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Master's thesis

Moduli spaces of Riemannian metrics of positive Ricci curvature on homotopy spheres of dimension $4k-1$

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Abstract

In this paper, we study the topology of the moduli spaces of Riemannian metrics of positive Ricci curvature on certain homotopy spheres of dimension $4k-1$.

In the 1950s, Milnor constructed some of the first examples of exotic spheres using a bilinear pairing

$$\beta : \pi_{4m-1}(SO_{4n-1}) \otimes \pi_{4n-1}(SO_{4m-1}) \to \Gamma_{4m+4n-1}$$

into the group of $(4m+4n-1)$-dimensional twisted spheres $\Gamma_{4m+4n-1}$. We show, following Wraith and Reiser, that the spheres in the image group $\operatorname{im} \beta \subset \Gamma_{4m+4n-1}$, $1 \leq m \leq n < 2m$, can be endowed with a metric of positive Ricci curvature. The aim of this paper is to show that the moduli spaces of Riemannianian metrics of positive Ricci curvature of the homotopy spheres obtained using the bilinear pairing β have infinitely many path-components.

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Chapter 1

Introduction

One of the most famous achievements in recent memory is the solution by Grigori Perelman in the early 2000s to the famous Poincaré conjecture, which states that every simply connected, compact, 3-dimensional manifold without boundary is homeomorphic to the standard 3-sphere. This means that in a sense, every space which looks somewhat like the 3-sphere *must* actually be the 3-sphere itself. What is even more surprising is that it was already known beforehand for every dimension $n \neq 3$ thanks to the works of Freedman for the dimension 4 and Smale for all the remaining dimensions $n \geq 5$. Even though this was not known to topologists of the first half of the 20th century, they wondered whether any two spaces which were enough alike, were also the same in a smooth sense, that is, whether any two homeomorphic smooth manifolds were also diffeomorphic. The first to answer this question was Milnor in the 1950s, using the theory of fiber bundles to construct the first *exotic sphere*: a smooth manifold which is homeomorphic, but not diffeomorphic, to the standard 7-sphere. He then showed that there exist at least 7 distinct differentiable structures on S^7; in fact, he showed some years later with Kervaire, that the distinct differentiable structures on S^7 form a group of order 28 if one considers orientation - and a set of size 15 if not (there is no longer a group operation in this case).

Towards the end of the 20th century, David Wraith studied these exotic spheres through the lens of Riemannian geometry. One might namely wonder what types of Riemannian metrics these manifolds can support and, more specifically, how many metrics of each type they can have. Moreover, one might ask if any two metrics of the same type are equivalent, in the sense that they can be deformed into one another through a path of metrics of the same type. The collection of such equivalence classes of metrics can be used to form the *moduli space of metrics* (with a given type). In this context, David Wraith showed in [Wra97, Theorem 1] that the subgroup bP_{n+1} of exotic n-spheres which bound parallelizable $(n+1)$-dimensional manifolds carry metrics of positive Ricci curvature and in [Wra11, Theorem A] that the moduli spaces of metrics with positive Ricci curvature of any such exotic sphere has infinitely many path-components, provided n is one less than a multiple of 4, i.e. $n = 4k - 1$ for some integer $k > 1$. The goal of this thesis is to use Wraith's approach to show the main theorem:

Theorem (Main theorem)**.** *The moduli space of Riemannian metrics $\mathcal{M}_{Ric>0}(M)$ of positive Ricci curvature of any homotopy sphere $M \in \Xi_{4m+4n-1}$ has infinitely many path-components, where $m \leq n < 2m$.*

Here, the group $\Xi_{4m+4n-1}$ is a subgroup of the group of twisted spheres $\Gamma_{4m+4n-1}$ defined as the image of the pairing β defined by Milnor in [Mil07, Chapter 4, pp. 209-210].

In this paper, we will first review some Riemannian geometry and we will present some general notions of cohomology and notably the concept of *Poincaré duality.* Next, we will review some notions of the theory of vector bundles, before introducing fundamental cohomological tools known as *characteristic classes.* As their name suggests, they play a major role in the general theory of vector bundles, since they will let us associate classes to vector bundles which characterize them. They will also let us compute the L- and $\hat{A}$-*genera* of a smooth $4k$-manifold, which we will define using the formalism developed by Hirzebruch in the 1950s.

We will then introduce an operation between smooth manifolds called the *connected sum* and study some of its properties. In particular, we will see how it relates to the differentiable structures on the sphere. However, since these are particularly hard to grasp and their study is quite difficult, we will instead investigate so-called *twisted spheres*, which turn out to be good potential candidates to find exotic spheres. Next, we will show that some of these twisted spheres can be realized as boundaries of higher dimensional manifolds, which are obtained by *plumbing* together disk bundles in a suitable way. We will study their topological properties and show that they carry Riemannian metrics which have specific curvature conditions. Finally, after having defined the notion of *moduli space of Riemannian metrics*, we will give the proof of the main theorem.

1.1 Review of Riemannian geometry

While we start with a review of the basic notions of Riemannian geometry, we will assume that the reader is already acquainted with the topic as well as the theory of manifolds at large. The sources for this section is Do Carmo's book [DF92] and the following lecture notes [Des24].

We first recall the following definitions.

Let M be a smooth manifold of dimension n, $p \in M$ and let $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to M$ smooth curves with $\gamma_1(0) = \gamma_2(0) = p$. Define an equivalence relation by $\gamma_1 \sim \gamma_2$ if and only if there exists a chart $h : M \supset U \to U' \subset \mathbb{R}^n$ around p such that $(h \circ \gamma_1)'(0) = (h \circ \gamma_2)'(0)$. We write $[\gamma]$ for the equivalence class of a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$.

Definition 1.1. A **tangent vector at** p is an equivalence class of such curves. The set of all tangent vectors at p form a vector space T_pM called the **tangent space of** M **at** p.

One can check using the chain rule that this equivalence relation is well-defined, that is, if any two curves are equivalent with respect to one chart of M, then they also are equivalent with respect to all of the other charts of M.

Definition 1.2. The disjoint union TM of all tangent spaces T_pM over all $p \in M$ is called the tangent bundle. Mathematically,

$$TM := \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM.$$

Using the charts of M, the set TM can be given the structure of a smooth manifold of twice the dimension of M. In fact, it can be given the structure of a smooth vector bundle, see [Des24, Lemma 2.9, Chapter 2].

Definition 1.3. A smooth map $X : M \to TM$ is called a **(smooth) vector field**. The collection of all vector fields is denoted by $\Gamma(M)$.

Definition 1.4. Let $f : M \to N$ be a smooth map between smooth manifolds. The **differential** of f is the linear map f_* defined by

$$f_* : T_pM \to T_{f(p)}N, \; f_*([\gamma]) = [f \circ \gamma],$$

for all $p \in M$.

Now that we've reviewed the tangent space and the differential of a smooth map, let us review some notions of Riemannian geometry.

Definition 1.5. A **Riemannian metric** or simply **metric** on M is a smooth choice of inner products $\{g_p\}_p$ on each tangent space T_pM for all $p \in M$. The choice is said to be smooth if for any two vector fields $X, Y \in \Gamma(M)$, the map $p \mapsto g_p(X(p), Y(p))$ is smooth as a map between the manifold M and $\mathbb{R}$. The pair (M, g) is called a **Riemannian manifold**.

If there is no confusion possible, we will simply drop the subscript $\cdot_p$ in order to lighten future expressions.

Recall that for each $p \in \mathbb{R}^n$ the partial derivatives $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ form a basis of $T_p\mathbb{R}^n$, see for example [Des24, Lemma 2.6, Chapter 2].

Examples 1.6.

1. $\mathbb{R}^n$ together with the canonical metric $g_{can}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}$, where δ_{ij} is the *Kronecker delta*, is a Riemannian manifold called the *standard Euclidean space*. This metric can be viewed as coming from the dot product on $\mathbb{R}^n$: If

$$\sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \text{ and } \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}$$

are tangent vectors at $p \in \mathbb{R}^n$, then

$$g_{can}\left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\right) = \sum_{i=1}^{n} a_i b_i.$$

2. Let $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the natural inclusion of the unit sphere. The *round metric* g_R on S^n is given by

$$g_R(u, v) := g_{can}(i_*(u), i_*(v)) \text{ for all } u, v \in T_pS^n.$$

3. Let $\mathbb{H}^n := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$. Define the *hyperbolic metric* g_{hyp} by

$$g_{hyp}(u, v) := \frac{1}{x_n^2} g_{can}(u, v),$$

where $u, v \in T_pM$ and $p = (x_1, \cdots, x_n)$. Together with this metric, the Riemannian manifold $(\mathbb{H}^n, g_{hyp})$ is called the *hyperbolic space of dimension n*.

Notice that the second example is an instance of a more general way of constructing new metrics out of old ones:

Definition 1.7. Let M, N be smooth manifolds with $\dim M \leq \dim N$ and let $i : M \to N$ be a smooth immersion. Assume N is given a metric g. The **pullback metric** i^*g on M is defined by

$$i^*g(u, v) := g(i_*(u), i_*(v)) \text{ for } u, v \in T_pM.$$

One easily verifies that this defines a Riemannian metric on M.

Remark 1.8. Any smooth manifold can be given a Riemannian metric by pulling back the canonical metric on $\mathbb{R}^n$ via the charts of its atlas and patching all of this local data to a well-defined metric on the whole manifold by using a partition of unity, see [DF92, Proposition 2.10, Chapter 1, p. 43].

This concept also lets us define the notion of sameness for Riemannian manifolds in this context:

Definition 1.9. Let $f : (M, g) \to (N, h)$ be a diffeomorphism between two Riemannian manifolds. We say that f is an **isometry** if $g = f^*h$. We also say that (M, g) and (N, h) are **isometric**.

Definition 1.10. A **connection** ∇ on a smooth manifold M is a map

$$\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(M),$$
$$(X, Y) \mapsto \nabla_X Y$$

which is:

1. $C^\infty(M)$-linear in the first component:

$$\nabla_{fX+gY} Z = f\nabla_X Y + g\nabla_Z Y,$$

2. linear in the second component:

$$\nabla_x Y + Z = \nabla_X Y + \nabla_X Z,$$

3. satisfies the following product rule:

$$\nabla_X(fY) = f \cdot \nabla_X Y + X(f) \cdot Y,$$

   for all smooth functions $f, g \in C^\infty(M)$ and all vector fields $X, Y, Z \in \Gamma(M)$.

The notion of the connection $\nabla_X Y$ was introduced in order to let us differentiate vector fields Y in direction X.

Remark 1.11. It can be shown that for a given connection ∇ on M, the value of $\nabla_X Y$ at $p \in M$ only depends on $X(p)$ and the value of Y along a curve through p in direction $X(p)$.

Let now (M, g) be a Riemannian manifold equipped with a connection ∇.

Definition 1.12. The connection is said to be **metric** if

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(M)$.
The connection is said to be **torsion-free** if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all $X, Y \in \Gamma(M)$.

Proposition 1.13 (Fundamental theorem of Riemannian geometry)**.** Let (M, g) be a Riemannian manifold. Then, there exists one and only one connection on M which is metric and torsion-free. It is called the *Levi-Civita connection on M.*

The proof can be found in [DF92, Theorem 3.6, Chapter 2, p. 55]. The main idea is to use *Koszul formula* to find an expression for $\nabla_X Y$ involving only the metric g.

Example 1.14. It can be shown that for the standard basis $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ of $T_p\mathbb{R}^n$ where $p \in \mathbb{R}^n$, the Levi-Civita connection satisfies $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$ for all $i, j \in \{1, \cdots, n\}$, see [DF92, p. 56].

The landscape of Riemannian geometry is quite expansive and there are still a lot of topics like geodesics, the exponential map, variational formulae for the energy and the length, the theorem of Hopf-Rinow, etc. which we will not cover at all. For more information on these topics, we refer to Do Carmo's book [DF92]. Let us now discuss some aspects of the notion of *curvature.*

Let R be the map

$$\Gamma(M) \times \Gamma(M) \times \Gamma(M) \to \Gamma(M)$$
$$(X, Y, Z) \mapsto R(X, Y)Z$$

defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

This can be seen as a measure of the noncommutativity of the connection ∇, and as such an obstruction to M being isometric to Euclidean $\mathbb{R}^n$. A proof of the following properties is found in [DF92, Propositions 2.2, 2.4, 2.5, Chapter 4, pp. 90-91]

Properties 1.15.
1. R is $C^\infty(M)$-trilinear,
2. It satisfies *Bianchi's identity*:
$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$
3. $g(R(X, Y)U, V) = -g(R(Y, X)U, V)$,
4. $g(R(X, Y)U, V) = -g(R(X, Y)V, U)$,
5. $g(R(X, Y)U, V) = g(R(U, V)X, Y)$,
for all $X, Y, Z, U, V \in \Gamma(M)$.

One can also check that the value of $R(X, Y)Z$ at $p \in M$ only depends on $X(p), Y(p)$ and $Z(p)$. This shows that the following is well-defined:

Definition 1.16. For $X, Y, Z \in T_pM$, let

$$R(X, Y)Z := (R(\tilde{X}, \tilde{Y})\tilde{Z})(p),$$

where $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(M)$ are vector fields on M with $\tilde{X}(p) = X, \tilde{Y}(p) = Y$ and $\tilde{Z}(p) = Z$. R is called the **curvature tensor**.

As noted before, this does not depend on the choice of the extensions $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$. The curvature tensor R naturally inherits the properties above.

Examples 1.17. For the standard Euclidean space $(\mathbb{R}^n, g_{can})$, the curvature tensor R vanishes, since $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ commute and $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. Thus,

$$R(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j})\tfrac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_j}} \underbrace{\nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_k}}_{=0} - \nabla_{\frac{\partial}{\partial x_i}} \underbrace{\nabla_{\frac{\partial}{\partial x_j}} \tfrac{\partial}{\partial x_k}}_{=0} + \nabla_{\underbrace{[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}]}_{=0}} \tfrac{\partial}{\partial x_k} = 0.$$

For this reason, $(\mathbb{R}^n, g_{can})$ is sometimes referred to as *flat* $\mathbb{R}^n$.

Definition 1.18. Let $E \subset T_pM$ be a 2-dimensional subspace and let $x, y \in E$ be an orthonormal basis of E. The **sectional curvature** of E is defined by

$$\sec(E) := g(R(x, y)x, y).$$

One can verify that this does not depend on the choice of the basis.

Properties 1.19. Let $\varphi : M \to M$ be an isometry, that is, φ is a diffeomorphism and $\varphi^* g = g$. Then, for the Levi-Civita connection ∇ on M, we have:

1. $\nabla_{\varphi_*(\tilde{X})}\varphi_*(\tilde{Y}) = \varphi_*(\nabla_{\tilde{X}}\tilde{Y})$ for all $\tilde{X}, \tilde{Y} \in \Gamma(M)$,
2. $R(\varphi_*(X), \varphi_*(Y))\varphi_*(Z) = \varphi_*(R(X, Y)Z)$ for all $X, Y, Z \in T_pM$ and all $p \in M$,
3. $g(R(\varphi_*(X), \varphi_*(Y))\varphi_*(U), \varphi_*(V)) = g(R(X, Y)U, V)$ for all $X, Y, U, V \in T_pM$ and all $p \in M$,
4. $\sec(\varphi_*(E)) = \sec(E)$ for all tangent planes $E \subset T_pM$ and all $p \in M$.

In particular, the fourth property shows that the sectional curvature is invariant under isometries. These properties are proved in [Des24, Lemma 8.11, Chapter 8]. Using an appropriate subgroup of the group of isometries acting transitively on the manifold, one can show the following:

Examples 1.20.

1. The round sphere (S^n, g_R) has constant sectional curvature equal to $+1$. The subgroup in question is the orthogonal group $O(n+1)$.

2. The standard Euclidean space $(\mathbb{R}^n, g_{can})$ has constant sectional curvature equal to 0. This is because R vanishes in this case, as noted above.

3. The hyperbolic space $(\mathbb{H}^n, g_{hyp})$ has constant sectional curvature equal to -1. Here, the subgroup is the special linear group $SL_n(\mathbb{R})$.

Definition 1.21. Let $(e_1, \cdots, e_n)$ be an orthonormal basis of T_pM, and let $x, y \in T_pM$ be two tangent vectors at p. Define $\mathrm{Ric}_p(x, y)$ by

$$\mathrm{Ric}_p(x, y) := \sum_{i=1}^{n} g(R(x, e_i)y, e_i).$$

The **Ricci curvature of** p **in direction** x is given by $\mathrm{Ric}_p(x) := \mathrm{Ric}_p(x, x)$. We say that a metric is **of positive Ricci curvature** if $\mathrm{Ric}(x) > 0$ for all $x \in T_pM$ and all $p \in M$.

One can check that the Ricci curvature does not depend on the choice of the orthonormal basis $(e_1, \cdots, e_n)$. Notice that the Ricci curvature can be seen as an average of the sectional curvature of planes in T_pM containing x. Taking the average of the Ricci curvatures at p yields the following notion:

Definition 1.22. The **scalar curvature at** p is

$$\mathrm{scal}(p) := \sum_{i=1}^{n} \mathrm{Ric}_p(e_i),$$

where $(e_1, \cdots, e_n)$ is an orthonormal basis of T_pM. Again, this does not depend on the choice of the basis.
We say that a metric is **of positive scalar curvature** if $\mathrm{scal}(p) > 0$ for all $p \in M$.

Notice that any metric which is of positive Ricci curvature is also of positive scalar curvature, but the converse is not true. In fact, we will construct such a metric in Chapter 5.

1.2 The fundamental class

We will assume some general knowledge of category theory as well as algebraic topology, in particular homology theory. The general reference for the following two sections is [MS74, Appendix A, pp. 257-280]. We will only discuss the compact case.
Let M be a smooth connected orientable compact n-dimensional manifold with or without boundary. It can be verified that M is orientable (in the classical sense) if and only if for the top homology group, we have $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ if M has no boundary or $H_n(M, \partial M;\mathbb{Z})$ if it does, see [Hat02, Theorem 3.26, Chapter 3, p. 236]. Note that if M has no boundary, $\partial M = \emptyset$, then $H_n(M, \partial M;\mathbb{Z}) = H_n(M;\mathbb{Z})$.

Let M be orientable, that is to say, $H_n(M;\mathbb{Z})$ (or $H_n(M, \partial M;\mathbb{Z})$) is isomorphic to the integers $\mathbb{Z}$.

Definition 1.23. The choice of a generator of $H_n(M;\mathbb{Z})$ (or of $H_n(M, \partial M;\mathbb{Z})$) is called an **orientation** for M and we say that M is **oriented** if an orientation has been specified on M. The chosen generator $[M] \in H_n(M;\mathbb{Z})$ (or in $H_n(M, \partial M;\mathbb{Z})$) is called the **fundamental class** of M.

If M has a nonempty boundary, one can check that $\partial[M] = [\partial M] \in$

$H_{n-1}(\partial M;\mathbb{Z})$ is a generator, where $\partial : H_n(M,\partial M;\mathbb{Z}) \to H_{n-1}(\partial M;\mathbb{Z})$ is the boundary map of homology. In particular, an orientation of M induces an orientation on the boundary ∂M which is compatible with that of M. Observe that if an orientation is chosen on M and if we denote by $-M$ the manifold with the opposite orientation, then $[-M] = -[M]$.

Remark 1.24. If follows from the *Universal coefficient theorem* that

$$H_k(M,\partial M;\mathbb{Q}) \cong H_k(M,\partial M;\mathbb{Z}) \otimes \mathbb{Q},$$

for all $k \in \mathbb{Z}$, see [Hat02, Section 3.1]. In particular, we can indiscriminately choose the fundamental class $[M]$ with integer or rational coefficients, since $H_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$ has no torsion.

Definition 1.25. A diffeomorphism $f : M \to N$ between smooth connected oriented compact n-manifolds is called **orientation-preserving** if $f_*[M] = [N]$, where f_* is the induced homomorphism on homology. It is **orientation-reversing** if $f_*[M] = -[N]$.

1.3 Quick survey of cohomology

In this section, we present in a succinct manner the theory of cohomology with the usual axiomatic definition using the *Eilenberg-Steenrod axioms* and its relation to homology, via notably the notion of *Poincaré duality.*

Cohomology can be seen as a dual theory of homology, where every arrow in homology is reversed. Surprisingly, this reversal gives the cohomology of a space the richer structure of a graded commutative ring.

Let R be a commutative ring with 1. We will mostly use $R = \mathbb{Z}_2, \mathbb{Z}$ and $\mathbb{Q}$.

Definition 1.26 (Eilenberg-Steenrod axioms for cohomology)**.** A **cohomology theory** is sequence of contravariant functors H^n, $n \in \mathbb{Z}$, from the category $\mathbf{Top_2}$ of topological pairs (X, A) to the category $\boldsymbol{R}$**−mod** of R-modules, along with natural transformations $d : H^n(A) \to H^{n+1}(X, A)$ called the **coboundary homomorphisms**, which satisfies the four following axioms:

1. **Homotopy:** Homotopic maps induce the same homomorphism on cohomology.

2. **Long exact sequence:** Given a pair (X, A), $i : A \to X$ and $j : X \to (X, A)$ the natural inclusions,
$$\cdots \xrightarrow{d} H^n(X, A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{d} H^{n+1}(X, A) \xrightarrow{j^*} \cdots ,$$
is an exact sequence of R-modules.

3. **Excision:** Given (X, B) and $A \subset B$ such that $\bar{A} \subset \mathring{B}$, the inclusion $(X \setminus A, B \setminus A) \to (X, B)$ induces an isomorphism
$$H^n(X, B) \xrightarrow{\cong} H^n(X \setminus A, B \setminus A),$$
for all $n \in \mathbb{Z}$.

4. **Dimension:** $H^n(\{*\}) \cong 0$ if $n \neq 0$ and $H^0(\{*\}) = R$, where $\{*\}$ is the one-point space.

The R-modules $H^n(X)$ (respectively $H^n(X, A)$) are called **cohomology groups** (respectively **relative cohomology groups**). If we need to put an emphasis on the ring R, we might also write $H^n(X; R)$ and $H^n(X, A; R)$.

Remarks 1.27. 1. $H^n(X)$ stands for $H^n(X, \emptyset)$ and similarly for $H^n(A)$.

2. Unlike homology which decreases the degrees of homology groups, cohomology increases the degrees of the cohomology groups.

3. All of the usual facts from homology which only need these four axioms can be directly adapted to cohomology if one reverses the arrows. In particular, there is a corresponding version of the *Mayer-Vietoris sequence:*
   Let $U, V \subset X$ be open and such that $U \cup V = X$. Then, there is a long exact sequence
   $$\cdots \to H^n(X) \xrightarrow{i_1^* \oplus i_2^*} H^n(U) \oplus H^n(V) \xrightarrow{k_1^* - k_2^*} H^n(U \cap V) \to H^{n+1}(X) \to \cdots,$$
   where $i_1 : U \hookrightarrow X$, $i_2 : V \hookrightarrow X$, $k_1 : U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ are the inclusions.

4. In the case of a smooth manifold M, it can be verified that $H^i(M) \cong 0$ for all $i > \dim M$.

For the next part, compare [MS74, pp. 263-265] or [Mun18, §48, Chapter 5, pp. 285-291]. Using *singular cohomology*, one can explicitly construct an operation

$$\smile : H^m(X) \times H^n(X) \to H^{m+n}(X),$$

called the *cup product*, which satisfies:

1. **Distributivity:** If $r \in R$, and if $x, \tilde{x} \in H^m(X)$ and $y, \tilde{y} \in H^n(X)$, then:
   (i) $(x + \tilde{x}) \smile y = x \smile y + \tilde{x} \smile y$,
   (ii) $x \smile (y + \tilde{y}) = x \smile y + x \smile \tilde{y}$,
   (iii) $(rx \smile y) = x \smile (ry) = r(x \smile y)$.

2. **Graded-commutativy:** $x \smile y = (-1)^{mn}(y \smile x)$ whenever $x \in H^m(X)$ and $y \in H^n(X)$.

3. **Functoriality:** If $f : X \to Y$ is continuous, $f^*(x \smile y) = f^*(x) \smile f^*(y)$, where $f^* : H^n(Y) \to H^n(X)$ is the induced homomorphism on cohomology.

This implies that the direct sum $H^*(X) = \bigoplus_{n\in\mathbb{Z}} H^n(X)$ equipped with the cup product $\smile$ is a graded-commutative ring.

Remarks 1.28. If one considers *de Rham cohomology* (and $R = \mathbb{R}$), the cup product is simply given by the wedge product $\wedge$ of differential forms.

The following theorem plays a fundamental role in the theory of smooth manifolds:

Theorem 1.29 (Poincaré duality)**.** Let M be a closed oriented manifold of dimension n. Then, there is a canonical isomorphism

$$D_M : H^k(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-k}(M;\mathbb{Z}),$$

for all $k \in \mathbb{Z}$. If M has a boundary, there is a corresponding isomorphism

$$D_M : H^k(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-k}(M,\partial M;\mathbb{Z}),$$

for all $k \in \mathbb{Z}$. In fact, there is also an isomorphism

$$H^k(M,\partial M;\mathbb{Z}) \xrightarrow{\cong} H_{n-k}(M,\mathbb{Z}),$$

for all $k \in \mathbb{Z}$.

The proof involves the fundamental class $[M]$, which is then paired with cohomology classes using the *cap product* $\frown$, see [MS74, Appendix A, pp. 276-280]. Poincaré duality also holds for coefficient in $\mathbb{Q}$ (or $\mathbb{Z}_2$).

Definition 1.30. The preimage of a homology class $X \in H_{n-k}(M;\mathbb{Z})$ (or $H_{n-k}(M,\partial M;\mathbb{Z})$ if M has a boundary) is called the **Poincaré dual** of X. We will write either x or $X^* \in H^k(M)$ for the corresponding Poincaré dual.

In the particular setting of smooth manifolds, we can interpret the cup product using Poincaré duality as a signed intersection number:
Let X and Y be smooth oriented transverse submanifolds of M^n of respective codimensions q and p, and denote again by $[X]$ and $[Y]$ the image of their fundamental class under the respective inclusions. Then, in terms of their Poincaré duals $[X]^* \in H^q(M;\mathbb{Z})$ and $[Y]^* \in H^p(M;\mathbb{Z})$, we have:

$$[X]^* \smile [Y]^* = [X \cap Y]^* \in H^{p+q}(M;\mathbb{Z}).$$

The transversality guarantees that $X \cap Y$ is again an oriented submanifold of M of dimension $m-p-q$. Observe that the codimensions and the degrees of the cohomology groups swap with each other. This is again true for rational coefficients. For more information, we refer to [Mun18, §69, Chapter 8, pp. 408-410].

To finish this introductory chapter, let us familiarize ourselves with the *Kronecker pairing* $\langle x, [M]\rangle$ which lets us integrate integral (or rational, $\mathbb{Z}_2$) cohomology classes $x \in H^n(M)$ (or $H^n(M,\partial M)$ if M has a boundary) on the fundamental class $[M]$, where $n = \dim M$.
The explicit definition is carried in [MS74, Appendix A, p. 259] or in [Mun18, §42, Chapter 5, p. 251] using singular (co-)homology or simplicial (co-)homology in the latter. One really should think of this pairing as integrating differential forms on the manifold (with $R = \mathbb{R}$); in fact, some authors use the notation $\langle x, [M]\rangle = \int_M x$ for the Kronecker pairing.

Properties 1.31. Let M^m, N^n be a smooth connected oriented manifolds and let $[M]$ be the fundamental class of M.

1. If $x \in H^{m-1}(M, \partial M)$, then $\langle dx, [M]\rangle = \langle x, [\partial M]\rangle$, see [Mun18, §42, Chapter 5, p. 251].

2. Let $f : M \to N$ be continuous. Then, for $x \in H^p(N)$ and $y \in H_p(M)$:

$$\langle x, f_*(y)\rangle = \langle f^*(x), y\rangle,$$

   see [Mun18, Exercise 2, Chapter 5, p. 278]. In particular, for $x \in H^m(N)$, we have that

$$\langle x, f_*([M])\rangle = \langle f^*(x), [M]\rangle.$$

3. **Cross product:** (See [Mun18, §61, Chapter 7, pp. 360-366] or [MS74, p. 266]) Consider the product manifold $M \times N$, with projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N$. Define the *cross product* of $x \in H^p(M)$ and $y \in H^q(N)$ by

$$x \times y := \pi_M^*(x) \smile \pi_N^*(y) \in H^{p+q}(M \times N).$$

   It is easily seen to be bilinear. Furthermore, it satisfies:

$$\langle x \times y, [M \times N]\rangle = \langle x, [M]\rangle \cdot \langle y, [N]\rangle.$$

Note that instead of defining the cross product in terms of the cup product, one can also first define the cross product and then use it to define the cup product, see [Mun18, Exercise 2, Chapter 7, p. 366].

Chapter 2

Vector bundles, characteristic classes

2.1 Fiber bundles and vector bundles

The notion of fiber bundles captures the idea of a topological space (e.g. a manifold) being locally the product of two spaces but not necessarily on a global scale. For example, the *Möbius band* in the illustration below has locally the structure of the Cartesian product $S^1 \times [-1, 1]$, but has a twist which can only be seen when looking at it in its entirety.

We first start by defining the general notion of fiber bundles. Since we will mainly work with the notion of vector bundles, we won't say too much about general fiber bundle and we instead specialize to the context of vector bundles.

Let F be a smooth manifold.

Definition 2.1. A **smooth fiber bundle with fiber** F is a smooth surjective map $\pi : E \to B$ between smooth manifolds, such that for all

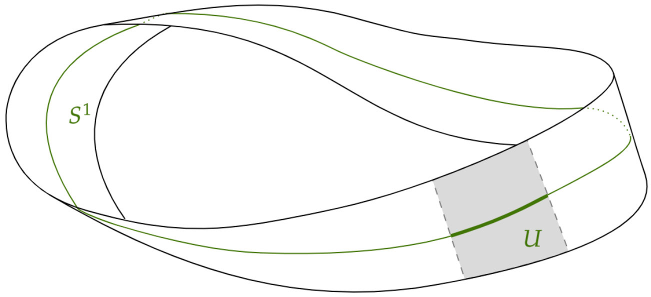

Figure 2.1: The *Möbius band*. The shaded area is homeomorphic to $U \times I$

.

$b \in B$, there exist an open set $U \subset B$, $b \in U$ and a diffeomorphism $\phi : \pi^{-1}(U) \xrightarrow{\cong} U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times F \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \mathrm{proj}_1} & \\ U & & \end{array}$$

where $\mathrm{proj}_1 : U \times F \to U$ is the projection onto the first factor.
E is called the **total space**, B the **base space**, π the **projection**, and ϕ is called a **local trivialization**. Since for any $b \in B$, the preimage $E_b := \pi^{-1}(b)$ is diffeomorphic to the model fiber F, we say that E_b is a **fiber** of the bundle. We might also say that the bundle is *over* the base space B.

We may sometimes refer to the bundle $\pi : E \to B$ by simply using the total space E. Notice that the *Möbius band* above can be realized as the total space of a smooth fiber bundle over the circle S^1 with model fiber $[-1, 1]$. The projection is then defined to be the mapping which associates every point (p, t) in the fiber over $p \in S^1$ to the point $(p, 0)$ corresponding to the middle of the fiber.

Let $\pi : E \to B$ be a smooth fiber bundle.

Definition 2.2. A smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$ is called a **section**. The collection of all sections is denoted by $\Gamma(E)$.

We will discuss this notion a little bit more in detail in the context of smooth vector bundles.

Now comes the important case when the fiber is a real or complex vector space, i.e. $F = \mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 2.3. A **smooth n-dimensional real vector bundle** is a surjective smooth map $\pi : E \to B$ between smooth manifolds, such that for all $b \in B$,

1. the **fiber** $E_b := \pi^{-1}(b)$ is a n-dimensional real vector space,
2. there exist an open set $U \subset B$, $b \in U$ and a diffeomorphism $\phi : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n$ which is fiberwise an isomorphism of vector spaces, i.e. for all $x \in U$, $\phi\big|_{E_x} \to \{x\} \times \mathbb{R}^n$ is an isomorphism of vector spaces.

E is called the **total space**, B the **base space**, π the **projection**, n the **rank** and ϕ is called a **local trivialization**. If U can be chosen to be all of B, so that there is only one local trivialization needed, we call the bundle **trivializable**. There is the related notion of a **smooth n-complex vector bundle** obtained by replacing **real** by **complex** and $\mathbb{R}$ by $\mathbb{C}$.

Examples 2.4.

1. The projection onto the first factor $\pi : B \times \mathbb{R}^n \to B$ defines a smooth n-dimensional vector bundle called the *trivial bundle* and denoted by $\varepsilon^n = \varepsilon^n_B$.

2. The tangent bundle TM of a smooth n-dimensional manifold M is a rank n real vector bundle over M.

3. The *real tautological line bundle* $\gamma^1(\mathbb{R}^{n+1})$ is defined by

$$\gamma^1(\mathbb{R}^{n+1}) := \{(L, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \in L\}$$

   This is a real vector bundle of rank 1 over $\mathbb{R}P^n$. For $n = 1$, it is essentially the *Möbius band*. In a similar way, the *complex tautological line bundle* $\gamma^1(\mathbb{C}^{n+1})$ is a complex 1-vector bundle over $\mathbb{C}P^n$ bundle defined by

$$\gamma^1(\mathbb{C}^{n+1}) := \{(L, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | v \in L\}.$$

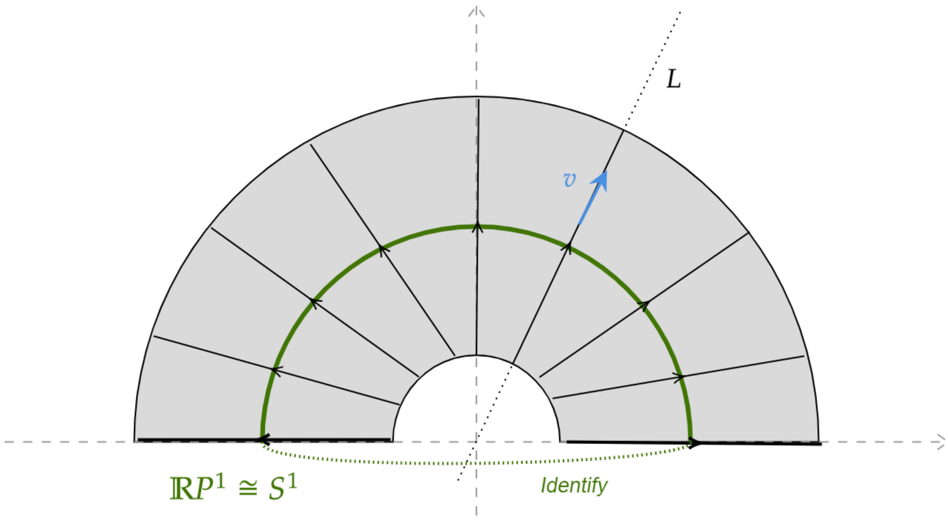

Figure 2.2: The Möbius band as a vector bundle

The following definition gives a notion of sameness for vector bundles.

Definition 2.5. Let $\pi : E \to B$ and $\pi' : E' \to B'$ be two smooth vector bundles and let $f : B \to B'$ be a smooth map. A smooth map $F : E \to E'$

is called **bundle map** if the following diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

commutes and F is fiberwise linear, i.e. for all $b \in B$, $F\big|_{E_b} : E_b \to E'_{f(b)}$ is linear. We say F **covers** f. The bundle map $F : E \to E'$ is called an **isomorphism** if $f : B \to B'$ is a diffeomorphism and F is fiberwise an isomorphism of vector spaces.

Definition 2.6. Let $f : B \to B'$ be smooth and $\pi : E \to B'$ be a smooth vector bundle of rank n. Then,

$$f^*E = \{(b, e) \in B \times E \mid f(b) = \pi(e)\} \subset B \times E,$$

together with the projection $f^*\pi : f^*E \to B, (b, e) \mapsto b$ is also a rank n smooth vector bundle, called the **pullback bundle** of E by f.

Example 2.7. A rank n vector space over a space B is trivializable $\iff$ it is isomorphic to ε^n_B $\iff$ it is isomorphic to a pullback bundle over the one-point space $\{*\}$ using the only map $B \to \{*\}$.

We will see that this is a particularly useful construction. It will notably imply that isomorphic bundles must have the same characteristic classes. Another useful construction is the following:

Definition 2.8. Given two vector bundles $\pi_i : E_i \to B$, $i = 1, 2$, over the same space B, one can construct the **Whitney sum** $E_1 \oplus E_2$, which is a vector bundle over B whose fibers are direct sums $E_{1,b} \oplus E_{2,b}$. Namely, $E_1 \oplus E_2 = \Delta^*(E_1 \times E_2)$, where $\Delta : B \to B \times B$ is the diagonal embedding and $\pi_1 \times \pi_2 : E_1 \times E_2 \to B \times B$ is the Cartesian product of bundles.

The concept of Whitney sum arises very naturally when studying smooth manifolds. Namely, for a smooth product manifold $M \times N$, it turns out that its tangent bundle $T(M \times N)$ is isomorphic to the Whitney sum $\pi_M^* TM \oplus \pi_N^* TN$, where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the evident projections.

Example 2.9. A vector bundle $\pi : E \to B$ is said to be *stably trivial* if the Whitney sum $E \oplus \varepsilon^k$ is trivial for some $k \geq 1$, where $\varepsilon^k = \varepsilon_B^k$ is the trivial vector bundle of rank k over B. For example, the tangent space of the sphere S^n is stably trivial. That is, if we consider the natural inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$, it is easy to see that $TS^n \oplus (TS^n)^\perp = T\mathbb{R}^{n+1}|_{S^n}$, where $(TS^n)^\perp$ is the set of all vectors normal to the sphere and $T\mathbb{R}^{n+1}|_{S^n}$ is the tangent bundle of $\mathbb{R}^{n+1}$ restricted to the unit sphere S^n. This is clearly a vector bundle over S^n and it can in fact be identified with the trivial bundle ε^1.

Several constructions for vector spaces (e.g. $V \otimes W$, V^*, $\mathrm{Bil}(V, W)$, $\Lambda^k V$, etc.) can be extended to vector bundles, see [MS74, Chapter 3].

Remark 2.10.

1. A section s is a smooth embedding of B into E.
2. If $s(b) = 0_b \in E_b$ for all b, s is called the *zero section*. Note that every vector bundle admits a section, namely the zero section.
3. A section $X : M \to TM$ is none other than a (smooth) vector field on M, that is, $\Gamma(M) = \Gamma(TM)$. Similarly, a k-differential form is a section of a certain smooth vector bundle $\Lambda^k T^* M$.
4. Let $f : B \to B'$ be smooth and $\pi : E \to B'$ a smooth rank n vector bundle. Any section $s : B' \to E$ defines a section $f^* s : B \to f^* E$, called the *pullback section* of s, given by $s(b) = (b, s(f(b))$ for $b \in B$.

2.2 Principal bundles and associated bundles

To conclude this introduction to vector bundles, let us introduce the two notions of a *principal bundle* and an *associated bundle*.
Given any smooth fiber bundle $\pi : E \to B$ with fiber F, one might wonder what happens if F is replaced by some other space F'. To make sense of this, let $\{U_i\}_i$ be a collection of trivializing subsets of $\pi : E \to B$ with respective local trivializations $\{\phi_i\}_i$. Then, for i and j such that $U_i \cap U_j \neq \emptyset$, we have the following commutative diagram:

$$\begin{array}{ccc} & \pi^{-1}(U_i \cap U_j) & \\ {}^{\phi_i}\swarrow & & \searrow^{\phi_j} \\ (U_i \cap U_j) \times F & \xrightarrow[\phi_j \circ \phi_i^{-1}]{} & (U_i \cap U_j) \times F \end{array}$$

The diffeomorphism $\phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ has the form $\phi_j \circ \phi_i^{-1}(b, v) = (b, t_{ij}(b)(v))$, where the maps $t_{ij} : U_i \cap U_j \to \mathcal{G} \subset \mathrm{Diff}(F)$ are called the *transition maps* of the bundle. The subgroup $\mathcal{G} \subset \mathrm{Diff}(F)$ is called the *structure group* of the bundle.
The transition maps are smooth in the sense that

$$(U_i \cap U_j) \times F \to F, (b, v) \mapsto t_{ij}(b)(v)$$

is a smooth map for all i, j.

Remarks 2.11. 1. The transition maps $\{t_{ij}\}_{i,j}$ satisfy the so-called *cocycle condition*, i.e. for all i, j, k with nonempty overlap, the following holds:

   (a) $t_{ii}(b) = id_F$ for all $b \in U_i$,

   (b) $t_{ij}(b) \circ t_{jk}(b) = t_{ik}(b)$ for all $b \in U_i \cap U_j \cap U_k$.

It turns out that, given a collection of maps $\{t_{ij}\}_{i,j}$ which satisfy the cocycle condition, there exists a (unique up to isomorphism) fiber bundle which has the maps t_{ij} as transition functions. This is formally known as the *fiber bundle construction theorem*, see [Bau22].

2. In the case of the tangent bundle TM of a smooth manifold M, one can verify that the structure group $\mathcal{G}$ is given by the general linear group $GL(n)$. If if M is orientable, it is given by $GL^+(n)$, i.e. the group of invertible matrices with positive determinant. Furthermore, if M is a Riemannian manifold, the structure group is the orthogonal group $O(n)$ and it is the special orthogonal group $SO(n) = SO_n = Gl^+(n) \cap O(n)$ if M is both orientable and has a Riemannian structure.

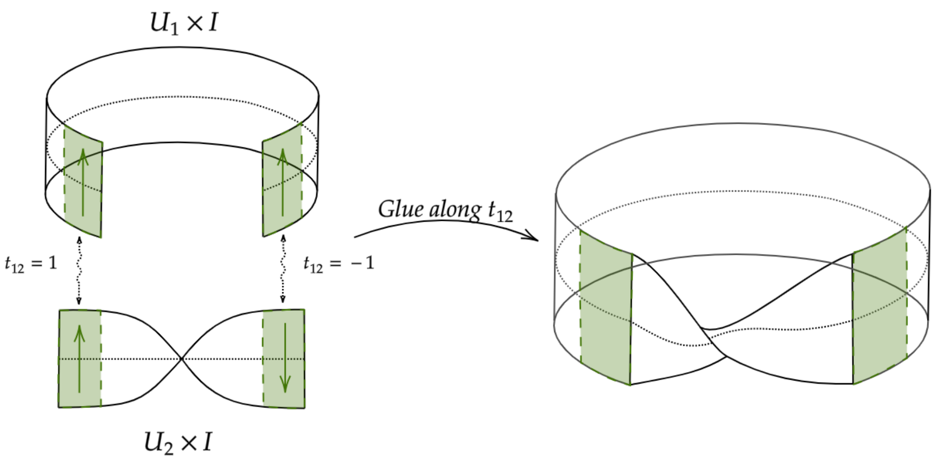

Figure 2.3: The transition functions for the Möbius band.

In the particular situation that the fiber F of a fiber bundle is equal to the p-dimensional disk D^p and the structure group is the general linear group $GL(p)$ (or one of its subgroups like SO_p) and not necessarily all of Diff(D^p), we call the bundle a *linear* D^p*- bundle*.

Given a fiber bundle $E \to B$ with fiber F, we know there exist transition maps $\{t_{ij}\}_{i,j}$ associated to this bundle acting on the fibers. Suppose furthermore that the transition maps act (in a suitable manner) on another space F'. By the remark above, one can construct a fiber bundle with the same base and total spaces as the original bundle but now with the fiber being F' instead of F. In this context, the next question arises quite naturally: *How does the fiber bundle with fiber F' relate to the original one with fiber F ?*

In the following, we formalize this idea which will come in handy later on.

Recall that a *Lie group* G is a smooth manifold which also has the structure of a group, that is, there exist a group law $\cdot$ on G such the inverse $(\cdot)^{-1} : G \to G$ and the multiplication $\cdot : G \times G \to G$ are smooth, or equivalently, if the map $G \times G \to G, (x, y) \mapsto x^{-1} \cdot y$ is smooth.

Let G be a Lie group and let $\pi : P \to B$ be a smooth fiber bundle equipped with a group action $P \times G \to G$, which:

1. is *smooth on the right*, i.e. for every $g \in G$, the multiplication on the right $R_g : P \to P, p \mapsto p \cdot g$ is smooth,

2. *preserves the fibers*, i.e. for any $b \in B$, if $p \in P_b$ is in the fiber over b, then so is $p \cdot g$ for every $g \in G$.

Definition 2.12. The bundle $\pi : E \to B$ is said to be a **(smooth) principal G-bundle** if the right action is:

1. **free**, i.e. if $p \cdot g = p$ for some $p \in P$, then $g = e$ is the identity of the group G,

2. **transitive** on the fibers, i.e. for any two $p, q \in P_b$, there exists an element $g \in G$ such that $p \cdot g = q$. The Lie group G is called the **structure group** of the principal bundle.

Note that one can show that any fiber of a principal G-bundle is diffeomorphic to G itself once a basis point is chosen and that the quotient E/G is diffeomorphic to B. We will see later that both notions of structure groups are related.

Example 2.13. The simplest but nonetheless important example of a principal G-bundle is that of the trivial bundle $X \times G$ equipped with the obvious right action. An important fact about this bundle is that any principal G-bundle which admits a global section is isomorphic to the trivial bundle $X \times G$, see [Bau22].

Example 2.14. Let $\pi : E \to B$ be a smooth vector bundle of rank n. Recall that for $b \in B$, the fiber E_b is an n-dimensional vector space.
An ordered basis of E_b is called a *frame*. The set of all frames at a point $b \in B$,

$$FE_b := \{(v_1, \cdots, v_n) \text{ is a basis of } E_b\},$$

is acted upon by right multiplication with invertible $n \times n$ matrices in a free and transitive manner, since there is a unique matrix in $GL(n)$ that sends any two bases from one to the other. The union

$$F(E) = F_{GL}(E) := \bigsqcup_{b \in B} FE_b = \bigcup_{b \in B} \{b\} \times FE_b$$

is called the *(unoriented) frame bundle* of E and can be given the structure of a principal $GL(n)$-bundle. Namely, the smooth right action is given by

$$F(E) \times GL(n) \to F(E) \tag{2.1}$$

$$((b, v_1, \cdots, v_n), G) \mapsto \left(b, \sum_{i=1}^{n} G_{i1} \cdot v_i, \cdots, \sum_{i=1}^{n} G_{in} \cdot v_i\right). \tag{2.2}$$

If M is a smooth manifold, it is usual to denote by $F(M)$ the frame bundle of the tangent bundle TM. Two further variations on this construction are the *orthonormal frame bundle* $F_O(M)$ and the *oriented orthonormal frame bundle* $F_{SO}(M)$, provided M carries a Riemannian metric for the former and is also oriented for the latter. They can be constructed in a completely analogous manner by respectively replacing $GL(n)$ by $O(n)$ or $SO(n)$ and choosing the bases to be orthonormal and also compatible with the orientation in the second case. They are all related by the notion of *reduction of the structure group*, which we will not discuss.

Now, let $\pi : P \to B$ be a smooth principal G-bundle. Recall that the fibers P_b are copies of the group G. We can finally answer the question which was asked previously: *What happens if one changes the fibers of P to a new set of fibers on which G acts?*
Let F be a smooth manifold on which G acts smoothly on the left, i.e. the left multiplication $L_g : F \to F, x \mapsto g \cdot x$ is smooth for all $g \in G$. Then, G acts freely on $P \times F$ (on the left) via the *diagonal action*:

$$P \times F \times G \to P \times F \tag{2.3}$$

$$(p, x, g) \mapsto (p \cdot g^{-1}, g \cdot x). \tag{2.4}$$

The quotient space $G\backslash(P \times F)$ of the diagonal action, together with the quotient map, forms a smooth fiber bundle over B with fiber F and structure group $\mathcal{G} = G$, see [GW09, p.92]. The quotient space is usually denoted by $P \times_G F$.

Definition 2.15. The bundle $P \times_G F$ is called the **bundle with fiber F associated to the principal bundle $\pi : P \to B$.**

2.3 Characteristic classes

We will now define the axioms of certain characteristic classes of vector bundles. These define topological invariants living in the cohomology ring of the base space and they can be seen as obstructions to a vector bundle being trivializable. In other words, they measure how "twisted" vector bundles are. A more detailed discussion as well as proofs of their existence can be found in the eponymous book by Milnor and Stasheff, see [MS74].

2.3.1 Chern classes

Definition 2.16 (Chern class)**.** To each complex vector bundle $\pi : E \to B$, one can associate an element $c(E) = 1 + c_1(E) + c_2(E) + \cdots \in H^{2*}(B;\mathbb{Z})$, where $c_i(E) \in H^{2i}(B;\mathbb{Z})$, satisfying the four following axioms:

1. **Naturality:** $c(E) = f^*c(E')$, if $f : B \to B'$ is a smooth map between the base spaces covered by a bundle isomorphism $F : E \to E'$,
2. **Whitney sum formula:** $c(E \oplus F) = c(E)c(F)$,
3. **Rank:** $c_i(E) = 0$ for all i strictly bigger than the rank of E,
4. **Normalization:** $c_1(\gamma^1) := -a$ is a generator of $H^2(\mathbb{C}P^1,\mathbb{Z}) \cong \mathbb{Z}$, where $a \in H^2(\mathbb{C}P^1,\mathbb{Z})$ is compatible with the natural orientation of $\mathbb{C}P^1$, i.e $\langle a, [\mathbb{C}P^1]\rangle = +1$. Here, $\gamma^1 = \gamma^1(\mathbb{C}^2)$ is the tautological line bundle over $\mathbb{C}P^1$.

The cohomology class $c_i(E) \in H^{2i}(B;\mathbb{Z})$ is called the ***i*-th Chern class** of the vector bundle E and the element $c(E) \in H^{2*}(B;\mathbb{Z})$ is called the **total Chern class** of E.

The Chern classes can be inductively constructed from the *Gysin sequence* as can be found in [MS74, Chapter 14, pp.157-158]. The naturality axiom implies both that isomorphic vector bundles have equivalent Chern classes and that $c(E) = 1$ for a trivializable vector bundle $\pi : E \to B$, see [MS74, Lemma 14.2, Chapter 14, p.158]. The Whitney sum formula can be

rewritten as $c_k(E \oplus F) = \sum_{i+j=k} c_i(E) \smile c_j(F)$. This in turn implies the *stability* of the Chern class, i.e., $c(E \oplus \varepsilon^k) = c(E)$ for all $k \geq 0$, where ε^k is the trivial bundle. This can also be proved without using the Whitney sum formula, as is done in [MS74, Lemma 14.3, Chapter 14, p.159]
We denote by $c(M)$ the total Chern class of the tangent bundle TM of a smooth complex manifold M. There exists another characteristic class, called the *Euler class*, which is only defined for oriented real vector bundles in [MS74, Chapter 9. p 98] using the *Thom isomorphism*. For more details on this theorem, we refer to [MS74, Chapter 10]. Since the complex plane $\mathbb{C}$ has a natural orientation as the 2-dimensional real plane $\mathbb{R}^2$, it can be shown ([MS74, Lemma 14.1, Chapter 14, p.155]) that a complex vector bundle induces a preferred orientation on its underlying real vector bundle, whose real rank is twice the complex rank of the original complex vector bundle. We give some properties of the Chern class without proof, see [MS74, Lemma 14.9 and Theorem 14.10, Chapter 10, pp.168-169].

Properties 2.17. Let $\pi : E \to B$ be a smooth complex vector bundle of (complex) rank n.

1. Then, $c_n(E) = e(E_{\mathbb{R}})$, where $E_{\mathbb{R}}$ is the underlying $2n$-real vector bundle of E,

2. $c_i(\bar{E}) = (-1)^k c_i(E)$, where $\bar{E}$ denotes the conjugate bundle of E,

3. $c(\mathbb{C}P^n) = (1+a)^{n+1}$ for all $n \geq 1$.

We want to point out that in the construction of the Chern classes carried out in Milnor and Stasheff's book, one *first* defines the top Chern class to be equal to the Euler class and *then* works one's way down recursively. In this sense, the first property is actually a definition.

2.3.2 Pontrjagin classes

Since the Chern classes were only defined for complex vector bundle, let us define similar characteristic classes for real vector bundles, which were originated by Pontrjagin.
Let $\pi : E \to B$ be a real smooth vector bundle rank n and consider its complexification $E \otimes \mathbb{C} \cong E \oplus iE$, see [MS74, Chapter 15, p.173]. One can check ([MS74, Lemma 15.1, Chapter 15, p.173]) that $E \otimes \mathbb{C}$ is a smooth complex vector bundle of complex rank n and is isomorphic to its conjugate bundle. This means in particular, using the second property of Properties 2.17 above, that:

$$
\begin{aligned}
c(E \otimes \mathbb{C}) &= 1 + c_1(E \otimes \mathbb{C}) + \cdots + c_n(E \otimes \mathbb{C}) \\
= c(\overline{E \otimes \mathbb{C}}) &= 1 - c_1(E \otimes \mathbb{C}) + \cdots + (-1)^n c_n(E \otimes \mathbb{C}).
\end{aligned}
$$

All odd Chern classes of $E \otimes \mathbb{C}$ are therefore of order 2 and are not of interest. This motivates the following definition:

Definition 2.18 (Pontrjagin class)**.** To each real vector bundle $\pi : E \to B$, one can associate an element $p(E) = 1 + p_1(E) + p_2(E) + \cdots \in H^{4*}(B, \mathbb{Z})$, where $p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(B, \mathbb{Z})$. We call $p_i(E)$ the ***i*-th Pontrjagin class** of E and the element $p(E)$ the **total Pontrjagin class** pf E.

Again, we write $p(M) := p(TM)$ for the tangent bundle of a smooth manifold M. They inherit directly by definition the following properties:

Properties 2.19.
1. **Naturality:** $p(E) = f^*p(E')$, if $f : B \to B'$ is a smooth map between the base spaces covered by a bundle map $F : E \to E'$,

2. **Whitney sum formula:** $p(E \oplus F) = p(E)p(F)$ modulo 2-torsion, that is, for all $k \geq 0$,

$$p_k(E \oplus F) = \sum_{i+j=k} p_i(E) \smile p_j(F) \quad \text{modulo 2-torsion,}$$

3. **Rank:** $p_i(E) = 0$ if $2i$ is strictly bigger than the rank of E,

4. **Stability:** $p(E \oplus \varepsilon^k) = p(E)$ for all $k \geq 0$.

The naturality again guarantees both that isomorphic vector bundles have (pretty much) the same Pontrjagin classes and that $p(E) = 1$ for a trivial vector bundle $\pi : E \to B$.

Example 2.20 (Pontrjagin classes of S^n)**.** Recall from Example 2.9 that $TS^n \oplus \varepsilon^1$ is trivial. Then, by the stability property and the remark just above, $p(TS^n) = p(TS^n \oplus \varepsilon^1) = 1$. The Pontrjagin classes of S^n are therefore all trivial.

Example 2.21 (Pontrjagin classes of $\mathbb{C}P^n$)**.** Since the complexification is $(T\mathbb{C}P^n)_\mathbb{R} \otimes \mathbb{C} \cong T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}$, we have using the Whitney sum formula:

$$c((T\mathbb{C}P^n)_\mathbb{R} \otimes \mathbb{C}) = c(T\mathbb{C}P^n)c(T\overline{\mathbb{C}P^n}) = (1+a)^{n+1}(1-a)^{n+1} = (1-a^2)^{n+1}.$$

Notice that the odd Chern class of $(T\mathbb{C}P^n)_\mathbb{R} \otimes \mathbb{C}$ must vanish. Then, for $p_i(\mathbb{C}P^n) := p_i((T\mathbb{C}P^n)_\mathbb{R})$,

$$1 - p_1(\mathbb{C}P^n) + \cdots + (-1)^n p_n(\mathbb{C}P^n) = c((T\mathbb{C}P^n)_\mathbb{R} \otimes \mathbb{C}) = (1-a^2)^{n-1},$$

and therefore $p(\mathbb{C}P^n) = 1 + p_1(\mathbb{C}P^n) + \cdots + p_n(\mathbb{C}P^n) = (1+a^2)^{n+1}$.

Lastly, let us consider a smooth oriented closed manifold M of dimension $4n$.

Definition 2.22. Let $I : k_1 + k_2 + \cdots + k_r = n$ be a partition of n. Then,

$$p_I[X] := \langle p_{k_1} \smile p_{k_2} \smile \cdots \smile p_{k_r}, [M] \rangle,$$

is called the ***I*-th Pontrjagin number**.

The Pontrjagin numbers along with similarly defined *Stiefel-Whitney numbers* play a key role in the theory of oriented cobordism, see [MS74, Chapters 17 & 18].

Example 2.23 (Pontrjagin numbers of $\mathbb{C}P^{2n}$)**.** If $I : k_1 + k_2 + \cdots + k_r = n$ is a partition of n, then by Example 2.21

$$p_I[\mathbb{C}P^{2n}] = \binom{2n+1}{k_1}\binom{2n+1}{k_2}\cdots\binom{2n+1}{k_r}.$$

In particular, $\mathbb{C}P^{2n}$ cannot have an orientation-reversing diffeomorphism: if it did, we would have $p_I[\mathbb{C}P^{2n}] = 0$ since $-p_I[\mathbb{C}P^{n}] = p_I[-\mathbb{C}P^{2n}] = p_I[\mathbb{C}P^{2n}]$, but this is clearly not the case.

Chapter 3

The Hirzebruch formalism

3.1 Genus of a multiplicative sequence

We introduce the notion of multiplicative sequence as defined by Hirzebruch in [HBS66].

Consider a commutative ring B with 1 such as $\mathbb{Q}$ and the polynomial ring $\mathfrak{B} = B[p_1, p_2, \dots]$ with indeterminates $p_1, p_2, \dots,$[1] where we define p_i to have weight i and $p_{i_1} p_{i_2} \cdots p_{i_m}$ to have weight $i_1 + i_2 + \cdots + i_m$. This makes $\mathfrak{B}$ into a graded ring with $\mathfrak{B} = \bigoplus_{k \geq 0} \mathfrak{B}_k$ with $\mathfrak{B}_0 = B$ and $\mathfrak{B}_r \mathfrak{B}_s \subset \mathfrak{B}_{r+s}$, where $\mathfrak{B}_k$ is the B-module of homogeneous polynomials in $\mathfrak{B}$ of weight k. Let $K_0, K_1, K_2, \dots$ be a sequence of polynomials with $K_0 = 1$ and $K_j = K_j(p_1, p_2, \dots, p_j) \in \mathfrak{B}_j$.

Definition 3.1. The sequence $\{K_j\}$ is called a **multiplicative sequence** if whenever we have for an indeterminate z

$$1 + p_1 z + p_2 z^2 + \cdots = (1 + q_1 z + q_2 z^2 + \cdots)(1 + r_1 z + r_2 z^2 + \cdots),$$

[1] The notation $p_1, p_2, \dots$ is suggestively chosen as to remind the reader of the Pontrjagin classes since we will later evaluate the polynomial using Pontrjagin classes.

we also have

$$\sum_{j=0}^{\infty} K_j(p_1, \ldots, p_j)z^j = \sum_{j=0}^{\infty} K_j(q_1, \ldots, q_j)z^j \sum_{j=0}^{\infty} K_j(r_1, \ldots, r_j)z^j.$$

In other words, setting

$$K(1 + p_1 z + p_2 z^2 + \cdots) = 1 + K_1(p_1)z + K_2(p_1, p_2)z^2 + \cdots,$$

the sequence $\{K_j\}$ is multiplicative if and only if

$$\begin{aligned} &K((1 + p_1 z + p_2 z^2 + \ \cdots)(1 + q_1 z + q_2 z^2 + \ \cdots)) \\ &\qquad = K(1 + p_1 z + p_2 z^2 + \ \cdots)K(1 + q_1 z + q_2 z^2 + \ \cdots). \end{aligned}$$

The power series $Q(z) := K(1 + z)$ is called the **characteristic series** of the sequence $\{K_j\}$.

Example 3.2. Let $K_j(p_1, \ldots, p_j) = \lambda^j p_j$ for $\lambda \in B$. Then,

$$K(1 + p_1 z + p_2 z^2 + \cdots) = 1 + \lambda p_1 z + \lambda^2 p_2 z^2 + \cdots,$$

and moreover,

$$\begin{aligned} &K((1 + p_1 z + p_2 z^2 + \cdots)(1 + q_1 z + q_2 z^2 + \cdots)) \\ &\qquad = K\left(\sum_{k=0}^{\infty}\left(\sum_{i+j=k} p_i q_j\right) z^k\right) = \sum_{k=0}^{\infty} \lambda^k \left(\sum_{i+j=k} p_i q_j\right) z^k, \end{aligned}$$

as well as

$$\begin{aligned} &K(1 + p_1 z + p_2 z^2 + \cdots)K(1 + q_1 z + q_2 z^2 + \cdots) = \\ &\qquad \left(\sum_{i=0}^{\infty} \lambda^i p_i z^i\right)\left(\sum_{j=0}^{\infty} \lambda^j p_j z^j\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} \lambda^k p_i q_j\right) z^k. \end{aligned}$$

Since both of these two quantities are equal, the polynomials $\{K_j\}$ form a multiplicative sequence. The corresponding characteristic series is therefore

$$Q(z) = K(1 + z) = \sum_{j=0} K_j(1, 0, \cdots, 0) = 1 + \lambda.$$

The characteristic series $Q(z) = K(1+z) = 1 + b_1 z + b_2 z^2 + \cdots$ with $b_j = K_j(1, 0, \ldots, 0)$ completely determines the multiplicative sequence $\{K_j\}$. The following proposition due to Hirzebruch shows that there is in fact a one-to-one correspondence between multiplicative sequences and formal power series with constant term equal to 1.

Proposition 3.3. To each formal power series $Q(z) = 1 + \sum_{k=1}^{\infty} b_j z^j$ with $b_j \in B$, there is an associated multiplicative sequence $\{K_j\}$ such that the coefficient of p_1^j in $K_j(p_1, \ldots .p_j)$ is precisely b_j for every $j \geq 1$, that is, $Q(z) = K(1+z)$.

The proof can be found in [HBS66, p. 10]. We give two important examples, which will play a major role in the following chapters (here B_k designates the k-th *Bernoulli number*, see Appendix A).

Example 3.4. The first terms of the multiplicative sequence $\{L_j\}$ associated to the power series

$$Q(z) = \frac{\sqrt{z}}{\tanh \sqrt{z}} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} 2^{2k} \frac{B_k}{(2k)!} z^k$$

are

$$\begin{aligned}
L_0 &= 1, \\
L_1 &= \frac{1}{3} p_1, \\
L_2 &= \frac{1}{45}(7p_2 - p_1^2), \\
L_3 &= \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3), \\
L_4 &= \frac{1}{14175}(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4).
\end{aligned}$$

Example 3.5. The first terms of the multiplicative sequence $\{\hat{A}_j\}$ associated to the power series

$$Q(z) = \frac{\sqrt{z}/2}{\sinh \sqrt{z}/2} = 1 + \sum_{k=1}^{\infty} (-1)^k 2^{-2k}(2^{2k-1} - 2)\frac{B_k}{(2k)!} z^k$$

are

$$\begin{aligned}
\hat{A}_0 &= 1, \\
\hat{A}_1 &= -\frac{1}{24} p_1, \\
\hat{A}_2 &= \frac{1}{5760}(-4p_2 + 7p_1^2), \\
\hat{A}_3 &= \frac{1}{967680}(-16p_3 + 44p_2p_1 - 31p_1^3), \\
\hat{A}_4 &= \frac{1}{464486400}(-192p_4 + 512p_3p_1 + 208p_2^2 - 904p_2p_1^2 + 381p_1^4).
\end{aligned}$$

The computations for the Taylor series expansions are found in Appendix A.

Let us now specialise to the case where $B = \mathbb{Q}$ and the indeterminates $p_1, p_2, \ldots$ are the Pontrjagin classes of a smooth closed manifold[2] M of dimension $4n$. Let $\{K_k\}$ be a multiplicative sequence with coefficients in $\mathbb{Q}$.

Definition 3.6. The K**-genus** of M^{4n} is given by evaluating $K_n(p_1, \ldots, p_n)$ on the fundamental class of M, i.e.

$$K[M] := \langle K_n(p_1, \ldots, p_n), [M] \rangle \in \mathbb{Q},$$

where $p_1, \ldots, p_n$ are the Pontrjagin classes of the manifold M, i.e. $p_i = p_i(M)$.

[2]If the dimension is not divisible by 4, we set $K[M] = 0$.

It can be shown that

- $K[M+N] = K[M] + K[N]$, where $M + N = M \cup N$ is the disjoint union of M and N,
- $K[M \times N] = K[M] \cdot K[N]$ for the product manifold $M \times N$,
- $K[M] = 0$ if M is a boundary,

see for example [HBS66, pp. 77-84].

Example 3.7. A fundamental example is the L-genus of the complex projective plan $\mathbb{C}P^{2n}$. Namely, the multiplicative sequence $\{L_j\}$ satisfies $L[\mathbb{C}P^{2n}] = 1$.
Recall from Example 2.21 that the total Pontrjagin class of $\mathbb{C}P^{2n}$ is given by $p(\mathbb{C}P^{2n}) = (1+a^2)^{2n+1}$, where $a \in H^2(\mathbb{C}P^{2n}; \mathbb{Z}$ is the preferred generator. Then,

$$\begin{aligned} L(p(\mathbb{C}P^{2n})) &= L((1+a^2)^{2n+1}) \\ &= (L(1+a^2))^{2n+1} = \left(\frac{\sqrt{a^2}}{\tanh\sqrt{a^2}}\right)^{2n+1} = \left(\frac{a}{\tanh a}\right)^{2n+1}. \end{aligned}$$

Since $H^*(\mathbb{C}P^{2n}, \mathbb{Z}) = \mathbb{Z}[a^2]/(a^{2n+1})$, it follows that $L_n(p_1, \ldots, p_n)$ is a monomial $\lambda_{2n} a^{2n}$, whose coefficient is equal to the coefficient before a^{2n} in the power series above. Therefore,

$$\langle L_n(p_1, \ldots, p_n), [\mathbb{C}P^{2n}]\rangle = \langle \lambda_{2n} a^{2n}, [\mathbb{C}P^{2n}]\rangle = \lambda_{2n}\langle a^{2n}, [\mathbb{C}P^{2n}]\rangle = \lambda_{2n}.$$

After having replaced a by $z \in \mathbb{C}$, we use the Cauchy integral formula to compute the coefficient of z^{2n} in the Laurent expansion of $\left(\frac{z}{\tanh z}\right)^{2n+1}$. Namely, integrating on a small circle around the origin:

$$\lambda_{2n} = \frac{1}{2\pi i}\oint \frac{(z/\tanh z)^{2n+1}}{z^{2n+1}} dz = \frac{1}{2\pi i}\oint \frac{1}{(\tanh z)^{2n+1}} dz,$$

and substituting $u = \tanh z$,

$$= \frac{1}{2\pi i} \oint \frac{1 + u^2 + u^4 + \dots}{u^{2n+1}} du = \frac{1}{2\pi i} \oint \frac{1}{u} du = 1,$$

hence $L[\mathbb{C}P^{2n}] = 1$.

3.2 The Hirzebruch signature theorem

Let M be a smooth oriented closed $4n$-dimensional manifold. The cup product of cohomology defines a bilinear map

$$H^{2n}(M;\mathbb{Q}) \times H^{2n}(M;\mathbb{Q}) \to \mathbb{Q},\ (\alpha, \beta) \mapsto \langle \alpha \smile \beta, [M] \rangle.$$

Definition 3.8. The index $\sigma(M)$ of the quadratic form associated to this bilinear form is called the **signature** of M.

Example 3.9 (Signature for $\mathbb{C}P^{2n}$)**.** Remember that for $a \in H^2(\mathbb{C}P^{2n};\mathbb{Q})$, a^n is the generator of $H^{2n}(\mathbb{C}P^{2n};\mathbb{Q}) \cong \mathbb{Q}$ compatible with the preferred orientation, that is, $\langle a^n \smile a^n, [\mathbb{C}P^{2n}] \rangle = 1$, hence $\sigma(\mathbb{C}P^{2n}) = 1$.

Theorem 3.10 (Hirzebruch signature theorem)**.** For a smooth oriented closed manifold M of dimension $4n$, one has

$$\sigma(M) = L[M] = \langle L_n(p_1(M), \dots, p_n(M)), [M] \rangle.$$

Sketch of the proof. We will use the theory of cobordism defined by Thom in [Tho54]. See also the section on cobordism later.
Two smooth oriented closed manifolds M and N of dimension n are called *oriented cobordant* if there exists a smooth compact oriented manifold W such that $\partial W = M \cup -N$, where $-N$ is the manifold N with the opposite orientation. This defines an equivalence relation and we write Ω_n the collection of all cobordism classes of dimension n. Then, the direct sum

$\Omega_* = \bigoplus_{n=0}^{\infty} \Omega_n$ forms a graded ring, called the *oriented cobordism ring*, where the addition is given by the disjoint union and the multiplication by the Cartesian product. By killing torsion in Ω_* by tensoring with $\mathbb{Q}$, Thom showed [Tho54, p. 84] that $\Omega_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \dots]$ is a polynomial ring. Furthermore, $M \mapsto \sigma(M)$ and $M \mapsto L[M]$ are ring homomorphisms $\Omega_* \to \mathbb{Q}$. We thus only need to check the formula for the generators $\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \dots$ But by Example 3.7 and Example 3.9, we have $\sigma(\mathbb{C}P^{2n}) = 1 = L[\mathbb{C}P^{2n}]$. □

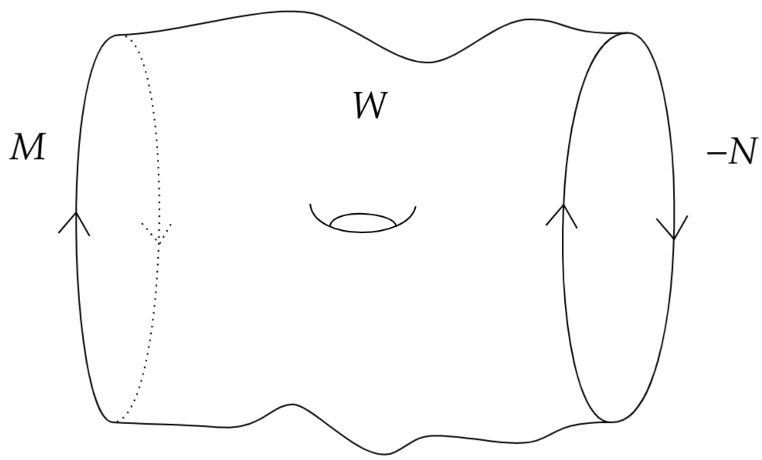

Figure 3.1: An oriented cobordism W between M and N.

Definition 3.11. The **$\hat{A}$-genus** of a smooth $4n$-dimension manifold M is

$$\hat{A}[M] = \langle \hat{A}_n(p_1, \dots, p_n)), [M] \rangle.$$

The following theorem is a consequence of the *Atiyah-Singer index theorem.* Namely, the $\hat{A}$-genus is actually the topological index of a certain Dirac operator. For more information, see [Bes07, pp. 169-170]. The Hirzebruch signature theorem 3.10 can also be shown using the theory of Atiyah-Singer.

Theorem 3.12 (Lichnerowicz)**.** If M is a compact connected spin Riemannian manifold of dimension 4k with positive scalar curvature, then $\hat{A}[M] = 0$.

We do not want to define precisely what a *spin* manifold is. We only note that an oriented manifold with $H^2(M;\mathbb{Z}_2) \cong 0$ can be shown to be spin.

3.3 Some computations

Proposition 3.13. Let s_k and a_k be the leading coefficient of p_k in $\{L_k\}$ and $\{\hat{A}_k\}$, respectively. Then,

$$s_k = 2^{2k}(2^{2k-1}-1)\frac{B_k}{(2k)!} \quad \text{and} \quad a_k = -\frac{B_k}{2(2k)!}.$$

Proof. Following [HBS66, p. 11], the coefficients s_k can be found using:

$$1 - z\frac{d}{dz}\log\frac{\sqrt{z}}{\tanh\sqrt{z}} = \sum_{k=0}^{\infty}(-1)^k s_k z^k.$$

Computing the left-hand side, we obtain

$$\begin{aligned}
1 - z\frac{d}{dz}\log\frac{\sqrt{z}}{\tanh\sqrt{z}} &= 1 - z\left(\frac{1}{2\sqrt{z}} - \frac{1-\tanh^2\sqrt{z}}{\tanh\sqrt{z}}\frac{1}{2\sqrt{z}}\right) \\
&= \frac{1}{2} + \frac{\sqrt{z}}{2}\frac{1}{\sinh\sqrt{z}\cosh\sqrt{z}} \\
&= \frac{1}{2} + \frac{1}{2}\frac{2\sqrt{z}}{\sinh 2\sqrt{z}} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k 2^{2k}(2^{2k-1}-1)\frac{B_k}{(2k)!}z^k,
\end{aligned}$$

where the last equality comes from Lemma A.2 in Appendix A. This directly yields

$$s_k = 2^{2k}(2^{2k-1} - 1)\frac{B_k}{(2k)!}.$$

We now compute a_k by using the corresponding formula for the series $\frac{\sqrt{z}/2}{\sinh \sqrt{z}/2}$:

$$1 - z\frac{d}{dz}\log\frac{\sqrt{z}/2}{\sinh\sqrt{z}/2} = \sum_{k=0}^{\infty}(-1)^k a_k z^k.$$

Similarly, using Lemma A.1, the LHS is

$$\begin{aligned}
1 - z\frac{d}{dz}\log\frac{\sqrt{z}/2}{\sinh\sqrt{z}/2} &= \frac{1}{2} + \frac{1}{2}\frac{\sqrt{z}/2}{\tanh\sqrt{z}/2} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{2k}}{2\cdot 2^{2k}}\frac{B_k}{(2k)!}z^k \\
&= 1 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{2(2k)!}z^k.
\end{aligned}$$

Comparing coefficients, we get

$$a_k = -\frac{B_k}{2(2k)!}.$$

□

The next proposition will be helpful later on.

Proposition 3.14. The coefficient of $p_m p_n$ in L_{m+n} is equal to

$$\begin{aligned}
s_m s_n - s_{m+n} \quad &\text{if} \quad m \neq n, \\
(s_m s_n - s_{m+n})/2 \quad &\text{if} \quad m = n.
\end{aligned}$$

Proof. Compare [MS74, Chapter 16] or [HBS66, pp. 76–77]. Let M^{4m} and N^{4n} be smooth oriented closed manifolds whose only non-zero Pontrjagin

numbers are $p_m[M]$ and $p_n[N]$, respectively. Recall that $T(M \times N) \cong \pi_M^*(TM) \oplus \pi_N^*(TN)$, where π_M and π_N are the obvious projections.
Let us now relate the Pontrjagin classes of M and N to those of $M \times N$ using the Properties 2.19. Namely, we have

$$\begin{aligned} p_k(M \times N) &= p_k(T(M \times N)) \\ &= p_k(\pi_M^*(TM) \oplus \pi_N^*(TN)) \\ &= \sum_{i+j=k} p_i(\pi_M^*(TM)) \smile p_j(\pi_N^*(TN)) \quad \text{mod 2-torsion} \\ &= \sum_{i+j=k} \pi_M^*(p_i(M)) \smile \pi_N^*(p_j(N)) \quad \text{mod 2-torsion} \\ &= \sum_{i+j=k} p_i(M) \times p_j(N) \quad \text{mod 2-torsion.} \end{aligned}$$

Using the third property of Properties 1.31, this implies that

$$p_{m+n}[M \times N] = p_m[M]p_n[N],$$

and

$$p_m p_n[M \times N] = \begin{cases} p_m[M]p_n[N] & \text{if } m \neq n, \\ 2p_m[M]p_n[N] & \text{if } m = n. \end{cases}$$

All other Pontrjagin numbers are zero. Making use of the multiplicativity of the L-genus, that is, $L[M \times N] = L[M] \cdot L[N]$, one gets for $s_{m,n}$ the coefficient of $p_m p_n$ in L_{m+n}:

$$\begin{aligned} &s_{m,n} p_m p_n[M \times N] + s_{m+n} p_{m+n}[M \times N] \\ &\qquad = s_{m,n} p_m[M]p_n[N] + s_{m+n} p_m[M]p_n[N] = s_m p_m[M] \cdot s_n p_n[N], \end{aligned}$$

if $m \neq n$ and

$$\begin{aligned} &s_{m,n} p_m p_n[M \times N] + s_{m+n} p_{m+n}[M \times N] \\ &\qquad = 2s_{m,n} p_m[M]p_n[N] + s_{m+n} p_m[M]p_m[N] = s_m p_m[M] \cdot s_n p_n[N], \end{aligned}$$

if $m = n$. Rearranging the equations and simplifying by $p_m[M]p_n[N] \neq 0$ finishes the proof. $\square$

Remark 3.15. The same proof shows that the same relations still hold for the coefficient $a_{m,n}$ of $p_m p_n$ in $\hat{A}_{m+n}$.

Chapter 4

Twisted spheres

In the following chapter, we will closely follow Milnor and especially chapters 2, 3 and 4 of [Mil07].

4.1 Collar neighbourhoods

To start this chapter, let us recall the following definition:

Definition 4.1. Let M be a smooth manifold with boundary $\partial M \neq \emptyset$ and let $U \subset M$ be an open neighbourhood of ∂M. We say that U is a **collar neighbourhood** of ∂M if U is diffeomorphic to the cylinder $\partial M \times [0, 1)$.

One can always chose the diffeomorphism to be the identity on $\partial M \times \{0\}$. The following theorem gives a sufficient condition for a manifold to admit a collar neighbourhood:

Theorem 4.2. Assume that the boundary $\partial M \neq \emptyset$ is compact. Then, there exists a collar neighbourhood of ∂M within M.

An idea of the proof can be found in [Mil07, Theorem 2.9, Chapter 2, p. 202].

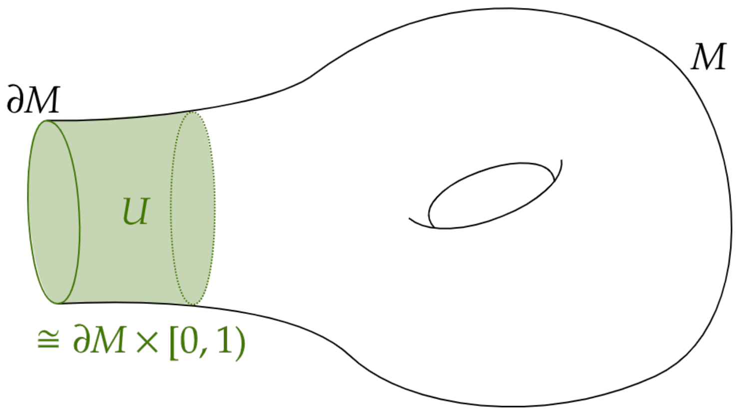

Figure 4.1: U is a collar neighbourhood of the smooth manifold M

The existence of collar neighbourhoods is very important since it lets us construct new manifolds by pasting together the boundaries of different manifolds, provided the boundaries are diffeomorphic.
Let M and N be smooth manifolds with nonempty boundaries ∂M and ∂N. Assume also that ∂M and ∂N are compact and connected.

Corollary 4.3. Let $h : \partial M \to \partial N$ be a diffeomorphism. Then, the space

$$M \cup_h N := M \sqcup N / \sim,$$

where $p \in \partial M$ is identified with $h(p) \in \partial N$, can be given the structure of a smooth manifold. As a smooth manifold, the space $M \cup_h N$ contains both M and N as smooth submanifolds.

If M and N are oriented, and if $h : \partial M \to \partial N$ is an orientation-preserving diffeomorphism, one should reverse the orientation of N and then glue M and N via h, i.e. the appropriate space would by $M \cup_h -N$, where $-N$ stands for the smooth manifold N with the opposite orientation.

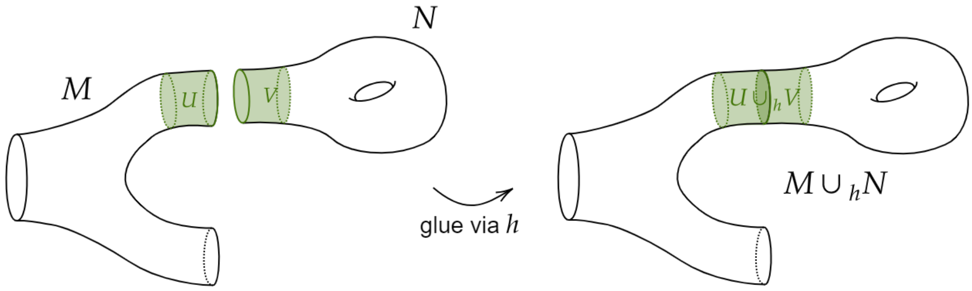

Figure 4.2: Gluing a *pair of pants* to a punctured torus

Proof. The space $M \cup_h N$ already has the structure of a smooth manifold everywhere but at the points of the identified boundary. Since the boundaries are compact, it follows from Theorem 4.2 that there exist collar neighbourhoods U and V of M and N respectively. i.e. $U \cong M \times [0, 1)$ and $V \cong N \times (-1, 0]$. Pasting both U and V via h yields a space $U \cup_h V$ which can be endowed with a smooth structure and is diffeomorphic (as a smooth manifold) to $\partial M \times (-1, 1)$. Hence, the space $M \cup_h N$ can be (globally) given a smooth structure. □

Remark 4.4. If the boundaries ∂M and ∂N are *not* connected, one can still make sense of the corollary, provided that both boundaries contain a connected component which are diffeomorphic. One can then paste together several boundary components as in the figure below.

4.2 Connected sum

Let M and N be connected oriented n-dimensional smooth manifolds. Let $f : D^n \to \mathring{M}$ be an orientation-preserving smooth disk embedding and let $g : D^n \to \mathring{N}$ be an orientation-reversing smooth disk embedding. Define

$$h : f(\mathring{D}^n \smallsetminus \{0\}) \to g(\mathring{D}^n \smallsetminus \{0\}),\ f(tu) \mapsto g((1-t)u)$$

for $u \in S^{n-1}$ and $0 < t < 1$. Note that this makes h into an orientation-preserving diffeomorphism between the embedded disks in M and N where their centers have been removed. Define $M\#N$ by gluing M and N via h, i.e.

$$M\#N := \Big(M \smallsetminus f(0)\Big) \cup_h \Big(N \smallsetminus g(0)\Big),$$

and give it the orientation compatible with that of M.

Definition 4.5 (Connected sum)**.** The oriented smooth n-dimensional manifold $M\#N$ is called the **connected sum** of M and N.

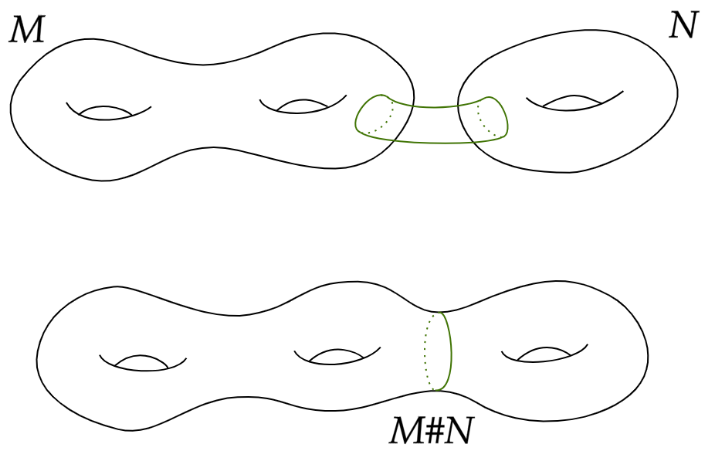

Figure 4.3: Illustration of the connected sum.

Lemma 4.6. The connected sum $M\#N$ is unique up to orientation-preserving diffeomorphism.

Proof. This follows from Palais' disk theorem[1], see [KM63, Lemma 2.1, Chapter 2, p. 505]. Namely, if $\tilde{f}, \tilde{g}$ are smooth disk embeddings defined similarly as in the definition and if $\tilde{h}$ is the corresponding orientation-preserving diffeomorphism, then the connected sum obtained using $\tilde{h}$ is diffeomorphic to the connected sum obtained using h via an orientation-preserving diffeomorphism. □

Remark 4.7.

1. If M and N are compact, then $M\#N$ is also compact.
2. $M\#N$ is connected, provided $n \geq 2$. This is not true in dimension 1, for example $\mathbb{R}\#\mathbb{R} \cong \mathbb{R} \cup \mathbb{R}$.

See [Mil07, Chapter 3, p. 203]

The connected sum operation can be understood as cutting open a disk in each of M and N and then connecting them along a tube going from one disk to the other. With this interpretation in mind (see Figure 4.3), one checks ([Mil07, Lemma 3.1, Chapter 3, p. 203]):

Properties 4.8. Let M, N, P be connected and oriented smooth n-dimensional manifolds, $n \geq 2$.

1. $M\#S^n \cong M$,
2. $M\#N \cong N\#M$,
3. $M\#(N\#P) \cong (M\#N)\#P$,

[1] It states that given two disk embeddings $f, g : D^n \to M$ with compatible orientations, one can find a diffeomorphism $h : M \to M$ with $h \circ f = g$ and $h\big|_{M \setminus K} = id_M$ outside a compact set $K \subset M$.

4. $M\#\mathbb{R}^n \cong M \setminus \{point\}$,
5. $M\#D^n \cong M \setminus f(\mathring{D}^n)$, where $f : D^n \to \mathring{M}$ is a smooth disk embedding,
6. The connected sum of a surface of genus g and a surface of genus h is diffeomorphic to a surface of genus $g+h$.

Notice that # almost defines a group operation on the class of n-dimensional manifolds. In fact, define $\mathfrak{M}_n$ to be the set of equivalence classes of connected, oriented, closed n-dimensional smooth manifolds under orientation-preserving diffeomorphism. Observe that the connected sum makes $\mathfrak{M}_n$ into a commutative monoid[2], where the neutral element is given by the diffeomorphism class of the n-sphere $[S^n]$. Write $\mathcal{A}_n$ for the group of invertible elements of $\mathfrak{M}_n$. One can show (see [Mil07, Chapter 3, p. 204]) that for $n \leq 4$:

- $\mathfrak{M}_1$ is trivial, since S^1 is, up to diffeomorphism, the only connected oriented closed smooth 1-dimensional manifold.
- $\mathfrak{M}_2$ is a free monoid generated by the torus.
- $\mathfrak{M}_3$ is a free monoid generated by infinitely many generators.
- $\mathfrak{M}_4$ is not free.

It follows that $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ contain only the neutral element given by the diffeomorphism class of the sphere. Elements in $\mathcal{A}_n$ are characterized by the following theorem:

[2] A *monoid* is a set together with an associative law, for which there exists a neutral element. This is thus very similar to the definition of a group, where the existence of inverses is not required.

Theorem 4.9 (Mazur)**.** If a smooth oriented connected closed n-dimensional manifold is invertible under the connected sum #, then it is homeomorphic to the standard sphere S^n.

Corollary 4.10. If $\mathcal{A}_n$ contains a diffeomorphism class other than that of the sphere, then there are at least $|\mathcal{A}_n|$ distinct differentiable structures on the n-sphere.

Mazur's theorem can be easily proved by introducing the notion of an *infinite* connected sum.
Let M_i, $i = 1, 2, 3, \dots$ be smooth oriented connected n-dimensional manifolds. Let $f_1 : D^n \to M_1$ be a smooth orientation-preserving disk embedding and let, for all $i \geq 2$, $f_i, g_i : D^n \to M_i$ be smooth disk embeddings such that the f_i preserve orientation and the g_i reverse it, and such that $f_i(D^n) \cap g_i(D^n) = \emptyset$.
The *infinite connected sum* is the quotient space

$$\Big(M_1 \setminus f_1(0)\Big) \uplus \Big(M_2 \setminus \{f_2(0), g_2(0)\}\Big) \uplus \Big(M_3 \setminus \{f_3(0), g_3(0)\}\Big) \uplus \cdots \Big/ \sim,$$

where the equivalence relation is given by $f_i(tu) \sim g_{i+1}((1-t)u)$, for $i = 1, 2, 3, \dots$, $u \in S^{n-1}$ and $0 < t < 1$. One can also check that this is well-defined and associative, at least up to orientation-preserving diffeomorphism. As an example, the infinite connected sum of n-spheres is somewhat surprisingly oriented diffeomorphic the Euclidean space $\mathbb{R}^n$, see Figure 4.4:

$$S^n \# S^n \# S^n \# \cdots \cong \mathbb{R}^n.$$

Proof. Let M and N be smooth oriented connected closed n-dimensional manifolds with $M \# N \cong S^n$. Consider the infinite connected sum of M and N:

$$M \# N \# M \# N \# M \# N \cdots$$

On the one hand,

$$(M\#N)\#(M\#N)\#\cdots \cong S^n\#S^n\#\cdots \cong \mathbb{R}^n,$$

and on the other,

$$M\#(N\#M)\#(N\#M)\#\cdots \cong M\#S^n\#S^n\#\cdots \cong M\#\mathbb{R}^n \cong M \setminus \{point\},$$

where the last equivalence comes from the fourth property of Properties 4.8. By the associativity of the infinite connected sum, we finally get $M \setminus \{point\} \cong \mathbb{R}^n$. Since M is compact, M is the one-point compactification of $\mathbb{R}^n$ and M is therefore homeomorphic to S^n. □

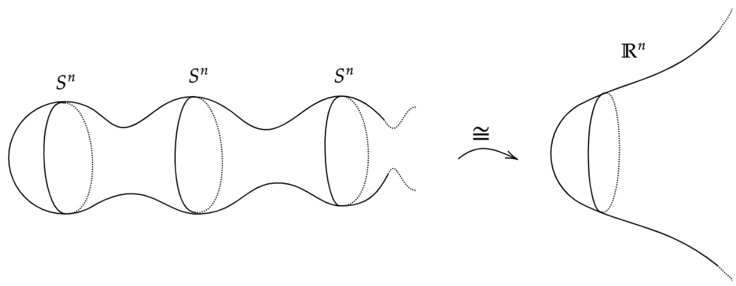

Figure 4.4: The infinite connected sum of n-spheres is diffeomorphic to $\mathbb{R}^n$.

4.3 Twisted spheres

In order to find smooth manifolds which are homeomorphic but a priori not necessarily diffeomorphic to the standard sphere, one might recall that S^n can be obtained by gluing two n-disks along their boundary via the identity on $\partial D^n = S^{n-1}$. Mimicking this construction but replacing the gluing map by an orientation-preserving diffeomorphism $f : S^{n-1} \to S^{n-1}$ gives rise to a *twisted sphere* $D^n \cup_f D^n$. However, we need to take care that the resulting space is smooth along the equator S^{n-1}. Let us therefore work with the Euclidean space $\mathbb{R}^n$ instead of D^n. We will follow [Mil07, Chapter 4, pp. 206-210].

Consider $f : S^{n-1} \to S^{n-1}$ an orientation-preserving diffeomorphism of the $(n-1)$-sphere and define $F : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ by $F(tu) := f(u)/t$, where $u \in S^{n-1}$ and $t > 0$. One checks that F is an orientation-*reversing* diffeomorphism.

Definition 4.11. The **twisted sphere** $\Sigma(f)$ with twist f is obtained by gluing two copies of $\mathbb{R}^n$ via the map F, that is

$$\begin{aligned}\Sigma(f) &:= \mathbb{R}^n \cup_F \mathbb{R}^n, \\ &:= \big(\mathbb{R}^n \mathbin{\dot\cup} \mathbb{R}^n\big)/\sim,\end{aligned}$$

where $tu \sim F(tu) = f(u)/t$ for all $t > 0$ and all $u \in S^{n-1}$. Together with the orientation induced by the first copy of $\mathbb{R}^n$, the twisted sphere $\Sigma(f)$ is an oriented smooth n-dimensional manifold which is homeomorphic to the standard n-sphere.

Remark 4.12. To prove that a twisted sphere is topologically a sphere, one could use either Reeb's theorem[3] or the fact that a twisted sphere is obtained by adding one point to a copy of $\mathbb{R}^n$ [Theorem 3.6 Mil07, p. 205].

One now might wonder how the choice of the twist impacts the resulting space. For this reason, we recall the following definition:

Definition 4.13. Two diffeomorphisms $f, g : M \to N$ are **smoothly isotopic**, $f \simeq g$, if there exists a smooth map $h : M \times \mathbb{R} \to N$ such that

[3]Reeb's theorem states that if a compact n-dimensional smooth manifold admits a Morse function with exactly two nondegenerate critical points, then it must be homeomorphic to S^n.

- $h_t = f$ for all $t \leq 0$,
- $h_t = g$ for all $t \geq 1$,
- $h_t : M \to N$ is a diffeomorphism for all $t \in \mathbb{R}$.

We will denote by $\pi_0\mathrm{Diff}(M)$ the set of smooth isotopy classes of diffeomorphisms of M to itself and by $\pi_0\mathrm{Diff}^+(M)$ if they preserve orientation.

Remark 4.14.
1. A smooth isotopy between two diffeomorphisms is a path between them in the space of diffeomorphisms between the spaces.
2. Together with the composition of diffeomorphisms, $\pi_0\mathrm{Diff}(M)$ and $\pi_0\mathrm{Diff}^+(M)$ can be given the structure of a group, which is nonabelian in general.
3. Palais' disk theorem further states that any two orientation-preserving disk embeddings of M are equivalent under a diffeomorphism of M which is smoothly isotopic to the identity, as noted by Milnor in [Mil07, Lemma 4.3, Chapter 4, p. 207].

Using this last remark, we prove:

Lemma 4.15. The group $\pi_0\mathrm{Diff}^+(S^n)$ is abelian for all n.

Proof. Denote by D^n_+ the northern hemisphere and by D^n_- the southern hemisphere of the n-sphere. Consider $f_1, f_2 \in \pi_0\mathrm{Diff}^+(S^n)$ and observe that $f_1|_{D^n_+}$ and $f_2|_{D^n_-}$ are two smooth orientation-preserving disk embeddings. By the third point of Remark 4.14, there exist two diffeomorphisms $h_1, h_2 : S^n \to S^n$ isotopic to id_{S^n} such that $h_2 \circ f_2|_{D^n_-} = id_{D^n_-}$ and $h_1 \circ f_1|_{D^n_+} = id_{D^n_+}$. Observe that $h_1 \circ f_1$ commutes with $h_2 \circ f_2$ since they map bijectively each hemisphere to itself. Finally,

$$f_1 \circ f_2 \simeq (h_1 \circ f_1) \circ (h_2 \circ f_2) = (h_2 \circ f_2) \circ (h_1 \circ f_1) \simeq f_2 \circ f_1,$$

which implies that $\pi_0\mathrm{Diff}^+(S^n)$ is abelian. □

We therefore denote the composition with $+$ in $\pi_0\mathrm{Diff}^+(S^n)$. The following proposition shows that $\Sigma(f)$ depends only on the smooth isotopy class $(f) \in \pi_0\mathrm{Diff}^+(S^{n-1})$.

Proposition 4.16. The map $\pi_0\mathrm{Diff}^+(S^{n-1}) \to \mathcal{A}_n$ which sends the isotopy class (f) to the equivalence class of the twisted sphere $\Sigma(f)$ is a homomorphism.
We call the image $\Gamma_n \subset \mathcal{A}_n$ the *group of twisted n-spheres.*

Proof. We first check that this map is well-defined. Let $f, g : S^{n-1} \to S^{n-1}$ be orientation-preserving diffeomorphisms with $f \simeq g$, we need to check that $\Sigma(f) \cong \Sigma(g)$. For this, let us consider the smooth isotopy $h : S^{n-1} \times \mathbb{R} \to S^{n-1}$ between $id_{S^{n-1}}$ and $g^{-1} \circ f$ given by $h_t = id_{S^{n-1}}$ for all $t \leq 1$, $h_t = g^{-1} \circ f$ for all $t \geq 2$. Next, define a mapping from $\Sigma(f) = \mathbb{R}^n \cup_F \mathbb{R}^n$ to $\Sigma(g) = \mathbb{R}^n \cup_G \mathbb{R}^n$ by

$$\begin{cases} tu & \mapsto \quad t \cdot h_t(u) \quad \text{on the first copy of } \mathbb{R}^n \setminus \{0\}, \\ f(u)/t & \mapsto \quad g \circ h_t(u)/t \quad \text{on the second copy of } \mathbb{R}^n \setminus \{0\}. \end{cases}$$

Since tu and $f(u)/t$ are equivalent in $\Sigma(f)$, so are their images $t \cdot h_t(u)$ and $g \circ h_t(u)/t$. But this is precisely the identification used for $\Sigma(g)$. The mapping is consequently a well-defined orientation-preserving diffeomorphism between $\Sigma(f)$ and $\Sigma(g)$.
Now, let us prove that $\Sigma : \pi_0\mathrm{Diff}^+(S^{n-1}) \to \mathcal{A}_n$ is a homomorphism. Choose $f, g : S^{n-1} \to S^{n-1}$ two orientation-preserving diffeomorphisms. We need to prove that $\Sigma(f \circ g) \cong \Sigma(f)\#\Sigma(g)$.
Consider the connected sum[4] $\Sigma(f)\#\Sigma(g)$ obtained by identifying the second

[4]This is not exactly how the connected sum was defined but one easily sees that it is equivalent to our definition. One can namely think of $\mathbb{R}^n$ as a disk of infinite radii.

copy of $\mathbb{R}^n$ in $\Sigma(f)$ with the first copy of $\mathbb{R}^n$ in $\Sigma(g)$ by $f(u)/t \leftrightarrow tf(u)$ for $t > 0$ and $u \in S^{n-1}$. The identification is therefore given by the composition:

$$tu \longmapsto f(u)/t \xmapsto{\text{identify}} tf(u) \longmapsto g(f(u))/t.$$

This is just $\mathbb{R}^n \cup_{G\circ F} \mathbb{R}^n = \Sigma(g \circ f)$, which is diffeomorphic to $\Sigma(f \circ g)$ by Lemma 4.15 and the first part of the proof. □

4.4 The pairing β

We hence know how to construct twisted spheres from diffeomorphisms of the sphere. Let us now give a recipe to find nonstandard diffeomorphism using higher homotopy groups. The following construction is based on [Mil07, Chapter 4, pp. 209-210].

Consider the special orthogonal group

$$SO_q = \Big\{ M \in \mathcal{M}(q \times q, \mathbb{R}) \Big| MM^t = M^t M = I_q, \det(M) = 1 \Big\}.$$

Instead of working directly with homotopy classes of maps $S^p \to SO_q$, we will use the fact that each homotopy class in $\pi_p(SO_q)$ can be represented by a smooth function $\phi : \mathbb{R}^p \to SO_q$ with compact support, i.e. $\phi(x) = I_q$ is the identity matrix for large enough $||x||$. The group law is then given by matrix multiplication in SO_q, i.e.

$$\phi_1 \cdot \phi_2(x) = \phi_1(x)\phi_2(x).$$

This is completely equivalent to the usual definition of the homotopy groups $\pi_p(SO_q)$, see [SW51, p. 26].

Remark 4.17. The group $(\pi_p(SO_q), \cdot)$ is abelian. One can namely deform any two maps by a homotopy so as to have maps with disjoint supports, which implies that both maps commute with each other.

For smooth maps $\phi : \mathbb{R}^p \to SO_q$ and $\psi : \mathbb{R}^p \to SO_q$ with compact support, let us define orientation-preserving diffeomorphisms $\Phi, \Psi : \mathbb{R}^{p+q} \to \mathbb{R}^{p+q}$ by:

$$\begin{aligned} \Phi(x,y) &:= (x, \phi(x)y), \\ \Psi(x,y) &:= (\psi(y)x, y), \end{aligned}$$

for $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^{p+q}$.
Consider the commutator $[\Phi, \Psi] = \Phi \circ \Psi \circ \Phi^{-1} \circ \Psi^{-1}$ of Φ and Ψ. This commutator is also an orientation-preserving diffeomorphism of $\mathbb{R}^{p+q}$ and is the identity matrix whenever $||x||$ or $||y||$ is large enough:
Suppose that $\phi(x) = I_q$ for all $||x|| \geq R$ for some radius $R > 0$. Since $\psi(y) \in SO_p$ is an isometry for all y, we get that $||\psi^{-1}(y)x|| = ||x|| \geq R$ and

$$\begin{aligned} [\Phi, \Psi](x,y) &= \Phi \circ \Psi \circ \Phi^{-1}(\psi^{-1}(y)x, y), \\ &= \Phi \circ \Psi(\psi^{-1}(y)x, \overbrace{\phi^{-1}(\psi^{-1}(y)x)}^{=I_q} y), \\ &= \Phi(\overbrace{\psi(y)\psi^{-1}(y)}^{=I_p} x, y), \\ &= (x, \overbrace{\phi(x)}^{=I_q} y), \\ &= (x,y), \end{aligned}$$

Using stereographic projection, we can uniquely extend $[\Phi, \Psi]$ to an orientation-preserving diffeomorphism $\widetilde{[\Phi, \Psi]}$ of S^{p+q}. Finally, define $\beta(\phi, \psi) \in \pi_0\text{Diff}^+(S^{p+q})$ to be the isotopy class of the extension $\widetilde{[\Phi, \Psi]}$.

Lemma 4.18. The pairing $\beta : \pi_p(SO_q) \otimes \pi_q(SO_p) \to \pi_0\text{Diff}^+(S^{p+q})$ is a well-defined group homomorphism. It is bilinear in the following sense:

$$\beta(\phi_1 \cdot \phi_2, \psi) = \beta(\phi_1, \psi) + \beta(\phi_2, \psi),$$

for all $\phi_1, \phi_2 \in \pi_p(SO_q)$ and for all $\psi \in \pi_q(SO_p)$ and

$$\beta(\phi, \psi_1 \cdot \psi_2) = \beta(\phi, \psi_1) + \beta(\phi, \psi_2),$$

for all $\phi \in \pi_p(SO_q)$ and for all $\psi_1, \psi_2 \in \pi_q(SO_p)$.

Remark 4.19. Remember that the + sign in the lemma stands for the group law of $\pi_0\mathrm{Diff}^+(S^{p+q})$. Moreover, $\pi_p(SO_q) \otimes \pi_q(SO_p)$ stands for the tensor product of these two abelian groups as $\mathbb{Z}$-modules. In other words, it is the abelian group given by elements $\phi \otimes \psi$ modulo the respective group relations in $\pi_p(SO_q)$ and in $\pi_q(SO_p)$. We have for $\phi, \phi_1, \phi_2 \in \pi_p(SO_q)$, $\phi, \phi_1, \phi_2 \in \pi_p(SO_q)$ and $n \in \mathbb{Z}$:

$$\begin{aligned}
\phi \otimes (\psi_1 + \psi_2) &= \phi \otimes \psi_1 + \phi \otimes \psi_2, \\
(\phi_1 + \phi_2) \otimes \psi &= \phi_1 \otimes \psi + \phi_2 \otimes \psi, \\
n \cdot (\phi \otimes \psi) &= (n \cdot \phi) \otimes \psi = \phi \otimes (n \cdot \psi).
\end{aligned}$$

Proof. Let us show that the isotopy class $\beta(\phi, \psi)$ only depends on the homotopy class of ϕ and ψ.
To prove this, let $h : \mathbb{R}^p \times \mathbb{R} \to SO_q$ be a smooth homotopy between ϕ_1 and ϕ_2 such that $h_t = \phi_1$ for all $t \leq 1$, $h_t = \phi_2$ for all $t \geq 2$, and $h_t : \mathbb{R}^p \to SO_q$ is smooth with compact support for all t. Then, the map $H : \mathbb{R}^{p+q} \times \mathbb{R} \to \mathbb{R}^{p+q}$ defined by $H_t(x, y) = (x, h_t(x)y)$ is a smooth isotopy between Φ_1 and Φ_2. One shows similarly that β only depends on the homotopy class of ψ, making β into a well-defined pairing.
Now, let $\phi_1, \phi_2 \in \pi_p(SO_q)$ and $\psi \in \pi_q(SO_p)$. Let us assume , without loss of generality, that $\mathrm{supp}(\phi_1)$ is contained inside the unit ball and $\mathrm{supp}(\phi_2)$ outside it. If, on the one hand, $||x|| \geq 1$, then $\phi_1(x) = I_q$ and $\beta(\phi_1, \psi)$ is the isotopy class of the identity map, i.e. $\beta(\phi_1, \psi) = 0$. Moreover,

$$\Phi_1 \circ \Phi_2(x, y) = (x, \overbrace{\phi_1(x)}^{=I_q} \cdot \phi_2(x) y) = \Phi_2(x, y),$$

and

$$\beta(\phi_1 \cdot \phi_2, \psi) = \beta(\phi_2, \psi) = \overbrace{\beta(\phi_1, \psi)}^{=0} + \beta(\phi_2, \psi).$$

On the other hand, if $||x|| < 1$, then $\phi_2(x) = I_q$ and $\beta(\phi_2, \psi) = 0$. Moreover,

$$\Phi_1 \circ \Phi_2(x, y) = (x, \phi_1(x) \cdot \overbrace{\phi_2(x)}^{=I_q} y) = \Phi_1(x, y),$$

and

$$\beta(\phi_1 \cdot \phi_2, \psi) = \beta(\phi_1, \psi) = \beta(\phi_1, \psi) + \overbrace{\beta(\phi_2, \psi)}^{=0}.$$

The proof for the linearity in the second component is completely analogous.

□

To conclude this section, we detail a slight variation found in [Mil59, Section 1, pp. 962-964] on the construction of twisted spheres, where $D^p \times D^q$ is used in place of a single disk D^{p+q}.

Recall that the boundary $\partial(D^p \times D^q)$ is equal to $D^p \times S^{q-1} \cup S^{p-1} \times D^q$. For this reason, start with the disjoint union $D^p \times S^{q-1} \uplus S^{p-1} \times D^q$, whose boundary consists of two disjoint copies of $S^{p-1} \times S^{q-1}$. We now paste these two spaces via an orientation-preserving diffeomorphism $f : S^{p-1} \times S^{q-1} \to S^{p-1} \times S^{q-1}$. As before, let us work with the Euclidean spaces $\mathbb{R}^p$ and $\mathbb{R}^q$ instead of D^p and D^q.

Let $f : S^{p-1} \times S^{q-1} \to S^{p-1} \times S^{q-1}$ be an orientation-preserving diffeomorphism and write $(u', v') = f(u, v)$. Define $F : \mathbb{R}^p \times S^{q-1} \to S^{p-1} \times \mathbb{R}^q$ by $(tu, v) \mapsto (u', v'/t)$, for all $(u, v) \in S^{p-1} \times S^{q-1}$ and $t > 0$. Identifying $\mathbb{R}^p \times S^{q-1}$ and $S^{p-1} \times \mathbb{R}^q$ via F, we obtain a smooth closed $(p + q - 1)$-dimensional manifold which we will denote $M(f)$.

Let $v = (v_1, \cdots, v_q) \in S^{q-1}$ and let $h(v) = v_q$ be the projection onto the last coordinate. Note that $h : S^{q-1} \to [-1, 1]$ has only two critical points, see [Mil59, Section 1, p. 963].

Lemma 4.20. If $h(v) = h(v')$ for all $(u, v) \in S^{p-1} \times S^{q-1}$, the manifold $M(f)$ is homeomorphic to the standard sphere S^{p+q-1}.

Proof. We construct a Morse function $g : M(f) \to [-1, 1]$ with only two nondegenerate critical points by

$$(tu, v) \mapsto \frac{h(v)}{\sqrt{1+t^2}}, \qquad \text{on } \mathbb{R}^p \times S^{q-1},$$
$$(u', v'/t) \mapsto \frac{h(v')/t}{\sqrt{1+(1/t)^2}}, \qquad \text{on } S^{p-1} \times \mathbb{R}^q.$$

This definition is compatible with the identifications, provided $h(v) = h(v')$:

$$g(tu, v) = \frac{h(v)}{\sqrt{1+t^2}} = \frac{h(v')}{\sqrt{1+t^2}} = \frac{h(v')}{t\sqrt{1+(1/t)^2}} = \frac{h(v')/t}{\sqrt{1+(1/t)^2}} = g(u, v'/t).$$

One can indeed check that g has only two nondegenerate critical points. Reeb's theorem implies that $M(f)$ is homeomorphic to S^{p+q-1}. □

The next part is similar in spirit to the construction of the pairing β above, see [Mil59, Section 1, p. 963].
Again, let $f_1 : S^{p-1} \to SO_q$ and $f_2 : S^{q-1} \to SO_p$ be smooth maps into the special orthogonal groups[5]. Let us define a diffeomorphism $S^{p-1} \times S^{q-1} \to S^{p-1} \times S^{q-1}$ by

$$v' = f_1(u) \cdot v, \; u' = f_2(v')^{-1} \cdot u = f_2(f_1(u) \cdot v)^{-1} \cdot u,$$

for all $u \in S^{p-1}, v \in S^{q-1}$ with inverse

$$u = f_2(v') \cdot u', \; v = f_1(u)^{-1} \cdot v' = f_1(f_2(v') \cdot u')^{-1} \cdot v',$$

and let us denote by $M(f_1, f_2)$ be the manifold obtained using this diffeomorphism. Then, by the Lemma 4.20, $M(f_1, f_2)$ is a twisted sphere if

[5]In particular, f_1 and f_2 are representatives of the homotopy classes $(f_1) \in \pi_{p-1}(SO_q)$ and $(f_2) \in \pi_{q-1}(SO_p)$

$h(v) = h(v') = h(f_1(u) \cdot v)$ for all (u, v). That is, if for every $u \in S^{p-1}$, $f_1(u)$ is a rotation around the axis going through the north and south poles of S^{q-1}, which in turn implies that $f_1(S^{p-1})$ is contained in $SO_{q-1} \subset SO_q$. Making the same argument for f_2 shows that $M(f_1, f_2)$ is a twisted sphere if $f_2(S^{q-1})$ is contained in $SO_{p-1} \subset SO_p$.

Remark 4.21. Compare [Mil07, Theorem 6.1, Chapter 6, pp. 218-219]. If both f_1 and f_2 carry S^{p-1} and S^{q-1} into $SO_{q-1} \subset SO_q$ and $SO_{p-1} \subset SO_p$ respectively, let us denote by $\phi_1 \in \pi_{p-1}(SO_{q-1})$ and $\phi_2 \in \pi_{q-1}(SO_{p-1})$ the homotopy classes which respectively map to (f_1) and to (f_2) under the inclusions. The construction above parallels that of the pairing β. Let us indeed define two diffeomorphisms $F_1(u, v) = (u, f_1(u) \cdot v)$ and $F_2(u, v) = (f_2(v) \cdot u, v)$ for $(u, v) \in S^{p-1} \times S^{q-1}$. Up to homotopy, $f_1(u)$ and $f_2(v)$ are the identity matrices except around small disk neighbourhoods $N_1 \subset S^{p-1}$ and $N_2 \subset S^{p-1}$ of the respective north poles. Then, as before,

$$[F_1, F_2](u, v) = (u, v) \text{ unless } (u, v) \in N_1 \times N_2,$$

where $N_1 \times N_2$ is a subset of the $(p + q - 1)$-sphere of radius $\sqrt{2}$. After scaling accordingly and using $[F_1, F_2]$ as a gluing map in the construction above, the resulting manifold $M(f_1, f_2)$ can be identified with the image of $\beta(\phi_1, \phi_2)$ under the homomorphism

$$\beta : \pi_{p-1}(SO_{q-1}) \otimes \pi_{q-1}(SO_{p-1}) \to \pi_0 \mathrm{Diff}^+(S^{p+q-2}) \to \Gamma_{p+q-1}.$$

4.5 Cobordism, homotopy spheres and twisted spheres

To finish Chapter 4, let us investigate the links between different categories of spheres, the first of which is due to the theory of cobordism, pioneered by Thom in the 1950s. Let us start with the following definition:

Definition 4.22. Let M and N be smooth closed n-dimensional manifolds and suppose there exists a smooth compact $(n+1)$-manifold W with boundary $\partial W = M \cup N$. Then, M and N are called **cobordant** and W is called a **cobordism** between M and N. If M and N are oriented, then W should be oriented with boundary $\partial W = M \cup -N$. If the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalence, we then say that M and N are h-**cobordant** and that W is a h-**cobordism**.

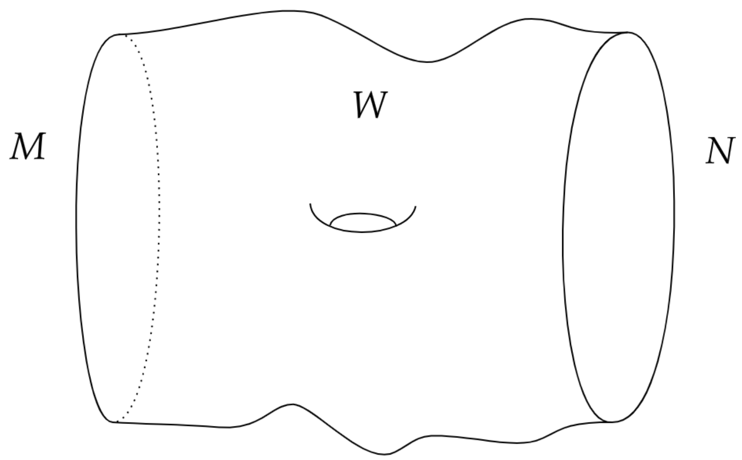

Figure 4.5: W is a cobordism between M and N.

Observe that being (oriented/h-)cobordant defines an equivalence relation. Namely, the reflexivity is easily checked by considering the cylinder $M \times [0,1]$ and the symmetry is self-evident. For the transitivity, one should consider two cobordisms (W, M, N) and (W', N, M'). By pasting them both together along their common boundary N, one obtains a cobordism between M and M'.

Let Ω_n^{un} denote the set of all (unoriented) cobordism classes of dimension n. Notice that M and $\emptyset$ are cobordant if and only if M is the boundary of

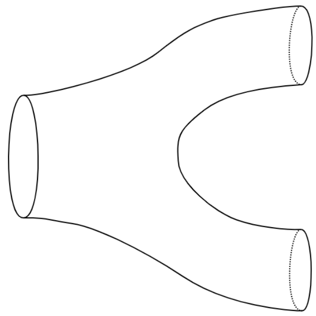

Figure 4.6: A *pair of pants*. This is a cobordism between the circle S^1 and two disjoint copies of itself.

a compact manifold.
Now, together with the addition $+$ given by taking the disjoint union and the cartesian product $\times$, the cobordism classes define a graded commutative ring $\Omega_*^{un} = \bigoplus_{n=0}^{\infty} \Omega_n^{un}$ with zero element $[\emptyset]$. The unoriented cobordism ring is actually a vector space over $\mathbb{Z}_2$ since for any $[M] \in \Omega_*^{un}$ we have

$$[M] + [M] = [M \cup M] = [\partial(M \times [0,1])] = [\emptyset].$$

We've already encountered the oriented cobordism ring Ω^* in the proof of the Hirzebruch signature theorem 3.10. Namely, Thom showed in [Tho54] that by killing the torsion, the oriented cobordism ring is given by a polynomial ring, i.e.

$$\Omega_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \dots].$$

Definition 4.23. We call a smooth n-dimensional manifold a **homotopy n-sphere** if it is homotopy equivalent to the standard n-sphere. We denote by Θ_n the h-cobordism group of oriented homotopy n-spheres.

Using the classification of manifolds in dimension 1 and 2, it is easily verified for these dimensions that any homotopy sphere is also homeomorphic to either S^1 or S^2.
The infamous generalized Poincaré conjecture states that any homotopy sphere is not only homotopy equivalent but even *homeomorphic* to the standard sphere for any dimension. This massive achievement is due to the works of Perelman for $n = 3$, Freedman for $n = 4$ and Smale for all dimensions $n \geq 5$. The latter case is an (almost) direct consequence of *Smale's h-cobordism theorem*:

Theorem 4.24. (Smale's h-cobordism theorem) Let M and N be smooth simply connected n-manifolds, $n \geq 5$, and let W be a simply connected h-cobordism between them. Then, W is diffeomorphic to the cylinder $M \times [0, 1]$.

In the theorem, one can actually choose the diffeomorphism to be the identity on $M \times \{0\}$.

The group Θ_n admits a cyclic subgroup bP_{n+1}, which consists of h-cobordism classes of homotopy n-spheres bounding parallelizable manifolds, i.e. manifolds with a trivializable tangent bundle. By considering the index $(\Theta_n : bP_{n+1})$ and the so-called J-homomorphism, Kervaire and Milnor showed in the 1960s that Θ_n is finite, see [KM63, Theorem 1.2, Section 1, p. 504].

The above table[6] is a computation of the orders of Θ_n and its subgroup bP_{n+1} which comes from [KM63, Section 4, p. 512]. See also the sequence A001676 on the OEIS. If the index $(\Theta_n : bP_{n+1})$ is equal to 1, e.g. $n = 1, \dots, 6, 7, 11, 12, 13$, then any twisted sphere in $\Gamma_n \cong \Theta_n$ bounds a parallelizable manifold (of dimension $n + 1$). Notice also that bP_{n+1} seems

[6]In the paper of Kervaire and Milnor, $n = 19$ is wrong by a factor of 2.

Dimension n	$1-6$	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Order of Θ_n	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24
Order of bP_{n+1}	1	28	1	2	1	992	1	1	1	8128	1	2	1	261632	1
$(\Theta_n : bP_{n+1})$	1	1	2	4	6	1	1	3	2	2	2	8	16	2	24

Table 4.1: Orders of the groups Θ_n and bP_{n+1} and their index $(\Theta_n : bP_{n+1})$

to be trivial whenever n is even - this is in fact the case.

Recall from Lemma 4.16 that the group of twisted spheres is defined as the image of the homomorphism $\Sigma : \pi_0\mathrm{Diff}^+(S^{n-1}) \to \Gamma_n$. It turns out that if $n \geq 6$, the group $\pi_0\mathrm{Diff}^+(S^{n-1})$ is isomorphic to the group of twisted spheres Γ_n, see [Cer70, Theorem 0, Corollary 2 (2°)].

All in all, we can summarize all the relations between all these different notions in the following proposition:

Proposition 4.25. The following groups are all finite and abelian, and are all isomorphic to each other:

1. The group of oriented smooth structures $\mathcal{A}_n$ of the n-sphere, $n \neq 4$,
2. The group Θ_n of h-cobordism classes of oriented n-homotopy spheres,
3. The group of h-cobordism classes of oriented n-spheres,
4. The group Γ_n of twisted spheres.
5. The group $\pi_0\mathrm{Diff}^+(S^{n-1})$, provided $n \geq 6$.

Chapter 5

The plumbing construction

We want to show that $M(f_1, f_2)$ constructed in the last chapter can be realized as boundary of a higher dimensional compact manifold, which we will denote $W(f_1, f_2)$. The first part of this chapter is based on Milnor's work in [Mil07, Chapter 6, pp. 217-219] and in [Mil59, Section 1, p. 964].

Let us start again by taking two smooth maps $f_1 : S^{p-1} \to SO_q$ and $f_2 : S^{q-1} \to SO_p$. Consider the union of three copies $(D^p \times D^q)_i$, $i = 1, 2, 3$ and let us write $(u_i, v_i) \in (D^p \times D^q)_i$. By pasting together $(S^{p-1} \times D^q)_1$ and $(S^{p-1} \times D^q)_2$ via

$$(u_1, v_1) \sim (u_2, v_2) :\Longleftrightarrow \ u_1 = v_2, \ v_2 = f_1(u_1) \cdot v_1,$$

we obtain a D^q-bundle over S^p with characteristic map[1] f_1, see [Ste99, Chapter 18]. Similarly, identifying $(D^p \times S^{q-1})_2$ and $(D^p \times S^{q-1})_2$ under

[1]Recall that this is similar to the notion of vector bundle, where one uses the fiber D^q instead of a vector space. It turns out that there is a one-to-one correspondence between the isomorphism classes of the bundles arising in this way and the homotopy classes $(f_1) \in \pi_{p-1}(SO_q)$.

the correspondence

$$(u_2, v_2) \sim (u_3, v_3) :\iff u_2 = v_3,\ u_3 = f_2(v_3)^{-1} \cdot u_2,$$

results in a D^p-bundle over S^q with characteristic map f_2. We will denote these bundles by $\xi(f_1)$ and $\xi(f_2)$, respectively.
Next, let $W(f_1, f_2)$ be the union $(D^p \times D^q)_1 \cup (D^p \times D^q)_2 \cup (D^p \times D^q)_3$ under these equivalences. This makes $W(f_1, f_2)$ into a $p + q$-dimensional topological manifold with boundary:

$$\partial W(f_1, f_2) = (D^p \times S^{q-1})_1 \cup (S^{p-1} \times D^q)_3.$$

Then, the intersection $(D^p \times S^{q-1})_1 \cap (S^{p-1} \times D^q)_3$ is equal to $(S^{p-1} \times S^{q-1})_1 = (S^{p-1} \times S^{q-1})_3$, which are identified via

$$(u_1, v_1) \sim (u_2, v_2) \sim (u_3, v_3),$$

i.e.

$$u_3 = f_2(v_3)^{-1} \cdot u_2 = f_2(v_3)^{-1} \cdot u_1,\ v_3 = v_2 = f_1(u_1) \cdot v_1.$$

If we relabel $(u_1, v_1) \rightsquigarrow (u, v)$ and $(u_3, v_3) \rightsquigarrow (u', v')$, we see that this is the identification used to define $M(f_1, f_2)$ and $\partial W(f_1, f_2)$ is therefore homeomorphic to $M(f_1, f_2)$. Smoothing out the corners along $S^{p-1} \times S^{q-1}$ in the boundary of $W(f_1, f_2)$, we obtain a smooth $p+q$-dimensional manifold $W(f_1, f_2)$ with boundary diffeomorphic to $M(f_1, f_2)$, see [Mil59, Section 1, p. 964].

5.1 Properties of $W(f_1, f_2)$

Let us now investigate the topological properties of the manifold $W(f_1, f_2)$. Since $W(f_1, f_2)$ is a (plumbed) union of the two disk bundles $\xi(f_1)$ and $\xi(f_2)$, the following lemma implies that $W(f_1, f_2)$ is compact.

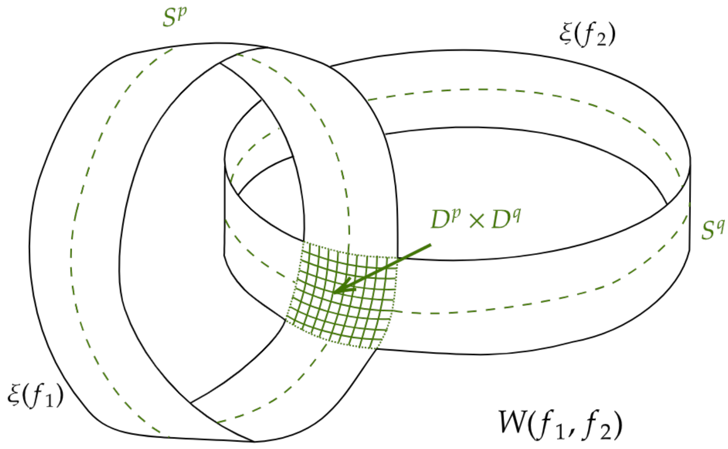

Figure 5.1: The manifold $W(f_1, f_2)$ is obtained by *plumbing* together the disk bundles $\xi(f_1)$ and $\xi(f_2)$.

Lemma 5.1. Let $\pi : E \to B$ be a fibre bundle with compact fiber F over a compact Hausdorff space B. Then, E is compact.

Proof. Since B is compact and Hausdorff, one can choose a finite open cover $\{U_1, \dots, U_m\}$ of B such that $\pi^{-1}(V_i) \approx V_i \times F$ where V_i are open neighbourhoods with $\bar{U}_i \subset V_i$. Then, $\pi^{-1}(\bar{U}_i) \approx \bar{U}_i \times F$ is compact, since $\bar{U}_i \subset B$ is closed, thus compact. Therefore, $E = \pi^{-1}(\bar{U}_1) \cup \dots \cup \pi^{-1}(\bar{U}_n)$ is also compact. $\square$

Observation 5.2. According to [Mil07, Chapter 6, pp. 217-218], the following conditions are equivalent:

1. The disk bundle $\xi(f_1)$ admits a nowhere vanishing section.

2. The associated vector bundle $E(f_1)$ splits as the Whitney sum of a $(q-1)$-vector bundle and a trivial line bundle over S^p.

3. The image $f_1(S^{p-1})$ lies in the subgroup $SO_{q-1} \subset SO_q$, up to homotopy. There is evidently a similar statement for f_2.

As a consequence of Lemma 4.20, if any one of the bundles $\xi(f_1)$ and $\xi(f_2)$ admits a nowhere vanishing section, then $M(f_1, f_2)$ is a twisted sphere. If this is the case for both, then this twisted sphere can be seen as the image of the pairing β, see Remark 4.21.

Let us now study the cohomology of $W = W(f_1, f_2)$. Let us write $M = M(f_1, f_2)$ the boundary of $W = W(f_1, f_2)$. Notice that the zero sections $s_1 : S^p \to \xi(f_1)$ and $s_2 : S^q \to \xi(f_2)$ intersect transversally in W. Evidently, the two bundles have respectively the homotopy types of S^p and S^q. One also easily verifies that W has the homotopy type of $S^p \vee S^q$. As an easy but important consequence we have the following:

Lemma 5.3. $H^2(W; \mathbb{Z}_2) \cong 0$ provided $p, q > 2$. W is therefore spin.

From now on, we consider (co-)homology with integer coefficients. Without loss of generality, assume $0 < p \leq q$. If $p = q$, we will assume that the bundles $\xi(f_1)$ and $\xi(f_2)$ both admit a nowhere vanishing section. Clearly, S^p and S^q embed as orientable smooth submanifolds into W via the trivial sections s_1 and s_2. Using the Mayer-Vietoris sequence for homology, we have the following isomorphisms

$$H_k(W) \xleftarrow{(i_1)_* - (i_2)_*} H_k(\xi(f_1)) \oplus H_k(\xi(f_2)) \xleftarrow{((s_1)_*, (s_2)_*)} H_k(S^p) \oplus H_k(S^q)$$

for all $k > 0$. Here, $i_k : \xi(f_k) \to W$ denotes the inclusions for $k = 1, 2$. In other words, the zero section $S^p \subset \xi(f_1) \subset W$ and the zero section $S^q \subset \xi(f_2) \subset W$ correspond under these isomorphisms to generators Y of $H_p(W)$ and X of $H_q(W)$, respectively. We can therefore describe the integer homology of W as the following:

$$H_*(W) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}\langle Y\rangle \oplus \mathbb{Z}\langle X\rangle, & \text{if } p \neq q, \\ \mathbb{Z} \oplus \mathbb{Z}\langle Y, X\rangle, & \text{if } p = q. \end{cases}$$

Applying Poincaré duality, the cohomology ring (with integer coefficients) of W is equal to

$$H^*(W, M) = \begin{cases} \mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle \oplus \mathbb{Z}\langle x \smile y\rangle, & \text{if } p \neq q, \\ \mathbb{Z}\langle x, y\rangle \oplus \mathbb{Z}\langle x \smile y\rangle, & \text{if } p = q. \end{cases}$$

where $x \in H^p(W, M)$ is the Poincaré dual to the homology class defined by $[S^q]$ and $y \in H^q(W, M)$ is the Poincaré dual corresponding to $[S^p]$ under the identifications above. Notice that $\langle x \smile x, [W]\rangle = 0$:
If on the one hand $p \neq q$, this is just because $H^{2p}(W, M) \cong 0$. But on the other hand, if $p = q$, we can slightly perturb $S^p \subset W$ such that the perturbed submanifold does not intersect with the zero section. The reason for this is that $\xi(f_1)$ admits (by assumption) a nowhere vanishing section. Similarly, for the generator $y \in H^q(W, M)$ we have $\langle y \smile y, [W]\rangle = 0$.
Moreover, the number $\langle x \smile y, [W]\rangle \in \mathbb{Z}$ is equal to the oriented number of intersections of S^p and S^q in W. This is therefore clearly equal to ± 1. We choose the orientations of S^p and S^q in W so as to have $\langle x \smile y, [W]\rangle = +1$.

Let us now specialize to the case $p = 4m$ and $q = 4n$ where $m, n \geq 1$ are positive integers. Let us write $M = M(f_1, f_2)$ the boundary of $W = W(f_1, f_2)$.
Recall that the isomorphism type of both of the bundles $\xi(f_1)$ and $\xi(f_2)$ only depend on the homotopy classes $(f_1) \in \pi_{4m-1}(SO_{4n})$ and $(f_2) \in \pi_{4n-1}(SO_{4m})$ and the same is true for the vector bundles $E(f_1)$ and $E(f_2)$ associated to $\xi(f_1)$ and $\xi(f_2)$, respectively (see Definition 2.15). Define the integers $\hat{p}_m(f_1) := \langle p_m(E(f_1)), [S^{4m}]\rangle \in \mathbb{Z}$ and $\hat{p}_n(f_2) := \langle p_n(E(f_2)), [S^{4n}]\rangle \in \mathbb{Z}$.

Lemma 5.4. The Pontrjagin class $p_m(\xi(f_1))$ is equal to $\hat{p}_m(f_1)$ times a generator of $H^{4m}(\xi(f_1)) \cong \mathbb{Z}$. Similarly, the Pontrjagin class $p_m(\xi(f_2))$ is equal to $\hat{p}_m(f_2)$ times a generator of $H^{4n}(\xi(f_2)) \cong \mathbb{Z}$.

Proof. Recall that the zero section $s_1 : S^{4m} \to \xi(f_1)$ induces an isomorphism on cohomology (since it is a homotopy equivalence). The tangent space $T\xi(f_1) \cong E \oplus N$ splits as a Whitney sum of the bundle E of vectors tangent to the fibres and the bundle N of vectors normal to the fibres. Restricting the tangent bundle to the zero section shows that

$$T\xi(f_1)\big|_{S^{4m}} = s_1^* T\xi(f_1) \cong s_1^* E \oplus s_1^* N = E\big|_{S^{4m}} \oplus N\big|_{S^{4m}}.$$

In particular, since $N\big|_{S^{4m}} = TS^{4m}$ is stably trivial (see Example 2.20), we have by the naturality axiom that

$$s_1^*(p_m(N)) = p_m(N\big|_{S^{4m}}) = p_m(TS^{4m}) = 0.$$

Hence, the Pontrjagin classes of N are therefore trivial. Restricting this time E to zero section, the bundle $E\big|_{S^{4m}}$ is precisely $E(f_1)$, the associated vector bundle to $\xi(f_1)$. Putting everything together, we have that

$$\begin{aligned} s_1^* p_m(\xi(f_1)) &= p_m(E\big|_{S^{4m}} \oplus N\big|_{S^{4m}}) = p_m(E\big|_{S^{4m}}) \\ &= p_m(E(f_1)) = \hat{p}_m(f_1) \cdot [S^{4m}]^*, \end{aligned}$$

and hence $p_m(\xi(f_1)) = \hat{p}_m(f_1) \cdot (s_1^*)^{-1}[S^{4m}]^*$, which is what we needed to prove. The proof for $p_n(\xi(f_2))$ is similar. □

Suppose now that at least one of $\xi(f_1)$ or $\xi(f_2)$ has a nowhere vanishing section, so that M is a twisted sphere. Since M has the same homology groups as $S^{4m+4n-1}$, the inclusion $j : W \to (W, M)$ induces an isomorphism $j^* : H^{4k}(W,M) \to H^{4k}(W)$ for all $0 < k < m+n$. We can therefore pull back the Pontrjagin classes of W to classes $q_k(W) := (j^*)^{-1} p_k(W) \in H^{4k}(W, M)$ for $0 < k < m + n$.

Definition 5.5. Apart from $q_{m+n}[W]$, the **(relative) Pontrjagin numbers** of W are defined to be $q_{k_1} \cdots q_{k_r}[W] := \langle q_{k_1} \smile \cdots \smile q_{k_r}(W), [W]\rangle$. Furthermore, the **signature** $\sigma(W)$ is the signature of the quadratic form

$$H^{2m+2n}(W, M; \mathbb{Q}) \to \mathbb{Q}, \ \alpha \mapsto \langle \alpha \smile \alpha, [W]\rangle.$$

Lemma 5.6. Suppose at least one of $\xi(f_1)$ or $\xi(f_2)$ admits a section which is nowhere zero, then:

1. The signature $\sigma(W)$ is equal to 0.
2. $q_m q_n[W] = \hat{p}_m(f_1)\hat{p}_n(f_2)$ if $m \neq n$ and twice this number if $m = n$. All other Pontrjagin numbers are trivial.

Recall that if $m = n$, we assumed that both $\xi(f_1)$ and $\xi(f_2)$ have a nowhere vanishing section.

Proof. **Ad 1:** Recall that $\langle x \smile y, [W]\rangle = +1$. Also: $\langle y \smile x, [W]\rangle = +1$. The quadratic form associated to the cup product is then given by the symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This matrix is thus diagonalizable with eigenvalues $+1$ and -1, hence the signature $\sigma(W)$ is 0.
Ad 2: Compare [Mil59, Lemma 3 & 4, Section 3, pp. 967-969]. Recall that $i_1 : \xi(f_1) \to W$ and $i_2 : \xi(f_2) \to W$ were the natural inclusions. It follows easily from the cohomology ring of W that the only relevant Pontrjagin number is $q_m q_n[W]$. Let us first start with the case $m \neq n$. By the naturality axiom, we have

$$\begin{aligned} i_1^*(p_m(W)) = p_m(i_1^* TW) = p_m(TW\big|_{\xi(f_1)}) &= p_m(\xi(f_1)) \\ &= \hat{p}_m(f_1) \cdot (s_1^*)^{-1}([S^{4m}]^*). \end{aligned}$$

By construction, the Poincaré dual $[S^{4m}]^*$ is the image of the generator x of $H^{4m}(W, M)$ under the homomorphisms

$$H^{4m}(W, M) \xrightarrow{\ j^*\ } H^{4m}(W) \xrightarrow{\ i_1^*\ } H^{4m}(\xi(f_1)) \xrightarrow{\ s_1^*\ } H^{4m}(S^{4m}),$$

and hence

$$(j^*)^{-1}p_m(W) = \hat{p}_m(f_1)\cdot(j^*)^{-1}\circ(i_1^*)^{-1}\circ(s_1^*)^{-1}[S^{4m}]^* = \hat{p}_m(f_1)\cdot x.$$

One shows in a completely analogous manner that $(j^*)^{-1}p_n(W) = \hat{p}_n(f_2)\cdot y$, where y is the generator of $H^{4n}(W,M)$. Then,

$$\begin{aligned} q_m q_n[W] &= \langle (j^*)^{-1}p_m(W) \smile (j^*)^{-1}p_n(W), [W]\rangle \\ &= \hat{p}_m(f_1)\hat{p}_n(f_2)\langle x \smile y, [W]\rangle \\ &= \hat{p}_m(f_1)\hat{p}_n(f_2). \end{aligned}$$

For the case $m = n$, considering the following sequence of isomorphisms

$$\begin{aligned} H^{4m}(W,M) \xrightarrow{j^*} H^{4m}(W) & \\ \xrightarrow{i_1^*\oplus i_2^*} H^{4m}(\xi(f_1)) \oplus H^{4m}(\xi(f_2)) & \\ \xrightarrow{s_1^*\oplus s_2^*} H^{4m}(S^{4m}) \oplus H^{4m}(S^{4m}), & \end{aligned}$$

yields

$$q_m(W) = (j^*)^{-1}(p_m(W)) = \hat{p}_m(f_1)\cdot x + \hat{p}_m(f_2)\cdot y.$$

Then, since $x \smile x = 0$ and $y \smile y = 0$,

$$\begin{aligned} (q_m)^2[W] &= \langle (\hat{p}_m(f_1)\cdot x + \hat{p}_m(f_2)\cdot y)^2, [W]\rangle \\ &= \hat{p}_m(f_1)^2\cancel{\langle x^2, [W]\rangle} + 2\hat{p}_m(f_1)\hat{p}_m(f_2)\langle x \smile y, [W]\rangle \\ &\quad + \hat{p}_m(f_2)^2\cancel{\langle y^2, [W]\rangle} \\ &= 2\hat{p}_m(f_1)\hat{p}_m(f_2), \end{aligned}$$

which finishes the proof. □

To conclude this subsection, we need to know what values occur in $\hat{p}_n(f_1)$. For more details, see [Mil07, Section 3, p. 970] and [Mil59, Lemma 6.3, Chapter 6, p. 220].

Lemma 5.7. The image under the Pontrjagin homomorphism

$$\hat{p}_m : \pi_{4m-1}(SO_q) \to \mathbb{Z}$$

is zero if $q \leq 2m$ and is generated by a multiple of $(2m-1)!$ with all prime divisors less than $2m$ if $q > 2m$. Moreover, if $q \geq 4m$, the generator is $(2m-1)!$ if m is even or $2(2m-1)!$ is m is odd.

Remark 5.8. It can be shown using the *Bott periodicity theorem* that $\pi_{4m-1}(SO_q)$ is infinite cyclic whenever $q \geq 4m$. The lemma augments this result to the case $q > 2m$.

Assume $m \leq n < 2m$. For any pair of integers $k, l \in \mathbb{Z}$, there exist homotopy classes (f_1) and (f_2) such that

$$\hat{p}_m(f_1) = k(2m-1)! \cdot a \quad \text{and} \quad \hat{p}_n(f_2) = l(2n-1)! \cdot b,$$

where a is equal to 1 or 2 according to the parity of m and all prime divisors of b are less than $2n$ as in the lemma above. In particular, a and b only depend on the dimensions m and n and not on (f_1) and (f_2). As a consequence, if $m \neq n$, the sole non-trivial Pontrjagin number of $W = W(f_1, f_2)$ is given by

$$q_m q_n[W] = kl(2m-1)!(2n-1)!\, ab,$$

or twice this if $m = n$.

5.2 Curvature of $W(f_1, f_2)$

The following proposition is a consequence of a theorem by Vilms, see [Bes07, Theorem 9.59, p. 249] or [GW09, Proposition 2.7.1, p. 97], as well as the work of Wraith, see [Wra11, Corollary 6.4, p. 2013].

Proposition 5.9. The smooth compact manifold W admits a Riemannian metric which has positive scalar curvature, is of product form near the boundary $\partial W = M$ and can be deformed inside positive scalar curvature to a metric which has positive Ricci curvature on $\partial W = M$.

Proof. Since the special orthonormal group $SO(q)$ acts on D^q and S^{q-1}, let us consider the following total spaces of the bundles associated to $\xi(f_1)$ (see Example 2.14 and Definition 2.15) :

$$P(f_1) := F_{SO}(\xi(f_1)) \text{ the oriented orthonormal frame bundle,}$$
$$D(f_1) := D(\xi(f_1) = P(f_1) \times_{SO(q)} D^q,$$
$$S(f_1) := S(\xi(f_1)) = P(f_1) \times_{SO(q)} S^{q-1},$$

where S^{q-1} is given the round metric g_R, D^q is given a torpedo metric g_{tor} and a connection has been chosen on $P(f_1) \to S^p$. We denote π_P, π_D and π_S the respective projections. Notice also that $D(f_1) = \xi(f_1)$.

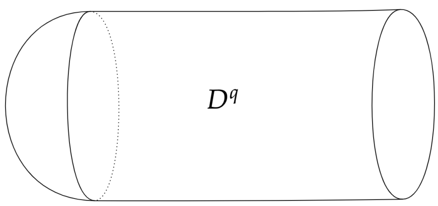

Figure 5.2: Giving the disk the metric g_{tor} makes it indeed look like a torpedo.

Let $\bar{g}$ be the (unique) metric on $D(f_1) = P(f_1) \times_{SO(q)} D^q$ given by the theorem of Vilms and let as well g be the corresponding metric on $S(f_1)$ given by Vilms. One can verify that $\bar{g}\big|_{S(f_1)} = g$.

Consider the vertical bundle $V = \ker(\pi_D)_*$ on $D(f_1)$ and let H be the horizontal bundle, i.e. $TD(f_1) = V \oplus H$, and $g(v,h) = 0$ for all $v \in V$ and $h \in H$.

The *canonical variation* $(\bar{g}_t)_{t\geq 0}$ of the metric $\bar{g}$ is defined by

$$\begin{aligned} \bar{g}(v,\tilde{v}) &= t \cdot \bar{g}(v,\tilde{v}) && \text{for } v,\tilde{v} \in V, \\ \bar{g}_t(h,\tilde{h}) &= \bar{g}(h,\tilde{h}) && \text{for } h,\tilde{h} \in H, \\ \bar{g}_t(v,h) &= 0 && \text{for } v \in V, h \in H. \end{aligned}$$

One easily sees that the vertical and horizontal vectors for $\bar{g}$ stay respectively in V and in H for all $\bar{g}_t$. This can be seen as shrinking the fibers. Then, for $u = u^V + u^H$ and $w = w^V + w^H$ in $T_xD(f_1) = V_x \oplus H_x$,

$$\bar{g}_t(u,w) = t \cdot g_{tor}(u^V, w^V) + \pi_D^* g_R(u,w), \tag{$*$}$$

Restricting to $S(f_1) \subset D(f_1)$, $(\bar{g}_t\big|_{S(f_1)})_{t\geq 0}$ is equal to to the canonical variation of g with vertical bundle $V_S = \ker(\pi_S)_* \subset V^*$ and horizontal bundle $H_S \subset H$. Now, since the base space and the fibers of $S(f_1)$ are compact and have positive Ricci curvature, it follows from [GW09, Theorem 2.7.3, p. 100] that the metric g_t on $S(f_1)$ has positive Ricci curvature for small $t > 0$. Following , the scalar curvature of the canonical variation can be computed by

$$\mathrm{scal}_{\bar{g}_t} = \frac{1}{t}\overbrace{\mathrm{scal}_{g_{tor}}}^{>0} + \overbrace{\mathrm{scal}_{g_R}}^{>0} \circ \pi_D - t|A|^2,$$

where $A : H \times H \to V$ is a particular tensor field on $D(f_1)$ determined by $\bar{g}$, see [Bes07, p. 9.53]. Choosing $t > 0$ sufficiently small, the metrics $\bar{g}_t$ have positive scalar curvature on all of $D(f_1)$, positive Ricci curvature on $S(f_1) = \partial D(f_1)$ and are of product form near the boundary by the equation $(*)$ above.

In an analogous way, one can show that $\xi(f_2)$ admits a metric of positive scalar curvature which is if product form near the boundary and has positive Ricci curvature on the boundary.

We have now constructed metrics of the desired form (positive scalar curvature, product form near the boundary and positive Ricci curvature on the boundary) on each disk bundle, which can be used to define a metric on the plumbed manifold W. While this approach lets us define such metrics quite easily for each bundle, special care has to be given where the two bundles are plumbed together in order to obtain a metric on W which is compatible with the identifications in the plumbing construction and such that the resulting metric is of the desired form. This is quite difficult and involves advanced notions of surgery theory, which is why we refer to Wraith's work in [Wra11, Corollary 6.4, p.2013] and to Philip Reiser's work in [Rei23, Theorem A with $M = \xi(f_1)$ and $E = \xi(f_2)$] to claim that this is possible. □

Chapter 6

Moduli space of metrics

The general reference for this chapter is the first chapter of Wilderich Tuschmann and David Wraith's book [TW15, Chapter 1, p. 5].

Let M be a compact smooth manifold and let $\mathcal{R}(M)$ be the space of all Riemannian metrics on M, which we equip with the C^∞-topology. We denote by $\mathcal{R}_{Ric>0}(M)$ and $\mathcal{R}_{scal>0}(M)$ the subspaces of Riemannian metrics with positive Ricci curvature and positive scalar curvature, respectively. The group $\mathrm{Diff}(M)$ of diffeomorphisms on M acts on $\mathcal{R}(M)$ by pulling back metrics, i.e. for a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ and a Riemannian metric $g \in \mathcal{R}(M)$, the *pullback metric* $\varphi^* g$ defined by

$$\varphi^* g(v, w) := g(\varphi_*(v), \varphi_*(w)), \quad \text{for all } v, w \in T_pM, p \in M,$$

is again a Riemannian metric on M, i.e. $\varphi^* g \in \mathcal{R}(M)$. This action is not free in general.

Definition 6.1. The **moduli space of Riemannian metrics** on M is the quotient space $\mathcal{M}(M) := \mathcal{R}(M)/\mathrm{Diff}(M)$ equipped with the quotient topology.

If we limit ourselves to metrics with positive Ricci curvature and positive scalar curvature, we obtain the respective moduli spaces $\mathcal{M}_{Ric>0}(M)$, and $\mathcal{M}_{scal>0}(M)$.

Remark 6.2. Since any convex combination of Riemannian metrics on M is again a Riemannian metric on M, the space $\mathcal{R}(M)$ is contractible.

Since we will be mainly interested in the path-connectedness of the moduli spaces $\mathcal{M}_{Ric>0}(M)$, and $\mathcal{M}_{scal>0}(M)$, the following theorem plays a role of fundamental importance:

Theorem 6.3. Suppose $\gamma : [0,1] \to \mathcal{M}(M)$ is a path connecting $\gamma(0) = [g_0]$ and $\gamma(1) = [g_1]$. Then, we can lift the path γ to a path $\tilde{\gamma} : [0,1] \to \mathcal{R}(M)$ such that $\tilde{\gamma}(0) = g_0$ and $\pi \circ \tilde{\gamma} = \gamma$, where $\pi : \mathcal{R}(M) \to \mathcal{M}(M)$ is the projection.

In particular, any path in $\mathcal{M}_{Ric>0}(M)$ (respectively in $\mathcal{M}_{scal>0}(M)$) can be lifted in $\mathcal{R}_{Ric>0}(M)$ (respectively in $\mathcal{R}_{scal>0}(M)$).
Note that in the theorem, the endpoint $\tilde{\gamma}(1)$ of the lift is not necessarily g_1, but lies in its isometry class, that is $\tilde{\gamma}(1) = \psi^* g_1$ for a diffeomorphism $\psi : M \to M$. The proof relies on Ebin's slice theorem [Ebi70], see for example [CK19, Proposition 4.6].

Chapter 7

Main Theorem

7.1 Preparations

Let us assume, as in Chapter 5, that $1 \leq m \leq n < 2m$.
Consider two $(4m + 4n)$-dimensional manifolds W_0 and W_1 obtained by plumbing disk bundles and suppose that at least one in each pair of disk bundles admits a nowhere zero section. The boundaries $M_0 := \partial W_0$ and $M_1 := \partial W_1$ are therefore $(4m + 4n - 1)$-dimensional twisted spheres. Let the relative Pontrjagin numbers be given by

$$q_m q_n[W_0] = k_0 l_0 (2m-1)!(2n-1)!ab,$$
$$q_m q_n[W_1] = k_1 l_1 (2m-1)!(2n-1)!ab,$$

with $k_0, l_0, k_1, l_1 \in \mathbb{Z}$ (compare with 5.1).
Suppose that their boundaries M_0 and M_1 are oriented diffeomorphic. Then, there exists an orientation-preserving diffeomorphism $\phi : M_0 \xrightarrow{\cong} M_1$, that is, they both represent the same differentiable structure on $S^{4m+4n-1}$, i.e. $[M_0] = [M_1] \in \Gamma_{4m+4n-1}$. Define a smooth closed oriented manifold $X =$

$W_0 \cup_\phi -W_1$ of dimension $4m+4n$ by gluing W_0 and W_1 along their boundary and giving X the orientation of W_0.

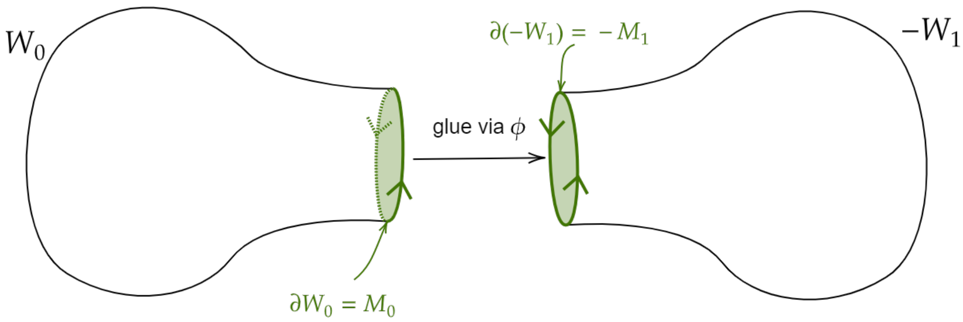

Figure 7.1: Picture of the manifold X obtained by gluing W_0 and $-W_1$ together via ϕ

Lemma 7.1. 1. Apart from the top Pontrjagin number $p_{m+n}[X]$, the Pontrjagin numbers of X are given by

$$p_{k_1} \cdots p_{k_r}[X] = q_{k_1} \cdots q_{k_r}[W_0] - q_{k_1} \cdots q_{k_r}[W_1].$$

2. For the signature of X, we have $\sigma(X) = \sigma(W_0) - \sigma(W_1)$.

Proof. Compare [Mil07, Lemma 5.4, Chapter 5, pp. 213-215] and [Mil56, Lemma 1, Section 1, pp. 400-401]. Let us denote by $M \subset X$ the common boundary of W_0 and W_1. By construction, the inclusion $i_0 : W_0 \to X$ is orientation-preserving while the inclusion $i_1 : W_1 \to X$ is orientation-reversing. Let in addition $j : X \to (X, M)$,

$$j_0 : W_0 \to (W_0, M_0) \quad \text{and} \quad j_1 : W_1 \to (W_1, M_1)$$

as well as

$$h_0 : (W_0, M_0) \to (X, M) \quad \text{and} \quad h_1 : (W_1, M_1) \to (X, M)$$

be the natural inclusions. Recall that the inclusions j_0 and j_1 induce isomorphisms on cohomology in all the degrees $4k$, where $0 < k < m+n$. The relative Pontrjagin classes of W_0 and W_1 are therefore given by $q_k(W_0) := (j_0^*)^{-1}p_k(W_0) \in H^{4k}(W_0, M_0)$ and $q_k(W_1) := (j_1^*)^{-1}p_k(W_1) \in H^{4k}(W_1, M_1)$ for all $0 < k < m+n$, see 5.1.

Consider strong open deformation retracts U and V of W_0 and W_1 respec-

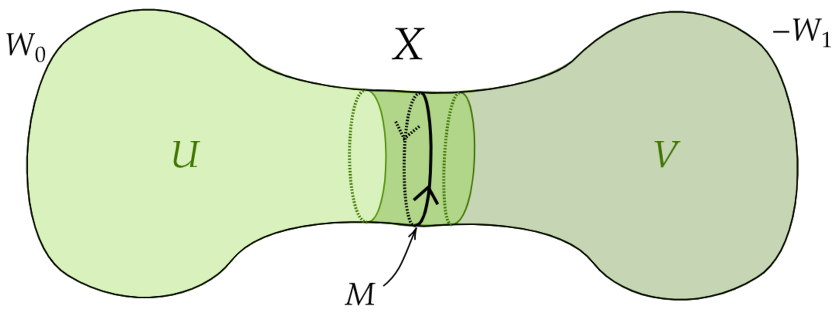

Figure 7.2: $U$and V are strong deformation retracts of W_0 and W_1 with intersection $U \cap V \simeq M$.

tively with intersection $U \cap V \simeq M$. Then, the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{4k-1}(U\cap V) & \longrightarrow & H^{4k}(X) & \longrightarrow & H^{4k}(U)\oplus H^{4k}(V) & \longrightarrow & H^{4k}(U\cap V) & \longrightarrow \cdots \\ & \downarrow\cong & & \downarrow = & & \downarrow\cong & & \downarrow\cong & \\ \cdots \longrightarrow & H^{4k-1}(M) & \longrightarrow & H^{4k}(X) & \xrightarrow[i_0^*\oplus i_1^*]{} & H^{4k}(W_0)\oplus H^{4k}(W_1) & \longrightarrow & H^{4k}(M) & \longrightarrow \cdots \end{array}$$

The first line is the Mayer-Vietoris sequence for the triad (X, U, V) and the vertical arrows correspond to the respective isomorphisms. In turn, this implies that $i_0^* \oplus i_1^* : H^{4k}(X) \to H^{4k}(W_0) \oplus H^{4k}(W_1)$ is an isomorphism for $0 < k < m + n$.

In particular, for the Pontrjagin classes, we have that

$$\begin{aligned}(i_0^* \oplus i_1^*)(p_k(X)) &= i_0^*(p_k(X)) \oplus i_1^*(p_k(X)) \\ &= i_0^*(p_k(TX)) \oplus i_1^*(p_k(TX)) \\ &= p_k(i_0^*TX) \oplus p_k(i_1^*TX)) \\ &= p_k(TW_0) \oplus p_k(TW_1) \\ &= p_k(W_0) \oplus p_k(W_1).\end{aligned}$$

A similar argument as above using the relative version of the Mayer-Vietoris sequence shows that $H^*(X, M) \cong H^*(W_0, M_0) \oplus H^*(W_1, M_1)$ via the isomorphism $h_0^* \oplus h_1^*$. The same is true if we rather consider homology. Putting everything together, we obtain commutative diagrams

$$\begin{array}{ccc} H^{4k}(W_0, M_0) \oplus H^{4k}(W_1, M_1) & \xleftarrow{h_0^* \oplus h_1^*} & H^{4k}(X, M) \\ \Big\downarrow{\scriptstyle j_0^* \oplus j_1^*} & & \Big\downarrow{\scriptstyle j^*} \\ H^{4k}(W_0) \oplus H^{4k}(W_1) & \xleftarrow[i_0^* \oplus i_1^*]{} & H^{4k}(X) \end{array}$$

where every morphism is an isomorphism for $0 < k < m + n$. Next note that $j_* : H_{4m+4m}(X) \to H_{4m+4n}(X, M)$ is injective. In particular, we have $j_*^{-1} \circ ((h_0)_* \oplus (h_1)_*)([W_0] - [W_1]) = [X]$ by the choice of the orientation of

X. The Pontrjagin numbers of X are therefore given by

$$\begin{aligned}
p_{k_1}\cdots p_{k_r}[X] &= \langle p_{k_1} \smile \cdots \smile p_{k_r}(X), [X]\rangle \\
&= \langle p_{k_1} \smile \cdots \smile p_{k_r}(X), j_*^{-1} \circ ((h_0)_* \oplus (h_1)_*)([W_0] - [W_1])\rangle \\
&= \langle (h_0^* \oplus h_1^*)(j^*)^{-1}(p_{k_1} \smile \cdots \smile p_{k_r}(X)), [W_0] - [W_1]\rangle \\
&= \langle (h_0^* \oplus h_1^*)(j^*)^{-1}(p_{k_1}(X)) \smile \cdots \\
&\qquad \cdots \smile (h_0^* \oplus h_1^*)(j^*)^{-1}(p_{k_r}(X)), [W_0] - [W_1]\rangle \\
&= \langle (j_0^* \oplus j_1^*)^{-1}(i_0^* \oplus i_1^*)(p_{k_1}(X)) \smile \cdots \\
&\qquad \cdots \smile (j_0^* \oplus j_1^*)^{-1}(i_0^* \oplus i_1^*)(p_{k_r}(X)), [W_0] - [W_1]\rangle \\
&= \langle (j_0^* \oplus j_1^*)^{-1}(p_{k_1}(W_0) \oplus p_{k_1}(W_1)) \smile \cdots \\
&\qquad \cdots \smile (j_0^* \oplus j_1^*)^{-1}(p_{k_r}(W_0) \oplus p_{k_r}(W_1)), [W_0] - [W_1]\rangle \\
&= \langle (j_0^*)^{-1}(p_{k_1}) \smile \cdots \smile (j_0^*)^{-1}(p_{k_r})(W_0), [W_0]\rangle \\
&\qquad - \langle (j_1^*)^{-1}(p_{k_1}) \smile \cdots \smile (j_1^*)^{-1}(p_{k_r})(W_1), [W_1]\rangle \\
&= q_{k_1}\cdots q_{k_r}[W_0] - q_{k_1}\cdots q_{k_r}[W_1],
\end{aligned}$$

as long as $k_1, \cdots, k_r < m + n$. In a similar manner, one can show that the diagram above still holds if one considers the degree $2m + 2n$ instead of $4k$ and rational coefficients. The bilinear form on the middle cohomology consequently splits as the difference of the bilinear forms on W_0 and W_1. The signature is therefore

$$\operatorname{sign}(X) = \operatorname{sign}(W_0) - \operatorname{sign}(W_1).$$

□

As a consequence, we have:

Corollary 7.2. All Pontrjagin numbers of X but $p_m p_n[X]$ and $p_{m+n}[X]$

vanish and the signature of X is equal to 0. Moreover,

$$\begin{aligned} p_m p_n[X] &= k_0 l_0 (2m-1)!(2n-1)!ab - k_1 l_1 (2m-1)!(2n-1)!ab \\ &= (k_0 l_0 - k_1 l_1)(2m-1)!(2n-1)!ab, \end{aligned}$$

if $m \neq n$ and

$$p_m^2[X] = 2(k_0 l_0 - k_1 l_1)(2m-1)!^2 ab,$$

if $m = n$.

Remark 7.3. The signature and the Pontrjagin numbers of X do not depend on the choice of the gluing map ϕ.

Recall from Proposition 5.9 that W_i admits a metric of positive scalar curvature which is of product form near the boundary M_i and can be deformed inside positive scalar curvature to a metric of positive Ricci curvature on M_i, for $i = 0, 1$. Denote by g_i the restriction of these metrics on M_i. Let $\phi_i : M \to M_i$ be an orientation-preserving diffeomorphism and let $h_i = \phi_i^*(g_i)$ be the pullback metric on M, $i = 0, 1$. This defines elements $[h_0]$ and $[h_1]$ in $\mathcal{M}_{Ric>0}(M)$.

Lemma 7.4. Suppose that the isometry classes $[h_0]$ and $[h_1]$ can be joined by a path of metrics in the moduli space $\mathcal{M}_{Ric>0}(M)$. Then, the $\hat{A}$-genus of X vanishes, i.e. $\hat{A}(X) = 0$.

Proof. Let $\gamma : [0,1] \to \mathcal{M}_{Ric>0}(M)$ be a path between $\gamma(0) = [h_0]$ and $\gamma(1) = [h_1]$ of metrics with positive Ricci curvature on M. By Theorem 6.3, we can lift this path to a path $\tilde{\gamma} : [0,1] \to \mathcal{R}_{Ric>0}(M)$ with $\tilde{\gamma}(0) = h_0$ and $\tilde{\gamma}(1)$ lies in the same Diff(M)-orbit as h_1, i.e. $\tilde{\gamma}(1) = \psi^* h_1$ for some diffeomorphism $\psi : M \to M$. We want ψ to be orientation-preserving, so if $\psi : M \to M$ is orientation reversing, we need to replace g_1 by its pullback under an orientation-reversing diffeomorphism of M_1 (the pullback of g_1 by

this orientation-reversing diffeomorphism still gives a representative of $[h_1]$ in $\mathcal{M}_{Ric>0}(M)$) to make up for this. We can therefore assume that ψ is orientation-preserving.
By construction, the Riemannian manifolds (M_i, g_i) and $(M, \tilde{\gamma}(i))$ are isometric via an orientation-preserving diffeomorphism for $i = 0, 1$.
Since positive Ricci curvature implies positive scalar curvature, the path $\tilde{\gamma}$ also lies in the space $\mathcal{R}_{scal>0}(M)$. Up to reparametrization, we can assume that the path is constant near the endpoints $\tilde{\gamma}(0) = h_0$ and $\tilde{\gamma}(1) = \psi^* h_1$.
The family of metrics $\{\tilde{\gamma}(t)\}_{t\in[0,1]}$ defines a metric on $M \times [0, 1]$ which is of positive scalar curvature on each slice $M \times \{t\}$ and is of product form near the boundary (since the path $\tilde{\gamma}$ is constant there). We can assume that the metric on $M \times [0, 1]$ is not only fiberwise but even globally of positive scalar curvature. Otherwise, one can stretch the interval $[0, 1]$ to an interval $[0, a]$ with $a >> 0$ very large, such that the metric on $M \times [0, a]$ obtained by the metrics $\tilde{\gamma}(t/a)$ is of positive scalar curvature, see [GL80, Lemma 3, Section 1, p. 430].
Since the metric on W_i and the metric on $M \times [0, 1]$ are of product form near their respective boundaries and these are isometric, the following space

$$\tilde{X} := W_0 \cup_{\phi_0^{-1}} (M \times [0, 1]) \cup_{\phi_1 \circ \psi} -W_1,$$

is a well-defined oriented Riemannian manifold without boundary of dimension $4m + 4n$ and the induced metric has positive scalar curvature. Furthermore, $\tilde{X}$ is compact and is spin because $H^2(\tilde{X}; \mathbb{Z}_2) \cong 0$ by using an appropriate Mayer-Vietoris sequence argument. It is also easily verified that $\tilde{X}$ and X are oriented diffeomorphic and therefore $\hat{A}(\tilde{X}) = \hat{A}(X)$. By Theorem 3.12, we conclude $\hat{A}(\tilde{X}) = \hat{A}(X) = 0$.

□

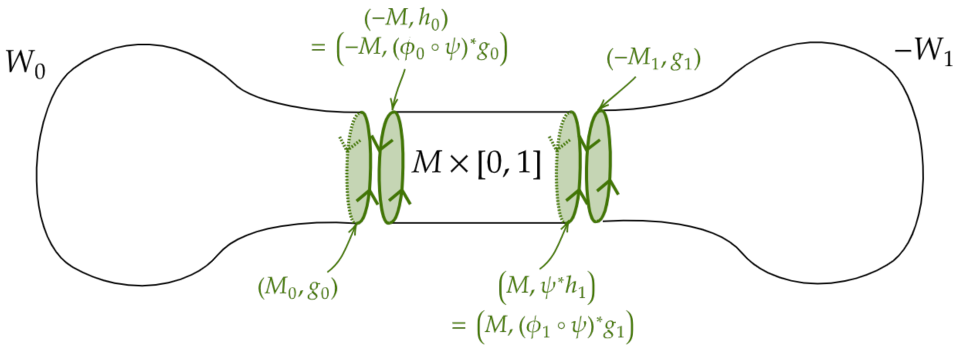

Figure 7.3: The Riemannian manifold $\tilde{X}$

Proposition 7.5 (Computation of $\hat{A}(X)$)**.**

$$\hat{A}(X) = c_{m,n}(k_0 l_0 - k_1 l_1)(2m-1)!(2n-1)!ab,$$

where $c_{m,n} \neq 0$ only depends on m and n.

Proof. Using the Hirzebruch signature theorem 3.10, we may express the top Pontrjagin number $p_{m+n}[X]$ as a linear combination of the other Pontrjagin numbers:

$$p_{m+n}[X] = (1 - \frac{s_m s_n}{s_{m+n}}) p_m p_n[X],$$

since by Corollary 7.2, $\sigma(X) = 0$.

Moreover, since almost all Pontrjagin numbers vanish, we have for the $\hat{A}$-genus of X:

$$\hat{A}(X) = \langle \hat{A}_{m+n}(p_1, \ldots, p_{m+n}), [X] \rangle = a_{m,n} p_m p_n[X] + a_{m+n} p_{m+n}[X].$$

For $m \neq n$, it follows from Proposition 3.13 that,

$$\hat{A}(X) = (a_m a_n - a_{m+n}) p_m p_n[X] + a_{m+n}(1 - \frac{s_m s_n}{s_{m+n}}) p_m p_n[X],$$

and then simplifying yields

$$\hat{A}(X) = (a_m a_n - a_{m+n}\frac{s_m s_n}{s_{m+n}})p_m p_n[X],$$

or half this quantity if $m = n$.

Let us compute the above quantity using Proposition 3.13:

$$\begin{aligned}
c_{m,n} &:= a_m a_n - a_{m+n}\frac{s_m s_n}{s_{m+n}} \\
&= \frac{B_m}{2(2m)!}\frac{B_n}{2(2n)!} + \\
&\frac{\cancel{B_{m+n}}}{\cancel{2(2m+2n)!}}\frac{\cancel{2^{2m}}(2^{2m-1}-1)B_m}{(2m)!}\frac{\cancel{2^{2n}}(2^{2n-1}-1)B_n}{(2n)!}\frac{\cancel{(2m+2n)!}}{\cancel{2^{2m+2n}}(2^{2m+2n-1}-1)\cancel{B_{m+n}}}, \\
&= \frac{B_m B_n}{2^2(2m)!(2n)!} + \frac{(2^{2m-1}-1)(2^{2n-1}-1)}{2(2^{2m+2n-1}-1)}\frac{B_m B_n}{(2m)!(2n)!}, \\
&= \frac{B_m B_n}{(2m)!(2n)!}\Big(\frac{1}{4} + \frac{(2^{2m-1}-1)(2^{2n-1}-1)}{2(2^{2m+2n-1}-1)}\Big).
\end{aligned}$$

We then easily see that this is nonzero.

In the case that $m = n$, the factor $1/2$ coming from Proposition 3.13 cancels against the 2 coming from Corollary 7.2. We therefore have in all cases:

$$\hat{A}(X) = c_{m,n}(k_0 l_0 - k_1 l_1)(2m-1)!(2n-1)!ab.$$

□

7.2 Main result

Recall from the Lemma 4.18 in Chapter 4 that we have a group homomorphism

$$\beta : \pi_{4m-1}(SO_{4n-1}) \otimes \pi_{4n-1}(SO_{4m-1}) \to \pi_0\mathrm{Diff}^+(S^{4m+4n-2}) \to \Gamma_{4m+4n-1}.$$

We denote by $\Xi_{4m+4n-1}$ the image group $im(\beta) \subset \Gamma_{4m+4n-1}$. By the Observation 5.2, every twisted sphere $M \in \Xi_{4m+4n-1}$ can be viewed as the boundary of a plumbed manifold W.

Theorem 7.6 (Main theorem)**.** The moduli space of Riemannian metrics $\mathcal{M}_{Ric>0}(M)$ of positive Ricci curvature of any homotopy sphere $M \in \Xi_{4m+4n-1}$ has infinitely many path-components, $1 \leq m \leq n < 2m$.

Proof. Recall from Proposition 4.25 and Remark 5.8 that $\Gamma_{4m+4n-1}$ is finite but the tensor group $\pi_{4m-1}(SO_{4n-1}) \otimes \pi_{4n-1}(SO_{4m-1})$ is not. It follows from the first isomorphism theorem that every twisted sphere M in $\Xi_{4m+4n-1}$ can be viewed as the boundary of a infinitely many plumbed manifolds $W_{k,l}$ with Pontrjagin number $q_m q_n[W_{k,l}] = kl(2m-1)!(2n-1)!ab$ which carry a metric with positive scalar curvature that is of product form near $\partial W_{k,l} = M$ and that can be deformed inside positive scalar curvature to a metric with positive Ricci curvature on $\partial W_{k,l} = M$ (see Proposition 5.9). Since there are infinitely many such plumbed manifolds, there are infinitely many integers k_0, l_0 and k_1, l_1 for which the manifolds W_{k_0,l_0} and W_{k_1,l_1} have a boundary diffeomorphic to M. In particular, there are infinitely many such integers with $k_0 l_0 \neq k_1 l_1$.
Now, let X_{k_0,l_0,k_1,l_1} be obtained by gluing W_{k_0,l_0} and $-W_{k_1,l_1}$ as in the beginning of this chapter. By the Proposition 7.5,

$$\hat{A}(X) = c_{m,n}(k_0 l_0 - k_1 l_1)(2m-1)!(2n-1)!ab,$$

where $c_{m,n}, a, b$ are nonzero.
As a consequence, there are infinitely many choices of manifolds W_{k_0,l_0} and W_{k_1,l_1} with boundary diffeomorphic M, such that $\hat{A}(X_{k_0,l_0,k_1,l_1}) \neq 0$. Lemma 7.4 implies that the metrics induced on M by W_{k_0,l_0} and W_{k_1,l_1} do not lie in the same path-component in $\mathcal{M}_{Ric>0}(M)$. This shows that $\mathcal{M}_{Ric>0}(M)$ admits infinitely many path-components. □

Since the equivalence class of metrics in $\mathcal{M}_{Ric>0}(M)$ also have positive scalar curvature, the moduli space $\mathcal{M}_{scal>0}(M)$ has infinitely many path-components as well.

On a final note, one might ask if our result extends that of David Wraith, which states that the moduli space $\mathcal{M}_{Ric>0}(\Sigma^{4k-1})$ has infinitely many path-components where Σ^{4k-1} is any homotopy sphere bounding a parallelizable manifold of dimension $4k$ with $k > 1$, that is to say any element $\Sigma^{4k-1} \in bP_{4k}$. Even though it is true that for any sphere $M \in \Xi_{4m+4n-1}$ with $1 \leq m \leq n < 2m$, there are infinitely many plumbed manifolds $W_{k,l}$ which M bounds and where at least one of k or l is nonzero - and as such $W_{k,l}$ has at least one non-trivial Pontrjagin class and cannot therefore be parallelizable -, this does *not* imply that M cannot bound a parallelizable manifold. Namely, if we refer to Table 4.1, any of the 28 different twisted spheres in $\Gamma_7 \cong \Theta_7$ and any of the 992 twisted spheres in $\Gamma_{11} \cong \Theta_{11}$ are boundaries of a parallelizable manifold. On the other hand, if both $k = l = 0$, all of the Pontrjagin classes of the manifold $W_{k,l}$ are trivial, but this does not imply that $W_{k,l}$ is parallelizable. It is therefore not clear whether the spheres in $\Xi_{4m+4n-1}, 1 \leq m \leq n < 2m$ are new examples of spheres for which the moduli space of metrics with positive Ricci curvature has infinitely many path-components.

Appendix A

Bernoulli numbers and Taylor series

We have seen that the Bernoulli numbers have cropped up all throughout this paper, in particular appearing in the power series defining the L- and $\hat{A}$-genera. Their number-theoretic properties turn out to play a role of utmost importance in the study of the diffeomorphism structures of the sphere and the oriented cobordism ring. As an example, recall from section about the h-cobordism that the h-cobordism group Θ_n of homotopy spheres has a cyclic subgroup bP_{n+1} consisting of homotopy spheres bounding parallelizable manifolds. Milnor and Kervaire showed in [KM63] that for $n \geq 2$, the order of the subgroup bP_{4n} is equal to

$$2^{2n-2}(2^{2n-1}-1) \text{ numerator } \left(\frac{4B_n}{n}\right).$$

For a more detailed discussion of the Bernoulli numbers, we refer to [MS74, Appendix B].

Recall that the Bernoulli numbers are defined to be the coefficients appearing in the Taylor expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k}, \quad |z| < 2\pi,$$

where the first Bernoulli numbers are

$$B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ B_4 = \frac{1}{30},\ B_5 = \frac{5}{66}, \ldots$$

Unlike the usual convention for Bernoulli numbers, this definition has the advantage that no Bernoulli number is zero and and each of them is always positive.

Let us now compute the Taylor series needed for the L- and $\hat{A}$-genera.

Lemma A.1. The Taylor series expansion for $\frac{z}{\tanh z}$ is given by

$$\frac{z}{\tanh z} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} 2^{2k} \frac{B_k}{(2k)!} z^{2k},$$

for $|z| < \pi$.

Proof. Notice that for $|z| < \pi$, we have

$$\frac{z}{\tanh z} = z \frac{e^{2z} + 1}{e^{2z} - 1} = \frac{2z}{e^{2z} - 1} + z \frac{e^{2z}}{e^{2z} - 1} - \frac{z}{e^{2z} - 1} = \frac{2z}{e^{2z} - 1} + z.$$

Therefore, using the above Taylor series for $\frac{2z}{e^{2z}-1}$, we obtain

$$\begin{aligned} \frac{z}{\tanh z} &= z + 1 - \frac{2z}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} (2z)^{2k} \\ &= 1 + \sum_{k=1}^{\infty} (-1)^{k-1} 2^{2k} \frac{B_k}{(2k)!} z^{2k}, \end{aligned}$$

which is precisely what was stated. □

Lemma A.2. The Taylor series expansion for $\frac{z}{\sinh z}$ is given by

$$\frac{z}{\sinh z} = 1 + \sum_{k=1}^{\infty} (-1)^k (2^{2k} - 2) \frac{B_k}{(2k)!} z^{2k},$$

for $|z| < \pi$.

Proof. As before, if $|z| < \frac{\pi}{2}$, we have

$$\begin{aligned}
\frac{1}{\tanh z} - \frac{1}{\tanh 2z} &= \frac{e^z + e^{-z}}{e^z - e^{-z}} - \frac{e^{2z} + e^{-2z}}{e^{2z} - e^{-2z}} \\
&= \frac{(e^z + e^{-z})(e^{2z} - e^{-2z}) - (e^{2z} + e^{-2z})(e^z - e^{-z})}{(e^z - e^{-z})(e^{2z} - e^{-2z})} \\
&= \frac{2e^z - 2e^{-z}}{(e^z - e^{-z})(e^{2z} - e^{-2z})} \\
&= \frac{2}{e^{2z} - e^{-2z}} \\
&= \frac{1}{\sinh 2z}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{2z}{\sinh 2z} &= \frac{2z}{\tanh z} - \frac{2z}{\tanh 2z} \\
&= 2 + 2\sum_{k=1}^{\infty}(-1)^{k-1}2^{2k}\frac{B_k}{(2k)!}z^{2k} - 1 - \sum_{k=1}^{\infty}(-1)^{k-1}2^{2k}\frac{B_k}{(2k)!}(2z)^{2k} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^{k-1}2^{2k}\frac{B_k}{(2k)!}z^{2k}(2 - 2^{2k}) \\
&= 1 + \sum_{k=1}^{\infty}(-1)^{k}(2^{2k} - 2)\frac{B_k}{(2k)!}2^{2k}z^{2k} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^{k}(2^{2k} - 2)\frac{B_k}{(2k)!}(2z)^{2k}.
\end{aligned}$$

Replacing $2z \rightsquigarrow z$ yields the desired formula. □

As a consequence, the power series associated to the L-genus is

$$\frac{\sqrt{z}}{\tanh\sqrt{z}} = 1 + \sum_{k=1}^{\infty}(-1)^{k-1}2^{2k}\frac{B_k}{(2k)!}z^k,$$

and the power series associated to the $\hat{A}$-genus is

$$\frac{\sqrt{z}/2}{\sinh\sqrt{z}/2} = 1 + \sum_{k=1}^{\infty}(-1)^{k}2^{-2k}(2^{2k} - 2)\frac{B_k}{(2k)!}z^k.$$

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