Cours on Lie groups

(fall 2020)

Below you find an incomplete list of DEFINITIONS, *examples* and theorems from the course on Lie groups:

REVIEW: TOPOLOGICAL MANIFOLDS, SMOOTH MANIFOLDS, SMOOTH MAPS, SUBMANIFOLDS, REGULAR VALUE

pre-image of a regular value is a submanifold (without proof)

LIE GROUPS, HOMOMORPHISM BETWEEN LIE GROUPS

G a Lie group, $H \subset G$ a subgroup which is also a submanifold $\Longrightarrow$ H is a Lie group

S^1, T^n, $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $O(n)$, $U(n)$, $SO(n)$, $SU(n)$, symplectic group $Sp(n)$, dimension of these Lie groups

REVIEW: TANGENT VECTORS, DERIVATIONS, TANGENT SPACE, TANGENT BUNDLE, DIFFERENTIAL OF A SMOOTH MAP, VECTOR FIELDS

LEFT INVARIANT VECTOR FIELDS

Lie group is parallizable, Lie bracket $[X, Y]$ of two left invariant vector fields is left invariant

LIE ALGEBRA $(LG, [\ ,\ ])$, JACOBI IDENTITY, INTEGRAL CURVES, ONE-PARAMETER GROUPS, EXPONENTIAL MAP $\exp : LG \to G$

$(\exp_*)_0 = id_{LG}$, naturality of the exponential map, exp for a matrix group $G \subset GL(n,\mathbb{C})$

Lie algebras of classical groups, $gl(n,\mathbb{R}), gl(n,\mathbb{C}), sl(n,\mathbb{R}), o(n), so(n), u(n), su(n), sp(n)$

G connected $\Longrightarrow$ a homomorphism $f : G \to H$ is determined by $f_* : LG \to LH$.
G a Lie group and H a closed abstract subgroup $\Longrightarrow$ H is a submanifold (and, hence, H a Lie (sub)group).
G, H Lie groups and $f : G \to H$ a continuous homomorphism of groups $\Longrightarrow$ f is smooth (and, hence, a homomorphism of Lie groups).

ADJOINT $\mathrm{Ad}: G \to Aut(LG)$, $\mathrm{ad}: LG \to End(LG)$

$[X,Y] = \mathrm{ad}(X)(Y)$. G abelian $\implies [\ ,\ ] = 0$.
For G connected holds: G abelian $\iff \exp: LG \to G$ is surjective.
G abelian connected $\implies G \cong T^k \times \mathbb{R}^l$.
G compact abelian $\implies G \cong T^n \times B$ for B a finite abelian group.

COMPLEX G-REPRESENTATION $\rho: G \times V \to V$, REPRESENTATION SPACE V, ASSOCIATED HOMOMORPHISM $G \to Aut(V)$, MATRIX REPRESENTATION, COMPLEX GROUP RING $\mathbb{C}[G]$, COMPLEX G-MODULES, MORPHISM BETWEEN G-REPRESENTATIONS, ISOMORPHIC/EQUIVALENT REPRESENTATIONS, UNITARY REPRESENTATIONS

standard $U(n)$-representation $\mathbb{C}^n$, one-dimensional representations of S^1 and of T^n, complex adjoint representation, direct sum of representations, tensor product of representations, dual representation, conjugate representation, exterior algebra, symmetric algebra, representations $\Lambda^k V$, $S^k V$

SUBREPRESENTATION, IRREDUCIBLE REPRESENTATION, COMPLETELY REDUCIBLE REPRESENTATION

Schur's lemma. G abelian and V irreducible G-representation $\implies \dim V = 1$.

every unitary G-representation is completely reducible, G compact $\implies$ there exists a left invariant normalized integral $\int: C^0(G) \to \mathbb{R}$, every G-representation has a G-invariant scalar product and every G-representation is completely reducible.

CHARACTER, IRREDUCIBLE CHARACTER, CLASS FUNCTION

properties of the character: 2.19 and Theorem 2.20

character of S^1- and T^n-representations

$Irr(G;\mathbb{C})$, $d_W: Hom_G(V,W) \otimes W \to V$, MULTIPLICITY FOR AN IRREDUCIBLE REPRESENTATION IN A GIVEN REPRESENTATION

$d := \oplus d_W : \bigoplus_{W \in Irr(G;\mathbb{C})} Hom_G(V,W) \otimes W \to V$ is an isomorphism of G-representations.

From now on all groups are assumed to be compact!

The isomorphism type of a G-representation G is uniquely determined by its character.
$\langle \chi_V, \chi_V \rangle = 1 \implies V$ is irreducible.
V irreducible G-representation, W irreducible H-representation $\implies V \otimes W$ irreducible $(G \times H)$-representation.
Every $(G \times H)$-representation is sum of representations of the form $V \otimes W$, where V is an irreducible G-representation and W is an irreducible H-representation.

representations of $SU(2)$ and $SO(3)$

$Irr(SU(2); \mathbb{C}) = \{V_0, V_1, V_2, \ldots\}$, $Irr(SO(3); \mathbb{C}) = \{V_0, V_2, V_4, \ldots\}$
Clebsch-Gordan formulas

GROTHENDIECK CONSTRUCTION, (COMPLEX) REPRESENTATION RING $R(G)$

$R(S^1)$, $R(SU(2)) \cong \mathbb{Z}[V_1]$
G and H compact Lie groups $\implies R(G \times H) \cong R(G) \otimes R(H)$.

$R(T^r)$, $R(SU(2) \times S^1)$

orthogonality relations 3.1

REPRESENTATIVE FUNCTIONS, $\mathcal{T}(G; \mathbb{C})$

$\mathcal{T}(G; \mathbb{C})$ is a subalgebra of $C^0(G; \mathbb{C})$ and $\overline{\mathcal{T}(G; \mathbb{C})} = \mathcal{T}(G; \mathbb{C})$

$\mathcal{T}(S^1; \mathbb{C})$

SUPREMUM NORM $| \ |_1$, L^2-NORM $|| \ ||$, $L^2(G)$

(Peter-Weyl) G compact $\implies \mathcal{T}(G; \mathbb{C})$ dense in $(L^2(G), || \ ||)$.
$\mathcal{T}(G; \mathbb{C})$ dense in $(C^0(G; \mathbb{C}), | \ |_1)$.
The irreducible characters generate a dense subspace in the vector space of all continuous class functions (wrt. $| \ |_1$).
Every compact Lie group G admits a faithful representation.
Every compact Lie group G is isomorphic to a closed subgroup of $U(n)$ for some $n \gg 0$.

maximal torus of $U(n)$, $SO(n)$

MAXIMAL TORUS T OF G, TOPOLOGICAL GENERATOR, NORMALIZER $N(T)$, WEYL GROUP $W(G) := N(T)/T$

maximal tori exist
Suppose $H \subset G$ is a subgroup. Then: H is a maximal torus $\iff$ H is a maximal connected abelian subgroup,
$N(T)_e = T$.

$f : G/T \to G/T$, $f(xT) := gxT$, has a fixed point $xT \iff g$ is conjugated to an element in T.

Lefschetz fixed point formula (without proof)

Suppose G is connected and T is a maximal torus. Then:
$W(G)$ is a finite group
every element is conjugate to an element in T and
all maximal tori are conjugate to each other.

RANK $\mathrm{rk}(G)$ OF A COMPACT LIE GROUP

For G connected the exponential map is surjective.
For G connected, S a connected abelian subgroup of G and $g \in Z_G(S)$ exists a maximal torus which contains S and g.

Suppose G is connected, $\widetilde{T}$ is a maximal torus and S is a connected abelian subgroup. Then:
$Z_G(\widetilde{T}) = \widetilde{T}$,
$Z_G(S)$ is the union of all maximal tori which contain S and
the center $Z(G)$ is the intersection of all maximal tori.

For G connected the Weyl group acts effectively on T,
two elements $x, y \in T$ are conjugated in $G \iff \exists\, w \in W(G)$ with $w(x) = y$.

SPACE OF CONJUGACY CLASSES $Con(G)$

$\kappa : T/W \to Con(G)$ is a homeomorphism.

$R(T)^{W(G)} := \mathrm{im}(R(G) \to R(T))$

Suppose G is connected. Then: $R(G) \to R(T)^{W(G)}$ is an isomorphism and
$R(G)$ is isomorphic to a subring of $\mathbb{Z}[\lambda_1, \lambda_1^{-1}, \ldots, \lambda_r, \lambda_r^{-1}]$, where $r := \mathrm{rk}(G)$.

$R(T)^{W(G)}$ for $G = SU(2)$ and $G = U(n)$