i)^*(p)\big)\Big] \\ &= \textstyle\sum_{i=1}^d \alpha_i\beta_i\Big[b\big(S(p)x_i, x_i\big) + tb\big(S(p)x_i, S(p)x_i\big)\Big] \\ &\overset{b(x_i,x_i)=1}{=} \textstyle\sum_{i=1}^d \alpha_i\beta_i\lambda_i(1+\lambda_i t). \end{aligned}$$

We finally compare :

$$\begin{array}{rcl}\left|h_t^G(v^{\mathcal{V}},w)\right| & = & \left|\sum_{i=1}^d \alpha_i\beta_i\lambda_i(1+\lambda_i t)\right| \\ & \overset{1+\tilde{\lambda}t>1}{\leqslant} & \left|\sum_{i=1}^d \alpha_i\beta_i\lambda_i(1+\lambda_i t)(1+\tilde{\lambda})\right| \\ & \leqslant & \left|\sum_{i=1}^d \alpha_i\beta_i\lambda_i(1+\lambda_i t)^2\right| \\ & = & \left|h\left(v^{\mathcal{V}},w\right)\right|.\end{array}$$

<u>*Ad (ii)*</u>*:*

The idea is to express h_t^G as a function of h for the orthonormal basis $(x_1,...,x_2)$ to then compare $h_{t_2}^G$ with $h_{t_1}^G$, $0<t_1=:t<t_2$, for general vectors $v,w\in T_pM$. We consider the same quantities as in step ③ but with $x:=*^{-1}\big(v^{\mathcal{V}}\big)\in\mathfrak{m}_p$ and $y:=*^{-1}\big(w^{\mathcal{V}}\big)\in\mathfrak{m}_p$.

We still have that $\alpha_i,\beta_i\in\mathbb{R}$ are the coefficients in $x=\sum_{i=1}^d\alpha_i x_i$ and $y=\sum_{i=1}^d\beta_i x_i$ and thus $X^*(p)=\sum_{i=1}^d\alpha_i X_i^*(p)$ and $Y^*(p)=\sum_{i=1}^d\beta_i X_i^*(p)$.

Define $\kappa_i:=\frac{1+\lambda_i t_1}{1+\lambda_i t_2}\in(0,1)$ for $i=1,...,d$, and $\tilde{\kappa}:=\max\{\kappa_i \mid i=1,...,d\}$. For $i\neq j$, we already know that $h_t^G\big(X_i^*(p),X_j^*(p)\big)=0$. And for $i=j$:

$$\begin{array}{rcl}h_t^G\big(X_i^*(p),X_i^*(p)\big) & = & h\big(C_tX_i^*(p),X_i^*(p)\big) \\ & = & b\left((Id+tS(p))^{-1}x_i,S(p)x_i\right) \\ & = & \frac{1}{1+\lambda_i t}b(x_i,S(p)x_i) \\ & = & \frac{1}{1+\lambda_i t}h\big(X_i^*(p),X_i^*(p)\big).\end{array}$$

Then,

$$\begin{array}{rcl}\left|h_{t_2}^G\big(v^{\mathcal{V}},w^{\mathcal{V}}\big)\right| & = & \left|\sum_{i=1}^d\alpha_i\beta_i h_{t_2}^G\big(X_i^*(p),X_i^*(p)\big)\right| \\ & = & \left|\sum_{i=1}^d\frac{1}{1+\lambda_i t_2}\alpha_i\beta_i h\big(X_i^*(p),X_i^*(p)\big)\right| \\ & = & \left|\sum_{i=1}^d\underbrace{\frac{1+\lambda_i t_1}{1+\lambda_i t_2}}_{=\kappa_i}\alpha_i\beta_i h\big(X_i^*(p),X_i^*(p)\big)\right| \\ & \leqslant & \tilde{\kappa}\cdot\left|\sum_{i=1}^d\alpha_i\beta_i h_{t_1}^G\big(X_i^*(p),X_i^*(p)\big)\right| \\ & \overset{\kappa<1}{\leqslant} & \left|h_{t_1}^G\big(v^{\mathcal{V}},w^{\mathcal{V}}\big)\right|.\end{array}$$

■

The last result shows that Cheeger deformations preserve the length of vectors perpendicular to the orbits while shrinking the lengths of tangent ones. The following section illustrates this notion by a concrete example.

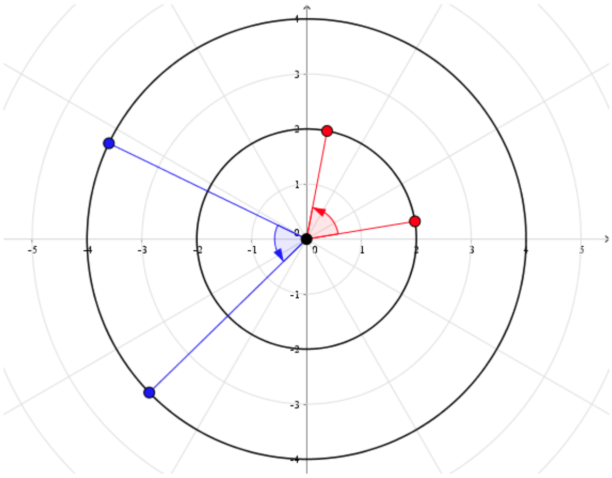
3.4 Example : $S^1 \curvearrowright \mathbb{C}$

This example is inspired by the work of Lawrence Mouillé, with a clear explanation in [Mou17a] but with other calculations in [Mou17b]. We will use the polar coordinates (r,θ) to describes the complex elements.

Consider the circle $(S^1,\cdot)$ acting on the complex plane $(\mathbb{C},\cdot)$ by multiplication, representing rotation about the origin 0 :

$$\begin{array}{cccc} \mu & : & S^1\times\mathbb{C} & \to & \mathbb{C} \\ & & \left(e^{i\phi}, re^{i\theta}\right) & \mapsto & re^{i(\phi+\theta)}. \end{array}$$

The following illustration shows the orbits, which are the origin 0 and the concentric circles centered at 0 :



Think of the associated vector fields related to polar coordinates (r,θ) : $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$.

Let's denote by h the metric on the complex plane $\mathbb{C}$ given by the euclidean scalar product, i.e. the real part of the hermitian scalar product :

$$h\left(r_1e^{i\theta_1}, r_2e^{i\theta_2}\right) := Re\left(r_1r_2e^{i(\theta_1-\theta_2)}\right) \ ;$$

b the induced metric from h on $S^1\subset\mathbb{C}$, which is biinvariant ;

$h_t^{S^1}$, $t\geqslant 0$, the metrics on $\mathbb{C}$ resulting from the Cheeger construction.

Theorem 3.9 tells us that in radial directions, the vectors's lenght is preserved and thus $\|\frac{\partial}{\partial r}(p)\|_{h_t^{S^1}} = 1$, for any $t\geqslant 0$ and for all complex point $p\in\mathbb{C}$. On the contrary, the vectors tangents to the orbits will shrink more and more. Let's calculate it accurately by following the Cheeger construction.

According to Lemma 3.1, (S^1,b) acts on $\left(\mathbb{C}\times S^1, h+\frac{1}{t}b\right)$ freely, by isometries and with closed orbits, for all $t\in\mathbb{R}_{>0}$:

$$\begin{array}{cccc} \psi & : & S^1\times\left(\mathbb{C}\times S^1\right) & \to & \mathbb{C}\times S^1 \\ & & \left(e^{i\phi_1}, \left(re^{i\theta}, e^{i\phi_2}\right)\right) & \mapsto & \left(re^{i(\phi_1+\theta)}, e^{i(\phi_1+\phi_2)}\right). \end{array}$$

ψ generates an orbital submersion :

$$\begin{array}{rcccl} \rho & : & \left(\mathbb{C}\times S^1, h+\frac{1}{t}b\right) & \to & \left(\mathbb{C}, h_t^{S^1}\right) \\ & & \left(re^{i\theta}, e^{i\phi}\right) & \mapsto & re^{i(\theta-\phi)}. \end{array}$$

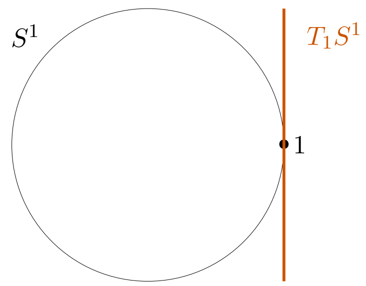
For any given $t>0$, the computation of the induced metric $h_t^{S^1}$ requires to know the vertical spaces $\mathcal{V}_p \subset T_p\mathbb{C}$ for the initial action μ, the horizontal spaces $\mathcal{H}^t_{(p,1)} \subset T_{(p,1)}(M\times G)$ for ψ, and also the differential of ρ at $(p,1)$, for all $p \in \mathbb{C}$.

Let $t>0$;
$p := re^{i\theta} \in \mathbb{C}$.

Vertical space $\mathcal{V}_p \subset T_p\mathbb{C} = \mathbb{C}$

(i) Lie algebra of S^1

We may observe it directly from the drawing of S^1 on the complex plane :



$\Rightarrow \mathfrak{s}^1 := T_1S^1 = i\mathbb{R} \subset \mathbb{C}$.

Let $x := i\lambda \in \mathfrak{s}^1$ fixed for the next steps.

(ii) Exponential map on S^1

The integral curve $\gamma_x : \mathbb{R} \to S^1$ of the left invariant vector field $X_L \in \mathfrak{X}(S^1)^L$ generated by x is

$$\gamma_x(s) = e^{i\lambda s}.$$

Then,

$$\exp(x) = \gamma_x(1) = e^{i\lambda} = e^x.$$

(iii) Action fields X^* on M

$$X^*\Big(\underbrace{re^{i\theta}}_{=p}\Big) = \frac{d}{ds}\Big|_{s=0}\mu\Big(exp(sx), re^{i\theta}\Big) = \frac{d}{ds}\Big|_{s=0} re^{i(s\lambda+\theta)} = i\lambda re^{i(0+\theta)} = i\lambda re^{i\theta}$$

Hence,

$$\boxed{\mathcal{V}_p = \left\{ i\lambda re^{i\theta} \;\middle|\; \lambda \in \mathbb{R}\right\}}.$$

As a direct consequence,

$$\boxed{\mathcal{H}_p = \mathcal{V}_p^{\perp} = \left\{ \lambda re^{i\theta} \;\middle|\; \lambda \in \mathbb{R}\right\}}.$$

Differential $d\rho_{(p,1)}$

Let $t > 0$;
$(v, x) := (v, i\lambda) \in T_{(p,1)}\big(\mathbb{C} \times S^1\big) \cong T_p\mathbb{C} \times T_1 S^1$.

(i) Method 1 : use Proposition 3.3 (iii)

$$\boxed{d\rho_{(p,1)}(v, x) = v - X^*(p) = v - i\lambda re^{i\theta}}$$

(ii) Method 2

Consider the following curve :

$$\begin{array}{rccc} \gamma & : & \mathbb{R} & \to & \big(\mathbb{C} \times S^1\big) \\ & & s & \mapsto & \Big(re^{i\left(\frac{v}{ire^{i\theta}}s+\theta\right)}, e^{i\lambda s}\Big). \end{array}$$

Then,

$$d\rho_{(p,1)}(v, x) = (\rho \circ \gamma)'(0) = \left(ir\left(\frac{v}{ire^{i\theta}} - \lambda\right)e^{i\left(\frac{v}{ire^{i\theta}}s+\theta-\lambda s\right)}\right)\Big|_{s=0} = v - i\lambda re^{i\theta}.$$

Horizontal space $\mathcal{H}^t_{(p,1)}$ with respect to $\rho : (\mathbb{C} \times S^1, h + \frac{1}{t}b) \to (\mathbb{C}, h^{S^1}_t)$ at $(p, 1)$

(i) Orbit tensor S

Let $x, y \in \mathfrak{s}^1$.

Then, since the metric b is induced by h,

$$h\big(S(p)x, y\big) = b\big(S(p)x, y\big) = h\big(X^*(p), Y^*(p)\big) = h\big(xre^{i\theta}, yre^{i\theta}\big) = \underbrace{\|re^{i\theta}\|^2_h}_{=r^2} \cdot h(x, y).$$

Hence,

$$\boxed{S(p) = S(re^{i\theta}) = r^2}.$$

(ii) Horizontal space $\mathcal{H}^t_{(p,1)}$

We simply use Proposition 3.3 (ii) :

$$\boxed{\mathcal{H}^t_{(p,1)} = \Big\{\big(i\lambda_1 re^{i\theta} + \lambda_2 re^{i\theta}, -i\lambda_1 tr^2\big) \mid \lambda_1, \lambda_2 \in \mathbb{R}\Big\}}.$$

To determine $\|\frac{\partial}{\partial\theta}(p)\|_{h^{S^1}_t}$ at any point $p \in M$, it suffices to focus on the points $r \in \mathbb{R}_{>0}$ since this norm stays constant on the orbits $[r]$. So let fix $r \in \mathbb{R}_{>0}$. We now compute in two steps the length of $\frac{\partial}{\partial\theta}(r) = ir \in T_r\mathbb{C} = \mathbb{C}$ in the Cheeger metrics $h^{S^1}_t$.

(i) Horizontal lift $(v, x) := (i\lambda_1 r + \lambda_2 r, -i\lambda_1 tr^2) \in \mathcal{H}^t_{(r,1)}$ of ir with respect to ρ

The coefficients $\lambda_1, \lambda_2 \in \mathbb{R}$ should satisfy

$$d\rho_{(p,1)}(v, x) = i\lambda_1 r + \lambda_2 r + i\lambda_1 tr^3 = ir$$

$$\Leftrightarrow \begin{cases} \lambda_1 = \frac{1}{1+tr^2} \\ \lambda_2 = 0 \end{cases}.$$

Therefore, the horizontal lift of ir is $\left(\frac{ir}{1+tr^2}, -\frac{itr^2}{1+tr^2}\right) \in \mathcal{H}^t_{(r,1)}$.

(ii) Length of $\frac{\partial}{\partial\theta}(r)$ with respect to $h_t^{S^1}$

$$\begin{aligned}
\|\tfrac{\partial}{\partial\theta}(r)\|^2_{h_t^{S^1}} &= h_t^{S^1}(ir, ir) \\
&\overset{\substack{\rho \text{ orb.}\\ \text{subm.}}}{=} \left(h + \tfrac{1}{t}b\right)\big((v,x),(v,x)\big) \\
&= h\left(\tfrac{ir}{1+tr^2}, \tfrac{ir}{1+tr^2}\right) + \tfrac{1}{t}b\left(-\tfrac{itr^2}{1+tr^2}, -\tfrac{itr^2}{1+tr^2}\right) \\
&= \tfrac{r^2}{(1+tr^2)^2} + \tfrac{1}{t}\cdot\tfrac{t^2r^4}{(1+tr^2)^2} \\
&= \tfrac{r^2}{1+tr^2}
\end{aligned}$$

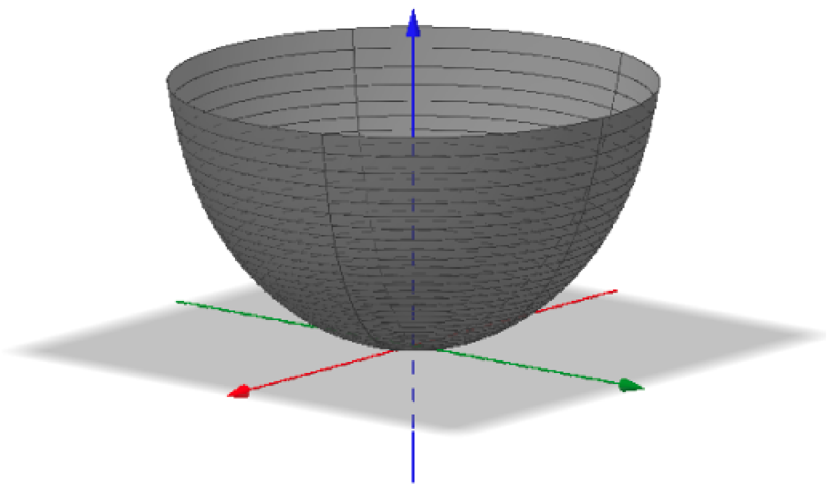
Hence,

$$\boxed{\|\tfrac{\partial}{\partial\theta}(r)\|_{h_t^{S^1}} = \tfrac{r}{\sqrt{1+tr^2}}}.$$

As a conclusion, the Cheeger process is a continuous transformation of the initial metric $h := h_0^{S^1}$ which shrinks vectors tangent to the orbits :

$$\lim_{t\to 0}\|\tfrac{\partial}{\partial\theta}(r)\|_{h_t^{S^1}} = r = \|\tfrac{\partial}{\partial\theta}(r)\|_h \quad \text{and} \quad \lim_{t\to+\infty}\|\tfrac{\partial}{\partial\theta}(r)\|_{h_t^{S^1}} = 0$$

We visualize the initial Riemannian manifold $(\mathbb{C}, h)$ as an imbedding in $\mathbb{R}^3$ which takes up the (x,y)-plane. As for the transformed manifolds $\left(\mathbb{C}, h_t^{S^1}\right)$, they bend upwards since the squeezed orbits are forced to get closer to the z-axis :



We now aim to calculate the sectional curvature $sec_{h_t^{S^1}}$ of $\mathbb{C}$ at $p \in \mathbb{C}$ with respect to the Cheeger metrics $\left(h_t^{S^1}\right)_{t>0}$. The strategy consists in the use of the Gray-O'Neill formula on the orthonormal basis $\left(\frac{\partial}{\partial r}(p), \frac{\frac{\partial}{\partial\theta}(p)}{\|\frac{\partial}{\partial\theta}(p)\|_{h_t^{S^1}}}\right)$ of $T_p\mathbb{C}$.

We will make computations with Levi-Civita connection more convenient by working with partial derivative vector fields on S^1 and $\mathbb{C}$.

Let fix again $\quad t > 0$;
$\quad p := re^{i\theta} \in \mathbb{C}$.

<u>Orthonormal basis of $T_{(p,1)}(\mathbb{C}\times S^1)$ with respect to $h+\frac{1}{t}b$ composed of the horizontal lifts of our basis vectors of $T_p\mathbb{C}$</u>

(i) Generators of $\mathcal{H}^t_{(p,1)}$

$\mathcal{H}^t_{(p,1)} = \left\{\left(i\lambda_1 re^{i\theta}+\lambda_2 re^{i\theta}, -i\lambda_1 tr^2\right) \mid \lambda_1,\lambda_2\in\mathbb{R}\right\}$ is a two-dimensional vector subspace of $T_{(p,1)}(\mathbb{C}\times S^1)$, generated by $\widehat{\partial_r^p} := \left(\frac{\partial}{\partial r}(p),0\right)$ and $\widehat{\partial_\theta^p} := \left(\frac{\partial}{\partial\theta}(p), -tr^2\frac{\partial}{\partial\phi}(1)\right)$, where ϕ is the angular coordinate in S^1.

$$\Rightarrow\ \mathcal{H}^t_{(p,1)} = \operatorname{span}_{\mathbb{R}}\left(\widehat{\partial_r^p},\widehat{\partial_\theta^p}\right)$$

(ii) Horizontal lifts of $\frac{\partial}{\partial r}(p)$ and $\frac{\frac{\partial}{\partial\theta}(p)}{\|\frac{\partial}{\partial\theta}(p)\|_{h_t^{S^1}}}$ at $(p,1)$

From definition of ρ, we deduce easily its differential at $(p,1)$ for partial derivatives $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\phi}$:

$$\begin{array}{rcl} d\rho_{(p,1)}\left(\left(\frac{\partial}{\partial r}(p),0\right)\right) &=& \frac{\partial}{\partial r}(p)\\ d\rho_{(p,1)}\left(\left(\frac{\partial}{\partial\theta}(p),0\right)\right) &=& \frac{\partial}{\partial\theta}(p)\\ d\rho_{(p,1)}\left(\left(0,\frac{\partial}{\partial\phi}(1)\right)\right) &=& -\frac{\partial}{\partial\theta}(p)\end{array}$$

We search the coefficients $\alpha_{\widehat{\partial_r^p}}, \alpha_{\widehat{\partial_\theta^p}}, \beta_{\widehat{\partial_r^p}}, \beta_{\widehat{\partial_\theta^p}}\in\mathbb{R}$ such that

$$\left\{\begin{array}{rcl} d\rho_{(p,1)}(\alpha_{\widehat{\partial_r^p}}\,\widehat{\partial_r^p}+\alpha_{\widehat{\partial_\theta^p}}\,\widehat{\partial_\theta^p}) &=& \frac{\partial}{\partial r}(p)\\ d\rho_{(p,1)}(\beta_{\widehat{\partial_r^p}}\,\widehat{\partial_r^p}+\beta_{\widehat{\partial_\theta^p}}\,\widehat{\partial_\theta^p}) &=& \frac{\frac{\partial}{\partial\theta}(p)}{\|\frac{\partial}{\partial\theta}(p)\|_{h_t^{S^1}}}\end{array}\right.$$

$$\Leftrightarrow\left\{\begin{array}{rcl} \alpha_{\widehat{\partial_r^p}}\,\frac{\partial}{\partial r}(p)+\alpha_{\widehat{\partial_\theta^p}}\,(1+tr^2)\,\frac{\partial}{\partial\theta}(p) &=& \frac{\partial}{\partial r}(p)\\ \beta_{\widehat{\partial_r^p}}\,\frac{\partial}{\partial r}(p)+\beta_{\widehat{\partial_\theta^p}}\,(1+tr^2)\,\frac{\partial}{\partial\theta}(p) &=& \frac{\sqrt{1+tr^2}}{r}\,\frac{\partial}{\partial\theta}(p)\end{array}\right.$$

$$\Leftrightarrow\left\{\begin{array}{rcl} \alpha_{\widehat{\partial_r^p}} &=& 1\\ \alpha_{\widehat{\partial_\theta^p}} &=& 0\\ \beta_{\widehat{\partial_r^p}} &=& 0\\ \beta_{\widehat{\partial_\theta^p}} &=& \frac{1}{r\sqrt{1+tr^2}}\ .\end{array}\right.$$

$\Rightarrow \widehat{\partial_r^p}$ and $\frac{1}{r\sqrt{1+tr^2}}\widehat{\partial_\theta^p}$ are the horizontal lifts of $\frac{\partial}{\partial r}(p)$ and $\frac{\frac{\partial}{\partial\theta}(p)}{\|\frac{\partial}{\partial\theta}(p)\|_{h_t^{S^1}}}$ at $(p,1)$, respectively, and form an orthonormal basis of $T_{(p,1)}(\mathbb{C}\times S^1)$, since ρ is a Riemannian submersion.

At this step, the Gray-O'Neill Formula 1.8 leads to :

$$\begin{array}{rcl} sec_{h_t^{S^1}}(T_p\mathbb{C}) &=& sec_{h_t^{S^1}}\left(\frac{\partial}{\partial r}(p),\frac{\frac{\partial}{\partial\theta}(p)}{\|\frac{\partial}{\partial\theta}(p)\|_{h_t^{S^1}}}\right)\\ &=& sec_{(h+\frac{1}{t}b)}\left(\widehat{\partial_r^p},\frac{1}{r\sqrt{1+tr^2}}\widehat{\partial_\theta^p}\right)+\frac{3}{4}\left\|\left[\widehat{\partial_r^p},\frac{1}{r\sqrt{1+tr^2}}\widehat{\partial_\theta^p}\right]^{\mathcal{V}}\right\|^2_{h+\frac{1}{t}b} \qquad (\star)\\ &\overset{\substack{\mathbb{C}\text{ is flat}\\ \text{w.r.t. } h}}{=}& \frac{3}{4r^2(1+tr^2)}\left\|\left[\widehat{\partial_r^p},\widehat{\partial_\theta^p}\right]^{\mathcal{V}}\right\|^2_{h+\frac{1}{t}b}.\end{array}$$

Lie Bracket term $\left[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\right]$

The following formula permits to compute $\left[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\right]$ by using components of $\widehat{\partial}^p_r$ and $\widehat{\partial}^p_\theta$ in the basis $\mathcal{B} := \Big(\big(\frac{\partial}{\partial r}(p), 0\big), \big(\frac{\partial}{\partial \theta}(p), 0\big), \big(0, \frac{\partial}{\partial \phi}(1)\big)\Big)$ of $T_{(p,1)}(\mathbb{C} \times S^1) \cong T_{(r,\theta,0)}\mathbb{R}^3 \cong \mathbb{R}^3$:

$$\widehat{\partial}^p_r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}} \qquad \widehat{\partial}^p_\theta = \begin{pmatrix} 0 \\ 1 \\ -tr^2 \end{pmatrix}_{\mathcal{B}}$$

(i) Coordinate formula for the Lie Bracket[3]

Let $X, Y \in \mathfrak{X}(M)$ for a smooth manifold M ;
$n := \dim M$;
$(x^i)_{i=1,\dots,n}$ some local smooth coordinates for M.

X, Y can be expressed in coordinates in the local charts :

$$X =: \sum_{i=1}^n \lambda_i \frac{\partial}{\partial x^i} \qquad Y =: \sum_{i=j}^n \mu_j \frac{\partial}{\partial x^j}$$

Then the Lie Bracket $[X, Y]$ has the following coordinate expression :

$$[X, Y] = \sum_{i,j=1}^n \left(\lambda_i \frac{\partial \mu_j}{\partial x^i} - \mu_i \frac{\partial \lambda_j}{\partial x^i}\right) \frac{\partial}{\partial x^j}.$$

(ii) Compute $\left[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\right]$

We directly make the computation at the point $(p, 1)$:

$$\boxed{\left[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\right] = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial}{\partial r}(-tr^2) \end{pmatrix}_{\mathcal{B}} = \left(0, -2tr\frac{\partial}{\partial \phi}(1)\right)}.$$

Projection $\left[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\right]^{\mathcal{V}}$ on the vertical space $\mathcal{V}_{(p,1)}$

(i) Vertical space $\mathcal{V}_{(p,1)}$ with respect to ρ and its generator v

$\dim \mathcal{V}_{(p,1)} = \dim T_{(p,1)}(\mathbb{C} \times S^1) - \dim \mathcal{H}^t_{(p,1)} = 3 - 2 = 1$ which means that a vector $v =: (v_1, v_2) =: \Big(\alpha_r \frac{\partial}{\partial r}(p) + \alpha_\theta \frac{\partial}{\partial \theta}(p), \beta \frac{\partial}{\partial \phi}(1)\Big) \in T_{(p,1)}(\mathbb{C} \times S^1)$ generates $\mathcal{V}_{(p,1)}$. Two conditions determine v :

$$\begin{cases} \left(h + \frac{1}{t}b\right)\left(\widehat{\partial}^p_r, v\right) &= 0 \\ \left(h + \frac{1}{t}b\right)\left(\widehat{\partial}^p_\theta, v\right) &= 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} h\left(\frac{\partial}{\partial r}(p), v_1\right) + \frac{1}{t}b(0, v_2) &= 0 \\ h\left(\frac{\partial}{\partial \theta}(p), v_1\right) + \frac{1}{t}b(-tr^2\frac{\partial}{\partial \phi}(1), v_2) &= 0 \end{cases}$$

[3] See [Lee12, Theorem 8.26]

$$\Leftrightarrow \left\{ \begin{array}{lcl} \alpha_r \underbrace{h\Big(\frac{\partial}{\partial r}(p), \frac{\partial}{\partial r}(p)\Big)}_{=1} + \alpha_\theta \underbrace{h\Big(\frac{\partial}{\partial r}(p), \frac{\partial}{\partial \theta}(p)\Big)}_{=0} & = & 0 \\ \alpha_r \underbrace{h\Big(\frac{\partial}{\partial \theta}(p), \frac{\partial}{\partial r}(p)\Big)}_{=0} + \alpha_\theta \underbrace{h\Big(\frac{\partial}{\partial \theta}(p), \frac{\partial}{\partial \theta}(p)\Big)}_{=r^2} - \beta r^2 \underbrace{b\Big(\frac{\partial}{\partial \phi}(1), \frac{\partial}{\partial \phi}(1)\Big)}_{=1} & = & 0 \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} \alpha_r & = & 0 \\ \alpha_\theta & = & \beta\,. \end{array} \right.$$

We can thus define $v := \big(\frac{\partial}{\partial \theta}(p), \frac{\partial}{\partial \phi}(1)\big)$ and conclude that

$$\boxed{\mathcal{V}_{(p,1)} = \mathrm{span}_{\mathbb{R}}(v) = \mathrm{span}_{\mathbb{R}}\Big(\Big(\frac{\partial}{\partial \theta}(p), \frac{\partial}{\partial \phi}(1)\Big)\Big)}\,.$$

(ii) Projection of $\Big[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\Big]$ on $\mathcal{V}_{(p,1)}$

$$\begin{array}{rcl} \Big[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\Big]^{\mathcal{V}} & = & \frac{\big(h+\frac{1}{t}b\big)\Big(\Big[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\Big], v\Big)}{\|v\|^2_{h+\frac{1}{t}b}}\, v \\ & = & \frac{h\Big(0, \frac{\partial}{\partial \theta}(p)\Big) + \frac{1}{t} b\Big(-2tr \frac{\partial}{\partial \phi}(1), \frac{\partial}{\partial \phi}(1)\Big)}{h\Big(\frac{\partial}{\partial \theta}(p), \frac{\partial}{\partial \theta}(p)\Big) + \frac{1}{t} b\Big(\frac{\partial}{\partial \phi}(1), \frac{\partial}{\partial \phi}(1)\Big)} \Big(\frac{\partial}{\partial \theta}(p), \frac{\partial}{\partial \phi}(1)\Big) \\ & = & \frac{0-2r}{r^2+\frac{1}{t}} \Big(\frac{\partial}{\partial \theta}(p), \frac{\partial}{\partial \phi}(1)\Big) \end{array}$$

Hence,

$$\boxed{\Big[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\Big]^{\mathcal{V}} = \frac{-2tr}{1+tr^2} \Big(\frac{\partial}{\partial \theta}(p), \frac{\partial}{\partial \phi}(1)\Big)}\,.$$

We continue the sectional curvature computation of the two-plane $T_p\mathbb{C}$ in the Cheeger metric $h_t^{S^1}$:

$$sec_{h_t^{S^1}}\big(T_p\mathbb{C}\big) \overset{(\star)}{=} \frac{3}{4r^2(1+tr^2)} \left\| \Big[\widehat{\partial}^p_r, \widehat{\partial}^p_\theta\Big]^{\mathcal{V}} \right\|^2_{h+\frac{1}{t}b} = \frac{3}{4r^2(1+tr^2)} \frac{4t^2r^2}{(1+tr^2)^2} \Big(r^2 + \frac{1}{t}\Big)$$

Finally,

$$\boxed{sec_{h_t^{S^1}}\big(T_p\mathbb{C}\big) = \frac{3t}{(1+tr^2)^2}}\,.$$

Depending on $t > 0$, the Cheeger sectional curvature :

- tends to the original one when t gets closer to 0 : $\lim_{t\to 0} sec_{h_t^{S^1}}\big(T_p\mathbb{C}\big) = 0 = sec_h\big(T_p\mathbb{C}\big)$;
- increases endlessly as t grows at the origin $0 \in \mathbb{C}$: $\lim_{t\to +\infty} sec_{h_t^{S^1}}\big(T_p\mathbb{C}\big) = +\infty$.

Chapter 4

Some properties of Cheeger deformations

4.1 Cheeger deformation through a Riemannian submersion

A Cheeger construction $\hat{h} \underset{G}{\to} \hat{h}^G_t$ on a Riemannian manifold $(\hat{M}, \hat{h})$ can be transmitted to a Riemannian submersed manifold (M, h), $h \underset{G}{\to} h^G_t$, while preserving the isometric relation between $(\hat{M}, \hat{h})$ and (M, h) :

Theorem 4.1

Let $\pi : (\hat{M}, \hat{h}) \to (M, h)$ be a Riemannian submersion ;
G be a compact Lie group acting *isometrically* on $(\hat{M}, \hat{h})$ through a smooth left action $\hat{\mu} : G \times \hat{M} \to \hat{M}$ which *preserves the fibers*,
i.e. for all $p \in M$, $g \in G$ if $\hat{p}_1, \hat{p}_2 \in \pi^{-1}(p)$ then $\pi\big(\hat{\mu}(g, \hat{p}_1)\big) = \pi\big(\hat{\mu}(g, \hat{p}_2)\big)$;
b be a biinvariant metric on G ;
$\hat{h} \underset{G}{\to} \hat{h}^G_t$ be the Cheeger construction induced by (G, b) on $(\hat{M}, \hat{h})$.

Then :

(i) G acts canonically and isometrically, on (M, h) ;

(ii) With the Cheeger construction $h \underset{G}{\to} h^G_t$ resulting from $(G, b) \rightharpoondown (M, h)$,
$\pi : (\hat{M}, \hat{h}^G_t) \to (M, h^G_t)$ remains a Riemannian submersion for any $t \in \mathbb{R}_{>0}$.

We first require an intermediate result :

Proposition 4.2

Let $\hat{f} : (\hat{M}_1, \hat{h}_1) \to (\hat{M}_2, \hat{h}_2)$ and $f : (M_1, h_1) \to (M_2, h_2)$ be Riemannian submersions ;
$\pi_1 : (\hat{M}_1, \hat{h}_1) \to (M_1, h_1)$ be a Riemannian submersion *preserving the fibers*, i.e.
if $\hat{p}_1, \hat{q}_1 \in \hat{M}_1$ satisfy $\hat{f}(\hat{p}_1) = \hat{f}(\hat{q}_1)$, then $f\big(\pi_1(\hat{p}_1)\big) = f\big(\pi_1(\hat{q}_1)\big)$.

Then π_1 induces a Riemannian submersion $\pi_2 : (\hat{M}_2, \hat{h}_2) \to (M_2, h_2)$ such that the following diagramm commutes :

$$\begin{tikzcd} (\hat{M}_1, \hat{h}_1) \arrow[r, "\pi_1"] \arrow[d, "\hat{f}"] & (M_1, h_1) \arrow[d, "f"] \\ (\hat{M}_2, \hat{h}_2) \arrow[r, dashed, "\pi_2"] & (M_2, h_2) \end{tikzcd}$$

Proof 4.2:

Let $\hat{p}_2 \in \hat{M}_2$;
$\hat{p}_1 \in \hat{f}^{-1}(\hat{p}_2) \subseteq \hat{M}_1$;
$\hat{x}_1 \in \ker\big((d\pi_1)_{\hat{p}_1}\big) \subseteq T_{\hat{p}_1}\hat{M}_1$.

We simply construct π_2 *by* $\pi_2(\hat{p}_2) := f\big(\pi_1(\hat{p}_1)\big)$.

π_2 *is :*

- *well-defined since* π_1 *preserves the fibers ;*
- *a submersion since it is formulated through a composition of submersions.*

Concerning the π_1*-vertical vector* $\hat{x}_1$*, we observe that*

$$d\pi_2\big(d\hat{f}(\hat{x}_1)\big) = df\big(\underbrace{d\pi_1(\hat{x}_1)}_{=0_{\pi_1(\hat{p}_1)}}\big) = 0_{\pi_2(\hat{p}_2)}.$$

$\Rightarrow d\hat{f}(\hat{x}_1)$ *is a* π_2*-vertical vector in* $T_{\hat{p}_2}\hat{M}_2$.

Let's study π_2*-horizontal vectors* $\hat{\xi}_2, \hat{\eta}_2 \in \ker(d\pi_2)^\perp \subseteq T_{\hat{p}_2}\hat{M}_2$ *and their* $\hat{f}$*-horizontal lift* $\hat{\xi}_1, \hat{\eta}_1 \in \ker(d\hat{f})^\perp \subseteq T_{\hat{p}_1}\hat{M}_1$ *:*

- $\underline{\hat{\xi}_1 \text{ and } \hat{\eta}_1 \text{ are } \pi_1\text{-horizontal}}$: $\hat{h}_1(\hat{\xi}_1, \hat{x}_1) = \hat{h}_2\big(\hat{\xi}_2, d\hat{f}(\hat{x}_1)\big) \overset{\perp}{=} 0$;
- $\underline{d\pi_1(\hat{\xi}_1) \text{ and } d\pi_1(\hat{\eta}_1) \text{ are } f\text{-horizontal}}$: *consider* $y_1 \in \ker\big(df_{\pi_1(\hat{p}_1)}\big) \subseteq T_{\pi_1(\hat{p}_1)}M_1$. *Then,* $h_1\big(d\pi_1(\hat{\xi}_1), y_1\big) = h_2\big((df \circ d\pi_1)(\hat{\xi}_1), \underbrace{df(y_1)}_{=0_{\pi_2(\hat{p}_2)}}\big) = 0.$

Hence,

$$\begin{aligned} \hat{h}_2(\hat{\xi}_2, \hat{\eta}_2) &\overset{\substack{\hat{f}\ Riem.\\ subm.}}{=} \hat{h}_1(\hat{\xi}_1, \hat{\eta}_1) \\ &\overset{\substack{\pi_1\ Riem.\\ subm.}}{=} h_1\big(d\pi(\hat{\xi}_1), d\pi(\hat{\eta}_1)\big) \\ &\overset{\substack{f\ Riem.\\ subm.}}{=} h_2\big((df \circ d\pi_1)(\hat{\xi}_1), (df \circ \pi_1)(\hat{\eta}_1)\big) \\ &= h_2\big((d\pi_2 \circ d\hat{f})(\hat{\xi}_1), (d\pi_2 \circ d\hat{f})(\hat{\eta}_1)\big) \\ &= h_2\big(d\pi_2(\hat{\xi}_2), d\pi_2(\hat{\eta}_2)\big). \end{aligned}$$

We proved then that $d\pi_2$ *is a linear isometry on the horizontal spaces, which means that* π *is Riemannian submersion.*

■

Proof 4.1:

Ad (i) : *Using the fiber preservation property, we define*

$$\begin{array}{cccc} \mu & : & G \times M & \to & M \\ & & (g,p) & \mapsto & \pi\big(\hat{\mu}(g,\hat{p})\big), \end{array}$$

where $\hat{p} \in \hat{M}$ is any point of the fiber $\pi^{-1}(p)$.

We verify easily that μ is a well-define smooth action.

Let $g \in G$;
$p \in M$;
$\hat{p} \in \pi^{-1}(p) \subseteq \hat{M}$;
$v, w \in T_pM$;
$\hat{v}, \hat{w} \in \mathcal{H}_{\hat{p}} = \ker(d\pi_{\hat{p}})^{\perp} \subseteq T_{\hat{p}}\hat{M}$.

Since $\hat{\mu}\big|_{\mathcal{H}_{\hat{p}}}$ is an isometry, its differential sends π-horizontal vectors on π-horizontal vectors, which implies that $(d\hat{\mu}_g)_{\hat{p}}(\hat{v}) \in \mathcal{H}_{\hat{\mu}_g(\hat{p})}$. Therefore $(d\hat{\mu}_g)_{\hat{p}}(\hat{v})$ is the horizontal lift of $(d\mu_g)_p(v)$ at $\hat{p}$ by chain rule and unicity of horizontal lift. The same holds for w instead of v.

Hence,

$$\begin{aligned} h_p(v,w) &\overset{\substack{\pi\ \text{Riem.}\\ \text{subm.}}}{=} \hat{h}_{\hat{p}}(\hat{v},\hat{w}) \\ &\overset{\hat{\mu}\ \text{isom.}}{=} \hat{h}_{\hat{\mu}_g(\hat{p})}\big((d\hat{\mu}_g)_{\hat{p}}(\hat{v}), (d\hat{\mu}_g)_{\hat{p}}(\hat{w})\big) \\ &\overset{\substack{\pi\ \text{Riem.}\\ \text{subm.}}}{=} h_{\mu_g(p)}\Big(d\pi_{\hat{\mu}_g(\hat{p})}\big((d\hat{\mu}_g)_{\hat{p}}(\hat{v})\big), d\pi_{\hat{\mu}_g(\hat{p})}\big((d\hat{\mu}_g)_{\hat{p}}(\hat{w})\big)\Big) \\ &= h_{\mu_g(p)}\big((d\mu_g)_p(v), (d\mu_g)_p(w)\big), \end{aligned}$$

which proves that $G \curvearrowright (M,h)$ isometrically.

✓

Ad (ii) : *(G,b) generates a Cheeger deformation $\big(h_t^G\big)_{t \geqslant 0}$ on (M,h).*

Since $\pi \times id_G$, $\hat{\rho}$ and ρ are Riemannian submersions and that $\pi \times id_G$ preserves the fibers, we directly use Proposition 4.2 to conclude that the following diagram holds for any $t \in \mathbb{R}_{>0}$:

$$\begin{tikzcd} (\hat{M} \times G, \hat{h} + \frac{1}{t}b) \arrow[r, "\pi \times id_g"] \arrow[d, "\hat{\rho}"] & (M \times G, h + \frac{1}{t}b) \arrow[d, "\rho"] \\ (\hat{M}, \hat{h}_t^G) \arrow[r, dashed, "\bar{\pi}"] & (M, h_t^G) \end{tikzcd}$$

✓

■

4.2 Key features concerning sectional and scalar curvatures

For the rest of this chapter, we consider the same isometric action $(G, b) \rightharpoondown (M, h)$ and the associated Cheeger deformation $(M, h_t^G)_{t \geqslant 0}$ as in Chapter 3. We also use the following notations :

- $sec_h, sec_b, sec_{h+\frac{1}{t}b}$ and $sec_{h_t^G}$ for the sectional curvatures of the Riemannian manifolds (M, h), (G, b), $\left(M \times G, h + \frac{1}{t}b\right)$ and (M, h_t^G), respectively ;
- $k_h, k_b, k_{h+\frac{1}{t}b}$ and $k_{h_t^G}$ for the numerator term of their sectional curvature[1], respectively ;
- $R_h, R_b, R_{h+\frac{1}{t}b}$ and $R_{h_t^G}$ for their curvature tensor, respectively.

The three following results can be found in [DG19, Propositions 5.1 to 5.3].

The sectional curvature of $(h_t^G)_{t \geqslant 0}$ is non-decreasing, which allows lower bounds on $sec_{h_t^G}$:

Proposition 4.3 - Lower bound for $sec_{h_t^G}$

Let $p \in M$;
$v, w \in T_pM$;
$x := *^{-1}\left(v^{\mathcal{V}}\right) \in \mathfrak{m}_p$ and $y := *^{-1}\left(w^{\mathcal{V}}\right) \in \mathfrak{m}_p$.

Then :

(i) There exists a map $\alpha : \mathbb{R}_{\geqslant 0} \to (0, 1]$ such that the following holds for all $t \geqslant 0$:

$$sec_{h_t^G}\left(C_t^{-1}v, C_t^{-1}w\right) \geqslant \alpha(t) \cdot sec_h(v, w) \ ;$$

(ii) $sec_h \geqslant 0 \;\Rightarrow\; sec_{h_t^G} \geqslant 0$, and $sec_h > 0 \;\Rightarrow\; sec_{h_t^G} > 0$;

(iii) If $[S(p)x, S(p)y] \neq 0$, we obtain even that

$$\lim_{t \to +\infty} sec_{h_t^G}\left(C_t^{-1}v, C_t^{-1}w\right) = +\infty.$$

Proof 4.3:

Ad(i): The case $t = 0$ being clear, we suppose $t > 0$.

Recall from Theorem 3.6 (i) that the ρ-horizontal lift of $C_t^{-1}v$ (respectively $C_t^{-1}w$) at $(p, e) \in M \times G$ is $(v, -tS(p)x)$ (respectively $(w, -tS(p)y)$). Based on this, we deduce that

$$span\Big(\underbrace{(v, -tS(p)x)}_{=:f(v)}, \underbrace{(w, -tS(p)y)}_{=:f(w)}\Big),$$

is the ρ-horizontal lift of the plane $span\left(C_t^{-1}v, C_t^{-1}w\right)$.

[1] See Definition A.29

We define two maps α, $\beta : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *by :*

$$\alpha(t) := \frac{\|v \wedge w\|_h^2}{\|f(v) \wedge f(w)\|^2_{h+\frac{1}{t}b}},$$

and

$$\beta(t) := \frac{\|(-tS(p)x) \wedge (-tS(p)y)\|^2_{\frac{1}{t}b}}{\|f(v) \wedge f(w)\|^2_{h+\frac{1}{t}b}}.$$

The denominator is the sum of the two numerators :

$$\|f(v) \wedge f(w)\|^2_{h+\frac{1}{t}b} = \|v \wedge w\|_h^2 + \|(-tS(p)x) \wedge (-tS(p)y)\|^2_{\frac{1}{t}b}.$$

Hence : $0 < \alpha(t) \leqslant 1$, $0 \leqslant \beta(t)$ *and* $\lim_{t\to\infty} \beta(t) = 1$.

Let's then compute the sectional curvature of the horizontal lift plane, with the idea to use the Gray-O'Neill formula later :

$$\begin{aligned}
sec_{h+\frac{1}{t}b}\big(f(v), f(w)\big) &\overset{def}{=} \frac{\left(h+\frac{1}{t}b\right)\left(R_{h+\frac{1}{t}b}(f(v),f(w))f(w),f(v)\right)}{\|f(v)\wedge f(w)\|^2_{h+\frac{1}{t}b}} \\
&\overset{Prop.\ A.35}{=} \frac{h(R_h(v,w)w,v)}{\|f(v)\wedge f(w)\|^2_{h+\frac{1}{t}b}} \\
&\quad + \frac{\frac{1}{t}b\left(R_{\frac{1}{t}b}(-tS(p)x,-tS(p)y)-tS(p)y,-tS(p)x\right)}{\|f(v)\wedge f(w)\|^2_{h+\frac{1}{t}b}} \\
&= \frac{\|v\wedge w\|_h^2 \cdot sec_h(v,w)}{\|f(v)\wedge f(w)\|^2_{h+\frac{1}{t}b}} \\
&\quad + \frac{||(-tS(p)x)\wedge(-tS(p)y)||^2_{\frac{1}{t}b} \cdot sec_{\frac{1}{t}b}\left(-tS(p)x,-tSp(y)\right)}{\|f(v)\wedge f(w)\|^2_{h+\frac{1}{t}b}} \\
&= \alpha(t)\cdot sec_h(v,w) + \beta(t)\cdot sec_{\frac{1}{t}b}\big(-tS(p)x, -tS(p)y\big) \\
&\overset{Lemma\ A.30}{=} \alpha(t)\cdot sec_h(v,w) + \beta(t)\cdot t\cdot sec_b\big(S(p)x, S(p)y\big),
\end{aligned}$$

where we also use in the last equality that

$$span\big(-tS(p)x, -tS(p)y\big) = span\big(S(p)x, S(p)y\big).$$

Finally, by Gray-O'Neill formula and since (G, b) *carries a non-negative sectional curvature*[2] *:*

$$sec_{h_t^G}\left(C_t^{-1}v, C_t^{-1}w\right) \geqslant sec_{h+\frac{1}{t}b}\big(f(v), f(w)\big) \geqslant \alpha(t)\cdot sec_h(v,w).$$

✓

<u>*Ad(ii)*</u> *: Direct consequence of the last equation (i), since* C_t *is surjective.*

✓

[2]See Corollary B.15

Ad(iii): $[S(p)x, S(p)y] \neq 0 \Rightarrow \left|\left|[S(p)x, S(p)y]\right|\right|_b^2 \gneq 0.$

$\Rightarrow \lim\limits_{t\to+\infty} \frac{1}{4} \cdot \beta(t) \cdot t \cdot \frac{\left|\left|[S(p)x,S(p)y]\right|\right|_b^2}{\left|\left|S(p)x\wedge S(p)y\right|\right|_b^2} = +\infty.$

$\Rightarrow \lim\limits_{t\to+\infty} sec_{h_t^G}\left(C_t^{-1}v, C_t^{-1}w\right) = +\infty,$ *since* $\alpha(t) \cdot sec_h(v,w)$ *stays bounded.*

✓

■

We obtain even some information on the scalar curvatures[3] of the Cheeger metrics $(h_t^G)_{t\geqslant 0}$ given specific properties of (M, h):

Proposition 4.4 - Some conditions for the positivity of $scal_{h_t^G}$

We can easily identify situations for which (M, h_t^G) have strictly positive scalar curvature:

(i) If $sec_h \geqslant 0$ and $scal_h > 0$, then $scal_{h_t^G} > 0$ for all $t > 0$;

(ii) If $sec_h \geqslant 0$ and $\forall p \in M\ \exists\, x, y \in T_eG = \mathfrak{g}$ such that $\left[S(p)x, S(p)y\right] \neq 0$, then $scal_{h_t^G} > 0$ for all $t > 0$;

(iii) Let K be a compact subset of M.
If $\forall p \in K\ \exists\, x, y \in \mathfrak{g}$ such that $\left[S(p)x, S(p)y\right] \neq 0$,
then $\exists t_0 > 0$ with the property that $scal_{h_t^G}(p) > 0$, for all $p \in K$, $t > t_0$.

Proof 4.4:

Let $t > 0$ *;*
$p \in M$ *;*
$(e_1, ..., e_n)$ *be an h-orthonormal basis of* T_pM*.*

Ad(i) : It implies that

$$scal_h(p) = 2 \sum_{1\leqslant i<j\leqslant n} sec_h(e_i, e_j) > 0.$$

Therefore there exists a 2-plane span$(e_k, e_l) \subset T_pM$, $k \neq l$, *with* $sec_h(e_k, e_l) > 0$, *and by Proposition 4.3 (i) :*

$$sec_{h_t^G}\left(C_t^{-1}e_k, C_t^{-1}e_l\right) \geqslant \alpha(t) \cdot sec_h(e_k, e_l) \overset{\alpha(t)>0}{>} 0.$$

We finally construct a h_t^G*-orthonormal basis* $(\tilde{e}_1, ..., \tilde{e}_n)$ *of* T_pM *by completing* $(\tilde{e}_l, \tilde{e}_k) := \left(\frac{C_t^{-1}e_k}{\left|\left|C_t^{-1}e_k\right|\right|_{h_t^G}}, \frac{C_t^{-1}e_l}{\left|\left|C_t^{-1}e_l\right|\right|_{h_t^G}}\right)$. *Since, by Proposition 4.3 (ii),* $sec_{h_t} \geqslant 0$ *:*

$$scal_{h_t^G}(p) = 2 \sum_{1\leqslant i<j\leqslant n} sec_{h_t^G}(\tilde{e}_i, \tilde{e}_j) > 0.$$

✓

[3] See Definition A.39

Ad(ii) : *We denote by $x, y \in T_eG$ vectors such that $\big[S(p)x, S(p)y\big] \neq 0$.*

In the last proof, we calculated for $v := X^*(p)$, $w := Y^*(p)$:

$$sec_{h_t^G}\big(C_t^{-1}v, C_t^{-1}w\big) = \underbrace{\alpha(t)\cdot sec_h(v,w)}_{\geqslant 0} + \underbrace{\frac{1}{4}\cdot\beta(t)\cdot t}_{>0}\cdot\underbrace{\frac{\big|\big|[S(p)x, S(p)y]\big|\big|_b^2}{\big|\big|S(p)x \wedge S(p)y\big|\big|_b^2}}_{>0} + \underbrace{\zeta_t}_{\geqslant 0} > 0.$$

We proceed as in the first case by completing $\left\{\frac{C_t^{-1}v}{\big|\big|C_t^{-1}v\big|\big|_{h_t^G}}, \frac{C_t^{-1}w}{\big|\big|C_t^{-1}w\big|\big|_{h_t^G}}\right\}$ to obtain an orthonormal basis $(\tilde e_1, \dots, \tilde e_n)$ of T_pM and compute

$$scal_{h_t^G}(p) = 2\sum_{1\leqslant i<j\leqslant n} sec_{h_t^G}(\tilde e_i, \tilde e_j) > 0.$$

✓

Ad(iii) : *For all $p \in M$, define $\kappa(p) := \min\{sec_h(v,w) \mid v, w \in T_pM\}$, which exists since T_pM has finite dimension.*

The compactness of K allows us to find a minimal κ among all $p \in K$, which means that

$$sec_h \geqslant \kappa \text{ on } K.$$

If $\kappa > 0$, the statement follows directly from the definition of scalar curvature. So assume $\kappa \leqslant 0$.

For a given $p \in K$, consider $x, y \in \mathfrak{g}$ with $[S(p)x, S(p)y] \neq 0$. By Proposition 4.3, there exists $t_0^p \in \mathbb{R}$ such that

$$sec_{h_t}^G\big(C_t^{-1}X^*(p), C_t^{-1}W^*(p)\big) > \Big(\sum_{i=1}^{n-1} i\Big)\cdot(-\kappa) > 0,$$

for all $t > t_0^p$.

Let's fix $t > 0$. We can easily assume that $e_1 := C_t^{-1}X^*(p)$ and $e_2 := C_t^{-1}Y^*(p)$ are h_t^G-orthonormal to each other, since the sectional curvature only depends on the generated 2-plane. We complete (e_1, e_2) by an orthonormal basis $(e_1, \dots, e_n)$ of T_pM and obtain :

$$\begin{aligned} scal_{h_t^G}(p) &= 2\textstyle\sum_{1\leqslant i<j\leqslant n} sec_{h_t^G}(e_i, e_j) \\ &> 2\left(\big(\textstyle\sum_{i=1}^{n-1} i\big)\cdot(-\kappa) + \sum_{\substack{1\leqslant i<j\leqslant n \\ (i,j)\neq(1,2)}}\kappa\right) \\ &\geqslant 0 \end{aligned}$$

By compactness of K, we can define a lower bound $t_0 := \min_{p\in K} t_0^p$ such that the statement holds for all $t > t_0$.

✓

■

4.3 $sec_{h_t^G}$ as a function of sec_h

In the proof of Proposition 4.3 (i), we partially calculated the numerator $k_{h_t^G}$ of the sectional curvature of the Cheeger metric $sec_{h_t^G}$ with respect to the initial one k_h. Let's finish this calculation and then examine more accurately the additional term ζ_t observed in the Gray-O'Neill Formula. We use the same notation :

Let $t > 0$;
$p \in M$;
$v, w \in T_pM$ generating a 2-plane ;
$x := *^{-1}(v^{\mathcal{V}}) \in \mathfrak{m}_p$, and $y := *^{-1}(w^{\mathcal{V}}) \in \mathfrak{m}_p$.

Recall that $f(v) := (v, -tS(p)x)$ and $f(w) := (w, -tS(p)y)$ are the ρ-horizontal lifts of $C_t^{-1}v$, respectively $C_t^{-1}w$.

By Gray-O'Neill Formula 1.8 :

$$\begin{aligned}
k_{h_t^G}\big(C_t^{-1}v \wedge C_t^{-1}w\big) &= k_{h+\frac{1}{t}b}\big(f(v) \wedge f(w)\big) + \underbrace{\frac{3}{4}\left|\left|\big[f(v), f(w)\big]^{\mathcal{V}}\right|\right|^2_{h+\frac{1}{t}b}}_{=:\zeta_t(v,w)=:\zeta_t} \\
&\overset{\substack{\text{Prop.}\\ A.35}}{=} \big(h + \tfrac{1}{t}b\big)\Big(\big(R_h(v,w)w, R_b(-tS(p)x, -tS(p)y)(-tS(p)y)\big), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \big(v, -tS(p)x\big)\Big) + \zeta_t \\
&= h\big(R_h(v,w)w, v\big) \\
&\quad + \tfrac{1}{t}b\big(R_b(-tS(p)x, -tS(p)y)(-tS(p)y), -tS(p)x\big) + \zeta_t \\
&= k_h(v \wedge w) + (-t)^4 \cdot \tfrac{1}{t}b\big(R_b(S(p)x, S(p)y)S(p)y, S(p)x\big) + \zeta_t \\
&\overset{\substack{\text{Prop.}\\ B.14(iii)}}{=} k_h(v \wedge w) + \tfrac{t^3}{4}\left|\left|\big[S(p)x, S(p)y\big]\right|\right|_b^2 + \zeta_t.
\end{aligned}$$

In order to calculate the sectional curvature $sec_{h_t^G}\big(C_t^{-1}v, C_t^{-1}w\big)$ we also need the denominator term $\left|\left|C_t^{-1}v \wedge C_t^{-1}w\right|\right|^2_{h_t^G}$. W.l.o.g, suppose v and w are h-orthonormal, which means that $k_h(v \wedge w) = sec_h(v,w)$. Indeed, the 2-plane $C_t^{-1}\sigma \subset T_pM$ generated by $C_t^{-1}v$ and $C_t^{-1}w$ only depends on the one σ generated by v and w and we can always find an orthonormal basis of σ.

Then,

$$\begin{aligned}
\left|\left|C_t^{-1}v \wedge C_t^{-1}w\right|\right|^2_{h_t^G} &= h_t^G\big(C_t^{-1}v, C_t^{-1}v\big) \cdot h_t^G\big(C_t^{-1}w, C_t^{-1}w\big) \\
&\quad - \Big(h_t^G\big(C_t^{-1}v, C_t^{-1}w\big)\Big)^2 \\
&\overset{\substack{\rho\text{ Riem.}\\ \text{subm.}}}{=} \big(h + \frac{1}{t}b\big)\big((v, -tS(p)x), (v, -tS(p)x)\big) \\
&\quad \cdot \big(h + \frac{1}{t}b\big)\big((w, -tS(p)y), (w, -tS(p)y)\big) \\
&\quad - \Big(\big(h + \frac{1}{t}b\big)\big((v, -tS(p)x), (w, -tS(p)y)\big)\Big)^2
\end{aligned}$$

$$\begin{aligned}
&= h(v,v)\cdot h(w,w) - \big(h(v,w)\big)^2 \\
&\quad + t\cdot\big(h(v,v)\cdot b(S(p)y,S(p)y) - 2\cdot h(v,w)\cdot b(S(p)x,S(p)y) \\
&\qquad\qquad\qquad\qquad + h(w,w)\cdot b(S(p)x,S(p)x)\big) \\
&\quad + t^2\Big(b(S(p)x,S(p)x)\cdot b(S(p)y,S(p)y) - \big(b(S(p)x,S(p)y)\big)^2\Big) \\
&= 1 + t\cdot\big(\|S(p)x\|_b^2 + \|S(p)y\|_b^2\big) + t^2\cdot\|S(p)x\wedge S(p)y\|_b^2.
\end{aligned}$$

Hence, the Cheeger sectional curvature is expressed with the initial one through

$$\boxed{sec_{h_t^G}\big(C_t^{-1}v, C_t^{-1}w\big) = \frac{sec_h(v,w) \,+\, \frac{t^3}{4}\,\big\|\big[S(p)x,S(p)y\big]\big\|_b^2 + \zeta_t(v,w)}{1 \,+\, t\cdot\big(\|S(p)x\|_b^2 + \|S(p)y\|_b^2\big) \,+\, t^2\cdot\|S(p)x\wedge S(p)y\|_b^2}}. \tag{4.1}$$

4.4 Analysis of the additional term ζ_t

To reformulate the additional term $\zeta_t(v,w) := 3\,\big\|A_{f(v)}f(w)\big\|_{h+\frac{1}{t}b}^2 = \frac{3}{4}\,\Big\|\big[f(v),f(w)\big]^{\mathcal{V}}\Big\|_{h+\frac{1}{t}b}^2$, we introduce a new concept related to action fields, which are in particular Killing vector fields :

Definition 4.5 - Killing form ω_x

In the context of $G \curvearrowright (M,h)$ by isometries, the **Killing form** corresponding to a tangent vector $x \in \mathfrak{g} = T_eG$ is the following 1-form :

$$\begin{array}{rccc} \omega_x & : & \mathfrak{X}(M) & \to & C^\infty(M) \\ & & W & \mapsto & \frac{1}{2}h(X^*,W). \end{array}$$

Proposition 4.6 - Exterior derivative of the Killing form $d\omega_x$

Let $x \in \mathfrak{g}$.

Then :

(i) The exterior derivative of the Killing form ω_x is the following 2-form :

$$\begin{array}{rccc} d\omega_x & : & \mathfrak{X}(M)\times\mathfrak{X}(M) & \to & C^\infty(M) \\ & & (V,W) & \mapsto & \frac{1}{2}\Big(h\big(\nabla_V X^*,W\big) - h\big(\nabla_W X^*,V\big)\Big). \end{array}$$

(ii) In particular, for horizontal vector fields $V^{\mathcal{H}}, W^{\mathcal{H}} \in \mathfrak{X}(M)^{\mathcal{H}}$ with respect to the orbits of the G-action on (M,h) :

$$d\omega_x\big(V^{\mathcal{H}},W^{\mathcal{H}}\big) = -h\big(A_{V^{\mathcal{H}}}W^{\mathcal{H}},X^*\big).$$

Remark 4.7. Note that the dimension of horizontal spaces $\mathcal{H}_p$, $p \in M$, may differ from a point to another. Therefore, the use of horizontal vector fields in (ii) is sometimes an abuse of langage. However, according to [Mü87, page 10], one can construct a different sort of O'Neill tensor A in the context of an isometric action of a Lie group G on (M,h). This permits to use Riemannian submersion arguments viewed in Chapter 1 in a consistent way for the second statement.

Proof 4.6:

Let $V, W \in \mathfrak{X}(M)$.

Ad (i) : According to [Lee12, Proposition 14.29], the exterior derivative of ω_x can be computed as follows :

$$\begin{aligned} d\omega_x(V,W) &:= V\big(\omega_x(W)\big) - W\big(\omega_x(V)\big) - \omega_x\big([V,W]\big) \\ &\overset{\substack{\nabla \text{ compatible} \\ \text{with } h \text{ and} \\ \text{symmetric}}}{=} \tfrac{1}{2}\Big(h\big(\nabla_V X^*, W\big) + h\big(X^*, \nabla_V W\big) - h\big(\nabla_W X^*, V\big) \\ &\qquad - h\big(X^*, \nabla_W V\big) - h\big(X^*, \nabla_V W\big) + h\big(X^*, \nabla_W V\big)\Big) \\ &= \tfrac{1}{2}\Big(h\big(\nabla_V X^*, W\big) - h\big(\nabla_W X^*, V\big)\Big). \end{aligned}$$

✓

Ad (ii) : For the horizontal vector fields :

$$\begin{aligned} d\omega_x(V^{\mathcal{H}}, W^{\mathcal{H}}) &\overset{(i)}{=} \tfrac{1}{2}\Big(h\big(\nabla_{V^{\mathcal{H}}} X^*, W^{\mathcal{H}}\big) - h\big(\nabla_{W^{\mathcal{H}}} X^*, V^{\mathcal{H}}\big)\Big) \\ &\overset{\text{Lemma 1.7 (iii)}}{=} \tfrac{1}{2}\Big(h\big(\nabla_{W^{\mathcal{H}}} V^{\mathcal{H}}, X^*\big) - h\big(\nabla_{V^{\mathcal{H}}} W^{\mathcal{H}}, X^*\big)\Big) \\ &\overset{\nabla \text{ symmetric}}{=} -\tfrac{1}{2} h\big(\big[V^{\mathcal{H}}, W^{\mathcal{H}}\big], X^*\big) \\ &\overset{X^* \text{ vertical}}{=} -\tfrac{1}{2} h\big(\big[V^{\mathcal{H}}, W^{\mathcal{H}}\big]^{\mathcal{V}}, X^*\big) \\ &\overset{\text{Prop. 1.6}}{=} -h\big(A_{V^{\mathcal{H}}} W^{\mathcal{H}}, X^*\big). \end{aligned}$$

✓

■

We'll also need the following Linear Algebra result :

Lemma 4.8

In the context of an euclidean space $(V, \langle\cdot,\cdot\rangle)$, i.e. a $\mathbb{R}$-vector space endowed with an inner product, we observe the following property $\forall v \in V$:

$$\langle v, v\rangle = \max_{\substack{w \in V \\ w \neq 0}} \left\{ \frac{\langle v, w\rangle^2}{\langle w, w\rangle} \right\}.$$

Proof 4.8:

Let $v, w \in V$, $w \neq 0$.

The outcome results directly from the Cauchy-Schwarz inequality :

$$\langle v, w\rangle^2 \leqslant \langle v, v\rangle \cdot \langle w, w\rangle \qquad \Leftrightarrow \qquad \frac{\langle v, w\rangle^2}{\langle w, w\rangle} \leqslant \langle v, v\rangle.$$

■

We come back to the context of G acting isometrically on $(M \times G, h + \frac{1}{t}b)$ and having the quotient map being the orbital submersion $\rho : (M \times G, h + \frac{1}{t}b) \to (M, h_t^G)$, for a given $t > 0$.

Theorem 4.9 - New expression for the additional term ζ_t

Let $t > 0$;
$p \in M$;
$v, w \in T_pM$ generating a 2-plane ;
$x := *^{-1}\big(v^{\mathcal{V}}\big) \in \mathfrak{m}_p$, and $y := *^{-1}\big(w^{\mathcal{V}}\big) \in \mathfrak{m}_p$.

Then,

$$\zeta_t(v,w) := 3\left|\left|A_{f(v)}f(w)\right|\right|^2_{h+\frac{1}{t}b} = 3t \max_{\substack{z\in\mathfrak{g}\\ z\neq 0}} \left\{ \frac{\big(d\omega_z(v,w) + \frac{t}{2}b([S(p)x, S(p)y], z)\big)^2}{th\big(Z^*(p), Z^*(p)\big) + 1} \right\},$$

where $\omega_z(v,w) := \omega_z(V,W)(p)$.

Proof 4.9:

We denote by SX_R, SY_R, $Z_R \in \mathfrak{X}(G)^R$ *the right-invariant vector fields generated by* $S(p)x$, $S(p)y$ *and* z, *respectively ;*
$SX_L, SY_L \in \mathfrak{X}(G)^L$ *the left-invariant vector fields generated by* $S(p)x$ *and* $S(p)y$, *respectively ;*
Z^* *the action field generated by* z *on* $M \times G$ *;*
$\tilde{\omega}^t_z := \big(h + \frac{1}{t}b\big)(Z^*, \cdot)$ *the Killing form related to* $z \in \mathfrak{g}$ *;*
$\bar{\omega}_z := b(z, \cdot)$ *the Killing form related to* $z \in \mathfrak{g}$, *defined on* T_eG.

Remember that $f(v) = (v, -tS(p)x)$ *and* $f(w) = (w, -tS(p)y)$ *are* $\big(h + \frac{1}{t}b\big)$-*horizontal vectors in* $T_{(p,e)}(M \times G)$.

From the computation formula of the exterior derivative of one forms, seen in [Lee12, Proposition 14.29], we deduce easily :

$$\begin{aligned}
d\omega^t_z\big(f(v), f(w)\big) &= d\omega_z(v,w) + \tfrac{1}{t}d\bar{\omega}_z(-tSX_R, -tSY_R)(e)\\
&\overset{\text{Prop. 4.6}}{=} d\omega_z(v,w) - t\, b(A_{SX_R}SY_R(e), z)\\
&\underset{z=Z_R(e)\ \text{vertical}}{\overset{\text{Prop. 1.6}}{=}} d\omega_z(v,w) - \tfrac{t}{2}\, b([SX_R, SY_R](e), z)\\
&\overset{\text{Prop. B.7}}{=} d\omega_z(v,w) + \tfrac{t}{2}\, b([SX_L, SY_L](e), z)\\
&= d\omega_z(v,w) + \tfrac{t}{2}\, b([S(p)x, S(p)y], z).
\end{aligned}$$

Then,

$$\begin{aligned}
\zeta_t &= 3\big(h + \tfrac{1}{t}b\big)\big(A_{f(v)}f(w), A_{f(v)}f(w)\big)\\
&\overset{\text{Lemma 4.8}}{=} 3\max_{\substack{z\in\mathfrak{g}\\ z\neq 0}} \left\{ \frac{\Big(h+\frac{1}{t}b\Big)\Big(A_{f(v)}f(w), Z^*(p,e)\Big)^2}{\Big(h+\frac{1}{t}b\Big)\Big(Z^*(p,e), Z^*(p,e)\Big)} \right\}\\
&\overset{\text{Prop. 4.6}}{=} 3\max_{\substack{z\in\mathfrak{g}\\ z\neq 0}} \left\{ \frac{d\omega^t_z\Big(f(v), f(w)\Big)^2}{h\Big(Z^*(p), Z^*(p)\Big) + \frac{1}{t}b(z,z)} \right\}\\
&= 3t \max_{\substack{z\in\mathfrak{g}\\ \|z\|_b=1}} \left\{ \frac{\Big(d\omega_z(v,w) + \frac{t}{2}\, b([S(p)x, S(p)y], z)\Big)^2}{t\, h\Big(Z^*(p), Z^*(p)\Big) + 1} \right\}.
\end{aligned}$$

■

Thanks to the last statements, we know that a manifold M of non-negative curvature always admits equal or less flat 2-planes for its Cheeger metrics h_t^G, $t > 0$, than for its initial metric h.

Corollary 4.10 - Flat 2-planes in the Cheeger metrics

Let $t > 0$;
$p \in M$;
$v, w \in T_pM$ linearly independent ;
$\sigma := \text{span}(v, w)$;
$x := *^{-1}\left(v^{\mathcal{V}}\right) \in \mathfrak{m}_p$, and $y := *^{-1}\left(w^{\mathcal{V}}\right) \in \mathfrak{m}_p$.

If M has *non-negative curvature*, $sec_{h_t^G}\left(C_t^{-1}\sigma\right)$ $\left(sec_{h_t^G}\left(C_t^{-1}v, C_t^{-1}w\right)\right) = 0$ if and only if the three following conditions are fulfilled :

(i) $sec(\sigma) = 0$;

(ii) $\left[S(p)x, S(p)y\right] = 0$;

(iii) $d\omega_z(v, w) = 0$ for all $z \in \mathfrak{g}$.

Proof 4.10: *Direct consequence of the last theorem and the equation (4.1).*

■

Chapter 5

Lawson-Yau theorem on positive scalar curvature

In 1974, Lawson and Yau[1] discovered the existence of a Riemannian metric of positive scalar curvature on a *compact* manifold with *non-abelian symmetry*, i.e. on which a *compact*, *connected* and *non-abelian* Lie group acts smoothly and *effectively*. In their recent paper[2], Cavenaghi and Sperança used the Cheeger deformation process to prove this result in a more intuitive way.

From now on, the considered connected manifold M is assumed to be *compact* as well as the Lie group G acting isometrically on (M, h).

5.1 Ricci and scalar curvatures in the Cheeger metrics

We first introduce the horizontal Ricci curvature in the initial metric h, which is the part of Ricci curvature at a vector $v \in T_pM$ related to horizontal space $\mathcal{H}_p$.

Definition 5.1 - Horizontal Ricci curvature

Let $p \in M$;
$v \in T_pM$;
$n := \dim T_pM$;
$n - l := \dim \mathcal{H}_p$;
$(w_{l+1}, ..., w_n)$ a h-orthonormal basis for $\mathcal{H}_p$.

We call **horizontal Ricci curvature** at v the following real number :

$$Ric^{\mathcal{H}}(v) := \sum_{i=l+1}^{n} h\big(R(v, w_i)w_i, v\big).$$

Remark 5.2. The linearity of the map $* : \mathfrak{m}_p \to \mathcal{V}_p$ permits to consider the metric tensor C_t also in $\mathfrak{g}$: $C_t x = *^{-1}\big(C_t X^*(p)\big) = \big(Id_{\mathfrak{g}} + tS(p)\big)^{-1} x, \ \forall x \in \mathfrak{g}$.

[1]See [LY74]
[2]See [CS18]

Lemma 5.3 - Ricci curvature in the Cheeger metrics

Let $t > 0$;
$p \in M$;
$v = X^*(p) + \xi \in T_pM$ with $x \in \mathfrak{m}_p$ and $\xi \in \mathcal{H}_p$;
$n := \dim T_pM$.

There exists a b-orthonormal basis $(y_1, ...y_l)$ of $\mathfrak{m}_p$, scalars $\lambda_1, ..., \lambda_l > 0$ and a h-orthonormal basis $(w_1, ..., w_n)$ of T_pM such that the Ricci curvature $Ric_{h_t^G}$ in the Cheeger metric h_t^G satisfies :

(i) $Ric_{h_t^G}(v) = Ric_h^{\mathcal{H}}(C_t v) + \sum_{i=1}^n \zeta_t\big(C_t^{1/2} w_i, C_t v\big)$
$\qquad + \sum_{i=1}^l \frac{1}{1+t\lambda_i}\Big(k_h\big(w_i \wedge C_t v\big) + \frac{\lambda_i t^3}{4}\Big\|\Big[y_i, \frac{S(p)}{1+tS(p)}x\Big]\Big\|_b^2\Big)$;

(ii) $\lim_{t\to\infty} Ric_{h_t^G}(v) = Ric_h^{\mathcal{H}}(\xi) + \lim_{t\to\infty} \sum_{i=1}^n \zeta_t\big(C_t^{1/2} w_i, C_t v\big) + \frac{1}{4}\sum_{i=1}^l \|[y_i, x]\|_b^2$.

Proof 5.3:

Ad (i) : Consider an b-orthonormal basis of eigenvectors of $S(p)$ in $\mathfrak{m}_p$, $(y_1, ..., y_l)$, with corresponding eigenvalues $\lambda_1 \leqslant ... \leqslant \lambda_l$. Recall from Theorem 3.9 that $\lambda_i > 0$ $\forall i = 1, ..., l$.

Define $w_i := \frac{1}{\lambda_i^{1/2}} Y_i^*(p)$ for $i = 1, ..., l$, which build a h-orthonormal basis of $\mathcal{V}_p$:

$$h(w_i, w_j) = \frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}} b\big(S(p)y_i, y_j\big) = \frac{\lambda_i}{\lambda_i^{1/2}\lambda_j^{1/2}} b(y_i, y_j) = \delta_{ij}.$$

Complete it with a h-orthonormal basis $(w_{l+1}, ..., w_n)$ of $\mathcal{H}_p$ to obtain an h-orthonormal basis of T_pM : $(w_1, ..., w_n)$.

Observe that

$$C_t^{-1/2} w_i = \begin{cases} (1+t\lambda_i)^{1/2} w_i, & i \leqslant l \ ; \\ w_i & i > l\,. \end{cases}$$

Hence $\big(C_t^{-1/2} w_1, ..., C_t^{-1/2} w_n\big)$ forms a h_t^G-orthonormal basis of T_pM :

$$h_t^G\big(C_t^{-1/2} w_i, C_t^{-1/2} w_j\big) \quad = \quad h\big(C_t^{1/2} w_i, C_t^{-1/2} w_j\big) = \frac{\mathbb{1}_{j\leqslant k}(1+t\lambda_j)^{\frac{1}{2}} + \mathbb{1}_{j>k}}{\mathbb{1}_{i\leqslant k}(1+t\lambda_i)^{\frac{1}{2}} + \mathbb{1}_{i>k}} h(w_i, w_j) = \delta_{i,j}.$$

We finally compute :

$$\begin{aligned} Ric_{h_t^G}(v) &= \sum_{i=1}^n h_t^G\big(R_{h_t^G}(C_t^{-1/2} w_i, v)v, C_t^{-1/2} w_i\big) \\ &= \sum_{i=1}^n k_t^G\big(C_t^{-1/2} w_i \wedge v\big) \\ &\overset{Equ.\ (4.1)}{=} \sum_{i=1}^n \Big(k_h\big(C_t^{1/2} w_i \wedge C_t v\big) + \zeta_t\big(C_t^{1/2} w_i, C_t v\big)\Big) \\ &\quad + \sum_{i=1}^l \frac{t^3}{4}\Big\|\Big[S(p)\frac{1}{\lambda_i^{1/2}(1+t\lambda_i)^{1/2}} y_i, S(p)C_t^{-1} x\Big]\Big\|_b^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{l} k_h\Big(\frac{1}{(1+t\lambda_i)^{1/2}} w_i \wedge C_t v\Big) + \sum_{i=l+1}^{n} k_h\big(w_i \wedge C_t v\big) \\
&\quad + \sum_{i=1}^{n} \zeta_t\big(C_t^{1/2} w_i, C_t v\big) \\
&\quad + \sum_{i=1}^{l} \frac{t^3}{4} \left\|\left[\frac{\lambda_i^{1/2}}{(1+t\lambda_i)^{1/2}} y_i, S(p) C_t^{-1} x\right]\right\|_b^2 \\
&= Ric_h^{\mathcal{H}}\big(C_t v\big) + \sum_{i=1}^{n} \zeta_t\big(C_t^{1/2} w_i, C_t v\big) \\
&\quad + \sum_{i=1}^{l} \frac{1}{1+t\lambda_i}\Big(k_h\big(w_i \wedge C_t v\big) + \frac{\lambda_i t^3}{4}\left\|\left[y_i, \frac{S(p)}{1+tS(p)} x\right]\right\|_b^2\Big).
\end{aligned}
$$

✓

<u>Ad (ii)</u> : We calculate separately the different limits :

- $\lim_{t\to\infty} C_t v = \lim_{t\to\infty} \big(Id + tS(p)\big)^{-1} X^*(p) + \xi = \xi$;
- *for* $i = 1, ..., l$: $\lim_{t\to\infty} \frac{1}{1+t\lambda_i} k_h\big(w_i \wedge C_t v\big) = \lim_{t\to\infty} \frac{1}{1+t\lambda_i} k_h\big(w_i \wedge \xi\big) = 0$;
- *for* $i = 1, ..., l$: $\lim_{t\to\infty} \frac{\lambda_i t^3}{4(1+t\lambda_i)} \left\|\left[y_i, \frac{S(p)}{1+tS(p)} x\right]\right\|_b^2 = \frac{1}{4} \lim_{t\to\infty} \left\|\left[y_i, \frac{tS(p)}{1+tS(p)} x\right]\right\|_b^2$
  $= \frac{1}{4} \|[y_i, x]\|_b^2$.

Hence,

$$
\lim_{t\to\infty} Ric_{h_t^G}(v) = Ric_h^{\mathcal{H}}\big(\xi\big) + \lim_{t\to\infty} \textstyle\sum_{i=1}^{n} \zeta_t(C_t^{1/2} w_i, C_t v) + \frac{1}{4} \sum_{i=1}^{l} \|[y_i, x]\|_b^2 .
$$

✓

■

Such an expression exists also for scalar curvature in Cheeger metrics :

Lemma 5.4 - Scalar curvature in the Cheeger metrics

Let $t > 0$;
$p \in M$;
$n := \dim T_p M$.

There exists a b-orthonormal basis $(y_1, ...y_l)$ of $\mathfrak{m}_p$, scalars $\lambda_1, ..., \lambda_l > 0$ and a h-orthonormal basis $(w_1, ..., w_n)$ of T_pM such that the scalar curvature $scal_{h_t^G}$ in the Cheeger metric h_t^G at point p satisfies :

$$
\begin{aligned}
scal_{h_t^G}(p) &= \textstyle\sum_{i,j=1}^{n} \Big(k_h\big(C_t^{1/2} w_i \wedge C_t^{1/2} w_j\big) + \zeta_t\big(C_t^{1/2} w_i, C_t^{1/2} w_j\big)\Big) \\
&\quad + \textstyle\sum_{i,j=1}^{l} \frac{\lambda_i \lambda_j t^3}{4(1+t\lambda_i)(1+t\lambda_j)} \|[y_i, y_j]\|_b^2 .
\end{aligned}
$$

Proof 5.4:

As in the last proof :

- *we construct a basis of* $\mathfrak{m}_p$, $(y_1, ...y_l)$, *composed of* b*-orthonormal eigenvectors of* $S(p)$ *with eigenvalues* $0 < \lambda_1 \leqslant ... \leqslant \lambda_l$ *;*
- *we define* $w_i := \frac{1}{\lambda_i^{1/2}} Y_i^*(p)$ *for* $i = 1, ..., l$*, that we complete with a* h*-orthonormal basis* $(w_{l+1}, ..., w_n)$ *of* $\mathcal{H}_p$ *to obtain an* h*-orthonormal basis of* T_pM *:* $(w_1, ..., w_n)$*.*

We simply apply Definition A.39 of the scalar curvature $scal_{h_t^G}$ *to the Riemannian manifold* (M, h_t^G)*. We consider an intermediate result (the third equality) in the proof of Lemma 5.3 (i) and replace* v *simultaneously by* $C_t^{-1/2} w_j$, $j = 1, ..., n$*, in Lemma 5.3 (i) since they build a* h_t^G*-orthonormal basis of* T_pM*.*

Notice the following identity we'll need in the Lie bracket term :

$$*^{-1}\big(w_j^{\mathcal{V}}\big) = 0, \qquad \forall j > l. \qquad (\star)$$

Then,

$$\begin{aligned}
scal_{h_t^G}(p) &= \textstyle\sum_{j=1}^n Ric_{h_t^G}\big(C_t^{-1/2} w_j\big) \\
&\overset{(\star)}{=} \textstyle\sum_{i,j=1}^n \Big(k_h(C_t^{1/2} w_i \wedge C_t^{1/2} w_j) + \zeta_t(C_t^{1/2} w_i, C_t^{1/2} w_j)\Big) \\
&\quad + \textstyle\sum_{i,j=1}^l \frac{t^3}{4} \left|\left|\left[S(p)\frac{1}{\lambda_i^{1/2}(1+t\lambda_i)^{1/2}} y_i, S(p)\frac{1}{\lambda_j^{1/2}(1+t\lambda_j)^{1/2}} y_j\right]\right|\right|_b^2 \\
&= \textstyle\sum_{i,j=1}^n \Big(k_h(C_t^{1/2} w_i \wedge C_t^{1/2} w_j) + \zeta_t(C_t^{1/2} w_i, C_t^{1/2} w_j)\Big) \\
&\quad + \textstyle\sum_{i,j=1}^l \frac{\lambda_i \lambda_j t^3}{4(1+t\lambda_i)(1+t\lambda_j)} \left|\left|[y_i, y_j]\right|\right|_b^2 .
\end{aligned}$$

■

5.2 Orbit types and isotropy representation

The behavior of $p \in M$ in the G-action may affect the last term of $scal_{h_t^G}$ in Lemma 5.4 in the sense that a lower dimension l of $\mathfrak{m}_p$ would decrease the number of summands. That's why we need a criterium, the orbit type, to distinguish between different kinds of points. Many notions appearing in this section are explained in the fourth part of Appendix A.

Definitions 5.5 - Principal and singular orbits

The **maximum orbit type of G on M** is the quotient space ${}^{\mathrm{G}}\!/_{\mathrm{H}}$ where $H \subseteq G$ is a subgroup conjugated to a subgroup of each isotropy group $G_p \subseteq G$, $p \in M$ (exists according to the next result).

An orbit $G \cdot p$ is said :

- **principal** if its orbit type is maximum, i.e. if $G \cdot p \cong {}^{G}\!/_{H}$, which means $G \cdot p$ is of highest dimension for an orbit ;
- **singular** if it has a smaller dimension.

Points of such orbits are called **principal** and **singular**, respectively.

We define the **regular part of M** as

$$M^{reg} := \{p \in M \mid G \cdot p \text{ principal orbit}\}.$$

Theorem 5.6

Let a *compact* Lie group G act smoothly on a *connected* manifold M, which isn't obligatory compact.

Then :

(i) A maximum orbit type ${}^{G}\!/_{H}$ for G on M exists ;

(ii) The regular part of M, M^{reg} is *open* and *dense* in M.

Proof 5.6: *See [Bre72, Chapter IV, Theorem 3.1].*

■

In order to define a useful map from $\mathfrak{g}$ to T_pM, for a singular point $p \in M$, we now explore some properties of the "isotropy representation". Let's first define this concept :

Definition 5.7 - Representation of a Lie group

Let G a Lie group ;
V a vector space.

A **representation of G on V** is a group homomorphism from G to $Aut(V)$, the automorphism group on V.

In the context of our *isometric* action $G \curvearrowright (M, h)$, let $p \in M$ and $g \in G_p$. Then the differential $(d\mu_g)_p : T_pM \to T_pM$ of $\mu_g = \mu(g, \cdot) \in \mathrm{Iso}(M, h)$ preserves :

- the norm ;
- the orthogonality, i.e. it sends h-horizontal vectors on h-horizontal vectors, i.e. $(d\mu_g)_p(\mathcal{H}_p) \subseteq \mathcal{H}_p$.

Hence the following representation of the isotropy group G_p restricted to the orthogonal group of the horizontal space $O(\mathcal{H}_p)$ is well-defined :

Definition 5.8 - Isotropy representation

The **isotropy representation** of G for $p \in M$, restricted to the horizontal space $\mathcal{H}_p$ is :

$$\begin{array}{cccc} \phi_p & : & G_p & \to & O(\mathcal{H}_p) \\ & & g & \mapsto & (d\mu_g)_p\big|_{\mathcal{H}_p}. \end{array}$$

We will examine the differential of ϕ_p at the neutral element $e \in G$ in more details :

$$(d\phi_p)_e : \mathfrak{g}_p \to \mathfrak{o}(\mathcal{H}_p),$$

with $\mathfrak{g}_p$ and $\mathfrak{o}(\mathcal{H}_p)$ the Lie algebras of the Lie groups G_p and $O(\mathcal{H}_p)$, respectively.

Proposition 5.9

Let V be a $\mathbb{R}$-vector field of dimension n.

The Lie algebra of the orthogonal group on V, $O(V)$, is the vector space of skew-symmetric matrices :

$$\mathfrak{o}(V) = \left\{A \in Mat(n,\mathbb{R}) \mid A^T = -A\right\}$$

Proof 5.9:

The exponential map on the manifold $O(n)$ at E_n coincides with the ordinary exponential map for orthogonal matrices :

$$\begin{array}{cccc} \exp & : & \mathfrak{o}(V) & \to & O(V) \\ & & A & \mapsto & \sum_{i=0}^{+\infty} \frac{A^i}{i!} \end{array}.$$

Let's consider a curve through E_n in the direction $A \in O(V)$:

$$\begin{array}{cccc} \alpha & : & (-\epsilon,\epsilon) & \to & O(V) \\ & & t & \mapsto & t \cdot A \end{array}.$$

Then, for all $t \in (-\epsilon,\epsilon)$:

$$E_n = \exp(t\cdot A)\cdot\exp(t\cdot A)^T = \exp(t\cdot A)\cdot\exp(t\cdot A^T) = \exp\left(t\cdot(A+A^T)\right),$$

$$\Leftrightarrow\ 0 = \frac{d}{dt}\Big|_{t=0} E_n = \frac{d}{dt}\Big|_{t=0}\exp\left(t\cdot(A+A^T)\right) = (A+A^T)\cdot\exp(A+A^T),$$

$$\Leftrightarrow\ A + A^T = 0.$$

■

Corollary 5.10

The differential of the isometry representation $(d\phi_p)_e : \mathfrak{g}_p \to \mathfrak{o}(\mathcal{H}_p)$ sends vectors of $\mathfrak{g}_p$ on *skew-symmetric* endomorphisms of $\mathcal{H}_p$, i.e. the representing matrices are skew-symmetric.

Proof 5.10: *Direct consequence of Proposition 5.9.*

■

Theorem 5.11

Let $p \in M$;
$x \in \mathfrak{g}_p$;
$v \in \mathcal{H}_p$;
$V \in \mathfrak{X}(M)$ an extension of v, i.e. $V(p) = v$.

Then,

$$\left((d\phi_p)_e(x)\right)(v) = \left(\nabla_V X^*\right)(p).$$

Proof 5.11:

Let's denote by $\exp_p^M : T_pM \to M$ the exponential map on the manifold M at point $p \in M$ and by $\exp^G : \mathfrak{g} \to G$ the Lie exponential, so as not to confuse between the two maps.

Let γ_v^p *be the geodesic through* p *in direction* v*, i.e.* $\gamma_v^p(s) = \exp_p^M(s \cdot v)$*,* $\forall s \in \mathbb{R}$ *;*
$f : \mathbb{R}^2 \to M, (t,s) \mapsto \mu\big(\exp^G(t \cdot x), \gamma_v^p(s)\big)$*, be a parametrized surface*[3]*.*

Observe that :

- f *represents a variation of the geodesic* γ_v^p *by geodesics, which means that* $J := \frac{\partial}{\partial t}\Big|_{t=0} f(t,\cdot) = X^* \circ \gamma_v^p$ *is a Jacobi field*[4] *along* γ_v^p *;*
- $J(0) = X^*(p) = 0_p$*.*

Recall that the exponential map $\exp_p^M : T_pM \to M$ *is a local diffeomorphism. According to [Car92, Chapter 5, Corollary 2.5], since* $J(0) = 0_p$*, the following holds for all* $s \in \mathbb{R}$ *:*

$$J(s) = s \cdot \frac{D}{ds}\Big|_{s=0} J(s) = s \cdot \nabla_{\partial/\partial s} J(0) = s \cdot \nabla_V X^*(p),$$

after identification of the tangent spaces $T_{\gamma_v^p(s)}M$ *and* $T_{s\cdot v}T_pM \cong T_pM$ *through normal coordinates*[5]*.*

On the other hand, as $\gamma_v^p(s)$ *is identified to* $s \cdot v$*,*

$$J(s) = X^* \circ \gamma_v^p(s) = \big((d\phi_p)_e(x)\big)(s \cdot v) \overset{d\phi_p \text{ linear}}{=} s \cdot \big((d\phi_p)_e(x)\big)(v).$$

Hence,

$$\big((d\phi_p)_e(x)\big)(v) = \nabla_V X^*(p).$$

■

5.3 Further considerations on the differential of the isometry representation

Given a tangent vector $v \in T_pM$ at $p \in M$, Cavenaghi and Sperança generalizes the linear map $d\phi_p(\cdot)(v) : \mathfrak{g}_p \to T_pM, x \mapsto (\nabla_V X^*)(p)$ to any element of the Lie algebra $\mathfrak{g}$:

Definition 5.12 - Auxiliary linear map Q_V

Let $p \in M$;
$v \in T_pM$ with an extension $V \in \mathfrak{X}(M)$;
$\gamma := \gamma_v^p : \mathbb{R} \to M, s \mapsto \gamma(s) := \exp(s \cdot v)$ the geodesic through p in direction v.

We define a linear map :

$$\begin{array}{rccc} Q_V & : \mathfrak{g} & \to & T_pM \\ & x & \mapsto & (\nabla_V X^*)(p). \end{array}$$

[3] Since G and M are assumed to be compact, the geodesics are defined on $\mathbb{R}$.
[4] See Definition A.36
[5] See [GHL04, 2.89 bis]

This map has some interesting properties :

Lemma 5.13

For $p \in M$, if $v \in \mathcal{H}_p$, then :

(i) $Q_V(\mathfrak{g}_p) \subseteq \mathcal{H}_p$;

(ii) $\mathfrak{g}_p \cap \ker(Q_V) = \mathfrak{g}_V$,
where $\mathfrak{g}_V$ is the Lie algebra of $G_V := \{g \in G_p \mid \phi_p(g)(v) = v\}$.

Proof 5.13: *See [CS18, Lemma 2].*

■

Since G is equipped with a (biinvariant) metric b and $\mathfrak{g}_V$ is a subvector space of $\mathfrak{g}_p$, we can define the **b-orthogonal complement of $\mathfrak{g}_V$ in $\mathfrak{g}_p$** :

$$\boxed{\mathfrak{p}_v := (\mathfrak{g}_V)^{\perp} \subseteq \mathfrak{g}_p}.$$

Corollary 5.14

The auxiliary map restricted on $\mathfrak{p}_V$

$$Q_V\Big|_{\mathfrak{p}_V} : \mathfrak{p}_V \to \mathcal{H}_p,$$

is *injective* for any $p \in M$, $v \in \mathcal{H}_p$.

Proof 5.14: *Direct consequence of Lemma 5.13.*

■

Definition 5.15 - Fake horizontal vector with respect to v

Let $p \in M$;
$v \in \mathcal{H}_p$.

We call **fake horizontal vector with respect to v** any element of $Q_V(\mathfrak{p}_V) \subseteq \mathcal{H}_p$. Furthermore, for $w \in \mathcal{H}_p$, we denote the unique preimage of its h-orthogonal projection on $Q_V(\mathfrak{p}_V)$ by $w_{\mathfrak{p}_V} \in \mathfrak{p}_V$.

Remark 5.16. The fake horizontal vector $Q_V\left(w_{\mathfrak{p}_V}\right)$ is also the h_t^G-orthogonal projection on $Q_V(\mathfrak{p}_V)$ by the same argument as in Remark 3.7.

This fake horizontal vectors are crucial in the process of finding a positive lower bound for the additional term ζ_t implied in the scalar positive curvature formula given by Lemma 5.4, in case the last term vanishes. To prove the Lawson-Yau Theorem, we will need the following result for singular points even if the statement holds for any kind of point.

Proposition 5.17 - Lower bound of ζ_t

Let $t > 0$;
$p \in M$;
$v, w \in \mathcal{H}_p$.

Then, if w is not h-orthogonal to $Q_V(\mathfrak{p}_V)$, i.e. $w_{\mathfrak{p}_V} \neq 0$:

$$\zeta_t(v, w) \geqslant 3t \frac{\|Q_V(w_{\mathfrak{p}_V})\|_h^4}{\|w_{\mathfrak{p}_V}\|_b^2} > 0.$$

We first introduce an intermediate result dealing with the Killing form term in our situation :

Lemma 5.18

Let $p \in M$;
$v \in \mathcal{H}_p$, with an *horizontal* extension $V \in \mathfrak{X}(M)^{\mathcal{H}}$;
$\gamma_v^p : \mathbb{R} \to M, s \mapsto \exp_p^M(s \cdot v)$, be the geodesic going through p in direction v ;
$u = Q_V(u_{\mathfrak{p}_V}) \in Q_V(\mathfrak{p}_V)$, be a fake horizontal vector ;
$z \in \mathfrak{g}$.

We define a vector field along $\gamma_v^p\big|_{\mathbb{R}_{>0}}$:

$$\begin{array}{cccc} U & : & \mathbb{R}_{>0} & \to & TM \\ & & s & \mapsto & U(s) := \frac{1}{s} U_{\mathfrak{p}_V}^*\big(\gamma_v^p(s)\big) \in T_{\gamma_v^p(s)}M. \end{array}$$

Then :

(i) $\lim_{s \to 0^+} U(s) = u$, so U extends smoothly to $s = 0$;

(ii) $\lim_{s \to 0^+} d\omega_z\big(U(s), V(s)\big) = -h\big(u, \nabla_V Z^*(p)\big)$, where $V(s) := V\big(\gamma_v^p(s)\big)$.

In the following two proofs, we write $\widetilde{W}(s) := \widetilde{W}(\gamma_v^p(s))$, $s \in \mathbb{R}$, for any $\widetilde{W} \in \mathfrak{X}(M)$.

Proof 5.18:

Ad (i) : As in the proof of Theorem 5.11, we use normal coordinates to identify a neighborhood of $0_p = U_{\mathfrak{p}_V}^*(0)$ with a neighborhood of $\gamma_v^p(0) = p$.

The geodesic $\gamma_v^p \subset M$ can thus be identified with a geodesic $\widetilde{\gamma_v^p} \in T_pM$ and $U_{\mathfrak{p}_V}^*$ is a Jacobi field along $\widetilde{\gamma_v^p}$ which can be written as

$$U_{\mathfrak{p}_V}^*(s) = s \cdot \nabla_V U_{\mathfrak{p}_V}^*(0), \qquad \forall s > 0.$$

Hence,

$$\lim_{s \to 0^+} U(s) = \frac{1}{s} \lim_{s \to 0^+} s \cdot \nabla_V U_{\mathfrak{p}_V}^*(0) = \nabla_V U_{\mathfrak{p}_V}^*(0) = u.$$

✓

Ad (ii) :

Claim : $d\omega_z\big(U^*_{\mathfrak{p}_V}(s), V(s)\big) = -\frac{1}{2} V h\big(U^*_{\mathfrak{p}_V}(s), Z^*(s)\big)$, *for all* $s \geqslant 0$.

Proof of the claim :

Let's first restrict to regular points $\gamma_v^p(s)$, *for some* $s \geqslant 0$.

Since the Lie derivative[6] $\mathcal{L}_{C^*} h$ *of* h *vanishes in the direction of a Killing vector field* $C^* \in \mathfrak{X}(M)$ *(so in particular for action fields), we deduce the following identity :*

$$C^* h(A, B) = h\left([C^*, A], B\right) + h\left(A, [C^*, B]\right), \qquad \forall A, B \in \mathfrak{X}(M). \quad (\star)$$

We calculate the differential of $d\omega_z$ *as in the proof of Proposition 4.6 :*

$$\begin{aligned}
d\omega_z\big(U^*_{\mathfrak{p}_V}, V\big) &= U^*_{\mathfrak{p}_V}\omega_z(V) - V\omega_z(U^*_{\mathfrak{p}_V}) - \omega_z\left([U^*_{\mathfrak{p}_V}, V]\right) \\
&= \tfrac{1}{2}\Big(U^*_{\mathfrak{p}_V}\underbrace{h(V, Z^*)}_{=0} - Vh(U^*_{\mathfrak{p}_V}, Z^*) - h\big([U^*_{\mathfrak{p}_V}, V], Z^*\big)\Big) \\
&\overset{(\star)}{=} -\tfrac{1}{2}\Big(Vh(U^*_{\mathfrak{p}_V}, Z^*) + U^*_{\mathfrak{p}_V}\underbrace{h(V, Z^*)}_{=0} + h\big(V, [U^*_{\mathfrak{p}_V}, Z^*]\big)\Big) \\
&\overset{\text{Prop. 2.8}}{=} -\tfrac{1}{2}\Big(Vh(U^*_{\mathfrak{p}_V}, Z^*) - \underbrace{h\big(V, [U_{\mathfrak{p}_V}, Z]^*\big)}_{=0}\Big) \\
&= -\tfrac{1}{2}Vh(U^*_{\mathfrak{p}_V}, Z^*).
\end{aligned}$$

So the equality holds at s, *if* $\gamma_v^p(s)$ *is regular.*

Now, suppose $\gamma_v^p(s)$ *is a singular point for some* $s \geqslant 0$. *Recall that the set of all regular points* M^{reg} *is open and dense in* M. *As a consequence, for an infinity of horizontal directions* $\widetilde{V}(s) \in T_{\gamma_v^p(s)}M$ *there exists a sequence* $(p_i)_{p_i \in \mathbb{N}}$ *of regular points on the geodesic through* $\gamma_v^p(s)$ *in direction* $\tilde{V}(s)$ *converging to* $\gamma_v^p(s)$. *By continuity of* $d\omega_z$ *and* h, *the equality also holds in the direction* $V(s)$ *for* $\gamma_v^p(s)$.

Claim ■

Hence,

$$d\omega_z\big(U(s), V(s)\big) = -\frac{Vh\big(U^*_{\mathfrak{p}_V}(s), Z^*(s)\big)}{2s}, \qquad \forall s > 0. \qquad (\star\star)$$

And at $s = 0$, *we know that*

$$d\omega_z\big(\underbrace{U^*_{\mathfrak{p}_V}(0)}_{=0}, V(0)\big) = 0.$$

Observe that the numerator $Vh\big(U^*_{\mathfrak{p}_V}(s), Z^*(s)\big)$ *and denominator* $2s$ *of* $(\star\star)$ *both vanish at* $s = 0$, *because* $U^*_{\mathfrak{p}_V}(0) = 0$. *Therefore, we apply L'Hospital rule to estimate the limit of the indeterminate form* $(\star\star)$ *when* $s \to 0^+$. *Moreover, since* $\gamma_v^{p\prime}(0) = v = V(0)$, *the linear derivation* V *evaluated at* $h\big(U^*_{\mathfrak{p}_V}, Z^*\big) : M \to \mathbb{R}$

[6] See Definition A.15

corresponds to the derivative of $h\big(U^*_{\mathfrak{p}_V}, Z^*\big) \circ \gamma^p_v : \mathbb{R} \to \mathbb{R}$ *at* 0 *:*

$$
\begin{aligned}
\lim_{s\to 0^+} d\omega_z\big(U(s), V(s)\big) &= -\lim_{s\to 0^+} \frac{Vh\big(U^*_{\mathfrak{p}_V}(s), Z^*(s)\big)}{2s} \\
&= -\frac{1}{2}\frac{d}{ds}\Big|_{s=0} Vh\big(U^*_{\mathfrak{p}_V}(s), Z^*(s)\big) \\
&= -\frac{1}{2}\frac{d^2}{ds^2}\Big|_{s=0} h\big(U^*_{\mathfrak{p}_V}(s), Z^*(s)\big) \\
&= -\frac{1}{2}\Big(h\big(\tfrac{D^2}{ds^2}U^*_{\mathfrak{p}_V}(0), Z^*(0)\big) + h\big(U^*_{\mathfrak{p}_V}(0), \tfrac{D^2}{ds^2}Z^*(0)\big) \\
&\qquad + 2h\big(\tfrac{D}{ds}U^*_{\mathfrak{p}_V}(0), \tfrac{D}{ds}Z^*(0)\big)\Big).
\end{aligned}
$$

The last equality holds because derivating with respect to s *is equivalent to derivate in direction* V*, which allows to use the compatibility of* ∇ *with the metric* h *twice. Let's compute these covariant derivatives for the geodesic* γ^p_v *:*

- $\frac{D^2}{ds^2}U^*_{\mathfrak{p}_V}(0) = -R_h\big(v, U^*_{\mathfrak{p}_V}(0)\big)v = -R_h\big(v, 0_p\big)v = 0_p$ *because of Jacobi Equation and tri-linearity of the curvature tensor* R_h *;*
- $\frac{D^2}{ds^2}Z^*(0) = 0_p$ *by similar arguments ;*
- $\frac{D}{ds}U^*_{\mathfrak{p}_V}(0) = \nabla_V U^*_{\mathfrak{p}_V}(0) = Q_V(u_{\mathfrak{p}_V}) = u$ *;*
- $\frac{D}{ds}Z^*(0) = \nabla_V Z^*(0) = \nabla_V Z^*(p)$.

So the two first terms vanish and we obtain :

$$\lim_{s\to 0^+} d\omega_z\big(U(s), V(s)\big) = -h\big(u, \nabla_V Z^*(p)\big).$$

✓

■

Proof 5.17:

Recall the formula for the additional term obtained at the end of Chapter 4 for the Cheeger metric h^G_t *:*

$$\zeta_t(v,w) := 3\left|\left|A_{f(v)}f(w)\right|\right|^2_{h+\frac{1}{t}b} = 3t \max_{\substack{z\in\mathfrak{g} \\ z\neq 0}} \left\{ \frac{\big(d\omega_z(v,w) + \frac{t}{2}b([S(p)x, S(p)y], z)\big)^2}{th\big(Z^*(p), Z^*(p)\big) + 1} \right\} \geqslant 0,$$

where $x := *^{-1}(v^{\mathcal{V}}) = 0 \in \mathfrak{g}$ *and* $y := *^{-1}(w^{\mathcal{V}}) = 0 \in \mathfrak{g}$ *since* v *and* w *are horizontal. That's why the second term of the numerator will disappear.*

We define the unit vector $z := \frac{w_{\mathfrak{p}_V}}{\|w_{\mathfrak{p}_V}\|_b} \subseteq \mathfrak{g}_p$*, so* $Z^*(p) = 0$*, and obtain :*

$$\zeta_t(v,w) \geqslant 3t\, d\omega_z(v,w)^2. \tag{$\star$}$$

Let $\gamma^p_v : \mathbb{R} \to M, s \mapsto \exp^M_p(s\cdot v)$ *be the geodesic going through* p *in direction* v *;* $V, W \in \mathfrak{X}(M)$ *extensions of* v *and* w*, respectively.*

Then, by Lemma 5.18,

$$d\omega_z(v,w) = -\lim_{s\to 0^+} d\omega_z\big(W(s), V(s)\big) = h\big(w, \nabla_V Z^*(0)\big).$$

By definition, $\nabla_V Z^*(0) = Q_V(z) = \frac{Q_V(w_{\mathfrak{p}_V})}{\|w_{\mathfrak{p}_V}\|_b} \in T_pM$*, so* $(\star)$ *becomes*

$$\zeta_t(v,w) \geqslant 3t\,h\left(w, \frac{Q_V(w_{\mathfrak{p}_V})}{\|w_{\mathfrak{p}_V}\|_b}\right)^2 = 3t\,h\left(Q_V(w_{\mathfrak{p}_V}), \frac{Q_V(w_{\mathfrak{p}_V})}{\|w_{\mathfrak{p}_V}\|_b}\right)^2 = 3t\frac{\|Q_V(w_{\mathfrak{p}_V})\|_h^4}{\|w_{\mathfrak{p}_V}\|_b^2} > 0.$$

■

5.4 Lawson-Yau Theorem

Everything seen until this point permits to prove the following major result on some compact manifolds :

Theorem 5.19 - Lawson-Yau Theorem (1974)

Let (M,h) be a *compact* Riemannian manifold ;
G be a *compact*, *connected* and *non-abelian* Lie group acting *effectively* and *by isometries* on (M,h).

Then there exists some metric $\widetilde{h}$ on M with positive scalar curvature, i.e.

$$scal_{\widetilde{h}}(p) > 0, \qquad \forall p \in M.$$

We follow the Cavenaghi and Sperança approach to find a Cheeger metric h_t fulfilling this property[7].

Proof 5.19:

Claim 1 : Our G-action can be reduced to the case of an effective S^3- or $SO(3)$-action on (M,h), regardless of the dimension of G.

Proof of the claim 1 :

According to [Bum04, Theorem 19.1], there exists a Lie group homomorphism $\phi : S^3 \to G$ with finite kernel. Therefore, S^3 acts (almost) effectively on M through $\tilde{\mu} := \mu \circ \phi : S^3 \to Iso(M,h)$.

If $\ker(\tilde{\mu}) = \{1\}$, then $\tilde{\mu}$ is effective. In the other case, the induced action $\tilde{\mu}/_{\ker\tilde{\mu}}$ of the quotient set $S^3/_{\ker\tilde{\mu}} \cong SO(3)$ on M is effective.

Claim 1 ■

We now consider the Cheeger construction $h \underset{G}{\to} h_t^G$ on (M,h) induced by (G,b).

Let's recall the formula of scalar curvature at $q \in M$ developed in Lemma 5.4 for a Cheeger metric h_t^G, $t > 0$:

$$\begin{aligned} scal_{h_t^G}(q) = &\textstyle\sum_{i,j=1}^n \left(k_h\left(C_t^{1/2}w_i \wedge C_t^{1/2}w_j\right) + \zeta_t\left(C_t^{1/2}w_i, C_t^{1/2}w_j\right)\right) \\ &+ \textstyle\sum_{i,j=1}^l \frac{\lambda_i\lambda_j t^3}{4(1+t\lambda_i)(1+t\lambda_j)} \,||[y_i,y_j]||_b^2\,. \end{aligned} \qquad (\star)$$

[7] See [CS18, Theorem 3.3].

where $(y_1, ..., y_l)$ *is a b-orthonormal basis of* $\mathfrak{m}_q = (\mathfrak{g}_q)^\perp$ *composed of eigenvectors of* $S(q)$ *with the corresponding eigenvalues* $0 < \lambda_1 \leqslant ... \leqslant \lambda_l$ *;*
$(C_t^{1/2} w_1, ..., C_t^{1/2} w_n)$ *is a* h_t^G*-orthonormal basis of* $T_q M$*.*

Claim 2 : The term $\sum_{i,j=1}^n k_h\big(C_t^{1/2} w_i \wedge C_t^{1/2} w_j\big)$ *possess a uniform lower bound* L*.*

Proof of the claim 2 :

For all $q \in M$*, let's define*

$$L_q := \min_{1 \leqslant i,j \leqslant n} k_h(w_i \wedge w_j) = \min_{1 \leqslant i,j \leqslant n} sec_h(w_i, w_j).$$

Since $span\{w_i, w_j\} = span\{C_t^{1/2} w_i, C_t^{1/2} w_i\}$*, we obtain for all* $t > 0$ *and all* $1 \leqslant i, j \leqslant n$ *:*

$$\begin{aligned} k_h(C_t^{1/2} w_i \wedge C_t^{1/2} w_j) &= sec_h(w_i, w_j) \cdot \| C_t^{1/2} w_i \wedge C_t^{1/2} w_j \|_h^2 \\ &\geqslant L_q \cdot \underbrace{\| C_t^{1/2} w_i \wedge C_t^{1/2} w_j \|_h^2}_{\leqslant 1} \\ &\geqslant \min(0, L_q). \end{aligned}$$

Hence, for the whole term,

$$\sum_{i,j=1}^n k_h\big(C_t^{1/2} w_i \wedge C_t^{1/2} w_j\big) \geqslant \min\left(\binom{n}{2} \cdot L_q, 0\right)$$

By compactness argument, there exists a lower bound L *for this sum on the whole manifold* M *which holds for all* $t > 0$*.*

Claim 2 $\blacksquare$

By contraction, suppose that none of the Cheeger metric h_t^G *carries a positive scalar curvature. Then, there exists a sequence* $(p_n)_{n \in \mathbb{N}}$ *in* M *with* $scal_{h_n^G}(p_n) \leqslant 0$*,* $\forall n \in \mathbb{N}$*. By the compactness assumption and Bolzano-Weierstrass Theorem, a convergent subsequence exists and we'll denote by* $p \in M$ *its limit point.*

Claim 3 : There exists a $N \in \mathbb{N}$ *with the property that* $scal_{h_n^G}(p) \leqslant 0$*, for all* $n \geqslant N$*.*

Proof of the claim 3 :

We again argue by contradiction. Suppose that there exists a sequence $n_i \to +\infty$ *in* $\mathbb{N}$ *such that* $scal_{h_{n_i}^G}(p) > 0$*, for all* $i \in \mathbb{N}$*.*

For a given n_i*,* $i \in \mathbb{N}$*, there exists a neighborhood* U *of* p *on which*

$$scal_{h_{n_i}^G} > \epsilon > 0,$$

because of the continuity of the scalar curvature $scal_{h_{n_i}}$*. We denote by* $U^{reg} := U \cap M^{reg}$ *the open and dense subset of regular points in* U*.*

We know then that for all $q \in U^{reg}$*, the last term of* $(\star)$ *tends to infinity and therefore exceeds* L *for all sufficiently big* $t > 0$*. By monotonicity of* ζ_t*, the last term in* $(\star)$ *and continuity of scalar curvature, this is true on the whole neighborhood* U*, i.e. a common lowest* $T > 0$ *exists with the property that* $scal_{h_t^G}(q) > 0$*, for all* $t > T$*,* $q \in U$*.*

Therefore, for all $n > T$ *with* $p_n \in U$ *:*

$$scal_{h_n^G}(p_n) > 0. \qquad \lightning$$

Claim 3 $\blacksquare$

Claim 4 : *p is a singular point.*

Proof of the claim 4 :

Suppose p is regular. It suffices to show that there exists non-commuting $v, w \in \mathfrak{m}_p$ *to obtain a diverging last term in* $(\star)$ *and contradict this hypothesis.*

In our situation, $G \cong S^3$ *or* $G \cong SO(3)$ *and in both cases,* $\mathfrak{g} \cong so(3)$. *This denotes the Lie algebra of skew symmetric* (3×3)*-matrices* $\mathfrak{so}(3)$ *which is equipped with the Lie bracket defined by the commutator :*

$$[A, B] = AB - BA, \qquad \forall A, B \in \mathfrak{so}(3).$$

G acts effectively on M and p has principal orbit.

$\Rightarrow\ G_p \neq G.$

$\Rightarrow\ \mathfrak{g}_p \neq \mathfrak{g}.$

$\Rightarrow\ \dim \mathfrak{g}_p \geqslant 2.$

If G_p *was of dimension 2, this would mean that its Lie algebra* $\mathfrak{g}_p$ *would be a 2-dimensional Lie subalgebra of* $\mathfrak{so}(3)$. *However, any linear independent* $A, B \in \mathfrak{so}(3)$ *has the property that* $[A, B] \notin span(A, B)$. *In other words, it generates all* $\mathfrak{so}(3)$. *As a consequence,* $dim \mathfrak{g}_p = 0$ *or* 1.

$\Rightarrow \dim \mathfrak{m}_p \geqslant 2.$

We can then choose linearly independent $v, w \in \mathfrak{m}_p$ *skew-symmetric matrices, up to isomorphism. These* v, w *generates a linear independent* $u \in \mathfrak{so}(3)$ *through the Lie bracket :*

$$[v, w] = u \neq 0.$$

Hence, there exists some $t > 0$ *such that the last term of* $(\star)$ *exceeds L, which means that*

$$scal_{h_t^G}(p) > 0. \qquad ↯$$

Claim 4 ■

If $\dim \mathfrak{m}_p \geqslant 2$, *then we have the same contradiction as in Claim 4 and the statement follows.*

Consider the opposite situation $\dim \mathfrak{m}_p = 0$ *(*$\mathfrak{g}_p$ *can't have dimension 2 as demonstrated above). In this case, p is a G-fixed point.*

Since G acts effectively on M, it is the same for the G-action on T_pM *given by the isotropy representation. We choose a horizontal vector* $v \in \mathcal{H}_p$ *with respect to h in the open and dense subset of principal orbits. Recall the principal isotropy* G_V *has maximal dimension 1, since it can't be the whole G.*

$\Rightarrow\ \dim \mathfrak{g}_V \leqslant 1.$

$\Rightarrow\ \dim \mathfrak{p}_V \geqslant 2.$

Therefore, there exists a non-zero $w_{\mathfrak{p}_V} \in \mathfrak{p}_V$, *which is also sent to a non-zero fake horizontal vector* $Q_V\{w_{\mathfrak{p}_V}\} \in \mathcal{H}_p$. *Define* $w \in \mathcal{H}_p$ *h-orthogonal to v with fake* *horizontal vector $Q_V\{w_{\mathfrak{p}_V}\}$. By Proposition 5.17, the additional term is positive :*

$$\zeta_t(v,w) \geqslant 3t\frac{\|Q_V(w_{\mathfrak{p}_V})\|_h^4}{\|w_{\mathfrak{p}_V}\|_b^2} > 0.$$

Suppose v and w are normalized with respect to h and observe that the h-orthonormal basis $(w_{l+1},...,w_n)$ of $\mathcal{H}_p$ can be constructed by completing (v,w). As a consequence, w.l.o.g. $v = w_{l+1}$ and $w = w_{l+2}$. Finally :

$$scal_{h_t^G}(p) \geqslant L + 3t\frac{\|Q_V(w_{\mathfrak{p}_V})\|_h^4}{\|w_{\mathfrak{p}_V}\|_b^2} \underset{t\to+\infty}{\longrightarrow} +\infty.$$

$$\lightning$$

$\blacksquare$

Appendices

Appendix A

Some Riemannian Geometry aspects

This appendix is largely based on the Riemannian Geometry lecture given at the University of Fribourg by Dr. David González Álvaro during the academic year 2017-18 ([Gon17]).

A.1 Basic recalls

Let M be a smooth manifold of dimension n ;
$p \in M$;
$f \in C^\infty(U)$ with U an open neighborhood of p.

We define the **germ of f at p** as the equivalence class of (f, U) for the following equivalence relation :

$$(f, U) \sim (g, V) \Leftrightarrow f \equiv g \text{ on an open neighborhood of } p, W \subseteq U \cap V.$$

The set of all germs of smooth functions at p is denoted by $C_p^\infty(M)$.

Definitions A.1 - Tangent vector & tangent space

A **tangent vector** at p is a linear derivation of $C_p^\infty(M)$, i.e. $v : C_p^\infty(M) \to \mathbb{R}$ satisfying $\forall \lambda \in \mathbb{R}$, $\forall f, g \in C_p^\infty(M)$:

(i) $v(\lambda \cdot f) = \lambda \cdot v(f)$;

(ii) $v(f \cdot g) = f(p) \cdot v(g) + v(f) \cdot g(p)$.

The set of all tangent vectors forms the **tangent space of M at p**, forming a real vector field :

$$T_pM := \{v : C_p^\infty(M) \to \mathbb{R} \text{ tangent vector at p}\}.$$

But sometimes we rather interpret T_pM as a set of equivalent curves, since each smooth $\alpha : \mathbb{R} \to M$ with $\alpha(0) = p$ defines a linear derivation v_α through $v_\alpha(f) := (f \circ \alpha)'(0)$. This tangent vector is usually denoted by $\alpha'(0)$ instead of v_α. Hence we consider two curves α_1 and α_2 as equivalent if $\alpha_1'(0) = \alpha_2'(0)$. Conversely, each tangent vector v is related to a curve α through $v = \alpha'(0)$. In this context, the tangent space is written

$$T_pM := \{\alpha : \mathbb{R} \to M \mid \alpha(0) = p\}/_\sim.$$

Definition A.2 - Differential

Let $\phi : M \to N$ be a smooth map between manifolds.

The following linear map is named the **differential of ϕ at point p** :

$$\begin{array}{rcccc} d\phi_p & : & T_pM & \to & T_{\phi(p)}N \\ & & v = \alpha'(0) & \mapsto & v(\phi) := (\phi \circ \alpha)'(0). \end{array}$$

It respects the *chain rule*:

$$d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p,$$

for any smooth $\phi : M_1 \to M_2$, $\psi : M_2 \to M_3$ and $p \in M_1$.

Remark A.3. If $\phi : M \to N$ is a diffeomorphism, then $d\phi_p$ is an isomorphism.

A.2 Vector fields

Definitions A.4 - Vector field & Lie brackets

A **vector field** on M refers to any section of the projection map $\pi : TM \to M$, i.e. any smooth map $X : M \to TM$ respecting $X(p) \in T_pM$ $\forall p \in M$.

X admits another interpretation, generalizing the linear derivation $X(p) \in T_pM$:

$$\begin{array}{rcccc} X & : & C^\infty(M) & \to & C^\infty(M) \\ & & f & \mapsto & Xf, \end{array}$$

$$\begin{array}{lrcccc} \text{where} & Xf & : & M & \to & \mathbb{R} \\ & & & p & \mapsto & (Xf)(p) := (X(p))(f) = df(X(p)). \end{array}$$

The vector fields set $\mathfrak{X}(M)$ has a vector space structure and is even a Lie algebra considering the following **Lie brackets on $\mathfrak{X}(\mathbf{M})$** :

$$\begin{array}{rcccc} [\cdot,\cdot] & : & \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathfrak{X}(M) \\ & & (X, Y) & \mapsto & [X, Y] := XY - YX. \end{array}$$

We have seen that a tangent vector may be defined from a curve but on the other way we can construct curves from vector fields :

Definition A.5 - Integral curve

Let $X \in \mathfrak{X}(M)$.

An **integral curve of X** is a differential curve $\alpha : \underset{\mathbb{R}}{\overset{}{I}} \to M$ whose velocity at each point $\alpha(t) \in M$ equals the tangent vector $X(\alpha(t)) \in T_{\alpha(t)}M$:

$$\alpha'(t) = X(\alpha(t)).$$

Note that some integral curves may not be defined on $\mathbb{R}$.[1]

[1] See [Lee12, Examples 9.9 and 9.10]

Definition A.6 - Flow of a vector field

Let $X \in \mathfrak{X}(M)$.

We call **flow domain for M** any open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ such that

$$\mathcal{D}^p := \{t \in \mathbb{R} \mid (t,p) \in \mathcal{D}\} = (a_p, b_p),$$

with $a_p < 0 < b_p$, for all $p \in M$.

A **local flow of X** is a smooth map $\theta : \mathcal{D} \to M$ respecting for all well-defined $t, s \in \mathbb{R}$ and for all $p \in M$:

(i) $\theta(0,p) = p$;

(ii) $\theta\left(s, \theta(t,p)\right) = \theta(s+t, p)$;

(iii) $\frac{\partial}{\partial t}\Big|_{t=0} \theta(t,p) = X(p)$.

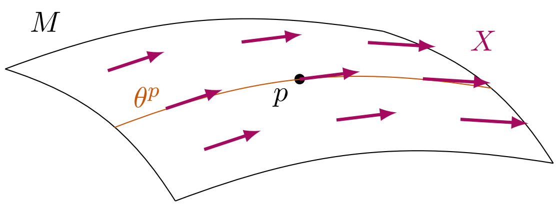
In the case $\mathcal{D} = \mathbb{R} \times M$, we speak of the **global flow of X**. Then, (i) and (ii) imply that θ becomes a smooth action of $\mathbb{R}$ over M, and θ generates a group homomorphism :

$$\begin{array}{rcccl} \tilde{\theta} & : & (\mathbb{R},+) & \to & (\mathrm{Diff}(M), \circ) \\ & & t & \mapsto & \begin{array}{rcccl} \theta_t & : & M & \to & M \\ & & p & \mapsto & \theta(t,p) \end{array} \end{array},$$

where $\mathrm{Diff}(M)$ is the diffeomorphism group of M.

Remark A.7. For all $p \in M$, the following is an integral curve :

$$\begin{array}{rcccl} \theta^p & : & (a_p, b_p) & \to & M \\ & & t & \mapsto & \theta(t,p). \end{array}$$



Definitions A.8 - ϕ-related, push-forward & ϕ-invariant

Let $\phi : M \to N$ be smooth.

Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are **ϕ-related** if for all $p \in M$:

$$d\phi_p\big(X(p)\big) = Y\big(\phi(p)\big).$$

Now suppose ϕ to be a diffeomorphism. Given $X \in \mathfrak{X}(M)$, we construct $\phi_* X \in \mathfrak{X}(N)$, called the **push-forward vector field**, which is the unique ϕ-related vector field to X:

$$(\phi_* X)\left(\phi(p)\right) := d\phi_p\big(X(p)\big), \qquad \forall p \in M.$$

For a diffeomorphism $\phi : M \to M$, $X \in \mathfrak{X}(M)$ is said **ϕ-invariant** if $\phi_* X = X$.

Proposition A.9 - Naturality of Lie Brackets

Let $\phi : M \to N$ be a smooth map ;
$X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ such that X_i is ϕ-related to Y_i, $i = 1, 2$.

Then $[X_1, X_2]$ is ϕ-related to $[Y_1, Y_2]$.

Proof A.9: *See [Lee12, Proposition 8.30].*

■

Corollary A.10 - Push-forward of Lie Brackets

Let $\phi : M \to N$ be a diffeomorphism ;
$X_1, X_2 \in \mathfrak{X}(M)$.

Then,

$$\phi_*[X_1, X_2] = [\phi_* X_1, \phi_* X_2].$$

Proof A.10:

Let $p \in M$.

Define $Y_i := \phi_* X_i$ the ϕ-related vector field to X_i, $i = 1, 2$.

Then

$$\begin{aligned}
\phi_*[X_1, X_2]\big(\phi(p)\big) &= d\phi_p\big(\,[X_1, X_2]\,(p)\big) \\
&\overset{Prop.\ A.9}{=} \big[d\phi_p(X_1), d\phi_p(X_2)\big](p) \\
&= \big[\phi_* X_1, \phi_* X_2\big]\big(\phi(p)\big).
\end{aligned}$$

■

A.3 Riemannian metric

Definitions A.11 - Riemannian metric & isometry

A **(Riemannian) metric** h **on M** is a map associating an inner product h_p on T_pM to each $p \in M$, for which $p \mapsto h_p\left(X(p), Y(p)\right)$ is smooth for all $X, Y \in \mathfrak{X}(M)$.
(M, h) is called a **Riemannian manifold**.

We sometimes use the norm $\|v\|_h := \sqrt{h(v, v)}$, for $v \in T_pM$.

Given an immersion $\phi : M \to N$, a metric h^N on N induces the **pullback metric** $h^M = \phi^* h^N$ **on M** :

$$\left(\phi^* h^N\right)_p (v, w) := \left(h^N\right)_{\phi(p)} \big(d\phi_p(v), d\phi_p(w)\big),$$

for all $p \in M$, $v, w \in T_pM$.

We define an **isometry** as a diffeomorphism $\phi : (M, h^M) \to (N, h^N)$ satisfying

$$h^M = \phi^* h^N.$$

The **isometry group of** (M, h) equipped with the map composition $\circ$ is

$$\mathrm{Iso}(M, h) := \{\phi : (M, h) \to (M, h) \text{ isometry}\}.$$

Note A.12. One often deletes the index p in the function $h_p(\ ,\ )$ when there is no possible confusion.

Examples A.13 - Riemannian manifolds

Common Riemannian manifolds are :

- $(\mathbb{R}^n, h_{euc})$ where $h_{euc}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}$ is called the *euclidean metric* ;
- $(\mathbb{H}^n, h_{hyp})$ where $h_{hyp_{(a_1,\dots,a_{n-1},b)}} = \frac{1}{b^2} \cdot h_{euc_{(a_1,\dots,a_{n-1},b)}}$ is called the *hyperbolic metric* ;
- $(\mathbb{S}^{n-1}, \iota^* h_{euc})$ for the canonical embedding $\iota : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$.

Proposition A.14

Every smooth manifold admits a Riemannian metric.

Proof A.14:

Idea : Use partition of unity and local charts arguments.

See [do Carmo, p.43] for details.

■

Analogously to smooth functions, we can derivate a metric in the direction of a vector field.

Definition A.15 - Lie derivative of a metric

Let (M, h) be a Riemannian manifold;
$V \in \mathfrak{X}(M)$.

We define the **Lie derivative of the metric h according to V** as follows :

$$(\mathcal{L}_V h)(X, Y) := V\big(h(X, Y)\big) - h\big([V, X], Y\big) - h\big(X, [V, Y]\big).$$

A.4 Covariant derivative and geodesic

This section focus on curves on a Riemannian manifold (M, h), and more particularly the curves minimizing the distance between two points, called geodesics. This requires some kind of derivation notion of vector fields along curves based on the following notion :

Definitions A.16 - AFFINE CONNECTION

An **(affine) connection on** M is any $\mathbb{R}-$bilinear map

$$\begin{array}{rcccc} \nabla & : & \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathfrak{X}(M) \\ & & (X, Y) & \mapsto & \nabla_X Y \ , \end{array}$$

satisfying $\forall X, Y, Z \in \mathfrak{X}(M), \forall f, g \in C^\infty(M)$:

(i) $\nabla_{f \cdot X + g \cdot Y} Z = f \cdot \nabla_X Z + g \cdot \nabla_Y Z$;

(ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$;

(iii) $\nabla_X (f \cdot Y) = f \nabla_X Y + Xf \cdot Y$ (**Leibniz Rule for** ∇).

Furthermore, ∇ is called :

- **compatible with the metric h** if

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z), \qquad \forall X, Y, Z \in \mathfrak{X}(M) \ ;$$

- **symmetric** if

$$\nabla_X Y - \nabla_Y X = [X, Y], \qquad \forall X, Y, Z \in \mathfrak{X}(M).$$

Theorem A.17 - LEVI-CIVITA

For a fixed Riemannian manifold (M, h), there exists a unique affine connection ∇ beeing compatible with h and symmetric. One names it **Levi-Civita connection**.

Proof A.17:

*Such a connection ∇ should satisfy the **Koszul formula**, which determines uniquely ∇ :*

$$\begin{array}{rcl} h(\nabla_X Y, Z) & = & \frac{1}{2}\{Yh(X, Z) + Xh(Z, Y) - Zh(Y, X) \\ & & -h([Y, Z], X) - h([X, Z], Y) - h([Y, X], Z)\}, \end{array} \qquad \forall X, Y, Z \in \mathfrak{X}(M).$$

$\blacksquare$

Remark A.18. Through the whole document, the "L-C"-mention may be implicit, since we only employ L-C connections.

Definition A.19 - Vector field along a curve

Let $\gamma : (a, b) \to M$ a curve on M.

A **vector field along** γ is a smooth map $V : (a, b) \to TM$ such that $V(t) \in T_{\gamma(t)}M$ $\forall t \in [a, b]$. The simplest example of such elements is the derivative of the curve $\gamma'(t)$.

One will denote by $\mathfrak{X}(\gamma)$ the set of vector fields along γ and keep in mind that such a map can be viewed as a restriction of a vector field $\widetilde{V} \in \mathfrak{X}(M)$ on $\gamma([a, b]) \subset M$:

$$V(t) = \widetilde{V}\big(\gamma(t)\big), \qquad \forall t \in [a, b].$$

Definition A.20 - Covariant derivative & parallel vector field

Let ∇ the Levi-Civita connection on (M, h) ;
$\gamma : [a, b] \to M$ a curve.

A **covariant derivative** on (M, h) is a map

$$\begin{array}{cccc} \frac{D}{dt} & : & \mathfrak{X}(\gamma) & \to & \mathfrak{X}(\gamma) \\ & & V & \mapsto & \frac{DV}{dt} \end{array},$$

such that the following holds for all $V \in \mathfrak{X}(\gamma)$:

(i) $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt} \qquad \forall W \in \mathfrak{X}(\gamma)$;

(ii) $\frac{D(fV)}{dt} = f\frac{DV}{dt} + \frac{df}{dt}V \qquad \forall f \in C^\infty\big([a, b]\big)$;

(iii) for vector fields extensions $\widetilde{V}, \widetilde{\gamma}' \in \mathfrak{X}(M)$ of V, respectively $\gamma'(t)$:

$$\frac{DV}{dt}(t) = \left(\nabla_{\widetilde{\gamma}'(t)}\widetilde{V}\right)\big(\gamma(t)\big), \qquad \forall t \in (a, b).$$

A vector field along γ, $V \in \mathfrak{X}(\gamma)$, is called **parallel** if $\frac{DV}{dt} = 0$, i.e. if its covariant derivative is the trivial vector field along γ.

The following type of curves is a key notion when studying Riemannian manifolds, because of their useful properties :

Definition A.21 - Geodesic

We call **geodesic** a curve $\gamma : (a, b) \to M$ if the derivative vector field $\gamma'(t)$ along γ is *parallel*, i.e. $\nabla_{\widetilde{\gamma}'}\widetilde{\gamma}'\big(\gamma(t)\big) = 0_{\gamma(t)} \in T_{\gamma(t)}M$, $\forall t \in (a, b)$.

Theorem A.22 - Existence and uniqueness of geodesics

Given $p \in M$ and $v \in T_pM$, there exists a unique geodesic (up to the definition set) $\gamma_v^p : (-\epsilon, \epsilon) \to M$ with $\gamma_v^p(0) = p$ and $\gamma_v^{p\prime}(0) = v$.

Proof A.22: *See for example [Car92, Lemma 2.3].* ∎

Definition A.23 - Exponential map of a Riemannian manifold

Let $p \in M$.

If $\Omega_p \subseteq T_pM$ refers to the set of vectors v one which $\gamma_v^p(1)$ is defined, the **exponential map** of M at p is :

$$\begin{array}{rcccc} \exp_p & : & \Omega_p \subseteq T_pM & \to & M \\ & & v & \mapsto & \gamma_v^p(1) \end{array}$$

Theorem A.24

For all $p \in M$, $\exp_p$ is a local diffeomorphismus from a neighborhood of $0_p \in T_pM$ to a neighborhood of $p \in M$.

Proof A.24: *See for example [GHL04, Proposition 2.88].* ■

Remark A.25. For Lie groups with a biinvariant metric, this definition of exponential map is equivalent to the Lie exponential map introduced at the beginning of Chapter 2 (Definition 2.2).

Properties A.26 - Geodesics

(i) Since ∇ is compatible with h, the norm of γ' is constant :

$$\frac{d}{dt}h\big(\gamma'(t),\gamma'(t)\big) = 2h\Big(\underbrace{\frac{D\gamma'}{dt}(t)}_{=0},\gamma'(t)\Big) = 0, \qquad \forall t \in [a,b].$$

That's why we often reparametrize the curve to obtain a **normal** geodesic, with norm of the derivative equal to 1.

(ii) Geodesics are curves minimizing distances between two points : for a normal ball $B \subset M$ (exponential image of a ball in T_pM, $p \in M$), if a geodesic $\gamma : [a,b] \to M$ is contained in B, then every curve $\alpha : [a,b] \to M$ joining $\gamma(a)$ and $\gamma(b)$ is longer than γ :

$$\text{length}(\gamma) := \int_a^b \|\gamma'(t)\|dt \leqslant \int_a^b \|\alpha'(t)\|dt =: \text{length}(\alpha).$$

(iii) All *compact* Riemannian manifolds (M,h) are **geodesically complete**, i.e. every geodesic is defined on the whole $\mathbb{R}$. It allows the exponential map $\exp_p$ to be defined on the whole tangent space T_pM, for all $p \in M$.

A.5 Sectional curvature

Locally, two Riemannian manifolds of the same dimension look topologically identical. However, the L-C connection ∇ leads to a useful map R permitting to distinguish them through their *curvature sec.*

Definition A.27 - Curvature tensor

The **curvature tensor of $(\mathbf{M}, \mathbf{h})$** is :

$$\begin{array}{rcccc} R & : & \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathfrak{X}(M) \\ & & (X, Y, Z) & \mapsto & R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,. \end{array}$$

As for the Euclidean space $(\mathbb{R}^n, h_{euc})$ $R \equiv 0$, we may interpret the curvature tensor as how the analyzed manifold deviates from the Euclidean case.

In fact, it appears that the vector $(R(X, Y)Z)(p) \in T_pM$ only depends on the vectors $X(p), Y(p), Z(p) \in T_pM$. This observation leads to the well-definition of the induced map :

$$R_p : T_pM \times T_pM \times T_pM \to T_pM.$$

Simple computations leads to a few useful characteristics :

Properties A.28

Let $X, Y, Z, W \in \mathfrak{X}(M)$.

(i) R is $C^\infty(M)$-tri-linear ;

(ii) **Bianchi identity** : $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$;

(iii) $h\left(R(X, Y)Z, W\right) = -h\left(R(Y, X)Z, W\right)$;

(iv) $h\left(R(X, Y)Z, W\right) = -h\left(R(X, Y)W, Z\right)$;

(v) $h\left(R(X, Y)Z, W\right) = h\left(R(Z, W)X, Y\right)$.

Definitions A.29 - Sectional curvature & space form

Let $p \in (M, h)$;
$\Pi_p = \mathrm{span}(v, w) \subset T_pM$ a 2-plane, i.e. a 2-dimensional vector subspace of T_pM with basis (v, w).

The **sectional curvature of $\mathbf{\Pi_p}$** refers to the following real value :

$$sec(v, w) = sec(\Pi_p) := \frac{k(v \wedge w)}{\|v \wedge w\|_h^2} := \frac{h(R(v, w)w, v)}{h(v, v) \cdot h(w, w) - h(v, w)^2}.$$

The denominator makes the sectional curvature of Π_p invariant to its basis.

Interpretation of sec :

- If $sec(v, w) \geqslant 0$, the distance between geodesics starting at the same point and with initial velocities in $span(v, w)$ grows slower than in the flat space $(\mathbb{R}^n, h_{euc})$;
- If $sec(v, w) \leqslant 0$, the geodesics move away from each other faster than in the euclidean case.

One calls a **space form** a Riemannian manifold having a constant sectional curvature, i.e. $sec \equiv c \in \mathbb{R}$.

Lemma A.30

For all $\lambda > 0$,

$$sec_{\lambda h} = \frac{1}{\lambda} sec_h.$$

Proof A.30:

Observe by the Koszul formula, that (M,h) and $(M,\lambda h)$ owns the same L.-C. connection :

$$\nabla^h = \nabla^{\lambda h}.$$

As a consequence, the curvature tensors are the same :

$$R_h = R_{\lambda h}.$$

Consider now $v, w \in T_pM$, for any $p \in M$ and compute :

$$\begin{aligned} sec_{\lambda h}(v,w) &= \tfrac{(\lambda h)\left(R_{\lambda h}(v,w)w,v\right)}{(\lambda h)(v,v)\cdot(\lambda h)(w,w)-(\lambda h)(v,w)^2} \\ &= \tfrac{\lambda}{\lambda^2}\tfrac{h(R(v,w)w,v)}{h(v,v)\cdot h(w,w)-h(v,w)^2} \\ &= \tfrac{1}{\lambda} sec_h(v,w). \end{aligned}$$

■

Examples A.31 - Space forms

The three common Riemannian manifolds in Examples A.13 are all space forms :

- $sec\ \mathbb{R}^n \equiv 0$ for h_{euc} ;
- $sec\ \mathbb{H}^n \equiv -1$ for h_{hyp} ;
- $sec\ \mathbb{S}^{n-1} \equiv 1$ for $\iota^* h_{euc}$.

Theorem A.32 - Classification of space forms, Hopf 1926

Let (M,h) be a *complete simply connected* Riemannian manifold, i.e. :

- complete : the exponential map exp_p is defined on all T_pM for all $p \in M$;
- simply connected : M is path-connected and the fundamental group (= set of homotopy classes) $\Pi_1(M) = \{id_M\}$.

If $sec\ M \equiv c$, for $c \in \mathbb{R}$, then :

- if $c = 0$, (M,h) is isometric to $(\mathbb{R}^n, h_{euc})$;
- if $c < 0$, $(M, \frac{1}{-c}h)$ is isometric to $(\mathbb{H}^n, h_{hyp})$;
- if $c > 0$, $(M, \frac{1}{c}h)$ is isometric to $(\mathbb{S}^{n-1}, \iota^* h_{euc})$.

Note A.33. If a complete Riemannian manifold doesn't fulfill the simple connectivity, the result holds for its universal covering $\bar{M}$ with the covering metric. This Riemannian manifold always exists according to [Lee12, Corollary 4.43].

Proof A.32: *See [GHL04, Theorem 3.82].*

■

Let's now see the particular case of a product manifold.

Product manifold

From two Riemannian manifolds, we can construct a new one :

Definition A.34 - Metric on a product manifold

Let (M_1, h_1) and (M_2, h_2) be Riemannian manifolds.

A possible metric on the product manifold $M_1 \times M_2$ is $h_1 + h_2$, simply defined as

$$(h_1 + h_2)\big((v_1, v_2), (w_1, w_2)\big) := h_1(v_1, w_1) + h_2(v_2, w_2),$$

for all $(v_1, v_2), (w_1, w_2) \in M_1 \times M_2$.

We call $(M_1 \times M_2, h_1 + h_2)$ **Riemannian product of M_1 and M_2.**

The curvature tensor R of $(M_1 \times M_2, h_1 + h_2)$ will take a form involving the ones R_1 of (M_1, h_1) and R_2 of (M_2, h_2) :

Proposition A.35 - Curvature tensor of a product manifold

Let $(p_1, p_2) \in M_1 \times M_2$;
$v := (v_1, v_2), w := (w_1, w_2), \zeta := (\zeta_1, \zeta_2) \in T_{(p_1,p_2)}(M_1 \times M_2)$.

Then :

(i) $R_1(v_1, w_1)\zeta_1 = R\big((v_1, 0), (w_1, 0)\big)(\zeta_1, 0)$;

(ii) $R_2(v_2, w_2)\zeta_2 = R\big((0, v_2), (0, w_2)\big)(0, \zeta_2)$;

(iii) $R(v, w)\zeta = \Big(R_1(v_1, w_1)\zeta_1, R_2(v_2, w_2)\zeta_2\Big)$,

if we identify any $\xi_1 \in T_{p_1}M_1$ with $(\xi_1, 0) \in T_{(p_1,p_2)}(M_1 \times M_2)$ and any $\xi_2 \in T_{p_2}M_2$ with $(0, \xi_2) \in T_{(p_1,p_2)}(M_1 \times M_2)$.

Proof A.35:

It comes out of the following identity, we easily prove with the Koszul formula :

$\nabla_X Y = \big(\nabla^1_{X_1} Y_1, \nabla^2_{X_2} Y_2\big)$, *for any* $X := (X_1, X_2), Y := (Y_1, Y_2) \in \mathfrak{X}(M_1 \times M_2)$.

See [AK03, Section 2] for details.

■

A.6 Jacobi fields

A Jacobi field measures how geodesics differ from each other in an infinitesimal way. In other words, they are variations of geodesics by geodesics. It is constructed through the derivative of a parametrized surface.

We fix a metric h on M.

Definition A.36

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a geodesic.

A vector field $J \in \mathfrak{X}(\gamma)$ is called **Jacobi field** if it satisfies the *Jacobi equation* for all $s \in (-\epsilon, \epsilon)$:

$$\frac{D^2 J}{ds^2} + R\big(\gamma'(s), J(s)\big)\gamma'(s) = 0.$$

Construction of Jacobi fields

Let $p \in M$;
$\tilde{v} \in T_pM$;
$w \in T_{\tilde{v}}T_pM \cong T_pM$;
$v : (a, b) \to M$ the geodesic in T_pM with $v(0) = \tilde{v}$ and $v'(0) = w$.

We define a parametrized surface on M :

$$\begin{array}{cccc} f & : & (-\epsilon, \epsilon) \times (a, b) & \to & M \\ & & (s, t) & \mapsto & \exp_p\big(s \cdot v(t)\big) . \end{array}$$

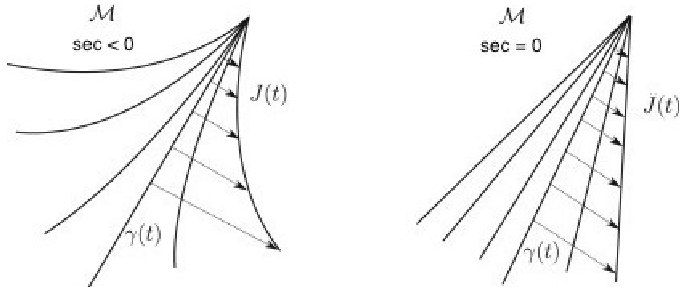
Then,

$$\begin{array}{cccc} J & : & (-\epsilon, \epsilon) & \to & TM \cong T_vTM \\ & & s & \mapsto & \left.\frac{\partial f}{\partial t}\right|_{t=0}(s, t) = (d\exp_p)_{s\cdot v}(s \cdot w) , \end{array}$$

is a Jacobi field along $\gamma_{\tilde{v}}^{p}$. See [Car92] for details.

Properties of Jacobi fields

Jacobi fields are used to describe how parallel vector fields change along geodesics on space forms and therefore how geodesics move away from each others :



Proposition A.37

Let γ be a geodesic on M ;
$V \in \mathfrak{X}(\gamma)$ a *parallel* vector field.

If (M, h) is a space form with $sec =: k \in \mathbb{R}$, then

$$J(s) := \begin{cases} \frac{\sin(s\cdot\sqrt{k})}{\sqrt{k}} \cdot V(s) & \text{if } k > 0 \ ; \\ s \cdot V(s) & \text{if } k = 0 \ ; \\ \frac{\sin(s\cdot\sqrt{-k})}{\sqrt{-k}} \cdot V(s) & \text{if } k < 0 \, . \end{cases}$$

forms a Jacobi field along γ.

A.7 Ricci curvature and scalar curvature

Information contained in the curvature tensor R can be summarized in new curvature tools :

Definition A.38 - Ricci curvature

Let $p \in M$;
$(e_1, ..., e_n)$ an orthonormal basis of T_pM ;
$v, w \in T_pM$.

The **Ricci curvature of span$(\mathbf{v}, \mathbf{w})$** is defined as the following real value :

$$Ric_p(v, w) := tr\,(h(R(\cdot, v)w, \cdot)) = \sum_{i=1}^{n} h(R(e_i, v)w, e_i).$$

The Ricci curvature is a *symmetric* and *bilinear* form.

One often looks at the Ricci curvature of one vector v :

$$Ric_p(v) := Ric_p(v, v).$$

Definition A.39 - Scalar curvature

Let $p \in M$;
$(e_1, ..., e_n)$ be an orthonormal basis of T_pM.

The **scalar curvature** is a function $scal : M \to \mathbb{R}$ given by the trace of Ric viewed as a tensor on T_pM :

$$scal(p) := tr(Ric_p) = \sum_{i=1}^{n} Ric_p(e_i) = 2 \sum_{1 \leqslant i < j \leqslant n} sec(e_i, e_j).$$

The last equality comes from the orthonormality of $(e_i)_{i=1,\dots,n}$ and

$$sec(e_i, e_i) = 0 \ \forall i = 1, ..., n.$$

Appendix B

Lie groups

This appendix focus on a very particular type of manifolds : the Lie groups. These topological objects possess properties that may notably affect their metrics and define interesting actions on smooth manifolds.

The elements defined here come for the most part from the two Lie groups lecture taught by Dr. Oliver Baues respectively Prof. Anand Dessai at the University of Fribourg during the fall semesters of 2016 and 2018 respectively ([Bau16] and [Des18]).

B.1 Basic recalls

Definitions B.1 - Lie group, & 1-parameter subgroup

A smooth manifold G with a group structure is called a **Lie group** if the internal law and the inverse operations are smooth maps.

In this category, the **Lie groups morphisms** $\phi : G_1 \to G_2$ are smooth group homomorphisms. When $G_1 = \mathbb{R}$, we speak of a **1-parameter subgroup of G** for ϕ.

Examples B.2 - Lie groups

(i) Every n-dimensional $\mathbb{R}$-vector space - for instance $(\mathbb{R}^n, +)$ - is an n-dimensional Lie group.

(ii) A torus $T^n := \underbrace{S^1 \times ... \times S^1}_{n \text{ factors}} \cong \mathbb{R}^n / \mathbb{Z}^n$ has a Lie group structure.

(iii) The isometry group $(\text{Iso}(M, h), \circ)$ of a Riemannian manifold (M, h) and any of its closed subgroups are Lie groups, according to the Myers-Steenrod Theorem [AB15, Theorem 2.12]. The last section of this appendix will focus on the 1-parameter subgroups of isometries which generate *Killing vector fields.*

Definitions B.3 - Lie brackets & Lie algebra

Let K be a field ;
V a K-vector space.

A **Lie product** or **Lie bracket** is any K-bilinear map $[\cdot, \cdot] : V \times V \to V$ satisfying $\forall X, Y, Z \in V$:

- *Anti-symmetry* : $[X, Y] = -[Y, X]$;
- *Jacobi identity* : $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$.

$X, Y \in V$ **commute** if $[X, Y] = 0$.

V equipped with this new operation becomes a **Lie algebra**, that we often denote by a lower-case fraktur letter : $\mathfrak{g} := (V, [\cdot, \cdot])$. In the Appendix A, the vector fields set $(\mathfrak{X}(M), [\cdot, \cdot])$ illustrates this concept.

A linear map between Lie algebras $\psi : (\mathfrak{g}_1, [\cdot, \cdot]_1) \to (\mathfrak{g}_2, [\cdot, \cdot]_2)$ is a **Lie algebra homomorphism** when $\psi([X, Y]_1) = [\psi(X), \psi(Y)]_2$, $\forall X, Y \in \mathfrak{g}$.

Given a Lie group G, we define the **Lie algebra of G** as the tangent space of G at the neutral element e :

$$\mathfrak{g} := \text{Lie } G := T_e G,$$

together with the Lie bracket being analog to the one for $\mathfrak{X}(G)$ after identification of $T_e G$ with specific vector fields, which we present just below.

B.2 Left- and right-invariant vector fields

Definition B.4 - Left-/right-invariant vector field

Let $g \in G$ and define the left-translation map $L_g : G \to G$ by $L_g(h) := g \cdot h$.

A vector field $X \in \mathfrak{X}(G)$ is said **left-invariant** if it is L_g-invariant for all $g \in G$, i.e. $(L_g)_* X = X \ \forall g \in G$.

We denote by $\mathfrak{X}(G)^L$ the set of left-invariant vector fields on G.

Analogously, we define the right-translation map R_g and **right-invariant** vector fields composing the set $\mathfrak{X}(G)^R$.

Remark B.5. Every $x \in \mathfrak{g}$ can be identified to a unique left-invariant (resp. right-invariant) vector field X_L (resp. X_R) :

$$X_L(e) := x,$$
$$X_L(g) := (dL_g)_e (x).$$

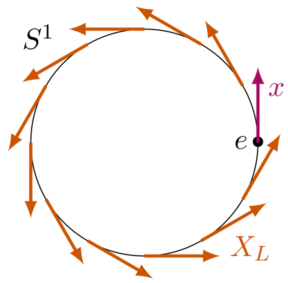
This observation reveals that $\mathfrak{g} = T_e G \cong \mathfrak{X}(G)^L \cong \mathfrak{X}(G)^R$.

In this context, Lie brackets on $T_e G$ are defined as follows for any $x, y \in T_e G$:

$$[x, y]_{T_e G} := [X_L, Y_L]_{\mathfrak{X}(G)}(e) = \big(X_L Y_L - Y_L X_L\big)(e).$$

Example B.6 - Left-invariant vector field

On $(S^1, \cdot)$, left-invariant vector fields take the following form :



The Lie brackets of left- and right-invariant vector fields generated by the same tangent vectors are linked :

Proposition B.7

Let G be a Lie group ;
$g \in G$;
$x, y \in \mathfrak{g}$.

Then,

$$[X_L, Y_L](e) = -[X_R, Y_R](e).$$

Proof B.7: *Based on [Ban06].*

We consider :

- *the inversion map :*

$$\begin{array}{cccc} I & : & G & \to & G \\ & & g & \mapsto & I(g) := g^{-1} \ ; \end{array}$$

- *the multiplication map :*

$$\begin{array}{cccc} \mu & : & G \times G & \to & G \\ & & (g_1, g_2) & \mapsto & g_1 \cdot g_2 \, . \end{array}$$

① *Claim : the differential of μ is the sum of differential of right- and left-translations. i.e. $d\mu_{(g_1,g_2)}(\xi_1, \xi_2) = dR_{g_1}(\xi_1) + dL_{g_2}(\xi_2)$, $\forall g_1, g_2 \in G$, $\forall \xi_1 \in T_{g_1}G, \xi_2 \in T_{g_2}G$.*

Let $g_1, g_2 \in G$;
$\xi_1 \in T_{g_1}G$;
$\xi_2 \in T_{g_2}G$.

We define the parametrized curve

$$\begin{array}{cccc} \alpha & : & \mathbb{R} & \to & G \times G \\ & & t & \mapsto & \alpha(t) := (c_1(t), c_2(t)) \, , \end{array}$$

with $c_1, c_2 : \mathbb{R} \to G$ such that

$$c_1(0) = g_1 \ ; \quad c_1'(0) = \xi_1 \ ;$$
$$c_2(0) = g_2 \ ; \quad c_2'(0) = \xi_2 \, .$$

Then,

$$(\mu \circ \alpha)(t) = \mu\left((c_1(t), c_2(t))\right) = c_1(t) \cdot c_2(t).$$

Hence,

$$\begin{aligned} d\mu_{(g_1,g_2)}(\xi_1, \xi_2) &= (\mu \circ \alpha)'(0) \\ &= \left(dR_{c_2(0)}\right)_{c_1(0)}(c_1'(0)) + \left(dL_{c_1(0)}\right)_{c_2(0)}(c_2'(0)) \\ &= \left(dR_{g_2}\right)_{g_1}(\xi_1) + \left(dL_{g_1}\right)_{g_2}(\xi_2). \end{aligned}$$

✓

② *Claim : the differential of I is $dI_g(\xi) = -(dL_g)^{-1}_{I(g)} \circ (dR_{I(g)})_g(\xi)$, for $g \in G, \xi \in T_gG$.*

Let $g \in G$;
$\xi \in T_gG$.

By definition of I, $\mu(g, I(g)) = e$. Therefore, by construction,

$$\begin{aligned} 0 &= d\mu_{(g,I(g))}(\xi, dI_g(\xi)) \\ &\overset{①}{=} \left(dR_{I(g)}\right)_g(\xi) + (dL_g)_{I(g)}\left(dI_g(\xi)\right). \end{aligned}$$

Hence,

$$dI_g(\xi) = -\left(dL_g\right)^{-1}_{I(g)} \circ \left(dR_{I(g)}\right)_g(\xi).$$

✓

Let $x \in T_eG = \mathfrak{g}$ and define X_L and X_R as in the proposition :

$$\begin{aligned} X_L(g) &= (dL_g)_e\,(x) \\ X_R(g) &= (dR_g)_e\,(x) \end{aligned} \qquad \forall g \in G.$$

③ *Claim : Push-forward of left-invariant vector field by I is its opposite : $I_*X_L = -X_R$*

Since I is a diffeomorphism, the push-forward vector field I_*X_L is well-defined.
Evaluated at $g \in G$, we obtain :

$$\begin{aligned} (I_*X_L)(g) &\overset{Def.\ A.8}{=} dI_{I^{-1}(g)}\left(X_L\left(I^{-1}(g)\right)\right) \\ &\overset{②}{=} -\left(\underbrace{\left(dL_{g^{-1}}\right)_e^{-1}}_{(dL_g)_e} \circ (dR_g)_{g^{-1}}\right)\left(X_L(g^{-1})\right) \\ &= -\left((dR_g)_e \circ (dL_g)_{g^{-1}}\right)\left(X_L(g^{-1})\right) \\ &= -\left(dR_g\right)_e\left(\underbrace{X_L(g \cdot g^{-1})}_{x}\right) \\ &= -X_R(g). \end{aligned}$$

✓

Let's now denote by $[X, Y]_R$ the right-invariant vector field generated by $[X_L, Y_L](e)$.
Rewrite also $[X, Y]_L := [X_L, Y_L]$ since it stays left-invariant.

④ $\underline{[X,Y]_R = -[X_R, Y_R]}$

$$\begin{aligned} -[X,Y]_R &\overset{③}{=} I_*[X,Y]_L \\ &\overset{Cor.\_A.10}{=} [I_*X_L, I_*Y_L] \\ &\overset{③}{=} [-X_R, -Y_R] \\ &= [X_R, Y_R]. \end{aligned}$$

✓

⑤ *Finally, at the identity $e \in G$:*

$$[X_R, Y_R](e) \overset{④}{=} -[X,Y]_R(e) = -[X_L, Y_L](e).$$

✓

■

B.3 Action of a Lie group on a smooth manifold

Definitions B.8 - Smooth action & Orbital space

Consider a Lie group $(G, *)$;
a smooth manifold M.

A **smooth (left) action of G on M**, written $G \curvearrowright M$, consists of a map

$$\mu : G \times M \to M,$$

fulfilling three properties :

(i) $\forall g \in G$ the map $\mu_g : M \to M, p \mapsto g \cdot p := \mu_g(p)$ is a diffeomorphism ;

(ii) $\mu_e = id_M$;

(iii) $\mu_{g_1} \circ \mu_{g_2} = \mu_{g_1 * g_2}$ $\forall g_1, g_2 \in G$.

ϕ induces an **equivalence relation on M** :

$$p \sim q \Leftrightarrow \exists g \in G \text{ such that } \mu_g(p) = q.$$

We define :

- G as a **transformation group of M** ;
- the **isotropy group at p** : $G_p := \{g \in G \mid g \cdot p = p\}$;
- the **orbit of p** : $[p] := G \cdot p = \{q \in M \mid p \sim q\}$;
- the **orbit map of p** : $\begin{array}{cccc} \mu^p & : & G & \to & M \\ & & g & \mapsto & g \cdot p \end{array}$;
- the **quotient space** or **orbital space**: ${}^{M}\!/_{G} := \{[p] \mid p \in M\}$;
- the **quotient map** as the canonical projection map $\begin{array}{cccc} \pi & : & M & \to & {}^{M}\!/_{G} \\ & & p & \mapsto & [p] \end{array}$.

The action is called :

- **effective** if $g \cdot p = p \; \forall p \in M \Rightarrow g = e$, i.e. $\bigcap_{p\in M} G_p = \{e\}$;
- **free** if $\forall p \in M : g \cdot p = p \Rightarrow g = e$, i.e. G_p are trivial $\forall p \in M$, which is a particular case of an effective action ;
- **transitive** if for each pair $p, q \in M$, there exists a $g \in G$ with $p = g \cdot q$.
  As a consequence, the quotient space contains only the neutral element $[e]$;
- **proper** if the map $\psi : G \times M \to M \times M, (g,p) \to (g \cdot p, p)$ is proper,
  i.e. if $K \subset M \times M$ is compact, then $\psi^{-1}(K)$ is also compact.

Let's now endow M with a metric h.

Definitions B.9 - Isometric action & homogeneous manifold

h is said **G-invariant metric** when $\mu_g \in \mathrm{Iso}(M,h) \; \forall g \in G$. In such a case, we speak of an **isometric action** or say that **G acts by isometries** on (M,h).

We call any Riemannian manifold (M,h) **homogeneous** if such a G exists and if G acts transitively on M. h is then called **homogeneous metric**.

B.4 Left- and right-invariant metrics

A Lie group $(G,\cdot)$, being in particular a smooth manifold and thus Riemannian, admits metrics. Due to the group structure of G, some of them take a particular form, whose properties may be useful in Riemannian geometry.

Definition B.10 - left-, right- & biinvariant metric

A metric l on G is called **left-invariant** if every left translation L_g is an isometry regarding l. In other words, l should satisfy :

$$l_{g_2}(X(g_2), Y(g_2)) = l_{g_1\cdot g_2}\Big(\left(dL_{g_1}\right)_{g_2} \left(X(g_1 \cdot g_2)\right), \left(dL_{g_1}\right)_{g_2} \left(Y(g_1 \cdot g_2)\right)\Big),$$

$$\forall g_1, g_2 \in G, \forall X, Y \in \mathfrak{X}(G).$$

We define a **right-invariant metric** r analogously.

If a metric is simultaneously right- and left-invariant, we name it **biinvariant**. We often denote such metric by b.

Remark B.11. Any Lie group $(G,\cdot)$ admits a left-, respectively right-invariant metric. It suffices to choose an inner product $\langle \cdot,\cdot \rangle_e$ on T_eG and enlarge it to the whole tangential bundle :

$$l_g(X(g), Y(g)) := \langle \left(dL_{g^{-1}}\right)_g (X(e)), \left(dL_{g^{-1}}\right)_g (Y(e)) \rangle_e,$$

$$\forall g \in G, \forall X, Y \in \mathfrak{X}(G).$$

Nonetheless, we can't construct a bi-invariant metric on any Lie group.

Proposition B.12

If a Lie group G is compact, then it owes a biinvariant metric b.

Proof B.12:

When a Lie group G is compact, it exists a unique left- and right-invariant integral on $C^\infty(G)$. Let's denote it by $\int_G \cdot\, \omega$, for any volume form $\omega \in \Omega^n(G)$. Considering a right-invariant metric r on G, we define b as follows :

$$b_{\tilde{g}}(X, Y) := \int_G r_{g*\tilde{g}}\left((dL_g)_{\tilde{g}}(X), (dL_g)_{\tilde{g}}(Y)\right) \omega,$$

$$\forall \tilde{g} \in G, \forall X, Y \in \mathfrak{X}(G).$$

The arguments that b is biinvariant can be found in [AB15, Proposition 2.24].

■

Lemma B.13

A Lie group G with a biinvariant metric b satisfies:

$$b\left(X, [Y, Z]\right) = b\big(\left[X, Y\right], Z\big),$$

$$\forall X, Y, Z \in \mathfrak{X}(G)^L.$$

Proof B.13: *See [AB15, Proposition 2.26 (i)]*

■

Probably the most interesting property of a Lie Group equipped with a biinvariant metric is its non-negative curvature which comes out of the following results :

Proposition B.14

Let (G, b) be a Lie group with a biinvariant metric;
$X, Y, Z \in \mathfrak{X}(G)^L$.

Then :

(i) $\nabla_X Y = \frac{1}{2}\left[X, Y\right]$;

(ii) $R(X, Y)Z = -\frac{1}{4}\big[[X, Y], Z\big]$;

(iii) $b\left(R(X, Y)Y, X\right) = \frac{1}{4}\|\left[X, Y\right]\|^2 \geqslant 0$.

Proof B.14:

<u>Ad (i)</u> : Let's consider the Koszul formula for any $Z \in \mathfrak{X}(G)^L$:

$$\begin{aligned} b(\nabla_X Y, Z) \quad = \quad & \tfrac{1}{2}\{Y b(X, Z) + X b(Z, Y) - Z b(Y, X) \\ & - b([Y, X], Z) - b([Y, Z], X) - b([X, Z], Y)\}. \end{aligned}$$

Since X, Y, Z are left-invariant vector fields and b is a biinvariant metric, the smooth maps $b(X,Z)$, $b(Z,Y)$ and $b(Y,X)$ are constant on M. Hence, the first three terms vanish.

Previous identity becomes

$$\begin{aligned} b(\nabla_X Y, Z) &= \tfrac{1}{2}\{-b([Y,X],Z) - b([Y,Z],X) - b([X,Z],Y)\} \\ &\overset{\text{Lemma B.13}}{=} \tfrac{1}{2}\{-b([Y,X],Z) + \cancel{b(Z,[Y,X])} + \cancel{b(Z,[X,Y])}\} \\ &\overset{\substack{\text{Anti-symmetry of} \\ \text{Lie brackets}}}{=} \tfrac{1}{2}b([X,Y],Z), \end{aligned}$$

This equality holds for all $Z \in \mathfrak{X}(G)^L$. Therefore,

$$\nabla_X Y = \frac{1}{2}[X,Y].$$

✓

Ad (ii) : The equality ensues from the definition of the curvature tensor :

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &\overset{(i)}{=} \tfrac{1}{2}\nabla_X [Y,Z] - \tfrac{1}{2}\nabla_Y [X,Z] - \tfrac{1}{2}[[X,Y],Z] \\ &\overset{(i)}{=} \tfrac{1}{4}[X,[Y,Z]] - \tfrac{1}{4}[Y,[X,Z]] - \tfrac{1}{2}[[X,Y],Z] \\ &\overset{\text{Rearranging}}{=} \tfrac{1}{4}\underbrace{\{[[X,Z],Y] + [[Z,Y],X] + [[Y,X],Z]\}}_{=0 \text{ by Jacobi identity}} + \tfrac{1}{4}[[Y,X],Z] \\ &= -\tfrac{1}{4}[[X,Y],Z]. \end{aligned}$$

✓

Ad (iii) :

$$\begin{aligned} b(R(X,Y)Y,X) &\overset{(ii)}{=} -\tfrac{1}{4}b([[X,Y],Y],X) \\ &= \tfrac{1}{4}b([XY,Y],X) - b([YX,Y],X) \\ &\overset{\text{Lemma B.13}}{=} -\tfrac{1}{4}b(XY,[Y,X]) + b(YX,[Y,X]) \\ &= -\tfrac{1}{4}b([X,Y],[Y,X]) \\ &= -\tfrac{1}{4}b([X,Y],[X,Y]) \\ &= \tfrac{1}{4}\|[X,Y]\|^2. \end{aligned}$$

✓

■

Corollary B.15

Let (G,b) be a Lie group with a biinvariant metric ;
$g \in G$.

Then every 2-plane of $T_g G$ has a non-negative sectional curvature.

Proof B.15: *Direct consequence of (iii) in Proposition B.14.*

■

B.5 Killing vector fields

In a word, Killing vector fields preserve the metric and distances of the Riemannian manifold on which they are defined. Let's now see what it means mathematically, supposing that every integral curve is defined on the whole $\mathbb{R}$, the other case being analog:

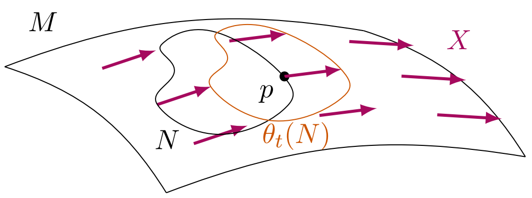
Definition B.16 - Killing vector field

A **Killing vector field** on a Riemannian manifold (M, h) describes a vector field $X \in \mathfrak{X}(M)$ whose flow is a 1-parameter subgroup of $\mathrm{Iso}(M, h)$. That is, referring to Definition A.6, the global flow $\theta : \mathbb{R} \times M \to M$ induces

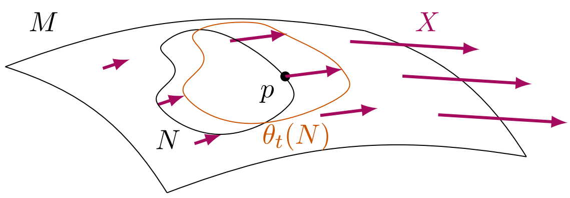
$$\begin{array}{rcccc} \tilde{\theta} & : & (\mathbb{R}, +) & \to & (\mathrm{Iso}(M), \circ) \\ & & t & \mapsto & \begin{array}{rcccc} \theta_t & : & M & \to & M \\ & & p & \mapsto & \theta(t, p)\,. \end{array} \end{array}$$

We intuitively interpret X as a "*displacement field*", in that if we consider any $N \subset M$, the map $\theta_t\big|_N$ preserves distances for any $t \in \mathbb{R}$:

$$d(p, q) = d\big(\theta_t(p), \theta_t(q)\big), \qquad \forall p, q \in M.$$



X Killing vector field



X not Killing vector field

Other characterizations of a Killing vector field, like the vanishing Lie derivative $\mathcal{L}_X h$ of h in the direction X, are explained in [Pet98, p. 164 and following], [Kob95, p. 42 and following] or [GHL04, Example 2.62].

The action fields studied in Chapter 2 are a typical example of Killing vector fields.

Bibliography

[AB15] Marcos M. Alexandrino and Renato G. Bettiol. *Lie Groups and Geometric Aspects of Isometric Actions.* Springer, 2015.

[AK03] M. Atceken and S. Keles. *On the product Riemannian manifolds.* Differential Geometry - Dynamical Systems (electronic journal), vol. 5. 2003.

[Ban06] Ravi N. Banavar. *Left and right-invariant vector fields.* Lecture notes. 2006.

[Bau16] Oliver Baues. *Lie Gruppen und Lie Algebren.* Unpublished lecture notes, University of Fribourg. 2016.

[Ber03] Marcel Berger. *A Panoramic View of Riemannian Geometry.* Springer, 2003.

[Bre72] Glen E. Bredon. *Introduction to compact transformation groups.* Pure and applied mathematics, a series of monographs and textbooks 46. Academic Press, 1972.

[Bum04] Daniel Bump. "Lie groups". In: Graduate Texts in Mathematics 225 (2004).

[Car92] Manfredo P. do Carmo. *Riemannian Geometry.* 1st. Birkhäuser Boston, 1992.

[Che73] Jeff Cheeger. *Some examples of manifolds of nonnegative curvature.* J. Differential Geom., Vol. 8, No 4, pp. 623-628. 1973.

[CS18] Leonardo F. Cavenaghi and Llohann D. Sperança. *Positive Ricci curvature through Cheeger deformation.* `https://arxiv.org/pdf/1801.06576.pdf`. 2018.

[Des18] Anand Dessai. *Lie groups.* Unpublished lecture notes, University of Fribourg. 2018.

[DG19] Anand Dessai and David González Álvaro. *Moduli space of metrics of nonnegative sectional or positive Ricci curvature on homotopy real projective spaces.* `https://arxiv.org/pdf/1902.08919.pdf`. 2019.

[GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry.* 3rd. Universitext. Springer, 2004.

[Gon17] David González Álvaro. *Riemannian Geometry.* Unpublished lecture notes, University of Fribourg. 2017-18.

[GW69] Detlef Gromoll and Meyer Wolfgang. *On complete open manifolds of positive curvature.* Annals of Mathematics, Vol. 90, pp.75-90. 1969.

[Kaw91] Katsuo Kawakubo. *The Theory of Transformation groups.* OUP Oxford, 1991.

[Kob95] Shoshichi Kobayashi. *Transformation groups in differential geometry.* Springer classics in mathematics. Springer, 1995.

[Lee12] John M. Lee. *Introduction to Smooth Manifolds*. 2nd. Graduate Texts in Mathematics 218. Springer-Verlag New York, 2012.

[Loh94] Joachim Lohkamp. *Metrics of negative Ricci curvature*. Annals of Mathematics, Series 2, Vol. 140, No 3., pp. 655-683. 1994.

[LY74] H. Blaine Jr. Lawson and Shing Tung Yau. *Scalar Curvature, Non-Abelian Group Actions, and the Degree of Symmetry of Exotic Spheres*. Commentarii Mathematici Helvetici, Vol. 49, pp. 232-244. 1974.

[Mou17a] Lawrence G. Mouillé. *Cheeger deformation*. `https://lawrencemouille.wordpress.com/2017/02/27/cheeger-deformations/`. Accessed on 2019-09-09. 2017.

[Mou17b] Lawrence G. Mouillé. *Cheeger deformation for S^1 action on $\mathbb{C}$*. `https://drive.google.com/file/d/0B9eKX5HNgjEEZGs5YjBoUlJmQlE/view`. 2017.

[Mü87] Michael Müter. "Krümmungserhöhende Deformationen mittels Gruppenaktionen". PhD thesis. Universität Münster, 1987.

[Pet98] Peter Peterson. *Riemannian Geometry*. 2nd ed. Graduate Texts in Mathematics 171. Springer New York, 1998.

[SA08] Flausino Lucas Neves Spindola and M. Alexandrino Alexandrino Marcos (sup.) "Grupos de Lie, ações próprias e a conjectura de Palais-Terng". MA thesis. Universidade de São Paulo, 2008.

[Vog15] David Vogan. *Introduction to Lie Groups, Isometric Actions and some Generalizations*. Lecture notes, `http://www-math.mit.edu/~dav/actionsvec.pdf`. 2015.

[Zil06] Wolfgang Ziller. *On M. Mueter's Ph.D. Thesis on Cheeger deformations*. Lecture notes. 2006.