Riemannian Geometry

Spring 2022

Below you find an incomplete list of key words for the material of the course on Riemannian geometry.

1. Smooth manifolds (review)

Topological space, open set, closed set, basis of a topology, open neighborhood, Hausdorff space, metric space, open ϵ-balls, standard topology of R^n, topological sum (disjoint union), Cartesian product, subspace topology, continuous map, homeomorphism,

n-dimensional topological manifold M, charts, change of charts, smooth atlas, smooth structure, smooth manifold, examples ($\mathbb{R}^n$, sphere S^n, torus T^n, topological sum, Cartesian product, open subsets of a smooth manifold $\dots$), smooth map, orientation, smooth submanifold, codimension, pre-image of a regular value is a smooth submanifold (without proof), examples (sphere, hyperboloid, torus, $GL_n(\mathbb{R})$, $SO(n)$ $\dots$)

2. Tangent bundle (in parts a review)

Tangent vector = equivalence classes of curves, tangent space, T_pM is a real vector space, algebra of germs of functions $\mathcal{E}(p)$, derivation $X : \mathcal{E}(p) \to \mathbb{R}$, at $p \in \mathbb{R}^n$ the partial derivatives $\partial/\partial x_1, \dots, \partial/\partial x_n$ form a basis of the vector space of derivations, differential h_* of a smooth function h between manifolds, description in terms of curves or in terms of derivations, every curve through $p \in M$ defines a derivation at p, T_pM is isomorphic to the vector space of derivations at p,
immersions, submersions, embeddings, tangent bundle TM of M, TM is a $2n$-dimensional smooth manifold (a smooth atlas $\{h_\alpha\}_\alpha$ for M defines a smooth atlas $\{(h_\alpha)_*\}_\alpha$ for TM), projection $\pi : TM \to M$,
vector field, local flow, integral curve, parallelizable manifold, Lie bracket of vector fields, Jacobi identity, description of Lie bracket in terms of flow, normal bundle of a submanifold

3. Riemannian manifolds

Riemannian metric (often just called metric), Riemannian manifold, standard (canonical) metric g_{can} on $\mathbb{R}^n$ (flat metric, flat Euclidean space) or on

an open subset of $\mathbb{R}^n$, induced metric on a submanifold of $\mathbb{R}^n$, product metric, metric g on $U \subset \mathbb{R}^n$ can be written in the form $\sum_{i,j} g_{ij} dx_i \otimes dx_j$, embedding theorems of Whitney and Nash (without proof),
length $L(c)$ of a smooth (or piecewise smooth) curve c, independence of parametrization, parametrization by arc length (unit speed curve), every curve admits a unit speed parametrization, pull-back metric with respect to a chart h, isometry, distance function $d(p, q)$ defined as infimum of the length of curves connecting p, q, for M connected $d : M \times M \to \mathbb{R}$ is a metric, the topology induced by the metric is the given topology of M (without proof), diameter diam, invariance under isometries, diameter is finite for M connected and compact,
construction of Riemannian metrics using partions of unity, every smooth manifold admits a Riemannian metric

4. Geodesics, connection/covariant derivative

Examples of covariant derivatives for flat $\mathbb{R}^n$ und for the round sphere S^n, lines (resp. big circles) solve the geodesic equation $\nabla_{\gamma'} \gamma' = 0$ for $\mathbb{R}^n$ (resp. S^n),
general covariant derivative/connection ∇, Levi-Civita (LC)-conection, metric connection, torsion free connection, on a Riemannian manifold there exists a unique LC-connection,
local description of a covariant derivative via Christoffel symbols Γ_{ij}^k, description of Γ_{ij}^k via the metric $\sum_{i,j} g_{ij} dx_i \otimes dx_j$, properties of Γ_{ij}^k, examples (∇, Γ_{ij}^k for flat $\mathbb{R}^n$, for the hyperbolic plane ...),
Riemannian submanifold $N \subset M$, relation between the L-C-connection of M and of N,
vector fields along a curve, covariant derivative D/dt along a curve, existence, uniqueness and properties of D/dt, parallel vector fields, parallel transport P, defining equation for geodesics, local description, examples (flat $\mathbb{R}^n$, round sphere, hyperbolic plane, flat torus ...), local isometry and its properties

5. Exponential map

Local existence of geodesics, the exponential map exp, $\Phi : \Omega \to M \times M$, $v \mapsto (\pi(v), \exp_{\pi(v)}(v))$, is a local diffeomorphism for some open neighborhood Ω of the zero-section $M \subset TM$, $\exp_x : B_\epsilon(0) \to M$ is a diffeomorphism for ϵ small,
Gauss lemma and reformulations, locally geodesics are shortest (=minimal)

connecting curves, shortest curves (locally or globally) are up to reparametrization geodesics, a unit speed curve c is a geodesic if and only if for any t there exists $\epsilon > 0$ such that $L(c_{|[t,t+\epsilon]}) = d(c(t), c(t+\epsilon))$, a piecewise smooth constant speed curve c is a geodesic if c is a minimal (=shortest) curve,
theorem of Hopf-Rinow, in a compact Riemannian manifold any two points can be joined by a minimal geodesic, normal coordinates, Christoffel symbols in normal coordinates

6. Excursion: Lie groups

Lie groups, homomorphism, theorem of Peter-Weyl (without proof), left (resp. right) invariant vector field, examples ($GL_n(\mathbb{C})$, $U(n)$) ...), left invariant metric, bi-invariant metric, example: $-\frac{1}{2}\mathrm{tr}(X \cdot Y)$ for $U(n)$, $\nabla_{\underline{X}}\underline{Y}$ and geodesics for bi-invariant metric, one-parameter subgroups algebraic exponential map

7. Curvature

$R(X,Y) : \Gamma(M) \to \Gamma M)$, properties of $R(X,Y)$, Riemannian curvature tensor $R : T_pM \times T_pM \times T_pM \to T_pM$, local description, first Bianchi identity, further properties, example: flat $\mathbb{R}^n$,
sectional curvature *sec*, formulas for ∇, R and *sec* under isometries, examples (flat $\mathbb{R}^n$, sectional curvature is constant for the round sphere or the hyperbolic plane), sectional curvature determines the curvature tensor (without proof), sectional curvature of Lie groups with bi-invariant metric is nonnegative, Hopf conjecture (without proof ☹)
isometric immersions $M \to \overline{M}$, second fundamental form $B(X,Y)$, H_η, self-adjoint endomorphism S_η, Gauss formula for sec_M and $sec_{\overline{M}}$, sectional curvature of round unit sphere is $+1$, principal curvatures for surfaces in $\mathbb{R}^3$, Gaussian curvature and relation to *sec*,

8. First and second variations, topological consequences

Energy E, variation H of a curve, variational field V, operator/covariant derivative $\overline{D}$ for vector fields along H, formulas for the first and second variation of energy and length, geodesics are critical points of the energy function under proper variations, relation between length and energy for curves and geodesics,
Ricci curvature, Theorem of Bonnet-Myers, fundamental group of a compact manifold with positive Ricci curvature is finite,

Theorem of Synge-Weinstein, fundamental group of an even dimensional orientable compact manifold with positive sectional curvature is trivial,

9. Jacobi fields

Variation of geodesic by geodesics, Jacobi fields, Jacobi equation,
relation between zeros of Jacobi fields and critical points of $\exp_p$,
Taylor-expansion of $|J(t)|^2$, geometric interpretation of sectional curvature in terms of Jacobi fields (spread of geodesics),
M complete of $sec \leq 0 \implies exp_p : T_pM \to M$ is a local diffeomorphism,
theorem of Cartan-Hadamard

Literature

Do Carmo: Riemannian Geometry, Birkhäuser
Gallot, Hulin, Lafontaine: Riemannian Geometry, Springer
Cheeger-Ebin: Comparison Theorems in Riemannian Geometry, North-Holland