Master's thesis

# The vanishing of the $\hat{A}$-genus for spin manifolds of positive scalar curvature

Émilien Bornet

**Supervisor:**
Prof. Dr. Anand Dessai

UNIVERSITÉ DE FRIBOURG
UNIVERSITÄT FREIBURG

Department of Mathematics
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**Summary**

In this master's thesis, we discuss some aspects of the strange interplay between the local Riemannian geometry and the global topology of smooth, compact and oriented manifolds. In particular, we turn our attention towards a celebrated theorem of Lichnerowicz which presents the so-called $\hat{A}$-genus as an obstruction to the existence of a Riemannian metric of positive scalar curvature on doubly-even dimensional compact spin manifolds.

The first chapter is dedicated to a review of the foundational notions of vector bundles, from their definition to the discussion of characteristic classes and the splitting principle. In the same spirit, the second chapter is intended as a reminder of various concepts and objects which we use extensively in the later parts of the thesis, such as the Riemannian curvature tensor and its properties. The following chapter serves as an introduction to the theory of genera and, in particular, to the $\hat{A}$-genus, which plays a crucial role in the main result. Indeed, its statement provides a direct link between this genus and the index of a specific differential operator, most notably through the Atiyah-Singer index theorem, to which the fourth chapter is dedicated. It is then in the fifth chapter that the spin geometric tools necessary to the proof, in particular Dirac operators and the Bochner identity, are developed. Lastly, we illustrate Lichnerowicz's result by applying it to the specific case of degree $d$ hypersurfaces of complex projective spaces.

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# 1. Vector bundles

We begin by an overview of the standard theory of vector bundles. Most of this material comes from the well-known books of Milnor and Stasheff [MS74], Hatcher [Hat17], Husemoller [Hus94] and Lawson and Michelsohn [BLM89]. Note however that, contrary to these sources and since our intention is to use these objects in the study of the curvature and topology of smooth manifolds, we will work almost exclusively within the smooth category.
This chapter may be safely skipped by a reader already familiar with these notions ; its role is that of providing references for the later parts of the thesis.

## 1.1 Basic notions

### 1.1.1 Real vector bundles

Let us first recall the definition of a (real and smooth) rank $d$ vector bundle. Consider two smooth manifolds $E$ and $M$.

**Definition 1.1** A smooth and surjective **projection** map $\pi : E \to M$ is a (real and smooth) **vector bundle** of **rank** $d$ (or **real** $d$**-plane bundle**) over the **base space** $M$ if for every point $x \in M$ the following conditions are met :

- the **fiber** $\pi^{-1}(\{x\}) = E_x$ carries the structure of a $d$-dimensional real vector space,
- there exists an open neighbourhood $U \subseteq M$ around $x$ and a diffeomorphism

  $$\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^d$$

  such that, if $p : U \times \mathbb{R}^d \to U$ denotes the canonical projection on $U$, then the following diagram commutes

  $$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{R}^d \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & U & \end{array}$$

  and $\varphi$ restricts to linear isomorphisms $\pi^{-1}(\{y\}) \to \mathbb{R}^d$ on each fiber $\pi^{-1}(\{y\})$ for $y \in U$.

The diffeomorphism $\varphi$ is often referred to as a **local trivialization** of the vector bundle, and the pair $(U, \varphi)$ as a **local coordinate system** around $x$.

In the context of the above definition, $E$ is called the **total space** of the vector bundle. It has, locally, the structure of a cartesian product but can overall be "twisted".

**Remark 1.2** In order to obtain the general definition of a continuous vector bundle over some topological space $B$, the only changes required are the following : the surjective projection map is only assumed to be continuous and the local trivializations are homeomorphisms. When working within the continuous category, we will denote the base space with $B$, rather than $M$.

If the local trivialization $U$ can be taken to be the whole base space $B$ however, the bundle is said to be **trivial**. We often denote the trivial real vector bundle of real rank $d$ by $\varepsilon^d$.

**Example 1.3** For a first classical example of a non-trivial vector bundle, we consider the real projective space $P^n(\mathbb{R})$, seen as the quotient of the sphere $S^n$ through the antipodal identification. We then define the set

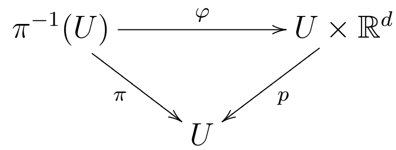
$$E_n^1 = E(\gamma_n^1) = \left\{(\{\pm x\}, v) \in P^n(\mathbb{R}) \times \mathbb{R}^{n+1} \mid v \in \mathbb{R}x\right\}$$

to serve as total space and a projection $\pi : E_n^1 \to P^n(\mathbb{R})$ by $(\{\pm x\}, v) \mapsto \{\pm x\}$. It is relatively easy to show that $\pi$ is indeed a vector bundle, and in particular that its total space $E_n^1$ is a smooth manifold. To prove that it is non-trivial requires the notion of a (global) section, and will therefore be done later (see 1.1.3). Notice however that in the case $n = 1$, the total space can be thought of as the well-known "Möbius band", and it is hence obviously "twisted".
Another well-known example is that of the tangent bundle of the sphere $S^2$. Proofs of its non-triviality usually also rely on sections of $TS^2$, arguing that none of them can be nowhere vanishing. This can be done for example by showing that such a section on $S^2$ would imply that the antipodal map is homotopic to the identity map, which is impossible because they do not have the same (topological) degree.

The notations $E_n^1$ and $E(\gamma_n^1)$ we used in the above example will make more sense in light of Section 1.1.4.

**Remark 1.4** Here, the nomenclature is often abused. Depending on the context, we may refer to either the projection map or to the total space as the vector bundle. In most cases, we write $\xi = (E, \pi, M)$ to collect all of the necessary data, and call the object $\xi$ the vector bundle.

Let $\xi = (E, \pi, M)$ be a rank $d$ vector bundle and suppose that $(U, \varphi)$ is a local coordinate system. If $p : U \times \mathbb{R}^d \to U$ denotes the standard projection on the first coordinate of the product, we know by definition that the following diagram




commutes. Now, if $(U, \varphi)$ and $(V, \psi)$ are two local coordinate systems of non-empty intersection, we may consider the map

$$\varphi \circ \psi^{-1} : (U \cap V) \times \mathbb{R}^d \to (U \cap V) \times \mathbb{R}^d.$$

Clearly, this is a diffeomorphism and, moreover, because of the above commutative diagram, for all $x \in U \cap V$ there must be a linear isomorphism $\Phi_{UV}(x) \in \mathrm{GL}_d(\mathbb{R})$ such that

$$\left(\varphi \circ \psi^{-1}\right)(x, v) = (x, \Phi_{UV}(x)(v)).$$

The smooth map $\Phi_{UV} : U \cap V \to \mathrm{GL}_d(\mathbb{R})$ is then known as the **transition function** from $(V, \psi)$ to $(U, \varphi)$. The isomorphisms $\Phi_{UV}(x) \in \mathrm{GL}_d(\mathbb{R})$ built in this way will be involved in other constructions in subsequent Sections. Notice that $\Phi_{UU}(x)$ corresponds to the identity on $\mathbb{R}^d$ for all $x \in U$ and that if $U, V$ and $W$ are intersecting domains of local trivializations of $\xi$, then

$$\Phi_{UV}(x) \circ \Phi_{VW}(x) = \Phi_{UW}(x)$$

for all $x \in U \cap V \cap W$.
In fact, we may adopt the "opposite" viewpoint to define vector bundles. Let us first consider an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of the same smooth manifold $M$. Suppose then that for any choice of indices $i, j \in I$ such that $U_i$ and $U_j$ intersect, we have a smooth map

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}_d(\mathbb{R})$$

and that the collection $\{g_{ij}\}$ of these maps satisfies the following conditions :

1. $g_{ii}(x) = \mathrm{id}_{\mathbb{R}^d}$, for all $i \in I$ and $x \in U_i$,
2. $g_{ij}(x) \circ g_{jk}(x) = g_{ik}(x)$, for all $i, j, k \in I$ and $x \in U_i \cap U_j \cap U_k$.

Note that these are precisely the properties of the transition functions that were introduced earlier.

**Definition 1.5** Such a family $\{g_{ij}\}$ is called a **cocycle** on the open cover $\mathcal{U}$.

We may now form the disjoint union

$$\coprod_{i \in I} U_i \times \mathbb{R}^d$$

and consider the following relation on this space : $(x, v) \sim (y, w)$ if and only if $x = y$ and $g_{ij}(x)(v) = w$, for some $i, j \in I$ such that $x \in U_i$ and $y \in U_j$. This is, actually, an equivalence relation.

**Proposition 1.6** Under the above hypotheses, the quotient by $\sim$ and the associated projection map form a (smooth) rank $d$ vector bundle over $M$, whose transition functions are given by the elements of the cocycle $\{g_{ij}\}$.

In our case, this bundle is in fact isomorphic to $\xi$.

Consider once again the $d$-plane bundle $\xi = (E, \pi, M)$ and an open cover of $M$ made out of local coordinate systems. The subgroup of $\mathrm{GL}_d(\mathbb{R})$ generated by the associated transition functions is called the **structure group** of the bundle (even though it depends on the often unmentioned choice of cover). For any smooth vector bundle however, it is possible to choose a cover in such a way that the structure group is limited to $\mathrm{O}(d)$ (see further down the paragraph about bundle metrics). This procedure is known as a **reduction of the structure group** and we will come back to it when introducing principal bundles in Section 1.3.

Most of the concept relating to vector spaces have an analogue in the setting of vector bundles. A first example of this principle can be seen by considering linear subspaces :

**Definition 1.7** Let $\eta = (F, \psi, M)$ and $\xi = (E, \pi, M)$ be vector bundles over the same base space $M$. If $F \subseteq E$ is a submanifold and if for every point $x \in M$ the fiber $\psi^{-1}(\{x\})$ can be identified with a linear subspace of $\pi^{-1}(\{x\})$, then $\eta$ is said to be a **subbundle** of $\xi$, which we denote by $\eta \subseteq \xi$.

Another instance this phenomenon arises when considering pullbacks :

**Definition 1.8** Let $\xi = (E, \pi, M)$ be a vector bundle, $N$ be a smooth manifold and $f : N \to M$ be a smooth map. We then form the **pullback bundle** $f^*\xi$ over $N$, with total space

$$f^*E = \{(x, e) \in N \times E \mid f(x) = \pi(e)\} \subseteq N \times E$$

and projection map $p : f^*E \to N$ given by $(x, e) \mapsto x$.

Note that $f^*E$ can be shown to be a submanifold of the product $N \times E$ and that therefore $f^*\xi$ is indeed a smooth vector bundle.

**Definition 1.9** Let $\xi = (E, \pi, M)$ and $\eta = (F, \psi, N)$ be vector bundles. A smooth map $\hat{f} : E \to F$ between the total spaces is called a **bundle morphism** if it restricts to isomorphisms on the fibers and if there exists another smooth map

$f : M \to N$ between the base spaces such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\hat{f}} & F \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \psi} \\ M & \xrightarrow[f]{} & N \end{array}$$

In this context, the morphism $\hat{f}$ is said to **cover** the map $f$.

Of course, the two bundles above are said to be **isomorphic**, which we write $\xi \cong \eta$, if there exists such a bundle morphism $\hat{f} : E \to F$ that is also a diffeomorphism. We usually denote by $\mathrm{Vect}^d(B)$ the set of isomorphism classes of $d$-plane bundles over some topological space $B$.

**Remark 1.10** A bundle morphism $\hat{f}$ between two bundles $\xi$ and $\eta$ can be more simply denoted by $\hat{f} : \xi \to \eta$.

A crucial property of the pullback bundle is then given in the following result :

**Lemma 1.11** Let $\xi = (E, \pi, M)$ and $\eta = (F, \psi, N)$ be vector bundles. If the map $\hat{f} : E \to F$ is a bundle morphism covering the smooth map $f : M \to N$, then $\xi \cong f^*\eta$.

**Proof :** This follows almost immediately from the Definitions 1.8 and 1.9. □

The precise topological nature of the base space plays an important role in the properties of vector bundles over it. Most often, as is the case in the more general treatment of the theory presented in [MS74], we place ourselves in the context of vector bundles with **paracompact** base spaces. Recall :

**Definition 1.12** A Hausdorff topological space $X$ is called **paracompact** if every open cover admits a locally finite refinement or, equivalently, if for every open cover there is a subordinate partition of unity.

Since we are only interested in smooth manifolds, the conditions above will always be satisfied. In the rest of this Section however, to clearly illustrate that paracompactness is the only required property for the results we cover, we will work within the continuous category.

**Remark 1.13** If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of a paracompact space $B$, then there is a subordinate partition of unity $\{f_i\}_{i \in I}$ sharing the same index set $I$. The proof of this fact is rather technical, although not particularly difficult. The idea is to take a partition of unity $\{g_j\}_{j\in J}$ subordinate to $\mathcal{U}$ and construct the partition $\{f_i\}_{i\in I}$ by setting

$$f_i = \sum_{\operatorname{spt}(g_j)\subset U_i} g_j.$$

In case there are no $j \in J$ such that $\operatorname{spt}(g_j) \subset U_i$ for some $i \in I$, we set $f_i \equiv 0$.

The first clear benefit of having a paracompact base space, is that we can endow any real vector bundle over it with a (bundle) **metric**. For now, it suffices to think of these objects as a way to continuously (or smoothly, if the bundle is smooth) assign a scalar product (i.e. a positive definite bilinear form) to each fiber.
Indeed, if $\xi = (E, \pi, B)$ is a real $d$-plane bundle over a paracompact topological space $B$, and $\{(U_i, \varphi_i)\}_{i\in I}$ a covering set of local coordinate systems for $\xi$, then the homeomorphisms

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^d$$

can be used to pull the standard inner product of $\mathbb{R}^d$ to inner products $\langle \cdot, \cdot \rangle_i$ on $\pi^{-1}(U_i)$. To do this, notice that $\varphi_i$ sends a vector $v$ to $(\pi(v), \psi_i(v))$, where the map

$$\psi_i : \pi^{-1}(U_i) \to \mathbb{R}^d,$$

known as a **local parametrization**, restricts to a linear isomorphism on each fiber. We then set :

$$\langle u, v \rangle_i = \langle \psi_i(u), \psi_i(v) \rangle_{\mathbb{R}^d}$$

whenever $\pi(u) = \pi(v) \in U_i$. In this way, we have endowed each fiber in $\pi^{-1}(U_i)$ with a scalar product. It may be however, that a given fiber $E_x$ is contained in many different such subsets $\pi^{-1}(U_i)$. We therefore consider a partition of unity $\{f_i\}_{i\in I}$ subordinate to the open cover $\{U_i\}_{i\in I}$ (see Remark 1.13 above), and use it to "patch the information together". More precisely, for two vectors $u, v \in E$ such that $\pi(u) = x = \pi(v)$, we define

$$\langle u, v \rangle_x = \sum_{i\in I} f_i(x) \langle u, v \rangle_i.$$

Note that, because of the defining properties of the partition of unity $\{f_i\}_{i\in I}$, this sum is locally finite. The assignment $x \mapsto \langle \cdot, \cdot \rangle_x$ is therefore also continuous (respectively, smooth). We will briefly come back to this fact under a more abstract approach in Section 1.2.4. Endowing a bundle $E$ with such a metric enables us to define the **orthogonal complement** $V^\perp$ of any subbundle $V \subset E$ : we simply take $V^\perp$ to be the collection of the orthogonal complements of the fibers of $V$ within $E$ relative to the metric. A proof of the local triviality can be found in [Hat17], Proposition 1.3 on page 12. The choice of a bundle metric also allows for the **reduction of the structure group** of any rank $d$ vector bundle over a paracompact base space from $\mathrm{GL}_d(\mathbb{R})$ to $\mathrm{O}(d)$, via the usual Gram-Schmidt process. We may therefore always assume that a vector bundle over a paracompact space has an orthogonal structure group.

Another convenient property of vector bundles over paracompact spaces is the following :

**Theorem 1.14** Let $E \to B$ be a vector bundle over some topological space $B$. Let $A$ be a paracompact space and suppose that $f_0, f_1 : A \to B$ are homotopic continuous maps. Then the pullback bundles $f_0^* E$ and $f_1^* E$ are isomorphic.

**Proof :** See [Hat17], Theorem 1.6 on page 20. □

This result is a direct consequence of the next proposition :

**Proposition 1.15** The restrictions of a vector bundle $E \to B \times [0,1]$ over $B \times \{0\}$ and $B \times \{1\}$ are isomorphic if $B$ is paracompact.

**Proof :** See [Hat17], Proposition 1.7 on page 20. □

Most of the time, we will work within the setting of (smooth) vector bundles over (smooth) compact manifolds, whose base space are thus obviously paracompact. In this case, we have the following useful property :

**Theorem 1.16** Let $\xi = (E, \pi, M)$ be an $n$-plane over a compact manifold $M$. Then there exists $N \in \mathbb{N}$ and an embedding of $E$ as a subbundle of $M \times \mathbb{R}^N$.

**Proof :** See [Hat17], Proposition 1.4 on page 13. □

### 1.1.2 Complex vector bundles

Since we will require knowledge of complex vector bundles when dealing with Chern classes later on, we briefly introduce the changes to Definition 1.1 that are required when working in the complex setting.

Let $M$ be a smooth manifold.

**Definition 1.17** A (smooth) **complex vector bundle** $\omega$ of complex **rank** $d$ (or **complex $d$-plane bundle**) over $M$ consists of a smooth manifold $E$ and a **projection map** $\pi : E \to M$, together with the structure of a $d$ dimensional complex vector space in each fiber and such that each point $x \in M$ admits an open neigbourhood $U$ with $\pi^{-1}(U) \approx U \times \mathbb{C}^d$ under a diffeomorphism that induces $\mathbb{C}$-linear maps $\pi^{-1}(\{y\}) \to \{y\} \times \mathbb{C}^d$ for every $y \in U$.

Note that we usually refer to a rank 1 complex bundle as a (complex) **line bundle**.

Under certain circumstances, it is possible to change the structure of the fibers of a real bundle $\xi$ of rank $2d$ to obtain a complex vector bundle of rank $d$. The procedure involves the concept of a complex structure :

**Definition 1.18** A **complex structure** on such a bundle $\xi = (E, \pi, M)$ is a smooth map $J : E \to E$ which sends each fiber to itself $\mathbb{R}$-linearly and such that $J \circ J = -\mathrm{id}_E$.

Given such a structure on $\xi$, we may transform each fiber into a complex vector space by setting :

$$(x + iy)v = xv + J(yv)$$

for every vector $v \in E$ and complex number $x + iy$. On the other hand, it is clear that if we are given a complex bundle $\omega$ of rank $d$, then we can form real bundle of rank $2d$ simply by forgetting the complex vector space structure of each fiber. The resulting bundle is called the **underlying real bundle** and denoted by $\omega_{\mathbb{R}}$.

Note that the conditions for a smooth manifold of even real dimension to admit a complex structure are extremely strict :

**Definition 1.19** A **complex structure on a manifold** $M$ of dimension $2d$ is a complex structure $J$ on its tangent bundle $TM$ such that every point $x \in M$ is contained within an open neighbourhoud that is diffeomorphic to an open subset $\mathbb{C}^d$ through a diffeomorphism whose differential is everywhere $\mathbb{C}$-linear.

Such manifolds, endowed with such structures, are called **complex manifolds**. For our purpose however, it is enough to consider **almost-complex manifolds**, that is to say manifolds whose tangent bundle can be seen as the underlying real bundle of some complex bundle.

There is still another (and in fact much easier) method to construct a new complex bundle out of a given real bundle. Indeed, if $\xi = (E, \pi, M)$ is a real $d$-plane bundle, then each of its fibers $F$ is a real $d$-dimensional vector space. Through complexification, we obtain a complex vector space $F \otimes \mathbb{C} \cong F \oplus iF$ of (complex) dimension $d$. We can then of course easily generalize this procedure to the full bundle as follows :

**Definition 1.20** The **complexification** $\xi \otimes \mathbb{C}$ of $\xi$ is the n-plane complex bundle constructed by replacing every fiber $F$ of $\xi$ by $F \otimes \mathbb{C}$.

One benefit of complex bundles is illustrated by the next result :

**Theorem 1.21** If $\omega$ is a complex vector bundle, then the underlying real vector bundle $\omega_{\mathbb{R}}$ has a canonically preferred orientation.

**Proof :** Consider a basis $\{a_1, \ldots, a_n\}$ of a fiber $F$. Then $\{a_1, ia_1, \ldots, a_n, ia_n\}$ forms an ordered basis of $F_{\mathbb{R}}$, which naturally induces an orientation on that real vector space. To show that this orientation does not depend on the original choice of a complex basis for $F$, we simply note that $\mathrm{GL}(n, \mathbb{C})$ is connected. We then extend this construction to every fiber of $\omega$. □

As an example, if we consider the particular case of tangent bundles, it follows that complex manifolds have a canonically preferred orientation. For more details on the orientation of vector bundles, see Section 1.1.6.

### 1.1.3 Sections

Consider a vector bundle $\xi = (E, \pi, M)$.

**Definition 1.22** A (smooth) **section** of $\xi$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

In other words, a section $s$ smoothly assigns to each point $x$ of the base manifold a vector $s(x)$ belonging to the fiber $\pi^{-1}(\{x\})$.

**Remark 1.23** We usually denote the set of all the sections of a bundle $\xi$ with total space $E$ by $\Gamma(E)$. It obviously forms a vector space (over $\mathbb{R}$ or $\mathbb{C}$, depending on the nature of $\xi$) although, as we will later see (for example in Section 2.1.2), it can be endowed with some further structure.

In the case of $E$ being the tangent bundle of $M$, the sections are usually known as **vector fields**. In Section 1.2, we will see that these are in fact only a particular instance of the more general notion of **tensor fields**.

Note that we can never simply assume that a given bundle admits non-trivial sections, by which we mean sections that are nowhere vanishing. We may, however, use the existence of specific sections to characterize trivial bundles :

**Proposition 1.24** A real $d$-plane bundle is trivial if and only if it admits $d$ pointwise linearly independent sections $s_1, \ldots, s_d$.

**Proof :** If $\xi = (E, \pi, M)$ is trivial and of rank $d$, we simply consider a global coordinate system

$$\varphi : E \to M \times \mathbb{R}^d$$

and set $s_j(p) = \varphi^{-1}(p, e_j)$, where $e_1, \ldots, e_d$ forms the standard basis of $\mathbb{R}^d$, for each point $p \in M$ and index $j \in \{1, \ldots, d\}$. These are then obviously pointwise linearly independent global sections.

Let us now assume that we have pointwise linearly independent sections $s_1, \ldots, s_d$ at our disposal for $\xi = (E, \pi, M)$. It follows that any vector $v \in E$ with $\pi(v) = p$ can be written in a unique manner as a linear combination

$$v = \sum_{i=1}^{d} v_i s_i(p),$$

where the coefficients $v_i \in \mathbb{R}$ also depend (smoothly) on $p$. We then build a global coordinate system by defining

$$\varphi\left(\sum_{i=1}^{d} v_i s_i(p)\right) = \left(p, \sum_{i=1}^{d} v_i e_i\right),$$

whenever $v \in E_p$. The map $\varphi$ is continuous and restricts to isomorphisms on each fiber of $\xi$. The fact that it is a homeomorphism, and therefore a bundle isomorphism is shown in details in [MS74], see Lemma 2.3 on page 27. □

This proposition helps us actually prove the already mentioned fact that the tautological line bundle $\gamma_n^1$ over $P^n(\mathbb{R})$ is non-trivial. Indeed, since a trivial line bundle would admit a nowhere vanishing section, let us assume that we have such an object $s : P^n(\mathbb{R}) \to E_n^1$ and work by contradiction. Note that $s$ sends an equivalence class $\{\pm x\}$ to an element

$$(\{\pm x\}, \lambda(x) \cdot x) \in E_n^1,$$

where $\lambda : S^n \to \mathbb{R}$ is a continuous maps such that $x \mapsto \lambda(x) \cdot x$ descends to $P^n(\mathbb{R})$. In other words, we expect it to satisfy the relation $\lambda(-x) = -\lambda(x)$. But then the composition $s \circ \pi : S^n \to \mathbb{R}$ being nowhere zero would represent a contradiction to the mean value theorem.

### 1.1.4 Classification of real vector bundles

As for any other mathematical construction, we are interested in classifying results for vector bundles. In fact, we will be able to show that for all integer $n$, there is a specific **classifying space** $\mathrm{BO}(n)$ and a so-called **tautological bundle** $E_n \to \mathrm{BO}(n)$ such that any $n$-plane bundle over a smooth manifold $M$ can be realized as a pullback along some (uniquely defined up to homotopy) smooth map $M \to \mathrm{BO}(n)$.

To this end, let $n$ and $k$ be integers and recall :

**Definition 1.25** The **Grassmannian** $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ is the set of all $n$-dimensional linear subspaces of $\mathbb{R}^{n+k}$.

This set is endowed with the quotient topology inherited from that of the **Stiefel manifold** $V_n(\mathbb{R}^{n+k})$ formed by of all the $n$-frames in $\mathbb{R}^{n+k}$, which itself is an open subset of the $n$-fold cartesian product

$$\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}.$$

The associated projection map simply takes an $n$-frame to the linear subspace it spans.

**Lemma 1.26** The Grassmannian $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ is a compact smooth manifold of dimension $nk$.

**Proof :** The proof of this is not complicated, but rather long and technical. It can be found for example in [MS74], see Lemma 5.1 on page 64. □

We may therefore refer to the spaces $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ as **Grassmann manifolds**.

**Remark 1.27** Clearly, $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ is the same as $P^n(\mathbb{R})$.

Note that each $n$-dimensional linear subspace $X \in \mathrm{Gr}_n(\mathbb{R}^{n+k})$ has an orthogonal complement $X^\perp$ in $\mathbb{R}^{n+k}$, which is a $k$-dimensional linear subspace. In fact :

**Lemma 1.28** The map defined by $X \mapsto X^\perp$ is a homeomorphism between the Grassmann manifolds $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ and $\mathrm{Gr}_k(\mathbb{R}^{n+k})$.

**Proof :** The proof of this results is also presented in [MS74], see Lemma 5.1 on page 64. □

A vector bundle is easily (and somewhat naturally) constructed over any Grassmann manifold $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ :

**Definition 1.29** The **tautological bundle** $\gamma_n^k$ over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ has total space

$$E_n^k = E(\gamma_n^k) = \left\{(X, v) \in \mathrm{Gr}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \mid v \in X\right\},$$

and its projection map is given by $(X, v) \mapsto X$.

A proof that this construction does indeed yield a vector bundle, especially that it is locally trivial, can be found in [MS74], see Lemma 5.2 on page 68.

Note that, for any integer $n$, we clearly have :

$$\mathrm{Gr}_n(\mathbb{R}^n) \subset \mathrm{Gr}_n(\mathbb{R}^{n+1}) \subset \cdots$$

If we use $\mathbb{R}^\infty$ to denote the real (infinite dimensional) vector space of sequences $\{x_i\}_{i\in\mathbb{N}}$ such that $x_i \neq 0$ for only a finite amount of indices $i \in \mathbb{N}$, then we may define :

**Definition 1.30** The **infinite Grassmann manifold** $\mathrm{Gr}_n = \mathrm{Gr}_n(\mathbb{R}^\infty)$ is the set of all the $n$-dimensional linear subspaces of $\mathbb{R}^\infty$.

We see $\mathrm{Gr}_n$ as the direct limit of the inclusions above ; in other words, a subset $U \subseteq \mathrm{Gr}_n$ is open if and only if its intersection with each $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ is open. We may now construct a **tautological bundle** $\gamma_n$ over $\mathrm{Gr}_n$ in much the same way we did for the finite dimensional case : set $E_n = E(\gamma_n)$ to be the set of pairs $(X, v)$, where $X$ is a linear subspace of dimension $n$ in $\mathbb{R}^\infty$ and $v$ a vector of $X$, then define the projection map $\theta : E(\gamma_n) \to \mathrm{Gr}_n$ by $(X, v) \mapsto X$.

**Definition 1.31** The bundle $(E_n, \theta, \mathrm{Gr}_n)$ is the **tautological bundle** over the infinite Grassmann manifold $\mathrm{Gr}_n$.

A proof of local triviality can be found in [Hat17, Lemma 1.15, page 28].

For the classification result announced at the beginning of this section, we only really need the base space to be paracompact. We will therefore once again present the relevant material in this context, although it is important to keep in mind that everything said here directly transfers to smooth bundles. The first result we mention already motivates the study of these tautological bundles we just introduced :

**Lemma 1.32** Any $n$-plane bundle $\xi$ over a paracompact base space admits a bundle morphism $\xi \to \gamma_n$.

**Proof :** See [MS74], Theorem 5.6 on page 73. □

We cannot expect of such morphisms to be unique, but we do have the following :

**Theorem 1.33** Any two bundle morphisms from an $n$-plane bundle over a paracompact base space to $\gamma_n$ are bundle-homotopic.

**Proof :** See [MS74], Theorem 5.7 on page 73. □

Recall that if $\xi$ and $\eta$ are vector bundles with respective total spaces $E$ and $F$, then two continuous maps $\hat{f}, \hat{g} : E \to F$ are called **bundle-homotopic** if there is a continuous map $\hat{h} : E \times [0, 1] \to F$ such that $\hat{h}(\cdot, t)$ is a bundle morphism for each $t \in [0, 1]$, $\hat{h}(\cdot, 0) = \hat{f}$ and $\hat{h}(\cdot, 1) = \hat{g}$.
An obvious consequence of Theorem 1.33 is the following :

**Corollary 1.34** Each $n$-plane bundle $\xi = (E, \pi, B)$ over a paracompact base space $B$ induces a unique homotopy class of continuous maps $B \to \mathrm{Gr}_n$.

We usually denote by $f_\xi : B \to \mathrm{Gr}_n$ any representative of this homotopy class, and refer to it as a **classifying map** of the bundle $\xi$. It is covered by a bundle morphism, which we write $\hat{f}_\xi$, such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\hat{f}_\xi} & E_n \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \theta} \\ B & \xrightarrow[f_\xi]{} & \mathrm{Gr}_n \end{array}$$

commutes. For all these reasons, the infinite Grassmann manifold $\mathrm{Gr}_n$ is also often called the **classifying space** $\mathrm{BO}(n)$.

**Remark 1.35** Note that the result above can be strengthened as follows : two $n$-plane bundles $\xi$ and $\eta$ over the same paracompact base space $B$ are isomorphic if and only if the induced maps $f_\xi$ and $f_\eta$ are homotopic.

In fact, we may go even further. Let us denote by $[B, Y]$ the set of homotopy classes of continuous maps $B \to Y$, where $B$ and $Y$ are topological spaces. Recall that, by Theorem 1.14, pullbacks along homotopic maps are isomorphic. Then :

**Lemma 1.36** If $B$ is paracompact, then the map $[f] \mapsto f^*(\gamma_n)$ induces a bijection $[B, \mathrm{Gr}_n] \to \mathrm{Vect}^n(B)$.

**Proof :** See [Hat17], Theorem 1.16 on page 29. □

The computation of $[B, \mathrm{Gr}_n]$ is, however, far too complicated to be carried out in most cases.

### 1.1.5 Classification of complex vector bundles

The case of complex vector bundles is very similar. For example, we consider the complex analogue of the real Grassmann manifolds :

**Definition 1.37** The **complex Grassmannian** $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ is the set of all $n$-dimensional complex linear subspaces of $\mathbb{C}^{n+k}$.

These are actually complex manifolds of dimension $nk$ and are therefore also called **complex Grassmann manifolds**. Once again, a complex vector bundle is easily constructed over $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ as follows :

**Definition 1.38** The **tautological bundle** $\nu_n^k$ over $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ has total space

$$E_n^k = E(\nu_n^k) = \left\{(X, v) \in \mathrm{Gr}_n(\mathbb{C}^{n+k}) \times \mathbb{C}^{n+k} \mid v \in X\right\},$$

and its projection map is given by $(X, v) \mapsto X$.

For $n$ fixed, the inclusions below are obvious

$$\mathrm{Gr}_n(\mathbb{C}^n) \subset \mathrm{Gr}_n(\mathbb{C}^{n+1}) \subset \cdots$$

and so we may define the direct limit $\mathrm{Gr}_n(\mathbb{C}^\infty)$, known as the **infinite complex Grassmannian**, over which we build the **tautological complex bundle** $\nu_n$ with total space $E_n$ exactly as we did in the real case. See [MS74], section 14.4 for the omitted details. As expected, it satisfies the following property :

**Theorem 1.39** Every complex $n$-plane bundle $\omega$ over a paracompact base space admits a bundle morphism $\omega \to \nu_n$.

**Proof :** See [MS74], Theorem 14.6 on page 169. □

It follows that if we have such a bundle $\omega = (E, \pi, B)$ and a **classifying** (base space) **map** $f_\omega : B \to \mathrm{Gr}_n(\mathbb{C}^\infty)$ covered by $\hat{f}_\omega : E \to E_n$, then $\omega$ is isomorphic to the pull-back $f_\omega^*(\nu_n)$. Note that, just as in the real case, the classifying maps induce a unique homotopy class. In fact, two complex bundles are isomorphic if and only if they have homotopic classifying maps.

We usually call $\nu_n$ the **universal complex $n$-plane bundle**, while its base space $\mathrm{Gr}_n(\mathbb{C}^\infty)$ is referred to as the **classifying space** for complex $n$-plane bundles, and therefore often denoted by $\mathrm{BU}(n)$ (see for example [MS74], page 169).

### 1.1.6 Orientation of vector bundles

There are many different, although of course equivalent, ways to define oriented vector bundles. In this section, we will introduce a "cohomological" notion of orientation but another (based on differential forms) will be presented in Section 1.2.3 below.

First, recall that for real vector spaces we already have an available definition of orientation :

**Definition 1.40** An **orientation** of a real vector space $V$ of dimension $n \geq 1$ is an equivalence class of bases under the following relation : two bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are seen as equivalent if the associated change of basis matrix has positive determinant.

There are, of course, only two such equivalence classes for every finite dimensional real vector space.

The notion of orientation in this setting can however also be expressed in the language of homology. Consider once again an $n$-dimensional real vector space $V$ and let $V_0 = V \backslash \{0\}$. Then we know that $H_n(V, V_0; \mathbb{Z}) \cong \mathbb{Z}$. There are therefore two possible generators of this top homology group, akin to the elements $\{\pm 1\}$ in $\mathbb{Z}$. We define the so-called **preferred generator**

$$\mu_V \in H_n(V, V_0; \mathbb{Z})$$

as the homology class of the image through a linear and orientation preserving embedding of the standard $n$-simplex $\Delta_n$ with canonically ordered vertices, such that the barycenter of $\Delta_n$ is mapped to 0. Now, and in a similar way, there exists a preferred generator $u_V$ of the top cohomology group $H^n(V, V_0; \mathbb{Z}) \cong \mathbb{Z}$ which is uniquely defined by the relation

$$\langle u_V, \mu_V \rangle = 1.$$

We now aim to extend this notion to vector bundles. Let $\xi = (E, \pi, M)$ be an $n$-plane bundle.

**Definition 1.41** An **orientation** of $\xi$ is the choice of a preferred generator $u_F \in H^n(F, F_0; \mathbb{Z})$ for all fibers $F$, in such a way that around each point $p \in M$ there is an open neighbourhood $X$ and a cohomology class

$$u \in H^n\left(\pi^{-1}(X), \pi^{-1}(X)_0; \mathbb{Z}\right)$$

with $u|_F = u_F$ for every fiber $F$ over any point $q \in X$.

Here, $\pi^{-1}(X)_0$ denotes the collection of the non-zero vector in $\pi^{-1}(X) \subseteq E$.

### 1.1.7 Thom isomorphism and Euler class

We quickly introduce the notion of a Thom class and its use in the definition of a so-called Euler class for oriented real vector bundles. This is intended as a preparation for the definition of Chern classes (see in particular Remark 1.72) and the general idea behind the splitting principle (see Section 1.5.2).

Let $\xi = (E, \pi, M)$ be an oriented $n$-plane bundle and let $E_0$ denote the space obtained by removing from $E$ the image of the zero section, that is, the space of all the non-zero vectors in $E$. Similarly, for a vector space $V$ we write $V_0$ to denote the set of non-zero vectors.

**Proposition 1.42** The cohomology group $H^i(E, E_0; \mathbb{Z})$ is trivial while $i < n$ and the group $H^n(E, E_0; \mathbb{Z})$ contains exactly one element $u$ whose restriction

$$u|_{(F,F_0)} \in H^n(F, F_0; \mathbb{Z})$$

is equal to the preferred generator $u_F$ for every fiber $F$ of $\xi$. Moreover, the map defined by

$$y \mapsto y \smile u$$

is an isomorphism from $H^k(E; \mathbb{Z})$ into $H^{k+n}(E, E_0; \mathbb{Z})$ for every integer $k$.

**Proof :** See Chapter 10 of [MS74] for the details. □

We can, however, prove the following easy lemma :

**Lemma 1.43** Let $\xi = (E, \pi, M)$ be an $n$-plane bundle. Then the pullback

$$\pi^* : H^*(M; \mathbb{Z}) \to H^*(E; \mathbb{Z})$$

is an isomorphism of graded rings.

**Proof :** The base manifold $M$ can be embedded into $E$ using the zero section. The submanifold $M \subseteq E$ is then clearly a deformation retract of $E$ with retraction map $\pi$, which immediately yields the expected result. □

Using this lemma in conjunction with Thom's isomorphism theorem, we see that the map

$$\phi : H^j(M; \mathbb{Z}) \to H^{j+n}(E, E_0; \mathbb{Z})$$

defined by

$$\phi(\alpha) = (\pi^*\alpha) \smile u$$

is also an isomorphism. We call it the **Thom isomorphism**.

**Definition 1.44** The unique class $u \in H^n(E, E_0; \mathbb{Z})$ satisfying the conditions of the Thom isomorphism theorem is called the **Thom class**.

We will now observe how this class $u$ in the top cohomology of the total space uniquely determines a particular class in $H^n(M; \mathbb{Z})$. Consider once again an oriented $n$-plane bundle $\xi = (E, \pi, M)$ with Thom class $u \in H^n(E, E_0; \mathbb{Z})$. The inclusion of pairs $(E, \emptyset) \to (E, E_0)$ induces a homomorphism

$$H^n(E, E_0; \mathbb{Z}) \to H^n(E; \mathbb{Z})$$

which we name the **restriction to** $E$. The image of an element $\alpha \in H^n(E, E_0; \mathbb{Z})$ will be denoted by $\alpha|_E$.

**Definition 1.45** The **Euler class** of $\xi$ is the cohomology class $e(\xi) \in H^n(M; \mathbb{Z})$ which corresponds to the restriction $u|_E$ under the isomorphism

$$\pi^* : H^n(M; \mathbb{Z}) \to H^n(E; \mathbb{Z}).$$

This new class satisfies a number of important properties, the most important for us being the following result. Let $\pi_0 : E_0 \to M$ denote the restriction of the projection map to $E_0$.

**Theorem 1.46 (Gysin sequence)** For every oriented real $n$-plane bundle $\xi$ there is a long exact sequence

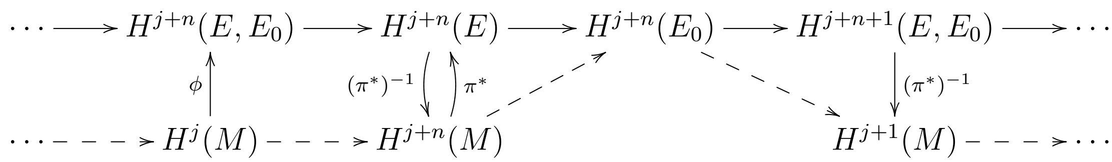
$$\cdots \longrightarrow H^i(M;\mathbb{Z}) \xrightarrow{\smile e(\xi)} H^{i+n}(M;\mathbb{Z}) \xrightarrow{\pi_0^*} H^{i+n}(E_0;\mathbb{Z}) \longrightarrow H^{i+1}(M;\mathbb{Z}) \longrightarrow \cdots$$

called the **Gysin sequence**.

**Proof :** If we admit the oriented Thom isomorphism theorem, the Gysin sequence is easily obtained by starting with the long exact sequence of cohomology for the pair $(E, E_0)$, that is

$$\cdots \longrightarrow H^{j+n}(E,E_0) \longrightarrow H^{j+n}(E) \longrightarrow H^{j+n}(E_0) \longrightarrow H^{j+n+1}(E,E_0) \longrightarrow \cdots$$

and "replacing" as shown in the following commutative diagram :



Here, we have omitted to denote that these cohomology groups are to be taken with integral coefficients. ☐

Let us finally simply illustrate why the Euler class and the **Euler characteristic** share a similar name. These two invariants are indeed linked as follows :

**Proposition 1.47** Let $M$ be a smooth compact oriented manifold. If $[M]$ denotes its fundamental class and $\chi(M)$ its Euler characteristic, then

$$\langle e(TM), [M] \rangle = \chi(M).$$

**Proof :** See [MS74], Corollary 11.12 on page 136. ☐

## 1.2 Functorial constructions

Using the cocycle definition introduced in the first section of this chapter, it is quite easy to transpose constructions on vector spaces to the context of vector bundles.

### 1.2.1 Tensor products, direct sums and dual bundle

For example, if $\{f_{ij}\}$ and $\{g_{ij}\}$ are the cocycles (associated to the same open cover of the base manifold $M$) induced by the vector bundles $\xi = (E, \pi, M)$ and $\eta$ respectively, then we construct their **tensor product** $\xi \otimes \eta$ through the process described in Proposition

1.6 using the cocycle $\{f_{ij} \otimes g_{ij}\}$. The fibers of $\xi \otimes \eta$ are given by the tensor product of the fibers.

Similarly, under the above hypotheses, we may build the **direct sum** $\xi \oplus \eta$ (also known as the **Whitney sum**) on the basis of the cocycle $\{f_{ij} \oplus g_{ij}\}$. Its fibers are the direct sums of the fibers in $\xi$ and $\eta$.

Lastly, the coycle $\{(f_{ij}^T)^{-1}\}$, where $f_{ij}^T$ is the transpose of the transition function $f_{ij}$, yields the **dual bundle** $\xi^*$ (or $E^*$), whose fibers are the dual spaces of the fibers of $\xi$. We may then build the **endomorphism bundle** $\mathrm{End}(E)$, given by the tensor product $E \otimes E^*$. One can check that its fiber over a point $p \in M$ is indeed formed by the endomorphisms $\pi^{-1}(\{p\}) \to \pi^{-1}(\{p\})$. The $\mathrm{Hom}(E, F) = E \otimes F^*$ bundle, associated to two bundles $E$ and $F$ over the same base manifold, can also be constructed using this type of cocycle.

### 1.2.2 Representations

Suppose that we are given a real $n$-plane bundle $\xi = (E, \pi, M)$ through a cocycle $\{g_{ij}\}$. Let $m$ be an integer. If we assume that the structure group of $\xi$ is reduced to the orthogonal group via a bundle metric and consider a representation

$$\rho : \mathrm{O}(n) \to \mathrm{O}(m),$$

then we may form an $m$-plane bundle over $M$ by applying Proposition 1.6 to the cocycle $\{\rho \circ g_{ij}\}$. This new bundle can be denoted by $\rho\xi$ or be referred to by its total space, which we write $\rho E$. In the case of a complex bundle $\omega$, we consider representations

$$\rho : U(n) \to U(m),$$

as their structure group can always be reduced to a unitary group through the choice of a **hermitian metric**.
The easiest example of this construction in the complex setting is maybe that of the **conjugate bundle** $\bar{\omega} = (\bar{E}, \pi, M)$, as one simply sets $\rho(U) = \bar{U}$ with $n = m$.

### 1.2.3 Exterior and symmetric products

Recall that if $V$ is a real vector space, we can form **tensor products**

$$T^j V = \underbrace{V \otimes \cdots \otimes V}_{j \text{ times}}$$

and the (graded) **tensor algebra**

$$T(V) = \bigoplus_{j \geq 0} T^j V.$$

The **exterior** and **symmetric algebras** $\Lambda(V)$, $S(V)$ are then constructed from $T(V)$ by taking quotients with appropriate ideals :

$$\Lambda(V) = T(V)/\langle\{x \otimes x \mid x \in V\}\rangle \quad \text{and} \quad S(V) = T(V)/\langle\{x \otimes y - y \otimes x \mid x, y \in V\}\rangle.$$

These are naturally endowed with the **exterior product** $\wedge$ and the **symmetric product** $\odot$ respectively, which in both cases are inherited from the tensor product on $T(V)$. For an integer $j$, we denote by $\Lambda^j V$ and $S^j V$ the respective linear subspaces formed by the elements of degree $j$.

We can now adapt these constructions to any $n$-plane bundle $\xi = (E, \pi, M)$. Consider the representations $\rho$ and $\zeta$ of $\mathrm{O}(n)$ such that :

$$\rho(U)(v_1 \wedge \cdots \wedge v_j) = (Uv_1) \wedge \cdots \wedge (Uv_j) \quad \text{and} \quad \zeta(U)(v_1 \odot \cdots \odot v_j) = (Uv_1) \odot \cdots \odot (Uv_j)$$

where $U \in \mathrm{O}(n)$ and $j \geq 1$ is an integer. We then define :

**Definition 1.48** The new bundles $\rho E = \Lambda^j E$ and $\zeta E = S^j E$ are known respectively as the ***j*-th exterior** and ***j*-th symmetric products**.

In the special case of the cotangent bundle, that is $E = T^*M$, the sections of the $j$-th exterior product $\Lambda^j T^*M$ are called the **differential *j*-forms** and we often write $\Omega^j(M)$ rather than $\Gamma(\Lambda^j T^*M)$.

**Remark 1.49** As announced at the start of Section 1.1.6, the orientability of a manifold $M$ of dimension $n$ can be rephrased in the language of differential forms. In this context, nowhere vanishing differential $n$-forms are called **volume forms**. We can define an equivalence relation $\sim$ on the set of these sections $\Gamma\left(\Lambda^n T^*M \backslash \{0\}\right)$. Note here that $\Lambda^n T^*M \backslash \{0\}$ is no longer a vector bundle, but a fiber bundle with fiber $\mathbb{R}\backslash\{0\}$ (see Section 1.3.1). We declare that $\alpha \sim \beta$ if and only if there exists a positive smooth function $f : M \to \mathbb{R}$ such that $\alpha = f\beta$. Then the quotient space can be shown to be made exactly of either one or two points. The latter case corresponds to an orientable manifold $M$.

Consider once again an $n$-dimensional real vector space $V$ and let $r$ and $s$ be integers. We define :

$$T^r_s V = \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ times}}.$$

Its elements are known as $(r, s)$**-tensors** or, alternatively, as **tensors of type** $(r, s)$. Of course, if we are given an $n$-plane bundle $E \to M$, we may build the so-called **tensor bundle of type** $(r, s)$, $T^r_s E \to M$. Its sections are then known as $(r, s)$**-tensor fields** on $M$. The most basic example of such a field is that of vector fields, which are section of the tensor bundle of type $(1, 0)$. One can also see the bundle metrics as a particular type of $(0, 2)$-tensor fields (see Section 1.2.5).

### 1.2.4 Sections and tensor products

Suppose that we are given two bundles $\xi = (E, \pi, M)$ and $\eta = (F, \psi, M)$ over the same compact base manifold $M$, of respective ranks $n$ and $m$. It is natural to wonder whether the sections of $\xi \otimes \eta$ can be expressed in terms of the sections of $\xi$ and $\eta$.

Note that there is a natural bilinear map $\Theta : \Gamma(E) \times \Gamma(F) \to \Gamma(E \otimes F)$ simply given by $(\sigma, \mu) \mapsto \sigma \otimes \mu$, where $(\sigma \otimes \mu)(p) = \sigma(p) \otimes \mu(p)$ for all $p \in M$. By the universal property of the tensor product, $\Theta$ induces another map $\theta : \Gamma(E) \otimes \Gamma(F) \to \Gamma(E \otimes F)$ such that $\theta(\alpha \otimes \beta) = \Theta(\alpha, \beta)$. In fact, we have the following :

**Theorem 1.50** The map $\theta$ is an isomorphism $\Gamma(E) \otimes \Gamma(F) \cong \Gamma(E \otimes F)$ of $\mathcal{C}^\infty(M)$-modules.

**Proof :** We only give an outline of the proof, which can be found in greater details in [Con93], Section 7.5.
Notice however that, in the case of trivial bundles, the result is quite obvious since $\Gamma(E \otimes F)$ is clearly a free $\mathcal{C}^\infty(M)$-module over the collection of elements $\sigma_i \otimes \mu_j$, for linearly independent sections $\sigma_1, \ldots, \sigma_n \in \Gamma(E)$ and $\mu_1, \ldots, \mu_m \in \Gamma(F)$.
If $E$ and $F$ are not trivial, we use Theorem 1.16 to find "completions" $E^\perp$ and $F^\perp$ such that $E \oplus E^\perp$ and $F \oplus F^\perp$ are trivial. □

This result will be helpful in Sections 2.2 and 2.3 when talking about curvature and in Chapter 4 for differential operators.

### 1.2.5 Bundle metrics

Using the constructions above, we can offer a more abstract reformulation for the definition of a bundle metric. Let us consider some smooth bundle $\pi : E \to M$. Recall that at any point $p \in M$, the metric $\langle \cdot, \cdot \rangle$ yields a scalar product $\langle \cdot, \cdot \rangle_p$ on the fiber $F = \pi^{-1}(\{p\})$. This product is, in particular, a bilinear form and therefore corresponds to an element of $(F \otimes F)^*$. Clearly, $(F \otimes F)^*$ is isomorphic to $F^* \otimes F^*$ and one can thus see the bundle metric $\langle \cdot, \cdot \rangle$ as a particular type of section of $E^* \otimes E^*$, that is, one which is positive definite and symmetric within each fiber.

## 1.3 Principal and associated bundles

When dealing with spin structures in later chapters, we will need the notion of **principal** and **associated bundles**. These are **fiber** (rather than vector) **bundles**, so we begin with a brief overview of these objects. Most of this material can be found in [Hus94] or in the classic book of Steenrod [Ste51].

### 1.3.1 Fiber bundles

Let $E$, $M$ and $F$ be smooth manifolds. Let $\pi : E \to M$ be a smooth and surjective map.

**Definition 1.51** The data $(E, \pi, M, F)$ forms a (smooth) **fiber bundle** over $M$ if for every point $x \in M$ there exists an open neighbourhood $U \subseteq M$ around $x$ and a diffeomorphism

$$\varphi : \pi^{-1}(U) \to U \times F$$

such that, if $p : U \times F \to F$ denotes the canonical projection to $F$, then the following diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\varphi\quad} & U \times F \\ & \underset{\pi}{\searrow} \quad \underset{p}{\swarrow} & \\ & U & \end{array}$$

is commutative. Clearly, for all $x \in M$, the preimage $\pi^{-1}(\{x\})$ is diffeomorphic to the model space $F$, which we call the **fiber**. The manifolds $E$ and $M$ are known respectively as the **total** and **base space**, while we usually refer to $(U, \varphi)$ as a **local coordinate system** or a **local trivialization**.

A fibre bundle $(E, \pi, M, F)$ is said to be **trivial** if there is a local trivialization $(M, \varphi)$ or, in other words, if it admits a **global trivialization**.

**Remark 1.52** The definition above can be adapted to the continuous case, that is, when we only assume that $E, M$ and $F$ are topological spaces. We simply replace every smooth map by a continuous analogue.

Obviously, every real or complex $d$-plane vector bundle (see Definitions 1.1 and 1.17) is in particular a fiber bundle, where $F = \mathbb{R}^d$ or $F = \mathbb{C}^d$. This new notion is, however, much broader :

**Example 1.53** The standard torus $T^2$ can be seen as a fiber bundle over the circle $S^1$, with fiber $S^1$ as well. This is in fact a trivial bundle.

The notion of fiber bundle also encompasses that of **covering spaces** :

**Example 1.54** Let $X$ be a topological space. A covering space $\pi : \tilde{X} \to X$ is then a (continuous) fiber bundle whose projection $\pi$ is in fact a local homeomorphism and the fiber is a discrete space.

All the notions we discussed when introducing vector bundles have direct analogues in this more general context, for example **bundle morphisms** and **sections**.

We can also define **transition functions** for fiber bundles. Indeed, if $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of the base space $M$ of some fiber bundle $(E, \pi, M, F))$ then, whenever $U_i$ and $U_j$ intersect, the map

$$\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

has to be of the form

$$\left(\varphi_i \circ \varphi_j^{-1}\right)(x, f) = (x, g_{ij}(x)(f))$$

where $g_{ij}(x) \in \text{Diff}(F)$. The collection of these diffeomorphisms $g_{ij}(x)$ for all $x \in M$ generates a subgroup of $\text{Diff}(F)$, which we once again call the **structure group** of the fiber bundle, and the maps $g_{ij} : U_i \cap U_j \to \text{Diff}(F)$ form a **cocycle** in the sense of Definition 1.5. As was already the case for vector bundles, all of the information necessary to construct a fiber bundle is found in such a cocycle (see Proposition 1.6). We will come back to this later when discussing principal bundles.

### 1.3.2 Principal bundles

To introduce the notion of principal bundle, we mainly follow the Appendix A in [BLM89] but adapt it to our usual smooth setting.

Let $(G, \cdot)$ be a Lie group and $M$ be a smooth manifold.

**Definition 1.55** A **principal $G$-bundle** over $M$ is a fiber bundle $\pi : P \to M$ together with a smooth right action on $P$ by $G$ which preserves the fibers and acts freely and transitively on them.

This particular formulation of the definition is quite dense, so we will now try to root out as much information as possible from it. Notice first that the properties of the action on the fibers ensure that they correspond to the orbits of $G$. From the local triviality of $\pi$, it follows that there is an open neighbourhood $U$ around every point $p \in M$ and a diffeomorphism

$$\Phi : \pi^{-1}(U) \to U \times G$$

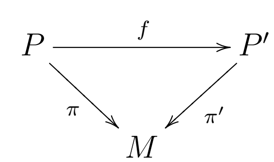
of the form $\Phi = \pi \times \varphi$, where $\varphi : \pi^{-1}(U) \to G$ is a smooth $G$-equivariant map. To fix the notation for the action of $G$ on $P$, we recall that this means

$$\varphi(p.g) = \varphi(p) \cdot g$$

for all $p \in \pi^{-1}(U)$ and $g \in G$.

**Remark 1.56** It can be shown that $M$ is diffeomorphic to the quotient $P/G$.

We declare two principal $G$-bundles $\pi : P \to M$ and $\pi' : P' \to M$ to be **equivalent** if there is a $G$-equivariant diffeomorphism $f : P \to P'$ such that the diagram




commutes. This is an equivalence relation and we denote by $\text{Prin}_G(M)$ the set of equivalence classes of principal $G$-bundles over $M$. Similarly, we define

**Definition 1.57** Let $\pi : P \to M$ be a principal $G$-bundle and $\pi' : P' \to N$ be a principal $G'$-bundle. A **morphism** of principal bundles between $P$ and $P'$ is then a pair $(F, f)$, where $F : P \to P'$ is smooth and $f : G \to G'$ is a group homomorphism, such that the following equivariance condition is satisfied

$$F(p.g) = F(p).f(g),$$

for all $p \in P$ and $g \in G$.

Note that the conditions above (in particular the fact that the structure groups act transitively on the fibers) ensure that the smooth map $F$ between the total spaces $P$ and $P'$ preserves the fibers. It follows that a morphism of principal bundles induces a map of base spaces $\tilde{F} : M \to N$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow[\tilde{F}]{} & N \end{array}$$

is commutative.

Observe that there is nothing to change in the usual definition of the **section** of a fiber bundle to adapt it to this new setting of principal bundles.

Let us now review some of the most useful examples of principal bundles. The easiest of these is of course the following :

**Example 1.58** Let $(G, \cdot)$ be a Lie group and $M$ a smooth manifold. Then the cartesian product $M \times G$ endowed with the obvious right action

$$(x, g).h = (x, g \cdot h),$$

and the natural projection $M \times G \to M$ forms a principal $G$-bundle, which we call the **trivial principal $G$-bundle over $M$**.

Note that we extend this terminology in the following way : any principal $G$-bundle over $M$ that is equivalent to $M \times G$ is said to be trivial.

**Remark 1.59** It can be shown that a principal $G$-bundle $P$ is trivial if and only if it admits a section.

This might appear to be a loosened condition compared to the case of vector bundles (see Proposition 1.24), but we will see why it still makes sense when discussing the next example.

**Example 1.60** Consider a real rank $d$ vector bundle $\pi : E \to M$. Its fibers $E_x$ are copies of $\mathbb{R}^d$, for which we may consider the the set of **frames**, that is, the set $FE_x$ of ordered bases $(v_1, \ldots, v_d)$ of $E_x$. We then define the disjoint union

$$FE = \coprod_{x \in M} FE_x,$$

which we consider as a subspace of the $d$-fold direct sum $E \oplus \ldots \oplus E$. There is an obvious projection $FE \to M$ and a natural right action of $\mathrm{GL}_d(\mathbb{R})$ on $FE$ given by

$$((v_1, \ldots, v_d), A) \mapsto (Av_1, \ldots, Av_d),$$

which make a principal $\mathrm{GL}_d(\mathbb{R})$-bundle out of $FE$. This construction is then called the **frame bundle** of $E$.

The idea of the frame bundle can be refined if there is additional structure on the original vector bundle $E$. For instance, in the case of an oriented bundle, we define the **bundle of oriented frames** $P_{\mathrm{GL}_d^+(\mathbb{R})}(E)$, which is a principal $\mathrm{GL}_d^+(\mathbb{R})$-bundle. Recall here that

$$\mathrm{GL}_d^+(\mathbb{R}) = \{A \in \mathrm{GL}_d(\mathbb{R}) \mid \det(A) > 0\}.$$

Finally, if $E$ is endowed with a bundle metric, we may consider the **bundle of orthonormal frames** $P_{\mathrm{O}(d)}(E)$ and the **bundle of oriented orthonormal frames** $P_{\mathrm{SO}(d)}(E)$, which are principal $\mathrm{O}(d)$ and $\mathrm{SO}(d)$-bundles respectively.

**Remark 1.61** Recall that, by Proposition 1.24, a real vector bundle $E \to M$ of rank $d$ is trivial if and only if we can find $d$ pointwise linearly independent sections $s_1, \ldots, s_d : M \to E$. Consider now the frame bundle $FE$ and notice that if it admits a section $s : M \to FE$, then $E$ is trivial. It is also easy to see that the converse is true. In other words, the triviality of $E$ as a real vector bundle is equivalent to the existence of a section of its frame bundle that is, using Remark 1.59, to the triviality of $FE$ as a principal $\mathrm{GL}_d(\mathbb{R})$-bundle.

Suppose that $\pi : P \to M$ and $\pi' : P' \to M$ are principal bundles over $M$ of respective structure groups $G$ and $G'$. Let $f : G \to G'$ be a Lie group homomorphism.

**Definition 1.62** An **$f$-reduction of the structure group** is a morphism of principal bundles $(F, f)$ such that $F : P \to P'$ induces the identity on the base space, that is, $\tilde{F} = \mathrm{id}_M$ in the notation introduced above.

The word "reduction" might be seen as a bad choice of terminology here, since $G$ might for example be "bigger" than $G'$ (and it will be, when we apply this construction to spin structures, see Section 5.2.1).

**Remark 1.63** In fact, the name stems from the more classical point of view, where $f$ is assumed to be injective and $F$, an embedding. This ensures the existence of a system of local coordinates on $M$ for the principal bundle $P'$ whose transition functions take their values in (the subgroup) $G$ rather than $G'$.

We end this section by an important comment on the equivalence classes of principal $G$-bundles over a manifold $M$.

**Remark 1.64** Principal $G$-bundles are, in particular, fiber bundles and, as such, can be given by an open cover $\mathcal{U} = \{U_i\}$ and transition functions $g_{ij} : U_i \cap U_j \to G$, which act on $G$ itself by left multiplication. In [BLM89], Appendix A, it is shown that the pair $(\mathcal{U}, \{g_{ij}\})$ can then be seen as a **Cech cocycle** and it can be further checked that if we are given two principal $G$-bundles $(\mathcal{U}, \{g_{ij}\})$ and $(\mathcal{U}, \{g'_{ij}\})$ over the same open cover $\mathcal{U}$, then they are equivalent if and only if there are maps $g_k : U_k \to G$ such that

$$g'_{ij} = g_i^{-1} \cdot g_{ij} \cdot g_j$$

over $U_i \cap U_j$ for all indices $i$ and $j$. This corresponds in **Cech cohomology** to a so-called **Cech coboundary condition**. The set of equivalence classes of this relation (that is, of principal $G$-bundles over $M$ which can be trivialized over the open sets of the cover $\mathcal{U}$) is denoted by $H^1(\mathcal{U}; G)$. It is possible (see [BLM89], Appendix A on page 372) to define the direct limit over the refinements of the open cover $\mathcal{U}$, which we write $H^1(M; G)$. This set can then be identified with $\mathrm{Prin}_G(M)$. Note moreover that, in the case of an abelian structure group $G$, $H^1(M; G)$ is the first **Cech cohomology** group of $M$ with coefficients in $G$. Thankfully, since $M$ is a smooth manifold (and therefore in particular has the homotopy type of a CW-complex), this corresponds to the more familiar singular cohomology.
For our purpose, the important consequence (see [BLM89], Example A5 on page 374) of these considerations is the existence of an isomorphism

$$c_1 : H^1(M; S^1) \to H^2(M; \mathbb{Z}).$$

which can be identified with the **first Chern class** (see Section 1.4.3). Here, $H^1(M; S^1)$ is to be understood as $\mathrm{Prin}_{S^1}(M)$.
Note finally that a complex line bundle can have its structure group $\mathrm{GL}_1(\mathbb{C})$ reduced to $U(1) = S^1$ through the choice of a hermitian bundle metric, which always exists. Therefore, if we denote by $L(M)$ the set of equivalence classes of complex line bundles over $M$, the first Chern class yields an isomorphism

$$(L(M), \otimes) \to (H^2(M; \mathbb{Z}), +).$$

The (abelian) group structure on $(L(M), \otimes)$ is given by the fact that $\bar{L} \otimes L$ is trivial for every $L \in L(M)$.

### 1.3.3 Associated bundles

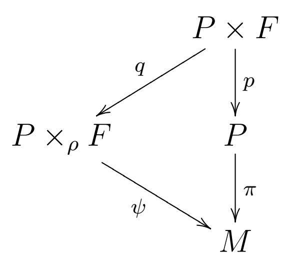
Suppose that we are given a principal $G$-bundle $\pi : P \to M$ and a left action $\rho$ of $G$ on a model manifold $F$, that is, a continuous group homomorphism

$$\rho : G \to \mathrm{Diff}(F),$$

where $\mathrm{Diff}(F)$ is endowed with a suitable topology. We can then define a right action of $G$ on the product space $P \times F$ as follows

$$(p, f).g = \left(p.g^{-1}, \rho(g).f\right),$$

for all $p \in P$, $f \in F$ and $g \in G$. We denote the quotient by $P \times_\rho F$. Note that this action of $G$ preserves the fibers of $P$. In fact, $P \times_\rho F$ is a fiber bundle over $M$, with fiber $F$ and projection $\psi$ given by the following commutative diagram




where $p$ is the projection on the first component and $q$ is the quotient map associated to the right action of $G$ on $P \times F$.

**Definition 1.65** We call $\psi : P \times_\rho F \to M$, the **fiber bundle associated to** $P$ **via** $\rho$.

In the more concrete case of a vector bundle $E \to M$, we can construct every tensor bundle (see Section 1.2.3) as an associated bundle of the frame bundle $FE$. For example, the tangent bundle $TM \to M$ can be written as

$$F(TM) \times_\gamma \mathbb{R}^d$$

if $M$ is of dimension $d$ and $\gamma : \mathrm{GL}_d(\mathbb{R}) \to \mathrm{GL}\left(\mathbb{R}^d\right)$ is the obvious isomorphism.

In fact, for every fiber bundle $\pi : E \to M$ with fiber $F$ and structure group $G$, there is an associated principal $G$-bundle $P_G(E)$. To build it, we first take an open cover of $M$ by domains of local coordinate systems $U_i$ and quotient the disjoint union

$$\coprod_i U_i \times F$$

through the identification given by the transition functions. This simply yields the total space $E$ (recall the exact construction of Proposition 1.6). We then create $P_G(E)$ by considering

$$\coprod_i U_i \times G$$

and gluing once again along the same transition functions, which act on $G$ from the left by multiplication. Note that this general construction does indeed coincide with our earlier definition of the frame bundle $P_{\mathrm{GL}_d(\mathbb{R})}(E)$ as well as our previous notations $P_{\mathrm{GL}_d^+(\mathbb{R})}(E)$, $P_{\mathrm{O}(d)}(E)$ and $P_{\mathrm{SO}(d)}(E)$.
On the other hand, being given $P_G(E)$, we may always recover the original bundle $E$ as an associated bundle $P_G(E) \times_\rho F$. The construction $F(TM) \times_\gamma \mathbb{R}^d$ above is only an example of this procedure.

## 1.4 Characteristic classes

### 1.4.1 General point of view

Let $\xi = (E, \pi, M)$ be a real $n$-plane bundle and suppose that we know the cohomology ring $H^*(\mathrm{BO}(n); G)$, where $G$ is some fixed coefficient group. Let $c \in H^*(\mathrm{BO}(n); G)$. Using a classifying map $f_\xi$ of $\xi$ (see Section 1.1.4), we may consider the pullback

$$c(\xi) = f_\xi^*(c) \in H^*(M; G)$$

since this only depends on the homotopy class of $f_\xi$ which, as we know, is uniquely determined.

**Definition 1.66** This element $c(\xi)$ is then known as the **characteristic cohomology class of $\xi$ determined by** $c$.

The construction above is natural with respect to bundle morphisms, that is, if $\hat{g} : \xi \to \eta$ is a bundle morphism covering a map $g$ of base spaces, then $g^* c(\eta) = c(\xi)$. Indeed, if $\xi = (E, \pi, M)$ and $\eta = (F, \psi, N)$, we have the following commutative diagram :

$$\begin{array}{ccccccc} E_n & \xleftarrow{\hat{f}_\xi} & E & \xrightarrow{\hat{g}} & F & \xrightarrow{\hat{f}_\eta} & E_n \\ \downarrow \theta & & \downarrow \pi & & \downarrow \psi & & \downarrow \theta \\ \mathrm{BO}(n) & \xleftarrow[f_\xi]{} & M & \xrightarrow[g]{} & N & \xrightarrow[f_\eta]{} & \mathrm{BO}(n) \end{array}$$

However, the composition $\hat{h} = \hat{f}_\eta \circ \hat{g}$ is also a bundle morphism $\xi \to \gamma_n$ with induced base space map $h = f_\eta \circ g$ and therefore, by Theorem 1.33, $\hat{h}$ and $\hat{f}_\xi$ are bundle homotopic. It follows that $h$ and $f_\xi$ are homotopic and so $h^* = f_\xi^*$ as maps from $H^i(\mathrm{BO}(n); G)$ to $H^i(M; G)$. Thus :

$$g^* c(\eta) = h^* c = f_\xi^* c = c(\xi).$$

Similarly, it is easy to show that for any association $\xi \to c(\xi)$ which is natural with respect to bundle morphisms, we have

$$c(\xi) = f_\xi^* c(\gamma_n).$$

It is therefore clear that the computation of the cohomology ring $H^*(\mathrm{BO}(n); G)$ is of crucial importance, for it is isomorphic to the ring formed by all the "**characteristic classes**" of $n$-plane bundles over paracompact spaces with coefficients in $G$. We will, however, not carry out these computations here (see for example the Chapters 5 and 6 of [MS74], which explore this problem in the case $G = \mathbb{Z}_2$).
We have, of course, corresponding notions in the context of complex bundles (replacing the classifying space $\mathrm{BO}(n)$ with $\mathrm{BU}(n)$), following the ideas laid out in Section 1.1.5.

We continue by introducing some particular kinds of characteristic classes which we will need to use in the following chapters.

### 1.4.2 Stiefel-Whitney classes

The first type of characteristic classes that we study in more detail are the so-called **Stiefel-Whitney classes**, which correspond to the case $G = \mathbb{Z}_2$. These classes will prove useful for a new characterization of orientability and, later on, to give a condition for the existence of a "spin structure" (see Section 5.2.1).

The Stiefel-Whitney classes can be constructed explicitly, for example using the Steenrod squares (see [MS74], Chapter 8), but we will not trouble ourselves with establishing them in this way. We simply describe them through a set of axioms, which we accept as fully and uniquely characterizing of actual elements in $H^*(M; \mathbb{Z}_2)$. The existence and uniqueness of cohomology classes satisfying these conditions is demonstrated, for example, in [MS74], Chapter 7 and 8.

- **Axiom 1 :** To each real vector bundle $\xi = (E, \pi, M)$ corresponds a sequence of cohomology classes $w_i(\xi) \in H^i(M; \mathbb{Z}_2)$, with $i \geq 0$. The class $w_0(\xi)$ is the unit element in $H^0(M; \mathbb{Z}_2)$ and, moreover, if $\xi$ is an $n$-plane bundle, then $w_i(\xi) = 0$ for all indices $i > n$.

- **Axiom 2 (Naturality) :** Let $\xi = (E, \pi, M)$ and $\eta = (F, \psi, N)$ be vector bundles. If $f : M \to N$ is covered by a bundle morphism $\xi \to \eta$, then :

$$w_i(\xi) = f^* w_i(\eta).$$

- **Axiom 3 (Whitney product theorem) :** If $\xi$ and $\eta$ are vector bundles over the same base manifold $M$, then :

$$w_j(\xi \oplus \eta) = \sum_{i=0}^{j} w_i(\xi) \smile w_{j-i}(\eta)$$

  for all $j \geq 0$.

- **Axiom 4 :** The class $w_1(\gamma_1^1)$ is non-zero.

Note that the symbol $\smile$ denoting the **cup product** (in the cohomology ring) will most often be omitted.

**Definition 1.67** The **total Stiefel-Whitney class** of a real $n$-plane bundle $\xi = (E, \pi, M)$ is the element

$$w(\xi) = 1 + w_1(\xi) + \ldots + w_n(\xi)$$

in the cohomology ring $H^*(M; \mathbb{Z}_2)$.

With this notation in mind, we may rewrite the Whitney product theorem as follows :

$$w(\xi \oplus \eta) = w(\xi)w(\eta).$$

Note finally that the Stiefel-Whitney classes of the universal bundle $\gamma_n$ over $\mathrm{BO}(n)$ freely generate the ring $H^*(\mathrm{BO}(n); \mathbb{Z}_2)$ as a polynomial algebra (for the proof, see [MS74], Theorem 7.1 on page 91).

### 1.4.3 Chern classes

We now turn to the **Chern classes**, which are defined for integer coefficients (that is, $G = \mathbb{Z}$) and on complex vector bundles. As is often the case, there are numerous ways to define them, for example through Chern-Weil theory or the use of an Euler class (see Remark 1.72 below). At first, we will simply introduce them through an axiomatic description, in much the same way we did for the Stiefel-Whitney classes.

- **Axiom 1 :** To each complex vector bundle $\omega = (E, \pi, M)$ corresponds a sequence of cohomology classes $c_i(\omega) \in H^{2i}(M; \mathbb{Z})$, with $i \geq 0$. The class $c_0(\omega)$ is the unit element in the cohomology ring $H^*(M; \mathbb{Z})$ and, moreover, if $\omega$ is a complex $n$-plane bundle, then $c_i(\omega) = 0$ for all indices $i > n$. In this case, we write
$$c(\omega) = 1 + c_1(\omega) + \ldots + c_n(\omega).$$
See Definition 1.68 below.

- **Axiom 2 (Naturality) :** Let $\omega = (E, \pi, M)$ and $\eta = (F, \psi, N)$ be complex vector bundles. If $f : M \to N$ is covered by a bundle morphism $\omega \to \eta$, then
$$c_i(\omega) = f^*(c_i(\eta)).$$

- **Axiom 3 (Whitney sum formula) :** If $\omega$ and $\eta$ are complex vector bundle of rank $n$ over the same base manifold $M$, then
$$c(\omega \oplus \eta) = c(\omega) \smile c(\eta).$$

- **Axiom 4 :** If $\omega = (E, \pi, M)$ is a complex line bundle, then $c(\omega) = 1 + e(\omega_{\mathbb{R}})$.

The naturality condition ensures, for instance, that the Chern class of a trivial bundle is trivial itself, that is, equal to the unit $1 \in H^*(M; \mathbb{Z})$.

Similarly to the total Stiefel-Whitney class, we define :

**Definition 1.68** The **total Chern class** of a complex rank $n$ bundle $\omega = (E, \pi, M)$ is the element

$$c(\omega) = 1 + c_1(\omega) + \ldots + c_n(\omega)$$

of the cohomology ring $H^*(M; \mathbb{Z})$.

It is rather easy to describe the effects of conjugation on the total Chern class of a complex bundle :

**Proposition 1.69** The Chern class $c_j(\bar{\omega})$ is equal to $(-1)^j c_j(\omega)$. It follows that if $\omega$ is of rank $n$, its total Chern class is given by :

$$c(\bar{\omega}) = 1 + \sum_{j=1}^{n} (-1)^j c_j(\omega).$$

**Proof :** See [MS74], Lemma 14.9 on page 173. Note that this version of proof relies on the alternative definition of Chern classes which we only introduce in Remark 1.72 below. □

**Remark 1.70** Recall, as mentioned in Remark 1.64, that the first Chern class induces an isomorphism

$$c_1 : (L(M), \otimes) \to (H^2(M; \mathbb{Z}), +).$$

Note however that we have not provided any proof of the relation of this isomorphism with the Chern classes as defined in this section (see [BLM89], Appendix A, for more details on this question). In particular, it follows that the first Chern class behaves in a pleasant way with respect to the tensor product of line bundles. Indeed, if $L_1$ and $L_2$ are complex line bundles over $M$, then

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2). \tag{1.1}$$

This fact is particularly useful when proving the multiplicative properties of the Chern character (see Section 1.5.1).

A useful example is the following computation :

**Theorem 1.71** The total Chern class of the tangent bundle $\tau^n$ of the complex projective space $P^n(\mathbb{C})$ is given by

$$c(\tau^n) = (1+a)^{n+1},$$

where $a$ is a suitably chosen generator of $H^2(P^n(\mathbb{C}); \mathbb{Z})$.

**Proof :** See [MS74], Theorem 14.10 on page 175 for the detailed proof. In fact, it turns out that $a = -c_1(E_n^1)$. Using Proposition 1.47 and the fact that the Euler characteristic of $P^n(\mathbb{C})$ is equal to $n+1$, we also deduce that $a$ is the generator compatible with the canonical orientation, that is, $\langle a^k, [P^k(\mathbb{C})]\rangle = 1$. □

Note that the formula above allows for an easy explicit computation of each Chern class $c_j(\tau^n)$.

**Remark 1.72** As mentioned at the beginning of this section, there is another, more explicit way of building the Chern classes relying on an Euler class (see Section 1.1.7). Recall that the existence of such a class depends on the orientability of the bundle and that, in the case of the underlying real bundle of a complex bundle, this condition is always canonically satisfied (see Theorem 1.21).
Let $\omega = (E, \pi, M)$ be a complex rank $n$ bundle and let $e(\omega_{\mathbb{R}}) \in H^{2n}(M; \mathbb{Z})$ be the Euler class of the underlying real bundle. Once again, we denote by $E_0$ the collection of all the non-zero vector in $E$ and aim to construct an $(n-1)$-bundle $\omega_0$ over $E_0$. Note that each element $v$ of $E_0$ is uniquely associated to some fiber $F$ over a point $x \in M$. We may therefore define the fiber of $\omega_0$ over $v$ to be the quotient complex vector space of $F$ by the (complex) line generated by $v$ in $F$.
If $\pi_0 : E_0 \to M$ is the restriction of the projection map to $E_0$, then the Gysin sequence (see Theorem 1.46) applied to the real rank $2n$ bundle $E_{\mathbb{R}}$ ensures that the pullback

$$\pi_0^* : H^i(M; \mathbb{Z}) \to H^i(E_0; \mathbb{Z})$$

is an isomorphism whenever $i < 2n-1$. We may now recover the Chern classes of $\omega$ inductively as follows :

$$c_j(\omega) = \begin{cases} (\pi_0^*)^{-1}(c_j(\omega_0)) & \text{if } j < n, \\ e(\omega_{\mathbb{R}}) & \text{if } j = n, \\ 0 & \text{if } j > n. \end{cases}$$

The construction in the remark above perfectly mirrors what we will do later when explaining the so-called splitting principle. For now, it is enough to note that if the complex vector bundle $\omega$ was endowed with a **hermitian metric** (that is, a bundle metric in the sense of Section 1.2.5, such that $|iv| = |v|$ for the induced norm on the total space $E$), then we could write

$$\pi_0^*(E) = L \oplus L^{\perp},$$

where $L$ is the obvious complex line bundle over $E_0$ and $L^{\perp}$ its complement with respect to the metric which, in fact, corresponds to $\omega_0$.

### 1.4.4 Pontrjagin classes

As we have seen, the Chern classes are only defined on complex bundles but if we are given a real $n$-plane bundle $\xi = (E, \pi, M)$, we may simply take its complexification $\xi \otimes \mathbb{C}$ and then consider the Chern classes of this new bundle. Note that we have the following :

**Lemma 1.73** The complexification $\xi \otimes \mathbb{C}$ is isomorphic to its conjugate $\overline{\xi \otimes \mathbb{C}}$.

**Proof :** Simply consider the homeomorphism $\hat{h} : E(\xi \otimes \mathbb{C}) \to E(\overline{\xi \otimes \mathbb{C}})$ given by $x + iy \mapsto x - iy$. □

In particular, the properties of the total Chern class yield $c(\xi \otimes \mathbb{C}) = c(\overline{\xi \otimes \mathbb{C}})$ and thus, using Proposition 1.69, we get :

$$1+c_1(\xi\otimes\mathbb{C})+c_2(\xi\otimes\mathbb{C})+\ldots+c_n(\xi\otimes\mathbb{C}) = 1-c_1(\xi\otimes\mathbb{C})+c_2(\xi\otimes\mathbb{C})-\ldots+(-1)^n c_n(\xi\otimes\mathbb{C})$$

and the odd Chern classes $c_{2j+1}(\xi \otimes \mathbb{C})$ are therefore elements of order 2 in their respective cohomology group . Ignoring these classes, we define :

**Definition 1.74** The $i$-th **Pontrjagin class** $p_i(\xi) \in H^{4i}(M; \mathbb{Z})$ is given by

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C})$$

and the **total Pontrjagin class** is

$$p(\xi) = 1 + p_1(\xi) + \ldots + p_{\lfloor \frac{n}{2} \rfloor}(\xi),$$

which is a unit of the cohomology ring $H^*(M; \mathbb{Z})$.

Note that the sign appearing in the definition of the Pontrjagin classes is only added to simplify some later formulas.

**Lemma 1.75** Let $\omega$ be a complex vector bundle. Then the complexification of its underlying real vector bundle $\omega_{\mathbb{R}} \otimes \mathbb{C}$ is canonically isomorphic to the Whitney sum $\omega \oplus \bar{\omega}$.

**Proof :** As already mentioned, the complexification $F_{\mathbb{R}} \otimes \mathbb{C}$ of the fibers of the underlying real bundle $\omega_{\mathbb{R}}$ (where $F$ is therefore a fiber of the complex vector bundle $\omega$) is isomorphic to $F_{\mathbb{R}} \oplus iF_{\mathbb{R}}$ and can therefore be identified with $F_{\mathbb{R}} \oplus F_{\mathbb{R}}$ endowed with the usual complex structure $J(x,y) = (-y,x)$. Using the map $F \to F_{\mathbb{R}} \oplus F_{\mathbb{R}}$ given by

$$v \mapsto (v, -iv)$$

which can be shown to be both complex and conjugate linear, we obtain a complex vector space isomorphism between $F_{\mathbb{R}} \otimes \mathbb{C}$ and $F \oplus \bar{F}$. This result is then extended to the full bundle. See [MS74], Lemma 15.4 on page 182 for the precise argument. □

Using the Lemmas 1.73 and 1.75 as well as the Whitney sum formula we get :

$$1 + \sum_{j=1}^{n} (-1)^j p_j(\omega_{\mathbb{R}}) = \left(1 + \sum_{j=1}^{n} c_j(\omega)\right)\left(1 + \sum_{j=1}^{n} (-1)^j c_j(\omega)\right). \tag{1.2}$$

In particular, if $\omega$ is a rank $2n$ complex bundle, the Chern classes $c_1(\omega), \dots, c_{2n}(\omega)$ determine the Pontrjagin classes $p_1(\omega_{\mathbb{R}}), \dots, p_n(\omega_{\mathbb{R}})$ as follows :

$$(-1)^\ell p_\ell(\omega_{\mathbb{R}}) = 2c_{2\ell}(\omega) + \sum_{j=1}^{\ell-1} (-1)^j c_{\ell-j}(\omega) c_j(\omega).$$

An important consequence is the following computation of the Pontrjagin classes of the complex projective spaces :

**Example 1.76** Using Equation (1.2) and the known formula for the total Chern class of $P^n(\mathbb{C})$ (see Theorem 1.71), we obtain

$$1 + \sum_{j=1}^{n} (-1)^j p_j(\tau^n_{\mathbb{R}}) = (1+a)^{n+1}(1-a)^{n+1} = \left(1 - a^2\right)^{n+1}$$

and therefore

$$p\left(\tau^n_{\mathbb{R}}\right) = \left(1 + a^2\right)^{n+1},$$

where we have once again denoted $TP^n(\mathbb{C})$ by $\tau^n$ and $a = -c_1(E^1_n) \in H^2(P^n(\mathbb{C}); \mathbb{Z})$.

Let $n \geq 0$ be an integer. Recall that we call any unordered sequence $I = \{i_1, \dots, i_r\}$ a **partition** of $n$ if $i_1 + \ldots + i_r = n$. There is a natural **concatenation** operation which makes $IJ$ a partition of $n+m$ if $I$ and $J$ are partitions of $n$ and $m$ respectively. From there, we define a **refinement** of a partition $I = \{i_1, \dots, i_r\}$ of $n$ to any partition which can be written as a concatenation $I_1 \cdots I_r$, where $I_j$ is a partition of $i_j$.
Suppose now that $M$ is a closed and oriented manifold of dimension $4n$. Let $I = \{i_1, \dots, i_r\}$ be a partition of $n$.

**Definition 1.77** The **$I$-th Pontrjagin number of** $M$ is the real value $p_I(M)$ given by

$$p_I(M) = \langle p_{i_1}(M) \cdots p_{i_r}(M), [M] \rangle,$$

where $[M]$ is the fundamental class of $M$ and $p_{i_j}(M)$ the $i_j$-th Pontrjagin class of its tangent bundle.

These numbers are powerful topological invariants of the manifold $M$. For example, if any one of them is non-zero, a result of Thom asserts that $M$ cannot be the boundary of a compact and oriented $(4n+1)$-dimensional manifold with boundary. In fact, the Pontrjagin numbers are even an oriented cobordism invariant. We will mention another use for these Pontrjagin numbers when discussing genera (see Section 3.2).

**Remark 1.78** The same idea can of course be adapted to Chern classes in order to define so-called **Chern numbers**.

# 1.5 Chern character and basic $K$-theory

## 1.5.1 The Chern character

Suppose that we are given a complex $n$-plane bundle $E$ over $M$ as a direct sum of $n$ (complex) line bundles $L_i$ :

$$E = L_1 \oplus \ldots \oplus L_n.$$

Let $x_i \in H^2(M;\mathbb{Z})$ denote the first (and therefore top) Chern class of $L_i$. It follows by the multiplicativity of the total Chern class that

$$c(E) = c(L_1) \cdot \ldots \cdot c(L_n) = (1+x_1) \cdot \ldots \cdot (1+x_n).$$

However, by definition, the following also has to hold :

$$1 + c_1(E) + \ldots + c_n(E) = (1+x_1) \cdot \ldots \cdot (1+x_n).$$

The Chern classes of $E$ can therefore be expressed as polynomial functions of the Chern classes $x_i$. Moreover, since the order of the line bundles in the direct sum is irrelevant to the isomorphism class of $E$, these functions also have to be symmetric. We therefore find polynomials $\sigma_j^n$ that are symmetric and homogeneous of degree $j$ such that :

$$c_j(E) = \sigma_j^n(x_1, \ldots, x_n).$$

Note that, since the $x_i$ are elements of $H^2(M;\mathbb{Z})$, $c_j(E)$ does indeed belong to $H^{2j}(M;\mathbb{Z})$.

**Definition 1.79** These functions $\sigma_j^n$ are called the **elementary symmetric polynomials**.

For example, by expanding the product $(1+x_1) \cdot \ldots \cdot (1+x_n)$, we obtain :

$$\sigma_1^n(x_1, \ldots, x_n) = x_1 + \ldots + x_n$$

$$\vdots$$

$$\sigma_n^n(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n.$$

In particular, every expression that is symmetric in the classes $x_i$ can be expressed as a function of the Chern classes of $E$.

Suppose now that $E$ is of the form

$$E = L_1 \oplus \ldots \oplus L_{2n}$$

and consider $E_{\mathbb{R}}$, the underlying real (rank $4n$) bundle of $E$. If we construct its complexification $E_{\mathbb{R}} \otimes \mathbb{C}$, which is isomorphic to $E \oplus \bar{E}$ (see Lemma 1.75), we may compute the Pontrjagin classes $p_1(E_{\mathbb{R}}), \ldots, p_n(E_{\mathbb{R}})$. Through the same argument that we used at the end of the previous section, we obtain

$$\begin{aligned} 1 - p_1(E_{\mathbb{R}}) + \ldots \pm p_n(E_{\mathbb{R}}) &= (1 + x_1) \cdot \ldots \cdot (1 + x_n) \cdot (1 - x_1) \cdot \ldots \cdot (1 - x_n) \\ &= \left(1 - x_1^2\right) \cdot \ldots \cdot \left(1 - x_n^2\right) \end{aligned}$$

where $x_i \in H^2(M; \mathbb{Z})$ is, again, the first Chern class of the complex line bundle $L_i$. We are then finally able to identify

$$p_j(E_{\mathbb{R}}) = \sigma_j^n \left(x_1^2, \ldots, x_n^2\right).$$

It follows, as above, that any expression that is symmetric in the $x_i^2$ can be written using the Pontrjagin classes.

Let us come back to a complex rank $n$ vector bundle $E$ over $M$, given as a direct sum of line bundles $L_1 \oplus \ldots \oplus L_n$ and let $x_i = c_1(L_i) \in H^2(M; \mathbb{Z})$.

**Definition 1.80** The **Chern character** $\mathrm{ch}(E)$ is the element of the rational cohomology ring of $M$ given by

$$\mathrm{ch}(E) = \mathrm{ch}(L_1 \oplus \ldots \oplus L_n) = \sum_{i=1}^{n} e^{x_i}.$$

The Chern character can easily be rewritten as follows

$$\begin{aligned} \mathrm{ch}(E) &= \sum_{i=1}^{n} \left(1 + x_i + \frac{x_i^2}{2} + \frac{x_i^3}{6} + \ldots\right) \\ &= n + \sum_{i=1}^{n} x_i + \frac{1}{2} \sum_{i=1}^{n} x_i^2 + \frac{1}{6} \sum_{i=1}^{n} x_i^3 + \ldots \end{aligned}$$

Note that each term in the last expression is obviously symmetric in the $x_i$ and can therefore be expressed using the Chern classes :

$$\mathrm{ch}(E) = n + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2} + \frac{c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)}{6} + \ldots$$

**Remark 1.81** Since all its Chern classes vanish, the Chern character of a trivial bundle is equal to the unit of $H^*(M;\mathbb{Q})$ multiplied by its rank.

It is clear that if both $E$ and $F$ are direct sums of line bundles over the same manifold $M$, then

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F).$$

One can moreover show that

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F).$$

Indeed, if we write $E = L_1 \oplus \ldots \oplus L_n$ and $F = L'_1 \oplus \ldots \oplus L'_m$, we get

$$E \otimes F = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (L_i \otimes L'_j)$$

and therefore, using (1.1), this yields

$$\mathrm{ch}(E \otimes F) = \sum_{i=1}^{n} \sum_{j=1}^{m} e^{c_1(L_i \otimes L'_j)} = \sum_{i=1}^{n} \sum_{j=1}^{m} e^{c_1(L_i) + c_j(L'_j)} = \mathrm{ch}(E) \cdot \mathrm{ch}(F).$$

Note that, up to this point, we have limited ourselves to the very convenient context of direct sums of (complex) line bundles, which may appear to be too restrictive to yield any interesting general result. The following section will, however, show why this is not the case.

### 1.5.2 Splitting principle

It is indeed not clear at all how one could even extend the Chern character defined above to all vector bundles, since there is already no apparent reason for their total Chern classes to split nicely as they do in the case of direct sums of line bundles. This problem is however taken care of through the so-called **splitting principle** :

**Theorem 1.82** Let $\pi : E \to M$ be a complex rank $n$ vector bundle. Then there exists a smooth manifold $F_E$ and a smooth map

$$p : F_E \to M$$

such that

1. the induced homomorphism $p^* : H^*(M; R) \to H^*(F_E; R)$ is injective for every coefficient ring $R$ and
2. the pullback bundle $p^*E$ splits as $p^*E = L_1 \oplus \ldots \oplus L_n$, where each $L_i$ is a complex line bundle over $F_E$.

**Proof :** In fact, we may simply consider the projectivization $p : PE \to M$ of $E$, whose fibers over $x \in M$ is the complex projective space associated to $E_x \cong \mathbb{C}^n$. The pullback $p^*E$ contains an obvious line bundle $L$ (assigning itself to any complex line in $PE$). Endowing $E$ with an hermitian metric, we are able, we may write

$$p^*E = L \oplus L^\perp,$$

for some sub-bundle $L^\perp$ of complex rank $n-1$. We then iterate this procedure to complete the proof. For more details, including the injectivity of the induced homomorphism $p^*$, we refer to [BLM89] (see Proposition 11.1 on page 225). □

Observe here the similarities with the construction of Remark 1.72.

The naturality of the Chern classes implies that $c(p^*E) = p^*(c(E))$. However, we know that if we denote $c_1(L_i) \in H^2(F_E;\mathbb{Z})$ by $x_i$, then

$$c(p^*E) = (1+x_1)\cdot \ldots \cdot (1+x_n).$$

It follows immediately that

$$p^*(c_i(E)) = c_i(p^*E) = \sigma_i(x_1, \ldots, x_n).$$

In fact, by the injectivity of $p^*$, every expression involving the Chern classes $c_i(E)$ can be formally written as a (symmetric, polynomial) function in the Chern roots $x_i$ of the pullback $p^*E$. By abuse of both terminology and notation, we often simply write

$$c(E) = (1+x_1)\cdot \ldots \cdot (1+x_n)$$

and call the classes $x_i$ the **formal Chern roots** of the complex bundle $E$.

There is a similar result for some particular types of real vector bundles :

**Theorem 1.83** Let $\pi : E \to M$ be an oriented real vector bundle of even rank $2n$. Then there is a smooth manifold $F_E$ and a smooth map

$$p : F_E \to M$$

such that

1. the induced homomorphism $p^* : H^*(M;R) \to H^*(F_E;R)$ is injective for every coefficient ring $R$ and
2. the pullback bundle $p^*(E\otimes\mathbb{C})$ splits as $p^*E = L_1 \oplus \bar{L}_1 \oplus \ldots \oplus L_n \oplus \bar{L}_n$, where each $L_i$ is a complex line bundle over $F_E$. In fact, there is even a splitting

$$p^*E = E_1 \oplus \ldots \oplus E_n,$$

where each $E_i$ is a real and oriented bundle of rank 2 and, for every $1 \le i \le n$, $E_i \otimes \mathbb{C} = L_i \oplus \bar{L}_i$.

**Proof :** See [BLM89], Proposition 11.2 on page 226. □

The idea of formal Chern roots also translates to this setting. Indeed, let $\pi : E \to M$ be a bundle such as in Theorem 1.83. Then, applying Theorem 1.82 to its complexification $E \otimes \mathbb{C}$, which is a complex bundle of rank $2n$, we obtain a space $F_{E\otimes\mathbb{C}}$ and a map $p : F_{E\otimes\mathbb{C}} \to M$ such that

$$p^*(E \otimes \mathbb{C}) = L_1 \oplus \ldots \oplus L_{2n}.$$

By Lemma 1.73, however, we know that $E \otimes \mathbb{C}$ is isomorphic to its conjugate $\overline{E \otimes \mathbb{C}}$ and it follows that the total Chern class of $E \otimes \mathbb{C}$ can be formally written as

$$c(E \otimes \mathbb{C}) = (1 + x_1) \cdot \ldots \cdot (1 + x_n) \cdot (1 - x_1) \cdot \ldots \cdot (1 - x_n).$$

The classes $x_1, \ldots, x_n$ are then usually called the **formal Chern roots of the bundle** $E$. It is then easy to check that

$$\sigma_i(x_1^2, \ldots, x_n^2) = (-1)^i \sigma_{2i}(x_1, \ldots, x_n, -x_1, \ldots, -x_n)$$

and therefore that the total Pontrjagin class of $E$ can be expressed in the formal roots

$$p(E) = (1 + x_1^2) \cdot \ldots \cdot (1 + x_n^2).$$

The splitting principle therefore allows us to assume, for the purpose of computing the Chern character, that every complex vector bundle decomposes formally as a direct sum of complex line bundles. In this sense, the Chern character extends naturally to these types of bundles and, in doing so, retains the additive and multiplicative properties mentioned in the previous section.

**Remark 1.84** Recall that in Remark 1.72, we mentioned that, for a complex $n$-plane bundle $\omega$, the Euler class $e(\omega_{\mathbb{R}})$ was equal to the top Chern class $c_n(\omega)$. However, using the Chern roots $x_1, \ldots, x_n$ to split the total Chern class, we observe that

$$c_n(\omega) = \prod_{i=1}^{n} x_i.$$

It follows immediately that the Euler class can be computed as the product of these (formal) Chern roots.

### 1.5.3 Basic notions in K-theory

This section is only intended as a basic overview of the subject, in which we only mention some interesting results and introduce the fundamental definitions for use in subsequent chapters, where $K$-theory plays a more technical and behind the scenes role. In particular, these notions will prove crucial in defining the analytical and topological index of differential operators.

Consider a compact manifold $M$, and let $V(M)$ denote the set of all the isomorphism classes of complex vector bundles over $M$. Note first that, when endowed with the operation induced by the direct sum, $V(M)$ forms an abelian semigroup. We then define $F(M)$ to be the free abelian group generated by $V(M)$, and $E(M)$ to be the subgroup given by the elements of the form $[X] + [Y] - ([X] \oplus [Y])$, where $+$ and $\oplus$ respectively denote the addition law in $F(M)$ and $V(M)$.

**Definition 1.85** The ***K*-group** of the manifold $M$ is defined by the following quotient :

$$K(M) = F(M)/E(M).$$

It is clearly an abelian group (whose law is denoted by $+$) and its elements are known as **virtual bundles** over $M$.

It is however possible to endow $K(M)$ with more structure. Indeed, if $X$ and $Y$ are complex vector bundles over $M$, we may consider their tensor product $X \otimes Y$, which is now a bundle over $M \times M$. Using the diagonal map $\Delta : M \to M \times M$, we pullback to define

$$[X] \cdot [Y] = \Delta^*([X \otimes Y]).$$

Then $(K(M), +, \cdot)$ forms a ring.

**Proposition 1.86** Every virtual bundle over $M$ can be represented as a formal difference $[X] - [Y]$, where $[X], [Y] \in V(M)$.

**Proof :** See [BLM89], Corollary 9.7 on page 60. □

Note that analogous results hold in the setting of real vector bundles over $M$. The **real *K*-ring** is then denoted by $KO(M)$.

Suppose that a $M$ comes with a distinguished point $* \in M$ and consider the natural inclusion $i : \{*\} \to M$. We may then use this map to pull the bundles over $M$ back to $\{*\}$, which, when extended to the $K$-rings, yields a ring homomorphism

$$i^* : K(M) \to K(\{*\}) \cong \mathbb{Z}.$$

**Definition 1.87** The **reduced *K*-ring** $\tilde{K}(M)$ of $M$ is defined as the kernel of the induced map $i^*$.

It is therefore obvious that $\tilde{K}(M)$ is an ideal of $K(M)$. Let now $N \subset M$ be non-empty and closed.

**Definition 1.88** The **relative $K$-group** is defined as

$$K(M,N) = \tilde{K}(M/N),$$

where the distinguished point of $M/N$ is obvious.

If the base manifold $M$ does not have a base point, we simply consider the disjoint union

$$M^+ = M \sqcup \{*\}$$

and identify

$$K(M) = \tilde{K}(M^+) = K(M,\emptyset).$$

Recall that for two base pointed manifold $(M,m)$ and $(N,n)$ we may define the **smash-product** $M \wedge N$ and the **wedge** $M \vee N$, as well as the **suspension**

$$\Sigma M = S^1 \wedge M.$$

By a simple iteration, we construct the **$i$-fold suspension** $\Sigma^i M$, and notice that it is homeomorphic to $S^i \wedge M$.

**Definition 1.89** For a compact base pointed manifold $M$ or a compact pair $(M,N)$, we define

$$\tilde{K}^{-i}(M) = \tilde{K}(\Sigma^i M)$$

and

$$K^{-i}(M,N) = \tilde{K}^{-i}(M/N) = \tilde{K}(\Sigma^i(M/N)).$$

If $M$ is only compact, we set

$$K^{-i}(M) = K^{-i}(M,\emptyset) = \tilde{K}(\Sigma^i M^+).$$

We denote the graded functor $(\tilde{K}^{-i})_{i\geq 0}$ by $\tilde{K}^{-*}$. Note that $K$ itself was a functor, although we did not mention it at the time.

**Proposition 1.90** Let $\{p\}$ be the one point space and $(M,x)$ be a base pointed manifold. Then $K^{-*}(\{p\})$ is a graded ring and $K^{-*}(M)$ is a graded module over $K^{-*}(\{p\})$.

**Proof :** This is a direct consequence of the existence of a pairing

$$\tilde{K}^{-i}(M) \otimes \tilde{K}^{-j}(N) \to K^{-(i+j)}(M \wedge N),$$

which is constructed in [BLM89], Proposition 9.17 on page 62. Here, both $M$ and $N$ are assumed to be base-pointed. □

One of the most significant (and surprising) result in this theory are the so-called **Bott periodicity** theorems. Here, we simply wish to present them without giving any proof. The interested reader is referred to [BLM89], Theorem 10.12 on page 222.

**Proposition 1.91** The ring $\tilde{K}^{-*}(\{p\})$ is polynomial algebra generated by an element $\xi \in \tilde{K}^{-2}(\{p\})$. In other words, there is a ring isomorphism $\tilde{K}^{-*}(\{p\}) \cong \mathbb{Z}[\xi]$.

This element $\xi$ can be identified as a particular virtual bundle over $S^2$, but we will not seek to do so here.

Results of a similar effect also hold in the real category :

**Proposition 1.92** The ring $KO^{-*}(\{p\})$ is generated by elements $\eta \in KO^{-1}(\{p\})$, $y \in KO^{-4}(\{p\})$ and $x \in KO^{-8}(\{p\})$ in such a way that there is a ring isomorphism

$$KO^{-*}(\{p\}) \cong \mathbb{Z}[\eta, y, x]/\langle 2\eta, \eta^3, \eta y, y^2 - 4x\rangle.$$

Note that $KO^{-*}(M)$ is a module over $KO^{-*}(\{p\})$ as well, in the same sense as in Proposition 1.90. In fact, we even have the following

**Proposition 1.93** The map $\mu_x : KO^{-i}(M) \to KO^{-(i+8)}(M)$ given by the module multiplication by $x$ is an isomorphism for every $i \geq 0$.

Let us now present a completely different point of view which, as we will see, turns out to be equivalent, but whose language makes it easier to discuss differential operators and their symbols in subsequent chapters. Once again, we will not be able to engage here in much details beyond the statement of basic definitions and results. The interested reader is referred to the corresponding sections of [BLM89].

We once again consider vector bundles, without making any distinction as to their nature (complex or real). Let $M$ be a compact manifold and $N \subset M$, a closed subspace. Let $n \geq 1$ be an integer.

**Definition 1.94** The set $\mathfrak{L}_n(M, N)$ is formed by all the elements

$$(V_0, V_1, \ldots, V_n; \sigma_1, \ldots, \sigma_n),$$

where $V_0, V_1, \ldots, V_n$ are vector bundles over $M$ and

$$0 \longrightarrow V_0|_N \xrightarrow{\sigma_1} V_1|_N \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} V_n|_N \longrightarrow 0$$

is an exact sequence of bundle morphisms.

Two such objects $(V_0, \ldots, V_n; \sigma_1, \ldots, \sigma_n)$ and $(W_0, \ldots, W_n; \mu_1, \ldots, \mu_n)$ are called **isomorphic**, if there are bundle morphisms $\varphi_i : V_i \to W_i$ such that the following ladderdiagram

$$\begin{array}{ccccccccccc}
V_0|_N & \xrightarrow{\sigma_1} & \cdots & \xrightarrow{\sigma_{i-1}} & V_{i-1}|_N & \xrightarrow{\sigma_i} & V_i|_N & \xrightarrow{\sigma_{i+1}} & \cdots & \xrightarrow{\sigma_n} & V_n|_N \\
\downarrow{\scriptstyle \varphi_0} & & & & \downarrow{\scriptstyle \varphi_{i-1}} & & \downarrow{\scriptstyle \varphi_i} & & & & \downarrow{\scriptstyle \varphi_n} \\
W_0|_N & \xrightarrow[\mu_1]{} & \cdots & \xrightarrow[\mu_{i-1}]{} & W_{i-1}|_N & \xrightarrow[\mu_i]{} & W_i|_N & \xrightarrow[\mu_{i+1}]{} & \cdots & \xrightarrow[\mu_n]{} & W_n|_N
\end{array}$$

commutes.

**Definition 1.95** An element $V = (V_0, \ldots, V_n; \sigma_1, \ldots, \sigma_n) \in \mathfrak{L}_n(M, N)$ is called **elementary** if there is an index $i$ such that $V_{i-1} = V_i$, $\sigma_i = \mathrm{id}_{V_{i-1}}$ and $V_j = 0$ for all $j \notin \{i-1, i\}$.

A natural direct sum can be defined on $\mathfrak{L}_n(M, N)$, which we simply denote by $\oplus$. Using this operation, we may compare elements in $\mathfrak{L}_n(M, N)$ as follows :

**Definition 1.96** Two elements $V$ and $W$ in $\mathfrak{L}_n(M, N)$ are said to be **equivalent** if there are elementary elements $E_1, \ldots, E_k$ and $F_1, \ldots, F_\ell$ such that

$$V \oplus E_1 \oplus \ldots \oplus E_k \cong W \oplus F_1 \oplus \ldots \oplus F_\ell.$$

This actually defines an equivalence relation on $\mathfrak{L}_n(M, N)$. The equivalence class of $V = (V_0, \ldots, V_n; \sigma_1, \ldots, \sigma_n)$ is then denoted by $[V_0, \ldots, V_n; \sigma_1, \ldots, \sigma_n]$, and the set of these classes by $L_n(M, N)$.

**Remark 1.97** The quotient $(L_n(M, N), \oplus)$ forms an abelian group.

It is clear that, given an element $V = (V_0, \ldots, V_n; \sigma_1, \ldots, \sigma_n)$ of $\mathfrak{L}_n(M, N)$, we may construct an element of $\mathfrak{L}_{n+1}(M, N)$ simply by adding a trivial bundle and map. This simple procedure yields an inclusion $\mathfrak{L}_n(M, N) \to \mathfrak{L}_{n+1}(M, N)$, given formally by

$$(V_0, \ldots, V_n; \sigma_1, \ldots, \sigma_n) \mapsto (V_0, \ldots, V_n, 0; \sigma_1, \ldots, \sigma_n, 0).$$

We now observe the consequence on the quotient :

**Proposition 1.98** For every $n \geq 1$, the map $L_n(M, N) \to L_{n+1}(M, N)$ induced by the inclusion above is an isomorphism.

Since $L_n(M, N) \cong L_{n+1}(M, N)$, we will simply denote these groups by $L(M, N)$ and note that it is sufficient to consider the basic case $n = 1$. Note that $L$ behaves as a functor, just like $K$. It remains now to see how these groups are related (and in fact isomorphic) to the earlier $K$-rings. For this, we have the next result :

**Proposition 1.99** There is an equivalence of functors $\chi : L(M, N) \to K(M, N)$, uniquely characterized by the property

$$\chi([V_0, \dots, V_n; \sigma_1, \dots, \sigma_n]) = \sum_{j=1}^{n} (-1)^j [V_j].$$

**Proof :** The proof relies on the construction of $\chi_1 : L_1(M, N) \to K(M, N)$. Details can be found in [BLM89], Proposition 9.25 on page 65. □

In light of this equivalence, we are now for example allowed to consider elements $[V_0, V_1; \sigma]$ as being in $K(M, N)$. This notation will prove particularly useful in Chapter 4, for the definition of the analytical and topological index.

There is still a relatively minor technical issue to be addressed. We have, up until now, only defined the $K$-rings of compact manifolds, but in the future we will require the ability to discuss the $K$-theory of vector bundles which, even when defined over compact base manifolds, are obviously not compact themselves.
Suppose then that $M$ is a non-compact manifold and let $M^+$ denote its usual one-point compactification. We then define

$$K_{\text{cpt}}(M) = \tilde{K}(M^+).$$

In fact, when it is clear that $M$ is not compact (see Remark 1.100 below for example), we will simply write $K(M)$ while referring to the **$K$-theory with compact support** of $M$.

**Remark 1.100** Let $E$ be a vector bundle over some manifold $M$ and suppose that it is endowed with a bundle metric $\langle \cdot, \cdot \rangle$. We may then construct the so-called **disk** and **sphere bundles** $DE$ and $SE$, given respectively by

$$DE = \{v \in E \mid ||v|| \leq 1\} \quad \text{and} \quad SE = \{v \in E \mid ||v|| = 1\},$$

where $||v|| = \sqrt{\langle v, v \rangle}$. It is easy to check that these are indeed still fiber bundles over $M$, using the adapted restriction of the original projection map. Note now that the quotient $DE/SE$ is naturally homeomorphic to the compactification $E^+$. It follows immediately (from the here unproved fact that $K$ is a functor) that

$$K_{\text{cpt}}(E) = \tilde{K}(E^+) = \tilde{K}(DE/SE) = K(DE, SE).$$

The quotient space $DE/SE$ is usually known as the **Thom space** of the bundle $E$ and often denoted by $\text{Th}(E)$. This construction is, for example, especially useful for the technical aspects such as the Atiyah-Bott-Shapiro isomorphisms (see [BLM89], Theorem 9.27 on page 69).

Let us finally mention that there is an analogue of the **Thom isomorphism** (see

Section 1.1.7) in the context of $K$-theory, of the form

$$i_! : K(M) \to K(E),$$

where $E \to M$ is a complex vector bundle. Here, of course, $K(E) = K_{\text{cpt}}(E)$ and similarly for $M$ if the base manifold is not compact either. See for example [Ama], or [BLM89], Theorem C.8 on page 387 for a more general statement and proof.

**Remark 1.101** Note that the Chern character defined Section 1.5.1 can readily be seen as a homomorphism from the $K$-ring of the total space of some vector bundle $E$ over $M$ to $H^*(M;\mathbb{Q})$.

# 2. Curvature

## 2.1 Basic notions

### 2.1.1 Riemannian metric

Let $M$ be a connected smooth (and compact) manifold and $TM$ be its tangent bundle. In this particular setting, a bundle metric $\langle \cdot, \cdot \rangle \in \Gamma(T^*M \otimes T^*M)$, also often written $g$, is known as a **Riemannian metric**. According to our discussion above (see Section 1.2.5), it smoothly assigns a positive-definite bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ to each point $p \in M$. This allows for the computation of the length of vectors in the tangent spaces $T_pM$, and therefore also of the length of smooth curves in $M$. Indeed, if $\mu : [a, b] \to M$ is such a curve, then we may define its length by

$$L(\mu) = \int_a^b \sqrt{g_{\mu(t)}(\mu'(t), \mu'(t))}\, dt.$$

It is then possible to define an actual **metric** $d$ on $M$ by declaring for example that the distance $d(p, q)$ between two points $p, q \in M$ is the infinimum among the lengths of all the smooth curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Note that, somewhat surprisingly, the metric topology that $M$ inherits in this way coincides with its original manifold topology.

**Definition 2.1** A **Riemannian manifold** $(M, g)$ is a smooth manifold $M$ endowed with a Riemannian metric $g$.

Note that a given smooth manifold $M$ may admit many different Riemannian metrics and, as we will see later in this chapter, the **curvature** is a feature of the particular metric with which the manifold (or rather, its tangent bundle) is endowed.

### 2.1.2 Lie bracket

Let $M$ be the same smooth and compact manifold and consider the set $\Gamma(TM)$ of all vector fields on $M$. It obviously forms a real (although typically infinite dimensional) vector space, but also admits some further structure through the so-called **Lie bracket** $[\cdot, \cdot]$, which we will review here.

**Theorem 2.2** For every such manifold $M$, there exists a unique bilinear and anti-symmetric map

$$[\cdot, \cdot] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

satisfying the Jacobi idendity

$$[X,[Y,W]]+[W,[X,Y]]+[Y,[W,X]]=0$$

for all $X, Y, W \in \Gamma(TM)$.

There are multiple ways to (more or less) explicitly define this operator, both intrinsically and in local coordinates, which accounts for the proof of its existence while uniqueness is usually shown by contradiction.
Here, thanks to the compactness of $M$, we may use the **global flows** $\Phi^X$ and $\Phi^Y$ induced by two given vector fields $X, Y \in \Gamma(TM)$ to express their Lie bracket $[X,Y] \in \Gamma(TM)$. These are smooth maps of the form $\mathbb{R}\times M \to M$, however it is common to write $\Phi^X_p(\cdot) = \Phi^X(\cdot, p)$ for $p \in M$ or $\Phi^X_t(\cdot) = \Phi^X(t,\cdot)$ for $t \in \mathbb{R}$, and thus to consider the **flow line** $\Phi^X_p : \mathbb{R} \to M$ of $X$ starting at $p$ or the **time-$t$ displacement map** $\Phi^X_t : M \to M$. Of course, the same would hold for $Y$.
With these notations in mind, recall that the flow $\Phi^W$ of a vector field $W \in \Gamma(TM)$ is uniquely determined by the following condition : for all $t \in \mathbb{R}$ and $p \in M$, we have

$$W\left(\Phi^W_p(t)\right) = \dot{\Phi}^W_p(t).$$

In other words, the tangent vectors along the flow line coincide with those given by the field itself. Note finally that the displacement maps mentioned above give rise to a 1-parameter group $\mathbb{R} \to \mathrm{Diff}(M)$ through the assignment $t \mapsto \Phi^W_t$. In particular, we have that $\Phi^W_t$ is a diffeomorphism for all $t \in \mathbb{R}$ and that its inverse is given by $\Phi^W_{-t}$.
Having made these preparations, we now observe the behaviour (more specifically the variation) of the field $Y$ along the flow of $X$. Let us fix a point $p \in M$. Then for any $t \in \mathbb{R}$, we obtain a point $\Phi^X_t(p)$ on the flow line of $X$ and may look at the vector $Y\left(\Phi^X_t(p)\right)$. To be able to compare it to $Y(p)$, we have to pull it back to the tangent space $T_pM$, which we do here by using the inverse of the differential

$$d_p\Phi^X_t : T_pM \to T_{\Phi^X_t(p)}M.$$

We know, however, that the inverse of the differential of $\Phi^X_t$ is equal to the differential (evaluated at the suitable point) of the inverse map $\Phi^X_{-t}$ and we therefore consider the vector

$$d_{\Phi^X_t(p)}\Phi^X_{-t}\left(Y\left(\Phi^X_t(p)\right)\right) \in T_pM.$$

The Lie bracket of $X$ and $Y$ at $p$ can then be computed as the derivative at $t = 0$ of the function

$$t \mapsto d_{\Phi^X_t(p)}\Phi^X_{-t}\left(Y\left(\Phi^X_t(p)\right)\right),$$

that is :

$$[X,Y](p) = \lim_{t\to 0}\frac{d_{\Phi^X_t(p)}\Phi^X_{-t}\left(Y\left(\Phi^X_t(p)\right)\right) - Y(p)}{t}.$$

Direct, although somewhat lengthy, computations show that the definition above does indeed satisfy the conditions laid out in Theorem 2.2 and, as announced earlier in this section, the Lie bracket provides $\Gamma(TM)$ with more structure than that of a simple vector space :

**Property 2.3** Endowing it with the Lie bracket makes a Lie algebra out of $\Gamma(TM)$.

In fact, this definition of the Lie bracket $[X, Y]$ corresponds to what is usually also called the **Lie derivative** of $Y$ with respect to $X$, denoted by $\mathcal{L}_X Y$, which can easily be extended to more general tensor fields on $M$.

## 2.2 Connections

It is well-known that vector fields on a manifold $M$ can be seen as derivations on the germs of smooth functions $M \to \mathbb{R}$. Under this point of view, a notion of directional derivative is easily defined for such functions, without the need of any additional structure on $M$. Note however, that this is not at all the case for the sections of a vector bundle. Indeed, if $\mu \in \Gamma(E)$ is a section of $\pi : E \to M$ and $\gamma : (-\varepsilon, \varepsilon) \to M$, a smooth curve in $M$, then, even for arbitrarily small values of $0 < t < \varepsilon$, the vectors $\mu(\gamma(t))$ and $\mu(\gamma(0))$ cannot be directly compared, since they do not belong to the same fiber. To compute some kind of variation of $\mu$ along $\gamma$, one has to choose a (non-canonical) way to connect the fibers over the points of its image (recall, for example, the construction of the Lie bracket above). **Connections**, as their name suggest, are the tools used to overcome this complication.
We start by introducing the concept in the wider context of principal and general vector bundles, before coming back to the more down to earth case of tangent bundles.

### 2.2.1 Connections on principal bundles

Recall that, broadly speaking, fiber bundles behave locally like a product of the base manifold $M$ with a model fiber $F$. It is therefore natural to wonder whether this product structure carries to the tangent bundle of the total space $E$, that is, if there is a splitting of each tangent space $T_uE = H_u \oplus F_u$, where $F_u$ would correspond to the vectors tangent to the fiber at $u \in E$. Unfortunately, it turns out that there is no canonical way of devising such a decomposition.

Let $G$ be a Lie group and $\pi : P \to M$ be a principal $G$-bundle (see Section 1.3). We denote by $\mathfrak{g}$ the Lie algebra of $G$. For a point $u \in P$, we denote by $G_u$ the linear subspace of $T_uP$ formed by the vectors tangent to the fiber. These can for example be thought of as the usual equivalence classes of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to P$ such that $\gamma(0) = u$ and $\gamma(t)$ stays in the same fiber as $u$ for all $t \in (-\varepsilon, \varepsilon)$.
Observe that any tangent vector $V \in \mathfrak{g}$ induces a vector field $X_V$ on $P$, which we define with the help of the exponential map as follows

$$X_V(u) = \frac{d}{dt}(u.\exp(tV))\bigg|_{t=0.}$$

It can be shown that the assignment $V \to X_V$ yields an isomorphism between $\mathfrak{g}$ and $G_u$. Suppose now that the base manifold $M$ is of dimension $n$.

**Definition 2.4** A **connection** on $P$ is the smooth assignment of an $n$-dimensional linear subspace $H_u \subset T_uP$ for every point $u \in P$, such that

1. the tangent space of $P$ at $u$ splits as $T_uP = H_u \oplus G_u$ and
2. if $R_g$ denotes the right multiplication by an element $g \in G$, then

$$d_u R_g(H_u) = H_{u.g}.$$

The linear spaces $H_u$ and $G_u$ are respectively known as the **horizontal** and **vertical subspaces** of $T_uP$. Every tangent vector $X \in T_uP$ has a unique decomposition $X = Y + Z$ such that $Y \in H_u$ and $Z \in G_u$ ; we call $Y$ its **horizontal component** and $Z$ its **vertical component**.

The second condition ensures that the choice of the subspaces $H_u$ is, in some sense, $G$-invariant. The smoothness of $u \mapsto H_u$ is to be understood as follows : if $X$ is a vector field on $P$, then so are its horizontal and vertical components.
Note that if we are given a connection on $P$, then we have a natural linear projection onto the vertical space

$$T_uP \to G_u$$

which corresponds to the quotient map $T_uP \to T_uP/H_u$. Using the isomorphism $G_u \cong \mathfrak{g}$ described above, we obtain a map $\omega_u : T_uP \to \mathfrak{g}$. This, finally, induces a $\mathfrak{g}$-valued 1-form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$, given by $\omega(u) = \omega_u$.

**Definition 2.5** The section $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ constructed above is called the **connection 1-form**.

The main properties of the connection 1-form $\omega$ are that $\omega(X_V) = V$ for every $V \in \mathfrak{g}$ and the fact the the following diagram

$$\begin{CD} T_uP @>{d_uR_g}>> T_{u.g}P \\ @V{\omega_u}VV @VV{\omega_{u.g}}V \\ \mathfrak{g} @>>{\mathrm{Ad}_{g^{-1}}}> \mathfrak{g} \end{CD}$$

commutes for every $u \in P$ and $g \in G$. A proof of this can be found in [KN63] (see Proposition 1.1 on page 75). Note that we recover the horizontal space $H_u$ from the connection 1-form as the kernel $\ker(\omega_u)$. In fact, connections and connection 1-forms are in one-to-one correspondence.

**Example 2.6** Suppose that $E$ is a real and oriented rank $n$ vector bundle, endowed with a Riemannian metric and consider the principal bundle $P_{\mathrm{SO}(n)}E$ of its oriented orthonormal frames. Then the Lie algebra of its structure group, $\mathfrak{so}(n)$, is formed by the real skew-symmetric matrices of size $n \times n$. A connection 1-form $\omega$ on $P_{\mathrm{SO}(n)}E$ can therefore be thought of as a $n \times n$ matrix of 1-forms $\omega_{ij}$ such that $\omega_{ij} = -\omega_{ji}$.

### 2.2.2 Connections and covariant derivatives on vector bundles

Consider a rank $n$ (real or complex) vector bundle $E$ over a smooth and compact manifold $M$.

**Definition 2.7** A **covariant derivative** on $E$ is an linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E),$$

where these spaces of sections are seen as real vector spaces, such that

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$

for every smooth funcion $f : M \to \mathbb{R}$ and section $\sigma \in \Gamma(E)$.

Note that, given a vector field $X$ on $M$, a covariant derivative $\nabla$ on $E$ induces a map

$$\nabla_X : \Gamma(E) \to \Gamma(E),$$

in the obvious way. It is then easy to show that for every $X \in \Gamma(TM)$ and $\sigma \in \Gamma(E)$, the value of the section $\nabla_X\sigma$ at any point $p \in M$ only depends on $X(p)$ and on $\sigma$ over a small smooth curve passing through $p$ in the direction of $X(p)$.

Suppose that $M$ is oriented and consider now the principal oriented frame bundle of $E$. Then the connection 1-forms on $P_{\mathrm{SO}(n)}E$ and the covariant derivatives on $E$ are intimately linked :

**Proposition 2.8** Every connection 1-form $\omega$ on $P_{\mathrm{SO}(n)}E$ determines a unique covariant derivative $\nabla$ on $E$ by setting

$$\nabla\sigma_i = \sum_{j=1}^{n} (\sigma^*\omega)_{ji} \otimes \sigma_j, \tag{2.1}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ is any local family of pointwise orthonormal sections of $E$ or, in other words, a section of $P_{\mathrm{SO}(n)}E$. Note then that this covariant derivative is such that

$$X\langle\mu, \sigma\rangle = \langle\nabla_X\mu, \sigma\rangle + \langle\mu, \nabla_X\sigma\rangle \tag{2.2}$$

for every vector field $X$ on $M$ and sections $\mu, \sigma \in \Gamma(E)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product on $E$. Conversely, any covariant derivative on $E$ satisfying (2.2) determines a connection 1-form through the relation (2.1).

**Proof :** See [BLM89], Proposition 4.4 on page 103. □

Any covariant derivative $\nabla$ on $E$ which satisfies the property (2.2) will be called **Riemannian**.

Notice that $T^*M = \Lambda^1(T^*M)$, and a covariant derivative on $E$ can therefore be seen as a map

$$\nabla : \Gamma(E) \to \Gamma(\Lambda^1(T^*M) \otimes E).$$

In fact, the Definition 2.7 can be extended to higher order differential forms on $M$

$$\widetilde{\nabla}^k : \Gamma(\Lambda^k(T^*M) \otimes E) \to \Gamma(\Lambda^{k+1}(T^*M) \otimes E)$$

if we set

$$\widetilde{\nabla}^k(\alpha \otimes \sigma) = d\alpha \otimes \sigma - \alpha \wedge \nabla\sigma$$

on fundamental tensors and extend linearly.

Consider once again an oriented real vector bundle $E$ of rank $n$ over $M$ with covariant derivative $\nabla$ induced by a connection 1-form $\omega$ on $P_{\mathrm{SO}(n)}E$ in the manner of Proposition 2.8.

**Definition 2.9** The **curvature operator** associated to the covariant derivative $\nabla$ is the map

$$R^\nabla : \Gamma(E) \to \Gamma(\Lambda^2(T^*M) \otimes E)$$

given by the following composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(\Lambda^1(T^*M) \otimes E) \xrightarrow{\widetilde{\nabla}^1} \Gamma(\Lambda^2(T^*M) \otimes E).$$

It can be shown that the curvature operator is linear, that is

$$R^\nabla(f\sigma) = fR^\nabla(\sigma),$$

for every smooth function $f : M \to \mathbb{R}$ and section $\sigma \in \Gamma(E)$. As a consequence, $R^\nabla$ can be seen as a section of the bundle $\mathrm{End}(E) \otimes \Lambda^2(T^*M)$. Another important property of this operator is the following :

**Proposition 2.10** For every (locally defined) vector fields $X$ and $Y$ over $M$ and every section $\sigma \in \Gamma(E)$, we have

$$(R^\nabla\sigma)(X,Y) = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma,$$

where $[\cdot,\cdot]$ is the Lie bracket defined in Section 2.1.2.

**Proof :** See [BLM89], Proposition 4.6 on page 105. □

Since we assume that $\nabla$ is Riemannian, it follows from the above proposition that

$$\langle R^\nabla_{X,Y}\sigma, \mu\rangle + \langle \sigma, R^\nabla_{X,Y}\mu\rangle = 0$$

for every $X, Y \in \Gamma(TM)$ and $\sigma, \mu \in \Gamma(E)$. It is also easy to show that the value at a point $p \in M$ of the expression $R^{\nabla}_{X,Y}\sigma$ only depends on $X(p), Y(p)$ and $\sigma(p)$. It follows directly from these considerations that, for every choice of tangent vectors $X, Y \in T_pM$, the restriction to the fiber

$$R^{\nabla}_{X,Y} : E_p \to E_p$$

is a skew-symmetric endomorphism, which we name the **curvature transformation**.

**Remark 2.11** We will denote the curvature transformation induced by tangent vectors $X$ and $Y$ over $p$ either by $R^{\nabla}_{X,Y}$ or $R^{\nabla}(X, Y)$.

## 2.3 Riemannian curvature tensor

We now come back to the simpler case of the tangent bundle $E = TM$ of some compact and oriented Riemannian manifold $M$ of dimension $n$. Suppose that $P_{\mathrm{SO}(n)}E$ is endowed with a connection 1-form and let $\nabla$ denote the associated covariant derivative. Let $p \in M$ be any point, and let $X$ and $Y$ be local fields around $p$. Then the expression

$$T_{X,Y} = \nabla_X Y - \nabla_Y X - [X, Y],$$

evaluated at $p$, can be shown to only depend on $X(p)$ and $Y(p)$. It is obviously bilinear and, through short computations, one easily proves that it is skew-symmetric. From this, we deduce that $T_{\cdot,\cdot}$ is a global 2-form taking its values in the tangent bundle, which we call the **torsion tensor**.
The following is a classical result in differential geometry :

**Theorem 2.12** There is a unique connection 1-form $\omega$ on $P_{\mathrm{SO}(n)}E$ whose torsion tensor vanishes everywhere.

This specific connection is then called the **canonical Riemannian connection** and the associated covariant derivative $\nabla$ on $E$ is known as the **Levi-Civita covariant derivative**. The latter is also Riemannian (in the sense of (2.2)) and torsion-free (that is, $T_{X,Y} \equiv 0$). It can be shown to exist more explicitly, for example using the well-known **Koszul formula**.

**Proposition 2.13** In the context described above, let $R = R^{\nabla}$ be the associated curvature operator and let $p \in M$ be any point. Then for every tangent vectors $W, X, Y, Z \in T_pM$ we have

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \tag{2.3}$$

and

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle. \tag{2.4}$$

Consider once again an $n$-dimensional, compact and oriented Riemannian manifold $M$. Let $\nabla$ be its Levi-Civita covariant derivative and $R = R^\nabla$ be the the associated curvature operator, which, from now on, we refer to as the **Riemannian curvature tensor**. In this setting, $R$ can be seen as a section of

$$\mathrm{End}(TM) \otimes \Lambda^2(T^*M) \subseteq T^*M \otimes TM \otimes T^*M \otimes T^*M.$$

In other words (in the spirit of Section 1.2.3), $R$ is a tensor of type $(1,3)$. However, we know that the Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$ induces a canonical bundle isomorphism $TM \cong T^*M$, and are therefore allowed to consider $R$ as a tensor of type $(0,4)$, that is

$$R \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes T^*M),$$

by defining

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle,$$

where of course $X, Y, Z, W \in \Gamma(TM)$. Under these assumptions, the Riemannian curvature tensor satisfies the following strong symmetries and other properties :

**Proposition 2.14** Let $X, Y, Z$ and $W$ be vector fields over $M$. Then

1. $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$
2. $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$
3. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

**Proof :** These are the result of direct computations. See for example [Des], Theorem 8.6 on page 102 for the proof of 3., which is usually known as the **first Bianchi identity**, and Lemma 8.7 on the same page for the other properties. $\square$

We can now use this object $R$ to derive more quantitative measures of the curvature. Suppose now that $M$ is of dimension greater or equal to 2 and let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric with which it is endowed.

**Definition 2.15** Let $p \in M$ be any point and suppose that the tangent vectors $u, v \in T_pM$ form an orthonormal basis of some plane $\Pi$ in the tangent space. Then the **sectional curvature** of $\Pi$ is defined by

$$\sec(\Pi) = \langle R(u,v)u, v\rangle \in \mathbb{R}.$$

Note that if we are any basis $\tilde{u}, \tilde{v} \in T_pM$ of the same plane $\Pi$, then

$$\sec(\Pi) = \frac{\langle R(\tilde{u},\tilde{v})\tilde{u}, \tilde{v}\rangle}{||\tilde{u}||^2 \cdot ||\tilde{v}||^2 - \langle \tilde{u}, \tilde{v}\rangle^2}.$$

In particular, the sectional curvature is independent of the choice of basis. It can be shown, although we will not do it here (see [Car92], Lemma 3.3 on page 94), that the values $\langle R(X,Y)Z,W\rangle$ can always be written as linear combinations of sectional curvatures. In other words, the sectional curvature fully determines the curvature tensor $R$, which is somewhat surprising.
Another important type of curvature is the following :

**Definition 2.16** For a point $p \in M$ and tangent vectors $u, v \in T_pM$, we define the **Ricci curvature** by

$$\mathrm{Ric}_p(u,v) = \sum_{j=1}^{n} R(u, e_j, v, e_j) = \sum_{j=1}^{n} \langle R(u, e_j)v, e_j\rangle,$$

where $\{e_1, \ldots, e_n\}$ forms an orthonormal basis of $T_pM$.

The Ricci curvature $\mathrm{Ric}_p(u,v)$ therefore corresponds to the trace of the endomorphism $T_pM \to T_pM$ given by $w \mapsto R(u,w)v$. From this point of view, it is clear why it does not depend on the choice of the orthonormal basis $\{e_1, \ldots, e_n\}$. It is easy to prove that this map $\mathrm{Ric}_p(\cdot,\cdot)$ is symmetric and bilinear, and it is therefore known as the **Ricci tensor**.

**Definition 2.17** The **scalar curvature** at the point $p \in M$ is then the trace of the Ricci tensor, that is

$$\kappa(p) = \sum_{j=1}^{n} \mathrm{Ric}_p(e_i, e_i),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_pM$.

Note that, if the Ricci curvature is always non-negative, by which we mean that for every point $p \in M$ and tangent vector $u \in T_pM$ we have $\mathrm{Ric}_p(u,u) \geq 0$, then $\kappa \geq 0$ as well.

## 2.4 Topological implications of positive curvature

All the different notions of curvature introduced in the previous section depend solely on local data and, most importantly, on the choice of a Riemannian metric, of which a given manifold may admit plenty. It is therefore somewhat surprising whenever curvature conditions are shown to have consequences on the global topology of the space. Such results are extremely scarce however, when it comes to positive (Ricci or scalar) curvature. The most famous example is the following :

**Theorem 2.18 (Bonnet-Myers theorem)** Let $M$ be an $n$-dimensional complete and connected Riemannian manifold. Suppose that there exists $r > 0$ such that

$$\mathrm{Ric}_p(u,u) \geq \frac{n-1}{r^2} > 0$$

for all points $p \in M$ and unit tangent vector $u \in T_pM$. Then $M$ is compact and its diameter satisfies $\mathrm{diam}(M) \leq \pi r$.

As a consequence, we have

**Corollary 2.19** Let $M$ be a compact Riemannian manifold with infinite fundamental group. Then $M$ does not admit any Riemannian metric of positive Ricci curvature, that is, such that $\mathrm{Ric}_p(u,u) > 0$ for every $p \in M$ and $0 \neq u \in T_pM$.

The proof of these results can be found in the lecture notes [Des] (see Theorem 9.10 and Corollary 9.11 on page 135).
These are, in fact, the only results concerning topological obstructions to the existence of metrics of positive Ricci curvature which can, in (relative) good faith, be called basic or, at least, classical. This scarcity has to be put in perspective against the following theorem, due to Lohkamp (see [Loh94]): every manifold of dimension greater or equal to 3 admits a Riemannian metric of negative Ricci curvature.

In terms of complexity, it seems that the next result in the context of positive curvature is the following :

**Theorem 2.20 (Lichnerowicz)** If a spin manifold of doubly even dimension admits a metric of positive scalar curvature, then its $\hat{A}$-genus vanishes.

Establishing this last result will be one of the main goals of this thesis, carried out over the next chapters. To do this however, we first need to study some basic **spin geometry** and, even before that, introduce some concepts in the general theory of **genera**.

# 3. Genera

In this short review, we aim to make sense of the basic concept in the theory of so-called genera. For this, we follow the formalism introduced by Hirzebruch as it is presented in [HBJ92]. Elliptic genera, which were first introduced in Ochanine's 1987 article [Och87], are of particular interest to us here because they include the $\hat{A}$-genus, but we will not have the time to delve into the extensive theory that they have since inspired nor to develop the deep links they share with classical elliptic functions (see for example the second chapter of [HBJ92]).

## 3.1 Cobordism

Let us begin with a quick review of some basic notions and useful results in the theory of oriented cobordism. Let $M$ and $N$ be smooth, closed and oriented $d$-dimensional manifolds.

**Definition 3.1** The manifolds $M$ and $N$ are called **cobordant**, which we write $M \simeq N$, if there is a compact and oriented $(d+1)$-dimensional manifold with boundary $X$ and an orientation preserving diffeomorphism

$$\psi : \partial X \to M \sqcup (-N).$$

Note that a connected and oriented manifold always admits precisely two distinct orientations. This choice is usually only implicit, but we will use the notation $-N$ to refer to the manifold $N$ endowed with the reversed orientation.

It is easy to check that the definition yields an equivalence relation on the set of closed and oriented $d$-dimensional manifolds. We usually denote by $\Omega_d^{\mathrm{SO}}$ the associated set of **oriented-cobordism classes**. When endowed with the law $+$ induced by the disjoint union, $\Omega_d^{\mathrm{SO}}$ forms an abelian group, whose neutral element is $[\emptyset]$. Notice that the inverse of a class $[M]$ is found by reversing the orientation of its representative. The cartesian product yields another well-defined operation

$$\cdot : \Omega_d^{\mathrm{SO}} \times \Omega_{d'}^{\mathrm{SO}} \to \Omega_{d+d'}^{\mathrm{SO}},$$

where we simply send $([M],[N])$ to $[M \times N]$. Now, if we form the direct sum

$$\Omega_*^{\mathrm{SO}} = \bigoplus_{d=0}^{\infty} \Omega_d^{\mathrm{SO}},$$

we obtain the **oriented cobordism ring** $(\Omega_*^{\mathrm{SO}}, +, \cdot)$. This object is, however, quite complicated. Through the work of Thom (see [MS74], Theorem 18.8 page 219), we know that the cobordism groups $(\Omega_d^{\mathrm{SO}}, +)$ are finitely generated when $d = 4k$ (the rank being then equal to the number of partitions of $k$) and even finite if 4 does not divide the dimension $d$. This description is greatly simplified if we consider the tensor product $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$, which is liberated of the torsion elements obfuscating our understanding. Indeed, we have the following result, due to Thom as well

**Theorem 3.2** The torsion-free oriented cobordism ring $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ is the rational polynomial ring generated by the classes $\left\{[P^{2k}(\mathbb{C})] \mid k \in \mathbb{N}\right\}$.

**Proof :** See [MS74], Corollary 18.9 on page 219. □

In particular, it follows that the groups $\Omega_{4k}^{\mathrm{SO}} \otimes \mathbb{Q}$ are generated by the classes

$$[P^{2i_1}(\mathbb{C}) \times \ldots \times P^{2i_r}(\mathbb{C})],$$

where $(i_1, \ldots, i_r)$ is any partition of $k$. For example, the first few groups are

$$\begin{aligned}
&\Omega_4^{\mathrm{SO}} \otimes \mathbb{Q} = \mathbb{Q}\left[[P^2(\mathbb{C})]\right], \\
&\Omega_8^{\mathrm{SO}} \otimes \mathbb{Q} = \mathbb{Q}\left[[P^2(\mathbb{C}) \times P^2(\mathbb{C})], [P^4(\mathbb{C})]\right], \\
&\Omega_{12}^{\mathrm{SO}} \otimes \mathbb{Q} = \mathbb{Q}\left[[P^2(\mathbb{C}) \times P^2(\mathbb{C}) \times P^2(\mathbb{C})], [P^4(\mathbb{C}) \times P^2(\mathbb{C})], [P^6(\mathbb{C})]\right].
\end{aligned}$$

Note that a sequence of oriented cobordism classes which generates $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$, such as the one formed by the complex projective spaces mentioned above, is sometimes also referred to as a **basis sequence**.

**Remark 3.3** There are many types of cobordism theories other than this oriented version, concerned with different structures. One such example is the spin-cobordism (see [BLM89], Chapter 2 on page 90).

## 3.2 Genera and multiplicative sequences

**Definition 3.4** A (multiplicative) **genus** is a map $\varphi$ which assigns an element of a commutative and unital $\mathbb{Q}$-algebra $(A, +, \cdot)$ to every (smooth) oriented closed manifold in such a way that the following properties are satisfied :

1. $\varphi(M^d \sqcup N^d) = \varphi(M^d) + \varphi(N^d)$,
2. $\varphi(M^d \times N^{d'}) = \varphi(M^d) \cdot \varphi(N^{d'})$,
3. if $M^d = \partial N^{d+1}$ is the oriented boundary of some compact manifold $N^{d+1}$, then $\varphi(M^d) = 0$.

Here, we have written $M^d$ to denote that the closed oriented manifold $M$ was of dimension $d$.

**Remark 3.5** For our purpose, the algebra $A$ could simply be $\mathbb{Q}$ itself. Notice however that it could for example be $\mathbb{R}$, $\mathbb{C}$ or even a polynomial ring such as $\mathbb{C}[x]$. In some applications, genera are made to take their values in rings of modular forms (see for example [HBJ92], Appendix I).

It is easy to deduce from the conditions above that such a genus $\varphi$ only depends on the oriented cobordism class of $M^d$, and therefore descends to a homomorphism

$$\varphi : \Omega_*^{\mathrm{SO}} \to A.$$

As already mentioned, the oriented cobordism ring is a very complicated object and we therefore rely on Thom's description of the tensor product $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ to extract some understanding of such objects. For example, since $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ is generated by the complex projective space of even complex dimension, it is clear that a genus $\varphi$ vanishes on manifolds whose dimension is not a multiple of 4. As we will see later on, the values $\varphi([P^n(\mathbb{C})])$ also play an important role in the characterization of $\varphi$.

**Remark 3.6** Of course, if $A = \mathbb{Q}$, a genus $\varphi$ descends to a map

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q},$$

which we (somewhat dangerously) also call a genus and denote with $\varphi$.

Thom even showed that any homomorphism $\Omega_*^{\mathrm{SO}} \to A$ can be written as a linear combination (with coefficients in $A$) of Pontrjagin numbers, which has important consequences for the particular case of multiplicative genera.

There is in fact a deep one-to-one correspondence between these genera and so-called **multiplicative sequence**, which we introduce now. Let

$$K_1(x_1), K_2(x_1, x_2), K_3(x_1, x_2, x_3), \ldots$$

be a sequence of polynomials with rational coefficients such that if the indeterminate $x_i$ is given a weight of $i$, then $K_n(x_1, \ldots, x_n)$ is homogeneous of weight $n$. Suppose that the (commutative and unital) $\mathbb{Q}$-algebra $A$ is graded, that is, there are linear subspaces $A^i$ such that

$$A = \bigoplus_{i \geq 0} A^i$$

and that the multiplication induces maps of the form

$$A^i \times A^j \to A^{i+j}$$

for each $i, j \geq 0$. Here, we assume that the unit 1 is an element of $A^0$. For each element $a \in A$ of the form

$$a = 1 + a_1 + a_2 + \ldots$$

with $a_i \in A^i$, we then define

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \ldots + K_n(a_1, \ldots, a_n) + \ldots$$

Obviously, the term $K_n(a_1, \ldots, a_n)$ is in $A^n$ and $K(a)$ as a whole belongs to $A$.

**Definition 3.7** Such a sequence of polynomials $\{K_n\}_{n\geq 1}$ is called a **multiplicative sequence** if the identity

$$K(ab) = K(a) \cdot K(b)$$

is satisfied for every $a, b \in A$ with leading coefficient $a_0 = 1 = b_0$ and every graded (commutative) $\mathbb{Q}$-algebra with unit $A$.

A first link with formal power series is established by the next result

**Proposition 3.8 (Hirzebruch)** Every formal even power series with rational coefficients

$$Q(x) = 1 + a_2x^2 + a_4x^4 + \ldots,$$

uniquely determines a multiplicative sequence $\{K_n\}_{n\geq 1}$ through the identity

$$Q(x) = K(1 + x^2).$$

**Proof :** See [MS74], Lemma 19.1 on page 223. □

These polynomials $\{K_n\}_{n\geq 1}$ form what we therefore call the **multiplicative sequence associated to** $Q$.

**Remark 3.9** Notice that in the proposition above, the fact that $Q(x) = K(1+x^2)$ means that the coefficient of $x_1^{2n}$ in $K_{2n}(x_1, \ldots, x_n)$ is $a_{2n}$.

We now consider a formal even power series $Q(x)$ with coefficients $a_{2k} \in \mathbb{Q}$, such that $a_0 = 1$ as before. Let $\{K_n\}_{n\geq 1}$ be its associated multiplicative sequence. Thanks to the defining properties of this sequence, we observe the following

$$\begin{aligned} Q(x_1) \cdot \ldots \cdot Q(x_n) &= K(1 + x_1^2) \cdot \ldots \cdot K(1 + x_n^2) \\ &= K\left((1 + x_1^2) \cdot \ldots \cdot (1 + x_n^2)\right) \\ &= K\left(1 + \sigma_1^n(x_1^2, \ldots, x_n^2) + \ldots + \sigma_n^n(x_1^2, \ldots, x_n^2)\right), \end{aligned}$$

where we have used the notation introduced in Definition 1.79 for the **elementary symmetric polynomials** $\sigma_j^n$. Recall now that if $M$ is a (closed, oriented) $4n$-dimensional manifold and $x_1, \ldots, x_{2n}$ are its formal Chern roots (see Section 1.5.2), then the total Pontrjagin class of its tangent bundle is given by

$$p = 1 + p_1 + \ldots + p_n = (1 + x_1^2) \cdot \ldots \cdot (1 + x_{2n}^2),$$

with $p_j = \sigma_j^n(x_1^2, \ldots, x_{2n}^2)$. We can therefore make sense of the value

$$K(M) = \langle K_n(p_1, \ldots, p_n), [M] \rangle \in \mathbb{Q},$$

where $[M] \in H_{4n}(M; \mathbb{Z})$ denotes the fundamental class of $M$.

**Remark 3.10** The notation $\langle \cdot, \cdot \rangle$ that we have used here refers to the natural pairing

$$H^{4n}(M;\mathbb{Q}) \times H_{4n}(M;\mathbb{Q}) \to \mathbb{Q},$$

but it is sometimes more appropriate to write

$$K(M) = K_n(p_1, \ldots, p_n)[M],$$

which we define to have the same meaning.

The construction above can in fact be extended to manifolds of dimensions that are not multiples of 4 :

**Definition 3.11** For any closed and oriented manifold $M$, we set

$$K(M) = \begin{cases} \langle K_n(p_1, \ldots, p_n), [M] \rangle & \text{if } \dim M = 4n, \\ 0 & \text{if } \dim M \not\equiv 0 \bmod 4. \end{cases}$$

In particular, if $\dim M = 4n$, we can make the dependence of $K(M)$ on the formal power series $Q$ explicit :

$$K(M) = \left( \prod_{j=1}^{2n} Q(x_j) \right) [M].$$

Note that $K(M)$ is a rational linear combination of **Pontrjagin numbers** of $M$. In fact, we can show that the assignment $M \mapsto K(M)$ induces a well-defined genus (see [HBJ92], page 14), which we refer to as either the **$K$-genus** or the **genus $\varphi_Q$ belonging to the power series $Q$**, depending on the context .

Let us consider once again a formal power series with rational coefficients of the form

$$Q(x) = 1 + a_2 x^2 + a_4 x^4 + \ldots.$$

Since it is normalized (that is, its leading coefficient is 1), it admits a multiplicative inverse which turns out to be of the same form, and we may define

$$f(x) = \frac{x}{Q(x)} = x + b_3 x^3 + b_5 x^5 + \ldots,$$

where the rational numbers $b_i$ can be expressed as functions of the coefficients $a_j$. Such power series always have a formal inverse with respect to the composition law, and we may thus set $g = f^{-1}$. This last power series is then known as the **logarithm** of the genus $\varphi_Q$.

**Lemma 3.12** The identity

$$g'(y) = \sum_{k=0}^{+\infty} \varphi_Q([P^k(\mathbb{C})]) \cdot y^k$$

holds for the logarithm $g$ of any genus $\varphi_Q$.

**Proof :** Recall that (see Example 1.76) the total Pontrjagin class of $P^k(\mathbb{C})$ is given by

$$p(P^k(\mathbb{C}) = \left(1 + a^2\right)^{k+1}.$$

Let $\{K_n\}_{n\geq 1}$ be the multiplicative sequence associated to the power series $Q$. It follows by definition that

$$K(p) = K\left(\left(1 + a^2\right)^{k+1}\right) = K\left(1 + a^2\right)^{k+1} = Q(a)^{k+1}.$$

Notice that

$$\varphi_Q([P^k(\mathbb{C})]) = K(P^k(\mathbb{C})) = K(p)[P^k(\mathbb{C})],$$

because the terms of order other than $2k$ in the cohomology class $K(p) \in H^*(P^k(\mathbb{C}); \mathbb{Z})$ vanish when evaluated on the fundamental class $[P^k(\mathbb{C})]$. We are therefore only looking for the coefficient of $a^k$ in

$$Q(a)^{k+1}[P^k(\mathbb{C})] = \left(\frac{a}{f(a)}\right)^{k+1} [P^k(\mathbb{C})]$$

since $a \in H^2(P^k(\mathbb{C}); \mathbb{Z})$. This corresponds directly to the coefficient of $a^k$ in the power series

$$\left(\frac{a}{f(a)}\right)^{k+1},$$

because we know that $\langle a^k, [P^k(\mathbb{C})]\rangle = 1$ (see the proof of Theorem 1.71). To compute it, we shall use the residue theorem and compute the residue of $\frac{Q(a)^{k+1}}{a^{k+1}} = \left(\frac{1}{f(a)}\right)^{k+1}$ at its simple pole 0. For a small curve $\mu$ going once around the origin of the complex plane, we obtain

$$\begin{aligned}\varphi_Q([P^k(\mathbb{C})]) = \mathrm{res}_0\left(\left(\frac{1}{f(a)}\right)^{k+1}\right) &= \frac{1}{2\pi i}\oint_\mu \left(\frac{1}{f(a)}\right)^{k+1} da \\ &= \frac{1}{2\pi i}\oint_{f(\mu)} \frac{g'(y)}{y^{k+1}}\, dy \\ &= \mathrm{res}_0\left(\frac{g'(y)}{y^{k+1}}\right),\end{aligned}$$

which obviously corresponds to the coefficient of $y^k$ in the power series $g'(y)$. Note that the equality of the residues holds whether the $f$ converges or not. □

We already know that every even power series $Q$ induces a genus $\varphi_Q$, but this formula makes it clear that the converse is also true. Indeed, setting values for a genus $\varphi$ on the oriented cobordism classes of the complex projective spaces $P^{2k}(\mathbb{C})$ determines $g'$ and, from there, we are able to recover $g, f$ and finally $Q$, which is the unique even power series such that $\varphi = \varphi_Q$.
There is a similar result for the quaternionic projective spaces :

**Lemma 3.13** Let $Q$ be an even power series and set $f(x) = \frac{x}{Q(x)}$. If we define

$$h(f(x)) = \frac{f(2x)}{2f(x)f'(x)},$$

then $\varphi_Q$ satisfies the identity

$$h(y) = \sum_{k=0}^{+\infty} \varphi_Q([P^k(\mathbb{H})]) \cdot y^{2k}.$$

**Proof :** See [HBJ92], page 15 and 16. □

The relation between the power series $f$ and $h$ is known as the **duplication formula**.

## 3.3 The $\hat{A}$-genus

The notion of elliptic genera, which we will introduce shortly, then stems from the requirement that

$$h(y) = \frac{1}{1-\varepsilon y^4} = 1 + \sqrt{\varepsilon}y^4 + \sqrt{\varepsilon}y^8 + \dots$$

for some $\varepsilon > 0$. Using Lemma 3.13, we would then obtain

$$\varphi_Q([P^k(\mathbb{H})]) = \begin{cases} 0 & \text{if } k \equiv 1 \bmod 2, \\ \varepsilon^{\frac{k}{2}} & \text{if } k \equiv 0 \bmod 2. \end{cases}$$

Note however that this condition is not enough to ensure the uniqueness of a solution $f$ (and therefore a genus $\varphi_Q$) to the equation

$$f(2x) = \frac{2f(x)f'(x)}{1-\varepsilon f(x)^4}. \tag{3.1}$$

In order to single out such a power series $f$, we need to fix the value of the associated genus $\varphi_Q$ on $[P^2(\mathbb{C})]$, which we often denote by a parameter $\delta$. In particular, the logarithm of $\varphi_Q$ would then be of the form

$$g'(y) = 1 + \delta y^2 + \dots.$$

From equation (3.1), we can deduce that

$$(f')^2 = 1 - 2\delta \cdot f^2 + \varepsilon f^4. \tag{3.2}$$

This allows the following

**Definition 3.14** A genus $\varphi_Q$ is called **elliptic** if $f(x) = \frac{x}{Q(x)}$ satisfies (3.2).

The terminology here is a reference to elliptic functions, which are related to these types of genera (see [HBJ92], Chapter 2).

**Remark 3.15** There are other equivalent ways of reformulating the condition (3.2), one could for example require that

$$f(x+y) = \frac{f(x)f'(y) + f'(x)f(y)}{1 - \varepsilon \cdot f(x)^2 f(y)^2}.$$

We now turn to some of the most important special cases of this type of genera.

**Example 3.16 (*L*-genus)** If $\delta = 1 = \varepsilon$, we obtain the so-called ***L*-genus** which corresponds to the power series

$$Q(x) = \frac{x}{\tanh(x)}.$$

Since $f(x) = \tanh(x)$, we know that the logarithm satisfies

$$g'(x) = 1 + x^2 + x^4 + \dots$$

It follows immediately from Lemma 3.12 that $L([P^{2k}(\mathbb{C})]) = 1$. We therefore know the behaviour of the $L$-genus on $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$, and this result can be used for example to prove that the $L$-genus coincides with the signature, another oriented cobordism invariant. We can also use the fact that

$$\frac{\tanh(2x)}{2\tanh(x)\tanh'(x)} = \frac{1}{1 - \tanh^4(x)}$$

that is,

$$h(y) = \frac{1}{1 - y^4}$$

in the duplication formula, as well as Lemma 3.13 to conclude that

$$L([P^k(\mathbb{H})]) = \begin{cases} 0 & \text{if } k \equiv 1 \bmod 2, \\ 1 & \text{if } k \equiv 0 \bmod 2. \end{cases}$$

The first few polynomials of the multiplicative sequence associated to the $L$-genus are

$$\begin{aligned} L_1(p_1) &= \frac{1}{3}p_1, \\ L_2(p_1,p_2) &= \frac{1}{45}(7p_2 - p_1^2), \\ L_3(p_1,p_2,p_3) &= \frac{1}{145}(62p_3 - 13p_1p_2 + 2p_1^3). \end{aligned}$$

The most important example for our purpose is the following :

**Example 3.17 ($\hat{A}$-genus)** If $\delta = -\frac{1}{8}$ and $\varepsilon = 0$, we obtain the $\hat{A}$**-genus** which corresponds to the power series

$$Q(x) = \frac{\frac{x}{2}}{\sinh\left(\frac{x}{2}\right)}.$$

Observe that, in this case, $f(x) = 2\sinh\left(\frac{x}{2}\right)$ and that $f(2x) = 2f(x)f'(x)$, thus $h(y) = 1$. By Lemma 3.13, we deduce that

$$\hat{A}([P^k(\mathbb{H})]) = 0$$

for all $k \geq 1$. In other words, the $\hat{A}$-genus vanishes on all the projective quaternionic spaces. The first polynomials of the associated multiplicative sequence are

$$\begin{aligned} \hat{A}_1(p_1) &= -\frac{1}{24}p_1, \\ \hat{A}_2(p_1,p_2) &= \frac{1}{5760}(-4p_2 + 7p_1^2), \\ \hat{A}_3(p_1,p_2,p_3) &= \frac{1}{967680}(-16p_3 - 44p_2p_1 - 31p_1^3). \end{aligned}$$

As we will later see, this genus is closely related to the index of Dirac operators (see Section 5.2.4).

# 4. The Atiyah-Singer index theorem

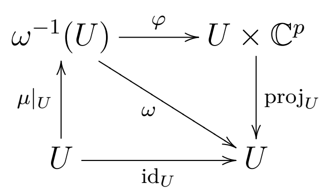
In this chapter, we cover the basic notions of the theory of differential operators over vector bundles and, in particular, of so-called elliptic differential operators in preparation of the Dirac operators (see 5.2.4). Assuming that it is impossible to give a satisfactory account of its proof in a timely manner such as is required of this thesis, we content ourselves with a quick overview of the main results and definitions leading up to it.

## 4.1 Differential operators

Let $E \xrightarrow{\omega} M$ and $F \xrightarrow{\eta} M$ be complex vector bundles of respective ranks $p$ and $q$ over some smooth and closed $n$-dimensional manifold $M$. Consider a $\mathbb{C}$-linear map

$$D : \Gamma(E) \to \Gamma(F)$$

and a smooth section $\mu \in \Gamma(E)$. Let $x \in M$ be any point and let $(U, \varphi)$ be a local coordinate system for $E$ around $x$. Clearly, the following diagram




is commutative. Hence, we may write $\varphi \circ \mu|_U = (\mathrm{id}_U, \mu_1, \ldots, \mu_p)$, where each of the $\mu_i : U \to \mathbb{C}$ is a smooth function. The same obviously holds for the image section $D(\mu) \in \Gamma(F)$ on any system of local coordinates $(V, \psi)$ for $F$. Note then that, $M$ being a smooth manifold, the domain $U$ can itself be identified to an open subset of $\mathbb{R}^n$. We may therefore simply assume that we have functions of the form $\mathbb{R}^n \supset U' \xrightarrow{\mu_i} \mathbb{C}$ and discuss the properties of $D$ at this local level.

**Definition 4.1** The $\mathbb{C}$-linear map $D$ is called a **differential operator of order** $m$ if it can be written locally (in the sense above) as

$$D = \sum_{|\alpha| \leq m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

where the $A^\alpha$ are $q \times p$ matrices of smooth, complex-valued functions and $A^\alpha \neq 0$ for at least one multi-index $\alpha$ such that $|\alpha| = m$.

Since this definition relies solely on local data, it is natural to wonder how it is impacted by changes of local trivializations or manifold coordinates. In fact, it turns out (see [BLM89, Page 167] for the details) that the matrices $i^m A^\alpha$ with $|\alpha| = m$ transform in such a way that they determine a (unique and well-defined) section of the bundle $\bigodot^m TM \otimes \mathrm{Hom}(E, F)$.

**Definition 4.2** This section $\sigma(D) \in \Gamma(\bigodot^m TM \otimes \mathrm{Hom}(E, F))$ is called the **principal symbol** of the operator $D$.

Recall that we may replace $\Gamma(\bigodot^m TM \otimes \mathrm{Hom}(E, F))$ with $\Gamma(\bigodot^m TM) \otimes \Gamma(\mathrm{Hom}(E, F))$ (see Section 1.2.4) and we are therefore allowed to write $\sigma(D)$ as a sum of elementary tensors, that is

$$\sigma(D) = \sum_{j=1}^{k} \lambda_j \alpha_j \otimes f_j,$$

where $\lambda_j \in \mathbb{R}$, $\alpha_j \in \Gamma(\bigodot^m TM)$ and $f_j \in \Gamma(\mathrm{Hom}(E, F))$. Let $x \in M$ be any point. Then, since the symmetric algebra $S(T_x M)$ is canonically isomorphic to the space of polynomials $\mathbb{R}[x_1, \ldots, x_n]$, where the elements $x_1, \ldots, x_n$ form any basis of the tangent space $T_x M$, we can identify each $\alpha_j(x) = \alpha_j^x$ with a homogeneous polynomial of degree $m$ in the $x_i$. It follows that any cotangent vector $\xi \in T_x^* M$ induces an element

$$\sigma_\xi(D) \in \mathrm{Hom}(E_x, F_x),$$

defined by $\sigma_\xi(D) = \sum_{j=1}^n \lambda_j \alpha_j^x(\xi) f_j(x)$. If $\{dx_1, \ldots, dx_n\} \subset T_x^* M$ is chosen as the basis dual to $\{x_1, \ldots, x_n\} \subset T_x M$ and if $\xi \in T_x^* M$ has coefficients $\xi_1, \ldots, \xi_n$ with respect to this basis, then we find

$$\sigma_\xi(D) = i^m \sum_{|\alpha|=m} A^\alpha(x) \xi^\alpha,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$.

### 4.1.1 Elliptic operators and their analytical index

**Definition 4.3** A differential operator $D : \Gamma(E) \to \Gamma(F)$ of order $m$ is called **elliptic** if the principal symbol $\sigma_\xi(D)$ is invertible for every non-zero cotangent vector $\xi$ over $M$.

Elliptic operators, being the subject of the Atiyah-Singer theorem, are of particular importance to us and we are therefore naturally interested in their properties. The result below lists two of their most important characteristics :

**Theorem 4.4** Let $D$ be an elliptic differential operator and $\sigma(D)$ be its principal symbol. Then

1. both $\ker D$ and $\mathrm{coker} D$ are finite dimensional and

2. the integer $\dim \ker D - \dim \mathrm{coker} D$ depends only on $\sigma(D)$.

**Proof :** The proof of the fact that the kernels and cokernels of elliptic operators is finite dimensional is given in [BLM89], Theorem 5.2 on page 193. For 2., see [BLM89], Corollary 7.9 on page 204. □

We may thus define

**Definition 4.5** The **analytical index** of an elliptic differential operator $D$ is given by

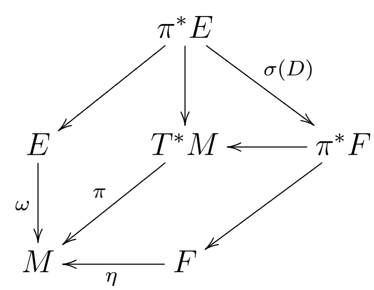
$$\text{a-Ind}(D) = \dim \ker D - \dim \mathrm{coker} D \in \mathbb{Z}.$$

Suppose now that we are given differential operators $D, D' : \Gamma(E) \to \Gamma(F)$ of the same order and $D'' : \Gamma(F) \to \Gamma(G)$, where $G$ is another complex vector bundle over $M$. It is then possible to show the properties below :

**Proposition 4.6** For all cotangent vector $\xi \in T^*M$ and real numbers $s, t \in \mathbb{R}$, we have $\sigma_\xi(sD + tD') = s\sigma_\xi(D) + t\sigma_\xi(D')$ and $\sigma_\xi(D'' \circ D) = \sigma_\xi(D'') \circ \sigma_\xi(D)$.

**Proof :** Using the explicit formulas above, it is easy to check that these hold indeed. □

We might therefore be interested in pulling back the bundles $E$ and $F$ back over to the cotangent bundle using the projection $\pi : T^*M \to M$ and see $\sigma(D)$ as a map $\pi^*E \to \pi^*F$. The following diagram summarizes the situation :




From this new point of view, it is clear that the symbol $\sigma(D)$ of a differential operator $D$ determines a class $i(D)$ in the $K$-theory of the cotangent bundle. Indeed, using the notations introduced in Section 1.5.3, we set

$$i(D) = [\pi^*E, \pi^*F; \sigma(D)] \in K(T^*M).$$

Note that this class will be of crucial importance for the rest of this section.

**Remark 4.7** It turns out that every single class of $K(T^*M)$ can be interpreted as the symbol of a **pseudo-differential operator**. However, to be properly explained, this result requires a lot of computations and technical results which we do not have the time to expose here. We refer the reader to the thirteenth paragraph of the third chapter in [BLM89]. Combining this and the facts that a-Ind($D$) only depends on $\sigma(D)$ and that the symbol of $D$ itself corresponds to a $K$-theory class $i(D)$, we may see the analytical index as a homomorphism

$$\text{a-Ind} : K(T^*M) \to \mathbb{Z}.$$

### 4.1.2 Topological index of elliptic operators

Suppose once again that $E$ is a complex vector bundle over a compact manifold $M$. Let $\pi : TM \to M$ be its tangent bundle and choose an embedding

$$f : M \to \mathbb{R}^N,$$

for some large enough integer $N$. Let $\nu$ denote the associated normal bundle of $M$ seen as a submanifold of $\mathbb{R}^N$. The differential of $f$ is a smooth map of the form

$$df : TM \to T\mathbb{R}^N.$$

Notice however that $T\mathbb{R}^N = \mathbb{R}^N \oplus \mathbb{R}^N$ can be canonically identified with $\mathbb{C}^N$ ; in this setting, the normal bundle of $df$ is a complex bundle corresponding to $\pi^*(\nu \otimes \mathbb{C})$. Consider now the induced Thom homomorphism on the level of the $K$-theory rings :

$$f_! : K_{\text{cpt}}(TM) \to K_{\text{cpt}}(T\mathbb{R}^N) \cong K_{\text{cpt}}(\mathbb{C}^N).$$

In fact, the special case of the inclusion $\{0\} \to \mathbb{R}^N$ yields the so-called Thom isomorphism

$$j_! : K_{\text{cpt}}(\{0\}) \xrightarrow{\cong} K_{\text{cpt}}(\mathbb{C}^N).$$

Let us write $q_! = (j_!)^{-1}$. Having made these preparations, we may give the following

**Definition 4.8** The **topological index** of an elliptic operator $D : \Gamma(E) \to \Gamma(E)$ is given by

$$\text{t-Ind}(D) = (q_! \circ f_!)\,(\sigma(D)) \in \mathbb{Z}.$$

Using the Thom isomorphism of Section 1.1.7, it is possible to derive more explicit formulas for the topological index, expressed in terms of the Chern character and other cohomological data. In the case of the tangent bundle $\pi : TM \to M$ of a compact manifold $M$ and of the embedding $i : M \to TM$ given by the zero section, it can be shown (see [BLM89], page 239) that $\pi_! = (i_!)^{-1}$ and

$$(i^* i_!)(u) = e(M)u$$

for every class $u$ in the cohomology ring of $M$, where $e(M)$ is the Euler class of the tangent bundle.
It follows that for any elliptic differential operator $D : \Gamma(TM) \to \Gamma(TM)$, we have

$$e(M)\pi_!(\text{ch}(\sigma(D))) = i^*(\text{ch}(\sigma(D))). \tag{4.1}$$

From there, one deduces the following cohomological formula for the topological index of the operator $D$ :

**Theorem 4.9** In the situation described above, we have

$$\text{t-Ind}(D) = (-1)^{\frac{n(n+1)}{2}} \left(\pi_! \left(\text{ch}(\sigma(D))\right) \cdot \hat{\mathcal{A}}(M)^2\right) [M], \tag{4.2}$$

where

$$\hat{\mathcal{A}}(M)^2 = \prod_{i=1}^{n} \left( \frac{x_i}{e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}} \right)^2 .$$

**Proof :** See [BLM89], Theorem 13.8 on page 255. □

Note that $\hat{\mathcal{A}}(M)[M] = \hat{A}(M)$. We will use this formula (4.2) in the proof of equation (5.6), which relates a particular elliptic operator (see Section 5.2.4) with the $\hat{A}$-genus (see Section 3.3).

## 4.2 Statement of the theorem

While its importance and the complexity of its proof cannot really be understated, the Atiyah-Singer index theorem is, at the surface level, deceptively easy to state :

**Theorem 4.10** The analytical and topological index of an elliptic differential operator $D$, that is

$$\text{a-Ind}(\sigma(D)) = \text{t-Ind}(\sigma(D)).$$

**Proof :** See the third chapter of [BLM89] or the original papers [AS68] of Atiyah and Singer. □

# 5. Notions of spin geometry

## 5.1 Overview of Clifford algebras

### 5.1.1 Definition and first properties

Let $V$ be a finite dimensional $\mathbb{K}$-vector space, which we suppose to be endowed with a quadratic form $q$. Here, $\mathbb{K}$ could be any commutative field, but we will only really work within the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Consider the tensor algebra $T(V)$ associated to $V$ (see Section 1.2.3) and the ideal $I_q(V)$ generated by the elements of the form

$$v \otimes v - q(v) \cdot 1,$$

where $v \in V$ and 1 is the unit in $T(V)$.

**Definition 5.1** The **Clifford algebra** $\mathrm{C}\ell(V, q)$ is the quotient

$$\mathrm{C}\ell(V, q) = T(V)/I_q(V).$$

Through the associated projection $\pi_q : T(V) \to \mathrm{C}\ell(V, q)$, it is clear that the vector space $V$ can be embedded in $\mathrm{C}\ell(V, q)$. In fact, the Clifford algebra is generated by 1 and the elements of $V$, using the relation given in the definition of $I_q(V)$.

**Remark 5.2** The operation induced on $\mathrm{C}\ell(V, q)$ by the tensor product is known as **Clifford multiplication**, and is denoted by $\cdot$.

The Clifford algebra $\mathrm{C}\ell(V, q)$ can also be defined in terms of a universal property. Indeed, any linear map $f$ from $V$ into an associative and unital $\mathbb{K}$-algebra $\mathcal{A}$ satisfying

$$f(v) \cdot f(v) = -q(v) \cdot 1$$

can be extended uniquely to a homomorphism of $\mathbb{K}$-algebras $\mathrm{C}\ell(V, q) \to \mathcal{A}$, and this holds for $\mathrm{C}\ell(V, q)$ alone (see [BLM89], Proposition 1.1 on page 8). Observe that if $(V, q)$ and $(W, p)$ are both $\mathbb{K}$-vector spaces endowed with quadratic forms, which some linear map $f : V \to W$ preserves, then $f$ induces a homomorphism

$$F : \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(W, p).$$

It is easy to deduce from this construction that the orthogonal group $\mathrm{O}(V, q)$ can be seen as a subgroup of $\mathrm{Aut}(\mathrm{C}\ell(V, q))$. In particular, we consider the automorphism

$$\alpha : \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q),$$

given by the extension of the map $v \mapsto -v$ defined on $V$, which obviously preserves $q$. Notice that $\alpha^2 = \mathrm{id}_{\mathrm{C}\ell(V,q)}$ and, therefore, we obtain a splitting

$$\mathrm{C}\ell(V,q) = \mathrm{C}\ell^0(V,q) \oplus \mathrm{C}\ell^1(V,q)$$

of the Clifford bundle into the eigenspaces

$$\mathrm{C}\ell^j(V,q) = \left\{\varphi \in \mathrm{C}\ell(V,q) \mid \alpha(\varphi) = (-1)^j \varphi\right\}$$

of $\alpha$, with $j \in \{0,1\}$. Note that, for $\varphi \in \mathrm{C}\ell^i(V,q)$ and $\psi \in \mathrm{C}\ell^j(V,q)$, we have

$$\varphi \cdot \psi \in \mathrm{C}\ell^{(i+j) \bmod 2}(V,q).$$

Indeed, $\alpha(\varphi \cdot \psi) = \alpha(\varphi) \cdot \alpha(\psi) = (-1)^{i+j} \varphi \cdot \psi$. Such a structure on $\mathrm{C}\ell(V,q)$ is known as a $\mathbb{Z}_2$**-grading**.

**Remark 5.3** It is obvious that the precise nature of the quadratic form $q$ on $V$, and thus also the choice of the field $\mathbb{K}$, greatly influences the Clifford algebra $\mathrm{C}\ell(V,q)$. See for example Section 5.1.3 below.

### 5.1.2 Pin and spin groups

Let $\mathrm{C}\ell^\times(V,q)$ denote the (multiplicative) group of units of the Clifford algebra. Note that, in particular, it contains all the vectors $v \in V$ such that $q(v) \neq 0$. The elements of this group act on $\mathrm{C}\ell(V,q)$ via the **adjoint representation**

$$\mathrm{Ad} : \mathrm{C}\ell^\times(V,q) \to \mathrm{Aut}(\mathrm{C}\ell(V,q))$$

which is given by

$$\mathrm{Ad}_\varphi(\psi) = \varphi \cdot \psi \cdot \varphi^{-1},$$

for all $\varphi \in \mathrm{C}\ell^\times(V,q)$ and $\psi \in \mathrm{C}\ell(V,q)$. This is obviously a group homomorphism. For any vector $v \in V \subset \mathrm{C}\ell(V,q)$ such that $q(v) \neq 0$, it holds that

$$-\mathrm{Ad}_v(w) = w - \frac{q(v+w) - q(v) - q(w)}{q(v)} v = w - 2\frac{q(v,w)}{q(v)} v$$

for all $w \in V$, where $q(v,w)$ refers to the polarization of $q$. Note that, on the right side of the above equation, we find an expression for the reflection through the hyperplane

$$v^\perp = \{w \in V \mid q(v,w) = 0\}.$$

The minus sign on the left will be dealt with later on. For now, we observe that, in particular, the identity above also implies that $\mathrm{Ad}_v(V) = V$ if $q(v) \neq 0$. Moreover, for such vectors, we notice that $\mathrm{Ad}_v$ preserves $q$. Therefore, if we denote by $P(V,q)$ the subgroup of $\mathrm{C}\ell^\times(V,q)$ generated by the elements $v \in V$ with $q(v) \neq 0$, we obtain a natural homomorphism

$$\mathrm{Ad} : P(V,q) \to \mathrm{O}(V,q), \tag{5.1}$$

simply by restricting the adjoint representation.

**Definition 5.4** The **pin group** $\mathrm{Pin}(V,q)$ is the subgroup of $P(V,q)$ generated by the elements $v \in V$ such that $q(v) \in \{\pm 1\}$ and the associated **spin group** is defined by

$$\mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap \mathrm{C}\ell^0(V,q).$$

In fact, the pin group can be seen as

$$\mathrm{Pin}(V,q) = \{v_1 \cdot \ldots \cdot v_k \in P(V,q) \mid q(v_i) \in \{\pm 1\}\},$$

while the spin group can be written as the set formed by all the products of the form

$$v_1 \cdot \ldots \cdot v_k \in \mathrm{Pin}(V,q),$$

where $k$ is even.

Using a modified version of 5.1 known as the twisted adjoint representation

$$\widetilde{\mathrm{Ad}} : \mathrm{C}\ell^\times(V,q) \to \mathrm{GL}(\mathrm{C}\ell(V,q)),$$

given by $\widetilde{\mathrm{Ad}}_\varphi(\psi) = \alpha(\varphi) \cdot \psi \cdot \varphi$, and under the additional assumption that the form $q$ is non-degenerate, it can then be shown that, in the real case ($\mathbb{K} = \mathbb{R}$), there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(V,q) \longrightarrow 0.$$

See [BLM89], Theorem 2.9 on page 19 for the proof. In other words, the spin group $\mathrm{Spin}(V,q)$ is a double cover of the special orthogonal group $\mathrm{SO}(V,q)$.

### 5.1.3 Real Clifford algebras

We are particularly interested in the following special case. Suppose that $V$ is an $n$-dimensional real vector space, endowed with a non-degenerate quadratic form $q$. It is always possible to choose a basis $e_1, \ldots, e_n$ of $V$ in such a way that $q$ becomes of the form

$$q(v) = q(v_1 e_1 + \ldots + v_n e_n) = v_1^2 + \ldots + v_r^2 - v_{r+1}^2 - \ldots v_{r+s}^2,$$

where $r + s = n$. Here, the integer $0 \leq r, s \leq n$ correspond respectively to the highest dimension reached by a linear subspace on which $q$ is positive or negative definite. It is a well-known result of linear algebra that the couple $(r,s)$ fully characterizes $q$, hence the associated Clifford algebra is denoted by $\mathrm{C}\ell_{r,s} = \mathrm{C}\ell(V,q)$.

**Remark 5.5** In this setting, the orthogonal and special orthogonal groups associated to $q = q_{r,s}$ are denoted respectively by $\mathrm{O}_{r,s}$ and $\mathrm{SO}_{r,s}$, while we write

$$\mathrm{Pin}_{r,s} = \mathrm{Pin}(V,q) \quad \text{and} \quad \mathrm{Spin}_{r,s} = \mathrm{Spin}(V,q)$$

for the pin and spin groups. In the special case that $r = n$, we simply write

$$\mathrm{C}\ell_n = \mathrm{C}\ell_{n,0}$$

and, similarly $\mathrm{SO}(n) = \mathrm{SO}_{n,0}$.

Here, an alternative description of the Clifford algebra $\mathrm{C}\ell(V,q)$ is possible. Indeed, if the vectors $e_1, \ldots, e_n$ form an orthonormal basis of $V$, then $\mathrm{C}\ell(V,q)$ is generated by $1, e_1, \ldots, e_n$ subject to the relation

$$e_i \cdot e_j + e_j \cdot e_i = \begin{cases} -2\delta_{ij} & \text{if } i \leq r \\ 2\delta_{ij} & \text{if } i > r. \end{cases} \tag{5.2}$$

It follows from the discussion at the end of the previous section that we have short exact sequences

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_{r,s} \longrightarrow \mathrm{SO}_{r,s} \longrightarrow 0.$$

In particular, it can be shown (see [BLM89], Theorem 2.10 on page 20) that when $n \geq 3$, the map

$$\widetilde{\mathrm{Ad}} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

is the universal covering of the special orthogonal group. This fact will play a central role in the definition of a **spin structure** (see Definition 5.18).

**Remark 5.6** For any couple of integers $(r,s)$ such that $r+s=n$, it turns out that the complexification of the real Clifford algebra $\mathrm{C}\ell_{r,s}$ coincides with the complex Clifford algebra induced by the complexified quadratic form :

$$\mathbb{C}\ell_{r,s} := \mathrm{C}\ell(\mathbb{C}^{r+s}, q_{r,s} \otimes \mathbb{C}) \cong \mathrm{C}\ell_{r,s} \otimes \mathbb{C}.$$

It is well-known however, that all the non-degenerate quadratic forms on $\mathbb{C}^n$ are equivalent over $\mathrm{C}\ell_n(\mathbb{C})$. We therefore simply set

$$q_{\mathbb{C}}(z_1 e_1 + \ldots + z_n e_n) = z_1^2 + \ldots + z_n^2$$

and write

$$\mathbb{C}\ell_n = \mathrm{C}\ell(\mathbb{C}^n, q_{\mathbb{C}}).$$

In a manner eerily reminiscent of the Bott Periodicity theorem (see the end of Section 1.5.3), the real and complex Clifford algebras exhibit a recurring pattern :

**Proposition 5.7** Whenever $n \geq 0$, we have the following isomorphism

$$\mathrm{C}\ell_{n+8,0} \cong \mathrm{C}\ell_{n,0} \otimes \mathrm{C}\ell_{8,0}$$
$$\mathrm{C}\ell_{0,n+8} \cong \mathrm{C}\ell_{0,n} \otimes \mathrm{C}\ell_{0,8}$$
$$\mathbb{C}\ell_{n+2} \cong \mathbb{C}\ell_n \otimes \mathbb{C}\ell_2.$$

See [BLM89], Theorem 4.3 on page 27 for the proof. This is, in fact, not a coincidence. The spinor bundles we will construct later in this chapter with the help of this Clifford algebras share deep links with the global topological data encoded in the $K$-theory of the base manifold and allow us the feat of relating them with local geometrical information such as the curvature. Note finally that this type of result is obviously of high interest for establishing a classification of Clifford algebras, which we will mention in the next section.

Let $V \cong \mathbb{R}^n = \mathbb{R}^{r+s}$ be once again endowed with the non-degenerate quadratic form $q_{r,s}$. Suppose that $V$ is oriented and let $e_1, \ldots, e_n$ denote an oriented and orthonormal basis.

**Definition 5.8** The (real) **volume element** $\omega$ associated to this basis is given by

$$\omega = e_1 \cdot \ldots \cdot e_n.$$

The relation given in (5.2) allows us to conclude that $\omega$ is, in fact, independent on the choice of the specific basis. A simple computation then shows that

$$\omega^2 = (-1)^{\frac{n(n+1)}{2}+s}.$$

Moreover, it can be shown (see [BLM89], Proposition 3.6 on page 23) that if $n$ is even and $\omega^2 = 1$, then any $\mathrm{C}\ell_{r,s}$-module $W$ decomposes as

$$W = W^+ \oplus W^-,$$

where $W^+$ and $W^-$ are the respective $\{\pm 1\}$-eigenspaces induced through Clifford multiplication by $\omega$.

### 5.1.4 Classification and representations

All of the real Clifford algebras $\mathrm{C}\ell_{r,s}$ can actually be identified as matrix algebras over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. See for example the table on page 29 of [BLM89]. We do not have the time to go over the details of this classification, and will therefore only refer to it for the purpose of introducing theory of the representations of Clifford algebras. Let us however simply mention the following useful isomorphism

**Proposition 5.9** For all integers $s$ and $r$, there are algebra isomorphisms

$$\mathrm{C}\ell_{r,s} \cong \mathrm{C}\ell^0_{r+1,s}$$

and, in particular, we have

$$\mathrm{C}\ell_n \cong \mathrm{C}\ell^0_{n+1}.$$

**Proof :** See [BLM89], Theorem 3.2 on page 23. □

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $V$ be a finite dimensional vector space over $\mathbb{K}$, endowed with a non-degenerate quadratic form $q$.

**Definition 5.10** A ($\mathbb{K}$-)**representation** of the Clifford algebra $C\ell(V, q)$ is a $\mathbb{K}$-algebra homomorphism of the form

$$\rho : C\ell(V, q) \to \mathrm{Hom}_{\mathbb{K}}(W, W),$$

where $W$ is a finite dimensional $\mathbb{K}$-vector space. It is also common place to name $W$ a **C$\ell(V, q)$-module** and we also often write

$$\rho(\varphi)(w) = \varphi \cdot w,$$

for all $\varphi \in C\ell(V, q)$ and $w \in W$. We refer to this external multiplication on the module $W$ as the **Clifford multiplication**.

Note that any complex representation of $C\ell_{r,s}$ (that is, when $\mathbb{K} = \mathbb{C}$ in the definition above) automatically extends to a representation of $C\ell_{r,s} \otimes \mathbb{C} = \mathbb{C}\ell_{r,s}$.

**Definition 5.11** A $\mathbb{K}$-representation $\rho : C\ell(V, q) \to \mathrm{Hom}_{\mathbb{K}}(W, W)$ is said to be **reducible** if the vector space $W$ splits as

$$W = W_1 \oplus W_2,$$

where each of the subspace $W_i$ is $\rho$-invariant, by which we mean that

$$\rho(\varphi)(W_i) \subseteq W_i$$

for all $\varphi \in C\ell(V, q)$. A representation is said to be **irreducible** if it is not reducible.

It is now almost obvious that through an iterative process, using the finiteness of the dimension of the module $W$, that we have the following

**Proposition 5.12** Every $\mathbb{K}$-representation of a Clifford algebra $C\ell(V, q)$ can be written as a direct sum of irreducible representations.

As is often the case, we are only interested in such representations up to a certain notion of equivalence, which we introduce now :

**Definition 5.13** Two representations $\rho_i : C\ell(V, q) \to \mathrm{Hom}_{\mathbb{K}}(W_i, W_i)$ are said to be **equivalent** if there is an isomorphism of $\mathbb{K}$-vector spaces

$$F : W_1 \to W_2$$

such that

$$F \circ \rho_1(\varphi) \circ F^{-1} = \rho_2(\varphi),$$

for all $\varphi \in \mathrm{C}\ell(V, q)$.

Recall that the classification of Clifford algebra asserts that they can all be built from matrix algebras over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The representations of these algebras are thankfully well-understood and rather simple : see [BLM89], Theorem 5.6 on page 31 for the details. In particular, it follows that there are never more than two non-equivalent irreducible real representations of $\mathrm{C}\ell_{r,s}$. The same conclusion holds for the number of non-equivalent irreducible complex representations of $\mathbb{C}\ell_n$.

We now come to the representations of the spin groups. Note first that, using 5.9, the representations of $\mathrm{Spin}_{r,s}$ are obtained by looking at the irreducible representations of the Clifford algebra $\mathrm{C}\ell_{r-1,s}$. We will, however, restrict our study to the specific Clifford algebra $\mathrm{C}\ell_n$. Suppose that $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n$. This defines, up to sign (that is, up to the choice of an orientation on $\mathbb{R}^n$), a real volume element $\omega$ in $\mathrm{C}\ell_n$ (see Definition 5.8).

**Proposition 5.14** Let $\rho : \mathrm{C}\ell_n \to \mathrm{Hom}_{\mathbb{R}}(W, W)$ be a real and irreducible representation, where $n = 4m$. Consider the splitting

$$W = W^+ \oplus W^-,$$

given by $W^\pm = (1 \pm \rho(\omega))W$. Then $W^+$ and $W^-$ are invariant under Clifford multiplication by the even sub-algebra $\mathrm{C}\ell_n^0$ and, using the isomorphism $\mathrm{C}\ell_n^0 \cong \mathrm{C}\ell_{n-1}$, these can be seen as the two non-equivalent and irreducible $\mathrm{C}\ell_{n-1}$-modules.

**Proof :** See [BLM89], Theorem 5.10 on page 35. □

Consider now an irreducible real representation $\rho : \mathrm{C}\ell_n \to \mathrm{Hom}_{\mathbb{R}}(S, S)$.

**Definition 5.15** The **real spinor representation** of $\mathrm{Spin}(n)$ is the homomorphism

$$\Delta_n : \mathrm{Spin}(n) \to \mathrm{GL}(S),$$

given by the restriction of $\rho : \mathrm{C}\ell_n \to \mathrm{Hom}_{\mathbb{R}}(S, S)$.

In the $n = 4m$-dimensional case, this representation $\Delta_n$ actually splits as a direct sum

$$\Delta_n = \Delta_n^+ \oplus \Delta_n^-,$$

of irreducible non-equivalent representations of $\mathrm{Spin}(n)$ (see [BLM89], Proposition 5.12 on page 35), known as the **half spin representations**. A similar result also hold for the complex case (see [BLM89], Proposition 5.15 on page 36).

**Remark 5.16** It can be shown (see [Bum04], Theorem 26.2 on page 179) that the roots of these representations are of the form

$$\frac{1}{2}(\pm x_1 \pm \ldots \pm x_{\frac{n}{2}}),$$

with $\Delta_n^+$ and $\Delta_n^-$ counting an even, respectively odd, number of coefficient $-1$. Here, the $x_i$ are the Chern roots of the spinor bundles $\mathbb{S}_{\mathbb{C}}^{\pm}$ introduced later (see the end of Section 5.2.4). This fact will prove very useful in deriving Equation (5.6) later on.

## 5.2 Overview of spin geometry

### 5.2.1 Spin structure on vector bundles

Let $M$ be a smooth manifold and $\pi : E \to M$ be a real rank $n$ vector bundle. As already discussed in Section 1.1.1, we may always assume that such a bundle comes endowed with a bundle metric. Its orientability or lack thereof can, however, not be similarly taken for granted. Recall (see Section 1.1.6) that an orientation on $E$ can be defined fiberwise by looking at equivalence classes of bases and let us therefore consider the orthonormal frame bundle $P_{\mathrm{O}(n)}E \to M$. We may then define the **bundle of orientations**

$$\mathrm{Or}(E) = P_{\mathrm{O}(n)}E/\mathrm{SO}(n),$$

where we use the natural action of $\mathrm{SO}(n)$ on the fibers to identify bases. This is a two-sheeted covering of the base manifold $M$ and one can show that $E$ is orientable if and only if $\mathrm{Or}(E)$ is trivial, that is, diffeomorphic to the disjoint union of two copies of $M$.

Let us now come back to our original bundle $\pi : E \to M$. Note that the set of equivalence classes of two-sheeted coverings over $M$ is in one-to-one correspondence with $H^1(M; \mathbb{Z}_2)$. This is indeed a particular case of the one-to-one correspondence between $\mathrm{Prin}_G(M)$ and the group $H^1(M; G)$, which we mentioned back in Remark 1.64. From the observations of the previous paragraph, it follows that the bundle of orientations $\mathrm{Or}(E)$ determines a particular cohomology class in $H^1(M; \mathbb{Z}_2)$ which turns out to be the first Stiefel-Whitney class $w_1(E)$ (see [BLM89], below Theorem 1.2 on page 79). We thus conclude :

**Proposition 5.17** A vector bundle $\pi : E \to M$ is orientable if and only if its first Stiefel-Whitney class $w_1(E)$ vanishes.

Note also that if $w_1(E) = 0$, then the different possible orientations on $E$ are determined by the elements of $H^0(M; \mathbb{Z}_2)$. In other words, there are precisely two orientations for each connected component of $M$.

Suppose now that the bundle $E$ is orientable. The choice of an orientation yields a $\mathrm{SO}(n)$-principal bundle of oriented orthonormal frames, which we write $P_{\mathrm{SO}(n)}E$. If

$n \geq 3$, then there is a two-sheeted universal covering

$$\xi_0 : \mathrm{Spin}(n) \to \mathrm{SO}(n),$$

whose kernel is isomorphic to $\mathbb{Z}_2$. In this case, we define

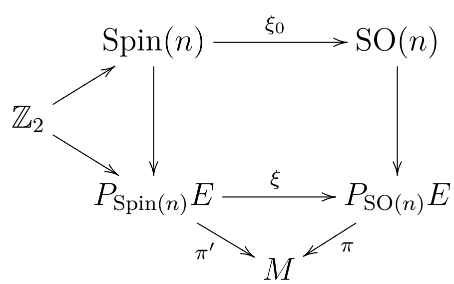
**Definition 5.18** A **spin structure** on $E$ is a principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}(n)}E$ over $M$ together with a $\mathrm{Spin}(n)$-equivariant two-sheeted covering

$$\xi : P_{\mathrm{Spin}(n)}E \to P_{\mathrm{SO}(n)}E.$$

Here, the $\mathrm{Spin}(n)$-equivariance of $\xi$ means that for any $p \in P_{\mathrm{Spin}(n)}(E)$ and $g \in \mathrm{Spin}(n)$, we have $\xi(p.g) = \xi(p)\xi_0(g)$.

**Remark 5.19** If $n = 2$, we replace the spin group with $\mathrm{SO}(2)$ and $\xi$ with the connected two-sheeted covering $\mathrm{SO}(2) \to \mathrm{SO}(2)$. If $n = 1$, the bundle of oriented orthonormal frames is diffeomorphic to $M$ itself and we therefore simply declare any two-sheeted covering to be a spin structure.

Note that, in any case, $\xi$ is a bundle morphism and that, restricted to the fibers, it coincides with $\xi_0$. The situation is illustrated by the following diagram of fibrations




There can be multiple different spin structures on a given oriented bundle $E$. Note however that they are in one-to-one correspondence with the two-sheeted coverings of $P_{\mathrm{SO}(n)}(E)$ which are not trivial in the fibers. Using this fact and the Serre spectral sequence associated to the fiber bundle

$$\mathrm{SO}(n) \xrightarrow{i} P_{\mathrm{SO}(n)}E \xrightarrow{\pi} M,$$

we get the following necessary condition for the existence of a spin structure on a given bundle :

**Proposition 5.20** An oriented vector bundle $\pi : E \to M$ admits a spin structure if and only if its second Stiefel-Whitney class vanishes.

**Proof :** See [BLM89], Theorem 1.7 on page 82. □

If $w_2(E) = 0$, then the spin structures that can exist over $M$ are in bijective correspondence with $H^1(M; \mathbb{Z}_2)$ (this is also proved in [BLM89], Theorem 1.7).

**Definition 5.21** A manifold $M$ is called a **spin manifold** if its tangent bundle admits a spin structure.

Using Proposition 5.20, this condition can be translated as follows : $M$ is a spin manifold if and only if $w_1(TM) = 0 = w_2(TM)$.

### 5.2.2 Clifford and spinor bundles

Supose once again that $\pi : E \to M$ is a real rank $n$ oriented vector bundle and let $P_{\mathrm{SO}(n)}E$ be its oriented frame bundle. Recall that, given any continuous homomorphism $\rho$ from the latter's structure group $\mathrm{SO}(n)$ into the diffeomorphism group of some other smooth manifold $F$, we may build the associated bundle $P_{\mathrm{SO}(n)}E \times_\rho F$ (see Section 1.3.3 for the detailed construction). Recall also that each element of O(n) induces an automorphism $\rho_n$ of the Clifford algebra $\mathrm{C}\ell_n$. This map $\rho_n$ then obviously restricts to a homomorphism

$$c\ell(\rho_n) : \mathrm{SO}(n) \to \mathrm{Aut}(\mathrm{C}\ell_n),$$

and we may therefore define

**Definition 5.22** The **Clifford bundle** $\mathrm{C}\ell(E)$ of the oriented bundle $E$ is the associated bundle

$$\mathrm{C}\ell(E) = P_{\mathrm{SO}(n)}E \times_{c\ell(\rho_n)} \mathrm{C}\ell_n.$$

The bundle $P_{\mathrm{SO}(n)}E$ depends on the bundle metric $g$ with which $E$ itself is endowed and, in fact, the fiber over $x \in M$ of the Clifford bundle $\mathrm{C}\ell(E)$ coincides with the Clifford algebra $\mathrm{C}\ell(E_x)$ constructed on $E_x$ using the quadratic form induced by $g_x$.

**Remark 5.23** The Clifford bundle $\mathrm{C}\ell(E)$ actually forms a bundle of algebras. Note that this additional structure on the fibers (known as **Clifford multiplication**) also translates to the level of sections.

Most of the interesting constructions associated to Clifford algebra, such as those we presented in Section 5.1, can be remodeled to fit the setting of the Clifford bundle. This is, of course, not surprising. The first example of this phenomenon is that we can decompose $\mathrm{C}\ell(E)$ as

$$\mathrm{C}\ell(E) = \mathrm{C}\ell^0(E) \oplus \mathrm{C}\ell^1(E).$$

For this, we need only realize that the algebra automorphism $\alpha$ described previously actually extends to a bundle automorphism

$$\alpha : \mathrm{C}\ell(E) \to \mathrm{C}\ell(E).$$

Recall now that another important aspect of the theory of Clifford algebras was the study of their (irreducible) representations (see Section 5.1.4). Although in our setting the Clifford bundle can always be constructed, we shall see that the of existence a bundle of irreducible Clifford modules depends on the vanishing of the second Stiefel-Whitney class.
Let $E$ be an oriented riemannian vector bundle of rank $n$ over some manifold $M$ and suppose that

$$\xi : P_{\mathrm{Spin}(n)}E \to P_{\mathrm{SO}(n)}E$$

is a spin structure on $E$.

**Definition 5.24** A **real spinor bundle** of $E$ is an associated bundle of the form

$$S(E) = P_{\mathrm{Spin}(n)}E \times_{\mu} A,$$

where $A$ is a left-module for $\mathrm{C}\ell_n$ and $\mu : \mathrm{Spin}(n) \to \mathrm{SO}(A)$ is the representation induced by the left multiplication by elements of $\mathrm{Spin}(n) \subseteq \mathrm{C}\ell_n^0$.

In a similar way, we define **complex spinor bundles** of $E$, simply by replacing $A$ with a complex $(\mathrm{C}\ell_n \otimes \mathbb{C})$-module $A_{\mathbb{C}}$ and adjusting the definition of the representation $\mu$ accordingly.

**Remark 5.25** If the module of the above definition is $\mathbb{Z}_2$-graded, we say that the spinor bundle itself is $\mathbb{Z}_2$-graded as well.

To form a first example of such a spinor bundle, we might consider $\mathrm{C}\ell_n$ as a module over itself and therefore build the following bundle

$$\mathrm{C}\ell_{\mathrm{Spin(n)}}E = P_{\mathrm{Spin}(n)}E \times_{\ell} \mathrm{C}\ell_n,$$

where $\ell : \mathrm{Spin}(n) \to \mathrm{SO}(\mathrm{C}\ell_n)$ corresponds to left multiplication on $\mathrm{C}\ell_n$.

**Remark 5.26** This example is not inconsequential. From the inclusion $\mathrm{Spin}(n) \subset \mathrm{C}\ell_n$, we deduce the existence of an embedding $P_{\mathrm{Spin}(n)}E \subset \mathrm{C}\ell_{\mathrm{Spin(n)}}E$. This holds for any bundle $E$ and it follows that every spinor bundle of $E$ stems from this "general prototype" $\mathrm{C}\ell_{\mathrm{Spin(n)}}E$.

Although they share the same model space for their fibers, the spinor bundle $\mathrm{C}\ell_{\mathrm{Spin(n)}}E$ and the Clifford bundle $\mathrm{C}\ell(E)$ are not the same. Recall that we defined $\mathrm{C}\ell(E)$ as an associated bundle of $P_{\mathrm{SO}(n)}E$, using the homomorphism

$$c\ell(\rho_n) : \mathrm{SO}(n) \to \mathrm{Aut}(\mathrm{C}\ell_n).$$

Note that we may think of $c\ell(\rho_n)$ as induced by the **adjoint representation**

$$\mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{Aut}(\mathrm{C}\ell_n),$$

given by $\mathrm{Ad}_g(\varphi) = g\varphi g^{-1}$ for $\varphi \in \mathrm{C}\ell_n$ and $g \in \mathrm{Spin}(n)$. In fact, we have the following

**Theorem 5.27** Let $S(E)$ be a spinor bundle over $E$. Then $S(E)$ is a bundle of modules over the bundle of algebras $\mathrm{C}\ell(E)$. In particular, the sections of the spinor bundle (known as **spinor fields**) form a module over the sections of the Clifford bundle.

**Proof :** Let $A$ be a left-module of $\mathrm{C}\ell_n$ and let the map

$$\mu : P_{\mathrm{Spin}(n)}E \times \mathrm{C}\ell_n \times A \to P_{\mathrm{Spin}(n)}E \times A$$

be defined in the natural way by $\mu(p, \varphi, a) = (p, \varphi \cdot a)$. For any element $g \in \mathrm{Spin}(n) \subset \mathrm{C}\ell_n$, we now consider the maps

$$\rho_g : P_{\mathrm{Spin}(n)}E \times \mathrm{C}\ell_n \times A \to P_{\mathrm{Spin}(n)}E \times \mathrm{C}\ell_n \times A$$

given by $(p, \varphi, a) \mapsto (p.g^{-1}, \mathrm{Ad}_g(\varphi), g \cdot a)$ and

$$\rho'_g : P_{\mathrm{Spin}(n)}E \times A \to P_{\mathrm{Spin}(n)}E \times A$$

sending $(p, \varphi, a)$ to $(p.g^{-1}, g{\cdot}a)$. Direct computations then clearly show that the diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}E \times \mathrm{C}\ell_n \times A & \xrightarrow{\mu} & P_{\mathrm{Spin}(n)}E \times A \\ \downarrow{\scriptstyle \rho_g} & & \downarrow{\scriptstyle \rho'_g} \\ P_{\mathrm{Spin}(n)}E \times \mathrm{C}\ell_n \times A & \xrightarrow{\mu} & P_{\mathrm{Spin}(n)}E \times A \end{array}$$

is commutative. Hence, $\mu$ descends to a map $\mathrm{C}\ell(E) \oplus S(E) \to S(E)$ which, as claimed, allows us to see $S(E)$ as a bundle of modules over the Clifford bundle. $\square$

**Remark 5.28** Analogous results hold also for the cases of complex and $\mathbb{Z}_2$-graded spinor bundles over $E$.

As always, we may be interested in the classification of spinor bundles, that is, using our last result, of bundle of $\mathrm{C}\ell(E)$-modules. With this aim in mind, we define

**Definition 5.29** Two spinor bundles of $E$ are said to be **equivalent** if they are equivalent as bundles of $\mathrm{C}\ell(E)$-modules. A spinor bundle is called **irreducible** if the fiber over each point $x$ of the base manifold is an irreducible $\mathrm{C}\ell(E_x)$-module.

Since every $\mathrm{C}\ell_n$-modules can be decomposed as a direct sum of irreducible modules (see Proposition 5.12), and since the equivalence classes of such objects are well-known, we easily conclude the following

**Proposition 5.30** Every spinor bundle of $E$ can be decomposed as a direct sum of irreducible ones.

In fact, provided that the base manifold $M$ is connected, we can even compute the number of equivalence classes of irreducible spinor bundles of $E$. This number depends solely on the dimension $n$ of $E$ and can be read from the following table

| $n$ mod 8 | Real ungraded | Complex ungraded | Real graded | Complex graded |
|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 1 |
| 2 | 1 | 1 | 1 | 2 |
| 3 | 2 | 2 | 1 | 1 |
| 4 | 1 | 1 | 2 | 2 |
| 5 | 1 | 2 | 1 | 1 |
| 6 | 1 | 1 | 1 | 2 |
| 7 | 2 | 2 | 1 | 1 |
| 8 | 1 | 1 | 2 | 2 |

Note, in particular, that there are never more than two of these classes. See [BLM89, Proposition 3.9, page 98] for more details.

The $\mathbb{Z}_2$-graded case will be of particular interest to us later on. Let us therefore already study some of its properties. We first notice that if

$$S(E) = S^0(E) \oplus S^1(E)$$

is a bundle of irreducible $\mathbb{Z}_2$-graded modules over $\mathrm{C}\ell(E) = \mathrm{C}\ell^0(E) \oplus \mathrm{C}\ell^1(E)$, then $S^0(E)$ is itself a bundle of irreducible modules over $\mathrm{C}\ell^0(E)$. Similarly, if we start with a bundle $S^0(E)$ of irreducible modules over $\mathrm{C}\ell^0(E)$, then we may define

$$S(E) = \mathrm{C}\ell(E) \otimes_{\mathrm{C}\ell^0(E)} S^0(E),$$

which turns out to be a bundle of irreducible modules over $\mathrm{C}\ell(E)$. Such splittings of spinor bundles will prove very useful in later sections. Their explicit construction, which we will present shortly (see Section 5.2.4), depends on the nature (complex or real) of the spinor bundle and relies on a **volume element** (see Definition 5.8).

For now, we have to turn to the study of connections on spinor bundles, which will play a crucial role for the introduction of Dirac operators in Section 5.2.4.

### 5.2.3 Connections on spinor bundles

In Section 2.2.1, we introduced the notion of a connection on a principal bundle. Suppose then that we are given a connection 1-form $\omega$ on a principal $G$-bundle $P$ over $M$. Let $\rho : G \to \mathrm{SO}(n)$ be a Lie group homomorphism which, for technical reasons, we assume to be injective and define the associated real vector bundle of rank $n$

$$E_\rho = P \times_\rho \mathbb{R}^n.$$

We may then build another principal bundle by setting

$$P_{\mathrm{SO}(n)} E_\rho = P \times_\rho \mathrm{SO}(n).$$

Here, recall that an element $g \in G$ acts on $\mathrm{SO}(n)$ through multiplication by $\rho(g)$. Observe now that $\omega$ can be trivially extended to $P \times \mathrm{SO}(n)$, and that $P_{\mathrm{SO}(n)}E$ is the quotient of this cartesian product by the action of $G$ we just described. Pushing forward with the projection, we obtain a connection 1-form $\omega_\rho$ on the principal bundle $P_{\mathrm{SO}(n)}E$. The map $i : P \to P_{\mathrm{SO}(n)}E$ given by $p \mapsto [(p, e)]$, where $[(p, x)]$ denotes the equivalence class of $(p, x) \in P \times \mathrm{SO}(n)$ under the projection to $P_{\mathrm{SO}(n)}E$ and $e \in G$ is the neutral element, then yields an embedding $P \subset P_{\mathrm{SO}(n)}E$.

**Proposition 5.31** Considering $P$ as embedded within $P_{\mathrm{SO}(n)}E$, we have

$$\omega_\rho|_P = \rho_* \omega,$$

where $\rho_* : \mathfrak{g} \to \mathfrak{so}(n)$ is the differential of $\rho$ at $e \in G$.

**Proof :** This follows from the easily checked fact that the embedding $i$ is $G$-equivariant. See [BLM89], Proposition 4.7 on page 106. □

Using this fact, we will now be able to transport the connections defined on the total space of vector bundles to the associated Clifford and spinor bundles.

Let $\pi : E \to M$ be a real and oriented vector bundle of rank $n$. Suppose that its oriented frame bundle $P_{\mathrm{SO}(n)}E$ comes with a connection 1-form $\omega$. Since the Clifford bundle of $E$ can be written as

$$\mathrm{C}\ell(E) = P_{\mathrm{SO}(n)}E \times_{c\ell(\rho_n)} \mathrm{C}\ell_n$$

that is, as an associated bundle of $P_{\mathrm{SO}(n)}E$, the proposition above ensures the existence and uniqueness of a corresponding connection $\omega'$ on $\mathrm{C}\ell(E)$.

**Remark 5.32** It can be shown that the covariant derivative $\nabla$ induced by this connection on the Clifford bundle acts as a derivation on the algebras of sections, and that it preserves the sub-bundles $\mathrm{C}\ell^0$ and $\mathrm{C}\ell^1$. See [BLM89], Proposition 4.8 and Corollary 4.9 on page 107 for more details.

Suppose that the oriented bundle $E$ admits a spin structure, with two-sheeted covering

$$\xi : P_{\mathrm{Spin}(n)}E \to P_{\mathrm{SO}(n)}E.$$

The previously described connection $\omega'$ on $P_{\mathrm{SO}(n)}E$ can be lifted through $\xi$ to a connection $\omega''$ on $P_{\mathrm{Spin}(n)}E$. Let now $S$ be a spinor bundle of $E$. Recall that, by definition, $S$ can be written as an associated bundle of $P_{\mathrm{Spin}(n)}E$ in the following way

$$S = P_{\mathrm{Spin}(n)}E \times_\mu A,$$

where $A$ is a left-module over $\mathrm{C}\ell_n$ and $\mu : \mathrm{Spin}(n) \to \mathrm{SO}(A)$ is a Lie group homomorphism. It follows that $\omega''$ determines a unique connection 1-form on $S$. We denote the associated covariant derivative and curvature operator by $\nabla^S$ and $R^S$ respectively.

**Remark 5.33** It turns out that $\nabla^S$ acts as a derivation with respect to the module structure on $C\ell(E)$. In other words :

$$\nabla^S(\varphi \cdot \mu) = \nabla(\varphi) \cdot \mu + \varphi \cdot \nabla^S(\mu), \tag{5.3}$$

whenever $\varphi \in \Gamma(C\ell(E))$ and $\mu \in \Gamma(S)$. See [BLM89], Proposition 4.11 on page 108 for the details. In fact, it can also be shown (see [BLM89], Proposition 5.16 on page 37), that

$$\langle e\mu, e\lambda \rangle = \langle \mu, \lambda \rangle \tag{5.4}$$

whenever $e$ is a unit vector in the tangent plane and $\mu, \lambda \in \Gamma(S)$.

The following result will be important for the computations necessary to prove Theorem 5.45 in Section 5.2.5 :

**Proposition 5.34** Let $p \in M$ be any point and let $X, Y \in T_pM$. Then the curvature transformation on the fiber $S_p$

$$R^S_{X,Y} : S_p \to S_p$$

is given by

$$\begin{aligned} R^S_{X,Y}(\sigma) &= \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle R_{X,Y}(e_i), e_j\rangle e_i \cdot e_j \cdot \sigma \\ &= \frac{1}{2}\sum_{i<j}\langle R_{X,Y}(e_i), e_j\rangle e_i \cdot e_j \cdot \sigma, \end{aligned}$$

where $R_{X,Y}$ is the curvature transformation on the fiber $E_p$.

**Proof :** See [BLM89], Proposition 4.15 on page 110. □

### 5.2.4 Dirac operators

We start by introducing these operators in the more general context of a bundle of left-modules, before studying the consequences in the particular case of spinor bundles.

Let $M$ be an $n$-dimensional Riemannian manifold and let $S$ be a bundle of left-modules over its Clifford bundle $C\ell(TM)$, that is, a vector bundle $S \to M$ such that the fiber $S_x$ over every point $x \in M$ is a left-module on the algebra $C\ell(TM)_x = C\ell(T_xM)$. Suppose that $S$ is endowed with a Riemmannian metric and a Riemannian connection and let $\nabla$ denote the covariant derivative on $S$ that they induce.

**Definition 5.35** The **Dirac operator** is a differential operator $D : \Gamma(S) \to \Gamma(S)$

of order 1 given by

$$D\sigma = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\sigma$$

for every $\sigma \in \Gamma(S)$, where $e_1, \dots, e_n$ form an orthonormal basis of $T_xM$ and $\cdot$ symbolizes the Clifford module multiplication. The composition

$$D^2 = D \circ D : \Gamma(S) \to \Gamma(S)$$

is then also a differential operator, which we call the **Dirac Laplacian**.

It is clear how the choice of a metric on any vector bundle $E \to M$ induces an isomorphism between $E$ and its dual $E^*$. In particular, since $M$ is assumed to be Riemannian, the tangent and cotangent bundles are isomorphic. We may therefore consider tangent vectors to discuss the principal symbol of the Dirac operator $D$, which we compute now.

**Lemma 5.36** For every $\xi \in TM \cong T^*M$, we have $\sigma_\xi(D) = i\xi$ and $\sigma_\xi(D^2) = ||\xi||$.

In other words, if we write $p : TM \to M$ for the tangent bundle projection, it follows that the map $\sigma_\xi(D) : p^*S \to p^*S$ corresponds to the Clifford multiplication on the left by $i\xi$ and that, similarly, the map $\sigma_\xi(D^2) : p^*S \to p^*S$ is given by the scalar multiplication with $||\xi||$. In particular, $D$ is elliptic.

**Proof :** The symbol of $D$ is computed by choosing an appropriate trivialization of $S$ aroung a point $p \in M$ and observing that the highest order terms in $\nabla_{e_j}$ is of order 1. Note that once it is established that $\sigma_\xi(D) = i\xi$, then

$$\sigma_\xi(D^2) = \sigma_\xi(D) \circ \sigma_\xi(D) = -\xi \cdot \xi = ||\xi||^2$$

as expected. □

We now combine the different notions presented above into the following

**Definition 5.37** A **Dirac bundle** over a Riemannian manifold $M$ is a bundle $S$ of left-modules over $\mathrm{C}\ell(TM)$, endowed with a Riemannian metric and an associated connection satisfying the conditions (5.3) and (5.4) above.

Note that every Dirac bundle comes with its own, canonically defined Dirac operator. Using the Riemannian metric $\langle \cdot, \cdot \rangle$ of the base manifold, we may also define an inner product $(\cdot, \cdot)$ on the space of sections $\Gamma(S)$ as follows

$$(\sigma, \mu) = \int_M \langle \sigma(x), \mu(x) \rangle_x \, dx.$$

Let us now point out some last properties satisfied by every Dirac bundle before we introduce the specific case we are interested in :

**Lemma 5.38** Let $D$ be the Dirac operator of any Dirac bundle $S$ over a Riemannian manifold $M$. Then

1. $D$ is (formally) self-adjoint, that is, for every compactly supported sections $\sigma, \mu \in \Gamma(S)$ we have $(D\sigma, \mu) = (\sigma, D\mu)$ and
2. the kernel of $D$ and that of the Dirac Laplacian are equal ($\ker D = \ker D^2$), both being finite dimensional.

**Proof :** The formal self-adjointness can be shown by direct computations, using the properties 5.3 and 5.4. See [BLM89], Proposition 5.3 on page 114 for a detailed argument. For 2., it is clear that $\ker D$ and $\ker D^2$ are finite dimensional since both $D$ and $D^2$ are elliptic (see Theorem 4.4). Now, if $\sigma \in \Gamma(S)$ is such that $D^2\sigma = 0$, then, by 1., we find

$$||D\sigma||^2 = (D\sigma, D\sigma) = (\sigma, D^2\sigma) = 0$$

and, therefore, $D\sigma = 0$ as well. Hence, $\ker D^2 \subseteq \ker D$. The reverse inclusion is obvious. □

Suppose now that $M$ is a spin manifold and that $S$ is a spinor bundle over $M$. Then, as shown in Theorem 5.27, $S$ is indeed a bundle of left-modules over the Clifford bundle $C\ell(TM)$, such as we have been considering for this whole section. Moreover, we known that it carries a canonical Riemannian connection which satisfies the conditions (5.3) and (5.4) (see Section 5.2.3). In this setting, the Dirac operator is known as the **Atiyah-Singer operator**.

The specific case we are interested in, are spinor bundles over $4k$-dimensional, compact and oriented manifolds. Let $M$ be such a space. Note first that the table given at the end of Section 5.2.2 assures us that there is precisely one complex (ungraded) spinor bundle of its tangent bundle (for $TM$ would then be $8k$-dimensional), which we denote by $\mathbb{S}_{\mathbb{C}}$. We write $\mathbb{D}_{\mathbb{C}}$ for the associated Dirac operator. We define a global section $\omega$ of the complexified Clifford bundle $C\ell(TM) \otimes \mathbb{C}$ by setting

$$\omega_{\mathbb{C}}(x) = (-1)^k e_1 \cdot \ldots \cdot e_{4k},$$

where $\{e_1, \ldots, e_{4k}\}$ forms an orthonormal basis of the tangent space over $x \in M$. This section is parallel, that is, $\nabla\omega_{\mathbb{C}} = 0$ (where $\nabla$ is the canonical Riemannian connection on the Clifford bundle), and also satisfies the following properties :

$$\omega_{\mathbb{C}}^2 = 1 \quad \text{and} \quad \omega_{\mathbb{C}} \cdot e = -e \cdot \omega_{\mathbb{C}},$$

for every $e \in TM$. This construction induces a natural bundle isomorphism

$$\lambda_{\omega_{\mathbb{C}}} : \mathbb{S}_{\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$$

given by the Clifford multiplication by $\omega_{\mathbb{C}}$, which splits $\mathbb{S}_{\mathbb{C}}$ as the direct sum of its $\pm 1$-eigenbundles. We may therefore write

$$\mathbb{S}_{\mathbb{C}} = \mathbb{S}_{\mathbb{C}}^+ \oplus \mathbb{S}_{\mathbb{C}}^-.$$

In other words, thanks to this volume element, we may always assume that the spinor bundle $\mathbb{S}$ is $\mathbb{Z}_2$-graded. In fact, we know that

$$\mathbb{S}^{\pm}_{\mathbb{C}} = (1 \pm \omega_{\mathbb{C}})\mathbb{S},$$

from which, using the properties above, we deduce that

$$e \cdot \mathbb{S}^{\pm}_{\mathbb{C}} \subseteq \mathbb{S}^{\mp}_{\mathbb{C}},$$

for every tangent vector $e$.

The clearest benefit of this additional grading structure on $\mathbb{S}_{\mathbb{C}}$ is that the Dirac operator $\mathbb{D}_{\mathbb{C}}$ itself can be decomposed as

$$\mathbb{D}_{\mathbb{C}} = \begin{pmatrix} 0 & \mathbb{D}^{-}_{\mathbb{C}} \\ \mathbb{D}^{+}_{\mathbb{C}} & 0 \end{pmatrix},$$

with $\mathbb{D}^{+}_{\mathbb{C}} : \mathbb{S}^{+}_{\mathbb{C}} \to \mathbb{S}^{-}_{\mathbb{C}}$ and $\mathbb{D}^{-}_{\mathbb{C}} : \mathbb{S}^{-}_{\mathbb{C}} \to \mathbb{S}^{+}_{\mathbb{C}}$, which turn out to be adjoints of each other (see Lemma 5.38). The ellipticity of $\mathbb{D}_{\mathbb{C}}$ descends to both operators $\mathbb{D}^{\pm}_{\mathbb{C}}$ and, in fact, their principal symbols

$$\sigma_{\xi}(\mathbb{D}^{\pm}_{\mathbb{C}}) : \mathbb{S}^{\pm}_{\mathbb{C}} \to \mathbb{S}^{\mp}_{\mathbb{C}}$$

are both given by the Clifford multiplication on the left by $i\xi$. Since $M$ is assumed to be compact, Theorem 4.4 ensures that the kernels and cokernels of $\mathbb{D}^{+}_{\mathbb{C}}$ and $\mathbb{D}^{-}_{\mathbb{C}}$ are finite dimensional. Note that their adjointness implies that

$$\ker \mathbb{D}^{-}_{\mathbb{C}} = \operatorname{coker} \mathbb{D}^{+}_{\mathbb{C}}$$

from which we deduce for example that the analytical index of $\mathbb{D}^{+}_{\mathbb{C}}$ can be rewritten as follows

$$\text{a-Ind}(\mathbb{D}^{+}_{\mathbb{C}}) = \dim \ker \mathbb{D}^{+}_{\mathbb{C}} - \dim \ker \mathbb{D}^{-}_{\mathbb{C}}. \tag{5.5}$$

It is however obvious that

$$\ker \mathbb{D}_{\mathbb{C}} = \ker \mathbb{D}^{+}_{\mathbb{C}} \oplus \ker \mathbb{D}^{-}_{\mathbb{C}}.$$

Thus, in the event that $\ker \mathbb{D}_{\mathbb{C}} = 0$, the kernels of both $\mathbb{D}^{+}_{\mathbb{C}}$ and $\mathbb{D}^{-}_{\mathbb{C}}$ are trivial as well, so that the equation (5.5) above yields $\text{a-Ind}(\mathbb{D}^{+}_{\mathbb{C}}) = 0$.

As a consequence of the expression for the roots of the half spin representations given in Remark 5.16, we have that

$$i^{*}(\operatorname{ch}(\sigma(\mathbb{D}^{+}_{\mathbb{C}}))) = \operatorname{ch}(\mathbb{S}^{+}_{\mathbb{C}}) - \operatorname{ch}(\mathbb{S}^{-}_{\mathbb{C}}) = \prod_{i=1}^{n} \left( e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}} \right),$$

where $i : M \to TM$ denotes the zero section of the tangent bundle of the manifold $M$ over which the spinor bundles $\mathbb{S}^{\pm}_{\mathbb{C}}$ are defined. Therefore, as the Euler class corresponds to the product of the (formal) Chern roots $x_1, \dots, x_n$ (see Remark 1.84), the Equation (4.1) gives us

$$\begin{aligned}\pi_!\left(\mathrm{ch}(\sigma(\mathbb{D}^+_{\mathbb{C}}))\right) \cdot \hat{\mathcal{A}}(M)^2 &= \frac{\prod_{i=1}^n \left(e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}\right)}{\prod_{i=1}^n x_i} \cdot \prod_{i=1}^n \left(\frac{x_i}{e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}}\right)^2 \\ &= \prod_{i=1}^n \frac{x_i}{e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}} \\ &= \hat{\mathcal{A}}(M).\end{aligned}$$

Now, since $\hat{\mathcal{A}}(M)[M] = \hat{A}(M)$, using Theorem 4.9, we conclude that

$$\text{t-Ind}(\mathbb{D}^+_{\mathbb{C}}) = \hat{A}(M).$$

Indeed, as we assumed that the dimension $n$ of $M$ was doubly even, we find that the coefficient $(-1)^{\frac{n(n+1)}{2}}$ in (4.2) is simply equal to 1.

Although, unfortunately, most of the magic behind this fact has to happen off page, Theorem 4.10 then immediately yields the following :

$$\text{a-Ind}(\mathbb{D}^+_{\mathbb{C}}) = \text{t-Ind}(\mathbb{D}^+_{\mathbb{C}}) = \hat{A}(M). \tag{5.6}$$

Note that the justification of this Atiyah-Singer theorem requires a lot of work and lengthy computations, which are, for example, carried out over the third chapter of [BLM89].

**Remark 5.39** The hypothesis on the dimension of the manifold ($\dim M = 4k$) that we made earlier can even be relaxed, and there is an extension of the fact above to the index of so-called $\mathrm{C}\ell_n$-linear operators (see [BLM89], Theorem 7.10 on page 149).

### 5.2.5 Bochner identity and Lichnerowicz formula

Let $M$ be a compact and oriented Riemannian manifold, whose tangent bundle is endowed with the Levi-Civita covariant derivative inherited from the canonical Riemannian connection 1-form $\omega$ on $P_{\mathrm{SO}(n)}TM$. Let $S$ be a spinor bundle over $M$, equipped with the covariant derivative $\nabla$ and curvature operator $R$ induced by the lifting of $\omega$ as described in Section 5.2.3.

**Definition 5.40** If $X$ and $Y$ are tangent vector fields on $M$, then the **invariant second derivative**

$$\nabla^2_{X,Y} : \Gamma(S) \to \Gamma(S)$$

is given by

$$\nabla^2_{X,Y}\mu = \nabla_X \nabla_Y \mu - \nabla_{\nabla_X Y}\mu.$$

Note that, somewhat confusingly, the term $\nabla_X Y$ in the definition above refers to the (Levi-Civita) covariant derivative on the tangent bundle, and therefore corresponds to a vector field on $M$. It can be shown that $(\nabla^2_{[X,Y]}\mu)(p)$ only depends on the values of $X, Y$ and $\mu$ at the point $p \in M$. On the other hand, if we fix a section $\mu \in \Gamma(S)$, we observe that

$$\nabla^2_{\cdot,\cdot}\mu \in \Gamma(T^*M \otimes T^*M \otimes S)$$

or, in other words, that $\nabla^2_{\cdot,\cdot}\sigma$ is a bilinear form on $TM$ taking its values in $S$.

**Definition 5.41** The **connection Laplacian**

$$\nabla^*\nabla : \Gamma(S) \to \Gamma(S)$$

is given by the opposite of the trace of the bilinear form induced by the invariant second derivative, that is

$$\nabla^*\nabla\mu = -\sum_{j=1}^{n} \nabla^2_{e_j,e_j}\mu$$

for every $\mu \in \Gamma(S)$, where the (local) vector fields $e_j$ form a local orthonormal frame of the tangent bundle $TM$.

Note that $\nabla^*\nabla$ is therefore a differential operator in the sense of Chapter 4. Its symbol at a given cotangent vector $\xi \in T^*M$ is given by $\sigma_\xi(\nabla^*\nabla) = ||\xi||^2$, and it follows immediately from this computation that the connection Laplacian is elliptic. Moreover, we have the following formal self-adjointness property :

**Proposition 5.42** Let $\mu, \lambda \in \Gamma(S)$. Then if at least one of these two sections has compact support on $M$, we have

$$(\nabla^*\nabla\mu, \lambda) = (\nabla\mu, \nabla\lambda).$$

**Proof :** First, let $X \in \Gamma(TM)$ be any vector field on $M$. Notice that, for $\mu, \lambda \in \Gamma(S)$ fixed, the condition

$$\langle Y, X\rangle = \langle \nabla_X \mu, \lambda\rangle$$

uniquely characterizes a well-defined vector field $Y \in \Gamma(TM)$. Let us now fix a point $p \in M$ and consider a local orthonormal frame $e_1, \ldots, e_n$ of $T_pM$, given by parallel (local) fields. Then, by definition of the connection Laplacian, we get

$$\langle \nabla^*\nabla\mu, \lambda\rangle = -\sum_{j=1}^{n}\langle \nabla^2_{e_j,e_j}\mu, \lambda\rangle = -\sum_{j=1}^{n}\langle \nabla_{e_j}\nabla_{e_j}\mu, \lambda\rangle,$$

since $\nabla_{e_j} e_j = 0$ by assumption. Using the fact that $\nabla$ is Riemannian, we find

$$\sum_{j=1}^{n} \langle \nabla_{e_j} \nabla_{e_j} \mu, \lambda \rangle = \sum_{j=1}^{n} \left( e_j \langle \nabla_{e_j} \mu, \lambda \rangle - \langle \nabla_{e_j} \mu, \nabla_{e_j} \lambda \rangle \right).$$

Recall now the definition of the field $Y$, and observe that

$$\operatorname{div}(Y) = \sum_{j=1}^{n} \langle \nabla_{e_j} Y, e_j \rangle = \sum_{j=1}^{n} e_j \langle Y, e_j \rangle = \sum_{j=1}^{n} e_j \langle \nabla_{e_j} \mu, \lambda \rangle,$$

where the second equality follows once again from the fact that $\nabla$ is Riemannian. By identification, we deduce that

$$\langle \nabla^* \nabla \mu, \lambda \rangle = -\operatorname{div}(Y) + \langle \nabla \mu, \nabla \lambda \rangle$$

and the statement of the theorem then follows by integrating over the whole base manifold $M$. Note that, here, we resorted to the **divergence theorem** and the fact that $\partial M$ is empty. □

In the compact case, which is our setting, this means that a section is globally parallel ($\nabla \mu = 0$) if and only if $\nabla^* \nabla \mu = 0$.

**Remark 5.43** The operator $D^2$ is sometimes called the **spinorial Laplacian** (in opposition to the connection Laplacian $\nabla^* \nabla$ we just introduced) and the elements of its kernel are therefore naturally known as **harmonic spinors**.

A priori, the difference between these Laplacians, $D^2 - \nabla^* \nabla$, has to be a differential operator $\Gamma(S) \to \Gamma(S)$ of order at most one. At the end of this section, however, we will show that it is actually of order 0 and, moreover, that it can be expressed in terms of the (scalar) curvature of $M$. To prove this, we must first consider the intermediate statement known as the Bochner identity.
Let $\mu \in \Gamma(S)$ and define

$$\mathfrak{R}(\mu) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} e_i \cdot e_j \cdot R_{e_i, e_j}(\mu),$$

where $R_{X,Y}$ is the curvature transformation of $S$ associated to the vector fields $X$ and $Y$ on $M$, the (local) vector fields $\{e_1, \ldots, e_n\}$ form a local orthonormal frame and $\cdot$ is the Clifford multiplication. This object $\mathfrak{R}$ can, more formally, be seen as a section of the bundle $\operatorname{Hom}(S, S)$. Note that, thanks to the symmetries of the curvature operator

$R$, we may write

$$\begin{aligned}\mathfrak{R} &= \frac{1}{2}\left(\sum_{k=1}^{n} e_k \cdot e_k \cdot R_{e_k,e_k} + \sum_{i\neq j} e_i \cdot e_j \cdot R_{e_i,e_j}\right)\\ &= \frac{1}{2}\left(0 + 2\cdot\sum_{i<j} e_i \cdot e_j \cdot R_{e_i,e_j}\right)\\ &= \sum_{i<j} e_i \cdot e_j \cdot R_{e_i,e_j}.\end{aligned}$$

We may now state this so-called Bochner identity :

**Theorem 5.44** Consider a Dirac bundle $S$ with connection laplacian $\nabla^*\nabla$ and Dirac operator $D$. Then the following relation holds :

$$D^2 = \nabla^*\nabla + \mathfrak{R}.$$

**Proof :** Let $e_1, \ldots, e_n$ be local parallel vector fields around some fixed point of the base manifold, such that they form a local orthonormal frame of its tangent bundle. Using the definition of the Dirac bundle $D$ and the fact that the connection $\nabla$ on $S$ acts as a derivation (see Remark 5.33), we find :

$$\begin{aligned}D^2 &= \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\left(\sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \cdot \nabla_{e_i}(e_j \cdot \nabla_{e_j})\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \cdot \left(\nabla_{e_i} e_j \cdot \nabla_{e_i} + e_j \cdot \nabla_{e_i}\nabla_{e_j}\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \cdot e_j \cdot \nabla_{e_i}\nabla_{e_j}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \cdot e_j \cdot \nabla^2_{e_i,e_j}\\ &= -\sum_{j=1}^{n} \nabla^2_{e_j,e_j} + \sum_{i<j} e_i \cdot e_j \cdot \left(\nabla^2_{e_i,e_j} - \nabla^2_{e_j,e_i}\right)\\ &= \nabla^*\nabla + \mathfrak{R}.\end{aligned}$$

Note that, in the computations above, we have used the fact that

$$\nabla_{e_i} e_j = 0,$$

and that

$$\nabla^2_{e_i,e_j} - \nabla^2_{e_j,e_i} = R_{e_i,e_j},$$

which holds because the Levi-Civita covariant derivative with which the tangent bundle is endowed is torsion free. □

Recall that we are working on a compact spin manifold $M$ with spinor bundle $S$, which we endow with its canonical Riemannian connection in the manner described in Section 5.2.3. In this setting, the Dirac operator is often called the Atiyah-Singer operator. We are now able to prove the so-called Lichnerowicz formula :

**Theorem 5.45** Let $D$ and $\nabla^*\nabla$ respectively denote the Atiyah-Singer operator and the connection laplacian on $S$. Then the following holds

$$D^2 = \nabla^*\nabla + \frac{1}{4}\kappa,$$

where $\kappa$ is the scalar curvature.

**Proof :** Recall that, in Proposition 5.34, we give a formula for the curvature transformation

$$R^S_{X,Y} : S_p \to S_p$$

over any point $p \in M$, where $X$ and $Y$ are in $T_pM$. Indeed, we have

$$R^S_{X,Y} = \frac{1}{4}\sum_{i,j}\langle R_{X,Y}(e_i), e_j\rangle e_i \cdot e_j$$

whenever $e_1, \ldots, e_n$ forms an orthonormal basis of the tangent space over $p$. It follows from the definition of the section $\mathfrak{R}$ that

$$\mathfrak{R} = \frac{1}{2}\sum_{i,j} e_i \cdot e_j \cdot R^S_{e_i,e_j} = \frac{1}{8}\sum_{i,j,k,\ell}\langle R_{e_i,e_j}(e_k), e_\ell\rangle e_i \cdot e_j \cdot e_k \cdot e_\ell.$$

We now separate the sum in two parts : one where all the indices are distinct, and the other where at least two are the same. Note that the first term has to vanish because of the first Bianchi identity (see Proposition 2.14). Using the fact that

$$e_i \cdot e_j \cdot e_i = e_j \quad \text{and} \quad e_i^2 = -1,$$

and the other properties of the Riemann curvature tensor, as well as the definition of the scalar curvature, we obtain

$$\mathfrak{R} = \frac{1}{4}\sum_{i,j,\ell}\langle R_{e_i,e_j}(e_i), e_\ell\rangle e_j \cdot e_\ell = \frac{1}{4}\kappa$$

where the bounds of the indices have been omitted in most of the sums. □

A direct consequence of the formula above is the following

**Corollary 5.46** Any compact spin manifold of positive scalar curvature admits no harmonic spinors.

**Proof :** Suppose that $\mu \in \Gamma(S)$ is a harmonic spinor, that is, $D\mu = 0$. It follows that

$$-\nabla^*\nabla\mu = \frac{1}{4}\kappa\mu,$$

and thus

$$-(\nabla^*\nabla\mu, \mu) = -(\nabla\mu, \nabla\mu) = \frac{1}{4}(\kappa\mu, \mu)$$

or, in other words

$$\int_M \kappa(x)||\mu(x)||^2\, dx = -\int_M ||\nabla\mu(x)||^2\, dx.$$

However, since we assume that $\kappa > 0$, it follows that $\nabla\mu = 0$ or, in other words, that $||\mu||$ is constant, and that $||\mu|| = 0$, from which we conclude that $\mu = 0$. □

It follows now, although this is less obvious since it relies on applying the Atiyah-Singer index theorem, that

**Theorem 5.47** If a compact spin manifold of doubly even dimension admits a metric of positive scalar curvature, then its $\hat{A}$-genus vanishes.

Indeed, by Corollary 5.46, there would be no harmonic spinor on such a manifold, which means that the kernel of $D^2$, which by the second part of Lemma 5.38 is equal to that of the Dirac operator $D$ itself, is trivial. We could then conclude using (5.6).

**Remark 5.48** In fact, the converse also holds for simply connected spin manifolds of doubly even dimension, that is, if the $\hat{A}$-genus of such a manifold vanishes, then it admits a metric of positive scalar curvature. This is a consequence of Stolz's proof of a conjecture of Gromov and Lawson (see [Sto92]).

There is also a much more generalized statement, involving the more refined $\alpha$-invariant defined by Hitchin in [Hit74]. It states that for a simply connected spin manifold $M$ of dimension $n \geq 5$, the vanishing of $\alpha(M) \in KO^{-n}(\{*\})$ guarantees the existence of a Riemannian metric of positive scalar curvature. Note that this is indeed a generalization of Lichnerowicz's result (Theorem 5.47), since in doubly even dimension, this $\alpha$-invariant is, up to a multiplicative constant, equal to $\hat{A}(M)$. A stable homotopy theoretic proof of this fact can be found in [Sto90].

# 6. Application to hypersurfaces

Here, we study the case of so-called **degree $d$ hypersurfaces** of the complex projective spaces $P^n(\mathbb{C})$. Using well-known formula for their total Chern classes, which we recall in Section 6.2, we derive their $\hat{A}$-genus and thus determine exactly in which case they admit a metric of positive scalar curvature.

## 6.1 Construction of degree $d$ hypersurfaces

Let us begin by reviewing the construction of these spaces. Let $m, d \geq 1$ be integers. Consider a homogeneous polynomial $P : \mathbb{C}^{m+1} \to \mathbb{C}$ of degree $d$ and suppose that

$$\nabla P(z_0, \dots, z_m) \neq 0$$

whenever $(z_0, \dots, z_m) \neq 0$ is such that $P(z_0, \dots, z_m) = 0$. Note that $P^{-1}(\{0\})$ can be seen as a subset of $P^{m+1}(\mathbb{C})$. Indeed, if $P(z_0, \dots, z_m) = 0$ then for every complex number $\lambda$ it also holds by the homogeneity of $P$ that

$$P(\lambda z_0, \dots, \lambda z_m) = \lambda^d P(z_0, \dots, z_m) = 0.$$

We may therefore define

$$V_m^d = \left\{(z_0 : \dots : z_m) \in P^{m+1}(\mathbb{C}) \mid P(z_0, \dots, z_m) = 0\right\}. \tag{6.1}$$

As the notation suggests, this subset only really depends on the degree $d$, and not on the specific polynomial $P$ itself. We will come back to this fact later (see Remark 6.1).

Let us first show that $V_m^d$ actually forms a complex submanifold of codimension one within $P^{m+1}(\mathbb{C})$. To see this, we consider the standard holomorphic atlas of the complex projective space given by the charts $h_i : U_i \to \mathbb{C}^m$, where

$$U_i = \{(z_0 : \dots : z_m) \mid z_i \neq 0\}$$

and

$$h_i(z_0 : \dots : z_m) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_m}{z_i}\right).$$

Next, we define the maps $P_i : \mathbb{C}^m \to \mathbb{C}$ by setting

$$P_i(w_1, \dots, w_m) = P(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m).$$

In the cases $i = 0$ or $i = m$, we simply have

$$P_0(w_1, \dots, w_m) = P(1, w_1, \dots, w_m)$$

and

$$P_m(w_1, \dots, w_m) = P(w_1, \dots, w_m, 1).$$

It is then easy to check that

$$h_i(V_m^d \cap U_i) = P_i^{-1}(\{0\})$$

for all indices $i \in \{0, \dots, m\}$. Our strategy is now to prove that $0 \in \mathbb{C}$ is a regular value of each polynomial $P_i$, from which we will deduce that $P_i^{-1}(\{0\})$ is a submanifold of $\mathbb{C}^m$ of codimension one and, hence, that $P^{-1}(\{0\})$ itself is a codimension one submanifold in each chart. Since being a submanifold is an inherently local condition, we would then be able to conclude.
Let $0 \leq i \leq m$ and suppose that $(w_1, \dots, w_m) \in \mathbb{C}^m$ is such that $P_i(w_1, \dots, w_m) = 0$. By definition, this means that $P(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m) = 0$ as well, although we must have

$$\nabla P(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m) \neq 0.$$

However, for $j \leq i$, we observe that

$$\frac{\partial P_i}{\partial w_j}(w_1, \dots, w_m) = \frac{\partial P}{\partial z_{j-1}}(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m),$$

while, for $j \geq i + 1$, we find

$$\frac{\partial P_i}{\partial w_j}(w_1, \dots, w_m) = \frac{\partial P}{\partial z_j}(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m).$$

Hence, if any of the derivatives $\frac{\partial P}{\partial z_k}(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m)$ is non-zero for some index $k \neq i$, then the differential of $P_i$ at $(w_1, \dots, w_m)$ is surjective and, thus, 0 is a regular value of $P_i$. In this case, we are obviously done.
Note however, that it is actually impossible to have $\frac{\partial P}{\partial z_i}(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m)$ be the only non-zero partial derivative of $P$ at $(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m)$. Indeed, since we know that $P$ vanishes on the whole line

$$\left\{\lambda(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m) \in \mathbb{C}^{m+1} \mid \lambda \in \mathbb{C}\right\},$$

its (directional) derivative in the direction of $(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m)$ must vanish. In other words, we must have

$$\langle \nabla P(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m), (w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m) \rangle = 0.$$

Now, if $\frac{\partial P}{\partial z_i}(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m)$ is the only component of the gradient to be non-zero, the equation above directly yields

$$\frac{\partial P}{\partial z_i}(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m) = 0,$$

which is an obvious contradiction. In fact, we have shown that at least two components $\frac{\partial P}{\partial z_k}(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_m)$ must be non-zero.

**Remark 6.1** It has been shown by Thom that any two homogeneous polynomials $Q_0$ and $Q_1$ of degree $d$ such as $P$ can be connected by a family $\{Q_t\}_{t\in[0,1]}$ of homogeneous polynomials of the same degree $d$, with

$$\nabla Q_t(z_1 0, \dots, z_m) \neq 0$$

whenever $Q_t(z_0, \dots, z_m) = 0$ and $(z_0, \dots, z_m) \neq 0$, such that the induced projection

$$P^{m+1}(\mathbb{C}) \times [0,1] \to [0,1]$$

restricts to a submersion

$$\bigcup_{t\in[0,1]} Q_t^{-1}(\{0\}) \to [0,1]$$

and that $Q_0^{-1}(\{0\})$ is diffeomorphic to $Q_1^{-1}(\{0\})$. In other words, the diffeomorphism type of the hypersurface defined in (6.1) only depends on the degree $d$. As a complex (sub)manifold, $V_m^d$ inherits a canonical orientation which, in fact, does not depend on the choice of the particular polynomial $P$, but only on its degree.

## 6.2 Chern and Pontrjagin classes of $V_m^d$

Consider now the fundamental class $[V_m^d] \in H_{2m}(V_m^d; \mathbb{Z})$ of some degree $d$ hypersurface $V_m^d$ in $P^{m+1}(\mathbb{C})$. Let $x \in H^2(P^{m+1}(\mathbb{C}); \mathbb{Z})$ be the generator mentioned in Theorem 1.71. In particular, $-x$ is the first Chern class of the tautological line bundle over $P^{m+1}(\mathbb{C})$. Using the inclusion map

$$i : V_m^d \to P^{m+1}(\mathbb{C})$$

we define the homology class $i_*([V_m^d]) \in H_{2m}(P^{m+1}(\mathbb{C}); \mathbb{Z})$. Since the projective spaces are compact, Poincaré duality yields isomorphisms of the form

$$H^k(P^{m+1}(\mathbb{C}); \mathbb{Z}) \cong H_{2m-k}(P^{m+1}(\mathbb{C}); \mathbb{Z}),$$

for every $0 \leq k \leq 2m$ and we may therefore consider the Poincaré dual $\mathrm{PD}(i_*([V_m^d]))$ of $i_*([V_m^d])$. This cohomology class turns out to be an integer multiple of $x$. In fact, we have

$$\mathrm{PD}(i_*([V_m^d])) = dx.$$

It can even be shown (see [Hir54], Section 2.1) that

$$\langle x^m, [V_m^d] \rangle = d, \tag{6.2}$$

where we have identified $x \in H^2(P^{m+1}(\mathbb{C}); \mathbb{Z})$ with its pullback $i^*(x) \in H^2(V_m^d; \mathbb{Z})$.

Let now $\nu$ be the normal bundle of $V_m^d$ within $P^{m+1}(\mathbb{C})$ associated to the inclusion map $i$. This forms a complex line bundle and it can be shown (see [MS74], Theorem 11.3 on page 119) that

$$c_1(\nu) = dx \tag{6.3}$$

as well. From there, it follows that the total Chern class of the degree $d$ hypersurface $V_m^d$ is given by

$$c(V_m^d) = (1+x)^{m+2}(1+dx)^{-1}. \tag{6.4}$$

Recall indeed that $c(P^{m+1}(\mathbb{C});\mathbb{Z}) = (1+x)^{m+2}$ ; the expression (6.4) above is then simply a consequence of the Whitney sum formula (see Section 1.4.3) and (6.3). Thanks to the method mentioned in Section 1.5.1, we deduce that its total Pontrjagin class is

$$p(V_m^d) = \left(1+x^2\right)^{m+2}\left(1+(dx)^2\right)^{-1}. \tag{6.5}$$

In particular, we can directly compute the first Chern class and obtain

$$c_1(V_m^d) = (2n+2-d)x, \tag{6.6}$$

which we will use at the end of the next section.

## 6.3 The $\hat{A}$-genus of $V_m^d$

Having obtained an explicit formula for their total Pontrjagin class, we are now able to compute the $\hat{A}$-genus of any degree $d$ hypersurface $V_m^d \subset P^{m+1}(\mathbb{C})$.

**Remark 6.2** Note here that $V_m^d$ is a complex manifold of dimension $m$ and, as such, a real manifold of dimension $2m$. In particular, we already know (see Definition 3.11) that its $\hat{A}$-genus vanishes if $m$ itself is not even. It is therefore quite natural to suppose that $m = 2n$ for some integer $n \geq 1$ and, hence, that $V_m^d$ is a $4n$-dimensional real manifold.

To simplify the notations in the following computations, let us write $X = V_{2n}^d$ and

$$p = 1 + p_1 + \ldots + p_n$$

for the (total) Pontrjagin classes of $X$. Recall that the $\hat{A}$-genus corresponds to the power series

$$Q(x) = \frac{\frac{x}{2}}{\sinh\left(\frac{x}{2}\right)}$$

and let us denote, as in Example 3.17, the associated multiplicative sequence by $\{\hat{A}_k\}_{k\geq 1}$. By multiplicativity, and thanks to formula (6.5), we find

$$\hat{A}(p) = \hat{A}\left(1+x^2\right)^{2n+2} \cdot \hat{A}\left((1+(dx)^2)\right)^{-1}. \tag{6.7}$$

Recall here that

$$\hat{A}\left(1+x^2\right) = Q(x)$$

and that the $\hat{A}$-genus of $X$ is given by

$$\hat{A}(X) = \langle \hat{A}_n(p_1,\ldots,p_n), [X]\rangle = \langle \hat{A}(p), [X]\rangle.$$

The second equality holds, because we may simply ignore the other, lower degree terms in the expression of $\hat{A}(p)$. We are therefore only really interested in the coefficient $\alpha_{2n}$ of $x^{2n}$ in equation (6.7) above. Indeed, using (6.2), we would get :

$$\hat{A}(X) = \alpha_{2n}\langle x^{2n}, [X]\rangle = \alpha_{2n} d.$$

In other words, we are looking for the coefficient of index $-1$ in the Laurent series around 0 of the meromorphic function $\frac{\hat{A}(p)}{x^{2n+1}}$. Using the residue theorem we find :

$$\begin{aligned}
\alpha_{2n} &= \frac{1}{2\pi i}\oint_\gamma \frac{Q(x)^{2n+2}}{Q(dx)\cdot x^{2n+1}}\,dx \\
&= \left(\frac{1}{2}\right)^{2n+1}\frac{1}{2d\pi i}\oint_\gamma \frac{\sinh\left(\frac{dx}{2}\right)}{\sinh\left(\frac{x}{2}\right)^{2n+2}}\,dx \\
&= \frac{1}{2d\pi i}\oint_\gamma \frac{e^{\frac{dx}{2}} - e^{-\frac{dx}{2}}}{\left(e^{\frac{x}{2}} - e^{\frac{x}{2}}\right)^{2n+2}}\,dx
\end{aligned}$$

where $\gamma$ is a suitably small curve going once around 0. We set $z = e^x$, which yields

$$\begin{aligned}
\alpha_{2n} &= \frac{1}{2d\pi i}\oint_\nu \frac{1}{z}\frac{z^{\frac{d}{2}} - z^{-\frac{d}{2}}}{\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)^{2n+2}}\,dz \\
&= \frac{1}{2d\pi i}\oint_\nu \frac{z^{n+\frac{d}{2}} - z^{n-\frac{d}{2}}}{(z-1)^{2n+2}}\,dz.
\end{aligned}$$

The curve $\nu$ now runs a single time around 1 in the complex plane. If we write $w = z-1$, we obtain

$$\alpha_{2n} = \frac{1}{2d\pi i}\oint_\mu \frac{(w+1)^{n+\frac{d}{2}}}{w^{2n+2}} - \frac{(w+1)^{n-\frac{d}{2}}}{w^{2n+2}}\,dw$$

for an appropriate curve $\mu$ circling once around 0. We may now apply the residue theorem for a second time. We thus get

$$\oint_\mu \frac{(w+1)^{n+\frac{d}{2}}}{w^{2n+2}}\,dw = 2\pi i\binom{n+\frac{d}{2}}{2n+1}$$

and

$$\oint_\mu \frac{(w+1)^{n-\frac{d}{2}}}{w^{2n+2}}\,dw = 2\pi i\binom{n-\frac{d}{2}}{2n+1}.$$

Therefore, the coefficient $\alpha_{2n}$ is given by

$$\alpha_{2n} = \frac{1}{d}\left[\binom{n+\frac{d}{2}}{2n+1} - \binom{n-\frac{d}{2}}{2n+1}\right]$$

which means that the $\hat{A}$-genus of $X$ is

$$\hat{A}(X) = \binom{n+\frac{d}{2}}{2n+1} - \binom{n-\frac{d}{2}}{2n+1}. \tag{6.8}$$

Here it is clear that the expression for the $\hat{A}$-genus is odd as a function of $d$. In order to deduce more information from it, we may develop as follows

$$\begin{aligned}\hat{A}(X) &= \frac{1}{(2n+1)!}\left[\prod_{k=0}^{2n}\left(n+\frac{d}{2}-k\right)-\prod_{k=0}^{2n}\left(n-\frac{d}{2}-k\right)\right]\\ &= \frac{1}{(2n+1)!}\left[\prod_{k=0}^{2n}\left(n+\frac{d}{2}-k\right)-\prod_{k=d}^{2n+d}\left(n+\frac{d}{2}-k\right)\right]\end{aligned}$$

Observe now that, for a fixed dimension, $\hat{A}(X)$ only depends on $d$ and is, in fact, a polynomial in $d$ of degree at most $2n+1$.

We may however, as we will see shortly, exhibit $2n+1$ explicit roots for $\hat{A}(X)$, from which it will obviously follow that the said polynomial is of degree $2n+1$ and thus factorizes completely, allowing us to write a surprisingly simple formula for $\hat{A}(X)$. For example, looking at (6.8), it is clear that $\hat{A}(X)=0$ if $d=0$. Notice that, under the hypothesis $0 \leq d \leq 2n$, the above expression becomes

$$\hat{A}(X) = \frac{1}{(2n+1)!}\underbrace{\prod_{k=d}^{2n}\left(n+\frac{d}{2}-k\right)}_{=L}\left[\prod_{k=0}^{d-1}\left(n+\frac{d}{2}-k\right)-\prod_{k=2n+1}^{2n+d}\left(n+\frac{d}{2}-k\right)\right].$$

For the highlighted term $L$, and therefore $\hat{A}(X)$ as well, to vanish, it now suffices for $d$ to be even and there are exactly $n$ such values that $d$ may take between 2 and $2n$. Moreover, since the polynomial $\hat{A}(X)$ is an odd function in $d$ (see (6.8)), we immediately obtain that the $n$ even integers $-2 \geq d \geq -2n$ are also roots of $\hat{A}(X)$. All in all, we have already found $2n+1$ roots of $\hat{A}(X)$ ; there can obviously be no other, as $\hat{A}(X)$ is of degree at most $2n+1$. We conclude from this that there has to be a constant $K(n)$, which may (and will) depend on the dimension, such that

$$\hat{A}(X) = K(n)\cdot\underbrace{(d-2n)\cdot\ldots\cdot(d-2)\cdot d\cdot(d+2)\cdot\ldots\cdot(d+2n)}_{=p(d)}.$$

To determine its value, we simply evaluate $p(d)$ at $d=2n+2$. Indeed, as the computations at the end of the next Section will show, the $\hat{A}$-genus of $X$ is, in this case, equal to 2. We have

$$p(d) = \prod_{j=-n}^{n}(d+2j)$$

and thus

$$p(2n+2) = 2^{2n+1}\prod_{j=-n}^{n}(n+1+j) = 2^{2n+1}\cdot(2n+1)!.$$

In the end we obtain

$$K(n) = \frac{1}{2^{2n}\cdot(2n+1)!},$$

and have therefore found the following explicit formula :

$$\hat{A}(X) = \frac{(d-2n)\cdot \ldots \cdot (d-2)\cdot d\cdot (d+2)\cdot \ldots \cdot (d+2n)}{2^{2n}\cdot (2n+1)!}. \tag{6.9}$$

The non-positive roots of this polynomial $p$ are of no real interest to us, since these solutions do not correspond to actual hypersurfaces. In other words, for $n$ fixed, the $\hat{A}$-genus vanishes only for the even values of $d$ with $d \leq 2n = \dim(X)$. As such, by Theorem 5.47 and Remark 5.48, the spin hypersurface $X = V^d_{2n}$ admits a metric of positive scalar curvature if and only if $d$ is even and less or equal to $2n$.

**Example 6.3** Given an integer $n \geq 1$, provided that we know the polynomial $\hat{A}_n$ and using the formula (6.5) to determine the Pontrjagin classes $p_1, \ldots, p_n$, it is possible to compute the $\hat{A}$-genus directly. In this way, for $n = 1$, we obtain

$$\hat{A}(X) = \frac{d^3 - 4d}{24},$$

whose only strictly positive root is $d = 2$. Similarly, for $n = 2$, we find

$$\hat{A}(X) = \frac{d^5 - 20d^3 + 64d}{1920},$$

and thus the positive solutions to $\hat{A}(X) = 0$ are $d = 2$ or $d = 4$. One can then easily check that these explicit results coincide with the general formula (6.9) given above.

Recall (see (6.6)) that the first Chern class of $X$ is given by

$$c_1(X) = (2n + 2 - d)x.$$

Let us write $c_1 = 2n + 2 - d \in \mathbb{Z}$, so that $c_1(X) = c_1 \cdot x$. Note that the results we have obtained in this Section can therefore be rephrased as follows :

**Theorem 6.4** The $\hat{A}$-genus of the degree $d$ hypersurface $X = V^d_{2n}$ vanishes if and only if $c_1$ is even and strictly positive.

## 6.4 A note on complete intersections

Finally, let us simply mention some particular results for so-called **complete intersections**. These are, as their name suggests, complex manifolds obtained as a result of transversal intersections of $r \geq 1$ hypersurfaces of respective degree $d_1, \ldots, d_r$. Such a space $X$ is then characterized by the dimension of the ambient complex projective space (for us, this will be $2n + r$) and the degrees $d_1, \ldots, d_r$.

If $x \in H^2(P^{2n+r}(\mathbb{C});\mathbb{Z})$ is once again the generator mentioned in Theorem 1.71, then we have

$$\langle x^{2n}, [X]\rangle = d_1 \cdot \ldots \cdot d_r.$$

The total Chern class is given by

$$c(X) = (1+x)^{2n+r+1} \cdot \prod_{j=1}^{r}(1+d_j x)^{-1},$$

from which we deduce that

$$p(X) = \left(1+x^2\right)^{2n+r+1} \cdot \prod_{j=1}^{r} \left(1+(d_j x)^2\right)^{-1}. \tag{6.10}$$

Let us write $d = d_1 \cdot \ldots \cdot d_r$. Then, using the same method as in the previous Section, we find

$$\hat{A}(X) = \langle \beta_{2n} x^{2n}, [X]\rangle = \beta_{2n} d,$$

where $\beta_{2n}$ is now the coefficient of $x^{2n}$ in the expression (6.10) above. In other words, we are looking for the coefficient of index $-1$ in the Laurent series around 0 of the meromorphic function $\frac{\hat{A}(p)}{x^{2n+1}}$. Using the residue theorem, we find

$$\begin{aligned} \hat{A}(X) &= \frac{d}{2\pi i} \oint_\gamma \frac{\hat{A}(p)}{x^{2n+1}}\, dx \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{\prod_{j=1}^{r}\left(e^{\frac{d_j x}{2}} - e^{-\frac{d_j x}{2}}\right)}{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{2n+r+1}}\, dx \end{aligned}$$

for a suitably small curve $\gamma$ circling once aroung the origin in the complex plane. We then substitute $u = e^x$, which yields

$$\hat{A}(X) = \frac{1}{2\pi i} \oint_\eta \frac{u^\ell}{(u-1)^{2n+r+1}} \prod_{j=1}^{r}\left(u^{d_j} - 1\right)\, du$$

where $\eta$ is now a small circle around 1 and

$$\ell = \frac{1}{2}\left(2n + r - 1 - D\right),$$

where $D = \sum_{j=1}^{r} d_j$.

We first suppose that $\ell \geq 0$. Note that, anyway, it has to be assumed at this point that $\ell$ is also an integer. Indeed, taking square roots along $\eta$ would require the choice of a branch, and the integral would therefore be ill-defined. We now set $\xi = u - 1$ and obtain

$$\begin{aligned} \hat{A}(X) &= \frac{1}{2\pi i} \oint_\mu \frac{(1+\xi)^\ell}{\xi^{2n+r+1}} \prod_{j=1}^{r}\left((1+\xi)^{d_j} - 1\right) \\ &= \frac{1}{2\pi i} \oint_\mu \frac{(1+\xi)^\ell}{\xi^{2n+r+1}} \prod_{j=1}^{r}\left(\sum_{k_j=1}^{d_j} \binom{d_j}{k_j} \xi^{k_j}\right)\, d\xi \end{aligned}$$

where $\mu$ is an appropriate curve going once around the origin in $\mathbb{C}$. We now observe that in the polynomial

$$(1+\xi)^\ell \prod_{j=1}^{r} \left( \sum_{k_j=1}^{d_j} \binom{d_j}{k_j} \xi^{k_j} \right)$$

the term of highest order is $\xi^{\ell+D}$. It follows that no monomial has order greater than

$$\ell + D - 2n - r - 1 = \frac{D - 2n - r - 3}{2}$$

in the integral above. However, since we assume that $\ell \geq 0$, we find

$$\frac{D - 2n - r - 3}{2} \leq -2$$

and we may therefore immediately conclude that, in this case, $\hat{A}(X) = 0$.

Let us now assume that $2n + r + 1 = D$. Observe that, in the integral above, the exponent $\ell$ simply becomes

$$\ell = \frac{2n + r - 1 - D}{2} = -1.$$

We may therefore replace the term $(1+\xi)^\ell$ by the power series

$$\frac{1}{1+\xi} = \sum_{k=0}^{+\infty} (-1)^k \xi^k.$$

It follows that

$$\frac{(1+\xi)^\ell}{\xi^{2n+r+1}} \prod_{j=1}^{r} \left( \sum_{k_j=1}^{d_j} \binom{d_j}{k_j} \xi^{k_j} \right) = \sum_{k=0}^{+\infty} (-1)^k \sum_{k_1=1}^{d_1} \cdots \sum_{k_r=1}^{d_r} \binom{d_1}{k_1} \cdots \binom{d_r}{k_r} \xi^{k-D+k_1+\ldots+k_r}$$

so the coefficient of the term $\xi^{-1}$, and hence the $\hat{A}$-genus of $X$, is

$$\hat{A}(X) = \sum_{k=0}^{D-r-1} (-1)^k \sum_{K=D-k-1} \binom{d_1}{k_1} \cdots \binom{d_r}{k_r}, \tag{6.11}$$

where we have written $K$ for the sum $k_1 + \ldots + k_r$.
This expression has now become quite complex. Let us therefore come back to a simpler setting, by assuming that $r = 1$, that is, by considering once again the case of a single degree $d$ hypersurface. In this case, it is clear that $d = D = 2n + 2$ and, hence, $d$ is

even. Adapting (6.11), we obtain

$$\begin{aligned}
\hat{A}(X) = \sum_{k=0}^{d-2}(-1)^k \binom{d}{d-k-1} &= \sum_{k=0}^{d-2}(-1)^k \binom{d}{k+1} \\
&= \sum_{j=1}^{d-1}(-1)^{j-1} \binom{d}{j} \\
&= -\sum_{j=1}^{d-1}(-1)^{j} \binom{d}{j} \\
&= -\sum_{j=0}^{d-1}(-1)^{j} \binom{d}{j} + \binom{d}{0} \\
&= -(-1)^{d-1} \binom{d-1}{d-1} + \binom{d}{0} = 1 + 1 = 2.
\end{aligned}$$

Note that this is consistent with our results of the previous Section. Indeed, although it is even, $d$ is strictly greater than $2n$ and, therefore, $\hat{A}(X)$ cannot be zero.

**Remark 6.5** Even though the computations leading to that result are much more complicated than the ones we have presented here, it can be shown that (6.11) also yields a value of 2 for $r > 1$.

These results on the $\hat{A}$-genus of degree $d$ hypersurfaces and complete intersections were actually first computed by Robert Brooks in [Bro83].

**Remark 6.6** Here, we have presented a very computational proof of the vanishing of the $\hat{A}$-genus for a spin hypersurface or complete intersection with positive first Chern class. There is, however, a more geometric way of arriving at the same conclusion. Indeed, in providing a solution for a conjecture of Calabi, Yau showed that Kähler manifolds with positive first Chern class, of which these complete intersections are examples, admit a metric of positive Ricci curvature (see [Bro83], page 529) and thus, by Theorem 5.47, their $\hat{A}$-genus vanishes.

# Bibliography


[Ama] Manuel Amann. “Genera, the Chern character and the Thom isomorphism”. Lecture notes of a course given at the University of Münster in the Winter semester of 2009-2010.

[AS68] Michael Francis Atiyah and Isadore Manuel Singer. “The Index of Elliptic Operators I, II and III”. In: *The Annals of Mathematics* (1968).

[BLM89] Herbert Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry*. Annals of Mathematics Studies. Princeton University Press, 1989.

[Bro83] Robert Brooks. “The $\hat{A}$-genus of Complex Hypersurfaces and Complete Intersections”. In: *Proceedings of the National Academy of Sciences of the United States of America, Volume 87* (1983).

[Bum04] Daniel Bump. *Lie Groups, First Edition*. Graduate Texts in Mathematics. Springer-Verlag, 2004.

[Car92] Manfredo Perdigão do Carmo. *Riemannian Geometry*. Birkhäuser, 1992.

[Con93] Lawrence Conlon. *Differentiable manifolds : A first course*. Springer-Verlag, 1993.

[Des] Anand Dessai. “Riemannian Geometry”. Lecture notes of a course given at the University of Fribourg in the Spring semester of 2024.

[Hat17] Allen Hatcher. *Vector Bundles and K-Theory*. 2017.

[Hir54] Friedrich Hirzebruch. “Der Satz von Riemann-Roch in Faisceau-theoretischer Formulierung einige Anwendungen und offene Fragen”. In: *Proceedings of the National Academy of Sciences of the United States of America* (1954).

[HBJ92] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung. *Manifolds and Modular Forms*. Graduate Texts in Mathematics. Max-Planck-Institue for Mathematics, 1992.

[Hit74] Nigel Hitchin. “Harmonic spinors”. In: *Advances in Mathematics* (1974).

[Hus94] Dale Husemoller. *Fibre bundles*. Graduate Texts in Mathematics. Springer-Verlag, 1994.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry : Volume I*. Interscience Publisher, 1963.

[Loh94] Joachim Lohkamp. “Metric of negative Ricci curvature”. In: *The Annals of Mathematics* (1994).

[MS74] John Willard Milnor and James Dillon Stasheff. *Characteristic Classes*. Annals of Mathematics Studies. Princeton University Press, 1974.

[Och87] Serge Ochanine. “Sur les genres multiplicatifs définis par des intégrales elliptiques”. In: *Topology* (1987).

[Ste51] Norman Steenrod. *The topology of fibre bundles*. Princeton University Press, 1951.

[Sto90] Stephan Stolz. “Harmonic spinors”. In: *Bulletin of the American Mathematical Society* (1990).

[Sto92] Stephan Stolz. “Simply connected manifolds of positive scalar curvature”. In: *Annals of Mathematics* (1992).