Variational properties of the discrete Hilbert-Einstein functional

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Colloquium on discrete curvature
CIRM, Luminy, November 18–22th, 2013

Outline

The functional

- Dimension 2: Gauss-Bonnet theorem
- Dimension 3

Infinitesimal rigidity of convex polyhedra

- Extrinsic definition
- Intrinsic reformulation

Alexandrov's embedding theorem

Rigidity of "weakly convex" polyhedra

The Hilbert-Einstein functional

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Keep topology fixed, vary the Riemannian metric on M.

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- For $n \geq 3$, critical points of S are Ricci-flat metrics: $\mathrm{Ric}_g = 0$.
- Critical points of S restricted to

$$Met_1(M) := \{g \mid \mathrm{Vol}(g) = \int_M \mathrm{dvol}_g = 1\}$$

  are characterized by $\mathrm{Ric}_g = \lambda g$ and called Einstein metrics.

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- In general relativity, Einstein metrics describe the geometry of spacetime.
- Existence of critical points is hard to prove, since S is neither convex nor concave.

Gauss-Bonnet theorem

If $\dim M = 2$, then $\mathrm{scal}_g = 2K_g$ is the Gauss curvature of the metric g, and the Gauss-Bonnet theorem says

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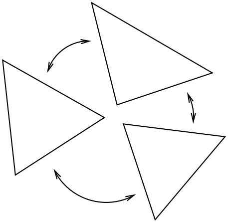
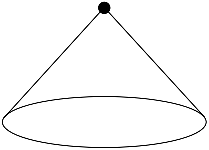
To make the Hilbert-Einstein functional interesting, need $\dim M \geq 3$.

Euclidean cone-metrics

Let dim $M = 2$. The discrete analog of a Riemannian metric on M is gluing M from Euclidean triangles (without bothering how and whether it can be realized in $\mathbb{R}^3$). There appear conical singularities.





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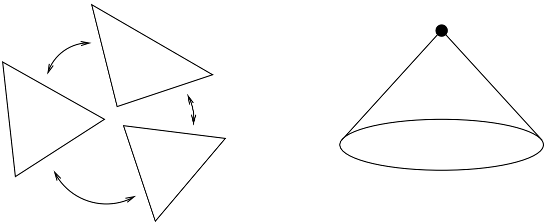

The curvature of a conical singularity is its angular defect $\kappa_i = 2\pi - \omega_i$. This intrinsic quantity has also an extrinsic interpretation:

$$\kappa_i = \text{the exterior solid angle at } i$$

(no matter how a neighborhood of the singularity is embedded in $\mathbb{R}^3$).
length(∂C) + area$(C^*) = 2\pi$ for every convex $C \subset \mathbb{S}^2$

The discrete Gauss-Bonnet theorem

Let M be equipped with a Euclidean cone-metric, κ_i be the defects of the cone singularities. Then

$$\sum_i \kappa_i = 2\pi\chi(M)$$

Proof: add angles in each triangle and apply Euler's formula $V - E + F = \chi(M)$.

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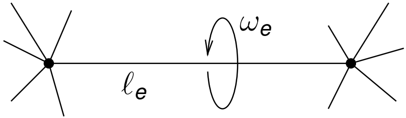
Extrinsic formulation (for a compact convex polyhedron):

$$\sum_i \text{ext. solid angle}_i = 4\pi = 2\pi\chi(\mathbb{S}^2)$$

Proof: bring the apices of all cones to a common origin.

The discrete Hilbert-Einstein functional

Fix a triangulation T of a compact 3-manifold M.
By assigning edge lengths $e \mapsto \ell_e$, get a Euclidean cone-metric on M: gluing of tetrahedra, ignoring closing conditions around the edges.
Let ω_e be the total angle around the edge e.



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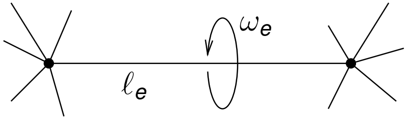
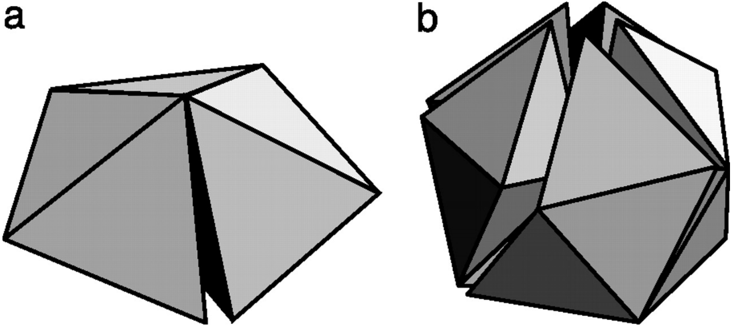


The Hilbert-Einstein functional on the space of Euclidean cone-metrics

$$S(\ell) = \sum_e \ell_e(2\pi - \omega_i)$$

Regge'61 *"General relativity without coordinates"* (the analog in dim 4).

Example of a cone-metric on $\mathbb{S}^3$: the 600-cell



Conway J H , and Torquato S PNAS 2006;103:10612-10617

Gluing can be done in $\mathbb{R}^4$; the polyhedron closes up and becomes the 600-cell.
Compare this to the icosahedron: 5 triangles at each vertex yield a 3-polyhedron with 20 faces.

Critical points of the discrete HE

$$S(\ell) = \sum_e \ell_e \kappa_e,$$

where $\kappa_e := 2\pi - \omega_e$ is the curvature at the edge e

Theorem (Schläfli; Regge)

$$\frac{\partial S}{\partial \ell_e} = \kappa_e$$

Thus, at a critical point $\kappa_e = 0$ for all e; the metric is flat.

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Two applications:

- Finding critical points $\rightsquigarrow$ flat metrics (or metrics of constant curvature).
- Non-degeneracy of critical points $\rightsquigarrow$ rigidity of flat metrics.

The boundary term

The HE functional on 3-manifolds with boundary

- smooth case (Riemannian metric g):

$$S(g) = \frac{1}{2} \int_M \mathrm{scal}_g \, \mathrm{dvol}_g + 2 \int_{\partial M} H \, \mathrm{darea}_g,$$

where $H = \frac{\kappa_1 + \kappa_2}{2}$ is the mean curvature.

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- polyhedral case (Euclidean cone-metric given by (T, ℓ):

$$S(T, \ell) = \sum_{e \in \mathcal{E}_i(T)} \ell_e (2\pi - \omega_i) + \sum_{e \in \mathcal{E}_\partial(T)} \ell_e (\pi - \theta_e),$$

  where θ_e is the dihedral angle at a boundary edge e.

The boundary term is the total mean curvature (cf. the Steiner formula).

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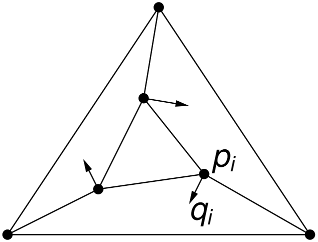
[Cheeger-Müller-Schrader'84]: every Riemannian g can be approximated by a sequence $(T^{(n)},\ell^{(n)})$ s. t. $S(T^{(n)},\ell^{(n)}) \to S(g)$.

Infinitesimal rigidity for frameworks and polyhedra

Infinitesimal isometric deformation:
a vector q_i at each vertex p_i such that

$$\langle q_i - q_j, p_i - p_j \rangle = 0 \text{ for every edge } p_i p_j$$

Deformation $(q_i)_{i=1}^n$ is called trivial if it is the derivative of a rigid motion.



Definition

A framework is called infinitesimally rigid if every its isometric deformation is trivial.

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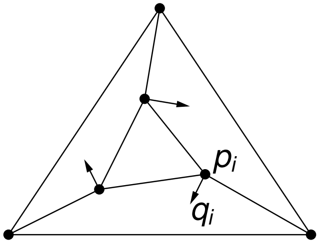

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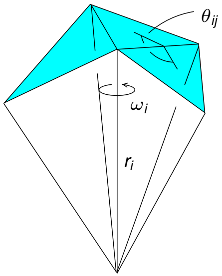
Theorem (Legendre-Cauchy-Dehn)

The skeleton of a convex polyhedron with triangular faces is infinitesimally rigid.

Infinitesimal rigidity: Reformulation

- Choose a point a inside P
- Subdivide P into triangular pyramids with apex a
- Vary the radial edge lengths r_i while keeping the boundary edge lengths ℓ_{ij} constant



The angles ω_i around the edges ap_i might change, so that curvatures $\kappa_i = 2\pi - \omega_i$ appear. Get Euclidean cone-metric inside P.

Infinitesimal rigidity: Reformulation

- Choose a point *a* inside *P*
- Subdivide *P* into triangular pyramids with apex *a*
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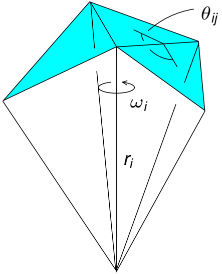

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One way to avoid curvatures: move *a* while keeping p_i fixed in $\mathbb{R}^3$. Any other way gives a non-trivial infinitesimal isometric deformation.

Corollary

P is infinitesimally rigid $\Leftrightarrow \dim\ker\left(\frac{\partial^2 S}{\partial r_i \partial r_j}\right) = \dim\ker\left(\frac{\partial \kappa_i}{\partial r_j}\right) = 3$

Proof of the infinitesimal rigidity

Theorem (I., *Canad. J. Math*, to appear)

The matrix $\left(\frac{\partial^2 S}{\partial r_i \partial r_j}\right)$ *has signature* $(+, 0, 0, 0, -, \ldots, -)$.

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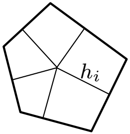
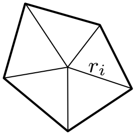
Duality:

$$\left(\frac{\partial^2 S}{\partial r_i \partial r_j}\right)(P) = \left(\frac{\partial^2 \mathrm{Vol}}{\partial h_i \partial h_j}\right)(P^*),$$

where P^* is the polar dual of P.





The signature of $\left(\frac{\partial^2 \mathrm{Vol}}{\partial h_i \partial h_j}\right)$ is provided by the second Minkowski inequality for mixed volumes (plus the characterization of the equality case proved by Bol).

The theorem

Theorem (Alexandrov'1942)

Every Euclidean cone-metric on the sphere with singular points of positive curvature can be embedded in $\mathbb{R}^3$ *(as a convex polyhedron).*

- The surface of a convex polyhedron can be developed into the plane; Alexandrov's theorem says: everything that "looks like" a development can be glued into a convex polyhedron.

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- The surface of a convex polyhedron can be developed into the plane; Alexandrov's theorem says: everything that "looks like" a development can be glued into a convex polyhedron.
- There is a smooth version (Weyl's problem): sphere with Riemannian metric of positive Gauss curvature embeds isometrically in $\mathbb{R}^3$ as a smooth convex surface.
- Alexandrov stated and proved his theorem in order to solve Weyl's problem by approximation.

Idea of the proof

Different proofs:

- Alexandrov's proof is non-constructive (but powerful).
- Volkov'1955: a constructive proof using Euclidean cone-metrics.
- Bobenko-I.'2008: a new constructive proof based on variational properties of the Hilbert-Einstein functional.

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Idea of the proof:

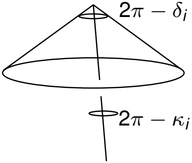
- Triangulate the sphere (vertices at singular points).
- Assign radius r_i to every vertex and glue a Euclidean cone-manifold from pyramids over faces.
- Deform radii so as to decrease the curvatures. Possibly need to change the triangulation in the process.

Where to start: Delaunay triangulation, $r_i = R$ large.

What makes the proof work

Lemma

If $0 < \kappa_i < \delta_i$ *for all i, then* $\det\left(\frac{\partial \kappa_i}{\partial r_j}\right) \neq 0$.



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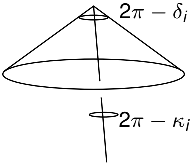

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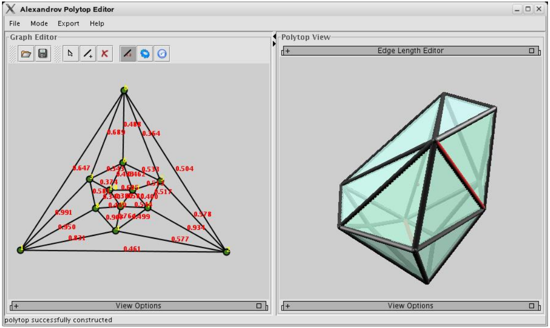
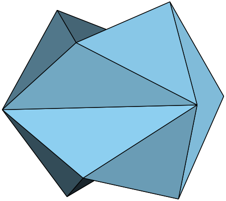
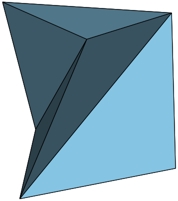


Screenshot of a program by Stefan Sechelmann.

Infinitesimal rigidity of weakly convex polyhedra

There exist nonconvex infinitesimally flexible polyhedra.





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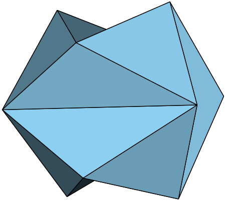

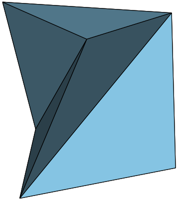

Theorem (I.-Schlenker’10)

If the vertices of a polyhedron P are in convex position and P and conv $P \setminus P$ *can be triangulated without adding new vertices, then P is infinitesimally rigid.*

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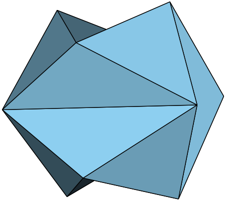

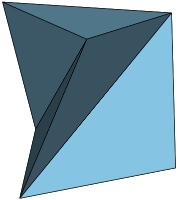

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Proof: $\left(\frac{\partial^2 S}{\partial r_i \partial r_j}\right)$ for the triangulation of conv P is positive definite.
In P, some interior edges become boundary edges $\Rightarrow$ restrict this quadratic form to a linear subspace.
The restriction is again positive, hence the matrix is non-degenerate.