Applications of elliptic genera to group actions and positive curvature

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Two fundamental classical operators

M oriented Riemannian manifold

signature operator $\rightsquigarrow$ index $sign(M)$ $\rightsquigarrow$ genus $sign : \Omega_*^{SO} \longrightarrow \mathbb{Q}$

M $Spin$-manifold

Dirac operator $\rightsquigarrow$ index $\hat{A}(M)$ $\rightsquigarrow$ genus $\hat{A} : \Omega_*^{SO} \longrightarrow \mathbb{Q}$

Equivariant operators

Let M be a smooth manifold and G a connected compact Lie group which acts smoothly on M (preserving the structure).

operator $\rightsquigarrow$ G-equivariant operator

index $\rightsquigarrow$ G-equivariant index

M oriented: signature $sign(M)$ $\rightsquigarrow$ equiv. signature $sign_G(M) \in R(G)$

M $Spin$: index of Dirac operator $\hat{A}(M)$ $\rightsquigarrow$ equiv. index $\hat{A}_G(M) \in R(G)$

Lefschetz fixed point formula for S^1-actions

Let M be a manifold with S^1-action, D an S^1-equivariant operator on M and $ind_{S^1}(D) \in R(S^1) = \mathbb{Z}[\lambda, \lambda^{-1}]$ the equivariant index.
By localizing the symbol of the operator one can associate to each connected component $X \subset M^{S^1}$ a local contribution $\nu_X(\lambda)$ of $ind_{S^1}(D)$.
$\nu_X(\lambda)$ is a meromorphic function with possible poles only in S^1, 0 or ∞.

Theorem (Lefschetz fixed point formula, Atiyah-Bott-Segal-Singer)

$ind_{S^1}(D)(\lambda) = \sum_X \nu_X(\lambda)$ *for any topological generator* $\lambda \in S^1$.

Corollary

S^1 *cannot act on an orientable manifold with precisely one fixed point.*

Idea of proof: Apply the Lefschetz fixed point formula to the signature operator.

Rigidity of the signature

Theorem (rigidity)

The signature is rigid, i.e. $sign_{S^1}(M)(\lambda) = sign(M)$ *for all* $\lambda \in S^1$*.*

Proof.

Apply the homotopy invariance of cohomology or the Lefschetz fixed point formula. □

The following properties are equivalent:

- The rigidity of the signature for S^1-actions.
- The rigidity of the signature for G-actions.
- (multiplicativity) $sign(E) = sign(M) \cdot sign(F)$ for any fibre bundle $E \to M$ with oriented fibre F and connected structure group G.

Signature and group actions

Two consequences of the rigidity:

- Let $\sigma \in S^1$ be the element of order 2 and $M^\sigma \circ M^\sigma$ a transversal self-intersection of the fixed point manifold M^σ.
  Then $sign(M) = sign(M^\sigma \circ M^\sigma)$. Proof:
  $sign(M) = sign_{S^1}(M)(\sigma) = sign(M^\sigma \circ M^\sigma)$. Then
  $sign(M) = sign(M^\sigma \circ M^\sigma)$.
- $sign_{S^1}(M) = sign(M^{S^1})$

The signature is the universal genus for any of these properties.

Idea of proof: $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \ldots]$.

An example

Example

Let $M = \mathbb{C}P^2$ and let $\lambda \in S^1$ act via

$$\lambda([z_0 : z_1 : z_2]) = [z_0 : \lambda \cdot z_1 : \lambda^2 \cdot z_2]$$

$$\implies \sigma([z_0 : z_1 : z_2]) = [z_0 : -z_1 : z_2]$$

$$\implies M^\sigma \cong \mathbb{C}P^1 \cup \mathbb{C}P^0$$

$$\implies sign(M^\sigma \circ M^\sigma) = sign(\mathbb{C}P^1 \circ \mathbb{C}P^1) = sign(pt) = 1$$

$$M^{S^1} = \{[1:0:0],[0:1:0],[0:0:1]\} \implies sign(M^{S^1}) = 1 - 1 + 1 = 1$$

Application to involutions

The identity $sign(M) = sign(M^\sigma \circ M^\sigma)$ directly yields

Corollary

$\dim M^\sigma < \frac{1}{2} \dim M \implies sign(M) = 0.$

These results are due to Hirzebruch and related to work of Conner-Floyd, Boardman ...

Signature and positive curvature

Betti number theorem (Gromov)

If M is of positive sectional curvature then the sum of the Betti numbers of M is bounded from above by a constant C which depends only on the dimension of M.

Corollary

M of positive sectional curvature $\implies |sign(M)| < C$.

Note: Positive Ricci curvature does not restrict the signature.

Examples of Sha-Yang and Perelman show: There are simply connected manifolds of positive Ricci curvature which do not admit a metric of positive sectional curvature.

Rigidity of the Dirac operator

Let M be a $Spin$-manifold with S^1-action.

$\hat{A}$-vanishing theorem (Atiyah-Hirzebruch)

If S^1 acts non-trivially on M then the equivariant $\hat{A}$-genus vanishes

$$\hat{A}_{S^1}(M) = \hat{A}(M) = 0.$$

Idea of proof: Apply the Lefschetz fixed point formula to the Dirac operator to conclude that the equivariant index $\hat{A}_{S^1}(M)$ extends to a holomorphic function on $\mathbb{C}$ which vanishes in ∞.

Conversely, if $\hat{A}(M) = 0$ then a multiple of M is $Spin$-bordant to a $Spin$-manifold with non-trivial S^1-action.

Rigidity of the Dirac operator

The following properties are equivalent:

- The vanishing of the equivariant $\hat{A}$-genus for non-trivial S^1-actions on $Spin$-manifolds.
- The vanishing of the equivariant $\hat{A}$-genus for non-trivial G-actions on $Spin$-manifolds.
- (multiplicativity) $\hat{A}(E) = 0$ for any fibre bundle $E \longrightarrow M$ with $Spin$-fibre and non-trivial connected structure group.

The $\hat{A}$-genus is the universal genus for any of these properties.
Idea of proof: $\Omega_*^{SO} \otimes \mathbb{Q} \cong \Omega_*^{Spin} \otimes \mathbb{Q} \cong \mathbb{Q}[K_3, \mathbb{H}P^2, \mathbb{H}P^3, \ldots]$

$\hat{A}$-genus and positive curvature

Theorem (Lichnerowicz)

If M is a $Spin$-manifold with positive scalar curvature then $\hat{A}(M) = 0$.

Theorem (Lichnerowicz-Hitchin)

If M is a $Spin$-manifold with positive scalar curvature then $\alpha(M) = 0$.

Rationally the converse is true (Gromov-Lawson).

Theorem (Stolz)

M simply connected $Spin$-manifold of dimension ≥ 5 with $\alpha(M) = 0 \implies M$ admits a Riemannian metric with positive scalar curvature.

Problem: Are there simply connected manifolds of positive scalar curvature which do not admit a metric of positive Ricci curvature?

The universally rigid genus for $Spin$-manifolds

The signature and the $\hat{A}$-genus are both rigid for connected group actions on $Spin$-manifolds.
A universal genus with this property is

$$\varphi_{univ} : \Omega_*^{SO} \to (\Omega_*^{SO}/I_*) \otimes \mathbb{Q},$$

where I_* is the ideal generated by fibre bundles with $Spin$-fibre, connected structure group and zero-bordant base.

Problem: Describe this universal genus geometrically.

This problem was studied in the 1980s by Landweber, Ochanine, Stong ...
... major breakthrough by Witten using the free loop space.

Orientation and $Spin$-structures on $\mathcal{L}M$

$\mathcal{L}M := Map(S^1, M)$ space of loops in M, M simply connected
$P \to M$ principal bundle with structure group G
$\rightsquigarrow \mathcal{L}P \to \mathcal{L}M$ principal bundle with structure group $\mathcal{L}G$

$\mathcal{L}M$ *orientable* $\overset{\text{def.}}{\iff}$ the structure group $\mathcal{L}SO$ can be reduced to the connected component of the identity.

$\mathcal{L}M$ orientable $\iff$ M is $Spin$

$\mathcal{L}M$ is $Spin$ $\overset{def.}{\iff}$ M is $Spin$ with $Spin$-structure $P \to M$ and the structure group $\mathcal{L}Spin$ of $\mathcal{L}P \to \mathcal{L}M$ can be lifted to the universal central extension by S^1.

$\mathcal{L}M$ is $Spin$ if M is a $String$-manifold, i.e. if M is $Spin$ and $\frac{p_1}{2}(M) = 0$.

Witten's heuristic

S^1 acts on $\mathcal{L}M$ by reparametrizing the loops.

$$(\mathcal{L}M)^{S^1} = \{ \text{ constant loops } \} = M$$

Witten's heuristic: Suppose

- $D^{\mathcal{L}M}$ is an S^1-equivariant operator on $\mathcal{L}M$.
- the S^1-equivariant index $ind_{S^1}(D^{\mathcal{L}M})$ could be defined.
- one could use the Lefschetz fixed point formula formally to the natural S^1-action on $\mathcal{L}M$ to compute from $ind_{S^1}(D^{\mathcal{L}M})$ an honest invariant of M.

Then $ind_{S^1}(D^{\mathcal{L}M})$ should be thought to be this invariant.

Witten's heuristic for the signature

M $Spin$-manifold $\rightsquigarrow \mathcal{L}M$ orientable
$D^{\mathcal{L}M}$ hypothetical S^1-equivariant signature operator on $\mathcal{L}M$
apply formally the Lefschetz fixed point formula to $ind_{S^1}(D^{\mathcal{L}M})$
$\rightsquigarrow$ one obtains the following honest invariant of M:
The *elliptic genus* (or *signature of the free loop space*) $sign(q, \mathcal{L}M)$
$sign(q, \mathcal{L}M)$ is a modular function of weight 0 for
$\Gamma_0(2) := \{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \bmod 2\}$.

The elliptic genus $sign(q, \mathcal{L}M)$

The elliptic genus can be expressed as a series of twisted signatures:

$$sign(q, \mathcal{L}M) = sign(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM)$$

$$= \sum_{m=0}^{\infty} sign(M, R_m) \cdot q^m = sign(M) + 2sign(M, TM) \cdot q + \dots,$$

where $S_t := \sum_i S^i \cdot t^i$ (resp. $\Lambda_t := \sum_i \Lambda^i \cdot t^i$) denotes the symmetric (resp. exterior) power operation,
$sign(M, E)$ is the index of the signature operator twisted with the complexified vector bundle $E \otimes \mathbb{C}$
and $R_0 = 1, R_1 = 2TM, R_2 = 2TM + TM \otimes TM + \Lambda^2 TM + S^2 TM, \dots$

Witten's heuristic for the Dirac operator

M $String$-manifold $\rightsquigarrow$ $\mathcal{L}M$ is $Spin$
$D^{\mathcal{L}M}$ hypothetical S^1-equivariant Dirac operator on $\mathcal{L}M$
apply formally the Lefschetz fixed point formula to $ind_{S^1}(D^{\mathcal{L}M})$
$\overset{Yoga}{\rightsquigarrow}$ the *Witten genus* $\varphi_W(M)$

$$\varphi_W(M) = q^{-\dim M/24} \cdot \hat{A}(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM)$$

$$= q^{-\dim M/24} \cdot (\hat{A}(M) + \hat{A}(M, TM) \cdot q + \hat{A}(M, TM + S^2 TM) \cdot q^2 + \ldots)$$

$\hat{A}(M, E) :=$ index of the Dirac operator twisted with $E \otimes \mathbb{C}$

$\varphi_W(M)$ is a modular function of weight 0 for $SL_2(\mathbb{Z})$

Rigidity of the elliptic genus $sign(q, \mathcal{L}M)$

Let M be an oriented manifold with smooth S^1-action. Then each twisted signature appearing in the definition of the elliptic genus refines to an S^1-equivariant twisted signature and $sign(q, \mathcal{L}M) \in \mathbb{Z}[[q]]$ refines to an S^1-equivariant elliptic genus $sign(q, \mathcal{L}M)_{S^1} \in R(S^1)[[q]]$

Rigidity theorem (Witten, Taubes, Bott-Taubes, Liu)

If M is a $Spin$-manifold then the elliptic genus is rigid, i.e. $sign(q, \mathcal{L}M)_{S^1}(\lambda) = sign(q, \mathcal{L}M)$ for all $\lambda \in S^1$.

Rigidity of the elliptic genus $sign(q, \mathcal{L}M)$

The following properties are equivalent:

- The rigidity of the elliptic genus for S^1-actions on $Spin$-manifolds.
- The rigidity of the elliptic genus for G-actions on $Spin$-manifolds.
- (multiplicativity) $sign(q, \mathcal{L}E) = sign(q, \mathcal{L}M) \cdot sign(q, \mathcal{L}F)$ for any fibre bundle $E \to M$ with $Spin$-fibre F and connected structure group.

The elliptic genus is the universal genus for any of these properties.
Idea of proof: Construct a sequence $(M_4, M_8, M_{12}, \ldots)$ of manifolds, where $M_4 = K_3$, $M_8 = \mathbb{H}P^2$ and each M_{4k}, $k \geq 3$, is an $\mathbb{H}P^2$-bundle over a zero-bordant base, such that

$$\Omega_*^{SO} \otimes \mathbb{Q} \cong \Omega_*^{Spin} \otimes \mathbb{Q} \cong \mathbb{Q}[M_4, M_8, M_{12}, \ldots].$$

Elliptic genus and group actions

Let M be a $Spin$-manifold with S^1-action, $\sigma \in S^1$ the involution and $M^\sigma \circ M^\sigma$ a transversal self-intersection.

Then $sign(q, \mathcal{L}M) = sign(q, \mathcal{L}(M^\sigma \circ M^\sigma))$.
Proof: $sign(q, \mathcal{L}M) = sign(q, \mathcal{L}M)_{S^1}(\sigma) = sign(q, \mathcal{L}(M^\sigma \circ M^\sigma))$.

The elliptic genus is the universal genus for this property.
Idea of proof: $\Omega_*^{SO} \otimes \mathbb{Q} \cong \Omega_*^{Spin} \otimes \mathbb{Q} \cong \mathbb{Q}[K_3, \mathbb{H}P^2, \mathbb{H}P^3, \ldots]$

The elliptic genus of homogeneous spaces

Recall that the elliptic genus is a series of twisted signatures

$$sign(q, \mathcal{L}M) = sign(M) + 2sign(M, TM) \cdot q + \dots$$

Strong rigidity theorem (Hirzebruch-Slodowy)

Let M be a homogeneous space. If M is $Spin$ then $sign(q, \mathcal{L}M) = sign(M)$.

Assume M is $Spin$ and S^1 acts on M with isolated fixed points.
Problem: Is $sign(q, \mathcal{L}M)$ strongly rigid, i.e. $sign(q, \mathcal{L}M) = sign(M)$?

Expansion in the $\hat{A}$-cusp

The elliptic genus $sign(q, \mathcal{L}M)$ is the Fourier expansion of a modular function for $\Gamma_0(2)$ in one of its cusps (the signature cusp). In the other cusp (the $\hat{A}$-cusp) the Fourier expansion of this modular function is given by a series $\Phi_0(M)$ of indices of twisted Dirac operators.

$$\Phi_0(M) = q^{-\dim M/8} \cdot \hat{A}(M, \bigotimes_{n=2m+1>0} \Lambda_{-q^n} TM \otimes \bigotimes_{n=2m>0} S_{q^n} TM)$$

$$= q^{-\dim M/8} \cdot (\hat{A}(M) - \hat{A}(M, TM) \cdot q + \hat{A}(M, \Lambda^2 TM + TM) \cdot q^2 + \ldots).$$

Applications to involutions

The identity $sign(q, \mathcal{L}M) = sign(q, \mathcal{L}(M^\sigma \circ M^\sigma))$ implies

$$\Phi_0(M) = \Phi_0(M^\sigma \circ M^\sigma)$$

Comparing the pole order of the left and right side gives

Theorem (Hirzebruch-Slodowy)

If the codimension of M^σ is $> 4r$ then the first $(r+1)$ coefficients in the expansion $\Phi_0(M)$ vanish.

Example

codim $M^\sigma > 0 \Rightarrow \hat{A}(M) = 0$, codim $M^\sigma > 4 \Rightarrow \hat{A}(M, TM) = 0$...

This can be generalized to elements σ of arbitrary finite order.

The elliptic genus and positive curvature

Theorem (D.)

Let M be a 2-connected manifold of dimension $\neq 8$. If M admits a metric of positive sectional curvature with effective isometric S^1-action then the first two coefficients in the expansion $\Phi_0(M)$ vanish, i.e. $\hat{A}(M, TM) = \hat{A}(M) = 0$.

Theorem (D.)

Let M be a $Spin$-manifold of dimension $> 12r - 4$. Suppose M admits a metric of positive sectional curvature and an effective isometric action by a torus T of rank $2r$. Then the first $(r + 1)$ coefficients in the expansion $\Phi_0(M)$ vanish.

The elliptic genus and positive curvature

Corollary

There are simply connected manifolds of arbitrary large dimension with non-trivial S^1-action and small Betti numbers which admit an S^1-equivariant metric of positive Ricci curvature but no S^1-equivariant metric of positive sectional curvature.

The only known examples of manifolds of positive sectional curvature are quotients of Lie groups (homogeneous spaces or biquotients). For the $Spin$-examples the elliptic genus is strongly rigid.

Problem: Let M be a $Spin$-manifold of positive sectional curvature. Is $sign(q, \mathcal{L}M) = sign(M)$?

Rigidity of the Witten genus $\varphi_W(M)$

Let M be a $Spin$-manifold with smooth G-action. Then each twisted Dirac index appearing in the definition of the Witten genus

$$\varphi_W(M) = q^{-\dim M/24} \cdot \hat{A}(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM) \in q^{-\dim M/24} \cdot \mathbb{Z}[[q]]$$

$$\varphi_W(M) = q^{-\dim M/24} \cdot (\hat{A}(M) + \hat{A}(M, TM) \cdot q + \hat{A}(M, TM + S^2 TM) \cdot q^2 + \ldots)$$

refines to an G-equivariant twisted Dirac index and $\varphi_W(M)$ refines to the G-equivariant Witten genus $\varphi_W(M)_G \in q^{-\dim M/24} \cdot R(G)[[q]]$.

$\hat{A}$-vanishing theorem for the loop space (D., Höhn, Liu)

Let M be a $String$-manifold. If S^3 acts non-trivially on M then the equivariant Witten genus vanishes $\varphi_W(M)_{S^3} = \varphi_W(M) = 0$.

Problem: Does the Witten genus vanish for S^1-actions?

Rigidity of the Witten genus $\varphi_W(M)$

The following properties are equivalent:

- The vanishing of the equivariant Witten genus for non-trivial S^3-actions on $String$-manifolds.
- The vanishing of the equivariant Witten genus for non-trivial semi-simple group actions on $String$-manifolds.
- (multiplicativity) $\varphi_W(E) = 0$ for any fibre bundle $E \to M$ with $String$-fibre and semi-simple structure group.

The Witten genus is the universal genus for any of these properties.
Idea of proof: Construct a sequence $(M_4, M_8, M_{12}, \ldots)$ of manifolds, where $M_4 = K_3$, M_{4k}, $k \geq 2$, is $String$ and each M_{4k}, $k \geq 4$, is a Cayley plane bundle.

Witten genus and positive curvature

Stolz' conjecture

Let M be a $String$-manifold. If M admits a metric of positive Ricci curvature then the Witten genus $\varphi_W(M)$ vanishes.

The conjecture is known to be true for

- homogeneous spaces (apply the vanishing theorem)
- biquotients $G/\!\!/H$, where G is a simple Lie group (D.)
- certain Kähler manifolds with positive Ricci curvature including complete intersections in $\mathbb{C}P^n$ (Landweber-Stong) or in exceptional complex symmetric spaces (Förster).

Problem: Does the Stolz' conjecture hold for Kähler manifolds with positive Ricci curvature?

The universally rigid genus for $String$-manifolds

The elliptic genus and the Witten genus are multiplicative for any fibre bundle $E \to M$ with $String$-fibre and semi-simple structure group. A universal genus with this property is

$$\varphi_{univ} : \Omega_*^{SO} \to (\Omega_*^{SO}/I_*) \otimes \mathbb{Q},$$

where I_* is the ideal generated by fibre bundles with $String$-fibre, semi-simple structure group and zero-bordant base.

Problem: Describe this universal genus geometrically.