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Master's Thesis

Riemannian submersions and positive Ricci curvature

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Abstract

In this thesis, we study the existence of Riemannian metrics of positive Ricci curvature in the context of Riemannian submersions, reviewing the works of O'Neill, Pro & Wilhelm, Gromoll & Walschap, Besse, and additionally providing further details.

We first summarise curvature formulas in the case of a Riemannian submersion, including O'Neills' six fundamental equations, and curvature formulas when the total space is a warped product.

In the next section, we show how a Riemannian submersion $\pi\colon M \to B$ needs not necessarily transport positive Ricci curvature from M to B. This is done by following Pro & Wilhelm's example and constructing a warped product manifold $M = S^2 \times_\nu F$ on which the metric of S^2 is modified so that the submersion $\pi\colon S^2 \times_\nu F \to S^2 = B$ is Riemannian, $S^2 \times_\nu F$ has positive Ricci curvature and S^2 has points of negative Ricci curvature. However, Riemannian submersions do transport some parts of positive Ricci curvature to the base space, for it cannot have globally non-positive Ricci curvature.

We then study a known result about lifting positive Ricci curvature, namely how positive Ricci curvature can be lifted from the base space B to the total space M, with additional assumptions such as π being a fiber bundle, M and B being compact, and the metric of fibers F of the bundle being invariant under the action of the structure group.

We finally show a new result, that starts with the same context and assumptions as the previous result, namely a fiber bundle $\pi\colon E \to M$ with fibers F, $\mathrm{Ric}_F > 0$ and $\mathrm{Ric}_M > 0$, with the structure group acting isometrically on F, and we modify this setting so that the submersion remains Riemannian with $\mathrm{Ric}_E > 0$ but M admits a new metric with points of negative Ricci curvature.

Acknowledgments

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Contents

1 Introduction

In this paper, we give an overview of different questions and results that have been worked on by various authors, going back to when Barrett O'Neill ([O'N66]) first introduced his now well known curvature equations in the case of a Riemannian submersion. On every Riemannian manifold (M, g), there is a unique Levi-Civita connection ∇, often called *covariant derivative* (see Appendix B.2), which allows for differentiation of vector fields in the direction of other vectors, extending the directional derivative in the Euclidean case. This differential operator allows, roughly, to measure how "curved" the manifold is, in the following way: we define the *curvature tensor* R as[1]

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for X, Y, Z vector fields on the manifold, and the *sectional curvature* $K(P_{XY})$, where P_{XY} is a plane tangent to M spanned by the linearly independent vectors X and Y as

$$K(P_{XY}) = \frac{\langle R(X, Y)Y, X\rangle}{\|X \wedge Y\|^2}$$

with $\|X \wedge Y\|$ denoting the area of the parallelogram spanned by X and Y. In particular, if M is an embedded submanifold of some $\mathbb{R}^n$, then the sectional curvature equals the extrinsic notion of Gaussian curvature. A more general notion of curvature is given by the *Ricci curvature*, denoted Ric and given by the formula

$$\mathrm{Ric}_p(Y, Z) = \mathrm{tr}(X \mapsto R(X, Y)Z) = \sum_{k=1}^{m} \langle R(v_k, Y)Z, v_k\rangle$$

where the vectors $v_1, \ldots, v_m$, $m = \dim M$, form an orthornomal basis of the tangent space T_pM at $p \in M$. This is indeed more "general" than the sectional curvature, in the sense that, if a manifold has positive sectional curvature at a point, then it also admits positive Ricci curvature at this point; while the converse is not true (in the Ricci sum, some of the factors, which represent sectional curvatures of specific planes, might be negative but compensated by other factors to yield overall positive Ricci curvature).

Now, if (M, g) and $(B, \breve{g})$ are Riemannian manifolds and $\pi\colon M \to B$ is a submersion in the usual sense (a smooth map with surjective differential at all points), with "fibers" given by $\pi^{-1}(b) = F_b$, $b \in B$, then we call this submersion *Riemannian* if the restriction $(\pi_*)_p\colon T_pF_b^\perp \to T_{\pi(p)}B$ is a Riemannian isometry for all $p \in \pi^{-1}(b)$, where the orthogonal complement is taken with respect to the metric g on M. Riemannian manifolds that admit a metric of positive sectional curvature are of particular interest, as this eminently geometric feature gives rise to various topological properties, shown for example in Myers' or Synge's theorems, or even in the famous Hopf conjecture: Myers' theorem states that if a complete and connected n-manifold M has a lower bound for its Ricci curvature of the sort $\mathrm{Ric}_M \geq (n-1)\frac{1}{r^2}$ for some $r > 0$, then any two points can be joined by a geodesic of length at most $\pi \cdot r$, hence resulting in the compactness of M, and the finiteness of the fundamental group $\pi_1(M)$; Synge's theorem gives a conclusion on the orientability, simply connectedness, and ultimately the fundamental group of a manifold M provided it has positive sectional curvature.

[1] There is also a different sign convention for the curvature tensor, used for example by Arthur Besse [Bes87] and Barett O'Neill [O'N66].

Section 2 is a detailed review of O'Neill's paper [O'N66], *The fundamental equations of a Riemannian submersion*, with primary objective to lay the groundwork for this paper, and to prove the six curvature equations. These formulas allow to conclude, for example, that if $\pi\colon M \to B$ is a Riemannian submersion, and if M has positive sectional curvature, then B does as well; the same result holds for positive Ricci curvature provided that the fibers $\pi^{-1}(b)$, $b \in B$, are totally geodesic.

Section 3 covers Pro and Wilhelm's paper *Riemannian submersions need not preserve positive Ricci curvature* [PW14], which constructs a counterexample to the naive idea that Riemannian submersions transport positive Ricci curvature from M to B. We prove the theorem:

Theorem ([PW14, Theorem 1])**.** For any $C > 0$, there exists a Riemannian manifold M and a Riemannian submersion $\pi\colon M \to B$ for which M is compact with positive Ricci curvature, while B contains points with Ricci curvature less than $-C$.

We construct an example where M is a warped product of a sphere and a positive Ricci curvature space F, $M = S^2 \times F$, and the Riemannian submersion is given by the projection onto the first factor; careful modification of the standard "round metric" on S^2 is to be made, with additional conditions on the warping function, to preserve positive Ricci curvature upstairs while artificially creating points of negative Ricci curvature downstairs. A second and more global result in Pro & Wilhelm's paper says:

Theorem ([PW14, Theorem 2])**.** Let M be a Riemannian manifold that is compact with positive Ricci curvature. Then there exists no Riemannian submersion $\pi\colon M \to B$ to a space of nonpositive Ricci curvature.

This suggests that Riemannian submersions do transport some parts of positive Ricci curvature from the total space to the base space.

Section 4 asks the opposite question: given that B admits a metric of positive Ricci curvature, can we find a metric on M for which the submersion $\pi\colon M \to B$ is Riemannian and M has positive Ricci curvature? Under some conditions, namely in the case of a fiber bundle and with a group G acting isometrically on the fibers of the bundle it is indeed possible:

Theorem. Let M and F be compact Riemannian manifolds with positive Ricci curvature, and $\pi\colon E \to M$ a fiber bundle with fiber F and structure group G. If the metric on F is G-invariant, then E admits a metric with positive Ricci curvature such that π is a Riemannian submersion.

We also mention a different and very interesting approach of the same result: a different construction of this theorem is given in [CS22] by Cavenaghi and Sperança, where they use the discriminant of a polynomial whose coefficients are given by the different Ricci components, to control the curvature.

Overall, lifting positive Ricci curvature has been a topic of interest in the field of Riemannian geometry for the past few decades, as shown in the papers [Poo75] by Poor, [Nas79] by Nash, and [GPT98] by Gilkey, Park & Tuschmann, who present preliminary results to the theorem we prove in Section 4.

This also gives rise to a natural question, similar in spirit to the Pro-Wilhelm counterexample, summarised in the following theorem:

Theorem. Let $\pi\colon E \to M$ be a fiber bundle from a compact manifold E to a compact manifold M that admits a metric g_M with $\mathrm{Ric}_M > 0$. Suppose that the fibers F admit a metric such that the structure group G acts isometrically on F, and that $\mathrm{Ric}_F > 0$. Then, there is a metric $\tilde{g}$ of positive Ricci curvature on E, and a metric $\tilde{g}_M$ such that $\pi\colon (E, \tilde{g}) \to (M, \tilde{g}_M)$ is a Riemannian submersion, and the corresponding metric $\tilde{g}_M$ has points of negative Ricci curvature on M.

We discuss this new result in Section 5, where we again artificially create points of negative Ricci curvature on the base space by modifying the metric locally so that the base manifold has a small neighbourhood of sectional curvature constantly $+1$; on this neighbourhood that can be viewed as a disc of a sphere $D^n \subset S^n$ of radius 1, we perform a similar construction as Pro & Wilhelm's one to modify the metric of D^n, so that the bundle projection π is a Riemannian submersion over D^n with parts of negative Ricci curvature on D^n but overall positive Ricci curvature on $\pi^{-1}(D^n)$ and on E.

The very last part is dedicated to an appendix, in which we present various definitions and results that we use throughout this paper.

2 Riemannian submersions

2.1 Preliminaries

Let (M, g) and $(B, \breve{g})$ be Riemannian manifolds, and $\pi\colon M \to B$ be a smooth submersion. By the regular value theorem, the fibers $\pi^{-1}(b)$ are submanifolds of M for all $b \in B$ (we use the term "fibers", although the projection π need not necessarily be a fiber bundle, see Remark 2.8). The tangent space of M at a point p on the fiber $F_b := \pi^{-1}(b)$, T_pM, decomposes into the so called *vertical* and *horizontal* parts, namely T_pF_b and $(T_pF_b)^\perp$. This decomposition is canonical once we fix a Riemannian metric on M, making the horizontal part the orthogonal complement of the vertical part, with respect to the metric.

Definition 2.1 (Riemannian submersion)**.** Let (M, g) and $(B, \breve{g})$ be Riemannian manifolds, and $\pi\colon M \to B$ a submersion in the usual sense. If π induces a Riemannian isometry $(\mathrm{d}\pi)_p : (T_pF_b)^\perp \to T_{\pi(p)}B$, then we say that it is a *Riemannian submersion.*

Example 2.2. *Quotient maps.* A large class of Riemannian submersions arise from quotient manifolds under a certain action of a Lie group. Let G be a Lie group and (M, g) a Riemannian manifold. Suppose that G acts isometrically, freely and properly on M. Then, the orbit space $B = M/G$ is again a Riemannian manifold (see [Lee13, Theorem 9.16]). If $\pi\colon M \to B$ is the quotient map, and if $b \in B$, then the "fibers" $\pi^{-1}(b) = G \cdot p$, for $b = \pi(p)$, are diffeomorphic to G because the action is free and proper. Moreover, π is a submersion in the usual sense, i.e. $\mathrm{d}\pi\colon T_pM \to T_{\pi(p)}B$ is surjective. Then, one can always equip the quotient manifold with a metric such that $\mathrm{d}\pi\colon T_p(G \cdot p)^\perp \to T_{\pi(p)}B$ is an isometry: define the quotient metric g_B as

$$g_B(v, w) = g_M(\tilde{v}, \tilde{w})_p, \quad v, w \in T_bB,\ [b] \in G \cdot p$$

where $\tilde{v}$, $\tilde{w}$ are the unique horizontal lifts of v and w, i.e. $\tilde{v}, \tilde{w} \in (T_p(G\cdot p))^\perp$ and $\mathrm{d}\pi(\tilde{v}) = v$, $\mathrm{d}\pi(\tilde{w}) = w$.
Hence, for example, the well known Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$, with the action of $S^1 \subset \mathbb{C}$ on $S^3 \subset \mathbb{C}^2 = \mathbb{R}^4$ (with the induced round metric) given naturally by

$$e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$$

is an example of a Riemannian submersion: S^1 is a Lie group and the above action is free, proper and isometric. Thus, one can always equip S^2 with the unique quotient metric that yields an isometry $(\mathrm{d}\pi)_p\colon (T_pF_b)^\perp \subset T_pS^3 \to T_{\pi(p)}S^2$ from the horizontal part of T_pS^3 to the tangent space of the base $T_{\pi(p)}S^2$.

The question of interest in this section is the relationship between the curvature of the base space B, the fibers F_b of the submersion, and M; more specifically, if B has positive sectional or Ricci curvature, does it follow that M has positive curvatures too? And vice versa, if M has positive curvature, can one expect the submersion to carry this property to the base? We review Barrett O'Neill's paper [O'N66], Sections 1 to 4, and give additional results regarding the corresponding Ricci curvature equations.

Definition 2.3 (Horizontal and vertical distributions)**.** Let $\pi\colon M \to B$ be a Riemannian submersion. The *horizontal* and *vertical* distributions are the distributions $\mathcal{H}$ and $\mathcal{V}$ induced from π in the following way: at a point $p \in M$,

$$\mathcal{V}_p = \ker(\pi_*)_p, \quad \mathcal{H}_p = (\mathcal{V}_p)^\perp$$

where the orthogonal complement is taken with respect to the metric g_M on M. The fact that $\mathcal{V}$ is a distribution tangent to a submanifold of M is a consequence of the regular value theorem. However, in general, the horizontal distribution is not the distribution tangent to a submanifold of M, see Remark 2.12.

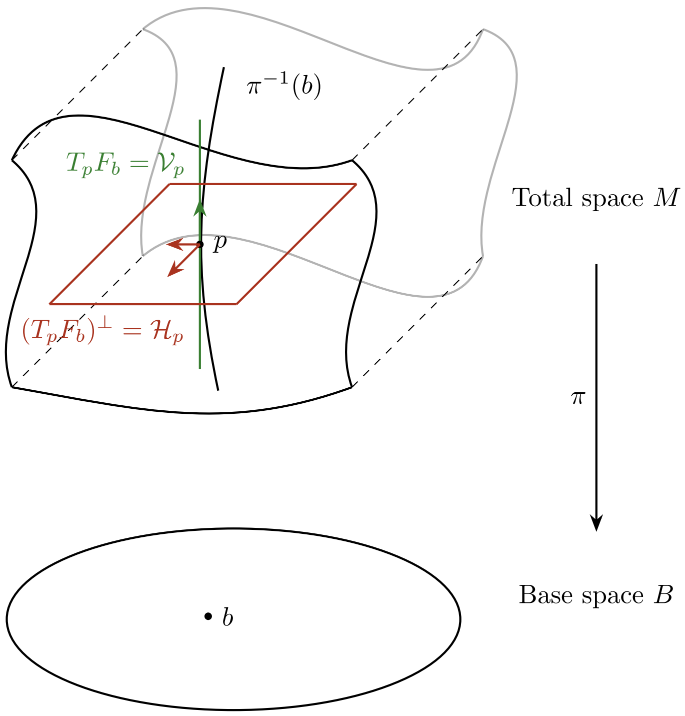

Figure 1: A Riemannian submersion $\pi\colon M \to B$, inducing a splitting of the tangent bundle TM into $\ker \pi_* \oplus \ker \pi_*^\perp = \mathcal{V} \oplus \mathcal{H}$, the vertical and horizontal spaces.

O'Neill defines two tensors of type $(1,2)$ (where we use the convention that a tensor of type (p,q) is a section of $TM^{\otimes p} \otimes T^*M^{\otimes q}$), the tensors T and A:

Definition 2.4 (A and T tensors)**.** Let E and F be arbitrary vector fields on the Riemannian manifold M. We define the (1,2)-tensors A and T as

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F),$$
$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F).$$

Here, ∇ is the Levi-Civita connection on M and $\mathcal{H}$, $\mathcal{V}$ denote respectively the projections onto the horizontal and vertical spaces. The same letters will denote the tangent distribution throughout the manifold. We have the following properties:

Lemma 2.5. *Let V and W be vertical vector fields, and let X be a horizontal vector field. Then,*

1. *At each point, T_E is a linear operator on the tangent space of M, and it reverses the horizontal and vertical subspaces.*

2. *T is vertical, i.e. $T_E = T_{\mathcal{V}E}$.*

3. *$T_V W = T_W V$.*

4. *$\langle T_V W, X\rangle = -\langle T_V X, W\rangle$.*

We will use $\langle\cdot,\cdot\rangle$ and $g(\cdot,\cdot)$ interchangeably; when there is possibly confusion between different metrics, we shall use $g(\cdot,\cdot)$, and otherwise prioritise $\langle\cdot,\cdot\rangle$.

Proof. For the first point, we show first that T_E is a linear operator; the fact that $T_E(F_1+F_2) = T_E F_1 + T_E F_2$ for F_1, F_2 arbitrary vector fields is immediate from the properties of the covariant derivative. Hence we show that $T_E(f\cdot F) = fT_E F$ for all $f \in \mathcal{C}^\infty(M)$. Let then F be a vector field and f a smooth function on the manifold. Then,

$$\begin{aligned} T_E(f\cdot F) &= \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}(f\cdot F)) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}(f\cdot F)) \\ &= \mathcal{H}(f\cdot\nabla_{\mathcal{V}E}\mathcal{V}F + D_{\mathcal{V}E}f\mathcal{V}F) + \mathcal{V}(f\cdot\nabla_{\mathcal{V}E}\mathcal{H}F + D_{\mathcal{V}E}f\mathcal{H}F) \\ &= f\cdot\mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + f\cdot\mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \\ &= f\cdot T_E F. \end{aligned}$$

Hence, T_E is $\mathcal{C}^\infty(M)$ linear. The second point if obvious. For the third one, we notice that, using torsion-freeness of the connection,

$$T_V W - T_W V = \mathcal{H}\nabla_V W - \mathcal{H}\nabla_W V = \mathcal{H}[V,W] = 0$$

because $[V,W]$ is again vertical since the distribution is integrable (see Theorem A.8; the vertical distribution is the distribution tangent to the fibers, which are submanifolds).

For the fourth property, we get

$$\langle T_V W, X\rangle = \langle\mathcal{H}\nabla_V W, X\rangle = \langle\nabla_V W, X\rangle = V\langle W, X\rangle - \langle W, \nabla_V X\rangle = -\langle W, T_V X\rangle,$$

using the metric compatibility and the fact that W and X are orthogonal. □

The T tensor relates to the second fundamental form in the following way: if E and F are vertical vector fields, then $T_E F$ reduces to $\mathcal{H}\nabla_E F$ which is exactly the projection onto the normal space of the vertical space, i.e. the horizontal space.

Furthermore, regarding A, one has:

Lemma 2.6. *Let X and Y be horizontal vector fields, and U a vertical vector field. Then,*

1. *At each point, A_E is a linear operator on the tangent space of M, and it reverses the horizontal and vertical subspaces.*

2. *A is horizontal, i.e. $A_E = A_{\mathcal{H}E}$.*

3. *$A_X Y = -A_Y X$.*

4. *$\langle A_X Y, U\rangle = -\langle A_X U, Y\rangle$.*

The third property isn't obvious and does not follow from an integrability argument, as the horizontal distribution is not necessarily integrable. Property 3 will be proved in Lemma 2.11. below, and properties 1 and 4 are the same as the T tensor.

We recall an important notion that will play a key role in simplifying formulas later on:

Definition 2.7 (Totally geodesic)**.** Let $\overline{M}$ be a Riemannian manifold with Levi-Civita connection $\overline{\nabla}$, and $M \subset \overline{M}$ a submanifold with Levi-Civita connection ∇. We say that M is *totally geodesic*, if every geodesic on M with respect to the induced metric is also a geodesic with respect to the metric on $\overline{M}$.

Remark 2.8. 1. This means in particular, if we note $\mathrm{II}\colon TM \times TM \to TM^{\perp}$ to be the second fundamental form, i.e. $\mathrm{II}(X,Y) := (\nabla_X Y)^{\perp} = \overline{\nabla}_X Y - \nabla_X Y$ (where $(\cdot)^{\perp}$ denotes the orthogonal projection onto the normal space), then this definition is equivalent to $\mathrm{II} \equiv 0$. In our case, we consider the family of submanifolds given by the fibers of the Riemannian submersion π. Similarly to the second fundamental form, we have that the fibers are totally geodesic if and only if $T \equiv 0$. To see this, note that as pointed above, if U, V are vertical, then $T_U V = \mathcal{H}\nabla_U V = \mathrm{II}(U,V)$.
2. As previously said, we use the word "fibers", while π might not be a fiber bundle projection. However, one has that $\pi\colon E \to M$ is a fiber bundle provided M is complete (see [GW09, Theorem 1.4.1]).

We also recall the Koszul formula: for all vector fields X, Y, Z on the manifold,

$$2g\left(\nabla_X Y, Z\right) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X).$$

This is particularly useful as the right-hand side only depends on the metric and its derivative, which can be easily controlled.

Definition 2.9 (Basic vector field)**.** Let X be a vector field on M. We call X *basic* if X is horizontal and π-related to a vector field X_* on B, i.e. $X_* \circ \pi = \pi_* \circ X$.

Assuming that π is surjective, for each vector field $\check{X} \in \Gamma(B)$ on B, there is exactly one basic vector field X on M that is π-related to $\check{X}$. Throughout this paper, we will often denote a vector field on B and its horizontal lift by the same letter. Note that the projections $\mathcal{V}$ and $\mathcal{H}$ are smooth maps and so horizontal lifts of vector fields on B are smooth on M.

Lemma 2.10. *If X and Y are basic vector fields on M, then*

1. $\langle X, Y\rangle = \langle X_*, Y_*\rangle \circ \pi$
2. *$\mathcal{H}[X,Y]$ is the basic vector field corresponding to $[X_*, Y_*]$*
3. *$\mathcal{H}\nabla_X Y$ is the basic vector field corresponding to $\nabla^*_{X_*}(Y_*)$, where ∇^* denotes the LC connection on the base B.*

Proof. 1. Denote g_B and g_M the Riemannian metrics on B and on M respectively. We have, for

$p \in M$, (we swap between $g(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ for clarification)

$$\begin{aligned}(\langle X_*, Y_*\rangle \circ \pi)\big|_p &= g_B(X_*\big|_{\pi(p)}, Y_*\big|_{\pi(p)}) \\ &= g_B(\pi_* \circ X\big|_p, \pi_* \circ Y\big|_p) \\ &= g_M(X\big|_p, Y\big|_p) \\ &= \langle X, Y\rangle\big|_p.\end{aligned}$$

2. Since $\pi_*[X,Y] = [X_*, Y_*]$, we have $[X_*, Y_*] = \pi_*[X,Y] = \pi_*(\mathcal{V}[X,Y] + \mathcal{H}[X,Y]) = \pi_*(\mathcal{H}[X,Y])$, using the fact that vertical vectors are in the kernel of π_*.
3. We use the Koszul formula above:

$$2g\left(\nabla_X Y, Z\right) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X).$$

Since $X_* \circ \pi = \pi_* \circ X$, and combining with 1. we get $X\langle Y, Z\rangle = X(\langle Y_*, Z_*\rangle \circ \pi) = X_*\langle Y_*, Z_*\rangle \circ \pi$. Similarly, $\langle X, [Y,Z]\rangle = \langle X_*, [Y_*, Z_*]\rangle \circ \pi$, and upon replacing every such term in the right-hand side above, by the Koszul formula again we must have that it is equal to $2\langle \nabla^*_{X_*} Y_*, Z_*\rangle \circ \pi$, and thus

$$\langle \nabla^*_{X_*} Y_*, Z_*\rangle \circ \pi = \langle \nabla_X Y, Z\rangle,$$

which means precisely that $\nabla^*_{X_*} Y_*$ is π-related to $\mathcal{H}\nabla_X Y$ by 1. □

Lemma 2.11. *If X and Y are horizontal vector fields, then $A_X Y = \frac{1}{2}\mathcal{V}[X,Y]$.*

Proof. Note first that the left-hand side of the equation is tensorial as shown in Lemma 2.6. For the right-hand side, we notice that since $[X,Y] = \nabla_X Y - \nabla_Y X$, we have the relation $\mathcal{V}[X,Y] = A_X Y - A_Y X$, hence $\mathcal{V}[X,Y]$ is tensorial as well. Another way to see this is to use the "product rule" of the Lie bracket:

$$[X, fY] = f[X,Y] + X(f)Y,$$

whenever f is a smooth function on M. This allows to see that $\mathcal{V}[X, fY] = f\mathcal{V}[X,Y]$, since the term $X(f)Y$ is horizontal and thus vanishes under the vertical projection. Then, $\mathbb{R}$-bilinearity of the Lie bracket of vector fields leads to the right-hand side of the equation $A_X Y = \frac{1}{2}\mathcal{V}[X,Y]$ being $\mathcal{C}^\infty(M)$-linear in both entries by skew-symmetry of the Lie bracket, hence tensorial. Therefore, we only need to prove the relation at a single point p on the manifold, regardless of how the tangent vector $X\big|_p$ is continued as a vector field. We thus may assume that the vector field X is basic. If the alternation property 3 is true for the A tensor, then the result follows, and the alternation property is equivalent to $A_X X = 0$ by linearity. Since X is basic, $\langle X, X\rangle = \langle X_*, X_*\rangle \circ \pi$ is constant (as a function on M) along the fibers, and thus $0 = V\langle X, X\rangle = 2\left\langle \nabla_V X, X\right\rangle$ for any vertical vector field V. Moreover, because X is horizontal, $A_X X = \mathcal{V}\nabla_X X$. But $[V, X] = \nabla_V X - \nabla_X V$ is vertical (since V is π-related to the zero vector field), hence $\langle \nabla_V X - \nabla_X V, X\rangle = 0$, and so

$$0 = \langle \nabla_V X, X\rangle = \langle \nabla_X V, X\rangle = \underbrace{X\langle X, V\rangle}_{0} - \langle V, \nabla_X X\rangle = -\left\langle V, A_X X\right\rangle.$$

Since V is arbitrary and $A_X X$ is vertical, this shows that $A_X X = 0$ and it proves the claim. □

Remark 2.12. By Frobenius' theorem (see Appendix A.8), we have that a distribution is integrable if and only if it is involutive. Now, with the lemma above, we just learned that the

A tensor evaluated at horizontal vector fields is nothing but the vertical component of their bracket. Combining these two facts, we finally get that if the A tensor vanishes, then there is no vertical part in the bracket of X and Y; in other words, the horizontal distribution is involutive and thus integrable. We therefore often call this tensor the *integrability* tensor.

Lemma 2.13. *Let X and Y be horizontal vector fields, and V and W vertical vector fields. Then*

1. $\nabla_V W = T_V W + \mathcal{V}\nabla_V W$,
2. $\nabla_V X = \mathcal{H}\nabla_V X + T_V X$,
3. $\nabla_X V = A_X V + \mathcal{V}\nabla_X V$,
4. $\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y$.

Furthermore, if X is basic, $\mathcal{H}\nabla_V X = A_X V$.

Proof. These properties follow from the definition of the tensors A and T. For the last remark, we note that

$$A_X V = \mathcal{V}\nabla_{\mathcal{H}X}(\mathcal{H}V) + \mathcal{H}\nabla_{\mathcal{H}X}(\mathcal{V}V) = \mathcal{H}\nabla_X V = \mathcal{H}(\nabla_V X + [X,V]).$$

But $[X,V]$ is vertical whenever X is basic and V vertical: $\pi_*[X,V] = [\pi_* X, \pi_* V] = 0$. □

Lemma 2.14. *Let X and Y be horizontal vector fields, V and W be vertical vector fields. Then,*

$$(\nabla_V A)_W = -A_{T_V W}, \quad (\nabla_X T)_Y = -T_{A_X Y}, \quad (\nabla_X A)_W = -A_{A_X W} \quad (\nabla_V T)_Y = -T_{T_V Y}.$$

Proof. These equations become relatively trivial using the following fact about the covariant derivative of a tensor: if T is a tensor field of type (p,q), and $\alpha^1, \alpha^2, \dots, \alpha^q$ are smooth sections of the cotangent bundle T^*M and $X_1, X_2, \dots, X_p$ are smooth sections of the tangent bundle TM, then the covariant derivative of T along Y is given by the formula (see [KN69] p.124)

$$\begin{aligned}(\nabla_Y T)(\alpha_1, \alpha_2, \cdots, X_1, X_2, \cdots) = \nabla_Y (T(\alpha_1, \alpha_2, \cdots, X_1, X_2, \cdots)) \\ - T(\nabla_Y \alpha_1, \alpha_2, \cdots, X_1, X_2, \cdots) - T(\alpha_1, \nabla_Y \alpha_2, \cdots, X_1, X_2, \cdots) - \dots \\ - T(\alpha_1, \alpha_2, \cdots, \nabla_Y X_1, X_2, \cdots) - T(\alpha_1, \alpha_2, \cdots, X_1, \nabla_Y X_2, \cdots) - \dots\end{aligned}$$

When T is for example the A-tensor, this gives, by evaluating on an arbitrary vector field E,

$$(\nabla_V A)_W E = \nabla_V (A_W E) - A_{\nabla_V W}(E) - A_W(\nabla_V E)$$

which is just $-A_{\nabla_V W} E$, since the other terms vanish by verticality of W. But $A_{\nabla_V W}$ is again just $A_{\mathcal{H}\nabla_V W}$ by the same argument, and $\mathcal{H}\nabla_V W = T_V W$ which ends the proof. □

Another useful set of formulas regarding the curvature tensors is presented in the following lemma:

Lemma 2.15. *For all vector fields E, vertical U, V, and horizontal X, Y,*

$$\langle (\nabla_E T)_U V, X \rangle = \langle (\nabla_E T)_V U, X \rangle \tag{2.1}$$
$$\langle (\nabla_E A)_X Y, U \rangle = -\langle (\nabla_E A)_Y X, U \rangle. \tag{2.2}$$

Proof. Using the covariant derivative of tensors formula above, we get

$$\begin{aligned}(\nabla_E T)_U V - (\nabla_E T)_V U &= \nabla_E(T_U V) - \nabla_E(T_V U)\\ &\quad - (T_{\nabla_E U} V - T_{\nabla_E V} U)\\ &\quad - (T_U(\nabla_E V) - T_V(\nabla_E U)).\end{aligned}$$

Symmetry of the T tensor gives $T_U V = T_V U$ so the first line vanishes. Now, $T_{\nabla_E U} V = T_{\mathcal{V}\nabla_E U} V = T_V \mathcal{V}\nabla_E U$. Since only the horizontal component matters (because of the inner product with X), this is just $T_V \nabla_E U$, and we find this term in the third line. Apply this argument one more time to the second term of the middle line and we get that (the horizontal projection of) each line vanishes, which gives the result.

The formula with the A tensor uses the same reasoning. □

Remark 2.16. 1. Covariant derivatives of tensors are again tensorial: for example,

$$(\nabla_Z A)_X Y = \nabla_Z(A_X Y) - A_{\nabla_Z X} Y - A_X(\nabla_Z Y)$$

and one can check $\mathcal{C}^\infty(M)$-linearity in the Y-argument. The second term of the right-hand side has already been verified since we know that A_E is tensorial for any vector field E. We expand the first term:

$$\begin{aligned}\nabla_Z(A_X fY) = \nabla_Z \mathcal{V}\nabla_X fY &= \nabla_Z(f\mathcal{V}\nabla_X Y + \mathcal{V}(D_X f \cdot Y))\\ &= \nabla_Z(f\mathcal{V}\nabla_X Y)\\ &= f\nabla_Z \mathcal{V}\nabla_X Y + D_Z f \cdot \mathcal{V}\nabla_X Y\\ &= f\nabla_Z(A_X Y) + D_Z f \cdot \mathcal{V}\nabla_X Y,\end{aligned}$$

and the third term

$$\begin{aligned}A_X(\nabla_Z fY) &= \mathcal{H}\nabla_X \mathcal{V}\nabla_Z fY + \mathcal{V}\nabla_X \mathcal{H}\nabla_Z fY\\ &= \mathcal{H}\nabla_X(f\mathcal{V}\nabla_Z Y + D_Z f \cdot \mathcal{V}Y) + \mathcal{V}\nabla_X(f\mathcal{H}\nabla_Z Y + D_Z f \cdot \mathcal{H}Y)\\ &= \mathcal{H}\nabla_X f\mathcal{V}\nabla_Z Y + \mathcal{V}\nabla_X(f\mathcal{H}\nabla_Z Y + D_Z f \cdot Y)\\ &= \mathcal{H}(f\nabla_X \mathcal{V}\nabla_Z Y + D_X f \cdot \mathcal{V}\nabla_Z Y)\\ &\quad + \mathcal{V}(f\nabla_X \mathcal{H}\nabla_Z Y + D_X f \cdot \mathcal{H}\nabla_Z Y + D_Z f \cdot \nabla_X Y)\\ &= fA_X(\nabla_Z Y) + D_Z f \cdot \mathcal{V}\nabla_X Y.\end{aligned}$$

Hence, the "error" term $D_Z f \cdot \mathcal{V}\nabla_X Y$ appears both in the first and third terms of the covariant derivative formula, and thus cancels upon subtraction.

2. Suppose $\pi\colon M \to B$ is a Riemannian submersion, and let $p \in M$. In a tensorial equation involving bracket terms like $[X, Y]$ for horizontal $X, Y \in \Gamma(M)$, one can always arrange that $[X, Y]$ is vertical. Because of tensoriality, it suffices to verify the equation tangent space by tangent space, thus considering vectors $X|_p$, $Y|_p$ regardless of their continuation as vector fields. For instance, if $q = \pi(p)$, consider the corresponding vectors $X_*|_q$, $Y_*|_q$ on B, and consider coordinates around the point $q \in B$. With their expression in local coordinates,

the vectors $X_*|_q$, $Y_*|_q$ can be extended to vector fields on B that satisfy

$$[X_*, Y_*]_q = 0.$$

Because X_* and Y_* lift to X and Y, $[X_*, Y_*]$ lifts to $[X, Y]$ via π which precisely means that $[X, Y]$ is π-related to the zero vector field on B and so it is vertical by definition.

Lemma 2.17. *Let X, Y, Z be horizontal vector fields, and V a vertical one. Then,*

$$\mathfrak{S}\langle(\nabla_Z A)_X Y, V\rangle = \mathfrak{S}\langle A_X Y, T_V Z\rangle$$

where $\mathfrak{S}$ denotes the cyclic sum over the vector fields X, Y, Z.

Proof. Since the inner product and the covariant derivative of the A and T tensors are tensorial, so is the equation; which means it holds at a point regardless of how the vectors X, Y, Z are continued as vector field throughout the manifold. Thus, we may pick X, Y, Z to be vectors that are restriction of wisely chosen vector fields, i.e. we may assume X, Y, Z to be basic with vertical brackets $[X, Y]$, $[Y, Z]$, $[X, Z]$. This is done as follows:

This means for example, using Lemma 2.11, that $\frac{1}{2}[X, Y] = A_X Y$. Thus,

$$\langle[\tfrac{1}{2}[X, Y], Z], V\rangle = \langle[A_X Y, Z], V\rangle = \langle\nabla_{A_X Y} Z - \nabla_Z(A_X Y), V\rangle,$$

where we used torsion-freeness for the second equality. Then,

$$\begin{aligned}
\langle\nabla_{A_X Y} Z, V\rangle &= \langle T_{A_X Y} Z, V\rangle && (A_X Y \textit{ is vertical}) \\
&= -\langle T_{A_X Y} V, Z\rangle && (\textit{property 4 of the } T\textit{-tensor}) \\
&= -\langle T_V(A_X Y), Z\rangle && (\textit{property 3}) \\
&= \langle T_V Z, A_X Y\rangle && (\textit{property 4}).
\end{aligned}$$

The Jacobi identity (see Appendix A.11) says that

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

and so the cyclic sum $\mathfrak{S}\langle[[X, Y], Z], V\rangle$ above vanishes, meaning that

$$\mathfrak{S}\langle\nabla_{A_X Y} Z, V\rangle = \mathfrak{S}\langle\nabla_Z(A_X Y), V\rangle$$

and so

$$\mathfrak{S}\langle T_V Z, A_X Y\rangle = \mathfrak{S}\langle\nabla_Z(A_X Y), V\rangle.$$

If we show that $\mathfrak{S}\langle\nabla_Z(A_X Y), V\rangle = \mathfrak{S}\langle(\nabla_Z A)_X Y, V\rangle$, the proof is over. To this end, consider the difference $\langle\nabla_Z(A_X Y), V\rangle - \langle(\nabla_Z A)_X Y, V\rangle$, which we may rewrite using the covariant derivative of tensors formula presented in Lemma 2.14

$$\begin{aligned}
\langle\nabla_Z(A_X Y), V\rangle - \langle(\nabla_Z A)_X Y, V\rangle &= \langle(\nabla_Z A)_X Y + A_{\nabla_Z X} Y + A_X(\nabla_Z Y), V\rangle - \langle(\nabla_Z A)_X Y, V\rangle \\
&= \langle A_{\nabla_Z X} Y, V\rangle + \langle A_X(\nabla_Z Y), V\rangle
\end{aligned}$$

Then, with property 3 of the A-tensor, the first term of the right-hand side is nothing but

$-\langle A_Y(\mathcal{H}\nabla_Z X), V\rangle$, and

$$-\langle A_Y(\mathcal{H}\nabla_Z X), V\rangle = -\langle A_Y(\mathcal{H}(\nabla_X Z + [X,Z])), V\rangle = -\langle A_Y(\nabla_X Z), V\rangle$$

since again we assumed that the bracket $[X,Z]$ was vertical (the projection $\mathcal{H}$ was dropped since A_Y sends the vertical part of $\nabla_X Z$ to a horizontal vector which vanishes in the inner product). Thus,

$$\mathfrak{S}\langle \nabla_Z(A_X Y), V\rangle - \mathfrak{S}\langle (\nabla_Z A)_X Y, V\rangle = -\mathfrak{S}\langle A_Y(\nabla_X Z), V\rangle + \mathfrak{S}\langle A_X(\nabla_Z Y), V\rangle = 0,$$

which finishes the proof. □

2.2 Fundamental equations of a submersion

The key result of O'Neill's paper is the following 6 equations regarding the curvature tensor of the manifold M (in case of a Riemannian submersion $\pi\colon M \to B$) and of the fibers $\pi^{-1}(b)$, that we list below. Note that there are two different sign conventions for the curvature tensor R. The two sources [O'N66] and [PW14] that serve as a reference for Sections 2 and 3 use different conventions. We made the choice here to modify the formulas in [O'N66] to match the convention used by Pro and Wilhelm.

Proposition 2.18. *Let R, $\hat{R}$ and $\check{R}$ denote the curvature tensors of M, of the fibers F_b and of B, respectively. Then,*

$$\begin{aligned}
\langle R(U,V)W,W'\rangle &= \langle \hat{R}(U,V)W,W'\rangle + \langle T_U W, T_V W'\rangle - \langle T_V W, T_U W'\rangle, &(2.3)\\
\langle R(U,V)W,X\rangle &= -\langle (\nabla_V T)_U W, X\rangle + \langle (\nabla_U T)_V W, X\rangle, &(2.4)\\
\langle R(X,U)Y,V\rangle &= -\langle (\nabla_X T)_U V, Y\rangle + \langle T_U X, T_V Y\rangle - \langle (\nabla_U A)_X Y, V\rangle &(2.5)\\
&\quad - \langle A_X U, A_Y V\rangle,\\
\langle R(U,V)X,Y\rangle &= -\langle (\nabla_U A)_X Y, V\rangle + \langle (\nabla_V A)_X Y, U\rangle - \langle A_X U, A_Y V\rangle &(2.6)\\
&\quad + \langle A_X V, A_Y U\rangle + \langle T_U X, T_V Y\rangle - \langle T_V X, T_U Y\rangle,\\
\langle R(X,Y)Z,U\rangle &= -\langle (\nabla_Z A)_X Y, U\rangle - \langle A_X Y, T_U Z\rangle + \langle A_Y Z, T_U X\rangle &(2.7)\\
&\quad + \langle A_Z X, T_U Y\rangle,\\
\langle R(X,Y)Z,Z'\rangle &= \langle \check{R}(X,Y)Z,Z'\rangle + 2\langle A_X Y, A_Z Z'\rangle - \langle A_Y Z, A_X Z'\rangle &(2.8)\\
&\quad + \langle A_X Z, A_Y Z'\rangle.
\end{aligned}$$

We have denoted above the curvature tensor $\check{R}$ on B and its horizontal lift by the same letter; in fact, they are related by the identity

$$\langle \check{R}(X,Y)Z,Z'\rangle = \langle \check{R}(\pi_* X, \pi_* Y)\pi_* Z, \pi_* Z'\rangle$$

(since we identify vector fields on B with their horizontal lift on M). These formulas above imply in particular for the sectional curvatures:

Corollary 2.19. *Let K, $\check{K}$ and $\hat{K}$ be the sectional curvatures on M, B, and the fibers respectively.*

Then, if X, Y are horizontal and V, W vertical,

$$K(P_{VW}) = \widehat{K}(P_{VW}) - \frac{\langle T_V V, T_W W\rangle - \|T_V W\|^2}{\|V \wedge W\|^2},$$
$$K(P_{XV})\|X\|^2\|V\|^2 = \langle (\nabla_X T)_V V, X\rangle + \|A_X V\|^2 - \|T_V X\|^2,$$
$$K(P_{XY}) = \check{K}(P_{X_*Y_*}) - \frac{3\|A_X Y\|^2}{\|X \wedge Y\|^2}, \text{ where } X_* = \pi_*(X),\ Y_* = \pi_*(Y).$$

This shows in particular that if M has positive sectional curvature in horizontal directions, i.e. $K(P_{XY}) > 0$ for all planes P_{XY} tangent to M, then B has positive sectional curvature too, since the quantity $\|A_X Y\|^2$ is always non negative; the second equation says precisely that, at least for mixed planes, M always has non negative curvature.

To get the analog equations for the Ricci curvature, we first need to define some quantities.

Definition 2.20. For $(X_i)_{i\in I}$ and $(U_j)_{j\in J}$ local orthonormal bases of $\mathcal{H}$ and $\mathcal{V}$, we define the following:

$$\langle A_X, A_Y\rangle := \sum_i \langle A_X X_i, A_Y X_i\rangle = \sum_j \langle A_X U_j, A_Y U_j\rangle, \tag{2.9}$$
$$\langle A_X, T_U\rangle := \sum_i \langle A_X X_i, T_U X_i\rangle = \sum_j \langle A_X U_j, T_U U_j\rangle, \tag{2.10}$$
$$\langle T_U, T_V\rangle := \sum_i \langle T_U X_i, T_V X_i\rangle = \sum_j \langle T_U U_j, T_V U_j\rangle, \tag{2.11}$$
$$\langle AU, AV\rangle := \sum_i \langle A_{X_i} U, A_{X_i} V\rangle,$$
$$\langle TX, TY\rangle := \sum_i \langle T_{U_i} X, T_{U_i} Y\rangle,$$

and if E is an arbitrary tensor field on M,

$$\check{\delta} E := -\sum_i (\nabla_{X_i} E)_{X_i},$$
$$\hat{\delta} E := -\sum_j (\nabla_{U_j} E)_{U_j},$$
$$\delta E := \check{\delta} E + \hat{\delta} E,$$
$$(\tilde{\delta} T)(U, V) := \sum_i \langle (\nabla_{X_i} T)_U V, X_i\rangle.$$

We also define the *mean curvature vector* N,

$$N := \sum_i T_{U_i} U_i.$$

To get the equalities in (2.9), (2.10) and (2.11), we develop the expression in the inner product and use the fourth property of the A tensor: for all vertical U, $\langle A_X Y, U\rangle = -\langle A_X U, Y\rangle$. This gives

$$\begin{aligned}\sum_i \langle A_X X_i, A_Y X_i\rangle &= \sum_i \left\langle \sum_j \langle A_X X_i, U_j\rangle\, U_j, \sum_k \langle A_Y X_i, U_k\rangle\, U_k \right\rangle \\ &= \sum_{i,j} \langle A_X X_i, U_j\rangle\, \langle A_Y X_i, U_j\rangle \\ &= \sum_{i,j} -\langle A_X U_j, X_i\rangle\, (-\langle A_Y U_j, X_i\rangle)\,,\end{aligned}$$

and from this last expression we easily recover $\sum_j \langle A_X U_j, A_Y U_j\rangle$. Equations (2.10) and (2.11) are similar. Note that these quantities arise as the trace of certain linear operators. For example, the quantity $\langle A_X, A_Y\rangle = \sum_i \langle A_X X_i, A_Y X_i\rangle$ above is the trace of the bilinear operator defined as

$$(E, F) \mapsto \langle A_X E, A_Y F\rangle.$$

Because traces are independent of the choice of basis, these definitions do not depend on the choice of horizontal and vertical bases.

Lemma 2.21. *If E is an arbitrary vector field, X and Y are horizontal, and V_k, $k = 1, \dots, n :=$ $\dim \mathcal{V}$ are an orthonormal frame of the vertical distribution, then*

$$\begin{aligned}\langle \nabla_E N, X\rangle &= \sum_{k=1}^{n} \langle (\nabla_E T)_{V_k} V_k, X\rangle \\ \frac{1}{2}\left(\langle \nabla_Y N, X\rangle - \langle \nabla_X N, Y\rangle\right) &= \sum_{k=1}^{n} \langle (\nabla_{V_k} A)_X Y, V_k\rangle\end{aligned}$$

For reference of Lemma 2.21, see [FIP04] p.15 or [Bes87] p.243. These identities are typically used in the proof of the Ricci formulas that we present below; let us denote by $\widehat{\mathrm{Ric}}$ and $\widetilde{\mathrm{Ric}}$ the Ricci curvature on the fibers and on the base space respectively. We now mention formulas regarding the relation between the Ricci curvature of the total space M, the base space B and the fibers, in the following proposition, which will have their equivalent with the Ricci curvatures of the canonical variation in Section 4:

Corollary 2.22. *Let* Ric, $\widehat{\mathrm{Ric}}$, $\widetilde{\mathrm{Ric}}$ *be the Ricci curvature of the total space, the fibers and the base space respectively (in particular,* $\widetilde{\mathrm{Ric}}$ *is such that* $\widetilde{\mathrm{Ric}}(X, Y) = \widetilde{\mathrm{Ric}}(\pi_* X, \pi_* Y)$*). Then, for all vertical U, V and horizontal X, Y,*

$$\mathrm{Ric}(U, V) = \widehat{\mathrm{Ric}}(U, V) - \langle N, T_U V\rangle + \langle AU, AV\rangle + (\tilde{\delta} T)(U, V), \tag{2.12}$$

$$\mathrm{Ric}(X, U) = \langle (\hat{\delta} T) U, X\rangle + \langle \nabla_U N, X\rangle - \langle (\check{\delta} A) X, U\rangle - 2\langle A_X, T_U\rangle, \tag{2.13}$$

$$\mathrm{Ric}(X, Y) = \widetilde{\mathrm{Ric}}(X, Y) - 2\langle A_X, A_Y\rangle - \langle TX, TY\rangle + \frac{1}{2}\big(\langle \nabla_X N, Y\rangle + \langle \nabla_Y N, X\rangle\big). \tag{2.14}$$

Proof. For the first equation, take a local orthonormal frame of M (we write $\dim M = m$ and $\dim(F_b) = n$ where F_b is any fiber) such that the first n vectors form an orthonormal frame of the vertical distribution, and the last $m - n$ vectors of the horizontal distribution. Denote this basis $(F_1, \dots, F_n, X_1, \dots, X_{m-n})$. By definition,

$$\mathrm{Ric}(U, V) = \sum_{k=1}^{n} g(R(F_k, U)V, F_k) + \sum_{l=1}^{m-n} g(R(X_l, U)V, X_l).$$

To retrieve the Ricci curvature from the curvature tensor R, we apply (2.3) and (2.5) with the vector fields indicated in the previous equation. This yields

$$\begin{aligned}\mathrm{Ric}(U,V) &= \widehat{\mathrm{Ric}}(U,V) + \sum_k (g(T_{F_k}U, T_V F_k) - g(T_U V, T_{F_k} F_k)) \\ &\quad + \sum_l (g((\nabla_{X_l} T)_V U, X_l) + g((\nabla_V A)_{X_l} X_l, U) \\ &\quad - g(T_V X_l, T_U X_l) + g(A_{X_l} V, A_{X_l} U)) \\ &= \widehat{\mathrm{Ric}}(U,V) - g(T_U V, N) + \tilde{\delta} T(U,V) + g(AU, AV)\end{aligned}$$

where we used that $g((\nabla_V A)_{X_l} X_l, U) = 0$ by (2.2), and $T_{F_k} U = T_U F_k$ together with equation (2.11) above.

For formula (2.14), take a sum over a vertical orthonormal basis in (2.5) and a sum over a horizontal orthonormal basis in (2.8). This immediately gives the terms $\widetilde{\mathrm{Ric}}(X,Y) - 2\langle A_X, A_Y\rangle - \langle TX, TY\rangle$. With Lemma 2.21 above we transform the remaining terms to obtain $1/2(\langle \nabla_Y N, Y\rangle + \langle \nabla_Y N, X\rangle)$.

The same argument applies for (2.13). Taking vertical sum in (2.4) and horizontal sum in (2.7), and together with formula (2.1), this immediately gives the terms $\langle A_X, T_U\rangle$, $-\langle (\check{\delta} A)X, U\rangle$, $\langle (\hat{\delta} T)U, X\rangle$ and $\langle \nabla_U N, X\rangle$. Afterwards, the remaining terms to deal with are $\sum_k -\langle A_{Z_k} Z_k, T_U X\rangle - \langle A_{Z_k} X, T_U Z_k\rangle$ (Z_k is an orthonormal horizontal frame). The first one vanishes by the alternating property of A and the second one is just, by the same argument, another copy of $\langle A_X, T_U\rangle$, hence the factor 2. □

In the case where V and W are orthonormal, and if we assume that the fibers are totally geodesic, the curvature equations simplify to the following:

$$\begin{aligned}K(P_{VW}) &= \hat{K}(P_{VW}) \\ K(P_{XV})\|X\|^2\|V\|^2 &= \|A_X V\|^2 \\ K(P_{XY}) &= \check{K}(P_{X_* Y_*}) - 3\|A_X Y\|^2, \\ \mathrm{Ric}(U,V) &= \widehat{\mathrm{Ric}}(U,V) + \langle AU, AV\rangle \\ \mathrm{Ric}(X,U) &= -\langle (\check{\delta} A)X, U\rangle \\ \mathrm{Ric}(X,Y) &= \widetilde{\mathrm{Ric}}(X,Y) - 2\langle A_X, A_Y\rangle\end{aligned}$$

Hence, in the totally geodesic case, one can improve Corollary 2.19 and find that the sectional vertical sectional curvatures of the total space equal the curvatures of the fibers. Lastly, if we assume further that the A tensor vanishes as well, then the "mixed" curvatures (involving horizontal and vertical vector fields) vanish, and the other curvatures equal their analog on the fibers or on the base.

Proof of Proposition 2.18. The first equation says:

$$\boxed{\langle R(U,V)W, W'\rangle = \langle \hat{R}(U,V)W, W'\rangle + \langle T_U W, T_V W'\rangle - \langle T_V W, T_U W'\rangle. \quad (2.3)}$$

Formula (2.3) relates the curvature tensor R of the total space to the curvature of the fibers $\hat{R}$.

By definition,

$$\begin{aligned} R(U,V)W &= \nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W, \quad \text{and} \\ \hat{R}(U,V)W &= \hat{\nabla}_U\hat{\nabla}_V W - \hat{\nabla}_V\hat{\nabla}_U W - \hat{\nabla}_{[U,V]}W. \end{aligned}$$

This implies after arranging the terms, that (2.3) holds if and only if

$$\begin{aligned} &\langle \nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W - (\hat{\nabla}_U\hat{\nabla}_V W - \hat{\nabla}_V\hat{\nabla}_U W - \hat{\nabla}_{[U,V]}W), W'\rangle \\ &\qquad = -\langle \mathcal{H}\nabla_V W, \mathcal{H}\nabla_U W'\rangle + \langle \mathcal{H}\nabla_U W, \mathcal{H}\nabla_V W'\rangle \end{aligned}$$

where we have replaced terms like $T_E F$ by $\mathcal{H}\nabla_E F$ by verticality of E and F. Now, by Appendix B.5, the Levi-Civita connection on the fibers $\hat{\nabla}$ is given by $\mathcal{V}\nabla$; in particular, the term $\langle \hat{\nabla}_{[U,V]}W - \nabla_{[U,V]}W, W'\rangle$ is 0, because $\nabla - \hat{\nabla}$ is always horizontal. What remains to show is the two following identities:

$$\begin{aligned} \langle \nabla_U\nabla_V W - \hat{\nabla}_U\hat{\nabla}_V W, W'\rangle &= -\langle \mathcal{H}\nabla_V W, \mathcal{H}\nabla_U W'\rangle \\ \langle \hat{\nabla}_V\hat{\nabla}_U W - \nabla_V\nabla_U W, W'\rangle &= \langle \mathcal{H}\nabla_U W, \mathcal{H}\nabla_V W'\rangle. \end{aligned}$$

Now, for the first one,

$$\nabla_U\nabla_V W - \hat{\nabla}_U\hat{\nabla}_V W = \nabla_U(\nabla_V - \hat{\nabla}_V)W + \mathcal{H}\nabla_U\hat{\nabla}_V W = \nabla_U(\mathcal{H}\nabla_V)W + \mathcal{H}\nabla_U\hat{\nabla}_V W.$$

Using the metric property of the Levi-Civita connection, we get that

$$\langle \nabla_U(\mathcal{H}\nabla_V)W + \mathcal{H}\nabla_U\hat{\nabla}_V W, W'\rangle = U\underbrace{\langle \mathcal{H}\nabla_V W, W'\rangle}_{0} - \langle \mathcal{H}\nabla_V W, \nabla_U W'\rangle + \underbrace{\langle \mathcal{H}\nabla_U\hat{\nabla}_V W, W'\rangle}_{0}$$

and the middle term becomes $-\langle \mathcal{H}\nabla_V W, \mathcal{H}\nabla_U W'\rangle$. The other computations are similar, and this shows the first formula.

The second formula says:

$$\boxed{\langle R(U,V)W, X\rangle = -\langle (\nabla_V T)_U W, X\rangle + \langle (\nabla_U T)_V W, X\rangle. \quad (2.4)}$$

For equation (2.4), using the formula in the proof of Lemma 2.14, the first inner product of the right-hand side develops to

$$-\langle \nabla_V(T_U W) - T_{\nabla_V U}W - T_U\nabla_V W, X\rangle.$$

Expanding the left part of this inner product, using the definition of the T tensor, we get

$$\begin{aligned} \nabla_V(T_U W) - T_{\nabla_V U}W - T_U\nabla_V W &= \nabla_V\mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}W + \nabla_V\mathcal{V}\nabla_{\mathcal{V}U}\mathcal{H}W \\ &\quad - (\mathcal{H}\nabla_{\mathcal{V}\nabla_V U}\mathcal{V}W + \mathcal{V}\nabla_{\mathcal{V}\nabla_V U}\mathcal{H}W) \\ &\quad - (\mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}\nabla_V W + \mathcal{V}\nabla_{\mathcal{V}U}\mathcal{H}\nabla_V W). \end{aligned}$$

The vertical terms, as well as the terms involving $\mathcal{H}W$, both vanish, after taking inner product with the horizontal vector field X, and because W is vertical. Altogether, looking at the right-hand side of (2.4), we obtain

$$\begin{aligned}\mathcal{H}\big[(\nabla_U T)_V W - (\nabla_V T)_U W\big] = \mathcal{H}\big[&\nabla_U \mathcal{H}\nabla_V W - \mathcal{H}\nabla_{\mathcal{V}\nabla_U V} W - \mathcal{H}\nabla_V \mathcal{V}\nabla_U W \\ &- (\nabla_V \mathcal{H}\nabla_U W - \mathcal{H}\nabla_{\mathcal{V}\nabla_V U} W - \mathcal{H}\nabla_U \mathcal{V}\nabla_V W)\big].\end{aligned}$$

Now, the left-hand side of the equation (2.4) becomes $\langle \nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W, X\rangle$, and

$$\nabla_U\nabla_V W = \nabla_U(\mathcal{H}\nabla_V + \mathcal{V}\nabla_V)W = \nabla_U\mathcal{H}\nabla_V W + \nabla_U\mathcal{V}\nabla_V W.$$

Then, since $\mathcal{H}(R(U,V)W) = \mathcal{H}(\nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W)$, we can, by moving every term except $\mathcal{H}\nabla_{[U,V]}W$ on the same side in (2.4), simplify the expression $\mathcal{H}\big[(\nabla_U T)_V W - (\nabla_V T)_U W - (\nabla_U\nabla_V W - \nabla_V\nabla_U W)\big]$ as the following quantity:

$$\mathcal{H}\big[(\nabla_U T)_V W - (\nabla_V T)_U W - (\nabla_U\nabla_V W - \nabla_V\nabla_U W)\big] = -\mathcal{H}\nabla_{\mathcal{V}\nabla_U V}W + \mathcal{H}\nabla_{\mathcal{V}\nabla_V U}W.$$

Therefore, it remains to show that this right-hand side is just $-\mathcal{H}\nabla_{[U,V]}W$. Torsion-freeness of the Levi-Civita connection gives $[U,V] = \nabla_U V - \nabla_V U$. Note that since both U and V are vertical, $[U,V]$ is again vertical, hence $[U,V] = \mathcal{V}[U,V] = \mathcal{V}\nabla_U V - \mathcal{V}\nabla_V U$. So $\nabla_{[U,V]} = \nabla_{\mathcal{V}\nabla_U V - \mathcal{V}\nabla_V U} = \nabla_{\mathcal{V}\nabla_U V} - \nabla_{\mathcal{V}\nabla_V U}$. Now take the horizontal part (since only the horizontal part contributes in the inner product with X) of this difference and apply it to W to obtain the desired result. This shows formula (2.4).

The fifth and sixth formulas say:

$$\langle R(X,Y)Z,U\rangle = -\langle (\nabla_Z A)_X Y, U\rangle - \langle A_X Y, T_U Z\rangle + \langle A_Y Z, T_U X\rangle + \langle A_Z X, T_U Y\rangle, \tag{2.7}$$
$$\langle R(X,Y)Z,Z'\rangle = \langle \check{R}(X,Y)Z,Z'\rangle + 2\langle A_X Y, A_Z Z'\rangle - \langle A_Y Z, A_X Z'\rangle + \langle A_X Z, A_Y Z'\rangle. \tag{2.8}$$

In (2.7) and (2.8), the equation is tensorial (curvature tensor and covariant derivative of A tensor). Because tensors are determined by their value at a specific point, we can pick vectors X, U, Y, V that are restrictions of wisely chosen vector fields on M. Thus, to simplify the proof we may assume that X, Y, Z are basic and whose brackets are vertical, i.e. $\mathcal{H}([X,Y]) = \mathcal{H}([X,Z]) = \mathcal{H}([Y,Z]) = 0$. Imposing this condition means in particular that $\frac{1}{2}[X,Y] = A_X Y$, and similarly for Z, by Lemma 2.11. Now by definition,

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

and $\nabla_{[X,Y]}Z$ reduces to $\nabla_{2A_X Y}Z$. Furthermore, $\nabla_{A_X Y}Z = \nabla_{\mathcal{V}\nabla_X Y}Z$ and $T_{A_X Y}Z = \mathcal{V}\nabla_{A_X Y}Z = \mathcal{V}\nabla_{\mathcal{V}\nabla_X Y}Z$. Also, $A_Z(A_X Y) = A_Z(\mathcal{V}\nabla_X Y) = \mathcal{H}\nabla_Z\mathcal{V}\nabla_X Y$. Using torsion-freeness, we have that $\nabla_Z\mathcal{V}\nabla_X Y = \nabla_{\mathcal{V}\nabla_X Y}Z + [Z, \mathcal{V}\nabla_X Y]$, and because Z is basic by assumption, this last bracket is vertical and thus vanishes under horizontal projection. So

$$\nabla_{[X,Y]}Z = \nabla_{2A_X Y}Z = 2\nabla_{\mathcal{V}\nabla_X Y}Z = 2(\mathcal{V}\nabla_{\mathcal{V}\nabla_X Y}Z + \mathcal{H}\nabla_{\mathcal{V}\nabla_X Y}Z) = 2A_Z(A_X Y) + 2T_{A_X Y}Z.$$

Up to identifying X, Y, Z with their corresponding vector fields X_*, Y_*, Z_* (which are $\pi_* X, \pi_* Y, \pi_* Z$), we can write, using Lemma 2.10, $\mathcal{H}\nabla_Y Z$ as $\nabla^*_Y Z$, and so $\nabla_Y Z = \nabla^*_Y Z + \mathcal{V}\nabla_Y Z = \nabla^*_Y Z + A_Y Z$.

This allows to develop $\nabla_X \nabla_Y Z$ as:

$$\begin{aligned}\nabla_X \nabla_Y Z &= \nabla_X (\nabla^*_Y Z + A_Y Z)\\ &= \nabla_X \nabla^*_Y Z + \nabla_X A_Y Z\\ &= \nabla^*_X \nabla^*_Y Z + \mathcal{V}\nabla_X \nabla^*_Y Z + \nabla_X A_Y Z\\ &= \nabla^*_X \nabla^*_Y Z + A_X(\nabla^*_Y Z) + \mathcal{H}\nabla_X A_Y Z + \mathcal{V}\nabla_X A_Y Z\\ &= \nabla^*_X \nabla^*_Y Z + A_X(\nabla^*_Y Z) + A_X A_Y Z + \mathcal{V}\nabla_X A_Y Z,\end{aligned}$$

where we have used that $\mathcal{H}\nabla_X A_Y Z = A_X A_Y Z$ because $A_Y Z$ is vertical. To retrieve the curvature tensor (that we split into horizontal and vertical part), we put together the above equations to get

$$\mathcal{H}R(X,Y)Z = [\nabla^*_X, \nabla^*_Y]Z - 2A_Z A_X Y + A_X A_Y Z - A_Y A_X Z, \tag{2.15}$$

$$\mathcal{V}R(X,Y)Z = -2\; T_{A_X Y}(Z) + \mathcal{V}\nabla_X \left(A_Y Z\right) - \mathcal{V}\nabla_Y \left(A_X Z\right) + A_X \left(\nabla^*_Y Z\right) - A_Y \left(\nabla^*_X Z\right). \tag{2.16}$$

Now, by definition, $\check{R}(X_*, Y_*)Z_* = -\nabla^*_{[X_*,Y_*]} Z_* + [\nabla^*_{X_*}, \nabla^*_{Y_*}]Z_*$, and by Lemma 2.10, $[X_*, Y_*]$ lifts to $\mathcal{H}[X,Y]$ on M which by our assumption is 0. Thus we rewrite (remember that we identify vector fields with their lift)

$$\check{R}(X,Y)Z = [\nabla^*_X, \nabla^*_Y]Z,$$

and taking inner product with a horizontal vector field Z' in (2.15) yields (2.8), using the skew-symmetry property of the A-tensor.

For (2.7), take inner product in (2.16) with a vertical vector field U to get

$$\begin{aligned}\langle R(X,Y)Z, U\rangle = &\; -2\,\langle\, T_{A_X Y}(Z), U\rangle + \langle \nabla_X \left(A_Y Z\right), U\rangle - \langle \nabla_Y \left(A_X Z\right), U\rangle\\ &+ \langle A_X \left(\nabla_Y Z\right), U\rangle - \langle A_Y \left(\nabla_X Z\right), U\rangle\,.\end{aligned} \tag{2.17}$$

Using property 3 and 4 of the T tensor, we obtain the following equality:

$$\langle T_{A_X Y} Z, U\rangle = -\langle Z, T_{A_X Y} U\rangle = -\langle Z, T_U(A_X Y)\rangle = \langle T_U Z, A_X Y\rangle,$$

and using Lemma 2.15,

$$\begin{aligned}\langle \nabla_X \left(A_Y Z\right), U\rangle - \langle \nabla_Y \left(A_X Z\right), U\rangle = &\; \langle (\nabla_X A)_Y\, Z, U\rangle - \langle (\nabla_Y A)_X\, Z, U\rangle\\ &+ \langle A_{\nabla_X Y} Z, U\rangle - \langle A_{\nabla_Y X} Z, U\rangle\\ &+ \langle A_Y \left(\nabla_X Z\right), U\rangle - \langle A_X \left(\nabla_Y Z\right), U\rangle\,.\end{aligned}$$

But the middle row terms $\langle A_{\nabla_X Y} Z, U\rangle - \langle A_{\nabla_Y X} Z, U\rangle$ reduce to $\langle A_{[X,Y]} Z, U\rangle$ by torsion-freeness, and since $[X,Y]$ is vertical by our assumption, this inner product vanishes. The two last terms of this right-hand side appear in (2.17) and thus simplify to 0. Thus, we get

$$\langle R(X,Y)Z, U\rangle = -2\langle T_U Z, A_X Y\rangle + \langle (\nabla_X A)_Y Z, U\rangle - \langle (\nabla_Y A)_X Z, U\rangle, \tag{2.18}$$

and, by Lemma 2.17, taking the cyclic sum and arranging the terms, we find

$$\begin{aligned}\langle A_X Y, T_U Z\rangle &= \langle (\nabla_Z A)_X Y, U\rangle + \langle (\nabla_Y A)_Z X, U\rangle + \langle (\nabla_X A)_Y Z, U\rangle \\ &\quad - \langle A_Z X, T_U Y\rangle - \langle A_Y Z, T_U X\rangle\end{aligned}$$

Thus,

$$\begin{aligned}\langle R(X,Y)Z, U\rangle &= -2\big\{\langle (\nabla_Z A)_X Y, U\rangle + \langle (\nabla_Y A)_Z X, U\rangle + \langle (\nabla_X A)_Y Z, U\rangle \\ &\quad - \langle A_Z X, T_U Y\rangle - \langle A_Y Z, T_U X\rangle\big\} + \langle (\nabla_X A)_Y Z, U\rangle - \langle (\nabla_Y A)_X Z, U\rangle \\ &= -2\big\{\langle (\nabla_Z A)_X Y, U\rangle + \langle (\nabla_Y A)_Z X, U\rangle - \langle A_Z X, T_U Y\rangle - \langle A_Y Z, T_U X\rangle\big\} \\ &\quad - \langle (\nabla_X A)_Y Z, U\rangle - \langle (\nabla_Y A)_X Z, U\rangle \\ &= -\langle (\nabla_Z A)_X Y, U\rangle - 2\langle (\nabla_Y A)_Z X, U\rangle + \langle A_Y Z, T_U X\rangle + \langle A_Z X, T_U Y\rangle \\ &\quad - \langle (\nabla_X A)_Y Z, U\rangle - \langle (\nabla_Y A)_X Z, U\rangle \\ &\quad - \langle (\nabla_Z A)_X Y, U\rangle - \langle A_X Y, T_U Z\rangle + \langle A_Y Z, T_U X\rangle + \langle A_Z X, T_U Y\rangle\end{aligned}$$

where we have rearranged the terms in the third equality so that the very last line matches the terms of the right-hand side of (2.7). Hence, to prove (2.7), it suffices to show that

$$\begin{aligned}&-\langle (\nabla_Z A)_X Y, U\rangle - 2\langle (\nabla_Y A)_Z X, U\rangle + \langle A_Y Z, T_U X\rangle \\ &\qquad + \langle A_Z X, T_U Y\rangle - \langle (\nabla_X A)_Y Z, U\rangle - \langle (\nabla_Y A)_X Z, U\rangle = 0.\end{aligned}$$

In fact, this follows from Lemma 2.17 and Lemma 2.15.

Third equation says:

$$\boxed{\langle R(X,U)Y, V\rangle = -\langle (\nabla_X T)_U V, Y\rangle + \langle T_U X, T_V Y\rangle - \langle (\nabla_U A)_X Y, V\rangle - \langle A_X U, A_Y V\rangle. \quad (2.5)}$$

Since the equation is again tensorial, we may pick X to be basic, and choose U such that the bracket $[X,U]$ vanishes at the point at which we are verifying the equation. We use computations in Lemma 2.15 to develop the covariant derivatives of the A and T tensors as

$$(\nabla_U A)_X Y = \nabla_U(A_X Y) - A_{\nabla_U X} Y - A_X(\nabla_U Y) \tag{2.19}$$

$$(\nabla_X T)_U V = \nabla_X(T_U V) - T_{\nabla_X U} V - T_U(\nabla_X V). \tag{2.20}$$

Calculations show:

$$\begin{aligned}\langle T_{\nabla_X U} V, Y\rangle = \langle T_{\mathcal{V}\nabla_X U} V, Y\rangle &= \langle T_V \mathcal{V}\nabla_X U, Y\rangle \\ &= -\langle T_V Y, \mathcal{V}\nabla_X U\rangle \\ &= -\langle T_V Y, \mathcal{V}\nabla_U X\rangle \\ &= -\langle T_V Y, T_U X\rangle\end{aligned}$$

where we have used that $\nabla_X U - \nabla_U X = [X,U] = 0$ by our assumption. In a similar manner, one shows that $\langle A_{\nabla_U X} Y, V\rangle = \langle A_X U, A_Y V\rangle$, where the sign is different from the T tensor because of the property 3 of the A tensor implicitly used in the calculations above. So, the middle terms in the RHS of (2.19) and (2.20) simplify to 0. Thus, (2.5) reduces to

$$\langle R(X,U)Y,V\rangle = -\langle \nabla_U(A_XY) - A_X(\nabla_UY), V\rangle - \langle \nabla_X(T_UV) - T_U(\nabla_XV), Y\rangle. \tag{2.21}$$

By definition, $R(X,U)Y = -\nabla_{[X,U]}Y + \nabla_X\nabla_UY - \nabla_U\nabla_XY$, and we have arranged that $\nabla_{[X,U]}$ vanishes so only the other two terms remain. Expanding the first inner product in equation (2.21), we get

$$\langle \nabla_U(A_XY) - A_X(\nabla_UY), V\rangle = \langle \nabla_U(\mathcal{V}\nabla_XY) - \mathcal{V}\nabla_X(\mathcal{H}\nabla_UY) - \mathcal{H}\nabla_X(\mathcal{V}\nabla_UY), V\rangle \tag{2.22}$$

$$= \langle \nabla_U(\mathcal{V}\nabla_XY) - \mathcal{V}\nabla_X(\mathcal{H}\nabla_UY), V\rangle. \tag{2.23}$$

On the other hand, using the metric compatibility,

$$\langle \nabla_X(T_UV), Y\rangle = X\langle T_UV, Y\rangle - \langle T_UV, \nabla_XY\rangle, \tag{2.24}$$

and by definition this last inner product is $-\langle \mathcal{H}\nabla_UV, \nabla_XY\rangle$. Since only the horizontal components matter in this inner product, we can write this as $-\langle \nabla_UV, \mathcal{H}\nabla_XY\rangle$. Another use of the metric compatibility yields

$$\langle \nabla_X(T_UV), Y\rangle = X\langle T_UV, Y\rangle - \underbrace{U\langle V, \mathcal{H}\nabla_XY\rangle}_{=0} + \langle V, \nabla_U\mathcal{H}\nabla_XY\rangle.$$

Notice that the term $\langle V, \nabla_U\mathcal{H}\nabla_XY\rangle$ matches the term $\langle V, \nabla_U\mathcal{V}\nabla_XY\rangle$ in (2.23) to give $\langle V, \nabla_U\nabla_XY\rangle$ when adding them together. Using similar reasoning, we get

$$\begin{aligned}
\langle T_U\nabla_XV, Y\rangle &= \langle \mathcal{H}\nabla_U\mathcal{V}\nabla_XV, Y\rangle \\
&= \langle \nabla_U\mathcal{V}\nabla_XV, Y\rangle \\
&= U\langle \mathcal{V}\nabla_XV, Y\rangle - \langle \mathcal{V}\nabla_XV, \nabla_UY\rangle \\
&= U\langle \mathcal{V}\nabla_XV, Y\rangle - \langle \nabla_XV, \mathcal{V}\nabla_UY\rangle \\
&= U\langle \mathcal{V}\nabla_XV, Y\rangle - X\langle V, \mathcal{V}\nabla_UY\rangle + \langle V, \nabla_X\mathcal{V}\nabla_UY\rangle, \qquad (2.25)
\end{aligned}$$

and $\langle V, \nabla_X\mathcal{V}\nabla_UY\rangle$ matches the term $\langle V, \nabla_X\mathcal{H}\nabla_UY\rangle$ in (2.23) to yield $\langle V, \nabla_X\nabla_UY\rangle$. In fact, the terms $U\langle \mathcal{V}\nabla_XV, Y\rangle - X\langle V, \mathcal{V}\nabla_UY\rangle$ overall end up being 0: the first one is by orthogonality and the second one is because when expanding the first inner product in the right-hand side of (2.24), we get

$$X\langle T_UV, Y\rangle = X\langle \mathcal{H}\nabla_UV, Y\rangle = X(U\langle V, Y\rangle) - X\langle V, \nabla_UY\rangle = -X\langle V, \nabla_UY\rangle$$

which adds up to 0 with the term in (2.25).

Altogether, we find that the RHS of (2.21) is nothing but $\langle V, \nabla_U\nabla_XY - \nabla_X\nabla_UY\rangle$, which equals the LHS by definition and thus proves (2.5).

Fourth equation says:

$$\begin{aligned}
\langle R(U,V)X,Y\rangle = &-\langle (\nabla_UA)_X Y, V\rangle + \langle (\nabla_VA)_X Y, U\rangle - \langle A_XU, A_YV\rangle \\
&+ \langle A_XV, A_YU\rangle + \langle T_UX, T_VY\rangle - \langle T_VX, T_UY\rangle. \qquad (2.6)
\end{aligned}$$

Equation (2.6) is just a consequence of the symmetries of the curvature tensor, stated in Appendix

B.7 and B.8. We show that both the left-hand side and the right-hand side can be obtained from the term $\langle R(X,U)Y,V\rangle - \langle R(X,V)Y,U\rangle$. On one hand, we claim that the right-hand side of (2.6) is indeed $\langle R(X,U)Y,V\rangle - \langle R(X,V)Y,U\rangle$, which we expand using (2.5) to

$$\begin{aligned}\langle R(X,U)Y,V\rangle - \langle R(X,V)Y,U\rangle = &-\langle(\nabla_X T)_U V, Y\rangle - \langle(\nabla_U A)_X Y, V\rangle \\ &+ \langle T_U X, T_V Y\rangle - \langle A_X U, A_Y V\rangle \\ &+ \langle(\nabla_X T)_V U, Y\rangle + \langle(\nabla_V A)_X Y, U\rangle \\ &+ \langle T_V X, T_U Y\rangle - \langle A_X V, A_Y U\rangle.\end{aligned} \tag{2.26}$$

By Lemma 2.15, we have $-\langle(\nabla_X T)_U V, Y\rangle + \langle(\nabla_X T)_V U, Y\rangle = 0$, so these terms simplify and we retrieve the right-hand side of (2.6). On another hand, the left-hand side can also be retrieved from $\langle R(X,U)Y,V\rangle - \langle R(X,V)Y,U\rangle$:

$$\begin{aligned}\langle R(X,U)Y,V\rangle - \langle R(X,V)Y,U\rangle &= -\langle R(X,U)V,Y\rangle + \langle R(X,V)U,Y\rangle \\ &= \langle R(U,X)V,Y\rangle + \langle R(X,V)U,Y\rangle \\ &= -\langle R(V,U)X,Y\rangle \\ &= \langle R(U,V)X,Y\rangle.\end{aligned}$$

by the Bianchi identity, which finishes the proof. □

2.3 Warped products

We define a common notion in the field of Riemannian geometry, which is particularly useful in our context:

Definition 2.23 (Products with varying metric on the fibers)**.** Let (B, g^1) be a Riemannian manifold, and g_b^2 be a family of smoothly varying metrics on another Riemannian manifold F, where the family is indexed by $b \in B$. Consider the product $M = B \times F$ with the metric $g = g^1 + g_b^2$, with values at $(b,x) \in B \times F$ given by

$$g|_{b,x} = \pi_1^*(g^1)|_b + \pi_2^*(g_b^2)|_x$$

where π_i are the projections onto the respective factors. We call this new manifold a *product with a varying metric on the fibers.*

A consequence of this is that the projection onto the first factor π_1 is a Riemannian submersion, and the horizontal distribution $\mathcal{H} = TB$ is integrable. This is indeed useful in the case of a warped product, since calculations will be made easier if the A tensor vanishes. A special case of a product with varying metric on the fibers is a warped product:

Definition 2.24 (Warped product)**.** Let (B, g_B) and (F, g_F) be two Riemannian manifolds. A *warped product* of B and F is the manifold $M = B \times F$ with metric defined as $g_M|_b := g_B|_b + f(b)^2 g_F|_b$ for a positive function f on the manifold B. We sometimes also write $(B \times F, g_B + f^2 g_F)$ to denote the warped product of B and F, or simply $B \times_f F$.

In the sections below we will consider another type of warped product $B \times F$ with metric $g_B + e^{2\nu} g_F$, that we will also denote $B \times_\nu F$ for simplicity.

Example 2.25. Let $M = \mathbb{R}^n$. The usual Euclidean "flat" metric is

$$g_{\text{flat}} = \sum_{i=1}^{n} \mathrm{d}x_i^2$$

in Cartesian coordinates. We may express this metric in polar coordinates: each point $x \in \mathbb{R}^n$ is determined by a modulus $\|x\| \in \mathbb{R}$ and $n-1$ angles $\theta_1, \ldots, \theta_{n-1} \in [0, 2\pi]$, hence the change of variable $x = r \cdot s$, $r \in \mathbb{R}$, $s \in S^{n-1}$. The standard metric thus becomes

$$\begin{aligned} g_{\mathbb{R}^n} = \sum_{i=1}^{n} \mathrm{d}x_i^2 &= \sum_{i=1}^{n} (\mathrm{d}(rs)_i)^2 \\ &= \sum_{i=1}^{n} (s_i \mathrm{d}r + r\mathrm{d}s_i)^2 \\ &= \sum_{i=1}^{n} s_i^2 \mathrm{d}r^2 + r^2 \mathrm{d}s_i^2 + 2rs_i \mathrm{d}r\mathrm{d}s_i \end{aligned}$$

where s_i denotes the i-th coordinate of the point $s \in S^{n-1} \subset \mathbb{R}^n$. The fact that $\sum s_i^2 = 1$ implies, by taking the differential, that $\sum s_i \mathrm{d}s_i = 0$, and so the last line is simply $\mathrm{d}r^2 + r^2 \mathrm{d}s_{n-1}^2$. This is nothing but a warped metric on the product space $\mathbb{R} \times_f S^{n-1}$ with warping function $f(r) = r$. This shows in particular that the product space's topology is deeply affected by the warping function: for $n = 2$, the space $[0, \infty) \times S^1$ with the usual product metric is an infinite cylinder with boundary; while the warped metric $\mathrm{d}r^2 + r^2 \mathrm{d}s_1^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2$ yields simply $\mathbb{R}^2$ (we allow the warping function to be 0 on the boundary). Similarly, restricting $r \in [0, 2\pi] = I$ gives a cylinder $I \times S^2$ and for $f(r) = \sin(r)^2$, the warping metric becomes $\mathrm{d}r^2 + \sin(r)^2 \mathrm{d}\theta^2$ which is the standard round metric on the sphere S^2, hence $I \times_{\sin^2} S^1 = S^2$ with the round metric. Lastly, the warped metric $\mathrm{d}r^2 + \sinh(r)^2 \mathrm{d}s_{n-1}$ on $[0, \infty) \times S^{n-1}$ yields the well known hyperbolic space of dimension n, $\mathbb{H}^n$.

The three following objects (gradient, Hessian, Laplacian) appear all throughout this paper and are common notions in Riemannian geometry:

Definition 2.26 (Gradient)**.** Let (M, g) be a Riemannian manifold and $f \in \mathcal{C}^\infty(M)$. The *gradient* of f, denoted ∇f, is the vector field on M determined by the relations

$$g(\nabla f, E) = \mathrm{d}f(E) = E(f)$$

for any vector field $E \in \Gamma(M)$, where $E(f)$ is the derivative of f in direction of E.

Next we define the Hessian of a real function on a Riemannian manifold. There are several different yet equivalent characterisations of the Hessian, the one we chose below being the definition put forward by [Pet98] p.48:

Definition 2.27 (Hessian)**.** Let (M, g) a Riemannian manifold with Levi-Civita connection ∇. Let $f \in \mathcal{C}^\infty(M)$ be a smooth function on M. The *Hessian* of f, denoted $\operatorname{Hess} f$, is the $(0, 2)$-tensor defined as

$$\operatorname{Hess} f = \frac{1}{2} L_{\nabla f}\, g$$

where L denotes the Lie derivative and ∇f is the gradient of f.

Hence, computing the Hessian comes down to computing the Lie derivative of the metric tensor. Using a standard formula for the Lie derivative of tensors (see [KN69] p.32), we find that, if $X, Y_1, \ldots, Y_n$ are vector fields and T is a tensor field (not necessarily O'Neill's T-tensor),

$$L_X(T(Y_1, \ldots, Y_n)) = (L_X T)(Y_1, \ldots, Y_n) + T((L_X Y_1), \ldots, Y_n) + T(Y_1, \ldots, (L_X Y_n)).$$

Applying this with the tensor $T = g$, one gets

$$L_X(g(Y_1, Y_2)) = (L_X g_\varphi)(Y_1, Y_2) + g(L_X Y_1, Y_2) + g(Y_1, L_X Y_2).$$

Further calculations lead to the following formula in coordinates for the Hessian of a function (calculations are shown in [Pet98] p.69):

$$\operatorname{Hess} f\,(\partial_i, \partial_j) = \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f.$$

Note that the Hessian may also be expressed using covariant differentiation ([Pet98] p.58):

$$\operatorname{Hess} f(X, Y) = g(\nabla_X \nabla f, Y) = (\nabla_X \mathrm{d} f)(Y).$$

Definition 2.28 (Laplacian)**.** Let (M, g) be a Riemannian manifold. The *Laplacian* of a function $f \in \mathcal{C}^\infty(M)$ is the scalar field defined as the trace of Hessian of f,

$$\Delta f = \operatorname{tr}(\operatorname{Hess} f).$$

Expressing the Laplacian Δf in terms of the metric g on M, we find

$$\Delta f = g^{ij} (\operatorname{Hess} f)_{ij} = g^{ij} \left(\partial_i \partial_j f - \Gamma_{ij}^k \partial_k f \right).$$

2.4 Ricci curvatures of a warped product $B \times_f F$

Corollary 2.22 dictates how the Ricci curvatures of the total space of a submersion $\pi \colon M \to B$ behave with regard to the curvatures of the fibers and the base. These formulas can be used to deduce the Ricci curvature in the case where the total space M arises from a product $B \times_f F$ with warped metric $g = g_B + f^2 g_F$, with π being the projection onto the first factor. To this end, one must determine the T-tensor in the new metric (the A-tensor being irrelevant here since it vanishes, see comment after Definition 2.23). Some fundamental computational results are presented in [Bes87], among which one finds Lemma 2.30 as well.

Lemma 2.29 ([Bes87, Corollary 9.105])**.** *Let $\pi \colon B \times_f F \to B$ be a Riemannian submersion with metric $g = g_B + f^2 g_F$ and with $\dim F = k$. Let U, V be vertical and X, Y be horizontal, and let*

E be an arbitrary vector field. Then, if T is the T-tensor for the warped metric,

$$g((\nabla_E T)_U V, X) = \frac{1}{k} g(U,V) g(\nabla_E N, X) \tag{2.27}$$

$$(\hat{\delta} T(U), X) = (\nabla_E N, X) = 0 \tag{2.28}$$

$$(\tilde{\delta} T)(U,V) = -\frac{1}{k} g(U,V) \check{\delta} N \tag{2.29}$$

$$g(\nabla_X N, Y) = k\left(-\frac{1}{f}\operatorname{Hess} f(X,Y) + \frac{1}{f^2} g(\nabla f, X) g(\nabla f, Y)\right) \tag{2.30}$$

$$\check{\delta} N = k\left(\frac{\Delta f}{f} + \frac{g_F(\nabla f, \nabla f)}{f^2}\right) \tag{2.31}$$

where the Hessian, gradient and Laplacian in the last two formulas are all with respect to the metric g_B.

The following Lemma is then a consequence of Corollary 2.22, together with the quantities of Lemma 2.29:

Lemma 2.30 ([O'N83, Section 7 Corollary 43], [Bes87, Proposition 9.106])**.** *Let $M = B \times_f F$ be a warped product with metric $g = g_B + f^2 g_F$, and let $k = \dim F > 1$. Let X, Y be horizontal vectors, and U, V vertical ones (with respect to the Riemannian submersion π given by the projection onto B). Then*

$$\operatorname{Ric}(X,Y) = \operatorname{Ric}_B(X,Y) - \frac{k}{f}\operatorname{Hess} f(X,Y) \tag{2.32}$$

$$\operatorname{Ric}(X,U) = 0 \tag{2.33}$$

$$\operatorname{Ric}(U,V) = \operatorname{Ric}_F(U,V) - f^2 g_F(U,V)\left[\frac{\Delta f}{f} + (k-1)\frac{g_F(\nabla f, \nabla f)}{f^2}\right] \tag{2.34}$$

where Δf is the Laplacian of f and ∇f is its gradient.

Note that in [Bes87] and [O'N66], the signs may differ since the two authors use different sign conventions for the Laplacian.

Proof of Lemma 2.30. Since everything is tensorial, assume that the horizontal vectors are basic. The Koszul formula says

$$2g(\nabla_X U, V) = X(g(U,V)) + U(g(X,V)) - V(g(X,U)) + g([X,U],V) - g([X,V],U) - g([U,V],X).$$

The second and third terms vanish by orthogonality of the vectors inside the inner product; the bracket $[U,V]$ is vertical and thus $g([U,V],X) = 0$; and finally, the bracket $[X,U]$ vanishes identically: since the vector field $X \in \Gamma(B)$ lifts to $X \in \Gamma(B \times F)$ (we use the same letter) via $\pi_1 \colon B \times F \to B$, X is π_2-related to the 0 vector field on F (where $\pi_2 \colon B \times F \to F$), and similarly V is π_1-related to the 0 vector field on B, and consequently $[X,U]$ is both π_1 and π_2-related to 0.

In the end, the only term remaining in the Koszul formula is $X(g(U,V))$. Using the Koszul formula again with a function f^2 attached to the vertical inner products (since the warped product metric is scaled by f^2 in the vertical direction), we find that $\mathcal{V}\nabla_U X = \frac{X(f)}{f}$ together with the property that $X(f^2) = 2fX(f)$. Now, note that we have, for the T tensor of the warped product

$B \times_f F$ with $\dim F = k$,

$$g(T_U V, X) = g(\nabla_U V, X) = -g(\nabla_U X, V) = -\frac{X(f)}{f} g(U, V).$$

But $X(f) = g(\nabla f, X)$, which leads to

$$g(T_U V, X) = -\frac{g(U, V)}{f} g(\nabla f, X).$$

Since X is an arbitrary horizontal (basic) vector field, this shows that

$$T_U V = -\frac{g(U, V)}{f} \nabla f. \tag{2.35}$$

Then, from [Bes87] p.266 we get $T_U V = \frac{g(U,V)}{k} N$, and thus $N = k \frac{\nabla f}{f}$. Applying Corollary 2.22 (2.12) to a warped metric yields

$$\mathrm{Ric}(U, V) = \mathrm{Ric}_F(U, V) - g(N, T_U V) + (\tilde{\delta} T)(U, V)$$

where we have removed the term containing the A-tensor because it vanishes. Combining all these equations together, and using (2.29) and (2.31), this gives

$$\begin{aligned} \mathrm{Ric}(U, V) &= \mathrm{Ric}_F(U, V) - g(k\frac{\nabla f}{f}, \frac{g(U, V)}{f} \nabla f) - g(U, V) \left(\frac{\Delta f}{f} + \frac{g_F(\nabla f, \nabla f)}{f^2} \right) \\ &= \mathrm{Ric}_F(U, V) - f^2 g_F(U, V) \left[\frac{\Delta f}{f} + (k-1) \frac{g_F(\nabla f, \nabla f)}{f^2} \right]. \end{aligned}$$

For the second equation, namely $\mathrm{Ric}(X, U) = 0$, we recall Corollary 2.22:

$$\mathrm{Ric}(X, U) = g((\hat{\delta} T)U, X) + g(\nabla_U N, X),$$

which is immediately 0 by Lemma 2.29.
Finally, for the first equation, from Corollary 2.22, we get

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}_B(X, Y) - g(TX, TY) + \frac{1}{2}\big(g(\nabla_X N, Y) + g(\nabla_Y N, X)\big).$$

Then,

$$\begin{aligned} g(TX, TY) = \sum_k g(T_{U_k} X, T_{U_k} Y) &= \sum_k \frac{1}{k^2} g(g(N, X) U_k, g(N, Y) U_k) \\ &= \sum_k k^2 \cdot \frac{1}{k^2} g\left(g\left(\frac{\nabla f}{f}, X\right) U_k, g\left(\frac{\nabla f}{f}, Y\right) U_k \right) \\ &= \sum_k \frac{1}{f^2} g(g(\nabla f, X) U_k, g(\nabla f, Y) U_k) \\ &= \sum_k \frac{1}{f^2} g(\nabla f, X) g(\nabla f, Y) \end{aligned}$$

since U_k are orthonormal. Thus, this term simplifies with the one in (2.30) to 0, and this finishes the proof by symmetry of X and Y in (2.30). □

3 Transporting Ricci curvature to the base

3.1 Main results

In this section, we review and provide details of an example constructed by Curtis Pro and Frederick Wilhelm in an article named "*Riemannian submersions need not preserve positive Ricci curvature*" ([PW14]). This is in fact a *counter*-example to the idea that Riemannian submersions need necessarily transport positive Ricci curvature from M to B, and this is done in Theorem 3.1.

Theorem 3.1 ([PW14, Theorem 1])**.** *For any $C > 0$, there exists a Riemannian manifold M and a Riemannian submersion $\pi\colon M \to B$ for which M is compact with positive Ricci curvature, while B contains points with Ricci curvature less than $-C$.*

Positive Ricci curvature means that for all points p on the manifold, and all non zero tangent vectors X based at p, $\mathrm{Ric}(X, X)_p > 0$. To obtain such points of negative Ricci curvature we will modify the metric on the base space $B = S^2$ smoothly, in a way that its curvature changes locally, while preserving the Riemannian submersion. We will in fact construct an example of a warped product $S^2 \times_\nu F$ with metric $g_B + e^{2\nu} g_F$ where F has positive Ricci curvature; the base space S^2 will be equip with a metric that is $\mathcal{C}^1$-close to the round metric.

Theorem 3.1 thus says that Riemannian submersions do not transport positive Ricci curvature in general; however the Ricci curvatures of the total and the base space are not completely unrelated, as states Theorem 3.2:

Theorem 3.2 ([PW14, Theorem 2])**.** *Let M be a Riemannian manifold that is compact with positive Ricci curvature. Then there exists no Riemannian submersion $\pi\colon M \to B$ to a space of nonpositive Ricci curvature.*

Hence, any Riemannian submersion on a compact manifold does transport at least some parts of positive Ricci curvature. There is one special case however where more information is passed down to the base space: if the metric on (M, g) is such that the Riemannian submersion $\pi\colon M \to B$ has totally geodesic fibers, then positivity is transported down to B. This is summarised in the following theorem:

Theorem 3.3. *Let (M, g) and $(B, \breve{g})$ be Riemannian manifolds with M compact, and let $\pi\colon M \to B$ be a Riemannian submersion such that the fibers $\pi^{-1}(b)$ are totally geodesic. Then, if M has positive Ricci curvature, B does as well.*

This is in fact a direct consequence of the formulas of Section 2. Since

$$\mathrm{Ric}(X, Y) = \breve{\mathrm{Ric}}(X, Y) - 2\langle A_X, A_Y\rangle$$

it follows that if $\mathrm{Ric} > 0$, then $\breve{\mathrm{Ric}} > 0$.

To prove Theorem 3.1 and Theorem 3.2 we need some preparations.

3.2 Ricci tensor of the warped sphere

We will now construct a manifold of the form $B \times F$ with a warped product metric, such that the product $B \times F$ has positive Ricci globally, and B has some points of negative Ricci curvature; the Riemannian submersion will be given by the projection on the first factor $\pi_1\colon B \times F \to B$. Our example will assume that F has positive Ricci curvature, and calculations below show that the "vertizontal" Ricci curvature vanishes. Since B contains points of negative curvature, the contribution of the warping function to the curvature of the product $B \times F$ must be "strong" enough to compensate these points.

Consider thus the sphere S^2 of radius 1, with spherical coordinates (r, θ), $r \in [0, \pi]$, $\theta \in [0, 2\pi)$, and let $r\colon S^2 \to [0, \pi]$ be the radial distance function, i.e. if q is a point of S^2, then $r(p) = \operatorname{dist}(p, q) = \inf_\gamma L(\gamma)$ for γ smooth curves joining p and q (in particular the differential of r is the component $\mathrm{d}r$ of usual metric $\mathrm{d}r^2 + \sin^2(r)\,\mathrm{d}\theta^2$). Choose $\varphi : [0, \pi] \to [0, \infty)$ so that S^2 with the metric $g_\varphi = \mathrm{d}r^2 + \varphi^2 \mathrm{d}\theta^2$ is a smooth Riemannian manifold denoted by S^2_φ. Calculations in the example above show how to retrieve a warped metric of this form starting from the standard metric expressed in Cartesian coordinates. This new metric essentially "smashes" both ends of the cylinder $I \times S^1$ to yield S^2, but it remains to ensure that the process is smooth at these endpoints. For this, one can show (see e.g. [Pet98] p.22) that for warped metric of the form $\mathrm{d}r^2 + \varphi^2(r)\mathrm{d}\theta^2$ to be smooth at $r = 0$ and $r = \pi$, one needs the following conditions

$$\varphi(0) = \varphi(\pi) = 0,\ \dot\varphi(0) = -\dot\varphi(\pi) = 1 \text{ and } \varphi^{(2n)}(0) = \varphi^{(2n)}(\pi) = 0.$$

Notice that the function $\varphi(r) \coloneqq \sin(r)$ is a function that satisfies these properties and gives rise to the well known round metric on the sphere. If $p\colon S^2 \to [0, \pi]$ denotes the "straight" projection of the sphere onto the interval, then in particular $p = r$ as maps, and the fibers are latitudinal circles whose sizes or lengths are dictated by this function φ (see Figure 1). In fact, if $\gamma\colon [0, 2\pi] \to S^2_\varphi$ is a curve joining a point $x \in S^2$ to itself, such that $\dot\gamma(t) = \partial_\theta$, then its length is

$$L(\gamma) = \int_0^{2\pi} g_\varphi(\partial_\theta, \partial_\theta) = \int_0^{2\pi} \varphi(r)^2 \cdot 1\,\mathrm{d}\theta = 2\pi \cdot \varphi(r)^2.$$

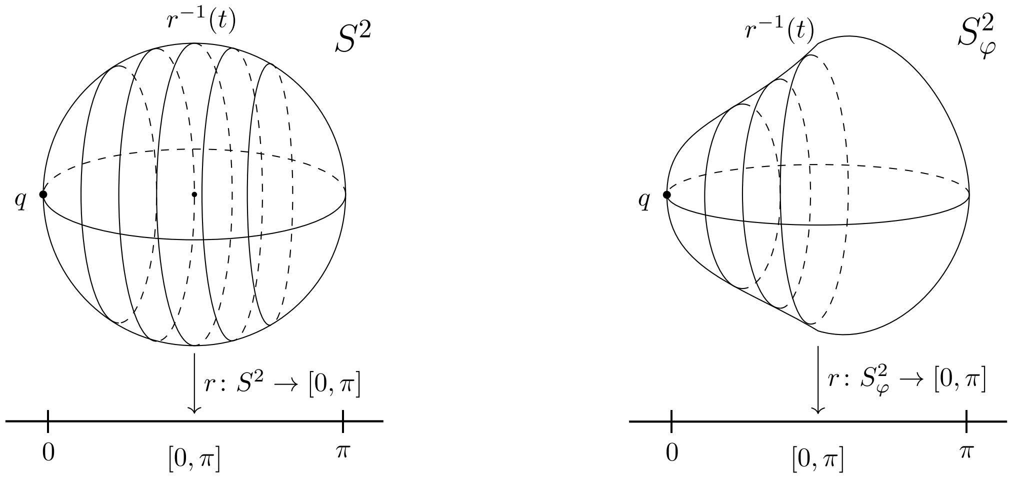

Figure 2: Round sphere (left) and warped sphere (right) with metric $\mathrm{d}r^2 + \varphi(r)^2\,\mathrm{d}\theta^2$. The fibers $r^{-1}(t)$ under the radial distance map from the point q, each have length $2\pi \sin(r)^2$ and $2\pi\varphi(r)^2$ respectively.

Let now $\nu\colon [0, \pi] \to \mathbb{R}$ be a function on S^2_φ that only depends on the colatitude r. Consider the warped product $S^2_\varphi \times_\nu F$ with metric $g_{S^2_\varphi} + e^{2\nu} g_F$, where (F, g_F) is any k-dimensional manifold ($k \geq 2$) with $\mathrm{Ric}_F \geq 1$.

Lemma 3.4. *If ν is the warping function on the manifold $S^2_\varphi \times_\nu F$, then*

$$\operatorname{Hess}\nu = \ddot{\nu}\,\mathrm{d}r^2 + \dot{\nu}\varphi\dot{\varphi}\,\mathrm{d}\theta^2.$$

Proof. Using the notation $\dot{\nu} = \partial_r \nu$, since ν only depends on r, its gradient becomes

$$\nabla \nu = \dot{\nu}\partial_r.$$

We have (see Definition 2.27)

$$\operatorname{Hess}\nu\,(\partial_i, \partial_j) = \partial_i\partial_j\nu - \Gamma^k_{ij}\partial_k\nu.$$

It remains to compute the Christoffel symbols of S^2_φ. Using the standard formula (see again [Pet98] p.66)

$$\Gamma^l_{ij} = \frac{1}{2}\sum_{k=1}^{m}\left(\frac{\partial}{\partial x_i}g_{jk} + \frac{\partial}{\partial x_j}g_{ki} - \frac{\partial}{\partial x_k}g_{ij}\right)\cdot g^{k,l},$$

we find $\Gamma^2_{12} = \frac{\dot{\varphi}}{\varphi}$, $\Gamma^1_{22} = -\varphi\dot{\varphi}$, while all the other symbols vanish. Therefore, the $\mathrm{d}r^2$ coefficient of the Hessian is

$$\operatorname{Hess}\nu(\partial_1, \partial_1) = (\partial_1\partial_1\nu - 0) = \ddot{\nu}$$

and similarly the $\mathrm{d}\theta^2$ coefficient is $\dot{\nu}\varphi\dot{\varphi}$. Thus the Hessian of ν is given by

$$\operatorname{Hess}\nu = \ddot{\nu}\mathrm{d}r^2 + \dot{\nu}\varphi\dot{\varphi}\mathrm{d}\theta^2.$$ □

We will use and prove a computational lemma regarding the Ricci curvature:

Lemma 3.5. *For the Ricci curvature, one has the following formula:*

$$\operatorname{Ric}_{ab} = \partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac} + \Gamma^d_{ab}\Gamma^c_{dc} - \Gamma^d_{ac}\Gamma^c_{db}$$

with the usual implicit summation over double indices.

Proof. To see that this is true, we use the definition of the curvature tensor, and compute its coordinate expression by evaluating it at $\partial_i' s$:

$$\begin{aligned} R(\partial_i, \partial_j)\partial_k &= \nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k - [\partial_i, \partial_j]\partial_k \\ &= \nabla_i\Gamma^l_{jk}\partial_l - \nabla_j\Gamma^l_{ik}\partial_l \\ &= \Gamma^l_{jk}\nabla_i\partial_l + (\partial_i\Gamma^l_{jk})\partial_l - \big(\Gamma^l_{ik}\nabla_j\partial_l + (\partial_j\Gamma^l_{ik})\partial_l\big) \\ &= \Gamma^l_{jk}\Gamma^n_{il}\partial_n + (\partial_i\Gamma^l_{jk})\partial_l - \Gamma^l_{ik}\Gamma^n_{jl}\partial_n - (\partial_j\Gamma^l_{ik})\partial_l. \end{aligned}$$

Hence, by choosing the standard orthonormal basis $\partial_1, \dots, \partial_m$ of T_pM (the choice of p does not matter, we thus omit it),

$$\begin{aligned}
\mathrm{Ric}(\partial_j, \partial_k) &= \sum_i \langle R(\partial_i, \partial_j)\partial_k, \partial_i\rangle \\
&= \sum_i \langle \Gamma^l_{jk}\Gamma^n_{il}\partial_n + (\partial_i \Gamma^l_{jk})\partial_l - \Gamma^l_{ik}\Gamma^n_{jl}\partial_n - (\partial_j \Gamma^l_{ik})\partial_l, \partial_i\rangle \\
&= \sum_i \langle \Gamma^l_{jk}\Gamma^n_{il}\partial_n + (\partial_i \Gamma^l_{jk})\partial_l, \partial_i\rangle - \langle \Gamma^l_{ik}\Gamma^n_{jl}\partial_n + (\partial_j \Gamma^l_{ik})\partial_l, \partial_i\rangle \\
&= \Gamma^l_{jk}\Gamma^n_{nl} + \partial_l \Gamma^l_{jk} - \Gamma^l_{nk}\Gamma^n_{jl} - \partial_j \Gamma^l_{lk}.
\end{aligned}$$

By setting $\mathrm{Ric}_{jk} \coloneqq \mathrm{Ric}(\partial_j, \partial_k)$, and upon changing the letters i, j, k, l to a, b, c, d for clarification, we get the desired formula. □

Lemma 3.6. *The Ricci (0,2)-tensor of the sphere S^2_φ is given by*

$$\mathrm{Ric}_{S^2_\varphi} = -\frac{\ddot{\varphi}}{\varphi} g_\varphi.$$

Proof. Applying Lemma 3.5 to the sphere S^2_φ, we find the numbers $\mathrm{Ric}_{11} = -\varphi\ddot{\varphi}$ and $\mathrm{Ric}_{22} = -\frac{\ddot{\varphi}}{\varphi}$:

$$\begin{aligned}
\mathrm{Ric}_{11} &= \partial_1 \Gamma^1_{12} + \partial_2 \Gamma^2_{12} - \partial_2 \Gamma^1_{11} - \partial_2 \Gamma^2_{12} \\
&\quad + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^1_{12}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^2_{22} \\
&\quad - \left(\Gamma^1_{11}\Gamma^2_{12} + \Gamma^1_{12}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^1_{22} + \Gamma^2_{12}\Gamma^2_{22}\right) \\
&= -\partial_1 \Gamma^2_{12} - \Gamma^2_{12}\Gamma^2_{12} \\
&= -\frac{\ddot{\varphi}\varphi - \dot{\varphi}^2}{\varphi^2} - \frac{\dot{\varphi}^2}{\varphi^2} = -\frac{\ddot{\varphi}}{\varphi},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}_{22} &= \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{21} + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^1_{21}\Gamma^1_{12} \\
&\quad + \partial_2 \Gamma^2_{22} - \partial_2 \Gamma^1_{22} + \Gamma^1_{22}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{12} \\
&\quad + \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{21} + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^2_{21}\Gamma^1_{22} \\
&\quad + \partial_2 \Gamma^2_{22} - \partial_2 \Gamma^2_{22} + \Gamma^2_{22}\Gamma^2_{22} - \Gamma^2_{22}\Gamma^2_{22} \\
&= \partial_1 \Gamma^1_{22} + \Gamma^1_{22}\Gamma^2_{12} - 2\Gamma^2_{12}\Gamma^1_{22} \\
&= -(\dot{\varphi}^2 + \varphi\ddot{\varphi}) + \dot{\varphi}^2 = -\ddot{\varphi}\varphi,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}_{12} &= \partial_1 \Gamma^1_{12} - \partial_2 \Gamma^1_{11} + \Gamma^1_{12}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{12} \\
&\quad + \partial_2 \Gamma^2_{12} - \partial_2 \Gamma^1_{12} + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{12} \\
&\quad + \partial_1 \Gamma^1_{12} - \partial_2 \Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{11} - \Gamma^2_{11}\Gamma^1_{12} \\
&\quad + \partial_2 \Gamma^2_{12} - \partial_2 \Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{12}\Gamma^2_{12} \\
&= 0
\end{aligned}$$

because all the terms vanish in this last computation. Thus, the Ricci tensor of S^2_φ is given as

$$\mathrm{Ric}_{S^2_\varphi} = -\frac{\ddot{\varphi}}{\varphi} g_\varphi = -\frac{\ddot{\varphi}}{\varphi}(\mathrm{d}r^2 + \varphi^2 \mathrm{d}\theta^2) = -\frac{\ddot{\varphi}}{\varphi}\mathrm{d}r^2 - \ddot{\varphi}\varphi \mathrm{d}\theta^2.$$ □

Note that alternatively the Ricci curvature formulas for warped products can be used to obtain the same result. Now, since the Ricci curvature is a function times the metric, and since the denominator has constant sign, it is relatively easy to find and control points where the manifold S^2_φ has positive or negative Ricci curvature.

If g is the metric on $M = S^2_\varphi \times F$, we denote by $g^{\mathcal{H}}$ and $g^{\mathcal{V}}$ the restrictions of g to $\mathcal{H}$ and $\mathcal{V}$, which are the distributions for the projection on the first factor $\pi_1 \colon S^2_\varphi \times F \to S^2_\varphi$. Consider as above the warped metric $g_\nu := e^{2\nu} g^{\mathcal{V}} + g^{\mathcal{H}}$ on M. Note that both $\mathcal{H}$ and $g^{\mathcal{H}}$ are unchanged, so π_1 is still a Riemannian submersion.

Remark 3.7. 1. In general, for a Riemannian manifold N and a Riemannian submersion $\pi \colon N \to B$ with fibers F, it is not true that N splits as a product $B \times F$ with a product metric, even if the horizontal distribution is integrable, i.e. $A = 0$. The very example of S^2_φ is indeed a warped product $I \times_\varphi S^1$, where I is an interval, and the projection on the first factor is a Riemannian submersion; the A tensor vanishes on S^2_φ by the comment after Definition 2.23, but S^2_φ is not isometric to the product $I \times S^1$. Other counterexamples are given by Riemannian submersions that are non-trivial fiber bundles as well.
2. If A vanishes identically, then locally the total space N is isometric to $B \times F$ with metric $g_B + (g_F)_b$ varying on the fibers. If both tensors T and A are zero, then N is in fact locally isometric to the product $B \times F$ with metric $g_B + g_F$ (see [Bes87] 9.26 or [GW09, Theorem 1.4.1]). The sphere S^2_φ is not locally isometric to $B \times F$, which suggests that the T tensor is non zero: indeed the fibers are latitude circles, on which no geodesic can be a geodesic on the sphere (only great circles are geodesics), except possibly at $r = \pi/2$.

However, if M does split as $B \times F$, i.e. $M = B \times_{e^{2\nu}} F$ with warped metric $g_\nu = g^{\mathcal{H}} + e^{2\nu} g^{\mathcal{V}}$ where g is the product metric, then Gromoll & Walschap showed ([GW09], Corollary 2.2.2)[2] the following equations regarding the Ricci curvature: for horizontal X, Y and vertical U, V, we have

$$\operatorname{Ric}_\nu(X,Y) = \operatorname{Ric}_B(X,Y) - k(\operatorname{Hess}\nu(X,Y) + g(\nabla\nu, X) g(\nabla\nu, Y)), \tag{3.1}$$

$$\operatorname{Ric}_\nu(X,U) = 0, \tag{3.2}$$

$$\operatorname{Ric}_\nu(U,V) = \operatorname{Ric}_F(U,V) - g(U,V) e^{2\nu}\left(\Delta\nu + k|\nabla\nu|^2\right) \tag{3.3}$$

These formulas can in fact also be retrieved from Lemma 2.30 by choosing the warping function to be $e^{2\nu}$. Having mixed terms being equal to 0 is convenient as it allows to only study the vertical and horizontal parts. We also identify vector fields on the base B and their horizontal lifts via $\pi_1 \colon B \times F \to B$.

Let $\operatorname{Ric}^{\mathcal{H}}_\nu$ and $\operatorname{Ric}^{\mathcal{V}}_\nu$ denote Ric_ν restricted to the horizontal and vertical distribution, respectively. With the results previously found, the equation (3.1) can be written as

$$-\operatorname{Ric}^{\mathcal{H}}_\nu = \left[\frac{\ddot\varphi}{\varphi} + k\left(\ddot\nu + \dot\nu^2\right)\right] \mathrm{d}r^2 + \varphi[\ddot\varphi + k\dot\nu\dot\varphi]\mathrm{d}\theta^2.$$

[2] Note that in [GW09], the third equation has a sign mistake, displaying $-k|\nabla\nu|^2$ instead of a positive quantity; the curvature tensor of the warped product $B^m \times_\nu F^k$ is given, for vertical V_i's, by

$$R(V_1, V_2) V_3 = R_F(V_1, V_2) V_3 - e^{2\nu} |\nabla\nu|^2 \left(\langle V_2, V_3\rangle V_1 - \langle V_1, V_3\rangle V_2\right)$$

which, when summed over a vertical orthonormal basis $\{W_j\}_j$, yields the vertical component of the Ricci curvature. Then,

$$|\nabla\nu|^2 \sum_j^k \langle\langle V_2, V_3\rangle W_j, W_j\rangle = \langle V_2, V_3\rangle k |\nabla\nu|^2$$

which has positive sign.

Indeed, $k\,\mathrm{Hess}\,\nu = k(\ddot{\nu}\,\mathrm{d}r^2 + \dot{\nu}\varphi\dot{\varphi}\,\mathrm{d}\theta^2)$, $\mathrm{Ric}_B = -(\frac{\ddot{\varphi}}{\varphi}\,\mathrm{d}r^2 + \ddot{\varphi}\varphi\,\mathrm{d}\theta^2)$ and

$$kg(\nabla\nu,\cdot)g(\nabla\nu,\cdot) = kg(\dot{\nu}\partial_r,\cdot)g(\dot{\nu}\partial_r,\cdot) = k\dot{\nu}^2\,\mathrm{d}r\otimes\mathrm{d}r = k\dot{\nu}^2\,\mathrm{d}r^2.$$

Similarly, the equation (3.3) can be written as

$$\mathrm{Ric}_\nu^{\mathcal{V}} = \mathrm{Ric}_F - e^{2\nu}\left(\ddot{\nu} + \frac{\dot{\varphi}\dot{\nu}}{\varphi} + k\dot{\nu}^2\right)g_F,$$

where we used that the Laplacian $\Delta\nu$, defined as positive[3] the trace of the Hessian, is given by

$$\Delta\nu = \mathrm{tr}(\mathrm{Hess}\,\nu) = g^{ij}(\mathrm{Hess}\,\nu)_{ij} = \mathrm{tr}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{\varphi^2}\end{pmatrix}\begin{pmatrix}\ddot{\nu} & 0\\ 0 & \dot{\nu}\varphi\dot{\varphi}\end{pmatrix} = \ddot{\nu} + \frac{\dot{\varphi}\dot{\nu}}{\varphi}.$$

Note that the Ricci curvature Ric_M, $M := S^2_\varphi \times_{\nu+\ln\lambda} F$, is positive if and only if both the horizontal and the vertical components $\mathrm{Ric}^{\mathcal{H}}$ and $\mathrm{Ric}^{\mathcal{V}}$ are positive. Notice that since $\mathrm{Ric}_F \geq 1$, if $\mathrm{Ric}_\nu^{\mathcal{H}}$ is positive, then these equations together with equation (3.2) imply that $S^2_\varphi \times_{\nu+\ln\lambda} F$ has positive Ricci curvature, provided λ is a sufficiently small positive constant; changing the warped metric with this small constant only affects the vertical component of the Ricci curvature, as it vanishes anywhere else under differentiation.

3.3 Construction of S^2_φ, and the warping function ν

With the explicit form of the Ricci tensor of the warped sphere S^2_φ, we can obtain parts of negative Ricci curvature on S^2_φ, by finding a point $p \in (0,\pi)$ so that $\ddot{\varphi}(p) > 0$. Then, the projection $\pi_1\colon S^2_\varphi \times_\nu F \longrightarrow S^2_\varphi$ is a Riemannian submersion (by the remark following Definition 2.23) for which the base has points of negative Ricci curvature, and with some additional conditions on the warping function ν, we wish to preserve positive Ricci curvature upstairs in the total space. Therefore, it suffices to construct functions φ and ν such that

1. S^2_φ is smooth and has points of negative Ricci curvature, that is,

   $$\ddot{\varphi}(p) = \eta > 0$$

   for some point $p \in (0,\pi)$ (recall that $\mathrm{Ric}_{S^2_\varphi} = -\frac{\ddot{\varphi}}{\varphi}g_\varphi$, and φ is a positive function),

2. $\mathrm{Ric}_\nu^{\mathcal{H}} > 0$, that is,

   $$\ddot{\nu} + \dot{\nu}^2 < -\frac{\ddot{\varphi}}{k\varphi}$$
   $$\dot{\varphi}\dot{\nu} < -\frac{\ddot{\varphi}}{k}$$

   and

3. ν is constant in a neighborhood of 0 and π.

These conditions are summarised in a lemma that Pro and Wilhelm used in order to build their counterexample, to which we give a slightly more detailed proof for readability. In fact, our purpose is to prove, in Section 5, a result in which we use a similar construction as this one but that requires the function φ to be equal to sin everywhere except on an arbitrarily small neighbourhood, and the

[3] Some authors define the Laplacian as the *negative* trace of the Hessian.

warping function ν needs to be made constant everywhere on $[0,\pi]$ except on a small neighbourhood around p. Therefore, we prove a stronger lemma than the one in [PW14].

Lemma 3.8. *For all $\eta > 0$ sufficiently small, and for all $p \in (0, \pi/4)$ and $\varepsilon > 0$, there exists $\delta > 0$ and smooth functions $\varphi, \nu \colon [0,\pi] \to [0,\infty)$ such that:*

1. *$\ddot{\varphi}$ is maximal at p with $\ddot{\varphi}(p) = \eta > 0$.*
2. *On $[p-\delta, p+\delta]$,*
$$\|\varphi - \sin\|_{\mathcal{C}^1} < \varepsilon$$
and for all r outside of $(p-\delta, p+\delta)$,
$$\varphi(r) = \sin(r).$$
3. *ν is constant outside of $(p-2\delta, p+2\delta)$.*
4. *On $(p-2\delta, p+2\delta)$,*
$$\ddot{\nu} + \dot{\nu}^2 < -\frac{\ddot{\varphi}}{k\varphi}, \text{ and } \dot{\nu}\dot{\varphi} < -\frac{\ddot{\varphi}}{k}.$$

We also define the $\mathcal{C}^k$ norm, for $\mathcal{C}^k$ functions $f \colon \mathbb{R} \to \mathbb{R}$ as

$$\|f\|_{\mathcal{C}^k} \coloneqq \sup_{i\le k} \sup_{x\in\mathbb{R}} \left\{ \left| f^{(i)}(x) \right| \right\}.$$

These conditions mean that the function φ is everywhere but on a small interval equal to sin; we take δ small enough so that the interval on which we perform the "curvature surgery" by making $\ddot{\varphi}$ far from $\frac{\mathrm{d}^2}{\mathrm{d}r^2}\sin(r) = -\sin(r)$ doesn't perturb the Ricci curvature of the warped product $S^2_\varphi \times_\nu F$. To do so, we also need conditions on the warping function ν to control the curvature on the warped product.

Thus, in some sense, the sphere S^2_φ is everywhere but on a small interval the round sphere S^2.

Proof. The idea is to construct such a function φ that achieves 1. and 2. by imposing conditions on $\ddot{\varphi}$, namely control the size of the interval on which it is different from $\frac{\mathrm{d}^2}{\mathrm{d}r^2}\sin(r) = -\sin(r)$, and then integrating twice. To do so, we need to construct $\ddot{\varphi}$ so that it reaches its maximum $\eta > 0$ at p. Since $-\sin$ is negative and concave up, in particular, around p, $\ddot{\varphi}$ must differ a lot from $-\sin$. To improve readability we separate the proof into different sections.

I. From $\ddot{\varphi}$ to $\dot{\varphi}$:
On $[0,\pi] \setminus (p-\delta, p+\delta)$, set $\ddot{\varphi} = -\sin$. Define a temporary function $\widetilde{\ddot{\varphi}}$ on $(p-\delta, p+\delta)$ as

$$\widetilde{\ddot{\varphi}}(t) = -\sin(t) + f(t)$$

where

$$f(t) = \begin{cases} (\eta + \sin(p))\left(1 - \frac{|t-p|}{\varepsilon\delta}\right) & t \in (p-\delta\varepsilon, p+\delta\varepsilon) \\ 0 & \text{otherwise} \end{cases}$$

is a continuous function reaching $\eta + \sin(p)$ at $t = p$ on $(p-\delta\varepsilon, p+\delta\varepsilon)$ with a very thin spike. Thus, $\widetilde{\ddot{\varphi}}$ is everywhere $-\sin$ except near p, where it achieves its maximum $\widetilde{\ddot{\varphi}}(p) = \eta > 0$ in an almost affine way, creating a very narrow "dent". The area of the spike, i.e. the integral of f can be easily computed with the formula for the area of triangles:

$$\int_{p-\delta\varepsilon}^{p+\delta\varepsilon} f(s)\,\mathrm{d}s = \frac{1}{2}\cdot 2\delta\varepsilon(\eta+\sin(p)) < \varepsilon$$

where the last inequality is true since we will choose δ and η sufficiently small, i.e. $\delta < 1$ and $\eta + \sin(p) < 1$ (recall that $p < \pi/4$, so $\sin(p) < \sqrt{2}/2$). However, we still need to correct the offset created by this dent, i.e. make it so that

$$\int_{p-\delta}^{p+\delta} \widetilde{\ddot{\varphi}}(s)\,\mathrm{d}s = \int_{p-\delta}^{p+\delta} -\sin(s)\,\mathrm{d}s$$

in order for $\dot{\varphi}(p+\delta)$ to be equal to $\cos(p+\delta)$. Let thus

$$A := \int_{p-\delta}^{p+\delta} \widetilde{\ddot{\varphi}}(s) + \sin(s)\,\mathrm{d}s = \int_{p-\delta\varepsilon}^{p+\delta\varepsilon} f(s)\,\mathrm{d}s = \delta\varepsilon(\eta+\sin(p))$$

and consider a bump function $\beta\colon [p-\delta, p+\delta] \to [0,\infty)$ such that $\beta(p\pm\delta) = \beta(p) = 0$, with vanishing derivatives at the endpoints, and with total integral $+1$. Define $\ddot{\varphi}(t) := \widetilde{\ddot{\varphi}}(t) - A\beta(t)$, and call $\widetilde{\dot{\varphi}}$ the integral of $\ddot{\varphi}$. Thus, the integral of $\ddot{\varphi}$ equals the integral of $-\sin$ by construction and since the integral of f is small, the number $A\beta$ is small as well and so $|\widetilde{\dot{\varphi}} - \cos| < \varepsilon$ on the interval $(p-\delta, p+\delta)$: for $s \in (p-\delta, p+\delta)$,

$$\begin{aligned}
|\widetilde{\dot{\varphi}}(s) - \cos(s)| &= |\widetilde{\dot{\varphi}}(p-\delta) + \int_{p-\delta}^{s} \widetilde{\ddot{\varphi}}(t) - A\beta(t)\,\mathrm{d}t - \cos(s)| \\
&= |\cos(p-\delta) + \int_{p-\delta}^{s} -\sin(t) + f(t) - A\beta(t)\,\mathrm{d}t - \cos(s)| \\
&= |\cos(p-\delta) + \cos(s) - \cos(p-\delta) - \cos(s) + \int_{p-\delta}^{s} f(t) - A\beta(t)\,\mathrm{d}t| \\
&= \left|\int_{p-\delta}^{s} f(t)\,\mathrm{d}t - A\int_{p-\delta}^{s} \beta(t)\,\mathrm{d}t\right| \\
&\le \left|\int_{p-\delta}^{s} f(t)\,\mathrm{d}t - A\right| \\
&\le \left|\int_{p-\delta}^{p+\delta} f(t)\,\mathrm{d}t\right| \\
&= \delta\varepsilon(\eta+\sin(p)) \\
&< \varepsilon,
\end{aligned}$$

assuming that $\delta < 1$ and $\eta + \sin(p) < 1$.

II. From $\dot{\varphi}$ to φ:

With the function $\ddot{\varphi}$ defined above, we can bound $|\varphi - \sin|$. First, we repeat the method from the previous point by choosing a bump function $\gamma\colon (p-\delta, p+\delta) \to [0,\infty)$, with vanishing derivatives at the endpoints and at the point $t = p$, and with total integral $+1$. Then, we correct the integral offset by considering

$$B := \int_{p-\delta}^{p+\delta} \widetilde{\dot{\varphi}}(s) - \cos(s)\,\mathrm{d}s$$

and by defining $\dot{\varphi}(s) := \widetilde{\dot{\varphi}}(s) - B\gamma(s)$, $s \in (p-\delta, p+\delta)$. The integral of $\dot{\varphi}$ now equals the integral of cos, i.e. sin, by construction. Similarly as before, we can achieve the bound $|\varphi - \sin| < \varepsilon$ on $(p-\delta, p+\delta)$:

$$\begin{aligned}
|\varphi(s) - \sin(s)| &= \left|\varphi(p-\delta) + \int_{p-\delta}^{s} \dot{\varphi}(t)\,\mathrm{d}t - \sin(s)\right| \\
&= \left|\sin(p-\delta) + \int_{p-\delta}^{s} \widetilde{\dot{\varphi}}(t) - B\gamma(t)\,\mathrm{d}t - \sin(s)\right| \\
&= \left|\sin(p-\delta) - \sin(s) + \int_{p-\delta}^{s} \widetilde{\dot{\varphi}}(t)\,\mathrm{d}t - B\int_{p-\delta}^{s} \gamma(t)\,\mathrm{d}t\right| \\
&\leq \left|\sin(p-\delta) - \sin(s) + \int_{p-\delta}^{s} \cos(t) + \varepsilon\,\mathrm{d}t - B\right|,
\end{aligned}$$

where we have used that $\widetilde{\dot{\varphi}}(t) < \cos(t) + \varepsilon$ by the previous calculations. Now, by the bound on $\widetilde{\dot{\varphi}}(t) - \cos(t)$ previously found, we have that B can be bounded above by $2\delta\varepsilon$, which leads to

$$\begin{aligned}
\left|\sin(p-\delta) - \sin(s) + \int_{p-\delta}^{s} \cos(t) + \varepsilon\,\mathrm{d}t - B\right| &\leq |(s - (p-\delta)) \cdot \varepsilon - 2\delta \cdot \varepsilon| \\
&= |\varepsilon \cdot (s - p - \delta)| \\
&< 2\delta\varepsilon \\
&< \varepsilon
\end{aligned}$$

as long as we pick $\delta < \frac{1}{2}$. Hence, we constructed a function φ that satisfies the conditions (1) and (2). At the points $t = p \pm \delta\varepsilon$, since the function $\ddot{\varphi}$ is only continuous and not differentiable, one might use a smoothing argument to retrieve differentiability.

III.1 Construction of ν | reduction of conditions (3) and (4):
Once we have constructed φ, we can construct the function ν so that it satisfies (3) and (4). To retrieve property (4), we look for sufficient conditions, that we split between the intervals $[0, p-\delta]$, $(p-\delta, p+\delta)$ and $(p+\delta, p+2\delta)$ (note that it is possible that $p - 2\delta < 0$, but we rule out this case by requiring δ to be small enough so that $p - 2\delta > 0$). For (4) to hold on $[0, p-\delta]$, it is enough to have

$$\nu|_{[0,p-2\delta]} \equiv 0, \tag{3.4}$$

$$(\ddot{\nu} + \dot{\nu}^2)|_{(p-2\delta,p-\delta]} < 0, \quad \text{and} \tag{3.5}$$

$$\dot{\nu}|_{[p-2\delta,p-\delta]} < 0. \tag{3.6}$$

Indeed, we need ν to be constant on a neighbourhood of 0. On $[0, p-\delta] \cap (p-2\delta, p+\delta) = (p-2\delta, p-\delta)$, $\ddot{\varphi}$ and φ have different signs and thus the quotient $-\ddot{\varphi}/(k\varphi)$ is positive; therefore it suffices to have $\ddot{\nu} + \dot{\nu}^2 < 0$. Moreover, since on the interval $(p-2\delta, p-\delta)$, $\varphi = \sin$, the second part of condition (4) reduces to

$$\dot{\nu} < \frac{\sin}{k\cos}$$

and since this right-hand side is positive, this inequality is automatically verified if (3.6) is true.

Now, for (4) to hold on the remaining interval, namely $(p-\delta, p+\delta)$, note that we do not necessarily have $\|\varphi - \sin\|_{\mathcal{C}^2} < \varepsilon$ on that interval. However, we can use that $\|\varphi - \sin\|_{\mathcal{C}^1} < \varepsilon$, and that $\ddot{\varphi}$ reaches its maximum at p. This is done as follows: we claim that if

$$(\ddot{\nu} + \dot{\nu}^2)|_{[p-\delta,p+\delta]} < -2\frac{\eta}{kp}, \quad \text{and} \tag{3.7}$$

$$\frac{\dot{\nu}|_{[p-\delta,p+\delta]}}{\sqrt{2}} < -2\frac{\eta}{k} \tag{3.8}$$

are satisfied on the indicated intervals, then the condition (4) is met. Indeed, since $\ddot{\varphi} \leq \eta$,

$$-2\frac{\eta}{kp} \leq -2\frac{\ddot{\varphi}}{kp},$$

and for the right-hand side to be less than $-\frac{\ddot{\varphi}}{k\varphi}$, we need that $\varphi > p/2$. Since φ is almost sin, and $\sin(x) > x/2$ on $[0, \pi]$, then up to ε we have that $\varphi(p) > p/2$, and this is true on $[p-\delta, p+\delta]$ as long as we pick δ sufficiently small.

Now, for the second inequality, we have that if $\dot{\nu}$ satisfies $\frac{\dot{\nu}|_{[p-\delta,p+\delta]}}{\sqrt{2}} < -2\frac{\eta}{k}$, then

$$\dot{\nu}\dot{\varphi} < -\frac{2\sqrt{2}\eta}{k}\dot{\varphi}$$

and for this right-hand side to be less than $-\frac{\ddot{\varphi}}{k}$, it suffices that it is less than $-\frac{\eta}{k}$ since $\ddot{\varphi}$ is maximal with value η. But this is equivalent to

$$\dot{\varphi} > \frac{1}{2\sqrt{2}}$$

which is always true near p since $\dot{\varphi} \approx \cos$.

III.2 Explicit formula for ν on $[p-\delta, p+\delta]$:
Thus choose ν so that it satisfies (3.4), and further impose that

$$\dot{\nu}(p-\delta) = -3\frac{\eta}{k}, \tag{3.9}$$

$$\ddot{\nu}\Big|_{[p-\delta,p+\delta]} = -4\frac{\eta}{kp}. \tag{3.10}$$

This completely determines $\dot{\nu}$ on $[p-\delta, p+\delta]$, and ν up to a constant:

$$\dot{\nu}(t)|_{[p-\delta,p+\delta]} = -4\frac{\eta}{kp}\cdot t - 3\frac{\eta}{k} + 4\frac{\eta}{kp}\cdot(p-\delta) \tag{3.11}$$

$$\nu(t)|_{[p-\delta,p+\delta]} = -2\frac{\eta}{kp}\cdot t^2 - \left(3\frac{\eta}{k} - 4\frac{\eta}{kp}\cdot(p-\delta)\right)\cdot t + C_1 \tag{3.12}$$

Then, (3.7) is verified provided η is sufficiently small; by noticing $-3\frac{\eta}{k} - 8\frac{\eta}{kp}\cdot\delta = \dot{\nu}(p+\delta) \leq \dot{\nu}(t)$

for $t \in (p-\delta, p+\delta)$, we retrieve equivalent conditions for (3.7) to be met:

$$\begin{aligned} & (\ddot{\nu}+\dot{\nu}^2)|_{[p-\delta,p+\delta]} < -2\frac{\eta}{kp} \\ \iff \quad & -4\frac{\eta}{kp} + \left(-3\frac{\eta}{k} - 8\frac{\eta}{kp}\cdot\delta\right)^2 < -2\frac{\eta}{kp} \\ \iff \quad & 9\frac{\eta^2}{k^2} + 64\frac{\eta^2\delta^2}{k^2p^2} + 48\frac{\eta^2\delta}{k^2p} < 2\frac{\eta}{kp} \\ \iff \quad & \frac{\eta}{k}\left(9 + 64\frac{\delta^2}{p^2} + 48\frac{\delta}{p}\right) < \frac{2}{p}. \end{aligned}$$

Condition (3.8) can be verified as well, using again that $\dot{\nu}(p+\delta) \leq \dot{\nu}(t)$:

$$\begin{aligned} & \frac{\dot{\nu}|_{[p-\delta,p+\delta]}}{\sqrt{2}} < -2\frac{\eta}{k} \\ \iff \quad & -3\frac{\eta}{k} - 8\frac{\eta}{kp} - \delta < -2\sqrt{2}\frac{\eta}{k} \\ \iff \quad & \frac{\eta}{k}\left(-3 - \frac{8}{p} + 2\sqrt{2}\right) < \delta. \end{aligned}$$

Hence, η needs to be sufficiently small, which also constitutes a lower bound for δ in terms of η. Thus, when δ needs to be picked small enough, one has to beforehand choose η small to allow for infinitesimally small values of δ.

III.3 Prescribing ν on $[p-2\delta, p-\delta]$:
Now, on $(p-2\delta, p-\delta)$, we need to prescribe ν so that it connects $\nu(p-\delta)$ with $\nu(p-2\delta) = 0$, and such that both $\dot{\nu}$ and $\ddot{\nu}$ connect their respective value at $p-2\delta$ with 0 smoothly. One way of achieving this is for example choosing a bump function χ such that $\chi(p-2\delta) = 0$ and $\chi(p-\delta) = 1$, with vanishing derivatives at the endpoints, and then considering the "convex" combination $\Psi(t) = (1-\chi(t))\cdot 0 + \chi(t)\dot{\nu}(t)$ on $(p-2\delta, p-\delta)$, where the function $\dot{\nu}$ is momentarily defined as the continuation of $\dot{\nu}|_{[p-\delta,p+\delta]}$ to the interval $(p-2\delta, p-\delta)$. Then, we get, by computing via the explicit expression of $\dot{\nu}$, $\dot{\nu}(p-2\delta) = 4\frac{\eta}{kp}\cdot\delta - 3\frac{\eta}{k}$, and we get the following condition for negativity:

$$\begin{aligned} & 4\frac{\eta}{kp}\cdot\delta - 3\frac{\eta}{k} < 0 \\ \iff \quad & \frac{4}{p}\cdot\delta < 3 \\ \iff \quad & \delta < \frac{3p}{4}. \end{aligned}$$

The function Ψ connects $\dot{\nu}$ at $p-\delta$ to the zero function at $p-2\delta$, and Ψ' connects in the same way $\ddot{\nu}$ to the zero function. To ensure that (3.5) is met, we compute $\Psi'(t)$:

$$\Psi' = \dot{\chi}\dot{\nu} + \chi\ddot{\nu} < 0$$

because $\dot{\chi}, \chi$ are positive and $\dot{\nu}$ is negative for $\delta < \frac{3p}{4}$, and $\ddot{\nu} = -4\frac{\eta}{kp}$ is negative as well on

$(p-2\delta, p-\delta)$. Now we estimate

$$
\begin{aligned}
(\ddot\nu + \dot\nu^2)|_{(p-2\delta,p-\delta)} = \dot\Psi + \Psi^2 &= \dot\chi\dot\nu + \chi\ddot\nu + (\chi\dot\nu)^2 \\
&\leq \dot\chi\dot\nu + \chi^2\ddot\nu + (\chi\dot\nu)^2 && (\chi \leq 1,\ \ddot\nu < 0) \\
&\leq \dot\chi\dot\nu + \chi^2(\ddot\nu + \dot\nu^2) && (\dot\chi\dot\nu \leq 0,\ \chi^2(\ddot\nu+\dot\nu^2) < 0) \\
&< 0
\end{aligned}
$$

and hence (3.5) is met.

To connect ν with the zero function, integrate $\dot\nu$ on $(p-2\delta, p-\delta)$, and choose the integration constant C_1 in ν such that the integral $\int_{p-2\delta}^{t} \dot\nu$ connects continuously with ν at $t = p-\delta$.

III.4 Prescribing ν on $[p+\delta, p+2\delta]$:
Lastly, we need to make ν constant outside of $(p-2\delta, p+2\delta)$. By the construction above, ν is already constant (in fact, constantly 0) on $[0, p-2\delta]$. We repeat a similar process as above by choosing a bump function $\chi\colon [p+\delta, p+2\delta] \to \mathbb{R}$ with vanishing derivatives at the endpoints and such that $\chi(p+\delta) = 1$, $\chi(p+2\delta) = 0$. Then, the function $\xi = \chi\dot\nu$ interpolates between $\dot\nu$ at $p+\delta$ and the zero function at $p+2\delta$, and similarly $\dot\xi$ connects $\ddot\nu$ with the zero function. The condition (4) reduces to $\ddot\nu + \dot\nu^2 < \frac{1}{k}$ because on this interval $\varphi = \sin$. Recall that $\dot\nu(p+\delta) = -3\frac{\eta}{k} - 8\frac{\eta}{kp} \cdot \delta$, which may be made arbitrarily small for very small values of η.

We compute as above

$$
\begin{aligned}
(\ddot\nu + \dot\nu^2)|_{[p+\delta,p+2\delta]} &= \dot\xi + \xi^2 \\
&= \chi\ddot\nu + \dot\chi\dot\nu + (\chi\dot\nu)^2 \\
&< \dot\chi\dot\nu + \chi^2\dot\nu^2
\end{aligned}
$$

and both of these terms approach 0 as η gets small, i.e. there is an η sufficiently small so that this last quantity is less than $\frac{1}{k}$, and condition (4) is met. Now, since $\varphi = \sin$ on $(p+\delta, p+2\delta)$, the second inequality of (4) is simply

$$\dot\nu < -\frac{\ddot\varphi}{\dot\varphi k} = \frac{\sin}{\cos \cdot k}$$

which is automatically satisfied since the interpolation function ξ is negative on this interval.

Then, we can integrate $\dot\nu$ on $(p+\delta, p+2\delta)$, and choose the integration constant so that the result connects continuously with ν at $p+\delta$. On $(p+2\delta, \pi]$, extend ν constantly by setting $\nu \equiv \nu(p+2\delta)$, and this finishes the proof. □

Let π_1 denote the projection $\pi_1\colon S^2 \times F \to S^2$. The following lemma is the last piece needed to prove Theorem 3.1:

Lemma 3.9. *For all $\eta > 0$ and for all $p \in (0, \pi/4)$, there is $\varepsilon = \varepsilon(p) > 0$, such that for φ and ν in Lemma 3.8,*

$$\mathrm{Ric}^{\mathcal{H}}_{\nu} \geq \frac{1}{2}$$

on $(r \circ \pi_1)^{-1}([0,\pi] \setminus (p-2\delta, p+2\delta))$.

For simplicity we refer to $r \circ \pi_1$ as just r, viewing r as a function on the product. Recall that $\mathrm{Ric}^{\mathcal{H}}_{\nu} \geq \frac{1}{2}$ means that all eigenvalues of the $(0,2)$-tensor $\mathrm{Ric}^{\mathcal{H}}_{\nu}$ are greater or equal to $1/2$, or

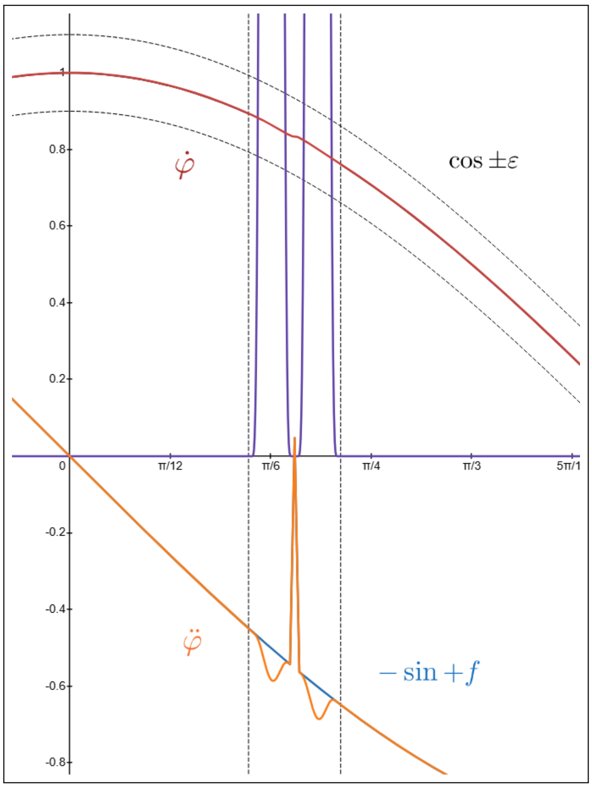

Figure 3: Affine modification of $-\sin$ to yield $\ddot{\varphi}$, and its integral $\dot{\varphi}$. Purple: bump function vanishing at $t = p$. For $0 < \delta \ll 1$, the value of the bump function $|\beta|$ becomes arbitrarily big, but it practically doesn't impact $\ddot{\varphi}$ because $\ddot{\varphi} = -\sin + f - A\beta$ and A decreases more rapidly than β increases. Values used: $p = \pi/4 - 0.2$, $\varepsilon = 0.1$, $\eta = 0.05$, $\delta = 0.1$.

equivalently that

$$\mathrm{Ric}^{\mathcal{H}}_{\nu}(X, X) \geq \frac{1}{2} g_{\varphi}(X, X)$$

for every vector $X \in TS^2_{\varphi}$, where g_{φ} is the metric on the warped sphere S^2_{φ}.

Proof. By Lemma 3.8, on the set $r^{-1}([0, \pi] \setminus (p - 2\delta, p + 2\delta))$, we have arranged that ν is constant, say $\nu = \widetilde{C}(\eta)$ (since in the construction the value of ν on $[p+2\delta, \pi]$ depends on η, $\nu|_{[p+2\delta,\pi]} = O(\eta)$), and that $\varphi = \sin$. Choosing a unit vector in the dr direction, we find

$$\mathrm{Ric}^{\mathcal{H}}_{\nu}(\partial_r, \partial_r) = -\frac{\ddot{\varphi}}{\varphi} - k\ddot{\nu} - k\nu^2 > -k \cdot \widetilde{C}(\eta)^2 + 1. \tag{3.13}$$

By picking η sufficiently small, we find that (3.13) is greater than $\frac{1}{2} g_{\varphi}(\partial_r, \partial_r) = \frac{1}{2}$. The second component of the Ricci tensor gives

$$\mathrm{Ric}^{\mathcal{H}}_{\nu}(\partial_\theta, \partial_\theta) = -\varphi\ddot{\varphi} - k\varphi\dot{\nu}\dot{\varphi} = \sin^2$$

which is always bigger than $\frac{1}{2} g_{\varphi}(\partial_\theta, \partial_\theta) = \frac{1}{2}\varphi^2 = \frac{1}{2}\sin^2$ since $1 \geq \frac{1}{2}$ and $\sin^2$ is positive. □

Proof of Theorem 3.1. Let $\eta > 0$. For $C' > 0$, choose $p \in (0, \pi/4)$ such that $-\eta/(2\sin(p)) < -C'$. Again, let φ and ν be the functions of Lemma 3.8. Recall that it suffices to have $\mathrm{Ric}^{\mathcal{H}} > 0$ in order to have positive Ricci curvature on the total space. By Lemma 3.9 this condition is satisfied everywhere except possibly on $(p - 2\delta, p + 2\delta)$. Recall that the horizontal Ricci tensor is given by

$$-\mathrm{Ric}^{\mathcal{H}}_{\nu} = \left[\frac{\ddot{\varphi}}{\varphi} + k\left(\ddot{\nu} + \dot{\nu}^2\right)\right] \mathrm{d}r^2 + \varphi[\ddot{\varphi} + k\dot{\nu}\dot{\varphi}]\mathrm{d}\theta^2,$$

and Lemma 3.8 has arranged that on $(p - 2\delta, p + \delta)$,

$$\ddot{\nu} + \dot{\nu}^2 < -\frac{\ddot{\varphi}}{k\varphi}, \text{ and } \dot{\nu}\dot{\varphi} < -\frac{\ddot{\varphi}}{k}$$

which means precisely that $\mathrm{Ric}^{\mathcal{H}} > 0$ on this interval.

It thus remains to find a point on the base space S^2_φ where the Ricci curvature is negative. On the fiber $r^{-1}(p) \subset S^2_\varphi$ (which represents a latitudinal circle), the Ricci tensor, as calculated above, is $-\ddot{\varphi}(p)/\varphi(p) \cdot g_\varphi$. But $\ddot{\varphi}(p) = \eta > 0$ and $\|\varphi(p) - \sin(p)\|_{\mathcal{C}^1} < \varepsilon$, and so we finally get that, for a unit length tangent vector $E \in T_q S^2_\varphi$, $q \in r^{-1}(p)$,

$$\mathrm{Ric}_{S^2_\varphi}(E, E) = -\frac{\ddot{\varphi}(p)}{\varphi(p)} < -\frac{\eta}{\sin(p) + \varepsilon} < -\frac{\eta}{2\sin(p)} < -C'$$

for some constant $C' > 0$, since $\sin(p) + \varepsilon > 0$. □

Remark 3.10. The size of the interval $(p - \delta, p + \delta)$ on which the curvature surgery is performed depends on the upper bound $-C'$ we aim to impose on the Ricci curvature of the base space S^2_φ. As the proof of Theorem 3.1 suggests, if p is chosen so that $-\eta/(2\sin(p)) < -C'$, then for very large (negative) values of C', p has to be chosen arbitrarily close to 0 (since η in the numerator cannot be made too big) which forces $(p - \delta, p + \delta)$ to be small.

3.4 Global result

We have just seen that it is possible that the total space of a Riemannian submersion $\pi\colon M \to B$ has positive Ricci curvature, while the base space has parts of negative Ricci curvature. This is a local result, and one can ask if it might be true globally, in the following sense: if M is a compact Riemannian manifold with positive Ricci curvature, and $\pi\colon M \to B$ is a Riemannian submersion, can B have negative Ricci curvature globally? The answer is given in Theorem 3.2, stated by Pro and Wilhelm ([PW14]), and suggests that it is not even true if you improve to nonpositive curvature. The proof uses the unit tangent bundle of M and its compactness (inherited from the compactness of M):

Definition 3.11 (Unit tangent bundle)**.** Let (M, g) be a Riemannian manifold with tangent bundle TM. The *unit tangent bundle*, denoted T^1M, is the set

$$T^1M := \bigcup_{p \in M} \{v \in T_pM \mid g_p(v, v) = 1\} \subset TM,$$

which corresponds to all tangent vectors that have norm 1 throughout the manifold. It is a topological space with the topology induced from TM.

Now, endow T^1M with a Riemannian metric g_{T^1M}, and achoose a metric (in the sense of a distance function) on T^1M that yields the same topology as the induced one. Since T^1M is again a smooth manifold, such a metric always exists (for example the distance defined by $d\colon T^1M \times T^1M \to \mathbb{R}$, $d(p, q) = \inf_\gamma L(\gamma)$ where γ is a piecewise smooth curve joining p and q, see Appendix).

Lemma 3.12. *If M is compact, then its unit tangent bundle T^1M is compact as well.*

Proof. Let $p\colon T^1M \to M$ be the bundle projection. If TM is trivial then the result is immediate; the idea when TM is not trivial remains similar in the sense that we will cover M with "trivialisable" open sets. Let thus U_q be an open set around some point q, small enough that the set T^1U_q splits as $T^1U_q \cong U_q \times S^{n-1}$ (this is equivalent to U_q having an orthonormal frame, and every smooth manifold admits a local orthonormal frame around any point). Choose U_q also small enough so that its closure $\overline{U}_q$ in the manifold topology splits as $T^1\overline{U}_q \cong \overline{U}_q \times S^{n-1}$. Cover the manifold M with neighbourhoods of this sort. This forms a cover and since M is compact, we can extract a finite subcover $\bigcup_{i=1}^k \overline{U}_i$ for some k. Then, since $M = \bigcup \overline{U}_i$, we have

$$T^1M = \bigcup_i^k p^{-1}(\overline{U}_i) = \bigcup_i^k \overline{U}_i \times S^{n-1}.$$

Therefore T^1M can be written as a finite union of compact sets, and thus is compact. □

Consider now a Riemannian submersion $\pi\colon M \to B$ with M compact. The following proposition will be used in the proof of Theorem 3.2:

Proposition 3.13. *Let d be a distance such that (T^1M, d) yields the topology of T^1M. For all $\varepsilon > 0$, there exists a horizontal unit speed geodesic $\gamma\colon [0,l] \to M$ such that $l \geq 1$, and*

$$d(\gamma'(0), \gamma'(l)) < \varepsilon.$$

Proof. Take any horizontal unit speed geodesic $\gamma\colon \mathbb{R} \to M$. Since T^1M is compact, the sequence $(\gamma'(n))_n$ contains a convergent (hence Cauchy) subsequence. This means, that given $\varepsilon > 0$, we find sufficiently big integers j, k such that

$$d(\gamma'(j), \gamma'(k)) < \varepsilon.$$

Consider the reparametrisation c of γ such that $c'(t) = \gamma'(t + j)$. Then, $c'(0) = \gamma'(j)$, $c'(k - j) = \gamma'(k)$ and since γ has unit speed, c has unit speed as well, and this proves the claim by setting $l := k - j$. □

Next, denote $\mathcal{H}^1$ the subset of T^1M consisting of horizontal unit vectors throughout M, for a given Riemannian submersion π. Let T be the O'Neill T tensor. Since T^1M is compact, so is $\mathcal{H}^1$, and thus the continuous map $\mathcal{T}\colon (\mathcal{H}^1, d) \to \mathbb{R}$, $\mathcal{T}(x) := \operatorname{tr}(V \mapsto T_V x)$ is uniformly continuous.

Proof of Theorem 3.2. By contradiction suppose that $\operatorname{Ric}_M > 0$ but $\operatorname{Ric}_B \leq 0$. If g denotes the metric on M, then by considering a sufficiently small number $\lambda > 0$, the manifold (M_λ, g_2), where $g_2 := \lambda \cdot g$, has Ricci curvature $\operatorname{Ric}_{M_\lambda} \geq 1$, because

$$\begin{aligned}
\operatorname{Ric}_{M_\lambda} &= \sum_k g_2 \left(R \left(\frac{1}{\sqrt{\lambda}} v_k, \frac{1}{\sqrt{\lambda}} X \right) \frac{1}{\sqrt{\lambda}} X, \frac{1}{\sqrt{\lambda}} v_k \right) \\
&= \sum_k \frac{\lambda}{\lambda^2} g(R(v_k, X)X, v_k) \\
&\geq \frac{1}{\lambda} \inf_{p,E} \operatorname{Ric}_p(E, E)
\end{aligned}$$

for any unit vectors E and X with respect to g. In particular since M is compact this infimum is non zero. So upon changing the metric, we can assume $\operatorname{Ric}_M \geq 1$. Let γ be a geodesic satisfying the condition in Proposition 3.13. Then,

$$l = \int_0^l 1\,\mathrm{d}t \leq \int_0^l \mathrm{Ric}_M(\gamma'(t), \gamma'(t))\,\mathrm{d}t.$$

Consider an orthonormal frame $\{V_k\}_k$ along the curve γ for the vertical distribution, such that each V_i is parallel along γ, and an orthonormal horizontal frame $\{X_k\}_k$. Then, using equation (2.14), we get

$$\begin{aligned}\mathrm{Ric}(\gamma', \gamma') &= \widetilde{\mathrm{Ric}}(\gamma', \gamma') - 2\langle A_{\gamma'}, A_{\gamma'}\rangle - \langle T\gamma', T\gamma'\rangle + \langle \nabla_{\gamma'} N, \gamma'\rangle \\ &= \widetilde{\mathrm{Ric}}(\gamma', \gamma') - 2\sum_i |A_{\gamma'} V_i|^2 - \sum_j |T_{V_j}\gamma'|^2 + \sum_k \langle \nabla_{\gamma'}(T_{V_k} V_k), \gamma'\rangle.\end{aligned}$$

Now, $\langle \nabla_{\gamma'}(T_{V_k} V_k), \gamma'\rangle = \gamma'\langle T_{V_k} V_k, \gamma'\rangle - \langle T_{V_k} V_k, \nabla_{\gamma'}\gamma'\rangle$ and this last inner product vanishes since γ is a geodesic. Since we assumed that the base had nonpositive Ricci curvature,

$$\widetilde{\mathrm{Ric}}(\gamma', \gamma') - 2\sum_i |A_{\gamma'} V_i|^2 - \sum_j |T_{V_j}\gamma'|^2 \leq 0,$$

and integrating yields

$$\begin{aligned} l \leq \int_0^l \mathrm{Ric}(\gamma', \gamma') &\leq \int_0^l \sum_k \gamma'\langle T_{V_k} V_k, \gamma'\rangle \\ &= \sum_k \langle T_{V_k} V_k, \gamma'\rangle\Big|_0^l. \end{aligned} \tag{A}$$

Property 4 of the T tensor lets us rewrite the inner products as $\sum_k -\langle V_k, T_{V_k}\gamma'\rangle$. But this is nothing else than $\mathrm{trace}(W \mapsto T_W \gamma')$ (or as written above, $\mathcal{T}(\gamma')$).

Let now $\varepsilon \coloneqq 1/2$. Since $\mathcal{T}$ is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |\mathcal{T}(x) - \mathcal{T}(y)| < \frac{1}{2}.$$

However small δ is, Proposition 3.13 ensures that we can always find a horizontal geodesic γ and a number $l \geq 1$ such that

$$d(\gamma'(0), \gamma'(l)) < \delta$$

and replacing x, y with $\gamma'(0), \gamma'(l)$ above yields a contradiction in (A), because $l \geq 1$. □

3.5 Ricci tensor of higher dimensional warped spheres S^n_φ

This subsection studies how warping the standard round metric of the sphere $S^n(\kappa)$ of radius $\kappa > 0$ affects its Ricci curvatures. If $S^n(\kappa) = \{x_0, \ldots, x_n \in \mathbb{R}^{n+1} \mid \sum_{k=1}^n x_k^2 = \kappa^2\}$ is parametrised by the coordinates

$$x_0 = \kappa^2 \cos r, \quad x_1 = \kappa^2 \sin\theta_1 \cos r, \quad \ldots \quad x_{n-1} = \kappa^2 \cos\theta_{n-1} \prod_{k=1}^{n-2} \sin\theta_k, \quad x_n = \kappa^2 \prod_{k=1}^{n} \sin\theta_k$$

where $r, \theta_1, \ldots, \theta_{n-1} \in [0, \pi]$ and $\theta_n \in [0, 2\pi]$, we get the round metric

$$\mathrm{d}s_n^2 = \kappa^2 \mathrm{d}r^2 + \kappa^2 \sin^2 r(\mathrm{d}\theta_1^2 + \sin^2\theta_1(\mathrm{d}\theta_2^2 + \cdots (\mathrm{d}\theta_{n-1}^2 + \sin^2\theta_{n-1})))$$

which is simply $\kappa^2 \mathrm{d}r^2 + \kappa^2 \sin(r)^2 \mathrm{d}s_{n-1}^2$. Then one can choose as above a function $\varphi\colon [0,\pi] \to [0,\infty)$ that depends on r, and consider the warped metric $\kappa^2(\mathrm{d}r^2 + \varphi(r)^2 \mathrm{d}s_{n-1}^2)$. Since the round metric of the sphere of radius $\kappa > 0$ is $\frac{1}{\kappa^2}$ times the metric of the sphere of radius 1, the same goes for the Ricci tensor and thus we only compute the radius 1 case.

We would like to generalise the construction of the previous subsection from S^2 to S^n:

Lemma 3.14. *One can construct manifolds of the form* $M = S^n_\varphi \times_\nu F$ *with* $\mathrm{Ric}_F \geq 1$*, such that the projection* $\pi_1\colon M \to S^n_\varphi$ *is a Riemannian submersion,* $\mathrm{Ric}_M > 0$ *but* S^n_φ *has points of negative Ricci curvature.*

Formulas from the previous subsection can be used to obtain the Ricci tensor of the warped n-sphere.

Lemma 3.15. *The Ricci tensor of the sphere* S^n_φ *with the metric* $g_\varphi = \mathrm{d}r^2 + \varphi(r)^2 \mathrm{d}s_{n-1}^2$ *is*

$$\mathrm{Ric}_{S^n_\varphi} = -(n-1)\frac{\ddot{\varphi}}{\varphi}\mathrm{d}r^2 + \left(\mathrm{Ric}_{S^{n-1}} - (\varphi\ddot{\varphi} + (n-2)\dot{\varphi}^2) g_{S^{n-1}}\right).$$

Since the sphere S^{n-1} with its usual round metric is an Einstein manifold, i.e. it has Ricci tensor equal to $(n-2)g_{S^{n-1}}$, the formula above further simplifies to

$$\mathrm{Ric}_{S^n_\varphi} = -(n-1)\frac{\ddot{\varphi}}{\varphi}\mathrm{d}r^2 + \left[((n-2) - \varphi\ddot{\varphi} - (n-2)\dot{\varphi}^2) g_{S^{n-1}}\right].$$

For $n = 2$, the round metric of $S^{n-1} = S^1$ is simply $\mathrm{d}\theta$, and we recover the warped metric of the previous subsection $\frac{\ddot{\varphi}}{\varphi}\mathrm{d}r^2 + \varphi\ddot{\varphi}\mathrm{d}\theta^2$.

Proof of Lemma 3.15. We might use the same process as the previous subsection, by computing the Christoffel symbols and then using the associated formula for each individual Ricci component. However, as the dimension n gets big, this process can become very tedious if one does not have a general and simple form for the Christoffel symbols. Thus we apply Lemma 2.30 to the warped product $I \times_\varphi S^{n-1}$ with the metric $g_\varphi = \mathrm{d}r^2 + \varphi(r)^2 \mathrm{d}s_{n-1}^2$. Since $\dim S^{n-1} = n-1$, the term $(k-1)$ becomes $(n-2)$ in Lemma 2.30. We thus rewrite the term $\frac{\Delta f}{f} + (k-1)\frac{g(\nabla f, \nabla f)}{f^2}$:

$$\begin{aligned}\frac{\Delta\varphi}{\varphi} + (n-2)\frac{g(\nabla\varphi, \nabla\varphi)}{\varphi^2} &= \frac{\mathrm{tr}(\mathrm{Hess}\,\varphi)}{\varphi} + (n-2)\frac{g(\dot{\varphi}\partial_r, \dot{\varphi}\partial_r)}{\varphi^2} \\ &= \frac{\ddot{\varphi}}{\varphi} + (n-2)\frac{\dot{\varphi}^2}{\varphi^2}.\end{aligned}$$

Then, for $i, j \in \{2, \dots, n\}^2$ (where the index 1 corresponds to $\mathrm{d}r^2$)

$$\begin{aligned}\mathrm{Ric}_{ij} = \mathrm{Ric}(\partial_i, \partial_j) &= \mathrm{Ric}_F(\partial_i, \partial_j) + g(\partial_i, \partial_j)\left(\frac{\ddot{\varphi}}{\varphi} + (n-2)\frac{\dot{\varphi}^2}{\varphi^2}\right) \\ &= (n-2)\mathrm{d}s_{n-1}^2(\partial_i, \partial_j) + \varphi^2 \mathrm{d}s_{n-1}^2(\partial_i, \partial_j)\left(\frac{\ddot{\varphi}}{\varphi} + (n-2)\frac{\dot{\varphi}^2}{\varphi^2}\right) \\ &= (n-2)\mathrm{d}s_{n-1}^2(\partial_i, \partial_j) + \mathrm{d}s_{n-1}^2(\partial_i, \partial_j)\left(\varphi\ddot{\varphi} + (n-2)\dot{\varphi}^2\right).\end{aligned}$$

This implies that the vertical part $\mathrm{Ric}^{\mathcal{V}}$ of the Ricci tensor on S^n_φ is

$$\mathrm{Ric}^{\mathcal{V}} = ((n-2) - \varphi\ddot{\varphi} - (n-2)\dot{\varphi}^2) g_{S^{n-1}}.$$

The horizontal part of the Ricci tensor is retrieved using the first formula of Lemma 2.30. Note however that the base space $B = I$ is one-dimensional, and thus there is no Ricci tensor on it. Therefore the only term left to compute is $(k/f)\,\mathrm{Hess}\, f$, which is simply $(n-1)\frac{\ddot\varphi}{\varphi}\mathrm{d}r^2$. Altogether, we finally obtain

$$\mathrm{Ric} = -(n-1)\frac{\ddot\varphi}{\varphi}\mathrm{d}r^2 + ((n-2) - \varphi\ddot\varphi - (n-2)\dot\varphi^2)g_{S^{n-1}}. \qquad \square$$

Consider now the warped product $S^n_\varphi \times_\nu F$ where F is as above a Riemannian manifold with $\mathrm{Ric} > 1$, and the warping metric is $g_\varphi + e^{2\nu} g_F$ for a function ν on S^n that only depends on r. Using the formula $\mathrm{Hess}\,\nu(\partial_i, \partial_j) = \partial_i\partial_j\nu - \Gamma^k_{ij}\partial_k\nu$, we find

$$\mathrm{Hess}\,\nu = \begin{pmatrix} \ddot\nu & 0 & \cdots & \cdots & 0 \\ 0 & \varphi\dot\varphi\dot\nu & & & \vdots \\ \vdots & & \varphi\dot\varphi\dot\nu\sin^2\theta_1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \varphi\dot\varphi\dot\nu\sin^2\theta_1\cdots\sin^2\theta_{n-1} \end{pmatrix}$$

which is exactly the same result as Lemma 3.4, which suggested that $\mathrm{Hess}\,\nu$ is nothing but $\ddot\nu\mathrm{d}r^2 + \varphi\dot\varphi\dot\nu\mathrm{d}s_{n-1}$. This is in fact immediate from the Christoffel symbols formula in Lemma 3.4, in which l has to be 1 (because only the ∂_1 derivative of ν is non zero) which in turn forces the sum index k to be 1 since the metric is diagonal, so the only term is $-\partial_1 g_{ii}$ for $i = 1, \dots, n$. We rewrite this Hessian in terms of the round metric $g_{S^{n-1}} = \mathrm{d}s_{n-1}$ on S^{n-1} as

$$\mathrm{Hess}\,\nu = \ddot\nu\mathrm{d}r^2 + \varphi\dot\varphi\dot\nu g_{S^{n-1}}.$$

Since, by (3.1),

$$\mathrm{Ric}_\nu(X,Y) = \mathrm{Ric}_B(X,Y) - k(\mathrm{Hess}\,\nu(X,Y) + g(\nabla\nu, X)g(\nabla\nu, Y)),$$

for horizontal vectors X and Y, and $g(\nabla\nu, X)g(\nabla\nu, Y) = \dot\nu^2\mathrm{d}r^2$, it follows that

$$\mathrm{Ric}^{\mathcal{H}}_\nu = (-\varphi\ddot\varphi + (n-2)(-\dot\varphi^2 + 1) - k\varphi\dot\varphi\dot\nu)g_{S^{n-1}} - ((n-1)\frac{\ddot\varphi}{\varphi} + k(\ddot\nu + \dot\nu^2))\mathrm{d}r^2.$$

We recall formula (3.3):

$$\mathrm{Ric}_\nu(U,V) = \mathrm{Ric}_F(U,V) - g(U,V)e^{2\nu}\left(\Delta\nu + k|\nabla\nu|^2\right),$$

where $\Delta\nu$ is the Laplacian:

$$\begin{aligned}\Delta\nu &= \mathrm{tr}(\mathrm{Hess}\,\nu) \\ &= g^{ij}(\mathrm{Hess}\,\nu)_{ij} \\ &= \mathrm{diag}\left(1, \frac{1}{\varphi^2}, \dots, \frac{1}{\varphi^2\sin^2\theta_1\dots\sin^2\theta_{n-1}}\right)\cdot\mathrm{diag}\left(\ddot\nu, \varphi\dot\varphi\dot\nu, \dots, \varphi\dot\varphi\dot\nu\sin^2\theta_1\dots\sin^2\theta_{n-1}\right) \\ &= \ddot\nu + \sum_{k=2}^n \frac{\dot\varphi\dot\nu}{\varphi} = \ddot\nu + (n-1)\frac{\dot\varphi\dot\nu}{\varphi},\end{aligned}$$

and $k|\nabla\nu|^2$ is simply $kg(\nabla\nu, \nabla\nu)$ where g is the product metric on $S^n_\varphi \times F$. Since $\nabla\nu = \dot\nu\partial_r$, this is nothing but $k\dot\nu^2$. Therefore

$$\mathrm{Ric}^{\mathcal{V}}_\nu = \mathrm{Ric}_F - e^{2\nu}(\ddot\nu + (n-1)\frac{\dot\varphi\dot\nu}{\varphi} + k\dot\nu^2)g_F,$$

which is positive if Ric_F is sufficiently big, e.g. $\mathrm{Ric}_F \geq 1$ and the other term is sufficiently small, e.g. as in the Pro-Wilhelm construction by warping by $\nu + \ln\lambda$ for a very small constant $\lambda > 0$. For the horizontal part of the Ricci tensor to be positive, we obtain the following inequalities:

$$-\varphi\ddot\varphi + (n-2)(1-\dot\varphi^2) - k\varphi\dot\varphi\dot\nu > 0$$
$$-(n-1)\frac{\ddot\varphi}{\varphi} - k(\ddot\nu + \dot\nu^2) > 0$$

which we rewrite, separating the ν's and the φ's,

$$\frac{-\varphi\ddot\varphi + (n-2)(1-\dot\varphi^2)}{k\varphi} > \dot\nu\dot\varphi \tag{A}$$
$$-\frac{(n-1)}{k}\frac{\ddot\varphi}{\varphi} > \ddot\nu + \dot\nu^2. \tag{B}$$

Since φ is $\mathcal{C}^1$ ε-close to sin, then in particular for small enough values of ε, $\dot\varphi \approx \cos$ and so φ can always be chosen so that $\dot\varphi^2 < 1$. This means $(n-2)(1-\dot\varphi^2) > 0$ and the inequality (A) is satisfied provided

$$\frac{-\ddot\varphi}{k} > \dot\nu\dot\varphi$$

which is, together with (B), the same as the estimates in Lemma 3.8 (4). Overall, we get the analog of Lemma 3.8:

Lemma 3.16. *For all $p \in (0, \pi/4)$, $\varepsilon > 0$, and $\eta > 0$ sufficiently small, there exists $\delta > 0$ and functions $\varphi, \nu \colon [0, \pi] \to \mathbb{R}$ such that*

1. *$\varphi = \sin$ outside of $(p-\delta, p+\delta)$, and $\|\varphi - \sin\|_{\mathcal{C}^1} < \varepsilon$ on $(p-\delta, p+\delta)$;*
2. *$\ddot\varphi$ is maximal at p and equals $\ddot\varphi(p) = \eta > 0$;*
3. *ν is constant outside $(p-2\delta, p+2\delta)$;*
4. *On $(p-2\delta, p+\delta)$,*
$$\frac{-\ddot\varphi}{k\dot\varphi} > \dot\nu \quad \textit{and} \quad -\frac{(n-1)}{k}\frac{\ddot\varphi}{\varphi} > \ddot\nu + \dot\nu^2.$$

Proof of Lemma 3.14. It is immediate from Lemma 3.16; take φ and ν as above and so we find that the manifold $M = S^n_\varphi \times F$, $\mathrm{Ric}_F \geq 1$ with projection $\pi_1 \colon M \to S^n_\varphi$ has $\mathrm{Ric}_M > 0$ but S^n_φ has points of negative Ricci curvature by Lemma 3.15. The fact that the second inequality in (4) has $(n-1)$ in the numerator does not affect the construction of Lemma 3.8: it gives in fact more "room" for the functions $\ddot\nu$ and $\dot\nu$ since the upper bound is $\ddot\nu + \dot\nu^2 < \frac{n-1}{k}$ instead of $\ddot\nu + \dot\nu^2 < \frac{1}{k}$. □

4 Lifting Ricci curvature

4.1 Canonical Variation

This subsection will study the so-called "canonical variation" of a metric g, that we define below. This is a way of making the metric vary in the vertical direction, to avoid losing the isometry between the horizontal distribution and the tangent spaces of B. This process will yield new formulas that help us understand the relation between the curvatures of (M, g), (M, g_t), and F_b.

Definition 4.1 (Canonical variation)**.** Let $\pi\colon M \to B$ be a Riemannian submersion and $t > 0$. The *canonical variation* g_t of the metric g on M is the modification of this metric given by

$$\begin{aligned} g_t \restriction \mathcal{V} &= t \cdot g \restriction \mathcal{V} \\ g_t \restriction \mathcal{H} &= g \restriction \mathcal{H} \\ g_t(\mathcal{V}, \mathcal{H}) &= 0. \end{aligned}$$

Notice that the horizontal part of the metric is left unchanged, so the submersion π remains Riemannian, even with respect to the canonical variation of the initial metric. This is summarised in the following:

Remark 4.1: 1. For all $t > 0$, $\pi\colon (M, g_t) \to (B, g^1)$ is a Riemannian submersion with the same horizontal distribution $\mathcal{H}$ for all t, and on each fiber F_b, $(g_t^2)_b = t g_b^2$ (here, g_b^2 is the smooth family of metrics on the fibers). Vertical, horizontal and basic vector fields are the same for all g_t's.
2. If g has totally geodesic fibers, then g_t does as well for any $t > 0$.

To prove this, we can use the formulas presented in Lemma 4.2 .

Lemma 4.2. *If A^t, T^t are the equivalent for g_t of the tensors A and T, then, for all vertical vector fields U, V, W and horizontal vector fields X, Y, Z, we have*

1. $A^t_X Y = A_X Y$, *and* $A^t_X U = t A_X U$,
2. $T^t_U V = t T_U V$, *and* $T^t_U X = T_U X$,
3. $[\langle (\nabla_X A)_Y Z, U\rangle]^t = t\langle (\nabla_X A)_Y Z, U\rangle$,
4. $[\langle (\nabla_U A)_X Y, V\rangle]^t = t\langle (\nabla_U A)_X Y, V\rangle + (t - t^2)[\langle A_X U, A_Y V\rangle - \langle A_X V, A_Y U\rangle]$,
5. $[\langle (\nabla_X T)_U V, Y\rangle]^t = t\langle (\nabla_X T)_U V, Y\rangle$,
6. $[\langle (\nabla_U T)_V W, X\rangle]^t = t\langle (\nabla_U T)_V W, X\rangle + (t - t^2)\langle T_V W, A_X U\rangle$,
7. $[\check{\delta} A]^t = \check{\delta} A$,
8. $[\langle (\hat{\delta} T)U, X\rangle]^t = \langle (\hat{\delta} T)U, X\rangle + (1 - t)\langle T_U, A_X\rangle$,
9. $(\tilde{\delta} T)^t = t(\tilde{\delta} T)$,
10. $N^t = N$.

Hence, since the T-tensors of the metric g and of the metric g_t are related in this way, it follows automatically that it vanishes for the original metric if and only if it vanishes for the canonical variation of this metric. Therefore the totally geodesic property is invariant under vertical variation of the metric.

Proof. We will prove a few of these formulas; the general strategy is the same. We can in fact use the Koszul formula to show the following:

$$\begin{aligned}
\mathcal{H}\nabla^t_U V &= t\mathcal{H}\nabla_U V, & \mathcal{V}\nabla^t_U V &= \mathcal{V}\nabla_U V,\\
\mathcal{H}\nabla^t_X U &= t\mathcal{H}\nabla_X U, & \mathcal{V}\nabla^t_X U &= \mathcal{V}\nabla_X U,\\
\mathcal{H}\nabla^t_U X &= t\mathcal{H}\nabla_U X, & \mathcal{V}\nabla^t_U X &= \mathcal{V}\nabla_U X,\\
\mathcal{H}\nabla^t_X Y &= \mathcal{H}\nabla_X Y, & \mathcal{V}\nabla^t_X Y &= \mathcal{V}\nabla_X Y.
\end{aligned}$$

This goes by showing that the right-hand side of the formula

$$2g\left(\nabla_X Y, Z\right) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X),$$

and its analog for g_t (replace g by g_t and ∇ by ∇^t) are either equal, or one is a factor t times the other.

For 1. and 2. we may assume that the horizontal vector fields are basic, and expanding the A tensor gives

$$A^t_X Y = \mathcal{V}\nabla^t_{\mathcal{H}X}\mathcal{H}Y + \mathcal{H}\nabla^t_{\mathcal{H}X}\mathcal{V}Y = \mathcal{V}\nabla^t_X Y.$$

Using the Koszul formula with an arbitrary vector field F, we obtain

$$\begin{aligned}
2g_t(A^t_X Y, F) = 2g_t(\mathcal{V}\nabla^t_X Y, F) &= 2g_t(\nabla^t_X Y, \mathcal{V}F)\\
&= X(g_t(Y,\mathcal{V}F)) + Y(g_t(X,\mathcal{V}F)) - \mathcal{V}F(g_t(X,Y))\\
&\quad + g_t([X,Y],\mathcal{V}F) - g_t([X,\mathcal{V}F],Y) - g_t([Y,\mathcal{V}F],X).
\end{aligned}$$

But $X(g_t(Y,\mathcal{V}F)) = 0 = X(tg(Y,\mathcal{V}F))$, and $Y(g_t(X,\mathcal{V}F)) = 0 = Y(tg(X,\mathcal{V}F))$. The term $\mathcal{V}F(g_t(X,Y))$ is constantly 0 because the map $p \mapsto g_t(X,Y)_p$ is constant along the fibers. For the brackets, we remark that $[X,F]$ are $[Y,F]$ are vertical and thus the inner products involving them vanish. The last term can be dealt by noticing that

$$g_t([X,Y],\mathcal{V}F) = g_t(\mathcal{V}[X,Y],\mathcal{V}F) = tg(\mathcal{V}[X,Y],\mathcal{V}F),$$

by definition of the canonical variation restricted to vertical vector fields. Thus, the Koszul formula above gives $2g_t(A^t_X Y, F) = 2tg(A_X Y, F)$. But $A^t_X Y$ is vertical and so $g_t(A^t_X Y, F) = tg(A^t_X Y, F)$. Hence,

$$2tg(A^t_X Y, F) = 2g_t(A^t_X Y, F) = 2tg(A_X Y, F)$$

and we conclude, since F is arbitrary, that $A^t_X Y = A_X Y$ and with similar calculations for $\mathcal{H}\nabla^t_X Y$, this shows $\nabla^t_X Y = \nabla_X Y$.

For the second part of 1. the Koszul formula reduces to

$$2g_t(A^t_X U, F) = -g_t([X,\mathcal{H}F],U),$$

and since U is vertical, this is nothing but $-tg([X,\mathcal{H}F],U)$. So $g(A^t_X U, F) = g_t(A^t_X U, F) = tg(A_X U, F)$ which proves the second formula of 1, and $\mathcal{H}\nabla^t_X U = t\mathcal{H}\nabla_X U$ as well.

Equation 2 as well as the formulas for the covariant derivative, all use the exact same reasoning as equation 1.

For 3. we use the formula from Lemma 2.14 to get

$$\begin{aligned} g_t((\nabla^t_X A^t)_Y Z, U) &= g_t(\nabla^t_X(A^t_Y Z) - A^t_{\nabla^t_X Y} Z - A^t_Y(\nabla^t_X Z), U) \\ &= g_t(\mathcal{V}\nabla^t_X(A^t_Y Z) - A^t_{\mathcal{H}\nabla^t_X Y} Z - A^t_Y(\mathcal{H}\nabla^t_X Z), U). \end{aligned}$$

We deal with the first term:

$$\begin{aligned} \mathcal{V}\nabla^t_X(A^t_Y Z) &= \mathcal{V}\nabla^t_{A^t_Y Z} X + \mathcal{V}[X, A^t_Y Z] \\ &= T^t_{A_Y Z} X + \mathcal{V}[X, A_Y Z] \\ &= T_{A_Y Z} X + \mathcal{V}[X, A_Y Z] \\ &= \mathcal{V}\nabla_X(A_Y Z). \end{aligned}$$

For the second term, proof of 1. and 2. shows that $\mathcal{H}\nabla^t_X Y = \mathcal{H}\nabla_X Y$, so $A^t_{\mathcal{H}\nabla^t_X Y} Z = A^t_{\mathcal{H}\nabla_X Y} Z = A_{\mathcal{H}\nabla_X Y} Z$. The same argument applies to the third term. Hence overall, there is no contribution of the covariant derivatives or of the tensors to a multiplicative t factor. So $g_t((\nabla^t_X A^t)_Y Z, U) = tg((\nabla_X A)_Y Z, U)$, by verticality of U.

For 4. we develop the covariant derivative of the A tensor similarly as in 3. to obtain

$$(\nabla^t_U A^t)_X Y = \nabla^t_U(A^t_X Y) - A^t_{\nabla^t_U X} Y - A^t_X(\nabla^t_U Y).$$

Because of the stretch in the vertical direction, $g_t((\nabla^t_U A^t)_X Y, V) = tg((\nabla^t_U A^t)_X Y, V)$. So, formula 4 is true if and only if

$$tg((\nabla^t_U A^t)_X Y, V) = tg((\nabla_U A)_X Y, V) + (t - t^2)(g(A_X U, A_Y V) - g(A_X V, A_Y U)).$$

The rightmost terms are dealt with remarking that

$$g(A_X U, A_Y V) = -g(\nabla_Y(A_X U), V)$$

using properties 3 and 4 of the A tensor. Similarly, $g(A_X V, A_Y U) = -g(\nabla_X(A_Y U), V)$, and hence every vector can be brought inside the same inner product. Thus, the equation 4 can be rewritten

$$g((\nabla^t_U A^t)_X Y, V) = g((\nabla_U A)_X Y + (1-t)(-\nabla_Y(A_X U) + \nabla_X(A_Y U)), V),$$

or equivalently,

$$\mathcal{V}((\nabla^t_U A^t)_X Y) = \mathcal{V}\big[(\nabla_U A)_X Y + (1-t)(-\nabla_Y(A_X U) + \nabla_X(A_Y U))\big].$$

Now, using that $\nabla_Y(A_X U) = -A_{\nabla_U X} Y$ and $\mathcal{V}\nabla_X(A_Y U) = A_X(\nabla_U Y)$, and expanding the A tensor again, this is in fact just

$$\mathcal{V}\big[\nabla^t_U(A^t_X Y) - A^t_{\nabla^t_U X} Y - A^t_X(\nabla^t_U Y)\big] = \mathcal{V}\big[\nabla_U(A_X Y) - tA_{\nabla_U X} Y - tA_X(\nabla_U Y)\big].$$

But since $\mathcal{H}\nabla_U X = t\mathcal{H}\nabla_U X$, this explains the t factor in $A^t_{\nabla^t_U X} Y = tA_{\nabla_U X} Y$ (we dropped the superscript t on the A tensor because $A^t_X Y = A_X Y$). Next, $A^t_X(\nabla^t_U Y) = tA_X(\nabla_U Y)$ by decomposing $\nabla^t_U Y$ into $\mathcal{V}\nabla^t_U Y + \mathcal{H}\nabla^t_U Y$: A^t_X applied to the first term gives a factor of t because

$A^t_X \restriction \mathcal{V} = tA_X \restriction \mathcal{V}$, and it also gives a factor of t applied to the second term because $\mathcal{H}\nabla^t_U Y = t\mathcal{H}\nabla_U Y$ and this finishes the proof.

The other formulas are similar. □

In [FIP04], there is an error for the covariant derivatives in the canonical variation. Indeed, on page 148 formula (5.7) indicates $\nabla^t_X U = \nabla_X U$ and $\nabla^t_U X = \nabla_U X$, which, if they were true, would imply in particular that $A^t_X U = A_X U$ in contradiction with Lemma 4.2 formula 1 among others.

We now give the Ricci curvature equations for the canonical variation, in case of totally geodesic fibers ($T \equiv 0$):

Proposition 4.3. *Let $\pi\colon M \to B$ be a Riemannian submersion with totally geodesic fibers, and let X, Y be horizontal vector fields and U, V be vertical vector fields. Then we get the following equations regarding the Ricci curvature Ric_t of the canonical variation g_t:*

$$\mathrm{Ric}_t(U,V) = \widehat{\mathrm{Ric}}(U,V) + t^2\langle AU, AV\rangle, \tag{4.1}$$
$$\mathrm{Ric}_t(X,Y) = \widetilde{\mathrm{Ric}}(X,Y) - 2t\langle A_X, A_Y\rangle, \tag{4.2}$$
$$\mathrm{Ric}_t(X,U) = t\langle(\check{\delta} A)(X), U\rangle. \tag{4.3}$$

In particular, as t tends to 0, these equations then become

$$\mathrm{Ric}_t(U,V) = \widehat{\mathrm{Ric}}(U,V), \quad \mathrm{Ric}_t(X,Y) = \widetilde{\mathrm{Ric}}(X,Y), \quad \mathrm{Ric}_t(X,U) = 0.$$

Proof. The formulas of Corollary 2.22 hold, upon replacing every expression by its g_t equivalent. Thus, it's just a matter of making the right-hand side of the formulas of Corollary 2.22 not depend on g_t but only on t and g. For (4.1), note that the metric g_t restricted to the fibers is the metric g_F rescaled by a factor t. Since the Levi-Civita is unchanged under rescaling of the metric, it follows from the definition

$$\hat{R}^t(U,V)W = \nabla^t_U \nabla^t_V W - \nabla^t_V \nabla^t_U W - \nabla^t_{[U,V]} W$$

that $\hat{R}^t = \hat{R}$, where $\hat{R}$ is the curvature tensor on the fibers. Then,

$$\begin{aligned}
\widehat{\mathrm{Ric}}_t(U,V) &= \sum g_t(\hat{R}^t(U, t^{-1/2}U_k)V, t^{-1/2}U_k) \\
&= \sum g_t(\hat{R}(U, t^{-1/2}U_k)V, t^{-1/2}U_k) \\
&= t\sum g(\hat{R}(U, t^{-1/2}U_k)V, t^{-1/2}U_k) \\
&= t\cdot\frac{1}{t}\widehat{\mathrm{Ric}}(U,V) \\
&= \widehat{\mathrm{Ric}}(U,V).
\end{aligned}$$

In (2.12), the terms involving the T-tensor vanish because we assumed the fibers were totally geodesic, and the term $g_t(AU, AV)$ is simply $t^2 g(AU, AV)$ because $A^t_X U = tA_X U$.

For (4.2), we compute $\widetilde{\mathrm{Ric}}_t(X,Y)$:

$$\widetilde{\mathrm{Ric}}_t(X,Y) = \sum_j g_t(R^*(X, Z_j)Y, Z_j) = \sum_j g(R(X,Z_j)Y, Z_j)$$

where R^* is the curvature tensor on the base space B, and $R^*(X,Y)Z = R(X,Y)Z$ by a similar argument as above. Hence, $\breve{\mathrm{Ric}}_t(X,Y) = \breve{\mathrm{Ric}}(X,Y)$ and equation (2.14) reduces to

$$\begin{aligned}\mathrm{Ric}_t(X,Y) &= \breve{\mathrm{Ric}}(X,Y) - 2g_t(A_X, A_Y)\\ &= \breve{\mathrm{Ric}}(X,Y) - 2tg(A_X, A_Y)\end{aligned}$$

by verticality of the vectors inside the inner product.

Lastly, (4.3) is immediate from (2.13) and Lemma 4.2 (7). □

4.2 In case of a fiber bundle

For this final section, we go over a well known result in the domain of Riemannian submersions, covered by Detlef Gromoll and Gerard Walschap in their book *Metric Foliations and Curvature* (see [GW09]). The previous sections taught us relations between the curvatures (curvature tensor R, Ricci and sectional curvatures) of the total space, the fibers, and the base space of a Riemannian submersion $\pi\colon M \to B$. In particular, if M is compact and has a metric of positive Ricci curvature, then, while it is possible that B contains points of negative Ricci curvature, globally B cannot have nonpositive Ricci. This says that any Riemannian submersion π will carry over the positivity of the Ricci curvatue of M to B, in a sense.

The converse question is also of interest: knowing that B admits a metric of positive Ricci curvature, can we lift this property to M? and under what condition? It is indeed true, at least in the case where π is fiber bundle with fibers F that admit a metric of positive Ricci curvature, and with structure group acting isometrically on the fibers. We review these results below.

Definition 4.4 (Fiber bundle, principal bundle)**.** Let M be a Riemannian manifold, and E, F topological spaces. A *fiber bundle* with fiber F is the data of a continuous surjection $\pi\colon E \to M$ such that for all $p \in M$, there exists an open neighbourhood U of p in M, and a homeomorphism $\varphi\colon \pi^{-1}(U) \to U \times F$ such that

$$\mathrm{proj}_U \circ \varphi = \pi|_{\pi^{-1}(U)}$$

where proj_U is the projection onto the open set U. In other words, the following diagram commutes:

$$\begin{array}{ccc} U \times F & \xleftarrow{\ \varphi\ } & \pi^{-1}(U) \\ {\scriptstyle \mathrm{proj}_U}\big\downarrow & \swarrow {\scriptstyle \pi} & \\ U. & & \end{array}$$

The homeomorphism φ is called *local trivialisation* of M at p.

A *principal bundle* is a fiber bundle $\pi\colon P \to M$, together with a topological group G and a continuous right action $P \times G \to P$ such that $y \in F_x$ implies $yg \in F_x$ for all $g \in G$ (we say that the action preserves the fibers $F_x = \pi^{-1}(x)$ at $x \in M$), and such that the action restricted to the fibers F is free and transitive. We call G the *structure group.*

As a consequence, the orbit space P/G is homeomorphic to M, and each fiber is homeomorphic to the structure group G; however, since there is no preferred choice of identity element, these sets are not by default isomorphic as groups.

Given a fiber bundle $\pi\colon E \to M$, and two open sets U, V of M with non empty intersection $U \cap V$ and local trivialisations $\varphi_U, \varphi_V : \pi^{-1}(U \cap V) \to (U \cap V) \times F$, the change of local trivialisation maps $\varphi_V^{-1} \circ \varphi_U : (U \cap V) \times F \to (U \cap V) \times F$ are of the form $(x,p) \mapsto (x, \psi_{U,V}(x)p)$, where $\psi_{U,V} : U \cap V \to \mathrm{Homeo}(F)$ is called a *transition function.* If all the transition functions are part of a group $G \subset \mathrm{Homeo}(F)$ then we call G the structure group of the fiber bundle. This group is not unique as it can in practice always be made bigger. As an example one can consider a vector bundle with structure group $O(n)$ the orthogonal group; of course, the transition functions are also part of the general linear group $\mathrm{GL}(n, \mathbb{R})$. The transition functions verify the three following conditions: $\psi_{U,U} = e$, $\psi_{U,V} = \psi_{V,U}^{-1}$, and $\psi_{U,W} = \psi_{U,V} \circ \psi_{V,W}$ (often called the cocycle condition).

Remark 4.5. If $\pi\colon E \to M$ is a fiber bundle with fibers F, one gets an open cover of the base space M by trivialisable sets U_α, i.e. open sets whose inverse image under π is $U_\alpha \times F$, and one gets homeomorphisms that constitute the transition functions of the bundle. Conversely, starting with this local information, namely an open cover of sets of the form $U_\alpha \times F$ and transition functions verifying the cocycle condition, we recover the full data of the fiber bundle by gluing together the local products $\{U_\alpha \times F\}_\alpha$ with these homeomorphisms. This proves to be particularly useful when constructing principal bundles from a fiber bundle: if $\pi\colon E \to M$ is a fiber bundle with structure group G, one gets an *associated principal G-bundle*, that we denote $P_G(E)$, by replacing the fibers F by G in the local products and gluing them by the same transition functions $\psi(x) = \psi_{U,V}(x)$, with these functions acting on the group G on the left by $p \mapsto \psi_{x,U,V} \cdot p$.

In fact, the original fiber bundle can be recovered from its associated principal G-bundle; the construction is given in Definition 4.10

The following theorem is the main goal of this section; we wish to retrieve conditions under which it is possible to lift positive Ricci curvature:

Theorem 4.6. *Let M and F be compact Riemannian manifolds with positive Ricci curvature, and $\pi\colon E \to M$ a fiber bundle with fiber F and structure group G. If the metric on F is G-invariant, then E admits a metric with positive Ricci curvature such that π is a Riemannian submersion.*

The theorem stated as such is found in [GW09, Theorem 2.7.3] , although John Nash's paper [Nas79] gives an early but similar version with proof. The proof we will provide differs from Gromoll & Walschap's one, as in their setup they lack explicit formulas for the Ricci curvatures of the canonical variations that are stated in Proposition 4.3. Before that, we need some machinery to tackle the proof.

Definition 4.7 (Riemannian isometry, isometry group)**.** Let (M, g) be a Riemannian manifold. A map $f\colon M \to M$ is called an *isometry*,if f is a diffeomorphism, and if for all $p \in M$, $v, w \in T_pM$,

$$g(f_*v, f_*w)_{f(p)} = g(v, w)_p.$$

In other words, the metric equals its pullback under f: $f^*g = g$. The set of isometries of a Riemannian manifold M under the composition of maps is a group called the *isometry group* of M, denoted $\mathrm{Isom}(M)$.

Definition 4.8 (Equivariant map)**.** Let X and Y be sets with a left action of G on X and Y. A map $f\colon X \to Y$ is called *equivariant*, or G-equivariant, if for all $g \in G$ and $x \in X$,

$$f(g \cdot x) = g \cdot f(x).$$

Equivariance is better depicted with a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{g\cdot} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g\cdot]{} & Y \end{array}$$

Remark 4.9. Let $H \subset \mathrm{Isom}(M)$ be a subgroup of the isometry group of M. If all the orbits under the action of H on M are equivariantly diffeomorphic (in the sense that each diffeomorphism is an equivariant map), then one can equip the orbit space M/G with a differentiable structure, for which the quotient projection $\pi\colon M \to M/G$ is a Riemannian submersion ([GW09] p.8).

We describe a way to obtain a fiber bundle, called the associated bundle, from a principal G-bundle, such that the two bundle projections are linked in a certain way:

Definition 4.10 (Associated bundle)**.** If $\pi_P\colon P \to M$ be a principal G-bundle, by the comment after Definition 4.4 we can write M as $M \cong P/G$, where the quotient denotes the orbit space. Now if F is a manifold with a left G-action, then the action

$$\begin{aligned} G \times P \times F &\longrightarrow P \times F \\ (g, x, y) &\longmapsto (xg^{-1}, gy) \end{aligned}$$

is free. We denote (see [GW09] p.92) the quotient space $G\backslash(P \times F)$ by $P \times_G F$, and call it the *the bundle with fiber F associated to the principal bundle π_P*. It has the structure of a manifold; moreover, it is the total space of a fiber bundle $\pi\colon P \times_G F \to M$, with fiber F and structure group G, where the map π acts as $[(x, y)] \mapsto \pi_P(x)$.

Then π_P and π are connected via the following commutative diagram:

$$\begin{array}{ccc} P \times F & \xrightarrow{\rho} & P \times_G F \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow[\pi_P]{} & M \end{array}$$

Here, ρ denotes the quotient projection and π_1 is the projection onto the first factor. If P and F have G-invariant metrics, then the G-action on both of these spaces is isometric, and thus the same goes for the action on the product $P \times F$. By Remark 4.9, $P \times_G F$ admits a metric such that the projection ρ is a Riemannian submersion, and same goes for $\pi_P\colon P \to M$. Remark 4.5 says that from any fiber bundle, one gets an associated principal G-bundle $P_G(E)$. The construction above in fact allows us to recover the original fiber bundle $\pi\colon E \to M$ from $P_G(E)$, by noticing that $E \cong P_G(E) \times_G F$ since the transition functions are the same. To simplify the notation, we write P instead of $P_G(E)$.

Next, we define a common notion in the theory of bundles:

Definition 4.11 (Connection, principal connection, [Bes87, 9.54])**.** Let $\pi\colon E \to M$ be a fiber bundle. A *connection* on the fiber bundle $\pi\colon E \to M$ is a vector-valued 1-form $\Phi \in \Omega^1(E, \ker \pi_*)$ with values in $\ker \pi_*$, such that Φ is idempotent, i.e. $\Phi \circ \Phi = \Phi$, and $\mathrm{Im}\,\Phi = \ker \pi_*$.

Let $\pi\colon P \to M$ be a principal bundle with structure group a Lie group G and associated Lie algebra $\mathfrak{g}$. Suppose that G acts on the left on itself (recall that by definition of a principal bundle, G acts on the right on the manifold). A *principal connection* Φ is a 1-form on P with values in $\mathfrak{g}$, such that:

1. For all $g \in G$, we have
$$(R_g)^*\Phi = \mathrm{Ad}(g^{-1}) \circ \Phi$$
where R_g denotes the right translation by g and $\mathrm{Ad}(g)(h) = ghg^{-1}$.

2. For any vector $V \in \mathfrak{g}$, we have at each point of P
$$\Phi(\widehat{V}) = V$$
where the vector field $\widehat{V}$ generates the one-parameter group of diffeomorphisms of P given by the action of the subgroup $\mathrm{Exp}(tV)$ of G, i.e.
$$\widehat{V}_g = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Exp}(tV) \cdot g \bigg|_{t=0}.$$

Since the fibers of the submersion $\pi\colon P \to M$ are the group G itself, any vector $V \in \mathfrak{g}$ is vertical by definition, and one may recover the vertical distribution $\mathcal{V} = \ker \pi_*$ from the vector fields $\widehat{V}$ by making V vary in $\mathfrak{g}$. Note that any principal connection induces a horizontal distribution complementary to $\mathcal{V} = \ker \pi_*$ by setting $\mathcal{H} = \ker \Phi$. This complement has full rank since Φ has full rank ($\mathrm{Im}\,\Phi = \ker \pi_*$).

In the more general case, a (not necessarily principal) connection also gives rise to a horizontal distribution. Indeed, at a point $u \in E$, the connection $\Phi_u\colon T_uE \to (\ker \pi_*)_u$ is a projection and thus we get the usual splitting $T_uE = \ker \Phi_u \oplus \mathrm{Im}\,\Phi_u = \ker \Phi_u \oplus (\ker \pi_*)_u$. For references about connections and principal connections, see [Mic08] sections 17, 18 and 19. In [GW09] p.4, the corresponding definitions denote the connection as $\mathcal{H}$ to emphasize the correspondence between connections and horizontal distributions.

Remark 4.12. If E and M are Riemannian manifolds and $\pi\colon E \to M$ is a principal bundle, then there are possibly many choices of horizontal distributions complementary to the vertical bundle $\ker \pi_*$. If $(\mathcal{H}^1_p)_{p\in U}$ is a distribution defined on a subset $U \subset M$, and $(\mathcal{H}^2_p)_{p\notin U}$ is another distribution defined outside of U, both complementary to $\ker \pi_*$ and such that $\mathcal{H}^i = \ker \Phi_i$ for principal connections Φ_i, $i = 1, 2$, then one can interpolate between the two with a bump function θ: if the function $\theta\colon M \to [0, 1]$ is smooth and such that $\theta = 1$ on an open set $V \subset U$, and $\theta = 0$ outside of U, then the new principal connection Φ_3 defined by
$$\Phi_3 = \theta\Phi_1 + (1 - \theta)\Phi_2$$
yields a horizontal distribution $\mathcal{H}^3 = \ker \Phi_3$ again complementary to $\ker \pi_*$ and interpolates between $\mathcal{H}^1$ inside of U and $\mathcal{H}^2$ outside of U.

Proposition 4.13 ([GW09, Proposition 2.7.1])**.** *Let M be a Riemannian manifold, $\pi_P\colon P \to M$ a principal G-bundle with connection Φ. If F is a Riemannian manifold on which G acts by isometries, then there exists a metric on $P \times_G F$ such that $\pi\colon P \times_G F \to M$ is a Riemannian submersion with totally geodesic fibers, isometric to F, and horizontal distribution $\mathcal{H} := \rho_*(\ker \Phi \times \{0\})$, where ρ denotes as above the quotient projection $\rho\colon P \times F \to P \times_G F$.*

We refer to this specific metric as a *connection metric.* Note that this metric is unique although we will only need that it exists. We provide a sketch of the proof given by Gromoll and Walschap; we will not show that the fibers are totally geodesic as the proof requires the use of holonomy

Jacobi fields. For the proof, we need to arrange that $\pi_*\colon T(P\times_G F)\to TM$ is a linear isometry on $\mathcal{H}$, by choosing a certain metric on $P\times_G F$ that yields the isometry. For this we specify inner products separately on the vertical and horizontal distributions; we cannot directly specify a metric on $P\times_G F$ that realises the isometry since the horizontal distribution is not necessarily integrable. We will "pull back" the inner products on TM via π to get the metric restricted to $\mathcal{H}$ on $P\times_G F$, and we will "push forward" the inner products on the factor F of $P\times F$ to obtain the metric restricted to the vertical distribution $\ker\pi_*$.

Proof. Given a connection Φ on P, we may transport its induced horizontal distribution to $P\times_G F$ by setting $\mathcal{H}:=\rho_*(\ker\Phi\times\{0\})$. Given $\mathcal{H}$ on $T(P\times_G F)$, the tangent bundle $T(P\times_G F)$ then splits as $\mathcal{H}\oplus\ker\pi_*$, and to make π_* an isometry at a point $\rho(p,m)\in P\times_G F$, $(p,m)\in P\times F$, equip $\mathcal{H}$ with the unique inner product that makes $\pi_*\colon\mathcal{H}_{\rho(p,m)}\to T_{\pi(\rho(p,m))}M=T_{\pi_P(p)}M$ an isometry. Next, let $h_p\colon F\to\rho(p,F)$ be the map defined by $h_p(m)=\rho(p,m)$. Note that $\rho(p,F)=\pi_P^{-1}(p)$ and $h_{p*}\colon T_mF\to\rho_*(\{0\}\times T_mF)$ is valued in the vertical distribution, on which we need to specify a metric. To do so, since F already has a metric, equip $\rho_*(\{0\}\times T_mF)$ with the unique inner product that makes h_{p*} an isometry. Since the diagram after Definition 4.10 commutes, i.e. $\pi\circ\rho=\pi_P\circ\pi_1$, we indeed have that $\mathcal{H}\oplus\ker\pi_*$ and $\ker\pi_*=\rho_*(\{0\}\times T_mF)$; moreover since G acts by isometries on F, the metric is well defined. Note that it only determines the metric $g=g_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j$ restricted to the vertical and horizontal distributions individually, so we are free to choose its g_{ij} coefficients so that these distributions are orthogonal. □

Now that we have defined the connection metric, we can upgrade the result already cited above in Remark 2.8:

Theorem 4.14 ([GW09, Theorem 2.7.2])**.** *Assume M is complete, and let $\pi\colon M\to B$ be a Riemannian submersion with totally geodesic fibers. Then π is a fiber bundle and the metric on M is a connection metric.*

Proof of Theorem 4.6. Let G be the structure group of the fiber bundle $\pi\colon E\to M$. Consider the principal G-bundle $\pi_P\colon P\to M$ associated to π as in Remark 4.5, and thus write E as $E=P\times_G F$. By Proposition 4.13, E admits a connection metric. Since the projection π from Proposition 4.13 is a Riemannian submersion, we get the usual decomposition of the bundle as $\mathcal{V}\oplus\mathcal{H}$. So the metric splits as a vertical component and a horizontal component; we can thus warp the metric in the vertical component with a constant function (see Definition 2.24 and Definition 4.1) $\varphi\colon E\to\mathbb{R}^+$, $\varphi(x):=t$. This is in fact exactly the canonical variation of the metric with factor t.

Proposition 4.3 gives the three components of the warped Ricci curvature; horizontal, vertical and "vertizontal" (mixed planes). Since we assumed that the fibers F had positive Ricci curvature, then (4.1) says $\mathrm{Ric}_t(U,U)=\widehat{\mathrm{Ric}}(U,U)+t^2g(AU,AU)$, which implies that E has positive Ricci, restricted to vertical vector fields. Now, since the base M has positive Ricci as well, $\widetilde{\mathrm{Ric}}(X,X)>0$ for any horizontal $X\neq 0$. In particular, since M is compact, there exists $\varepsilon>0$ such that $\widetilde{\mathrm{Ric}}(X,X)>\varepsilon$ everywhere. It suffices thus to take t sufficiently small so that

$$\varepsilon - tg(A_X,A_X)>0$$

for all $p\in M$ and for all $X\in T_pM$, where g is the metric on M. Take for example

$$t=\varepsilon/(1+\sup_{p,X} g(A_X,A_X)),$$

and compactness of M ensures that $\sup g(A_X, A_X) < \infty$ (for any unit vector field $X \in T^1M$, the assignment $(X,p) \mapsto g(A_X, A_X)_p$ is a continuous map from the compact set $M \times T^1M$ to $\mathbb{R}$). Therefore E has positive Ricci restricted to horizontal vector fields as well. If N is any vector field, it decomposes uniquely into a horizontal part and a vertical part, $N = X + V$ for some $X \in \mathcal{H}$, $V \in \mathcal{V}$. Thus, the quantity $\mathrm{Ric}(N,N)$ expands to

$$\begin{aligned}\mathrm{Ric}_t(N,N) &= \mathrm{Ric}_t(X,X) + 2\,\mathrm{Ric}_t(X,V) + \mathrm{Ric}_t(V,V)\\ &= \widetilde{\mathrm{Ric}}(X,X) - 2tg(AX,AX) + \widehat{\mathrm{Ric}}(V,V) + t^2 g(AV,AV) + tg((\check{\delta}A)(X),V)\\ &\geq \widetilde{\mathrm{Ric}}(X,X) + \widehat{\mathrm{Ric}}(V,V) - 2tg(AX,AX) + tg((\check{\delta}A)(X),V)\end{aligned}$$

and since $\widetilde{\mathrm{Ric}}(X,X)$ and $\widehat{\mathrm{Ric}}(V,V)$ are positive, this implies that for very small values of t, $\mathrm{Ric}_t(N,N) > 0$, which ends the proof. □

5 Reverse result

5.1 The result

This last part is dedicated to the construction of an example of a fiber bundle $\pi\colon E \to M$ that does not transport the positivity of the Ricci curvature. This is, in some sort, a "reverse result", in the sense that the assumptions are similar as Theorem 4.6. More precisely, we prove the theorem:

Theorem 5.1. *Let $\pi\colon E \to M$ be a fiber bundle from a compact manifold E to a compact manifold M that admits a metric g_M with $\mathrm{Ric}_M > 0$. Suppose that the fibers F admit a metric such that the structure group G acts isometrically on F, and that $\mathrm{Ric}_F > 0$. Then, there is a metric $\tilde{g}$ of positive Ricci curvature on E, and a metric $\tilde{g}_M$ such that $\pi\colon (E, \tilde{g}) \to (M, \tilde{g}_M)$ is a Riemannian submersion, and the corresponding metric $\tilde{g}_M$ has points of negative Ricci curvature on M.*

The example we will construct uses a specific principal connection, canonically defined for Lie groups:

Definition 5.2 (Maurer-Cartan form)**.** Let G be a Lie group. The *Maurer-Cartan form* $\omega \in \Omega^1(G, T_eG)$ is a 1-form on G whose action on tangent vectors $X \in T_gG$ is given by

$$\omega_g(X) = (L_{g^{-1}})_* X$$

where $e \in G$ is the identity element and L is the left translation.

On a Lie group G with Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, we denote by $[\omega \wedge \eta]$ the wedge product of two 1-forms $\omega, \eta \in \Omega^1(G, \mathfrak{g})$, defined by

$$[\omega \wedge \eta](X, Y) = [\omega(X), \eta(Y)] - [\omega(Y), \eta(X)].$$

Definition 5.3 (Curvature form)**.** Let G be a Lie group and $\pi_P\colon P \to M$ be a principal G-bundle, with principal connection Φ. The *curvature form* of Φ is the Lie algebra valued 2-form Ω given by

$$\Omega = \mathrm{d}\Phi + \frac{1}{2}[\Phi \wedge \Phi]$$

or more precisely, $\Omega(X, Y) = \mathrm{d}\Phi(X, Y) + \frac{1}{2}[\Phi \wedge \Phi](X, Y)$, where d denotes the exterior derivative.

The Maurer-Cartan form is of particular interest because its associated curvature form vanishes:

$$\frac{1}{2}[\omega \wedge \omega] = [\omega, \omega], \quad \mathrm{d}\omega = -[\omega, \omega].$$

Indeed, let $X, Y \in \mathfrak{g} = \mathrm{Lie}(G)$ be left-invariant vector fields. Then,

$$\mathrm{d}\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

where $[\cdot,\cdot]$ denotes the Lie bracket of vector fields. But $\omega(X) = X_e$ is constant, so $Y(\omega(X)) = 0$ and similarly $X(\omega(Y)) = 0$. Moreover, $(L_h)_*([X, Y]) = [(L_h)_*X, (L_h)_*Y]$ for all $h \in G$, so

$-\omega([X,Y]) = -[\omega(X),\omega(Y)]$. So

$$\begin{aligned}\Omega_\omega(X,Y) &= \mathrm{d}\omega(X,Y) + \frac{1}{2}[\omega\wedge\omega](X,Y)\\ &= -[\omega(X),\omega(Y)] + \frac{1}{2}\left([\omega(X),\omega(Y)] - [\omega(Y),\omega(X)]\right)\\ &= -[\omega(X),\omega(Y)] + [\omega(X),\omega(Y)]\\ &= 0,\end{aligned}$$

and since left-invariant vector fields span the tangent space of G at each point, and the objects above are tensorial, this holds globally on G.

Moreover, since left translations commute with right translations, the Maurer-Cartan form defines a principal connection. Indeed,

$$\mathrm{Ad}(g^{-1})(R^*_{g^{-1}}\omega_e)X = (R_g \circ L_{g^{-1}})_*\omega_e(R_{g^{-1}})_*X = (L_{g^{-1}})_*X = \omega_g(X).$$

To prove Theorem 5.1 we will need to deform the metric on a small open set of the base space to obtain constant sectional curvature 1 on this set while keeping the same metric outside of this open set, which allows essentially to identify this set with a geodesic ball on the sphere S^n of radius 1. This is made possible by the following lemma:

Lemma 5.4 ([Rei24, Lemma 4.2]). *Let (M,g) be a Riemannian manifold of positive Ricci curvature. Then, there exists a metric g' of positive Ricci curvature on M with the following properties:*

- *There exists $\varepsilon > 0$ such that on the geodesic ball $D_\varepsilon(p) \subseteq M$ which we identify with $D_\varepsilon(0) \subseteq T_pM$ via the exponential map, the metric g' coincides with the induced metric of a geodesic ball of radius ε in the sphere of radius 1;*
- *The metric g' coincides with g outside an arbitrary small neighbourhood of p.*

Modifying sectional curvature locally is a standard procedure in Riemannian geometry. However, the exact original reference is difficult to pinpoint; an early reference would be [GY86]. Lemma 5.4 is found in [Rei24] almost exactly as such, where Philipp Reiser cites [Wra02], paper in which David Wraith also cites Yau and Gao's work.

The second part of the proof of Theorem 5.1 combines the Pro-Wilhelm construction and the canonical variation of the metric; after modifying the sectional curvature on a small open set of the base space M, it remains to ensure that these two modifications of the metric connect smoothly. The next lemma helps achieving this:

Lemma 5.5. *Let (F, g_F) be a Riemannian manifold of positive Ricci curvature, and let D^n denote a geodesic ball inside S^n. Then, for all $R > 0$ and $t > 0$ sufficiently small, there exists a metric of positive Ricci curvature on $D^n \times F$ such that:*

- *The projection $\pi\colon (D^n \times F) \to D^n$ is a Riemannian submersion and the metric on D^n has points of negative Ricci curvature.*
- *On a neighbourhood of the boundary $\partial D^n \subset S^n$, the metric of $D^n \times F$ is of the form $\mathrm{d}r^2 + \sin(r)^2\mathrm{d}s^2_{n-1} + t^2 g_F$ with $r \in [R-\pi, R]$.*

Proof. We apply the Pro-Wilhelm construction to the sphere S^n instead of S^2. To achieve this, we modify the metric on S^n, by warping some of the components with a function φ, to yield S^n_φ and thus the modification will affect D^n as well. Recall from Section 3 that if we denote the standard round metric of the sphere S^{n-1} by $\mathrm{d}s_{n-1}^2$, the metric on the n-sphere can be derived from the formula

$$\mathrm{d}s_n^2 = \mathrm{d}r^2 + \sin(r)^2 \mathrm{d}s_{n-1}^2.$$

We then consider $\varphi\colon [0,\pi] \to \mathbb{R}$ smooth such that the metric

$$g_\varphi = \mathrm{d}r^2 + \varphi(r)^2 \mathrm{d}s_{n-1}^2$$

on S^n is smooth; denote it by S_φ^n. In the Pro-Wilhelm construction, the base space S_φ^2 has negative Ricci curvature along $r^{-1}(p)$. We thus choose the point $p \in (0, \pi/4)$ so that the geodesic ball D^n contains the $(n-1)$-sphere $r^{-1}(p) \in S_\varphi^n$. This means that along $r^{-1}(p)$, D^n has negative Ricci curvature, see Lemma 3.16.

Now let (F, g_F) be a Riemannian manifold of positive Ricci curvature. Consider the warped product $D^n \times_\nu F$ with a function ν as in Lemma 3.16. The warping function ν affects the metric with a factor of $e^{2\nu} g_F$, or more precisely the metric on the product $D^n \times_\nu F$ is

$$\mathrm{d}r^2 + \varphi(r)^2 \mathrm{d}s_{n-1}^2 + e^{2\nu} g_F$$

where the function ν only depends on r. Since in the construction of Lemma 3.16 we have arranged that $\varphi = \sin$ and $\nu = \text{const}$ everywhere except on an arbitrarily small neighbourhood of the point p, it follows that the metric near the boundary ∂D^n is

$$\mathrm{d}r^2 + \sin(r)^2 \mathrm{d}s_{n-1}^2 + e^{2\nu} g_F.$$

By the comment after Definition 2.23, the projection $\pi_1\colon D^n \times_\nu F \to D^n$ onto the first factor is a Riemannian submersion, with fibers F. For a sufficiently small number $\beta > 0$, the warped product $D^n \times_{\nu + \ln\beta} F$ then has positive Ricci curvature while D^n has points of negative Ricci curvature (see Section 3). Thus the metric is of the expected form, by setting $t^2 = \beta e^{2\nu}$. □

5.2 Proof of the result

In the Pro-Wilhelm construction, the Riemannian submersion is of the form $\pi\colon S_\varphi^2 \times_\nu F \to S_\varphi^2$, i.e. the projection on the first factor of a warped product. Here, after modifying the sectional curvature of a small open set $U \subset M$ so that it is constantly 1, that we view as a subset (in fact, a geodesic ball) D^n of the sphere S^n, we consider the warped product $D^n \times_\nu F$ for a manifold F as in the Theorem 4.6, and the submersion π being the projection onto D^n. We will perform a similar modification of the metric of $D^n \subset S^n$ with functions φ and ν as in Lemma 3.16, with the extra condition that φ and ν need to be such that the metric is smooth outside of the set $\pi^{-1}(D^n)$.

Proof of Theorem 5.1. Let then $\pi\colon E \to M$ be a fiber bundle with fiber F as in Theorem 4.6. Assume that $\mathrm{Ric}_M > 0$ and $\mathrm{Ric}_F > 0$. Let G be the structure group of the fiber bundle π and $\pi_P\colon P \to M$ the associated principal G-bundle, with fibers diffeomorphic to G; assume moreover that G acts isometrically on F. Then $E = P \times_G F$ and the following diagram commutes:

$$\begin{array}{ccc} P \times F & \xrightarrow{\ \rho\ } & P \times_G F \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow[\ \pi_P\ ]{} & M. \end{array}$$

For a point $x \in M$, consider a small enough (simply connected) neighbourhood $U_x \subset M$ of x on which we arrange that $\sec_M|_{U_x} = 1$. This is done by applying Lemma Lemma 5.4 to the manifold M; moreover, since the metric of Lemma 5.4 agrees with g_M outside of the neighbourhood, we indeed preserve positivity of the Ricci curvature on M. If n is the dimension of M, and S^n is the n-sphere of radius 1, then its sectional curvature is constantly 1. For some $\alpha > 0$ we denote by D^n the geodesic ball on S^n of radius α, where we choose the geodesic ball to be centered at $r^{-1}(0)$; after identifying D^n with U_x, we can embed isometrically D^n in M, where the isometry is a Riemannian isometry.

Since we have chosen the open set U_x to be sufficiently small, it trivialises over $\pi_P \colon P \to M$ as $U_x \times G$, that we now view as $D^n \times G$. We now prescribe a principal connection on the principal bundle π_p: on the product $D^n \times G$, consider the connection Φ defined by $\Phi\colon TD^n \times TG \to \mathfrak{g}$, $\Phi(X, Y) = \omega(Y)$, where ω is the Maurer-Cartan 1-form on G and $\mathfrak{g}$ is the Lie algebra associated to G (i.e. Φ is the Maurer-Cartan form extended constantly in the D^n direction). The connection outside of $D^n \times G$ does not matter; with a partition of unity we can simply choose it to be any principal connection outside of $D^n \times G$, see Remark 4.12. With this choice of principal connection, Φ is nothing but a projection onto $\mathfrak{g}$, and the horizontal distribution associated to Φ is thus $\ker \Phi = TD^n$. Now equip E with a connection metric as in Proposition 4.13, i.e. one that makes $\pi\colon E = P \times_G F \to M$ a Riemannian submersion, and such that the connection on the fiber bundle π is $\rho_*(\ker \Phi \times \{0\})$. Since the construction in Proposition 4.13 describes what the connection metric is on the total space $P \times_G F$ (the metric is specified on each of the two factors of $TD^n \oplus \ker \pi_* = T(P \times_G F) = TE$), it follows that we have a product metric over $\rho(D^n \times G \times F)$. Then, $\rho(D^n \times G \times F) = \pi^{-1}(D^n) = D^n \times F$ (assuming we picked U_x small enough so that it also trivialises over π) because the above diagram commutes. Since π is trivial over D^n, it is indeed the projection $\pi\colon D^n \times F \to D^n$, and the equations (3.1), (3.2) and (3.3) hold when applied to the metric $g_{D^n} + e^{2\nu} g_F$.

We now make use of Lemma 5.4 on $D^n \times F$, to obtain that the projection onto D^n is Riemannian and D^n has points of negative Ricci curvature, and the metric on $D^n \times F$ is of the form $\mathrm{d}r^2 + \sin(r)^2 \mathrm{d}s_{n-1}^2 + t^2 g_F$ near $\partial D^n \times F$ for a very small $t > 0$. In fact, on E we consider the canonical variation g_β for a sufficiently small number $\beta > 0$ such that outside of $\pi^{-1}(D^n) \subset E$, (E, g_β) has positive Ricci cuvature (see Theorem 4.6). Since by Lemma 5.4, the metric on M agrees with the metric g_M outside of D^n, it suffices that the warping function ν is constant on the boundary of D^n for the metric on $\pi^{-1}(D^n)$ to agree with the metric of E outside of $D^n \times F$; this is already achieved in Lemma 5.5. Now extend the warping function ν so that it is constant on the whole manifold E, and extend the warping metric constantly throughout the manifold E. By choosing $\lambda > 0$ to be sufficiently small, the warping function $\nu + \ln \lambda$ yields points of negative Ricci curvature on D^n but overall positive Ricci curvature on $D^n \times_{\nu + \ln \lambda} F$, and the canonical variation $g_{\beta + \lambda}$ yields positive Ricci curvature outside of $D^n \times F$. Then, for $t^2 = \beta \cdot \lambda e^{2\nu}$, we indeed get a metric of the form of Lemma 5.5, which is smooth on E, and this finishes the proof. □

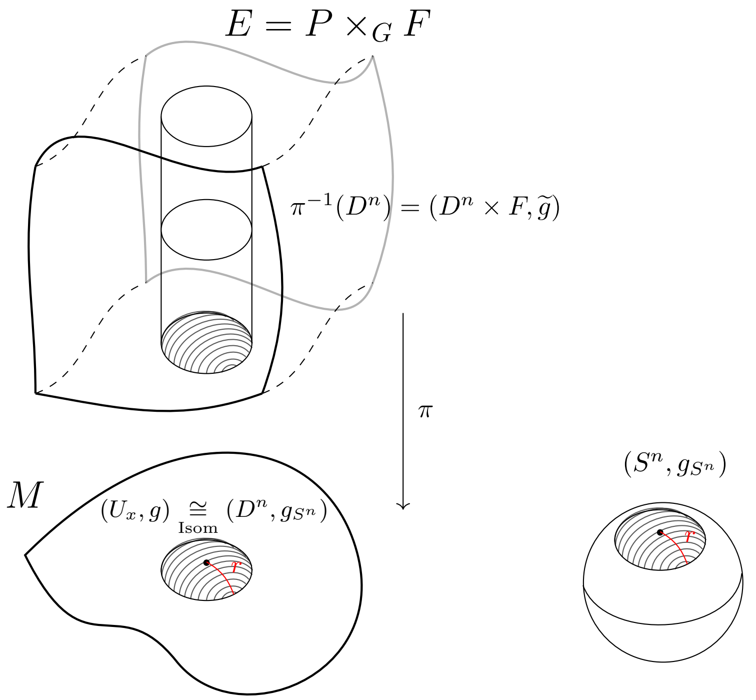

Figure 4: Fiber bundle $\pi \colon E \to M$ with modification of the metric on a small open set $U_x \subset M$ so that it has constant sectional curvature 1. The metric upstairs is chosen to be a connection metric, and U_x trivialises as $D^n \times F$ on which the metric has the form $\mathrm{d}r^2 + \sin(r)^2 \mathrm{d}s_{n-1}^2 + t^2 g_F$ near ∂D^n.

Appendix

A General notions of differential geometry

We recall here some basic notions of differential geometry that are indispensable to understand this thesis. This is deeply inspired by the Algebra, and Riemannian Geometry lectures given by Prof. Anand Dessai.

Definition A.1 (Manifold, smooth manifold)**.** Let M be a set. We say that M is a *manifold* or *topological manifold* of dimension m, if the following hold:

- M is a Hausdorff topological space,
- M is second countable,
- M is locally Euclidean, i.e. for any point $p \in M$, there is an open neighbourhood $U_p \subset M$, an open set $V \subset \mathbb{R}^m$ and a homeomorphism $\varphi\colon U_p \to V$.

If M is a manifold of dimension m, a *chart* is a pair (U, φ) where $U \subset M$ is open and $\varphi\colon U \to V$ is a homeomorphism onto a subset of $\mathbb{R}^m$. If (U, φ), $(\widetilde{U}, \psi)$ are two charts around the same point $p \in M$, then the composition $\psi \circ \varphi^{-1}\colon \varphi(U \cap \widetilde{U}) \to \psi(U \cap \widetilde{U})$ is called a *transition map*. Two charts are smoothly compatible if their transition map is a diffeomorphism in the usual Euclidean sense. Moreover, we call an *atlas* any collection of charts $\{U_\alpha, \varphi_\alpha\}_\alpha$ for which $M = \bigcup_\alpha U_\alpha$. An atlas is *smooth* if any two charts in the atlas are smoothly compatible. Finally, a smooth atlas is said to be *maximal* if it is not contained in any other smooth atlas.

We then say that M is a *smooth* manifold of dimension m, provided that M is a topological manifold of dimension m together with a maximal smooth atlas, which provides its smooth structure.

We would like to use analysis tools to study functions defined on smooth manifolds. In the Euclidean case, say $\mathbb{R}^n$, since the structure is linear, points (viewed as vectors) can be added and subtracted, thus simplifying the computation of the differential of a map. Since this setup is so easy to work with, and since manifolds ultimately resemble $\mathbb{R}^n$ locally, we may define differentiability in terms of the classical definition in the Euclidean case:

Definition A.2 (Smooth map)**.** Let M, N be two smooth manifolds of dimension m and n respectively. A map $f\colon M \to N$ is said to be smooth at $p \in M$, if there exist charts $\varphi\colon U_x \to V \subset \mathbb{R}^m$, and $\psi\colon U_{f(p)} \to V' \subset \mathbb{R}^n$, such that the map $\widetilde{f} := \psi \circ f \circ \varphi^{-1}\colon V \to V'$ is smooth as a map between open sets of $\mathbb{R}^m$ and $\mathbb{R}^n$.

The notion extending the one-dimension derivative of a single variable function is the *differential* of a map $f\colon \mathbb{R}^n \to \mathbb{R}^m$, often denoted $\mathrm{d}f$, or f_*. There is an equivalent for maps $f\colon M \to N$ defined on smooth manifolds, defined roughly as the best linear approximation of f at a certain point $p \in M$. Since it takes vectors as argument, we need to define an "overlying" space with a linear structure on top of the manifold:

Definition A.3 (Tangent vector, tangent space)**.** Let M be a smooth manifold and $p \in M$. Two curves $\gamma, \sigma\colon (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = \sigma(0) = p$ are equivalent if for any chart φ around

p, one has $(\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0)$. A *tangent vector* v at p is then defined as an equivalence class of curve $v = [\gamma]$. Hence, one should see tangent vectors as being the "arrows" tangent to curves defined on manifolds, despite the object $\gamma'(t)$, which in an Euclidean setting defines the velocity vector of the curve, being ill-defined on general manifolds due to the lack of linear structure (the Euclidean derivative as quotient of difference makes no sense in all generality). We denote by T_pM the set of tangent vectors at p and call it the *tangent space*.

If $\varphi\colon U \to \widetilde{U}$ is a chart, then its differential (see Definition A.4) $\varphi_*\colon T_pU \to T_{\varphi(p)}\widetilde{U}$ is bijective. One can show then that the map sending $[\gamma] \in T_{\varphi(p)}\widetilde{U}$ to $\gamma'(0) \in \mathbb{R}^m$ is bijective as well, allowing to identify $T_{\varphi(p)}\widetilde{U}$ with $\mathbb{R}^m$ as vector spaces. Since φ_* is bijective, one can endow T_pU with a vector space structure, and since every curve in $U \subset M$ is also a curve in M, this makes T_pM into a vector space of dimension $m = \dim(M)$.

Another useful interpretation of tangent vectors is given via the equivalent definition of derivations. If U, V are two neighbourhoods of a point $p \in M$, we say that two smooth functions f and g defined on U and V respectively are equivalent if there exists a neighbourhood $W \subset U \cap V$ of p such that $f|_W = g|_W$. Then, one might consider the $\mathbb{R}$-algebra of classes of functions defined on a neighbourhood U of p

$$\mathcal{E}(p) \coloneqq \{[f] \mid f\colon U \to \mathbb{R} \text{ smooth}\}.$$

Then, a *derivation* $X\colon \mathcal{E}(p) \to \mathbb{R}$ is a linear map satisfying

$$X(fg) = X(f)g(p) + f(p)X(g)$$

and we denote $\mathrm{Der}_p(M)$ the set of derivations at a point $p \in M$. One shows $\mathrm{Der}_p(M)$ is a vector space of dimension m and the key point is that it is isomorphic to T_pM. Note as well that the usual partial derivative operator $\partial_{x^i} \coloneqq \frac{\partial}{\partial x^i}$ in $\mathbb{R}^m$ defines a derivation via $f \mapsto \frac{\partial f}{\partial x^i}$. Then, one can show that a basis of $\mathrm{Der}_p(\mathbb{R}^m)$ is given by $(\partial_{x^1}, \ldots, \partial_{x^m})$. Viewing partial derivatives as directional derivatives, this translates to the fact that each curve gives a direction along which functions can be differentiated, hence the correspondence between tangent vectors and derivations. Then, one gets an isomorphism $\mathrm{Der}_p(M) \cong \mathrm{Der}_{\varphi(p)}(\mathbb{R}^m)$ via any chart φ. In fact, this gives a coordinate basis of $\mathrm{Der}_p(M) = T_pM$ by pushing forward the derivations $\partial_{x^1}, \ldots, \partial_{x^m}$ via φ_*^{-1}, basis that we often write again $(\partial_{x^1}, \ldots, \partial_{x^m})$. Thus, any tangent vector $v \in T_pM$ can be written

$$v = v^1\partial_{x^1} + \cdots + v^m\partial_{x^m}.$$

Additionally we define the tangent bundle of M as the set $TM \coloneqq \bigcup_{p\in M} T_pM$, where the union is disjoint. Elements of TM are pairs (p, v) specifying the tangent vector and the point it's based at. There is a natural projection map $\pi\colon TM \to M$ sending a pair (p, v) to its basepoint p. The tangent bundle TM is a smooth manifold of dimension $2\dim(M) = 2m$: for any chart (U, φ), that we write in coordinates $\varphi = (x^1, \ldots, x^m)$, and any tangent vector $v = v^1\partial_{x^1} + \cdots + v^m\partial_{x^m}$, the map $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2m}$ defined as

$$\widetilde{\varphi}(p, v) = (x^1(p), \ldots, x^m(p), v^1, \ldots, v^m)$$

is bijective onto its image, and one can recover an atlas of TM by taking chart domains U_α that cover M. One checks easily that the transition maps are smooth and hence we get a smooth structure on the tangent bundle TM. The fact that $\dim TM = 2m$ comes from the fact that $\dim \mathbb{R}^{2m} = 2m$.

Definition A.4 (Differential, pushforward)**.** Let M and N be smooth manifolds. The *differential* or *pushforward* of a map $f\colon M \to N$ at a point $p \in M$ is the linear map $(\mathrm{d}f)_p\colon T_pM \to T_{f(p)}N$ defined in the following way: if v is a tangent vector at p and $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is a curve such that $\gamma(0) = p$ and $\gamma'(0) = v$, then

$$\mathrm{d}f(v) = (f \circ \gamma)'(0).$$

The pushforward is functorial with respect to the composition of maps:

$$\mathrm{d}(f \circ g)_p = \mathrm{d}f_{g(p)} \circ \mathrm{d}g_p$$

for maps $g\colon M \to N$ and $f\colon N \to K$, which translates to the fact that we can multiply the Jacobi matrices when computing the differential of a composition in the Euclidean setting.

Definition A.5 (Vector field)**.** Let M be a smooth manifold and $U \subset M$ an open subset. A *vector field* on U is a continuous map $X\colon U \to TM$ such that $(\pi \circ X)(p) = p$ for all $p \in U$. This means in fact that $X(p) \in T_pU = T_pM$ for any point p. Since T_pM is spanned by the derivations $(\partial_{x^1}, \dots, \partial_{x^m})$ in a coordinate chart φ, locally any vector field can be written

$$X = X^1\partial_{x^1} + \cdots + X^m\partial_{x^m},$$

where $X^1, \dots, X^m$ are called the component functions of X in the chart φ. A useful characterisation of smoothness for vector fields is the following: if (U, φ) is a chart, a vector field X is smooth if and only if its component functions X^i, $i = 1, \dots, m$, are smooth. In practice, one can always take $U = M$ since any vector field can be extended on the whole manifold M with a partition of unity.

We denote by $\Gamma(TM)$ or $\Gamma(M)$ the collection of all smooth vector fields on M.

Definition A.6 (Distribution)**.** Let M be a smooth manifold of dimension m and TM its tangent bundle. A rank k *distribution* $\mathcal{D}$ on M is a choice of subspace $D_p \subset T_pM$ of dimension $k \leq m$ for each $p \in M$. The distribution is said to be *smooth*, if the disjoint union

$$\mathcal{D} = \bigcup_{p \in M} \mathcal{D}_p \subset TM$$

is a smooth subbundle of TM. One gets a vector field characterisation of smoothness: a rank k distribution $\mathcal{D}$ is smooth if and only if around each point $p \in M$, there is a neighbourhood U on which there are vector fields $X_1, \dots, X_k\colon U \to TM$ such that $X_1|_q, \dots, X_k|_q$ form a basis of $\mathcal{D}_q$ for any $q \in U$.

Definition A.7 (Integrable, involutive distribution)**.** Let M be a smooth manifold and $\mathcal{D}$ a smooth rank $k \leq m$ distribution on M. We say that $\mathcal{D}$ is *integrable*, if for each point $p \in M$ there exists a submanifold $N \subset M$ of dimension k such that $\mathcal{D}_p = T_pN$.

Moreover, we say that $\mathcal{D}$ is *involutive*, if the distribution is bracket-preserving, i.e. $[X, Y]_p \in \mathcal{D}_p$ whenever X and Y are vector fields around p such that $X|_p, Y|_p \in \mathcal{D}_p$.

Theorem A.8 (Frobenius)**.** *Let M be a smooth manifold and $\mathcal{D}$ a smooth distribution on M. Then, $\mathcal{D}$ is integrable if and only if $\mathcal{D}$ is involutive.*

Reference for this theorem is typically found in [Lee13] p.497.

Definition A.9 (Lie bracket of vector fields)**.** Let $X, Y \in \Gamma(M)$ be vector fields, that we view as derivations on functions defined on M. The *Lie bracket* of X and Y, denoted $[X, Y]$, is again a vector field and is defined via its action on functions $f \in \mathcal{C}^\infty(M)$ by

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Definition A.10 (Topological group, Lie group)**.** Let $(G, \cdot)$ be a group. We say that G is a *topological group* if there exists a topology $\mathcal{O}$ on G such that the maps

$$(g, h) \mapsto g \cdot h, \quad g \mapsto g^{-1}$$

are continuous with respect to $\mathcal{O}$, as maps $G \times G \to G$ and $G \to G$ (on $G \times G$ we put the product topology).

If the group G admits a smooth structure turning it into a smooth manifold, we say that it is a *Lie group* if the two aforementioned maps are smooth.

On a Lie group, there is a canonical structure called the *Lie algebra* of the Lie group G, often denoted $\mathfrak{g}$, which is a vector space with additional structure:

Definition A.11 (Left-invariant vector field, Lie algebra)**.** Let G be a Lie group. A vector field $X \in \Gamma(G)$ is said to be *left-invariant* if the following holds:

$$X_{gh} = (\mathrm{d}L_g)_h X_h$$

for all $g, h \in G$, where L_g denotes the left multiplication by g. If $\dim G = n$, then there are exactly n linearly independent left-invariant vector fields on G; this can easily be achieved at a point, say the identity $e \in G$, and then translated to any other point $g \in G$ by left-multiplication.

Now, a *Lie algebra* $\mathfrak{g}$ is a (real) vector space together with an extra operation $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket*, satisfying the following conditions: for all $X, Y, Z \in \mathfrak{g}$,

- $[X, Y] = -[X, Y]$ (alternating)
- $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, $[Z, aX + bY] = a[Z, X] + b[Z, Y]$ for all $a, b \in \mathbb{R}$

(bilinearity)

- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

The set of all left-invariant vector fields with the Lie bracket of vector fields forms a real Lie algebra and is called the *Lie algebra of the Lie group G*. There is a one-to-one correspondence between vectors at the identity $e \in G$ and left-invariant vector fields: any vector $v \in T_eG$ can be extended to a left-invariant vector field on G by simply setting

$$v|_g = (\mathrm{d}L_g)v|_e.$$

With this, the tangent space at the identity T_eG can be identified with the Lie algebra $\mathfrak{g}$ of G, where the bracket operation is defined as

$$[v, w] = [\widetilde{v}, \widetilde{w}]_e \in T_eG$$

where $\widetilde{v}$ and $\widetilde{w}$ denote the left-invariant vector fields obtained from v and w.

Definition A.12 (One-parameter (sub-)group, $\mathrm{Exp}(tX)$)**.** Let G be a topological group. A *one parameter (sub-)group* is a continuous homomorphism $\varphi\colon \mathbb{R} \to G$, i.e. it satisfies

$$\varphi(s + t) = \varphi(s) \cdot \varphi(t).$$

If G is a Lie group with Lie algebra $\mathfrak{g}$ we denote by $\mathrm{Exp}(tX)$ the one-parameter subgroup of G generated by the vector $X \in \mathfrak{g}$.

If M is a Riemannian manifold on which the Lie group G acts on the left, then $\mathrm{Exp}(tX)$ acts on M and yields a one-parameter group φ_t of diffeomorphisms of M defined by

$$\varphi_t(y) = \mathrm{Exp}(tX)y.$$

B Riemannian geometry

Definition B.1 (Riemannian manifold)**.** A Riemannian structure on a smooth manifold is a family $(g_p)_{p \in M}$ of inner products on the tangent space T_pM, such that for all fixed vector fields X, Y on M, the assignment $p \mapsto g_p(X, Y)$ is smooth as a map $M \to \mathbb{R}$. A *Riemannian manifold* is a choice of Riemannian structure on a smooth manifold.

Definition B.2 (Affine connection, covariant derivative)**.** Let M be a Riemannian manifold. An *affine connection* is a map $\nabla\colon \Gamma(M) \times \Gamma(M) \to \Gamma(M)$, $X, Y \mapsto \nabla(X, Y) =: \nabla_X Y$, such that for all $f \in \mathcal{C}^\infty(M)$,

1. $\nabla_{fX+Y} = f\nabla_X + \nabla_Y$,
2. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$, and

3. $\nabla_X(fY) = f\nabla_X Y + D_X f(Y)$

where $D_X f$ is the directional derivative, i.e. $D_X f = \langle \operatorname{grad} f, X\rangle$.

Definition B.3 (Metric compatibility, torsion-free)**.** Let (M, g) be a Riemannian manifold. A connection ∇ on M is called *compatible with the metric*, or just metric, if for all $X, Y, Z \in \Gamma(M)$,

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

We say that ∇ is *torsion-free*, if for all $X, Y \in \Gamma(M)$,

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Theorem B.4 (Fundamental theorem of Riemannian geometry)**.** *Let (M, g) be a Riemannian manifold. Then, there exists a unique affine connection ∇ that is both metric and torsion-free. We call this connection the Levi-Civita connection.*

For reference and proof, see [Lee06] Theorem 5.4.

Let now $(\overline{M}, \overline{g})$ be a Riemannian manifold with $\overline{\nabla}$ as Levi-Civita connection. Let $M \subset \overline{M}$ be a submanifold and let g be the metric on M induced by $\overline{g}$, i.e.

$$g(v, w) := \overline{g}(v, w) \quad \forall v, w \in T_pM,\ p \in M.$$

For $p \in M$, let $(\cdot)^T : T_p\overline{M} \to T_pM$ be the orthogonal projection with respect to the scalar product $\overline{g}_p$, with the splitting $T_p\overline{M} = T_pM \oplus T_pM^{\perp}$. A vector field $Z \in \Gamma(\overline{M})$ defines a vector field on M, $(Z\big|_M)^T \in \Gamma(M)$, by first restricting to M and then using $(\cdot)^T$. This defines a linear map

$$\begin{aligned} (\cdot)^T : \Gamma(\overline{M}) &\longrightarrow \Gamma(M) \\ Z &\longmapsto (Z\big|_M)^T \end{aligned}$$

with $(Z\big|_M)^T : M \to TM$, $p \mapsto (Z_p)^T$. On the other hand, every vector field $X \in \Gamma(M)$ on M can be extended to a vector field $\overline{X} \in \Gamma(\overline{M})$ on $\overline{M}$ using a partition of unity; this extension is not unique.

For $X, Y \in \Gamma(M)$, consider $\overline{\nabla}_{\overline{X}}\overline{Y}$, where $\overline{X}$ and $\overline{Y} \in \Gamma(\overline{M})$ are extensions of X, Y. In fact $\overline{\nabla}_{\overline{X}}\overline{Y}(p)$, $p \in M$ only depends on $\overline{X}(p) = X(p)$, and the restriction of $\overline{Y}$ to a curve γ with $\gamma'(0) = \overline{X}(p)$. Since $\overline{X}(p) = X(p)$, we can choose γ to be a curve $\gamma : I \to M$. Therefore, it only depends on $Y\big|_\gamma$.

Hence, $\overline{\nabla}_{\overline{X}}\overline{Y}(p)$ is independent of the choice of extensions $\overline{X}$, $\overline{Y}$.

Proposition B.5. *The map*

$$\begin{aligned} \nabla^M : \Gamma(M) \times \Gamma(M) &\longrightarrow \Gamma(M) \\ (X, Y) &\longmapsto \left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)^T \end{aligned}$$

is the Levi-Civita connection.

For reference and proof, see [GHL+90] Proposition 2.56. In particular, in case of a Riemannian submersion $\pi\colon M \to B$, the fibers F_b are submanifolds of M, and the Levi-Civita connection $\hat{\nabla}$ on the fibers F_b is given by the map $(U,V) \mapsto \left(\hat{\nabla}_{\hat{U}}\hat{V}\right)^T$; since the vectors U,V are typically chosen to be vertical, meaning they already belong to T_pF_b, and since the projection $(\cdot)^T$ in this context is nothing but the vertical projection $\mathcal{V}$, we get that the Levi-Civita connection on the fibers is $\mathcal{V}\nabla$.

Definition B.6 (Curvature tensors)**.** Let (M,g) be a Riemannian manifold. The *curvature tensor* R is the map $R\colon \Gamma(M)\times\Gamma(M)\times\Gamma(M)\to\Gamma(M)$, defined by

$$R(X,Y)Z \coloneqq R(X,Y,Z) \coloneqq \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Hence, $R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$.

The *Ricci curvature* is the trace of the curvature tensor, or more precisely,

$$\mathrm{Ric}(Y,Z) \coloneqq \mathrm{tr}(X \mapsto R(X,Y)Z).$$

On a specific fiber T_pM, for any orthonormal basis $(v_1,\ldots,v_m)$ of T_pM, this expression develops to

$$\mathrm{Ric}_p(Y,Z) = \sum_{k=1}^{m}\langle R(v_k,Y)Z, v_k\rangle.$$

The map R is indeed a tensor and restrict to a multilinear map on the fibers of the projection $\pi\colon TM \to M$. We state three simple but important identities regarding the curvature tensor:

Lemma B.7. *For any vector fields (not necessarily vertical or horizontal) X,Y,U,V, one has:*

1. $\langle R(X,Y)U,V\rangle = -\langle R(Y,X)U,V\rangle$,
2. $\langle R(X,Y)U,V\rangle = -\langle R(X,Y)V,U\rangle$,
3. $\langle R(U,X)Y,V\rangle = \langle R(Y,V)U,X\rangle$.

Lemma B.8 (Bianchi identity)**.** *For all vector fields X,Y,Z (not necessarily horizontal),*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

For references for Lemma B.7 and Lemma B.8, check for example Proposition 3.1.1 p.79 of [Pet98] (P.P. Riemannian Geometry).

For two linearly independent vectors V and W, let $V\wedge W$ denote the parallelogram spanned by V and W, and P_{VW} denote tangent plane spanned by V and W.

Definition B.9 (Sectional curvature)**.** Let M be a Riemannian manifold, $p\in M$ and $X,Y\in$

T_pM be linearly independent vectors. The *sectional curvature* of $P_{XY} < T_pM$ is the number

$$K(P_{XY}) \coloneqq K(X,Y) \coloneqq \frac{\langle R(X,Y)Y, X\rangle}{\|X \wedge Y\|^2}.$$

Definition B.10 (Vector field along a curve, parallel vector field)**.** Let $\gamma\colon I \to M$ be a smooth curve. A vector field $V : I \to TM$ is said to be *along the curve* γ, if $V(t) \in T_{\gamma(t)}M$ for all $t \in I$.

A vector field V along a curve γ is *parallel* if $\frac{\mathrm{D}V}{\mathrm{d}t} = 0$.

Proposition B.11 (Covariant derivative along a curve, see [Lee18])**.** *Let M be a smooth manifold with covariant derivative ∇. Then there exists a unique operator $\frac{\mathrm{D}}{\mathrm{d}t}$, which maps any vector field V along a curve γ to a vector field $\frac{\mathrm{D}V}{\mathrm{d}T}$ along γ, called the* covariant derivative of V along γ*, such that*

1. $\frac{\mathrm{D}}{\mathrm{d}t}(a \cdot V + W) = a \cdot \frac{\mathrm{D}}{\mathrm{d}t}(V) + \frac{\mathrm{D}}{\mathrm{d}t}(W)$ *with V, W vector fields along γ, $a \in \mathbb{R}$.*
2. $\frac{\mathrm{D}}{\mathrm{d}t}(f \cdot V) = f \cdot \frac{\mathrm{D}}{\mathrm{d}t}(V) + \frac{\mathrm{d}f}{\mathrm{d}t} \cdot V$ *with V a vector field, $f\colon I \to \mathbb{R}$ smooth.*
3. *If $V = Y \circ \gamma$ for a vector field $Y \in \Gamma(M)$ on M then*

$$\frac{\mathrm{D}V}{\mathrm{d}t}(t) = \nabla_{\gamma'(t)}Y.$$

Definition B.12 (Geodesic, speed of geodesic)**.** Let M be a Riemannian manifold, with LC connection ∇. A curve $\gamma\colon I \to M$, where I is any compact interval, is called *geodesic* if it satisfies the following equation:

$$D_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' = 0.$$

The *speed* of a geodesic is the number $|\gamma'|$. If this number is equal to 1, then we say that the geodesic has unit speed.

One useful characterisation of geodesics is the following, using coordinate representation:

Proposition B.13. *A curve $\gamma = (x_1, \ldots, x_n)$ on (M, g) is a geodesic if it satisfies locally the system of ODEs*

$$x_k'' + \sum_{i,j=1}^{m} x_i' \cdot x_j' \cdot \Gamma_{ij}^k = 0$$

for all $k = 1, \ldots, m$. For initial data $v \in T_pM$ there exists $\varepsilon > 0$ and a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, $\gamma'(0) = v$.

(See [GHL$^+$90] p.90). This implies that at any point $p \in M$, and for any tangent vector $v \in T_pM$, on a sufficiently small neighbourhood U of p there exists a unique geodesic γ_v starting at p, in the direction of v. Cover the manifold M with neighbourhoods of this sorts and let Λ denote the union of the sets $\mathcal{E}$ defined by

$$\mathcal{E} \coloneqq \{v \in T_pM \mid |v| < \varepsilon, p \in U\}$$

where $\varepsilon > 0$ is the number as in Proposition B.13. The set Λ is an open neighbourhood of the zero section.

Definition B.14 (Exponential map)**.** The *exponential map* is the map $\exp\colon \Lambda \to M$, which to any vector $v \in T_pM$, $|v| < \varepsilon$, maps its unique geodesic at time 1, $\exp(v) = \gamma_v(1)$.

Proposition B.15 ([Lee13, Corollary 13.29])**.** *Let (M, g) be a Riemannian manifold. Define the length functional of a curve $\gamma\colon [a, b] \to M$ by*

$$L(\gamma) \coloneqq \int_a^b |\gamma'(t)| \, \mathrm{d}t$$

and define the distance function on M by

$$d(p, q) \coloneqq \inf_{\gamma_{pq}} L(\gamma_{pq})$$

where γ_{pq} is a piecewise smooth curve joining p and q. Then, the distance function $d(\cdot, \cdot)$ generates the same topology as the manifold topology of M.

Definition B.16 (G-invariant metric, isometric action)**.** Let (M, g_M) be a manifold and G a group acting (on the left) on M. We say that the action is *isometric* and the metric g_M is said to be *G-invariant*, if for all $\varphi \in G$, the map

$$f_\varphi\colon M \to M, \quad x \mapsto \varphi \cdot x$$

is an isometry, i.e. $g_M((f_\varphi)_* v, (f_\varphi)_* w) = g_M(v, w)$ for all $p \in M$, $v, w \in T_pM$.

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